Romberg Type Cubature over Arbitrary Triangles

Guido Walz

Nr. 225/1997

Priv.-Doz. Dr. Guido Walz
Lehrstuhl für Mathematik IV
Universität Mannheim
D-68131 Mannheim
Germany
walz@math.uni-mannheim.de
http://www.math.uni-mannheim.de/~math4/walz.htm

Romberg Type Cubature over Arbitrary Triangles

Guido Walz

Abstract. We develop an extrapolation algorithm for numerical integration over arbitary non-standard triangles in $\mathbb{R}^2$, which parallels the well-known univariate Romberg method. This is done by a suitable generalization of the trapezoidal rule over triangles, for which we can prove the existence of an asymptotic expansion.

Our approach relies mainly on two ideas: The use of barycentric coordinates and the interpretation of the trapezoidal rule as the integral over an interpolating linear spline function.

Since our method works for arbitrary triangles, it yields – via triangulation – a method for cubature over arbitrary, possibly non-convex, polygon regions in $\mathbb{R}^2$. Moreover, also numerical integration over convex polyhedra in $\mathbb{R}^d$, $d > 2$, can be accomplished without difficulties.

Numerical examples show the stability and efficiency of the algorithm.

Keywords. Romberg method, cubature, numerical integration, triangulation, polygon region, asymptotic expansion, extrapolation

AMS (MOS) Classification. 65 D 30, 65 B 05, 41 A 60

Guido Walz
Department of Mathematics and Computer Science
University of Mannheim
D-68131 Mannheim, Germany
walz@math.uni-mannheim.de

0. Introduction

The aim of this paper is to present an extrapolation algorithm for numerical integration (cubature) over arbitary triangles in $\mathbb{R}^2$, which parallels almost completely the well-known unvariate Romberg method, introduced by W. Romberg in 1955 [16], and since then investigated by a lot of authors, e.g. [1,2,7,8,9]; see [3,4,5,17] for a historical survey.

Therefore, we first have to prove the existence of an asymptotic expansion for a suitable generalization of the trapezoidal rule over non-standard triangles; this rule is developed in Theorem 1.2, its asymptotic expansion is established in Theorem 1.3.

Having established the existence of the desired asymptotic expansion, we know that extrapolation can be applied in order to improve the convergence order of the method considerably (cf. [6,17]). Our method consists now in the application of *the same* extrapolation process as it is known from the univariate Romberg method, and therefore is very efficient and numericallly stable.

Our approach relies mainly on two ideas: We represent the points in the integration domain by their barycentric coordinates (which makes it coordinate-independent and therefore applicable to any triangle in $\mathbb{R}^2$), and we interpret the trapezoidal rule as the integral over a linear spline function, which interpolates the given function in the spline knots. This approach appears to be new, and in order to get the reader used with it we first apply it briefly to the univariate case. It turns out that the proofs can be shortened considerably, in particular no Euler summation is needed.

Since our method works for arbitrary triangles, it yields – via triangulation – a method for cubature over arbitrary, possibly non-convex, polygon regions in $\mathbb{R}^2$. Moreover, also numerical integration over convex polyhedra in $\mathbb{R}^d$, $d > 2$, can be accomplished by our approach, since its main ingredients (barycentric coordinates and interpolating splines) also exist in this situation.

Of course, the problem of integration over triangles by extrapolation was considered before; instead of presenting a list of the existing literature in this field, we refer the reader to the excellent survey paper by Lyness & Cools [14], where all information can be found. However, all known approaches seem to be designed for special standard triangles, and so it is for example difficult to use them for integration over polygon regions. Therefore, the approach to be given below is a certainly a step forward.

1. Asymptotic Expansion for Trapezoidal Rule over Triangles

As pointed out in the introduction, a main idea is to introduce barycentric coordinates and to interpret the trapezoidal rule as integral over the linear spline function, which interpolates the given function f in equispaced points. Since the reader might be unfamiliar with bivariate spline interpolation, we first very briefly illustrate the idea in the univariate case and give, in particular, a short self-contained proof for the existence of the asymptotic expansion.

