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Abstract. We introduce a family of trigonometric polynomials, denoted as Stancu polynomials, which covers as special cases the trigonometric Lagrange and Bernstein polynomials. This family depends only on one real parameter, denoted as *design parameter.* Our approach works for curves as well as for surfaces over triangles. The resulting Stancu curves resp. surfaces therefore establish a link between trigonometric interpolatory and Bernstein-Bézier curves resp. surfaces.

Keywords. Trigonometric Lagrange polynomials, trigonometric Bernstein polynomials, Stancu polynomials, design parameter.

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0. Introduction

Recently, trigonometric splines and polynomials have gained very much interest within CAGD, in particular curve design, see for example Alfeld, Neamtu, Schumaker (1995), Koch, Lyche, Neamtu, Schumaker (1995), Gonsor & Neamtu (1996), Peña (1996), Walz (1997). The aim of the present note is twofold: First, similarly as done in the (algebraic) polynomials case in Farin & Barry (1986) and Walz (1988, 1991), we introduce a one-parameter family of trigonometric polynomials, which provides a link between the trigonometric Lagrange polynomials (1.1) and Bernstein polynomials (1.2). Then, in section 2, this approach will be transfered to trigonometric polynomials (in barycentric coordinates) on a triangle.

1. Stancu Polynomials on an Interval

We consider, for even integer n, the $(n+1)-$ dimensional space of trigonometric polynomials

$$\mathcal{T}_m \ := \ \text{span}\{1, \cos(2\tau), \sin(2\tau), \ldots, \cos(n\tau), \sin(n\tau)\}$$

where $0 \leq \tau \leq 1$. The *Lagrange* polynomials (w.r.t. equidistant nodes) from $\mathcal{T}_m$ are the functions

$$l_\nu(\tau) \ = \ \prod_{\substack{j=0 \\ j\neq\nu}}^{n} \frac{\sin(\tau - \frac{j}{n})}{\sin(\frac{\nu-j}{n})}\,, \quad \nu = 0, \ldots, n\,, \tag{1.1}$$

whereas the corresponding *Bernstein* polynomials are

$$b_\nu(\tau) \ = \ \frac{\sin^\nu(\tau)\cdot\sin^{n-\nu}(1-\tau)}{\sin^n(1)}\,, \quad \nu = 0, \ldots, n\,. \tag{1.2}$$

Both sets of functions form a basis of $\mathcal{T}_m$. Moreover, since $l_\nu(\frac{\mu}{n}) \ = \ \delta_{\mu\nu}$ for $\mu, \nu = 0, \ldots, n$, the trigonometric polynomial (curve)

$$L(\tau) \ = \ \sum_{\nu=0}^{n} d_\nu l_\nu(\tau) \tag{1.3}$$

interpolates the points $d_0, \ldots, d_n$, whereas the trigonometric Bernstein-Bézier curve

$$B(\tau) \ = \ \sum_{\nu=0}^{n} d_\nu b_\nu(\tau) \tag{1.4}$$

possesses nice shape-preserving properties.

We now make the following construction: With a real parameter a, $0 \leq a \leq \frac{1}{n}$, and the variable τ we set

$$\varphi_k(\tau) \ := \ \prod_{j=0}^{k-1} \sin(\tau - ja)\,, \tag{1.5}$$

for $k \in I\!N$, and $\varphi_0 \equiv 1$. Then, for each $\nu \in \{0, \ldots, n\}$, the trigonometric polynomial

$$s_\nu(\tau, a) \ := \ \frac{\varphi_\nu(\tau, a)\cdot\varphi_{n-\nu}(1-\tau, a)}{\varphi_\nu(\tau_a, a)\cdot\varphi_{n-\nu}(\tau_a, a)}\,, \tag{1.6}$$

where $\tau_a := 1 - a(n+1)$, is denoted as ν^{th} *Stancu polynomial.*
Obviously, $s_\nu \in \mathcal{T}_m$. Moreover, the following result holds:

Theorem 1: *a) For each parameter value* $0 \leq a \leq \frac{1}{n}$, *it is*

$$\mathcal{T}_m = \operatorname{span}\{s_0(\cdot, a), \ldots, s_n(\cdot, a)\} .$$

b) For each $\nu \in \{0, \ldots, n\}$, *we have*

$$s_\nu(\tau, a) = \begin{cases} b_\nu(\tau) & \text{for } a = 0 \text{ , and} \\ l_\nu(\tau) & \text{for } a = \frac{1}{n} \text{ .} \end{cases}$$

Proof. We consider the relation

$$\sum_{\nu=0}^{n} c_\nu s_\nu(\tau, a) = 0 \quad \text{for } \tau \in [0,1] . \tag{1.7}$$

In particular, (1.7) must hold for $\tau = 0$. Since $s_\nu(0, a) = 0$ for $\nu = 1, \ldots, n$, it follows that c_0 must be zero. Analogously, setting $\tau = a, 2a, \ldots, (n-1)a$, it follows successively that $c_1 = \cdots c_n = 0$. This proves the linear independence of the functions s_ν and therefore statement a). Statement b) can be verified by straightforward calculation. □

Theorem 1 shows in particular that the *trigonometric Stancu Curve*

$$S(\tau, a) = \sum_{\nu=0}^{n} d_\nu s_\nu(\tau, a) \tag{1.8}$$

establishes a link between the interpolatory and the shape preserving curves (1.3) and (1.4). The real parameter a should be denoted as *design parameter.*

2. Stancu Polynomials over a Triangle

In this section we transfer the idea described in Section 1 to the case of bivariate trigonometric polynomials over a triangle. So, as a special case, we will also obtain trigonometric triangular Bernstein Bézier polynomials. The corresponding (algebraic) polynomials were introduced and investigated in (Farin, 1986) resp. (Walz, 1991).

