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YANG-MILLS AND DIRAC FIELDS WITH INHOMOGENEOUS BOUNDARY CONDITIONS

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ABSTRACT

Finite time existence and uniqueness of solutions of the evolution equations of minimally coupled Yang-Mills and Dirac system are proved for inhomogeneous boundary conditions. A characterization of the space of solutions of minimally coupled Yang-Mills and Dirac equations is obtained in terms of the boundary data and the Cauchy data satisfying the constraint equation. The proof is based on a special gauge fixing and a singular perturbation result for the existence of continuous semigroups.

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1. Introduction

Minimally interacting Yang-Mills and Dirac fields with homogeneous bag boundary conditions have been studied in [1] and [2]. In this paper we extend the results on the finite time existence and uniqueness of solutions for inhomogeneous boundary data. In this way we get a complete description of solutions of the evolution equations for minimally coupled Yang-Mills and Dirac fields in bounded domains.

We consider the Yang-Mills system with a compact structure group G on a spatially bounded region $\mathbb{R} \times M$ of the space-time, where M is a contractible bounded domain in $\mathbb{R}^3$. The 3+1 splitting of the space-time yields a splitting of the Yang-Mills potential A^μ into its spatial component A (treated as a time dependent vector field on M with values in the structure algebra $\mathfrak{g}$) and the time component A^0. This leads to a representation of the field strength $F_{\mu\nu}$ in terms of the "electric" component $E = E_i dx^i$ and the "magnetic" field $B = B_i dx^i$, where

$$E_i = F_{0i} \qquad \text{and} \qquad B_i = \tfrac{1}{2}\epsilon_{ijk}F_{jk} \ . \tag{1.1}$$

The Dirac field is described by time dependent 4-spinors Ψ with values in the vector space V_G of the fundamental representation of G. Let

$$\mathcal{D} = -\gamma^0(\gamma^k\partial_k + im) \tag{1.2}$$

be the free Dirac operator in $M \subset \mathbb{R}^3$. The minimally coupled Yang-Mills and Dirac equations split into the evolution equations

$$\partial_t A = E + \operatorname{grad} A^0 - [A^0, A] \ , \tag{1.3}$$

$$\partial_t E = -\operatorname{curl} B - [A\times, B] + [A^0, E] + J \ , \tag{1.4}$$

$$\partial_t \Psi = \mathcal{D}\Psi - (\gamma^0\gamma^k A^k + A^0)\Psi \ , \tag{1.5}$$

and the constraint equation

$$\operatorname{div} E + [A; E] = J^0 \ . \tag{1.6}$$

With $\{T^a\}$ denoting a basis of $\mathfrak{g}$,

$$B = \operatorname{curl} A + [A\times, A] \ , \quad J^0 = \Psi^\dagger(I \otimes T^a)\Psi T_a \ , \quad J^k = \Psi^\dagger(\gamma^0\gamma^k \otimes T^a)\Psi T_a \ . \tag{1.7}$$

The strategy for investigating the system (1.3) through (1.6) is as follows. Observing that the scalar potential A^0 is not fixed by the equations, we can eliminate it by means of an appropriate gauge fixing. Then we study the evolution equations in the space of the Cauchy data

$$(A, E, \Psi) \in \mathbf{P} = H^2(M, \mathfrak{g}) \times H^1(M, \mathfrak{g}) \times H^2(M, V_G \otimes \mathbb{C}^4) \tag{1.8}$$

where $H^k(M, V)$ is the Sobolev space of fields (either $\mathfrak{g}$-valued vector fields or V_G-valued spinors) which are square integrable together with their derivatives up to order k, [3]. Finally we show that the constraint (1.6) is preserved under the classical evolution.

Since the domain $I\!R \times M$ is contractible, we need not be concerned about the topology of the principal bundle of the theory. Gauge transformations can be described by maps $\Phi : I\!R \times M \to G$. The corresponding right action on the fields (A^0, A, E, Ψ) is given by

$$A^0 \mapsto \widetilde{A}^0 = \Phi^{-1} A^0 \Phi + \Phi^{-1} \partial_t \Phi \ , \tag{1.9}$$

$$A \mapsto \widetilde{A} = \Phi^{-1} A \Phi + \Phi^{-1} \mathrm{grad}\, \Phi \ , \quad E \mapsto \widetilde{E} = \Phi^{-1} E \Phi \ , \quad \Psi \mapsto \widetilde{\Psi} = \Phi^{-1} \Psi \ . \tag{1.10}$$

For the scalar potential A^0 the following gauge fixing can be established :

Theorem 1.

a) For each $E \in H^1(M, \mathfrak{g})$ one can chose $A^0 \in H^2(M, \mathfrak{g})$ as the unique solution of the Neumann problem

$$\Delta\ A^0 = -\mathrm{div}\, E \quad , \quad \mathrm{n}(\mathrm{grad}\, A^0) = -\mathrm{n}E \quad \text{with} \quad \int_M A^0\, d^3x = 0 \ . \tag{1.11}$$

Here n denotes the normal component on the boundary ∂M. For this gauge fixing there exists a constant $C > 0$ such that $\|A^0\|_{H^2} \le C \|E\|_{H^1}$. If the fields $(A, E, \Psi) \in \mathbf{P}$ satisfy the constraint equation (1.6), then $A^0 \in H^3(M, \mathfrak{g})$.

b) If a curve of configurations $(A(t), E(t), \Psi(t)) \in \mathbf{P}$ satisfies the constraint (1.6) and $A^0(t)$ is a differentiable curve in $H^3(M, \mathfrak{g})$, then there exists a maximal $\widetilde{T} > 0$ and a unique gauge transformation Φ such that :

i) $\Phi(0, x) = id$ for all $x \in M$,

ii) for all $t \in (-\widetilde{T}, \widetilde{T})$ the scalar potential $\widetilde{A}^0$, determined from A^0 by (1.9), satisfies the gauge condition $\Delta \widetilde{A}^0 = -\mathrm{div}\, \widetilde{E}$, $\mathrm{n}(\mathrm{grad}\, \widetilde{A}^0) = -\mathrm{n}\widetilde{E}$ and $\int_M \widetilde{A}^0\, d^3x = 0$.

Moreover $(\widetilde{A}(t), \widetilde{E}(t), \widetilde{\Psi}(t))$, obtained by the gauge transformation (1.10), is a curve of configurations in $H^2(M, \mathfrak{g}) \times H^1(M, \mathfrak{g}) \times H^2(M, V_G \otimes \mathbb{C}^4)$, which satisfy the constraint equation (1.6).

