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Abstract. We study harmonic maps $\phi : (M, g) \to (N, h)$ which are coupled to the metric g by the Einstein equation $\kappa Ric[g] = \phi^* h$, κ being a constant. Using Bochner's technique we find that this class of maps is rather restricted if (M, g) is a warped product. In case $Ric[g]$ is parallel the source manifold (M, g) decomposes into a (pseudo-) Riemannian product and ϕ factorizes. Any diffeomorphism ϕ from M to (N, h) can locally be made to a harmonic map by an appropriate choice of g.

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1. Introduction

A map $\phi : M \to N$ between (pseudo–) Riemannian manifolds (M, g) and (N, h) is called harmonic if it obeys the Euler–Lagrange equations to the Lagrangian

$$e[\phi] = \frac{1}{2} g^{\mu\nu} \phi^i_{,\mu} \phi^j_{,\nu} h_{ij}(\phi) \ ,$$

which is also called the energy density e of ϕ. Here and in the following we use standard tensor notations; in particular $\phi^i_{,\mu}$ means $\partial\phi^i/\partial x^\mu$ and the summation convention for repeated indices is applied.

In this paper we study the coupling of ϕ to Einstein gravitation in (M, g), that means we investigate the Euler–Lagrange equations to

$$L[g, \phi] = \kappa R[g] - e[g, \phi] \ ,$$

where $R[g]$ denotes the scalar curvature to $g = g_{\mu\nu} dx^\mu dx^\nu$, $\kappa \neq 0$ is some coupling constant and e is now taken as a functional of g, too. Variation with respect to ϕ just gives the harmonic map equation

$$\tau[\phi] := tr(\nabla d\phi) = 0 \ ,$$

where the tension field τ in local coordinates reads, with Christoffel symbols Γ:

$$\tau^k[\phi] = g^{\mu\nu} \nabla_\mu \phi^k_{,\nu} = g^{\mu\nu} (\phi^k_{,\mu\nu} - {}^M\Gamma^\rho_{\mu\nu} \phi^k_{,\rho} + {}^N\Gamma^k_{ls} \phi^l_{,\mu} \phi^s_{,\nu}) \ . \tag{1.1}$$

Variation with respect to g gives the Einstein equation

$$\kappa(Ric - \frac{R}{2} g) = \phi^* h - eg \ . \tag{1.2}$$

Here Ric is the Ricci tensor of g and $\phi^* h$ denotes the pullback of h with respect to ϕ, in local coordinates $(\phi^* h)_{\mu\nu} = h_{ij} \phi^i_{,\mu} \phi^j_{,\nu}$. The righthand side of (1.2) defines the energy momentum tensor $T := \phi^* h - eg$ of the "matter field" ϕ. It has been studied in some papers: T is divergence–free [BaEe] and it obeys the dominant energy condition of general relativity if g is Lorentzian [ScHi].

Contracting both sides of (1.2) with g gives $\kappa(R - (m/2)R) = 2e - me$, $m := dim(\mathrm{M})$. For $m \geq 3$ – the case we exclusively consider – it follows $\kappa R = 2e$ and this yields a remarkable cancellation in (1.2):

$$\kappa Ric = \phi^* h \ . \tag{1.3}$$

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Throughout the paper the source manifold (M, g) and the target manifold (N, h) are assumed to be smooth (i.e. C^∞) and connected. (N, h) is always assumed to be properly Riemannian while (M, g) can be Lorentzian as well. The latter case is physically motivated: the gravitational field g is coupled through the Einstein equation (1.2) – or equivalently (1.3) – to the harmonic map ϕ, taken as a matter field. The properly Riemannian case, actually the major case considered by physicists, e.g. [OmPe,Gh], can be motivated by the Euclidization procedure of quantum field theory. We take motives from physics, but are mainly interested in mathematical aspects of the system

$$\kappa Ric[g] = \phi^* h \ , \quad \tau[\phi] = 0 \ . \tag{1.4}$$

In our approach the target manifold (N, h) is assumed to be given while g and ϕ are two unknown objects which are coupled by (1.4). Let us present here some selected results of the paper:

• If (M, g) is a warped product $\hat{M} \times_f \check{M}$ then we find in several cases that $\kappa Ric = \phi^* h$ implies $f =$ const. and that $\phi(\hat{p}, \check{p})$ must be constant too, with respect to one of both variables. Notice that many physically relevant exact solutions in general relativity have the form of a warped product.