We are interested in the numerical computation of the integral

$$I^{[1]}(f) \; := \int_a^b f(x)\,dx \tag{1.1}$$

(the index 1 indicates that we are still in the univariate situation) for a sufficiently smooth function f. If we formally describe each point $x \in [a,b]$ by its *barycentric coordinates* (τ_1, τ_2), which are uniquely defined as the solutions of the system

$$\begin{aligned} \tau_1 a \;+\; \tau_2 b \;&=\; x\,, \\ \tau_1 \;+\; \tau_2 \;&=\; 1\,, \end{aligned} \tag{1.2}$$

then the *trapezoidal rule* (with n subintervals) can be written as

$$T_n^{[1]}(f) \;=\; \frac{b-a}{2n} \cdot \left(f(x_0) + 2\sum_{\nu=1}^{n-1} f(x_\nu) + f(x_n) \right), \tag{1.3}$$

where, for $\nu = 0, \ldots, n$, the point x_ν has the barycentric coordinates

$$x_\nu \;=\; \left(\frac{n-\nu}{n}, \frac{\nu}{n} \right). \tag{1.4}$$

Theorem 1.1: *If the function f is sufficiently smooth, then the elements of the sequence $\{T_n^{[1]}(f)\}$ possess an asymptotic expansion of the form*

$$T_n^{[1]}(f) \;=\; I^{[1]}(f) \;+ \sum_{\mu=1}^{r} \frac{c_\mu}{n^{2\mu}} \;+\; o(n^{-2r}) \qquad \text{for } n \to \infty\,. \tag{1.5}$$

Here, the number r depends on the smoothness of f.

We give a short proof of this well-known result, which has - in contrast to the classical ones – the advantage that it can easily be generalized to the bivariate case, see below.

Proof of Theorem 1.1. By $s_n^{[1]}$, we denote the unique linear spline function with knots x_ν, which interpolates f in these knots. Then

$$T_n^{[1]}(f) \;=\; I^{[1]}(s_n^{[1]}), \tag{1.6}$$

and consequently the quadrature error can be written as

$$I^{[1]}(f) - T_n^{[1]}(f) \;=\; I^{[1]}(f - s_n^{[1]})\,. \tag{1.7}$$

We now claim that, for each $x \in [a,b]$, the interpolation error has an asymptotic expansion of the form

$$f(x) - s_n^{[1]}(x) = \sum_{\mu=2}^{\rho} \frac{\gamma_\mu(x)}{n^\mu} + o(n^{-\rho}) \quad \text{for } n \to \infty \tag{1.8}$$

with some integer ρ, which depends on the smoothness of f, and coefficient functions $\gamma_\mu(x)$, which do not depend on n.

This can either be deduced from the result in [10] and [13], or seen directly as follows: Let, for each ν, p_ν denote the restriction of $s_n^{[1]}$ to the subinterval $[x_\nu, x_{\nu+1}]$. Since $p_\nu \in \Pi_1$ interpolates f in the knots x_ν and $x_{\nu+1}$, for each x there exists a point $\xi_x \in [x_\nu, x_{\nu+1}]$ such that

$$f(x) - p_\nu(x) = \frac{(x - x_\nu)(x - x_{\nu+1})}{2} \cdot f^{(2)}(\xi_x). \tag{1.9}$$

Since f is sufficiently smooth, we may use Taylor-expansion of $f^{(2)}$ about x_ν and write (1.9) as

$$f(x) - p_\nu(x) = \frac{(x - x_\nu)(x - x_{\nu+1})}{2} \cdot \left(\sum_{\mu=2}^{\rho} \frac{(\xi_x - x_\nu)^{\mu-2}}{(\mu-2)!} \cdot f^{(\mu)}(x_\nu) + o((\xi_x - x_\nu)^{\rho-2}) \right),$$

and since $x_{\nu+1} - x_\nu = (b-a)/n$ and $(\xi_x - x_\nu) \le (x_{\nu+1} - x_\nu)$, this leads to the estimate

$$|f(x) - p_\nu(x)| \le \frac{1}{n^2} \cdot \sum_{\mu=2}^{\rho} \frac{\tilde{\gamma}_\mu(\nu, x)}{n^{\mu-2}} + o(n^{-\rho}) \tag{1.10}$$

with some coefficient functions $\tilde{\gamma}_\mu$, which depend on x and ν, but not on n.

This proves the existence of the expansion (cf. e.g.[17, Lemma 1.4 or P1.1])

$$f(x) - p_\nu(x) = \sum_{\mu=2}^{\rho} \frac{\tilde{\gamma}_\mu(\nu, x)}{n^\mu} + o(n^{-\rho}), \quad \text{for } n \to \infty$$

and putting $\gamma_\mu(x) := \tilde{\gamma}_\mu(\nu, x)$ for $x \in [x_\nu, x_{\nu+1}]$, (1.8) is proved (note that some of the γ_μ's might be zero).