In order to describe the boundary conditions we introduce the following notation. For every vector field X on M, we denote by $\mathrm{n}X$ and $\mathrm{t}X$ the normal and the tangential component of X on ∂M, respectively. For the spinor fields we set

$$\Gamma(\Psi) := \left(\tfrac{1}{2}(Id - i\gamma^k \mathrm{n}_k)\Psi\right)|_{\partial M} \tag{1.12}$$

where n_k is the k-component of the outward pointing unit normal vector of ∂M in $I\!R^3$. The boundary conditions considered here consist of specifying $\mathrm{t}(\mathrm{curl}\, A)$, $\Gamma(\Psi)$ and $\Gamma(\mathcal{D}\Psi)$, where $\mathcal{D}$ is the Dirac operator (1.2). The existence and uniqueness result of [2] extends to the case of these inhomogeneous boundary conditions as follows :

Theorem 2.

a) Let a differentiable curve of boundary data

$$\big(\lambda(.), \mu(.), \nu(.)\big) : [0, T_0) \longrightarrow H^{1/2}(\partial M, \mathfrak{g}) \times H^{3/2}(\partial M, V_G \otimes \mathbb{C}^4) \times H^{1/2}(\partial M, V_G \otimes \mathbb{C}^4) \tag{1.13}$$

be given such that

$$(i\gamma^k \mathrm{n}_k + Id)\mu = 0 \qquad \text{and} \qquad (i\gamma^k \mathrm{n}_k + Id)\nu = 0 \ . \tag{1.14}$$

For every $(A_0, E_0, \Psi_0) \in H^2(M, \mathfrak{g}) \times H^1(M, \mathfrak{g}) \times H^2(M, V_G \otimes \mathbb{C}^4)$ satisfying

$$\mathrm{t}(\mathrm{curl}\, A_0) = \lambda(0) \ \ , \ \Gamma(\Psi_0) = \mu(0) \ \text{ and } \ \Gamma(\mathcal{D}\Psi_0) = \nu(0) \tag{1.15}$$

there exists a unique classical solution

$$(A(t), E(t), \Psi(t)) \in H^2(M, \mathfrak{g}) \times H^1(M, \mathfrak{g}) \times H^2(M, V_G \otimes \mathbb{C}^4) \tag{1.16}$$

of the evolution equations (1.3) through (1.5), defined for $t \in [0, T)$ with a maximal $T \in (0, T_0]$, which satisfies the initial conditions $A(0) = A_0$, $E(0) = E_0$ and $\Psi(0) = \Psi_0$, the boundary conditions

$$\mathrm{t}(\mathrm{curl}\, A(t)) = \lambda(t) \ \ , \ \ \Gamma(\Psi) = \mu(t) \ \text{ and } \ \Gamma(\mathcal{D}\Psi) = \nu(t) \ \ \forall\, t \in [0, T) \ . \tag{1.17}$$

b) If the initial condition $(A_0, E_0, \Psi_0) \in H^2(M, \mathfrak{g}) \times H^1(M, \mathfrak{g}) \times H^2(M, V_G \otimes \mathbb{C}^4)$ satisfies the constraint equation (1.6), then

$$\mathrm{div}\, E(t) + [A(t); E(t)] - J^0(t) = 0 \tag{1.18}$$

for all t in the interval $[0, T)$ of the existence of the dynamics.

By the trace theorem, [3], we conclude from $A \in H^2(M, \mathfrak{g})$ and $\Psi \in H^2(M, V_G \otimes \mathbb{C}^4)$ that $\mathrm{t}(\mathrm{curl}\, A) \in H^{1/2}(\partial M, \mathfrak{g})$ and that $\Gamma(\Psi) \in H^{3/2}(\partial M, V_G \otimes \mathbb{C}^4)$ and $\Gamma(\mathcal{D}\Psi) \in H^{1/2}(\partial M, V_G \otimes \mathbb{C}^4)$. This implies that the choice of boundary conditions (1.17) is consistent, provided condition (1.14) is satisfied. Our result gives a complete (local in time) characterisation of solutions of the Yang-Mills and Dirac equations (1.3) through (1.6) in the space **P** specified in (1.8).

In [2] we used the theory of Lipschitz perturbations of strongly continuous semigroups to obtain the existence and uniqueness theorems for the evolution equations under the homogeneous boundary conditions

$$\mathrm{n}E = 0 \ \ , \ \ \mathrm{n}A = 0 \ \ , \ \ \mathrm{tcurl}\, A = 0 \ \ , \ \ \Gamma\Psi = 0 \ \ , \ \ \Gamma D\Psi = 0 \ . \tag{1.19}$$

In the present paper we observe that our gauge condition enables us to drop the restrictions on $\mathrm{n}A$ and $\mathrm{n}E$. Moreover, using the result of [4] allows for less regular perturbations and makes it possible to consider also inhomogeneous boundary conditions. It should be noted that the invariant subspace determined by the boundary conditions (1.19) is contained in a bigger invariant subspace characterized the extra conditions $\mathrm{n}A = 0$ and $\mathrm{n}E = 0$ on the boundary.

The proof of Theorem 1 will be given in section 2. In section 3 we study the influence of the boundary conditions imposed by Theorem 2 and the linearized Cauchy problem corresponding to (1.3) through (1.5). In section 4 we state a generalization of Segal's theorem on non-linear semigroups in the singular case, cf. [4]. This implies the proof of Theorem 2.

2. Proof of Theorem 1.

The proof of this result, as well as some arguments in the context of Theorem 2 rely on a special version of the Helmholtz decomposition. The following has been shown in [5] :

Theorem 3.