• If $\kappa Ric = \phi^* h$ and additionally $\nabla Ric = 0$ is assumed on a complete, simply connected manifold (M, g) then (M, g) decomposes into $(M_1 \times M_2, g_1 \oplus g_2)$ and ϕ is constant on M_1. Moreover the Einstein equation reduces to the one on (M_2, g_2). This example provides an extra motive to study (1.4) for properly Riemannian manifolds, since (M, g) can be Lorentzian while (M_2, g_2) is properly Riemannian.

• If $\phi : M \to N$ is an arbitrary diffeomorphism then $\kappa Ric = \phi^* h$ already implies the harmonic map equation $\tau[\phi] = 0$. This fact has an interesting conclusion: every local diffeomorphism ϕ from M into (N, h) can be made to a local harmonic map. Namely, for such a ϕ the Einstein equation $\kappa Ric[g] = \phi^* h$ always admits a solution g defined on a neighborhood $U \subset M$, thus ϕ harmonically maps (U, g) into (N, h) .

The paper is organized as follows: In Section 2 we investigate implications from the Einstein equation $\kappa Ric = \phi^* h$; particular attention is devoted to the case $\nabla Ric = 0$. In Section 3 we mainly study decomposable manifolds $M = M_1 \times M_2$ and apply the Bochner technique. In the last section we compare our approach to the literature on the subject.

2. Implications from the Einstein equation

Whenever we write $\kappa Ric = \phi^* h$ we implicitly assume the situation $\phi : (M, g) \to (N, h)$. Since $\phi^* h(u, v) = h(d\phi(u), d\phi(v))$ we shall sometimes write $h(d\phi, d\phi)$ instead of $\phi^* h$. Let us discuss some elementary conclusions from (1.3).

Proposition 2.1 *Let $\kappa Ric = \phi^* h$. If Ric, taken as a bilinear form, vanishes on an integrable distribution of vector fields on M then ϕ is constant on each leaf of the corresponding foliation of M.*

Proof: Let coordinates x^μ on M be adapted to the situation, that means the distribution is spanned locally by the r first coordinate vectors $\partial_1, \dots, \partial_r$. By assumption $\kappa Ric(\partial_\mu, \partial_\nu) = 0$ for $\mu, \nu = 1, \dots, r$ and $\kappa Ric = \phi^* h$ implies $h(\phi_{,\mu}, \phi_{,\nu}) = 0$ for these μ, ν. Considering that h is positive definite we find $\phi_{,\mu} = 0$ for $\mu = 1, \dots, r$, so the result follows. ■

Another useful observation is the following:

Proposition 2.2 *If $\kappa Ric = \phi^* h$ then κRic is nonnegative. Moreover $Ric = 0$ if and only if ϕ =const., and then $R = 0$. In case (M, g) is properly Riemannian then $R = 0$ also implies that ϕ is constant.*

Proof: Only the last claim is not obvious. Notice that $2e = g^{\mu\nu} h_{ij} \phi^j_{,\nu} \phi^i_{,\mu}$ is positive definite in the variables $\phi^i_{,\mu}$, thus $\kappa R = 2e = 0$ implies $\phi^i_{,\mu} = 0$, so ϕ is constant. ■

For $\kappa > 0$ the Proposition 2.2 implies severe restrictions on ϕ in the following cases:

1) If (M, g) is a compact properly Riemannian manifold and $Riem[h] \leq 0$ – that means (N, h) has nonpositive sectional curvature – then the map ϕ is totally geodesic, i.e. $\nabla d\phi = 0$, and has constant energy density $e[\phi]$.

