Bivariate Segment Approximation

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In this note we state some problems on approximation by univariate splines with free knots, bivariate segment approximation and tensor product splines with variable knot lines.

There is a vast literature on approximation and interpolation by univariate splines with fixed knots (see e.g. the books of de Boor [1], Braess [2], DeVore & Lorentz [4], Powell [20], Schumaker [21], Nürnberger [13] and the book of Chui [3] on multivariate splines). On the other hand, numerical examples show that in general, the error is much smaller if variable knots are used for the approximation of functions instead of fixed knots. This is true for univariate splines as well as for bivariate splines. But approximation by splines with free knots leads to rather difficult nonlinear problems.

We first consider best approximation by univariate splines with free knots. We denote by $S_{m,k}$ the (non-convex) set of *splines* of degree m with k *free knots.* Given a function $f \in C[a,b]$, we call $s_f \in S_{m,k}$ a *best approximation* of f if $\|f - s\| = \min_{s \in S_{m,k}} \|f - s\|$, where $\|h\| = \max_{t \in [a,b]} |h(t)|$ denotes the *maximum norm* of $h \in C[a,b]$. This problem was first investigated by Johnson [5] in 1960. Since then, many results were proved on this topic by using advanced methods (see e.g. the books [2], [4], [13], [21] and the recent results in [6], [10], [11], [12], [14], [15], [16], [17], [22]). Nevertheless, there is no complete solution of the following problem.

Problem 1. Give characterizations of best approximations and unique best approximations from $S_{m,k}$.

It is well known that, in contrast to the fixed knot case, best approximations from $S_{m,k}$ cannot be characterized by alternation properties of the error function alone. But at present, it seems not to be known what the additional conditions could be which completely characterize best approximations.

The deeper reason that nonlinear problems of this type are unsolved is that there is no complete theory of nonlinear optimization and no algorithm for computing global minima of arbitrary nonlinear optimization problems.

On the other hand, an algorithm was developed for computing good spline approximations with free knots (Nürnberger, Sommer & Strauß [19], Meinardus, Nürnberger, Sommer & Strauß [8]). The spline approximations are computed in two steps.

Given a function $f \in C[a,b]$, first a segment approximation problem is solved. We denote by $PP_{m,k}$ the (non-convex) set of *piecewise polynomials* of degree m with k *free knots.* In the first step, a best approximation of f from $PP_{m,k}$ is computed such that the errors on all knot-intervals are the same. Therefore, the corresponding knots, denoted by $x_1, \ldots, x_k$ (which in general, are nonuniform), reflect the critical parts of the function f. The algorithm converges by starting with an arbitrary set of knots.

Then in the second step, by applying a Remez type algorithm (Nürnberger & Sommer [18]), a best approximation of f from the space $S_m(x_1, \ldots, x_k)$ of *splines* of degree m with k *fixed knots* is computed. Numerical examples show that in this way a good approximation from $S_{m,k}$ is obtained (see [8], [13]). The resulting error can be compared with the error for $PP_{m,k}$ which is a lower bound.

In the following, we consider similar problems in the bivariate case. First, we formulate a general spline approximation problem for variable knot lines.

Let a rectangle $T = [a, b] \times [c, d]$ be subdivided by knot lines $x = x_i$ and $y = y_i$, $i = 1, \ldots, k$, into $(k+1)^2$ subrectangles. Such a *partition* is called *of type I.* Moreover, let an approximation or interpolation method be given which yields for each function $f \in C(T)$ an approximation $A(f)$ from the tensor product spline space $S_m(x_1, \ldots, x_k) \otimes S_m(y_1, \ldots, y_k)$ (see e.g. [1], [13]). We consider the case when the knot lines are variable.

Problem 2. Determine a partition of type I for which the error $\|f - A(f)\|_T$ is relatively small.

As in the univariate case, it is natural to get a good partition as in Problem 2 by solving segment approximation problems for $\Pi_n \otimes \Pi_n$ (where Π_n denotes the space of univariate *polynomials* of degree n) of the following type.

We subdivide the rectangle T by k *horizontal* lines into $k + 1$ strips. Then we subdivide each strip by k vertical line segments into $k + 1$ subrectangles. The partitions of each strip are different, in general. Such a *partition* of T is called *of type II.* If we subdivide T by k *vertical* lines into $k + 1$ strips, and subdivide each strip by k horizontal line segments into $k + 1$ subrectangles, then the *partition* is called *of type III.*

We associate to each subrectangle T_j of a partition of type I, II or III a real number $d(T_j)$ which may be the error $\|f - B(f)\|_{T_j}$ of some approximation or interpolation method, where $B(f)$ is from the tensor product space $\Pi_n \otimes \Pi_n$. The real number may also be a lower or upper bound of this error. We consider the case when the line segments of the partition are variable.

Problem 3. Describe and determine the partitions of type I, II or III for which $\max_j d(T_j)$ is minimal.

There may be relations between optimal partitions of type I, II and III.

Problem 4. Do the horizontal lines of an optimal partition of type II combined with the vertical lines of an optimal partition of type III yield an optimal partition of type I (in the sense of Problem 3)?

In a recent paper, Meinardus, Nürnberger & Walz [9] developed an algorithm for computing optimal partitions of type II and III for the best approximation error and related functionals (Problem 3). They also showed that for some of these cases, Problem 4 has a positive answer. Their numerical examples show that tensor product polynomial interpolation at uniform points on the subrectangles is a suitable method for segment approximation (although this method does not quite fit into the general setting of the convergence results). An optimal partition of type I in the sense of Problem 3 yields a good partition for tensor product spline interpolation in the sense of Problem 2.

Although, first results on the above problems are known, several problems for various approximation methods with different function classes are unsolved at present. Efficient methods for solving these problems would be important for applications in practice.

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