Let M be a simply connected bounded domain in $\mathbb{R}^3$. A vector field $V \in H^{k+1}(M, \mathfrak{g})$ uniquely splits into

$$V = V^L + V^T \quad \text{where} \quad \begin{cases} V^L = \operatorname{grad} \Theta_V & \text{with } \Theta_V \in H^{k+2}(M, \mathfrak{g}) \\ V^T = \operatorname{curl} W_V & \text{with } W_V \in H^{k+2}(M, \mathfrak{g}) \ , \ \mathrm{t}W_V = 0 \ . \end{cases} \tag{2.1}$$

The scalar function Θ_V is unique up to a constant, which can be chosen so that $\int_M \Theta_V d_3x = 0$. The boundary condition on W_V implies that $\mathrm{n}V^T = 0$. The maps

$$\begin{aligned} \pi^T : H^k(M, \mathfrak{g}) &\longrightarrow H^k(M, \mathfrak{g}) \qquad \pi^T(V) = V^T \\ \pi^L : H^k(M, \mathfrak{g}) &\longrightarrow H^k(M, \mathfrak{g}) \qquad \pi^L(V) = V^L \end{aligned} \tag{2.2}$$

are continuous projections, and

$$\|\Theta_V\|_{H^{k+2}} \leq C \|\operatorname{div} V^L\|_{H^k} \leq C \|V^L\|_{H^{k+1}} \ . \tag{2.3}$$

Applying the decomposition (2.1) to the field E, determines a function Θ_E such that $\operatorname{grad} \Theta_E = E^L$. Then $A^0 := -\Theta_E$ is a unique solution of the boundary value problem

$$\Delta\, A^0 = -\operatorname{div} E \ , \quad \mathrm{n}(\text{grad } A^0) = -\mathrm{n}E \ \text{ and } \int_M A^0 d_3x = 0 \ . \tag{2.4}$$

From the estimate (2.3) we infer that

$$\|A^0\|_{H^{k+2}} \leq C \|\operatorname{div} E\|_{H^k} \leq C \|E\|_{H^{k+1}} \ . \tag{2.5}$$

If (A, E, Ψ) satisfy the constraint equation (1.6), we can estimate

$$\|\operatorname{div} E\|_{H^1} \leq C\big(\|A\|_{H^2} \|E\|_{H^1} + \|\Psi\|^2_{H^2}\big) \ . \tag{2.6}$$

With (2.5) this implies that $A^0 \in H^3(M, \mathfrak{g})$. This proves part a) of Theorem 1.

In view of the transformation law

$$A^0 \mapsto \Phi^{-1} A^0 \Phi + \Phi^{-1} \partial_t \Phi \tag{2.7}$$

we have to find a gauge transformation $\Phi : [0, \widetilde{T}) \to H^3(M, G)$ which satisfies the initial value problem

$$\Phi(0) = id \qquad \text{and} \qquad \partial_t \Phi(t) = \mathcal{G}\big(\Phi(t), A^0(t), E(t)\big) \ . \tag{2.8}$$

Here $id : M \to G$ maps all $x \in M$ to the identity in G and

$$\begin{aligned} &\mathcal{G}(\Phi, A^0, E) = -A^0\Phi + \Phi\,\Theta_{\widetilde{E}} \\ &\text{where}\ \ \Delta\Theta_{\widetilde{E}} = -\mathrm{div}\left(\Phi^{-1}E\Phi\right)\ \ \text{and}\ \ \mathrm{n}(\mathrm{grad}\,\Theta_{\widetilde{E}}) = -\mathrm{n}(\Phi^{-1}E\Phi)\ . \end{aligned} \tag{2.9}$$

Considering the structure group G as matrix group, $\Phi \in H^3(M,G)$ becomes a map from M to the space $gl(k)$ of $k \times k$ matrices. Since M and G are compact, there exists a finite cover of open sets $\mathcal{U}_i$ of $H^3(M,G)$ in $H^3(M, gl(k))$ such that

$$\|\Phi^{-1}\|_{H^3} < \infty \quad \text{for all} \quad \Phi \in \bigcup \mathcal{U}_i\ . \tag{2.10}$$

Therefore

$$\begin{aligned} &\|\Phi^{-1} - \widetilde{\Phi}^{-1}\|_{H^3} = \|\Phi^{-1}(\widetilde{\Phi} - \Phi)\widetilde{\Phi}^{-1}\|_{H^3} \le C\,\|\widetilde{\Phi} - \Phi\|_{H^3} \quad \text{and} \\ &\|\mathrm{grad}\,(\Phi^{-1} - \widetilde{\Phi}^{-1})\|_{H^2} \le C\,\|\widetilde{\Phi} - \Phi\|_{H^3}\ , \end{aligned} \tag{2.11}$$

where $\|\Phi - \widetilde{\Phi}\|_{H^3}$ is understood as the distance in $H^3(M, gl(k))$, and C depends on $\|\Phi^{-1}\|_{H^3}$ $\|\widetilde{\Phi}^{-1}\|_{H^3}$.

As far as the initial value problem (2.8) is concerned, we fix $(A^0, E) \in H^3(M, \mathfrak{g}) \times H^1(M, \mathfrak{g})$ and consider the map $\mathcal{G}(\Phi, A^0, E)$ as a map from $H^3(M,G)$ to $H^3(M, \mathfrak{g})$. In order to show that $\mathcal{G}$ has a flow, we have to prove that it is Lipschitz. We can do so, provided that the fields (A, E, Ψ) satisfy the constraint equation (1.6). For the gauge transformed field $\widetilde{E}$ the constraint equation reads

$$\mathrm{div}\left(\Phi^{-1}E\Phi\right) = \left(\mathrm{grad}\,\Phi^{-1}\right)\cdot E\Phi + \Phi^{-1}\left(-[A;E] + J^0\right)\Phi + \left(\Phi^{-1}E\right)\cdot \mathrm{grad}\,\Phi\ . \tag{2.12}$$

Using (2.11), the estimate (2.6) yields

$$\begin{aligned} \left\|\mathrm{div}\left(\Phi^{-1}E\Phi - \widetilde{\Phi}^{-1}E\widetilde{\Phi}\right)\right\|_{H^1} &\le \|\mathrm{div}\left((\Phi^{-1} - \widetilde{\Phi}^{-1})E\Phi\right)\|_{H^1} + \|\mathrm{div}\left(\widetilde{\Phi}^{-1}E(\widetilde{\Phi} - \Phi)\right)\|_{H^1} \\ &\le C\|\Phi - \widetilde{\Phi}\|_{H^3}\left(\|A\|_{H^2}\|E\|_{H^1} + \|\Psi\|_{H^2} + 2\|E\|_{H^1}\right)\ , \end{aligned} \tag{2.13}$$