2) If (M, g) is a non–compact, complete and properly Riemannian manifold and (N, h) is compact with $Riem[h] \leq 0$, then any harmonic map with total energy $\int_M e[\phi] dvol < \infty$ is constant. Here we denoted the Riemannian volume form by *dvol*.

These and further properties of harmonic maps satisfying $Ric[g] \geq 0$ and $Riem[h] \leq 0$ are discussed in [EeLe], p. 10–13.

Proposition 2.3 *If ϕ is a totally geodesic map such that $\kappa Ric = \phi^* h$ then the Ricci tensor of (M, g) is parallel: $\nabla Ric = 0$.*

The proof is obvious by the formula

$$\nabla(\phi^* h) = \nabla(h(d\phi, d\phi)) = h(\nabla d\phi, d\phi) + h(d\phi, \nabla d\phi) \ , \tag{2.1}$$

which follows from the chain rule. Note that ∇ on the left hand side means ordinary covariant differentiation on (M, g) while the meaning of $\nabla d\phi$ is given in (1.1).

By a *densely submersive map* $\phi : M \to N$ we shall mean a smooth map such that $d\phi_p : T_pM \to T_{\phi(p)}N$ is surjective for all p in a dense subset of M.

Proposition 2.4 *Let $\phi : M \to N$ be a smooth map. If $\kappa Ric = \phi^* h$ then $\tau[\phi](p) \in T_{\phi(p)}N$ is orthogonal to $d\phi(T_pM)$ for every $p \in M$. In particular, if additionally ϕ is densely submersive then ϕ is harmonic.*

Proof: In view of (2.1), taking the covariant derivative on both sides of $\kappa Ric = \phi^* h$ yields:

$$\kappa \nabla_\mu R_{\nu\rho} = \gamma_{\mu\nu\rho} + \gamma_{\mu\rho\nu} \ ,$$

where $\gamma_{\mu\nu\rho} := h_{ij}(\nabla_\mu \phi^i_{,\nu})\phi^j_{,\rho}$, and we notice the symmetry $\gamma_{\mu\nu\rho} = \gamma_{\nu\mu\rho}$. From this we find

$$\kappa(\nabla_\mu R_{\nu\rho} + \nabla_\nu R_{\rho\mu} - \nabla_\rho R_{\mu\nu}) = 2\gamma_{\mu\nu\rho} = 2h_{ij}(\nabla_\mu \phi^i_{,\nu})\phi^j_{,\rho} \ . \tag{2.2}$$

Contraction of (2.2) with $g^{\mu\nu}$ and application of the Bianchi identity yield the claimed orthogonality

$$\kappa(\nabla_\mu R^\mu_\rho + \nabla_\mu R^\mu_\rho - R_{,\rho}) = 0 = 2h(\tau[\phi], \phi_{,\rho}) \ .$$

In case that $d\phi_p$ is surjective for p in a dense subset of M this implies $\tau[\phi] = 0$. ■

Note that Proposition 2.4 holds for any signature of g. Also the same proof applies for arbitrary (nondegenerate) signature of h.

Example: By Proposition 2.4 it is obvious that the identical map $\phi = id$ from (M, g) to (M, h) solves (1.4) if and only if $\kappa Ric = h$. This shows that every manifold (M, g) with a nondegenerate Ricci tensor provides a solution of our problem.

An alternative proof of Proposition 2.4 runs as follows: $\kappa Ric = \phi^* h$ is equivalent to (1.2), and $Ric - (R/2)g$ is divergence–free. From Theorem 2.9 in [BaEe] then follows the assertion. However, the proof given above is shorter and more direct than the one in [BaEe] and in addition formula (2.2) allows to draw another conclusion which is a kind of converse of Proposition 2.3:

Corollary 2.5 *Let ϕ be densely submersive and $\kappa Ric = \phi^* h$. Then $\nabla Ric = 0$ implies that ϕ is totally geodesic.*

Example: Assume that $\kappa Ric = \phi^* h$ holds for an Einstein manifold (M, g), that means $Ric = c \cdot g$ with some constant c. Then either $\phi =$ const. (in case $c = 0$) or $\kappa c \cdot g = \phi^* h$, that means ϕ is a homothetic map. In the latter case $\nabla Ric = c\nabla g = 0$ and Corollary 2.5 implies that ϕ is also a totally geodesic map.