As usual, we describe a point P on a given fixed triangle $< T_1, T_2, T_3 >$ by its barycentric coordinates (τ_1, τ_2, τ_3), i.e.,

$$P = \tau_1 T_1 + \tau_2 T_2 + \tau_3 T_3 ,$$
$$1 = \tau_1 + \tau_2 + \tau_3 .$$

We will use the index set

$$N_n := \{(\nu_1, \nu_2, \nu_3) \in \mathbb{N}_0^3 \, ; \, \nu_1 + \nu_2 + \nu_3 = n\}$$

with $\binom{n+2}{2}$ elements. Then, with the auxiliary functions φ_k and the point τ_a from above, we define for each $(\nu_1, \nu_2, \nu_3) \in N_n$ the *Stancu polynomial* over a triangle as

$$s_{(\nu_1,\nu_2,\nu_3)}(P, a) := \frac{\varphi_{\nu_1}(\tau_1, a) \cdot \varphi_{\nu_2}(\tau_2, a) \cdot \varphi_{\nu_3}(\tau_3, a)}{\varphi_{\nu_1}(\tau_a, a) \cdot \varphi_{\nu_2}(\tau_a, a) \cdot \varphi_{\nu_3}(\tau_a, a)} . \tag{2.1}$$

The following theorem, which also can be proved by straightforward calculation, shows that this one-parameter family of functions contains the trigonometric Bernstein polynomials

$$b_{(\nu_1,\nu_2,\nu_3)}(P) \;=\; \frac{\sin^{\nu_1}(\tau_1)\cdot\sin^{\nu_2}(\tau_2)\cdot\sin^{\nu_3}(\tau_3)}{\sin^n(1)} \tag{2.2}$$

as well as the trigonometric Lagrange polynomials

$$l_{(\nu_1,\nu_2,\nu_3)}(P) \;=\; \prod_{j_1=0}^{\nu_1-1}\frac{\sin(\tau_1-\frac{j_1}{n})}{\sin(\frac{j_1+1}{n})}\cdot\prod_{j_2=0}^{\nu_2-1}\frac{\sin(\tau_2-\frac{j_2}{n})}{\sin(\frac{j_2+1}{n})}\cdot\prod_{j_3=0}^{\nu_3-1}\frac{\sin(\tau_3-\frac{j_3}{n})}{\sin(\frac{j_3+1}{n})} \tag{2.3}$$

over triangles as special cases.

Theorem 2: *For each* $(\nu_1,\nu_2,\nu_3)\in N_n$ *, we have*

$$s_{(\nu_1,\nu_2,\nu_3)}(P,a) \;=\; \begin{cases} b_{(\nu_1,\nu_2,\nu_3)}(P) & \text{for } a=0\ , \text{ and}\\ l_{(\nu_1,\nu_2,\nu_3)}(P) & \text{for } a=\frac{1}{n}. \end{cases}$$

Corollary: *For each set of* $\binom{n+2}{2}$ *points* $\{\, d_{(\nu_1,\nu_2,\nu_3)}\ ;\ (\nu_1,\nu_2,\nu_3)\in N_n \,\}$ *, the surface*

$$L(P) \;:=\; \sum_{(\nu_1,\nu_2,\nu_3)\in N_n} d_{(\nu_1,\nu_2,\nu_3)}\cdot l_{(\nu_1,\nu_2,\nu_3)}(P)$$

possesses for all $(i_1,i_2,i_3)\in N_n$ *the interpolation property*

$$L(\tfrac{i_1}{n},\tfrac{i_2}{n},\tfrac{i_3}{n}) \;=\; d_{(i_1,i_2,i_3)}\,.$$

Proof. Follows from the fact that for all (i_1,i_2,i_3) and $(\nu_1,\nu_2,\nu_3)\in N_n$,

$$l_{(\nu_1,\nu_2,\nu_3)}(\tfrac{i_1}{n},\tfrac{i_2}{n},\tfrac{i_3}{n}) \;=\; \begin{cases} 1, & \text{if } (i_1,i_2,i_3)=(\nu_1,\nu_2,\nu_3)\,,\\ 0, & \text{otherwise.}\end{cases}$$

□

References

Alfeld, P., Neamtu, M. & Schumaker, L.L. (1995), Circular Bernstein-Bèzier Polynomials, in: *Mathematical Methods for Curves and Surfaces*, M. Daehlen, T. Lyche & L.L. Schumaker eds., Vanderbilt University Presss, Nashville, 11 – 20.

Farin, G. (1986), Triangular Bernstein-Bézier Patches, Computer Aided Geom. Design 3, 83 – 127.

Farin, G. & Barry, P.J. (1986), Link between Bézier and Lagrange Curve and Surface Schemes, Computer-Aided Design 18, 525 – 528.

Gonsor, D. & Neamtu, M. (1996), Null Spaces of Differential Operators, Polar Forms and Splines, J. Approx. Theory 86, 81 – 107.

Koch, P.E., Lyche, T., Neamtu, M. & Schumaker, L.L. (1995), Control Curves and Knot Insertion for Trigonometric Splines, Adv. in Comp. Maths. 3, 405 – 424.

Peña, J.M. (1996), Shape Preserving Representations for Trigonometric Polynomial Curves, Preprint, University of Zaragoza.

Walz, G. (1988), On Generalized Bernstein Polynomials in CAGD, Mannh. Math. Manuskr. 86.

Walz, G. (1991), *Spline-Funktionen im Komplexen,* B.I.-Wissenschaftsverlag, Mannheim/ Wien/Zürich.

Walz, G. (1997), Some Identities for Trigonometric B-Splines, with Application to Curve Design, to appear in BIT.

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