where $C = C\left(\|\Phi\|_{H^3}, \|\Phi^{-1}\|_{H^3}, \|\widetilde{\Phi}\|_{H^3}, \|\widetilde{\Phi}^{-1}\|_{H^3}\right)$. Using (2.3), the Lipschitz estimate (2.13) for $\mathrm{div}\,(\Phi^{-1}E\Phi)$ implies a Lipschitz estimate for the term $\Phi\Theta_{\widetilde{E}}$ of $\mathcal{G}$. Using furthermore (2.5) to estimate the term $\|A^0(\Phi - \widetilde{\Phi})\|_{H^3}$ we obtain

$$\begin{aligned} &\|\mathcal{G}(\Phi, A^0, E) - \mathcal{G}(\widetilde{\Phi}, A^0, E)\|_{H^3} \le C\|\Phi - \widetilde{\Phi}\|_{H^3} \\ &\quad\text{where}\qquad C = C\left(\|\Phi^{-1}\|_{H^3}, \|\Phi\|_{H^3}, \|\widetilde{\Phi}\|_{H^3}, \|\widetilde{\Phi}^{-1}\|_{H^3}, \|A\|_{H^2}, \|E\|_{H^1}, \|\Psi\|^2_{H^2}\right)\ . \end{aligned} \tag{2.14}$$

This proves that $\mathcal{G}(., A^0, E)$ is (locally) Lipschitz. Therefore the Picard proof on the existence and uniqueness of solutions of initial value problems, cf. [6], applies to the case under consideration. This proves that there exists $\widetilde{T} > 0$ such that the gauge fixing (2.4) can be achieved by a gauge transformation

$$\Phi(t) \in C^1\left([-\widetilde{T}, \widetilde{T}), H^3(M,G)\right)\ , \tag{2.15}$$

provided that the fields $(A^0(t), A(t), E(t), \Psi(t))$ are of Sobolev class $H^3(M, \mathfrak{g}) \times H^2(M, \mathfrak{g}) \times H^1(M, \mathfrak{g}) \times H^2(M, V_G \otimes \mathbb{C}^4)$ and satisfy the constraint equation (1.6). The continuity of the gauge transformation (1.10) then follows from the standard estimates.

This proves Theorem 1.

3. Boundary conditions and linearisation.

Let $\lambda(t)$ be a differentiable curve in $H^{1/2}(\partial M, \mathfrak{g})$. Then there exists a differentiable curve $a(t) \in H^2(\partial M, \mathfrak{g})$ of solutions of the boundary value problem

$$\Delta a(t) = 0 \quad \text{and} \quad \mathrm{t}(\operatorname{curl} a(t)) = \lambda(t) \quad \forall\, t \in [0, T_0) \ . \tag{3.1}$$

This is a direct consequence of the solvability of a Neumann problem for vector fields on a simply connected domain M, [5].

As far as boundary data for the Dirac field are concerned we observe that a spinor field ρ on ∂M can be in the range of the boundary operator $\Gamma\frac{1}{2}(Id - i\gamma^k \mathrm{n}_k)\Psi|_{\partial M}$ only if

$$\big(i\gamma^k \mathrm{n}_k + Id\big)\rho = 0 \ . \tag{3.2}$$

Hence (1.14) give necessary conditions on the existence on an extension for the boundary data (μ, ν). By means of the trace theorem [7] for each differentiable curve $(\mu(t), \nu(t))$ in $H^{3/2}(\partial M, V_G \otimes \mathbb{C}^4) \times H^{1/2}(\partial M, V_G \otimes \mathbb{C}^4)$ there exist respective extensions $\psi_1(t)$ and $\psi_2(t)$ in $H^2(M, V_G \otimes \mathbb{C}^4)$ such that

$$\psi_1|_{\partial M} = \mu \quad \partial_{\mathrm{n}}\psi_1 = 0 \qquad \text{and} \qquad \psi_2|_{\partial M} = 0 \quad \partial_{\mathrm{n}}\psi_2 = \rho \ . \tag{3.3}$$

Using this, it is an easy algebraic construction to find for given $\mu(t)$ and $\nu(t)$ satisfying (3.2) an extension $\psi(t)$ in $H^2(M, V_G \otimes \mathbb{C}^4)$ such that

$$\Gamma(\psi(t)) = \mu(t) \qquad \text{and} \qquad \Gamma(\mathcal{D}\psi(t)) = \nu(t) \ . \tag{3.4}$$

With the fields a and ψ, constructed above, as a background we consider the fields

$$\widehat{A} := (A - a) \in H^2(M, \mathfrak{g}) \qquad \text{and} \qquad \widehat{\Psi} := (\Psi - \psi) \in H^2(M, V_G \otimes \mathbb{C}^4) \tag{3.5}$$

as the dynamical degree of freedom. By construction, these satisfy the boundary conditions (1.15) of Theorem 2 with homogeneous boundary data. Rewriting the Yang-Mills equations (1.3) and (1.4) in term of $\widehat{A}$ and splitting $(\widehat{A}, E)$ by means of the Helmholtz decomposition (2.1) into the longitudinal and transversal components we obtain

$$\begin{aligned}
\partial_t \widehat{A}^L &= E^L + \operatorname{grad} A^0 + \partial_t a^L - \pi^L([A^0, \widehat{A} + a]) \ , \\
\partial_t \widehat{A}^T &= E^T + \partial_t a^T - \pi^T([A^0, \widehat{A} + a]) \ , \\
\partial_t E^L &= -\pi^L(\operatorname{curl} B) - \pi^L\big([(\widehat{A} + a)\times, B] + [A^0, E] + J\big) \ , \\
\partial_t E^T &= -\pi^T(\operatorname{curl} B) - \pi^T\big([(\widehat{A} + a)\times, B] + [A^0, E] + J\big) \ .
\end{aligned} \tag{3.6}$$

The Hodge decomposition implies that $\operatorname{curl}(\operatorname{curl}\widehat{A}) = \pi^T(\Delta \widehat{A}^T)$. Moreover $\Delta a = 0$, and hence

$$\begin{aligned} \pi^T(\operatorname{curl} B) &= \Delta \widehat{A}^T + \pi^T\big(\operatorname{curl}[(\widehat{A}+a)\times, (\widehat{A}+a)]\big) \qquad \text{and} \\ \pi^L(\operatorname{curl} B) &= \pi^L\big(\operatorname{curl}[(\widehat{A}+a)\times, (\widehat{A}+a)]\big) \ . \end{aligned} \tag{3.7}$$