Using now (1.7), (1.8), and the mean value theorem of integration, we obtain that

$$\begin{aligned} I^{[1]}(f) - T_n^{[1]}(f) &= \sum_{\mu=2}^{\rho} \frac{1}{n^\mu} I^{[1]}(\gamma_\mu(x)) + o(n^{-\rho}) \quad \text{for } n \to \infty \\ &= \sum_{\mu=2}^{\rho} \frac{1}{n^\mu} \gamma_\mu(\xi_\mu) + o(n^{-\rho}) \quad \text{for } n \to \infty \end{aligned} \tag{1.11}$$

with $\xi_\mu \in [a,b]$. Since the trapezoidal rule is a symmetric rule, if we interpret it again as a spline collocation method (i.e., $T_n^{[1]} = T_{-n}^{[1]}$), we see that the terms with odd indices in (1.11) must be zero, and after setting

$$r = [\rho/2] \quad \text{and} \quad c_\mu = -\gamma_{2\mu}(\xi_{2\mu}) \quad \text{for } \mu = 1, \ldots, r,$$

Theorem 1.1 is established. □

Remark. In contrast to the standard proofs via Euler summation (cf. [17]), with this method of proof we do not get an explicit representation of the coefficients c_μ through the Bernoulli numbers. However, for the application of an extrapolation method this is not needed, since the *existence* of the expansion is information enough.

The (univariate) Romberg method consists in applying repeated Richardson extrapolation (cf. (2.2)) to the elements of the sequence $\{T_n^{[1]}(f)\}$, which is justified by Theorem 1.1.

We now transfer this approach to the bivariate case. Let there be given a non-degenerate triangle $\Delta = \Delta(v_1, v_2, v_3)$ in $\mathbb{R}^2$ with vertices v_1, v_2, v_3. By $|\Delta|$, we denote the area of Δ.

For each point z in $\mathbb{R}^2$, there exists a unique triple (τ_1, τ_2, τ_3) of real numbers, such that

$$\begin{aligned} \sum_{i=1}^{3} \tau_i v_i &= z\,, \\ \sum_{i=1}^{3} \tau_i &= 1\,, \end{aligned} \tag{1.12}$$

called the barycentric coordinates of z with respect to the points $\{v_1, v_2, v_3\}$; if $z \in \Delta$, then all barycentric coordinates of z are non-negative.

If there is no confusion possible, we also drop the v_i's and identify z with the triple (τ_1, τ_2, τ_3). Also, we will not distinguish between the two representations of a function as a function of $z = (x, y)$ or $z = (\tau_1, \tau_2, \tau_3)$.

Given a function $f \in C(\Delta)$, we are now interested in the numerical computation of the integral

$$I^{[2]}(f) := \int_\Delta f(x,y)\,dx\,dy \tag{1.13}$$

by a proper generalization of the trapezoidal rule (1.3).

To do this, we introduce for $n \in \mathbb{N}$ the index set

$$N_n := \{(\nu_1, \nu_2, \nu_3) \in \mathbb{N}_0^3\,;\ \nu_1 + \nu_2 + \nu_3 = n\}$$

with $\binom{n+2}{2}$ elements. Then, for each $(\nu_1, \nu_2, \nu_3) \in N_n$, we define the point $z_{(\nu_1,\nu_2,\nu_3)} \in \Delta$ by setting its barycentric coordinates to

$$z_{(\nu_1,\nu_2,\nu_3)} := \left(\frac{\nu_1}{n}, \frac{\nu_2}{n}, \frac{\nu_3}{n}\right). \tag{1.14}$$

Then the set of points

$$\left\{z_{(\nu_1,\nu_2,\nu_3)}\,;\ (\nu_1, \nu_2, \nu_3) \in N_n\right\} \tag{1.15}$$

defines an equispaced triangulation of the initial triangle Δ into n^2 subtriangles

$$\{\delta_j\}_{j=1,\dots,n^2}\,,$$

each of which has the same area $|\Delta|/n^2$.

It is well-known that there exists a uniquely determined piecewise linear spline function over this triangulation, which interpolates f in the vertices $z_{(\nu_1,\nu_2,\nu_3)}$ (e.g. [15]). Let us denote this bivariate spline by $s_n^{[2]}$; then we have the following result.

