Toward an Iterative Algorithm for Spline Interpolation

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In Richtung eines iterativen Spline-Interpolationsalgorithmus

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Summary. One of the fundamental results in spline interpolation theory is the famous Schoenberg-Whitney Theorem, which completely characterizes those distributions of interpolation points which admit unique interpolation by splines. However, until now there exists no iterative algorithm for the explicit computation of the interpolating spline function, and the only practicable method to obtain this function is to solve explicitly the corresponding system of linear equations. In this paper we suggest a method which computes iteratively the coefficients of the interpolating function in its B-spline basis representation; the starting values of our one-step iteration scheme are quotients of two low order determinants in general, and sometimes even just of two real numbers. Furthermore, we present a generalization of Newton's interpolation formula for polynomials to the case of spline interpolation, which corresponds to a result of G. Mühlbach for Haar spaces.

Zusammenfassung. Eines der fundamentalen Resultate in der Spline-Interpolations-Theorie ist der berühmte Satz von Schoenberg-Whitney, der eine vollständige Charakterisierung derjenigen Verteilungen von Punkten angibt, welche eindeutige Interpolation durch Splines zulassen. Allerdings gibt es bisher keinen iterativen Algorithmus zur expliziten Berechnung der interpolierenden Splinefunktion, und die einzig praktikable Methode zur Gewinnung dieser Funktion ist die explizite Lösung des zugehörigen linearen Gleichungssystems. In dieser Arbeit schlagen wir eine Methode vor, die auf iterative Weise die Koeffizienten des interpolierenden Splines in seiner B-Spline-Basis Darstellung berechnet. Die Startwerte unseres Einschritt-Iterationsverfahrens sind Quotienten zweier Determinanten von, im allgemeinen Fall, kleiner Reihenzahl, und in manchen Fällen sogar nur von zwei reellen Zahlen. Weiterhin geben wir eine Verallgemeinerung von Newton's Interpolationsformel für Polynome auf den Fall der Spline-Interpolation an, die einem Resultat von G. Mühlbach für den Haarschen Fall entspricht.

Key Words and Phrases: Spline Interpolation, Schoenberg-Whitney Theorem, Multistep Formula, Newton-Type Interpolation Formula.

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1. Introduction and Preliminaries

Let there be given a natural number m, $m \geq 2$, and a set of real numbers $K = \{x_j\}$, the so-called *knots*, with the property $x_j < x_{j+1}$ for all j. A real function s is called a *spline* of order m belonging to the knot sequence K, if

1. the restriction of s to each intervall $[x_j, x_{j+1}]$ belongs to the space Π_{m-1}, and
2. if $s \in C^{m-2}(\mathbb{R})$.

We denote the real vector space of all these splines by $S_m(K)$. It is well-known that, for each fixed index j, there is precisely one function $B_{mj} \in S_m(K)$ with the properties

$$B_{mj}(x) = 0 \quad \text{for } x \leq x_j \text{ or } x \geq x_{j+m} \quad \text{and} \tag{1.1}$$

$$\int_{-\infty}^{\infty} B_{mj}(x)\,dx = \frac{x_{j+m} - x_j}{m} . \tag{1.2}$$

The collection of these functions B_{mj}, the *B-splines*, form a comfortable basis of the spline space $S_m(K)$. They have the property $B_{mj}(x) > 0$ for $x_j < x < x_{j+m}$ and are normalized such that

$$\sum_{j=-\infty}^{\infty} B_{mj}(x) = 1 . \tag{1.3}$$

Now, for some $n \in \mathbb{N}$, let $S_m(K_n)$ denote the restriction of $S_m(K)$ to the interval $I := [x_0, x_n]$, i.e. to the finite knot sequence

$$K_n := x_0 < \cdots < x_n .$$

$S_m(K_n)$ is a finite-dimensional linear space, its dimension being $m+n-1$; furthermore, the restrictions of the B-splines $B_{m,-m+1}, \ldots, B_{m,n-1}$ to the interval I provide a basis of the space $S_m(K_n)$ (see any textbook on splines, e.g. [2,6,10,12]).