DeTurck [DeT] proved for the C^∞ and properly Riemannian case the local solvability of the equation

$$Ric[g] = K \ ,$$

with an arbitrarily prescribed, nondegenerate (as a quadratic form) tensor field K. More precisely, for such a C^∞–tensor K and for every point $p \in M$ there is a neighborhood $U(p)$ and a positive definite metric g on $U(p)$ such that $Ric[g] = K$. Let us combine this with Proposition 2.4:

Corollary 2.6 *Let ϕ be a local diffeomorphism from M into (N,h) and $p \in M$. Then there is an open neighborhood U of p and a properly Riemannian metric g on U such that ϕ harmonically maps (U,g) into (N,h) .*

3. Bochner and other techniques

Let us begin with an example that relates the numerical value of $\kappa > 0$ to the smallest eigenvalue λ_1 of $Ric[g]$ (with respect to g). We apply the following fact given in [EeLe], p. 16: *Let (M,g) be a compact, properly Riemannian manifold and (N,h) be the standard sphere $S^n \subset \mathbb{R}^{n+1}$. Then any harmonic map $\phi : M \to N$ satisfying $2e \le \lambda_1$ is constant.*

Now consider a non–constant harmonic map $\phi : M \to S^n$ which satisfies $\kappa Ric = \phi^* h$. Since $Ric \ge 0$ by Proposition 2.2, we have $\lambda_1 \ge 0$. From $\kappa R = 2e$ we find that $2e \le \lambda_1$ is equivalent to $\kappa R \le \lambda_1$. Since this cannot hold for non–constant ϕ we must have

$$\kappa > \frac{\inf\{\lambda_1(p)|p \in M\}}{\sup\{R(p)|p \in M\}} =: \kappa_0(g) \ .$$

Notice that $\kappa_0(g) > 0$ if $\lambda_1 > 0$. Thus we have the following non–existence result:

Proposition 3.1 *Let (M,g) be a compact Riemannian manifold with $\lambda_1 > 0$. If $0 < \kappa \le \kappa_0$ there is no harmonic map $\phi : M \to S^n$ which solves $\kappa Ric = \phi^* h$.*

For related results, cf. [Ghi,GhVi]. In the following we apply the "Bochner technique" for a closed (i.e. compact, connected, without boundary) manifold M. Its main ingredients are the arguments

$$\int_M \Delta u dvol = 0 \quad , \quad \int_M f dvol = 0 \ and \ f \ge 0 \ \Rightarrow \ f = 0 \ .$$

Theorem 3.2 *Let (M, g) be properly Riemannian, closed and conformal to a manifold (M', g') with vanishing scalar curvature. Then $\kappa Ric = \phi^*h$ implies that ϕ is constant.*

Proof: By assumption there exists a smooth function $u > 0$ on M such that g' is represented by $g' = u^p \cdot g$, $p = 4/(m-2)$, and g' has vanishing scalar curvature R', that means

$$0 = u^{\frac{m+2}{m-2}} R' = uR + 4\frac{m-1}{m-2}\Delta u \ ,$$

see [Bes], p. 59. Integration over M yields

$$\int_M uRdvol = 0 \ .$$

From $R \geq 0$ and $u > 0$ we conclude that $R = 0$, so ϕ is constant by Proposition 2.2. ■

Proposition 2.3 states: if ϕ is a totally geodesic map then $\nabla Ric = 0$. For such a map – in the properly Riemannian case – a factorisation $\phi = f \circ \gamma$ holds, see [Vil], and if M is closed and simply connected, M decomposes into $M_1 \times M_2$. We shall prove that already the condition $\nabla Ric = 0$ is sufficient for M to decompose. For preparation we need the following result.