Theorem 1.2: *Adopting the notations from above, the integral of the spline function* $s_n^{[2]}$ *over the triangle* Δ *is explicitly given by the formula*

$$\begin{aligned} I^{[2]}(s_n^{[2]}) &= \frac{|\Delta|}{3\cdot n^2}\cdot\Bigg\{ f(1,0,0) + f(0,1,0) + f(0,0,1) \\ &+ 3\cdot\sum_{\nu=1}^{n-1}\left(f\Big(\frac{\nu}{n},\frac{n-\nu}{n},0\Big) + f\Big(\frac{\nu}{n},0,\frac{n-\nu}{n}\Big) + f\Big(0,\frac{\nu}{n},\frac{n-\nu}{n}\Big)\right) \\ &+ 6\cdot\sum_{\nu_1=1}^{n-2}\sum_{\nu_2=1}^{n-\nu_1-1} f\Big(\frac{\nu_1}{n},\frac{\nu_2}{n},\frac{n-\nu_1-\nu_2}{n}\Big)\Bigg\}. \end{aligned} \tag{1.16}$$

Here, as usual, empty sums are set equal to zero.

Proof. Let δ_j denote an arbitrary subtriangle of the triangulation under consideration, and denote the vertices of δ_j for the moment by $\zeta_{j,1}, \zeta_{j,2}$, and $\zeta_{j,3}$.

There exists a unique linear polynomial p_j, which interpolates f in these vertices. The integral of p_j over δ_j is easily verified to be

$$\begin{aligned} \int_{\delta_j} p_j(x,y)\,dx\,dy &= \frac{|\delta_j|}{3}\cdot\Big(f(\zeta_{j,1}) + f(\zeta_{j,2}) + f(\zeta_{j,3})\Big) \\ &= \frac{|\Delta|}{3\cdot n^2}\cdot\Big(f(\zeta_{j,1}) + f(\zeta_{j,2}) + f(\zeta_{j,3})\Big). \end{aligned} \tag{1.17}$$

Thus

$$\begin{aligned} I^{[2]}(s_n^{[2]}) &= \sum_{j=1}^{n^2}\int_{\delta_j} p_j(x,y)\,dx\,dy \\ &= \frac{|\Delta|}{3\cdot n^2}\cdot\sum_{j=1}^{n^2}\Big(f(\zeta_{j,1}) + f(\zeta_{j,2}) + f(\zeta_{j,3})\Big), \end{aligned} \tag{1.18}$$

and formula (1.16) follows by re-ordering the sums in (1.18), taking into account that $z_{(\nu_1,\nu_2,\nu_3)}$ is the vertex of exactly six subtriangles δ_j, if $z_{(\nu_1,\nu_2,\nu_3)}$ lies in the interior of Δ, of exactly three subtriangles, if it lies on the boundary of Δ (but is not a vertex), and of exactly one subtriangle, if it is a vertex of Δ. $\square$

The idea is now to imitate the univariate situation and to take the right hand side of (1.16) as definition for the *trapezoidal rule over a triangle* Δ. We set

$$\begin{aligned} T_n^{[2]}(f) &:= \frac{|\Delta|}{3\cdot n^2}\cdot\Bigg\{ f(1,0,0) + f(0,1,0) + f(0,0,1) \\ &+ 3\cdot\sum_{\nu=1}^{n-1}\left(f\Big(\frac{\nu}{n},\frac{n-\nu}{n},0\Big) + f\Big(\frac{\nu}{n},0,\frac{n-\nu}{n}\Big) + f\Big(0,\frac{\nu}{n},\frac{n-\nu}{n}\Big)\right) \end{aligned}$$

$$+\ 6\cdot\sum_{\nu_1=1}^{n-2}\sum_{\nu_2=1}^{n-\nu_1-1} f\Big(\frac{\nu_1}{n},\frac{\nu_2}{n},\frac{n-\nu_1-\nu_2}{n}\Big)\Big\}\,. \tag{1.19}$$

Since $s_n^{[2]}$ converges to f, or more precisely,

$$s_n^{[2]} - f = O(n^{-2}) \quad \text{for } n\to\infty$$

(cf. [11]), we get that the sequence $\{T_n^{[2]}(f)\}$ converges to $I^{[2]}(f)$. But we can prove even more:

Theorem 1.3: *If the function f is sufficiently smooth, then the elements of the sequence $\{T_n^{[2]}(f)\}$ possess an asymptotic expansion of the form*

$$T_n^{[2]}(f) = I^{[2]}(f) + \sum_{\mu=1}^{r}\frac{d_\mu}{n^{2\mu}} + o(n^{-2r}) \qquad \textit{for } n\to\infty\,. \tag{1.20}$$

Here, the number r depends on the smoothness of f.