We now consider the following Lagrange type interpolation problem: Given a real function f and $m+n-1$ points

$$t_{-m+1} < t_{-m+2} < \cdots < t_{n-2} < t_{n-1}$$

on the interval I, does there exist a spline $s \in S_m(K_n)$ satisfying the interpolation conditions

$$s(t_j) = f(t_j) \quad \text{for } j = -m+1, \ldots, n-1 \,? \tag{1.4}$$

Since the space $S_m(K_n)$ does not satisfy the Haar condition, this problem cannot be expected to be solvable for each distribution of the points t_j, but there is a very nice characterization of those situations, where unique Lagrange interpolation is possible, namely the famous Schoenberg-Whitney-Theorem:

Theorem 1.1 (I.J.Schoenberg & A.Whitney [11]): *The following statements are equivalent:*

a) There exists exactly one spline function $s \in S_m(K_n)$ *satisfying*

$$s(t_j) \;=\; f(t_j) \quad \textit{for}\ j = -m+1, \ldots, n-1 \,.$$

b) For all $j \in \{-m+1, \ldots, n-1\}$, *we have*

$$B_{mj}(t_j) \;\neq\; 0 \,.$$

c) For all $j \in \{-m+1, \ldots, n-1\}$, *we have*

$$x_j \;<\; t_j \;<\; x_{j+m} \,.$$

While the equivalence of statements b) and c) is obvious from the B-splines' finite-support-property, the remaining implications are fundamental results within the spline theory. For the purposes of this paper, we would like to re-state it in the following form:

The matrix of the linear system of equations

$$\sum_{\mu=-m+1}^{n-1} a_\mu B_{m\mu}(t_j) \;=\; f(t_j)\,, \quad j = -m+1, \ldots, n-1\,, \tag{1.5}$$

is regular if and only if all elements on the main diagonal are different from zero.

This matrix, the so-called *B-spline collocation matrix*, possesses many important and interesting properties, see e.g. [1,2,5,10]. We only note in passing that it is *totally positive*, but we shall make no explicit use of this fact here.

We close this introductory section by giving the following stronger version of the above statement, which is due to C. deBoor:

Theorem 1.2 (C. deBoor [1]): *Let* $\{\nu_1, \ldots, \nu_p\}$ *with* $\nu_1 < \cdots < \nu_p$ *be any subsequence of* $\{-m+1, \ldots, n-1\}$. *Then the matrix of the linear system of equations*

$$\sum_{\mu=1}^{p} a_{\nu_\mu} B_{m\nu_\mu}(t_j) \;=\; f(t_j)\,, \quad j = 1, \ldots, p\,,$$

is regular if and only if all elements on the main diagonal are different from zero.

2. Iterative Computation of Determinantal Quotients

From now on let us fix the values of n and m; we assume that one (hence all) assertions of Theorem 1.1 are satisfied (we say that the interpolation points t_j are in *Schoenberg-Whitney position*), and concentrate on the explicit computation of the interpolating spline, i.e. of the coefficients a_μ in the representation (1.5). So far, the only practicable way to do this is the solution of the linear system (1.5); it was shown by C. deBoor and A. Pinkus [3] that this is possible in a numerically stable way by Gaussian elimination, but the matrix under consideration is usually quite large, and one would surely prefer to have an iterative procedure for the computation of the a'_μ s.

In the following, we want to suggest such a method; it reduces the original "large" problem essentially to that of computing a much smaller determinant (as initial value), in general, and in some cases even to that of computing only a 1×1-determinant, i.e. a real number.

We are going to use the following notation: For functions $\varphi_0, \ldots, \varphi_k$ and points $z_0, \ldots, z_k$, we set

$$D\begin{pmatrix} \varphi_0 & \varphi_1 & \cdots & \varphi_k \\ z_0 & \cdots & \cdots & z_k \end{pmatrix} := \det \begin{pmatrix} \varphi_0(z_0) & \cdots & \cdots & \varphi_k(z_0) \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_0(z_k) & \cdots & \cdots & \varphi_k(z_k) \end{pmatrix}.$$

Now let us be given a function f, and a fixed index $\mu \in \{-m+1, \ldots, n-1\}$; we introduce the following re-notation for the interpolation points t_j and the B-splines $B_{m,j}$: For $j = 0, \ldots, m+n-2$ put

$$\tau_j := \begin{cases} t_{j+\mu}, & \text{for } j = 0, \ldots, n-\mu-1, \\ t_{n-j-1}, & \text{for } j = n-\mu, \ldots, n+m-2, \end{cases}$$

and in the same way

$$P_j := \begin{cases} B_{m,j+\mu}, & \text{for } j = 0, \ldots, n-\mu-1, \\ B_{m,n-j-1}, & \text{for } j = n-\mu, \ldots, n+m-2. \end{cases}$$