Proposition 3.3 *Let g and ϕ solve (1.4). If $\nabla Ric = 0$ then for each $p \in M$ the subspace $ker(d\phi_p) \subset T_pM$ is invariant under the holonomy group at $p \in M$.*

Proof: It suffices to show that a vector $e_p \in ker(d\phi_p)$ is mapped to a vector $e_q \in ker(d\phi_q)$ under parallel transport along an arbitrary smooth curve $c : [0,1] \to M$ from p to q. Let e_t be the image of e_p under parallel transport at the point $c(t)$. Then

$$\begin{aligned} \frac{d}{dt}h(d\phi(e_t), d\phi(e_t)) &= \frac{d}{dt}\kappa Ric(e_t, e_t) \\ &= \kappa(\nabla_{\dot{c}(t)} Ric)(e_t, e_t) + 2\kappa Ric(\nabla_{\dot{c}(t)} e_t, e_t) = 0 \ , \end{aligned}$$

i.e. $h(d\phi(e_t), d\phi(e_t))$ does not depend on t. Since this function vanishes at $t = 0$ and h is positive definite it follows $d\phi(e_1) = d\phi(e_q) = 0$. ■

The de Rham decomposition theorem for complete, simply connected pseudo–Riemannian manifolds (M, g) states (see [Wu]): Let V be a nondegenerate subspace of T_pM (that means $V \neq \{0\}$ and $g|_V$ is regular) which is invariant under the holonomy group at p. Then (M, g) is isometrically isomorphic (denoted by $\cong$) to $(M_1 \times M_2, g_1 \oplus g_2)$. This leads to

Theorem 3.4 *Let (M,g) be complete and simply connected. Assume $\kappa Ric = \phi^*h$ and $\nabla Ric = 0$. If $ker(d\phi_p) \subset T_pM$ is nondegenerate then $(M,g) \cong (M_1 \times M_2, g_1 \oplus g_2)$. Moreover there is a splitting $\phi = \Phi \circ \pi$ of ϕ into $\pi : M_1 \times M_2 \to M_2$, the (harmonic) projection, and a harmonic immersion $\Phi : M_2 \to N$.*

Proof: The decomposition of (M,g) follows from the de Rham theorem and Proposition 3.3. The only thing to observe is that, in our situation, M_1 corresponds to the leaves of the foliation defined by the holonomy–invariant distribution, i.e. by $ker(d\phi)$. Thus ϕ is constant on the set $M_1 \times \{m_2\}$ for any $m_2 \in M_2$, hence $\Phi(m_2) := \phi(m_1, m_2)$ is a well defined smooth function on M_2 if we fix some $m_1 \in M_1$. Therefore $\phi = \Phi \circ \pi$. The harmonicity assertions are easily verified. ■

Next we shall study (1.4) under the assumption that (M,g) is a warped product manifold. Note that most of the cosmological models are warped products. Also in the mathematical literature it is wide–spread to study warped products as the next step after the study of direct Riemannian products. For the basic facts about warped products, and also for their relevance in physics, we refer to [O'Ne].

A warped product is a manifold $M = \hat{M} \times \check{M}$ equipped with a metric $g = \hat{g} \oplus_f \check{g}$, denoted $\hat{M} \times_f \check{M}$, where $f : \hat{M} \to I\!R$ is strictly positive. In the tangent space T_pM with base point $p = (\hat{p}, \check{p}) \in \hat{M} \times \check{M}$ the metric g is defined by

$$g((\hat{u}, \check{u}), (\hat{v}, \check{v})) = \hat{g}(\hat{u}, \hat{v}) + f^2(\hat{p})\check{g}(\check{u}, \check{v}))$$

where $(\hat{u}, \check{u}), (\hat{v}, \check{v}) \in T_pM = T_{\hat{p}}\hat{M} \oplus T_{\check{p}}\check{M}$. Since T_pM is a direct sum it is convenient to define $\hat{d}\phi := d\phi|_{T\hat{M}}$ and $\check{d}\phi := d\phi|_{T\check{M}}$. With this notation and with the decomposition of the Ricci tensor on a warped product, as given e.g. in [O'Ne], (1.3) splits into three parts:

$$\kappa(\hat{Ric} - \check{m} f^{-1}\hat{\nabla}\hat{d}f) = h(\hat{d}\phi, \hat{d}\phi) \ , \tag{3.1a}$$

$$0 = h(\hat{d}\phi, \check{d}\phi) \ , \tag{3.1b}$$

$$\kappa(\check{Ric} - \check{g}\frac{f^{2-\check{m}}}{\check{m}}\hat{\Delta}(f^{\check{m}})) = h(\check{d}\phi, \check{d}\phi) \ . \tag{3.1c}$$