Proof. We proceed in exact analogy to the proof of Theorem 1.1. Let p_j denote the restriction of the spline $s_n^{[2]}$ to an arbitrary subtriangle δ_j. Thus p_j is a linear polynomial, which interpolates f in the three vertices $\zeta_{j,1},\zeta_{j,2}$, and $\zeta_{j,3}$ of δ_j.

As a special case of Theorem 1 in [11] it follows that there exist points $\xi_{x,1}$, $\xi_{x,2}$, and $\xi_{x,3}$, such that for each $z\in\delta_j$ the representation

$$f(z)-p_j(z) = -\sum_{i=1}^{3} D^2 f(\xi_{x,i})(\zeta_{i,j}-z)^2\cdot l_i(z)/2 \tag{1.21}$$

holds. Here, for $i=1,2,3$, the point $\xi_{x,i}$ lies on the line segment connecting $\zeta_{j,i}$ and z, in particular $\xi_{x,i}\in\delta_j$, and $l_i(z)$ denotes the Lagrange polynomial w.r.t. the points $\{\zeta_{j,i}\}$.

Using (bivariate) Taylor-expansion of $D^2 f$, and due to the fact that the distance between any two points in δ_j is bounded by $const/n$, we again obtain the estimate (cf. (1.10))

$$|f(z)-p_j(z)| \le \frac{1}{n^2}\cdot\sum_{\mu=2}^{\rho}\frac{\tilde\alpha_\mu(j,z)}{n^{\mu-2}} + o(n^{-\rho})\,. \tag{1.22}$$

Thus

$$f(z)-p_j(z) = \sum_{\mu=2}^{\rho}\frac{\tilde\alpha_\mu(j,z)}{n^\mu} + o(n^{-\rho}), \qquad \text{for } n\to\infty$$

and therefore, for all $z\in\Delta$, the asymptotic expansion

$$f(z)-s_n^{[2]}(z) = \sum_{\mu=2}^{\rho}\frac{\alpha_\mu(z)}{n^\mu} + o(n^{-\rho}) \quad \text{for } n\to\infty \tag{1.23}$$

with some integer ρ, depending on the smoothness of f, and coefficient functions $\alpha_\mu(z)$, which do not depend on n, holds.

The rest of the proof is completely analoguous to that of Theorem 1.1, using the (bivariate) mean value theorem of integration, and the fact that the trapezoidal points $z_{(\nu_1,\nu_2,\nu_3)}$ are totally symmetric (w.r.t. their barycentric çoordinates). □

2. The Algorithm and Numerical Results

Having proved the existence of the asymptotic expansion (1.20), we can now state the following extrapolation algorithm for the numerical integration of a function f over the triangle Δ.

Choose natural numbers n_0 and K with $K \leq r$, and compute by the formula (1.19) the generalized trapezoidal values

$$y_i^{(0)} \;:=\; T_{n_i}^{[2]}(f)\;, \qquad i = 0, 1, 2 \ldots, \tag{2.1}$$

where $n_i := n_0 \cdot 2^i$ for all i. Now apply *linear extrapolation* , i.e., compute the improved approximations

$$y_i^{(k)} \;:=\; y_{i+1}^{(k-1)} + \frac{y_{i+1}^{(k-1)} - y_i^{(k-1)}}{4^k - 1}, \qquad \begin{cases} k = 1, 2, \ldots, K, \\ i = 0, 1, \ldots\ , \end{cases} \;. \tag{2.2}$$

It is clear that each of the sequences $\{y_i^{(k)}\}_{i \in I\!N}$ then possesses an asymptotic expansion of the form

$$y_i^{(k)} \;=\; I^{[2]}(f) \;+\; \sum_{\mu=k+1}^{r} \frac{d_\mu^{(k)}}{n_i^{2\mu}} + \; o(n_i^{-2r}) \quad \text{ for } \; n_i \to \infty \tag{2.3}$$

with coefficients $d_\mu^{(k)}$, which are independent of n_i. In particular, each of the sequences $\{y_i^{(k)}\}$ converges faster to the limit $I^{[2]}(f)$ than its predecessor.