The gauge fixing of Theorem 1 implies that $E^L = -\operatorname{grad} A^0$. Linearising the equations (3.6) and (1.5) in such a way that we also do not consider the affine contributions from the background fields (a, ψ) we obtain three uncoupled linear systems :

$$\partial_t \begin{pmatrix} A^L \\ E^L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \partial_t \begin{pmatrix} \widehat{A}^T \\ E^T \end{pmatrix} = \mathcal{T} \begin{pmatrix} \widehat{A}^T \\ E^T \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\Delta & 0 \end{pmatrix} \begin{pmatrix} \widehat{A}^T \\ E^T \end{pmatrix} \qquad \partial_t \widehat{\Psi} = \mathcal{D}\widehat{\Psi} \ . \tag{3.8}$$

In [1] and [2] we have shown the following : The operator $\mathcal{T}$ with domain

$$\mathbf{D}(\mathcal{T}) = \{(\widehat{A}^T, E^T) \in H^2(M, g) \times H^1(M, g) \mid \mathrm{t}(\operatorname{curl} \widehat{A}^T) = 0\} \tag{3.9}$$

is the infinitesimal generator of a one-parameter group of continuous transformations in the Hilbert space

$$\mathbf{H}_T = \{(\widehat{A}^T, E^T) \in H^1(M, g) \times L^2(M, g)\} \ . \tag{3.10}$$

The (free) Dirac operator $\mathcal{D}$, considered as an operator with the domain

$$\mathbf{D}(\mathcal{D}) = \{\widehat{\Psi} \in H^2(M, V_G \otimes \mathbb{C}^4) \mid \Gamma(\widehat{\Psi}) = 0 \text{ and } \Gamma(\mathcal{D}\widehat{\Psi}) = 0\} \tag{3.11}$$

is the infinitesimal generator of a one-parameter group of continuous transformations in the Hilbert space

$$\mathbf{H}(\mathcal{D}) = \{\widehat{\Psi} \in H^1(M, V_G \otimes \mathbb{C}^4) \mid \Gamma(\widehat{\Psi}) = 0\} \ . \tag{3.12}$$

4. Proof of Theorem 2.

Using the results on the linearized dynamics given above, the coupled non-linear system can be tackled by using the following generalisation, [4], of Segal's result on non-linear semigroups in the singular case:

Theorem 5.

Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be Banach spaces and $\exp(t\mathcal{S}) : \mathbf{B}_2 \to \mathbf{B}_2$ be a continuous one-parameter semigroup of bounded linear operators generated by an operator $\mathcal{S}$ with domain $\mathbf{D}(\mathcal{S}) \subset \mathbf{B}_2$. Assume that :

i) $\mathcal{F}_1 : \mathbf{B}_1 \times \mathbf{D}(\mathcal{S}) \to \mathbf{B}_1$ is a map, which is continuous and locally Lipschitz with respect to the norm

$$|||(V_1, V_2)|||_1 = \|V_1\|_{\mathbf{B}_1} + \|V_2\|_{\mathbf{B}_2} + \|\mathcal{S}V_2\|_{\mathbf{B}_2} \quad \text{where} \quad (V_1, V_2) \in \mathbf{B}_1 \times \mathbf{D}(\mathcal{S}) \ . \tag{4.1}$$

ii) $\mathcal{F}_2 : \mathbf{B}_1 \times \mathbf{D}(\mathcal{S}) \to \mathbf{B}_2$ is a map, which is continuous and differentiable with respect to the norm (4.1).

iii) The following derivative $\mathcal{K} : \mathbf{B}_1 \times \mathbf{D}(\mathcal{S}) \times \mathbf{B}_2 \to \mathbf{B}_2$ of $\mathcal{F}_2$ given by

$$\begin{aligned}&\mathcal{K}(V_1, V_2, v_2) := \mathcal{K}_1(V_1, V_2) + \mathcal{K}_2(V_1, V_2, v_2) \\ &\mathcal{K}_1(V_1, V_2) = D\mathcal{F}_2(V_1, V_2)\big(\mathcal{F}_1(V_1, V_2), 0\big) \quad \text{and} \quad \mathcal{K}_2(V_1, V_2, v_2) = D\mathcal{F}_2(V_1, V_2)\big(0, v_2\big)\end{aligned} \tag{4.2}$$

is locally Lipschitz with respect to the norm

$$|||(V_1, V_2, v_2)|||_2 = \|V_1\|_{\mathbf{B}_1} + \|V_2\|_{\mathbf{B}_2} + \|\mathcal{S}V_2\|_{\mathbf{B}_2} + \|v_2\|_{\mathbf{B}_2} \, . \tag{4.3}$$

Then, for every initial condition $(V_1(0), V_2(0)) \in \mathbf{B}_1 \times \mathbf{D}(\mathcal{S})$ there exists a maximal $T > 0$ such that the differential equation

$$\partial_t \begin{pmatrix} V_1(t) \\ V_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \mathcal{S}V_2(t) \end{pmatrix} + \begin{pmatrix} \mathcal{F}_1(V_1(t), V_2(t)) \\ \mathcal{F}_2(V_1(t), V_2(t)) \end{pmatrix} \tag{4.4}$$

has a unique classical solution $(V_1(t), V_2(t)) \in \mathbf{B}_1 \times \mathbf{D}(\mathcal{S})$ in the interval $[0, T)$, satisfying the initial condition.

To apply this theorem to the case under consideration we set

$$\mathbf{B}_1 = \mathbf{H}_L = \{(\widehat{A}^L, E^L) \in H^2(M, \mathfrak{g}) \times H^1(M, \mathfrak{g})\} \, . \tag{4.5}$$

The generator of the linear semigroup we choose to be $\mathcal{S} := \mathcal{T} + \mathcal{D}$. The Banach space $\mathbf{B}_2 = \mathbf{H}_T \times \mathbf{H}_D$ is normed by

$$\|(\widehat{A}^T, E^T, \Psi^T)\|_{\mathbf{B}_2} = \|\widehat{A}^T\|_{H^1} + \|E^T\|_{L^2} + \|\widehat{\Psi}\|_{H^1} \, . \tag{4.6}$$