Note that $\hat{\nabla}\hat{d}f$ has the components $\hat{\nabla}_\mu \hat{\partial}_\nu f = f_{,\mu\nu} - \hat{\Gamma}^\rho_{\mu\nu} f_{,\rho}$ for $\mu, \nu = 1, \ldots, \hat{m}$. If M is an ordinary product (i.e. $f = 1$) it seems at first sight that one obtains two independent Einstein equations, one on $\check{M}$ and one on $\hat{M}$. But this is not true because (3.1b) means that the tangent mapping of ϕ along $\hat{M}$ must map into the orthogonal space of the tangent mapping of ϕ along $\check{M}$. In general it is therefore not possible to extend a solution given on $(\check{M}, \check{g})$ in a non–trivial way to $(\hat{M}, \hat{g}) \times (\check{M}, \check{g})$:

Proposition 3.5 *Let* $(M, g) = (\hat{M} \times \check{M}, \hat{g} \oplus_f \check{g})$. *Assume that* $\phi(\hat{p}, \cdot)$ *is a submersion for each* $\hat{p} \in \hat{M}$ *and* $\kappa Ric = \phi^* h$. *Then* $\phi(\cdot, \check{p})$ *is constant for each* $\check{p} \in \check{M}$.

Proof: We evaluate (3.1b) on an arbitrary pair of basis vectors $\hat{\partial}_\mu \in T_{\hat{p}}\hat{M}$ and $\check{\partial}_\nu \in T_{\check{p}}\check{M}$: $h(\hat{\partial}_\mu \phi, \check{\partial}_\nu \phi) = 0$. Since $\{\check{\partial}_\nu \phi |\ \nu = \hat{m}+1, \ldots, \hat{m}+\check{m}\}$ spans $T_{\phi(\check{p})}N$ and h is regular it follows that $\hat{\partial}_\mu \phi = 0$ for all $\mu = 1, \ldots, \hat{m}$, which means $\hat{d}\phi = 0$ at each $\hat{p} \in \hat{M}$. ■

Let us derive a formula from (3.1a) by contraction with $\hat{g}$ and by multiplication with f:

$$\kappa f \hat{R} - \check{m} \hat{\Delta} f = 2 f \hat{e} \ , \tag{3.2}$$

where $\hat{e} = \frac{1}{2} \hat{g}^{\mu\nu} \phi^i_{,\mu} \phi^j_{,\nu} h_{ij}$. Analogously it follows from (3.1c):

$$\kappa f^{\check{m}-2} \check{R} - \kappa \hat{\Delta} f^{\check{m}} = 2 f^{\check{m}-2} \check{e} \ , \tag{3.3}$$

with $\check{e} = \frac{1}{2} \check{g}^{\mu\nu} \phi^i_{,\mu} \phi^j_{,\nu} h_{ij}$. The total energy density is given by $e = \hat{e} + f^2 \check{e}$.

Proposition 3.6 *Assume that* $(\hat{M}, \hat{g})$ *is a closed and properly Riemannian manifold, and let* $(M, g) = (\hat{M} \times \check{M}, \hat{g} \oplus_f \check{g})$. *Then* $\kappa Ric = \phi^* h$ *implies*

$$\kappa \int_{\hat{M}} f \hat{R} \hat{d}vol \geq 0 \ .$$

This integral vanishes if and only if f *is constant and* ϕ *is constant on each leaf* $\hat{M} \times \{\check{p}\}$.