Remark. Instead of the basis 2, one could take in (2.1) any natural number b (i.e., set $n_i := n_0 \cdot b^i$), and then replace the term 4^k in the denominator of (2.2) by b^{2k}. However, in order to imitate the classical Romberg process as far as possible, we consider here only the "classical" case above.

The algorithm given by (2.1), (2.2) establishes the desired Romberg type cubature method for an arbitrary triangle. As generally known for linear extrapolation , it is numerically stable and fast. The results (and similarly the errors) are usually displayed in a triangular array of the following form, called *Romberg tableau.*

$$
\begin{array}{ccccc}
y_0^{(0)} & & & & \\
 & y_0^{(1)} & & & \\
y_1^{(0)} & & \ddots & & \\
 & \vdots & & y_0^{(K-1)} & \\
\vdots & & \vdots & & y_0^{(K)} \\
 & \vdots & & y_1^{(K-1)} & \vdots \\
\vdots & & \iddots & \vdots & \\
 & y_{K-1}^{(1)} & \vdots & & \\
y_K^{(0)} & \vdots & & & \\
\vdots & & & &
\end{array}
$$

Table 1. A Romberg Tableau

We illustrate the efficiency of the method by the results of some numerical experiments. The domain of integration was the triangle Δ with vertices

$$v_1 \;=\; \begin{pmatrix}1\\0\end{pmatrix}, \; v_2 \;=\; \begin{pmatrix}0\\1\end{pmatrix}, \;\text{ and }\; v_3 \;=\; \begin{pmatrix}0\\2\end{pmatrix} \tag{2.4}$$

in euclidean coordinates.

As a first test, we applied the method to the bivariate polynomial

$$f_1(x,y) \;:=\; 3\,x\,y^2\,. \tag{2.5}$$

.3500 e(00)		
	.3750 e(-1)	
.1156 e(00)		.0000 e(1)
	.2344 e(-2)	
.3066 e(-1)		.0000 e(1)
	.1465 e(-3)	
.7776 e(-2)		.0000 e(1)
	.9155 e(-5)	
.1951 e(-2)		.0000 e(1)
	.5722 e(-6)	
.4881 e(-3)		.0000 e(1)
	.3576 e(-7)	
.1221 e(-3)		

Table 2. Errors in Approximating $I^{[2]}(f_1)$

The *errors* of the approximations $y_i^{(k)}$ of the true value $I^{[2]}(f_1) = 0.35$, computed by our method with $K = 2$, $n_0 = 1$, and $i = 0, \ldots, 6$, are shown in Table 2. As expected, the entries of the third column are identically zero, since f_1 is a polynomial of degree 2, and therefore the second extrapolation step already gives the exact result. As a second example we compute the integral of the function

$$f_2(x, y) \;:=\; \exp(x + y) \tag{2.6}$$

and again compare our numerical approximations with true value

$$I^{[2]}(f_2) = \exp(2) - 2\exp(1) = 1.95249244201\ldots\,.$$

This time, the errors (in absolute value) of the approximations computed by our method with $K = 3$, $n_0 = 4$, and $i = 0, \ldots, 6$ are shown (Table 3).

.1026 e(-1)			
	.2263 e(-4)		
.2548 e(-2)		.3280 e(-8)	
	.1417 e(-5)		.1088 e(-12)
.6359 e(-3)		.5135 e(-10)	
	.8863 e(-7)		.4258 e(-15)
.1589 e(-3)		.8028 e(-12)	
	.5540 e(-8)		.1664 e(-17)
.3972 e(-4)		.1255 e(-13)	
	.3463 e(-9)		.6501 e(-20)
.9931 e(-5)		.1960 e(-15)	
	.2164 e(-10)		
.2483 e(-5)			

Table 3. Errors in Approximating $I^{[2]}(f_2)$

In Table 4, finally, we have the *quotients* of two subsequent values in the columns of Table 3. As predicted by (2.3), the entries of the k^{th} column (starting with $k = 0$) converge to 4^{k+1}.

4.027			
	15.965		
4.007		63.867	
	15.991		255.492
4.002		63.967	
	15.998		255.873
4.000		63.992	
	15.999		255.968
4.000		63.998	
	16.000		255.992
4.000		63.999	
	16.000		
4.000			

Table 4. Quotients of the Entries of Table 3

This is of course only a selection of several numerical tests. All of them showed the asymptotic behaviour which was predicted by the theory above.