In view of the equations (3.6), (3.7) and (1.5) the components $\mathcal{F}_1 = \big((\mathcal{F}_1)_A, (\mathcal{F}_1)_E\big)$ and $\mathcal{F}_2 = \big((\mathcal{F}_2)_A, (\mathcal{F}_2)_E, (\mathcal{F}_2)_\Psi\big)$ read as

$$\begin{aligned}
&(\mathcal{F}_1)_A\big((\widehat{A}^L, E^L),(\widehat{A}^T, E^T, \widehat{\Psi})\big) = \partial_t a^L - \pi^L\big([A^0, (\widehat{A} + a)]\big) \\
&(\mathcal{F}_1)_E\big((\widehat{A}^L, E^L),(\widehat{A}^T, E^T, \widehat{\Psi})\big) = \\
&\qquad -\pi^L\Big(\mathrm{curl}\,[(\widehat{A} + a)\times, (\widehat{A} + a)] + [(\widehat{A} + a)\times, B] + [A^0, E] + J\Big) \\
&(\mathcal{F}_2)_A\big((\widehat{A}^L, E^L),(\widehat{A}^T, E^T, \widehat{\Psi})\big) = \partial_t a^T - \pi^T\big([A^0, (\widehat{A} + a)]\big) \\
&(\mathcal{F}_2)_E\big((\widehat{A}^L, E^L),(\widehat{A}^T, E^T, \widehat{\Psi})\big) = \\
&\qquad -\pi^T\Big(\mathrm{curl}\,[(\widehat{A} + a)\times, (\widehat{A} + a)] + [(\widehat{A} + a)\times, B] + [A^0, E] + J\Big) \\
&(\mathcal{F}_2)_\Psi\big((\widehat{A}^L, E^L),(\widehat{A}^T, E^T, \widehat{\Psi})\big) = \partial_t \psi - \gamma^0\gamma^k(\widehat{A}^k + a^k)\widehat{\Psi} - A^0\widehat{\Psi}
\end{aligned} \tag{4.7}$$

By assumption, the background fields $a(t)$ and $\psi(t)$ as well as their time derivatives are of Sobolev class H^2. By Theorem 3, the projections π^L and π^T to the components of the Helmholtz decomposition are continuous with respect to the Sobolev topology. The results of [1] and [2] then imply the following :

(A) The map $\mathcal{F}_1 : \mathbf{B}_1 \times \mathbf{D}(\mathcal{S}) \to \mathbf{B}_1$ is continuous an Lipschitz with respect to the norm given by (4.1).
(B) The map $\mathcal{F}_2 : \mathbf{B}_1 \times \mathbf{D}(\mathcal{S}) \to \mathbf{B}_2$ is continuous and differentiable with respect to the norm (4.1).
(C) The norms (4.1) and (4.3) are in the case under consideration equivalent to the respective norm

$$\left|\left|\left| ((\widehat{A}^L, E^L), (\widehat{A}^T, E^T, \widehat{\Psi})) \right|\right|\right|_1 = \|\widehat{A}\|_{H^2} + \|E\|_{H^1} + \|\widehat{\Psi}\|_{H^2} \tag{4.8}$$

$$\begin{aligned} &\left|\left|\left| ((\widehat{A}^L, E^L), (\widehat{A}^T, E^T, \widehat{\Psi}), (\alpha^T, \epsilon^T, \varphi)) \right|\right|\right|_2 = \\ &\left|\left|\left| ((\widehat{A}^L, E^L), (\widehat{A}^T, E^T, \widehat{\Psi})) \right|\right|\right|_1 + \|\alpha^T\|_{H^1} + \|\epsilon^T\|_{L^2} + \|\varphi\|_{H^1} \ , \end{aligned} \tag{4.9}$$

where $(\alpha^T, \epsilon^T, \varphi) \in \mathbf{H}_T \times \mathbf{H}_D$ be an arbitrary infinitesimal variation of $(\widehat{A}^T, \widehat{E}^T, \widehat{\Psi})$.
In view of later differentiation we observe that A^0 is a linear functional of E^L, and set

$$\delta_1 A^0 := DA^0((\mathcal{F}_1)_E) \ . \tag{4.10}$$

The respective differentials of $B = \operatorname{curl}(\widehat{A}^T + a^T) + [(\widehat{A}+a), \times(\widehat{A}+a)]$ and of the components J^k_b of the matter current read

$$\begin{aligned} \delta_1 B &:= DB(A^L, A^T)((\mathcal{F}_1)_A, 0) = 2\big[(\mathcal{F}_1)_A \times, (\widehat{A}+a)\big] \\ \delta_2 B &:= DB(A^L, A^T)(0, \alpha^T) = \operatorname{curl}\alpha^T + 2\big[\alpha^T \times, (\widehat{A}+a)\big] \\ (\delta_2 J)^k_b &:= D\pi^T(J^k_b(\widehat{\Psi}))(\varphi) = \pi^T\Big(\widehat{\Psi}^\dagger(\gamma^0\gamma^k \otimes T_b)\varphi + \varphi^\dagger(\gamma^0\gamma^k \otimes T_b)\widehat{\Psi}\Big) \ . \end{aligned} \tag{4.11}$$

Using these notations, the differential $\mathcal{K}_1\big((\widehat{A}^L, E^L), (\widehat{A}^T, E^T, \widehat{\Psi})\big)$ of $\mathcal{F}_2$ is the sum of the following terms

$$\begin{aligned} (\mathcal{K}_1)_A &= -\,\pi^T\big([\delta_1 A^0, (\widehat{A}+a)] + [A^0, (\mathcal{F}_1)_A]\big) \\ (\mathcal{K}_1)_E &= -\,\pi^T\big(2\operatorname{curl}[(\mathcal{F}_1)_A \times, (\widehat{A}+a)] + [(\mathcal{F}_1)_A \times, B] + [(\widehat{A}+a)\times, \delta_1 B]\big) \\ &\quad -\,\pi^T\big([\delta_1 A^0, E] + [A^0, (\mathcal{F}_1)_E]\big) \\ (\mathcal{K}_1)_\Psi &= -\,\gamma^0\gamma^k(\mathcal{F}_1)^k_A - \delta_1 A^0 \widehat{\Psi} \end{aligned} \tag{4.12}$$

Similarly we have for $\mathcal{K}_2\big((\widehat{A}^L, E^L), (\widehat{A}^T, E^L, \widehat{\Psi}), (\alpha^T, \epsilon^T, \varphi)\big)$:

$$\begin{aligned} (\mathcal{K}_2)_A &= -\,\pi^T\big([A^0, \alpha^T]\big) \\ (\mathcal{K}_2)_E &= -\,\pi^T\big(2\operatorname{curl}[\alpha^T \times, (\widehat{A}+a)] + [\alpha^T \times, B] + [(\widehat{A}+a)\times, \delta_2 B]\big) \\ &\quad -\,\pi^T\big([A^0, \epsilon^T] + \delta_2 J\big) \\ (\mathcal{K}_2)_\Psi &= -\,\gamma^0\gamma^k \alpha^T_k \widehat{\Psi} - \gamma^0\gamma^k(\widehat{A}^k + a^k)\varphi - A^0\varphi \end{aligned} \tag{4.13}$$

Lemma 6.

