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Limit Theorems for the Donsker Delta Function: An Example

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Abstract. Limit theorems of the type of the law of large numbers and the central limit theorem are established (in the sense of Hida distributions) for the composition of the Dirac distribution with the stochastic exponential of Brownian motion.

1. Introduction and Background

In [PS 92a, PS 92b] we have proved certain limit theorems for the composition $\delta_x \circ Y(t)$ of the Dirac distribution at $x \in I\!R$ with a real-valued process $Y(t), t > 0$. These limit theorems hold in the sense of Hida distributions and resemble the law of large numbers and the central limit theorem. Our motivation to study these limits is the attempt to understand propagation of chaos and the fluctuation problem for interacting diffusions in the framework of generalized random variables.

The paper [PS 92a] deals only with one dimensional Gaussian processes Y, while in [PS 92b] the general case of non-degenerate diffusions Y is treated. The purpose of the present note is to make the arguments in [PS 92b] more transparent in the simplest non-trivial example: the stochastic exponential

$$Y(t) =: e^{B(t)} :\equiv e^{B(t)-\frac{t}{2}}, \tag{1.1}$$

which is the solution of

$$dY(t) = Y(t)dB(t), \quad t > 0, \tag{1.2}$$

with $Y(0) = 1$.

We use the framework of white noise analysis (e.g., [HKP 92, Po 92]) for our discussion. In particular, we choose $(\mathcal{S}'(I\!R), \mathcal{B}, \mu)$ as the underlying probability space, where $\mathcal{S}'(I\!R)$ is the Schwartz space of tempered distributions, $\mathcal{B}$ is the σ-algebra generated by the cylinder sets, and μ is the centered Gaussian measure whose covariance is determined

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by the inner product $(\cdot,\cdot)$ of $L^2(I\!R, dt)$ (dt denoting Lebesgue measure). We denote by $X. : \mathcal{S}(I\!R) \to L^2(\mu), X_\xi(\omega) =<\omega,\xi>, \omega \in \mathcal{S}'(I\!R), \xi \in \mathcal{S}(I\!R)$, the canonical coordinate process. It is easy to see that $X.$ has a continuous extension from $\mathcal{S}'(I\!R)$ to $L^2(I\!R)$. In particular we may form $X_{1_{[0,t)}} \equiv B(t), t \geq 0$, which is a Brownian motion starting at zero.

It is clear that Donsker's delta function of Brownian motion $\delta_x(B(t))$, or more generally $\delta_x(Z)$, where Z is a certain random variable, can not be considered as a regular random variable. However, such expressions make sense as generalized random variables in various spaces. The two most prominent spaces of generalized random variables are the Meyer–Watanabe space $\mathcal{D}^*$ and the space $\mathcal{S}'(I\!R)$ of Hida distributions. (For the construction, properties and applications of $\mathcal{D}^*$ we refer to [Me 83, Wa 83]; – for a formulation over the white noise probability space cf. also [HKP 92]. The space $\mathcal{S}'(I\!R)$ was for example discussed in [PS 91], and we refer also to [HKP 92].) Here we note that $\mathcal{D}^* \subset \mathcal{S}'(I\!R)$. The LLN that we are aiming at can be discussed in $\mathcal{D}^*$ and $\mathcal{S}'(I\!R)$, while for the CLT it turns out that $\mathcal{S}'(I\!R)$ is the suitable space. Therefore we shall consider only $\mathcal{S}'(I\!R)$ here.

It is convenient to study elements Φ in $\mathcal{S}'(I\!R)$ through their S–transform given by

$$S\Phi(\xi) := e^{-\frac{1}{2}|\xi|_2^2} < \overline{\Phi}, e^{<\cdot,\xi>} >, \quad \xi \in \mathcal{S}'(I\!R),$$

where $\bar{\cdot}$ denotes complex conjugation, and $|\cdot|_2$ is the norm of $L^2(I\!R, dt)$. Moreover, $<\cdot,\cdot>$ denotes the canonical dual pairing between $\mathcal{S}'(I\!R)$ and $\to$. The following result has been shown in [PS 91].