Proof: We integrate (3.2) over $\hat{M}$ and obtain

$$\kappa \int_{\hat{M}} f \hat{R} \hat{d}vol = \int_{\hat{M}} 2 \hat{e} f \hat{d}vol \geq 0 \ .$$

The last integral only vanishes for $\hat{e} = 0$, i.e. for $\hat{d}\phi = 0$, so that $\phi(\cdot, \check{p})$ is constant. But then $\check{e}(\cdot, \check{p})$ is constant, so (3.3) yields that

$$\kappa \frac{\hat{\Delta} f^{\check{m}}}{f^{\check{m}-2}} = \kappa \check{R} - 2\check{e}$$

is constant on $\hat{M} \times \{\check{p}\}$. Since the lefthand side depends only on $\hat{p}$ this means that it is actually a constant c. Integrating the equality $\kappa \hat{\Delta} f^{\check{m}} = c f^{\check{m}-2}$ over $\hat{M}$ finally yields $c = 0$, hence f is constant. ■

Example: The one–dimensional closed manifold $\hat{M} = S^1$ is flat, that means $\hat{R} = 0$. Thus a solution of $\kappa Ric = \phi^* h$ on $S^1 \times_f \check{M}$ can only exist for $f =$ const., and then $\phi(\cdot, \check{p})$ must be constant, too.

By the same type of arguments we now derive properties of $\check{Ric}$ and $\check{R}$; but here we must assume that $\check{g}$ is definite, because only in this case we can conclude $\check{e} \geq 0$ or $\check{e} \leq 0$.

Theorem 3.7 *Let $\hat{g}$ and $\epsilon\check{g}$ for $\epsilon = 1$ or -1 be properly Riemannian and let $\hat{M} \times_f \check{M}$ have a closed first factor. Then $\kappa Ric = \phi^* h$ implies*

1) $\kappa\check{Ric} \geq 0$. If $\kappa\check{Ric}$ is indefinite at some point $\check{p} \in \check{M}$ then f is constant.

2) $\kappa\check{R}\epsilon \geq 0$. Moreover, $\kappa\check{R} = 0$ if and only if f and $\phi(\hat{p}, \cdot)$ are constant for all $\hat{p} \in \hat{M}$.

Proof: 1) Multiplication of (3.1c) with $f^{\check{m}-2}$ gives

$$\kappa f^{\check{m}-2}\check{Ric} - \frac{\kappa}{\check{m}}\check{g}\hat{\Delta} f^{\check{m}} = f^{\check{m}-2} h(\check{d}\phi, \check{d}\phi) \ . \tag{3.4}$$

We evaluate this on a fixed vector $\check{v}_{\check{p}}$ and integrate over $\hat{M}$:

$$\kappa\check{Ric}(\check{v}_{\check{p}}, \check{v}_{\check{p}}) \int_{\hat{M}} f^{\check{m}-2} \hat{d}vol = \int_{\hat{M}} f^{\check{m}-2} h(\check{d}\phi(\check{v}_{\check{p}}), \check{d}\phi(\check{v}_{\check{p}}))\hat{d}vol \ . \tag{3.5}$$

From this we read off $\kappa\check{Ric} \geq 0$. If there is a point $\check{v}_{\check{p}} \in T\check{M}$ such that $\check{Ric}(\check{v}_{\check{p}}, \check{v}_{\check{p}}) = 0$ then (3.5) implies $\check{d}\phi(\check{v}_{\check{p}}) = 0$. Evaluation of (3.4) at $\check{v}_{\check{p}}$ gives $\hat{\Delta} f^{\check{m}} = 0$, so f is constant.

2) Integration of (3.3) over $\hat{M}$ yields

$$\kappa\check{R} \int_{\hat{M}} f^{\check{m}-2} \hat{d}vol = \int_{\hat{M}} 2\check{e} f^{\check{m}-2} \hat{d}vol \tag{3.6}$$

which gives $\epsilon\check{R} \geq 0$ as well as $\check{R} = 0 \iff \check{e} = 0$. Hence $\check{R} = 0$ implies $\check{d}\phi = 0$, so $\phi(\hat{p}, \cdot)$ is constant and (3.3) then implies $\hat{\Delta} f^{\check{m}} = 0$, i.e. f is constant.