If W_i be finite dimensional vector spaces and $* : W_1 \times W_2 \to W_3$ is an algebraic product, then $* : H^1(M, W_1) \times H^2(M, W_2) \to H^1(M, W_3)$ such that

$$\|V_1 * V_2\|_{H^1} \leq C\|V_1\|_{H^1}\|V_2\|_{H^2} \qquad \text{and} \tag{4.14}$$

$$\|V_1 * V_2 - U_1 * U_2\|_{H^1} \leq C\big(\|V_2\|_{H^2} + \|U_1\|_{H^1}\big)\big(\|V_1 - U_1\|_{H^1} + \|V_2 - U_2\|_{H^2}\big). \tag{4.15}$$

Proof.

By definition of the H^1-norm

$$\|V_1 * V_2\|_{H^1} \leq \|V_1 * V_2\|_{L^2} + \|(\operatorname{grad} V_1) * V_2\|_{L^2} + \|V_1 * (\operatorname{grad} V_2)\|_{L^2} \; . \tag{4.16}$$

The first two terms on the right hand side can be estimated by $\|V_1\|_{H^1}\|V_2\|_{H^2}$ since $H^2(M, W_2) \subset C^0(M, W_2)$ by the Sobolev embedding theorem. Moreover, since $H^1(M, W_i) \subset L^4(M, W_i)$, the third term can be estimated by $\|V_1\|_{H^1}\|(\operatorname{grad} V_2)\|_{H^1}$. This proves the inequality (4.14). Since $V_1 * V_2 - U_1 * U_2 = (V_1 - U_1) * V_2 - U_1 * (U_2 - V_2)$, the estimate (4.15) follows from the triangle inequality. Q.E.D.

To derive the Lipschitz estimate for $\mathcal{K}_1$ we let $(\widehat{A}, E, \widehat{\Psi}), (\widetilde{A}, \widetilde{E}, \widetilde{\Psi}) \in \mathbf{B}_1 \times \mathbf{D}(\mathcal{S})$ and understand

$$\mathcal{F}_1 = \mathcal{F}_1(\widehat{A}, E, \widehat{\Psi}) \;\;,\;\; \widetilde{\mathcal{F}_1} = \mathcal{F}_1(\widetilde{A}, \widetilde{E}, \widetilde{\Psi}) \;\text{ and }\; \delta_1\widetilde{A}^0 = DA^0\big((\widetilde{\mathcal{F}_1})_E\big) \; . \tag{4.17}$$

Using this we apply Lemma 6 to estimate $(\mathcal{K}_1)_A$ in the norm (4.6) as

$$\begin{aligned} &\big\|(\mathcal{K}_1)_A\big((\widehat{A}^L, E^L), (\widehat{A}^T, E^T, \widehat{\Psi})\big) - (\mathcal{K}_1)_A\big((\widetilde{A}^L, \widetilde{E}^L), (\widetilde{A}^T, \widetilde{E}^T, \widetilde{\Psi})\big)\big\|_{H^1} \\ &\leq C\Big(\|\delta_1 A^0 - \delta_1\widetilde{A}^0\|_{H^1} + \|\widehat{A} - \widetilde{A}\|_{H^2} + \|A^0 - \widetilde{A}^0\|_{H^2} + \|(\mathcal{F}_1)_A - (\widetilde{\mathcal{F}}_1)_A\|_{H^1}\Big) \; , \end{aligned} \tag{4.18}$$

where the constant C depends on the norm in $\mathbf{B}_1 \times \mathbf{D}(\mathcal{S})$ of all the fields involved. With $\Delta A^0 = -\operatorname{div} E^L$ and $\Delta\big(\delta_1 A^0\big) = -\operatorname{div}(\mathcal{F}_1)_E$, the estimate (2.5) implies that

$$\begin{aligned} \|A^0 - \widetilde{A}^0\|_{H^2} &\leq C\|E^L - \widetilde{E}^L\|_{H^1} \qquad \text{and} \\ \|\delta_1 A^0 - \delta_1\widetilde{A}^0\|_{H^1} &\leq C\|(\mathcal{F}_1)_E - (\widetilde{\mathcal{F}}_1)_E\|_{L^2} \; . \end{aligned} \tag{4.19}$$

Moreover, Property (A) above states that the nonlinearity $\mathcal{F}_1 : \mathbf{B}_1 \times \mathbf{D}(\mathcal{S}) \to \mathbf{B}_1$ is locally Lipschitz. Therefore (4.18) implies that

$$\begin{aligned} &\big\|(\mathcal{K}_1)_A\big((\widehat{A}^L, E^L),(\widehat{A}^T, E^T, \widehat{\Psi})\big) - (\mathcal{K}_1)_A\big((\widetilde{A}^L, \widetilde{E}^L), (\widetilde{A}^T, \widetilde{E}^T, \widetilde{\Psi})\big)\big\|_{H^1} \\ &\qquad \leq C\big|\big\|\big((\widehat{A}^L, E^L), (\widehat{A}^T, E^T, \widehat{\Psi})\big) - \big((\widetilde{A}^L, \widetilde{E}^L), (\widetilde{A}^T, \widetilde{E}^T, \widetilde{\Psi})\big)\big\|\big|_1 \; . \end{aligned} \tag{4.20}$$

As far as the estimate for $(\mathcal{K}_1)_E$ is concerned we get from (4.15)

$$\begin{aligned} &\Big\|\operatorname{curl}\big([(\mathcal{F}_1)_A \times, (\widehat{A} + a)] - [(\widetilde{\mathcal{F}}_1)_A \times, (\widetilde{A} + a)]\big)\Big\|_{L^2} \\ &\qquad \leq C\big(\|(\mathcal{F}_1)_A - (\widetilde{\mathcal{F}}_1)_A\|_{H^1} + \|\widehat{A} - \widetilde{A}\|_{H^2}\big) \; . \end{aligned} \tag{4.21}$$