Theorem 1.1. (a) Let $\Phi \in \mathcal{S}'(I\!R)$ and set $F := S\Phi$. Then F has a ray entire extension, i.e., for all $\eta, \xi \in \mathcal{S}(I\!R)$, the mapping $\lambda \mapsto F(\eta + \lambda\xi), \lambda \in I\!R$, has an entire analytic extension. Moreover, there exist $K_1, K_2 > 0, \alpha, \beta \in I\!N_0$, so that for all $\xi \in \mathcal{S}'(I\!R), z \in \mathbb{C}$,

$$|F(z\xi)| \leq K_, e^{K_2|z|^2|\xi|^2_{\alpha,\beta}}, \tag{1.3}$$

where $|\cdot|_{\alpha,\beta}$ is a standard Schwartz space norm. Conversely, if F is a complex–valued function on $\mathcal{S}'(I\!R)$ which has a ray entire extension, and if there exist $K_1, K_2 > 0, \alpha, \beta \in I\!N_0$, so that (1.1) holds for all $\xi \in \mathcal{S}(I\!R), z \in \mathbb{C}$, then there exists a unique $\Phi \in \mathcal{S}'(I\!R)$ with $S\Phi = F$.

(b) . Assume that $\{F_n, n \in I\!N\}$ is a sequence of functions on $\mathcal{S}'(I\!R)$ which have ray entire extensions and which are pointwise Cauchy. Suppose furthermore that (1.3) holds for $F_n, n \in I\!N$, uniformly in $n \in I\!N$, and let $\Phi_n := S^{-1}F_n \in \mathcal{S}'(I\!R)$. Then there exists a unique $\Phi \in \mathcal{S}'(I\!R)$, so that $\Phi_n \to \Phi$ strongly in $\mathcal{S}'(I\!R)$.

If we want to apply part (b) of the preceding theorem to control the convergence of sums of terms like $\delta_x \circ Y_m, \{Y_m, m \in I\!N\}$ being a sequence of independent random variables, then we need a formula for $S(\delta_x \circ Y_m)$. (The fact that $\delta_x \circ Y_m \in \mathcal{S}'(I\!R)$ follows from Watanabe's result in [Wa 89] for non–degenerate $Y_m \in \mathcal{D}$.) A suitable formula can be derived by using a standard technique from Malliavin calculus. For the example of this paper this is carried out in Section 2. In Section 3 we use this formula in order to obtain the desired LLN and CLT in the sense of Hida distributions.

2. Computation of an S–Transform

Let $f \in L^2(\mathbb{R}, dt), f \neq 0$, and consider $\delta_x \circ Y_f, x > 0$, where

$$Y_f =: e^{X_f} :\equiv e^{<\cdot,f> - \frac{1}{2}|f|_2^2}.$$

It is straightforward to show that Y_f belongs to the space $\mathcal{D}$ of smooth functions in the sense of Meyer–Watanabe, and that Y_f has a non– degenerate Malliavin covariance matrix. Thus, by Watanabe's theorem [Wa 83] we have $\delta_x \circ Y_f \in \mathcal{D}^* \subset \mathcal{S}'(\mathbb{R})$. (A different proof that $\delta_x \circ Y_f \in \mathcal{S}'(\mathbb{R})$ is done by the following computation and Theorem 1.1.a.)
Let $g \in C_b^1(\mathbb{R})$ and denote by D_f the Gâteaux derivative in direction f. Then we obtain from the chain rule the relation

$$g' \circ Y_f = |f|_2^{-2} Y_f^{-1} D_f\, g \circ Y_f. \tag{2.1}$$

If $\varphi \in \rightarrow$, then we obtain the following formula

$$< g' \circ Y_f, \varphi >= |f|_2^{-2} < g \circ Y_f, Y_f^{-1}[(X_f + |f|_2^2)\varphi - D_f\varphi] >,$$

where we made use of the equation $D_f^* = X_f - D_f$ for the adjoint D_f^* of D_f in $L^2(\mu)$, and of the product and chain rules for D_f. Also, the fact that Y_f^{-1} belongs to $\mathcal{D}$ was used. Now choose $\varphi =: e^{X_\xi} :, \xi \in \mathcal{S}(\mathbb{R})$. Then

$$S(g' \circ Y_f)(\xi) = |f|_2^{-2} \int (g \circ Y_f) Y_f^{-1} (X_f + |f|_2^2 - (f, \xi)) : e^{X_\xi} : d\mu. \tag{2.2}$$

Note that in the sense of distributions we have $\delta_x = \frac{d}{dx} 1_{[x,+\infty]}$. Using this together with a simple limit argument (cf. [PS 92b] for details) we obtain the following formula:

$$S(\delta_x \circ Y_f)(\xi) = |f|_2^{-2} \int\limits_{Y_f \geq x} Y_f^{-1} (X_f + |f|_2^2 - (f, \xi)) : e^{X_\xi} : d\mu. \tag{2.3}$$

Since $Y_f = \exp(X_f - \frac{1}{2}|f|_2^2)$, the last integral is easily reduced to a one–dimensional integral by projecting ξ onto the subspace in $L^2(\mathbb{R}, dt)$ spanned by f:

Lemma 2.1. Let $f \in L^2(\mathbb{R}, dt), f \neq 0, x > 0$. Then for all $\xi \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned} S(\delta_x \circ Y_f)(\xi) &= |f|_2^{-2} e^{\frac{1}{2}(|f|_2^2 - |f|_2^{-2}(\xi,f)^2)} \,. \\ &\cdot \int_{\ln x + \frac{1}{2}|f|_2^2}^{\infty} (u + |f|_2^2 - (f,\xi)) e^{-(1 - |f|_2^{-2}(\xi,f))u - \frac{1}{2}|f|_2^{-2}u^2)} \frac{du}{\sqrt{2\pi |f|_2^2}}. \end{aligned} \tag{2.4}$$

Remark. Consider the general case where Y_f above is replaced by a random variable in $\mathcal{D}$ with non-degenerate Malliavin covariance matrix. Then up to equation (2.3) this case can be handled in the same way (with a slightly more complicated formula than (2.3)). However, a reduction to a one-dimensional integral like in (2.4) seems impossible. On the other hand, for the LLN and the CLT this is unnecessary, cf. [PS 92b]. Also we note that from (2.3) — and the corresponding formula for the general case — one obtains an expression for the S-transform for the composition of a non-degenerate random variable in $\mathcal{D}$ with an arbitrary continuous, polynomially bounded function, see [PS 92b].

3. LLN and CLT

In the classical LLN and CLT one considers sequences of independent random variables. Although there exist notions of independence for generalized random variables (see, e.g., [Am 92] and the contribution of S. Amine to these proceedings), we take a more heuristic standpoint here. Namely, since we work only with the composition of δ_x with a random variable Y, we shall consider for the LLN and CLT compositions of δ_x with independent copies of Y. It is convenient to produce these independent copies of Y on the same probability space. We have shown in [PS 92a] that for a CLT (in the sense of Hida distributions) one has to choose appropriate versions of these copies. We make the same choice in the present paper by setting for $n \in \mathbb{N}$, $m = 0, 1, ..., n-1$, $t \in [0, 1]$,

$$B_{n,m}(t) := \sqrt{n} < \cdot, 1_{[\frac{m}{n}, \frac{m+t}{n})} >,$$
$$Y_{n,m}(t) := e^{B_{n,m}(t) - t/2}.$$

It is easy to see that for $n \in \mathbb{N}$, $\{B_{n,m}(t),\ m = 0, 1, ..., n-1\}$ and $\{Y_{n,m}(t),\ m = 0, 1, ..., n-1\}$ are families of independent copies of $B(t)$, $Y(t)$ respectively.

From Lemma 2.1 we get immediately the following formula.

Lemma 3.1. Let $n \in \mathbb{N}$, $m = 0, 1, ..., n-1$, $x \geq 0$, $t \in (0, 1]$, and $\xi \in S(\mathbb{R})$. Then

$$\begin{aligned} S(\delta_x \circ Y_{n,m}(t))(\xi) &= t^{-1} e^{\frac{t}{2} - \frac{1}{2tn}(\int_0^t \xi_{n,m}(s)ds)^2} \cdot \\ &\cdot \int_{\ln x + \frac{1}{2}t}^{\infty} (u + t - \frac{1}{\sqrt{n}} \int_0^t \xi_{n,m}(s)ds) e^{-(1 - \frac{1}{t\sqrt{n}} \int_0^t \xi_{n,m}(s)ds)u - \frac{1}{2t} u^2} \frac{du}{\sqrt{2\pi t}}, \end{aligned} \tag{3.1}$$

where $\xi_{n,m}(s) = \xi(\frac{m+s}{n})$.

The fact that $\xi_{n,m}$ is uniformly bounded in n and m and the dominated convergence theorem allow to conclude from (3.1), that $S(\sigma_x \circ Y_{n,m}(t))(\xi)$ converges uniformly in m to $S(\delta_x \circ Y_{n,m}(t))(0) = \mathbb{E}\, \delta_x \circ Y_{n,m}(t) = \mathbb{E}\, \delta_x \circ Y(t)$ as n tends to infinity. Consequently we find that as n tends to infinity the S-transform of

$$\frac{1}{n} \sum_{m=0}^{n-1} \delta_x \circ Y_{n,m}(t) \tag{3.2}$$

converges to $\mathbb{E}\,\delta_x \circ Y(t)$. In order to obtain the strong convergence of (3.2) to $\mathbb{E}(\delta_x \circ Y(t))$ we want to apply Theorem 1.1.(b). To this end, we note that the ray analytic extension of (3.1) at zero is given by