Furthermore, define for $k = 0, \ldots, m+n-2$ and $\nu = 0, \ldots, m+n-2-k$:

$$T_\nu^k = T_\nu^k(f) = \frac{D\begin{pmatrix} f & P_1 & \cdots & P_k \\ \tau_\nu & \cdots & \cdots & \tau_{\nu+k} \end{pmatrix}}{D\begin{pmatrix} P_0 & P_1 & \cdots & P_k \\ \tau_\nu & \cdots & \cdots & \tau_{\nu+k} \end{pmatrix}}, \tag{2.1}$$

provided that the denominator is different from zero (we shall give sufficient conditions for this in the next section). Then the following result on the recursive computation of the interpolating spline's coefficients holds:

Theorem 2.1: *a) The number* $T_0^{m+n-2}(f)$ *always exists, and we have*

$$a_\mu = T_0^{m+n-2}(f) . \tag{2.2}$$

b) Assume that, for some $k \in \mathbb{N}$ *and* $\nu \in \mathbb{N}_0$ *, the values* T_ν^k *,* T_ν^{k-1} *and* $T_{\nu+1}^{k-1}$ *exist (i.e. that the corresponding denominators are different from zero).*
Then the following reccurence relation holds:

$$T_\nu^k(f) = \frac{T_{\nu+1}^{k-1}(P_k)T_\nu^{k-1}(f) - T_\nu^{k-1}(P_k)T_{\nu+1}^{k-1}(f)}{T_{\nu+1}^{k-1}(P_k) - T_\nu^{k-1}(P_k)} . \tag{2.3}$$

Proof. Assertion a) is a simple application of Cramer's rule, the denominator being different from zero due to the assumed Schoenberg-Whitney position of the interpolation points (note that $T_0^{m+n-2}(f)$ really depends on μ, through the indexing of the P_j's and t_j's).

In order to verify the second assertion, we first note that $T_\nu^k(f)$ is a linear functional on the space of real functions f, possessing the property

$$T_\nu^k(P_j) = \begin{cases} 1 & \text{for } j = 0, \text{ and} \\ 0 & \text{for } j = 1, \ldots, k , \end{cases} \tag{2.4}$$

as can readily be seen from (2.1), cf. also [1]. This implies that the B-splines $\{P_0, \ldots, P_k\}$ span a *characteristic space* of the linear functional T_ν^k (see [4]), and we can conclude by Theorem 4.1 in [4] that relation (2.3) holds true, if the regularity condition

$$D\begin{pmatrix} P_1 & \cdots & \cdots & P_{k-1} \\ \tau_{\nu+1} & \cdots & \cdots & \tau_{\nu+k-1} \end{pmatrix} \neq 0 \tag{2.5}$$

is satisfied.
Since T_ν^k exists, we conclude from (2.1) that

$$D\begin{pmatrix} P_0 & \cdots & \cdots & P_k \\ \tau_\nu & \cdots & \cdots & \tau_{\nu+k} \end{pmatrix} \neq 0 .$$

This implies, by means of de Boor's Theorem 1.2 that the main diagonal of this matrix is different from zero; but the matrix in (2.5) has no other diagonal elements, and so a second application of Theorem 1.2 yields the desired result. □

Here is a first a little *example*: Suppose we want to compute the linear spline interpolant (i.e. $m = 2$) on the interval defined by the knot sequence $x_0 = 0$, $x_1 = 3$, $x_2 = 5$. The dimension of this spline space is 3, and for simplicity of the exposition we choose the interpolation points

$$t_{-1} = 1 \,, \quad t_0 = 2 \,, \quad t_1 = 4 \,.$$

The collocation matrix is easily computed to be

$$\begin{pmatrix} 2/3 & 1/3 & 0 \\ 1/3 & 2/3 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix} . \tag{2.6}$$