Similarly,

$$\left\|[(\widehat{A}+a)\times,\delta_1 B]-[(\widetilde{A}+a)\times,\delta_1\widetilde{B}]\right\|_{L^2}\leq C\big(\|\delta_1 B-\delta_1\widetilde{B}\|_{H^1}+\|\widehat{A}-\widetilde{A}\|_{H^2}\big)\ , \tag{4.22}$$

where $\delta_1\widetilde{B}=2\big[(\widetilde{\mathcal{F}}_1)_A\times,\widetilde{A}+a\big]$. With Lemma 6 we can estimate

$$\|\delta_1 B-\delta_1\widetilde{B}\|_{H^1}\leq C\big(\|(\mathcal{F}_1)_A-(\widetilde{\mathcal{F}_1})_A\|_{H^1}+\|\widehat{A}-\widetilde{A}\|_{H^2}\big)\ . \tag{4.23}$$

Taking (4.19) into account, similar arguments as above apply to the remaining terms of $(\mathcal{K}_1)_E$. With the Lipschitz property (A) of $\mathcal{F}_1$ we then obtain

$$\begin{aligned}&\left\|(\mathcal{K}_1)_E\big((\widehat{A}^L,E^L),(\widehat{A}^T,E^T,\widehat{\Psi})\big)-(\mathcal{K}_1)_A\big((\widetilde{A}^L,\widetilde{E}^L),(\widetilde{A}^T,\widetilde{E}^T,\widetilde{\Psi})\big)\right\|_{L^2}\\&\leq C\Big(\|(\mathcal{F}_1)_A-(\widetilde{\mathcal{F}}_1)_A\|_{H^1}+\|\widehat{A}-\widetilde{A}\|_{H^2}+\|(\mathcal{F}_1)_E-(\widetilde{\mathcal{F}}_1)_E\|_{L^2}+\|E-\widetilde{E}\|_{H^1}\Big)\\&\leq C\left|\left|\left|\big((\widehat{A}^L,E^L),(\widehat{A}^T,E^T,\widehat{\Psi})\big)-\big((\widetilde{A}^L,\widetilde{E}^L),(\widetilde{A}^T,\widetilde{E}^T,\widetilde{\Psi})\big)\right|\right|\right|_1\ .\end{aligned} \tag{4.24}$$

The estimates for $(\mathcal{K}_1)_\Psi$ can be performed in the same way so that we end up with the local Lipschitz estimate

$$\begin{aligned}&\left|\left|\left|\mathcal{K}_1\big((\widehat{A}^L,E^L),(\widehat{A}^T,E^T,\widehat{\Psi})\big)-\mathcal{K}_1\big((\widetilde{A}^L,\widetilde{E}^L),(\widetilde{A}^T,\widetilde{E}^T,\widetilde{\Psi})\big)\right|\right|\right|_{\mathbf{B}_2}\\&\qquad\leq C\left|\left|\left|\big((\widehat{A}^L,E^L),(\widehat{A}^T,E^T,\widehat{\Psi})\big)-\big((\widetilde{A}^L,\widetilde{E}^L),(\widetilde{A}^T,\widetilde{E}^T,\widetilde{\Psi})\big)\right|\right|\right|_1\ .\end{aligned} \tag{4.25}$$

The terms of $\mathcal{K}_2$ can be tackled in literally the same way as we have done this for $\mathcal{K}_1$. The corresponding estimates are even more direct, since one need not use Lipschitz argument for the nonlinearity $\mathcal{F}_1$. That is, we can replace $\|\mathcal{F}_1-\widetilde{\mathcal{F}}_1\|_{\mathbf{B}_2}$ by

$$\|\alpha^T-\widetilde{\alpha}^T\|_{H^1}+\|\epsilon^T-\widetilde{\epsilon}^T\|_{L^2}+\|\varphi-\widetilde{\varphi}\|_{H^1}\ , \tag{4.26}$$

where ever it appears. Doing so, we end up with

$$\begin{aligned}&\left\|\mathcal{K}_2\big((\widehat{A}^L,E^L),(\widehat{A}^T,E^T,\widehat{\Psi}),(\alpha,\epsilon,\psi)\big)-\mathcal{K}_2\big((\widetilde{A}^L,\widetilde{E}^L),(\widetilde{A}^T,\widetilde{E}^T,\widetilde{\Psi})(\widetilde{\alpha},\widetilde{\epsilon},\widetilde{\psi})\big)\right\|_{\mathbf{B}_2}\\&\leq C\left|\left|\left|\big((\widehat{A}^L,E^L),(\widehat{A}^T,E^T,\widehat{\Psi})(\alpha,\epsilon,\psi)\big)-\big((\widetilde{A}^L,\widetilde{E}^L),(\widetilde{A}^T,\widetilde{E}^T,\widetilde{\Psi})\big)(\widetilde{\alpha},\widetilde{\epsilon},\widetilde{\psi})\big)\right|\right|\right|_1\end{aligned} \tag{4.27}$$

Together with the properties (A)-(C), stated above, this proves that the nonlinearity of the theory given by (4.7) all the prerequests of Theorem 5.

This proves part a) of Theorem 2.

The proof of part b) of Theorem 2, stating that the constraint equation is preserved under the classical dynamics is given in [2]. The argument literally applies to the case of inhomogeneous boundary conditions.

References

[1] J. Śniatycki and G. Schwarz, "The existence and uniqueness of solutions of the Yang-Mills equations with bag boundary conditions" *Comm. Math. Phys.*, **159** (1994) 593-604.

[2] G. Schwarz and J. Śniatycki, "Yang-Mills and Dirac fields in a bag, existence and uniqueness theorems", *Comm. Math. Phys.*, **168** (1995) 441-453.

[3] R.A. Adams, *Sobolev Spaces*, Academic Press, Orlando, Florida, 1975.

[4] J. Tafel and J. Śniatycki, "Nonlinear semigroups and the Yang-Mills equations with the metallic boundary conditions", Preprint.

[5] G. Schwarz, *Hodge Decomposition – A Method for Solving Boundary Value Problems*, Lecture Notes in Mathematics 1607, Springer-Verlag, Heidelberg, 1995.

[6] S. Lang, *Differential and Riemannian Manifolds*, Springer-Verlag, Heidelberg, 1995.

[7] J. Wloka, *Partial Differential Equations.* Cambridge University Press, Cambridge, 1987.