That $f =$ const. and $\phi(\hat{p}, \cdot) =$ const. imply $\check{R} = 0$ is obvious by (3.3). ∎

Note that $\kappa\check{Ric} \geq 0$ holds for any signature of $\check{g}$ because the condition $\epsilon\check{g} \geq 0$ is not needed for this part of the proof. Note also that (3.6) provides an interpretation for $\check{R}$ similar to the general equality $\kappa R = 2e$: The $\hat{m}$–form

$$\omega = \frac{f^{\check{m}-2}\hat{d}vol}{\int_{\hat{M}} f^{\check{m}-2}\hat{d}vol}$$

defines a normed measure on $\hat{M}$ and $\kappa\check{R}$ is the mean value of $2\check{e}$ with respect to it: $\kappa\check{R} = \int_{\hat{M}} 2\check{e}\omega$.

We finally give an example for Proposition 3.6 which is similar to the foregoing one.

Example: Let the product $\hat{M} \times \check{M}$ of a closed space manifold $\hat{M}$ and a time axis $\check{M} \subset \mathbb{R}$ be equipped with the static metric $g = \hat{g} \oplus_f (-dt^2)$. Then $\check{R} = 0$. Thus a solution of $\kappa Ric = \phi^* h$ can only exist if f is constant, and then ϕ does not depend on t.

4. Discussion

For fixed ϕ the field equation $\kappa Ric[g] = \phi^* h$ for g is highly nonlinear, while for fixed g the field equation $\tau[\phi] = 0$ for ϕ is semilinear. The coupling between g and ϕ is nonlinear too. More precisely, g enters $tr(\nabla d\phi) = 0$ by the so–called minimal coupling, that means through the Christoffel symbols in the covariant derivatives, while ϕ enters $\kappa Ric = \phi^* h$ as a matter source, that means through the right hand side $\phi^* h$.

The validity of the Einstein equation imposes conditions on g and conditions on ϕ. For instance, κRic has to be positive semi–definite (a condition on g) and $h(\tau, d\phi) = 0$ (a condition on ϕ). More generally, imposing a condition on the source manifold (M, g) implies – by the coupling – a condition on ϕ and vice versa. We have seen that mild conditions on (M, g) or ϕ imply stronger conditions on these objects if (1.4) is satisfied.

Note that in general relativity theory the problem to find necessary and/or sufficient conditions on (M, g) to satisfy the Einstein equation

$$\kappa(Ric - \frac{R}{2} g) = T$$

for an unspecified matter field of some given type is called "geometrodynamics". A part of our results can be interpreted in this way. A complete solution for the geometrodynamical problem for harmonic maps is not yet available.

A harmonic map coupled to Einstein gravitation has repeatedly been studied in the literature; let us cite [AFF,AuSa,Gh,GhCo,GhVi,GMZ1,GMZ2,IV1,IV2,OmPe,ScHi,Vis]. The authors, motivated by physical considerations, mainly discuss the following points:

- The field equations, sometimes extended by a "cosmological term", and elementary properties of that system.
- The size of κ as compared to other quantities, e.g. the cosmological constant.
- The relation between Killing vector fields of (M, g) and symmetries of ϕ.
- Solutions ϕ which are submersions and their application in the realm of Kaluza–Klein theory. In particular, the so–called "compactification induced by the matter field ϕ" is a major subject.
- Construction of special solutions. Typically, the authors take a known class of gravitational fields (e.g. plane–symmetric, cosmological models,...) and try to complete a given g from this class by some ϕ to obtain a solution (g, ϕ) of the coupled equations.

Note that physicists often prefer the name "sigma models" for harmonic maps. But, in a narrower sense, a sigma model has a homogeneous target space, that means $N = G/H$, where G is some Lie group, H is some Lie subgroup and h is a left–invariant metric on N.

Our approach differs from the quoted papers: we study the system from the mathematical point of view and without the assumption that (N, h) is a homogeneous space. There are enough open problems, physical and mathematical ones, which make harmonic maps coupled to Einstein gravitation a subject for further research.

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