3. Concluding Remarks

We have considered a Romberg type method for cubature over triangles in $\mathbb{R}^2$, which is completely coordinate-free and therefore independent of the special triangle. Thus, the following extensions resp. applications of the method are easily done:

– Numerical cubature over arbitrary polygon regions in $\mathbb{R}^2$. This is possible by precisely the same algorithm as above, since each polygon region can be subdivided into a finite number of triangles. One just has to apply the algorithm to each triangle separately, and then sum up.

– Development of (bivariate) Newton-Cotes formulae over triangles. For the construction of the trapezoidal rule in (1.16) resp. (1.19), we computed the integral of a spline function, which is piecewise a linear polynomial. Obviously, the same approach goes through for a continuous spline consisting of higher degree polynomial pieces, and the approximation order results for those polynomials given in [11] yield that we obtain Newton-Cotes type cubature formulae of arbitrarily high order. However, we do not follow this approach, since via extrapolation we reach the same goal (cubature of arbitrarily high order), and the numerical stability as well as the efficiency of extrapolation is simply unbeatable.

– Numerical integration over a convex polyhedron in $\mathbb{R}^d$, $d > 2$. Let Δ denote a polyhedron in $\mathbb{R}^d$ with exactly $d+1$ vertices. Then each point in Δ is uniquely determined by a $(d+1)-$ tuple of barycentric coordinates , and a generalization of our approach is straightforward.

References

[1] F. L. Bauer, La méthode de l'intégration numérique de Romberg, Colloque sur l'Analyse Numérique, Libraire Universitaire, Louvain 1961, 119 – 129

[2] F. L. Bauer, H. Rutishauser & E. Stiefel, New Aspects in Numerical Quadrature. In: *Proc. Symp. Applied Mathematics, Vol. 15.* American Mathematical Society, Providence, R. I 1963, 199 – 218

[3] C. Brezinski, Convergence Acceleration Methods: The Past Decade, J. Comp. Appl. Maths. **12&13** (1985), 19 – 36

[4] C. Brezinski, A Survey of Iterative Extrapolation by the E-Algorithm, Det Kong. Norske Vid. Selskab Skr. **2** (1989), 1 – 26

[5] C. Brezinski, Extrapolation Algorithms and Pade Approximations: A Historical Survey, Appl. Numer. Math. **20** (1996), 299 – 318

[6] C. Brezinski & M. Redivo Zaglia, *Extrapolation Methods, Theory and Practice*, North Holland, Amsterdam 1992

[7] R. Bulirsch, Bemerkungen zur Romberg-Integration, Numer. Math. **6** (1964), 6 – 16

[8] R. Bulirsch & J. Stoer, Fehlerabschätzungen und Extrapolation mit rationalen Funktionen bei Verfahren vom Richardson-Typus, Numer. Math. **6** (1964), 413 – 427

[9] R. Bulirsch & J. Stoer, Numerical Quadrature by Extrapolation, Numer. Math. **9** (1967), 271 – 278

[10] T.P. Chen, Asymptotic Error Expansions for Spline Interpolations, Sci. Sinica **5** (1983), 389 – 399

[11] P.G. Ciarlet and P.A. Raviart, General Lagrange and Hermite Interpolation in $\mathbb{R}^n$ with Applications to Finite Element Methods, Arch. Rat. Mech. Anal. **46** (1972), 177 – 199

[12] E. Hairer, S.P. Nørsett and G. Wanner, *Solving Ordinary Differential Equations I: Nonstiff Problems*, Springer, Berlin/Heidelberg 1987

[13] G. Han, Extrapolation of a Discrete Collocation-Type method of Hammerstein Equations, J. Comp. Appl. Maths. **61** (1995), 73 – 86.

[14] J.N. Lyness and R. Cools, A Survey of Numerical Cubature over Triangles, in: *Mathematics of Computation 1943-1993: A Half-Century of Computational Mathematics*, W. Gautschi ed., American Mathematical Society, Providence, RI 1994, 127-150.

[15] G. Nürnberger, *Approximation by Spline Functions*, Springer, Berlin/Heidelberg 1989

[16] W. Romberg, Vereinfachte Numerische Integration, Det Kong. Norske Vid. Selskab Forhdl. **28** (1955), 30 – 36

[17] G. Walz, *Asymptotics and Extrapolation*, Akademie-Verlag, Berlin 1996.