$$S(\delta_x \circ Y_{n,m}(t))(\xi) = t^{-1} e^{\frac{t}{2} - \frac{1}{2tn} z^2 (\int_0^t \xi_{n,m}(s)ds)^2} \cdot$$

$$\cdot \int_{\ln x + \frac{1}{2}t}^{\infty} (u + t - \frac{z}{\sqrt{n}} \int_0^t \xi_{n,m}(s)ds) e^{-(1 - \frac{z}{t\sqrt{n}} \int_0^t \xi_{n,m}(s)ds) u - \frac{1}{2t} u^2} \frac{du}{\sqrt{2\pi t}},$$

where $z \in \mathbb{C}, \xi \in \mathcal{S}(\mathbb{R})$. It is now straightforward to derive the following (rough but sufficient) estimate:

$$|S(\delta_x \circ Y_{n,m}(t))(z\xi)| \leq (t^{-1/2} + 1 + \frac{|z|}{\sqrt{n}} |\xi|_\infty) e^{\frac{5}{2}t} e^{\frac{5}{2}\frac{t}{n}|z|^2|\xi|^2_\infty}.$$

Clearly this implies that there exist $K_1(t) > 0$ and $K_2 > 0$ so that for all $z \in \mathbb{C}, \xi \in \mathcal{S}(\mathbb{R})$ and $n \in \mathbb{N}$,

$$|S(\frac{1}{n}\sum_{m=0}^{n-1} \delta_x \circ Y_{n,m}(t))(z\xi)| \leq K_1(t) e^{K_2 t |z|^2 |\xi|^2_\infty}.$$

Now we can apply Theorem 1.1.(b) to conclude the following result which resembles the LLN:

Theorem 3.1. Let $x > 0, t > 0$. Then

$$\frac{1}{n}\sum_{m=0}^{n-1} \delta_x \circ Y_{n,m}(t)$$

converges strongly in $\mathcal{S}'(\mathbb{R})$ to $\mathbb{E}(\delta_x \circ Y(t)) \equiv \langle \delta_x \circ Y(t), 1 \rangle$ as n tends to infinity.

Using the trivial fact that $\mathbb{E}(\delta_x \circ Y(t)) = S(\delta_x \circ Y(t))(0)$ and Taylor's theorem, we obtain from (3.1.)

$$S(\delta_x \circ Y_{n,m}(t))(\xi) - \mathbb{E}(\delta_x \circ Y(t)) = \frac{1}{\sqrt{n}\,t}\Big(\int_0^t \xi_{n,m}(s)ds\Big)\frac{1}{t}e^{t/2}.$$

$$\cdot \int_{\ln x + 1/2t}^{\infty} (u^2 + ut - t) e^{-u - \frac{1}{2t}u^2} \frac{du}{\sqrt{2\pi t}} + 0(\frac{1}{n}),$$

where the order symbol depends on ξ, t, x and m and is uniformly bounded in m. Let us denote

$$\sigma(t,x) := \frac{1}{t} e^{t/2} \int_{\ln x + \frac{1}{2t}}^{\infty} (u^2 + ut - t) e^{-u - \frac{1}{2t}u^2} \frac{du}{\sqrt{2\pi t}}. \tag{3.3}$$

Observe that

$$\sum_{m=o}^{n-1} \frac{1}{nt}(\int_0^t \xi_{n,m}(s)ds)$$

is a Riemann approximation to $\int_0^1 \xi(s)ds$, which is the S–transform of $B(1)$ at $\xi \in \mathcal{S}(\mathbb{R})$. Therefore, for every $\xi \in \mathcal{S}(\mathbb{R})$, $x > 0$, $t > 0$, we have that

$$S\Big(\frac{1}{\sqrt{n}} \sum_{m=0}^{n-1} [\delta_x \circ Y_{n,m}(t) - \mathbb{E}(\delta_x \circ Y_{n,m}(t))]\Big)(\xi) \tag{3.4}$$

converges to $S(\sigma(t,x)B(1))(\xi)$ as n tends to infinity. Similarly as in the case of the LLN one finds a bound like (1.3) for the ray analytic continuation of (3.4) (we leave the details to the interested reader). As a consequence of Theorem 1.1.(b) we get the following CLT–type result.

Theorem 3.2. Let $x > 0$, $t > 0$. Then

$$\frac{1}{\sqrt{n}} \sum_{m=0}^{n-1} [\delta_x \circ Y_{n,m}(t) - \mathbb{E}(\delta_x \circ Y_{n,m}(t))]$$

converges strongly in $\mathcal{S}'(\mathbb{R})$ to $\sigma(t,x)B(1)$, where $\sigma(t,x)$ is given in (3.3).

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