(Note that the row sums are all equal to 1, according to the normalization (1.4)). Now let us compute the coefficient a_0 in the representation (2.5) of the interpolating spline; of course, this can be done here directly by Cramer's rule, and for control reasons we do this and find

$$a_0 = \frac{D\begin{pmatrix} B_{2,-1} & f & B_{2,1} \\ t_{-1} & t_0 & t_1 \end{pmatrix}}{D\begin{pmatrix} B_{2,-1} & B_{2,0} & B_{2,1} \\ t_{-1} & t_0 & t_1 \end{pmatrix}}$$

$$= \frac{\frac{1}{3} f(t_0) - \frac{1}{6} f(t_{-1})}{\frac{1}{6}} = 2 f(2) - f(1) \,. \tag{2.7}$$

Now let us apply Theorem 2.1; we have to set

$$\tau_0 = t_0 = 2 \,, \quad \tau_1 = t_1 = 4 \,, \quad \tau_2 = t_{-1} = 1 \,,$$

and in the same way the P_j's. It turns out that for *all* values of k and ν the assumptions of the theorem are satisfied, and we can compute successively

$$\begin{array}{lll} T_0^0(f) = \frac{3}{2} f(2) & T_1^0(f) = 2 f(4) & T_2^0(f) = 3 f(1) \\ T_0^1(f) = \frac{3}{2} f(2) & T_1^1(f) = 3 f(1) & \\ T_0^2(f) = 2 f(2) - f(1) \,, & & \end{array}$$

in accordance with (2.7).
Needless to say that this example was only for illustration and has no practical relevance, since the recursive computation of the coefficients usually is worth only for larger matrices.

Obviously, our method for the iterative computation of interpolation coefficients is not restricted to splines, but can be applied to rather arbitrary linear spaces; in [7,8], G. Mühlbach studied the case of *complete Haar spaces (Chebyshev systems)*, where the non-vanishing of the denominators is automatically satisfied, and derived by direct methods the corresponding reccurence relations.

3. Criteria for Non-Zero Denominators. Strategies.

The situation as we had it in Theorem 2.1 is not quite the one you will meet in practice; there one usually has computed the values of the $T-$ sequence of some specific stage, say $k-1$, and wants to know if T_ν^k is computable via (2.3) without explicitly assuming its existence. So we need criteria for this; we begin by analyzing the special case $\mu = -m+1$, where it will turn out that no additional assumption is necessary:

Remark. Still keeping our notation from section 2, let us put $\mu := -m+1$, i.e.

$$P_j \;=\; B_{m,j-m+1} \quad \text{and} \quad \tau_j \;=\; t_{j-m+1} \qquad \text{for } j=0,\ldots,m+n-2\,,$$

and assume that T_ν^{k-1} and $T_{\nu+1}^{k-1}$ exist, hence

$$D\begin{pmatrix} B_{m,-m+1} & \cdots & \cdots, & B_{m,-m+k} \\ t_{\nu-m+1} & \cdots & \cdots & t_{\nu-m+k} \end{pmatrix} \neq 0 \tag{3.1}$$

and

$$D\begin{pmatrix} B_{m,-m+1} & \cdots & \cdots & B_{m,-m+k} \\ t_{\nu-m+2} & \cdots & \cdots & t_{\nu-m+k+1} \end{pmatrix} \neq 0\,. \tag{3.2}$$

We claim that in this situation the existence of $T_\nu^k(f)$ is automatically satisfied. To prove this, we have to show that

$$D\begin{pmatrix} B_{m,-m+1} & \cdots & \cdots & B_{m,-m+k+1} \\ t_{\nu-m+2} & \cdots & \cdots & t_{\nu-m+k+1} \end{pmatrix} \neq 0 \tag{3.3}$$

holds. But in view of (3.1) and using Theorem 1.2, (3.3) is true if the new diagonal element is different from zero, i.e.

$$B_{m,-m+1+k}(t_{\nu-m+1+k}) \neq 0\,. \tag{3.4}$$

This can be seen as follows: Using again Theorem 1.2, we can conlude from (3.2) that $B_{m,-m+k}(t_{\nu-m+1+k}) \neq 0$, i.e. $t_{\nu-m+1+k} < x_k$. On the other hand, the Schoenberg-Whitney condition implies $B_{m,\nu-m+1+k}(t_{\nu-m+1+k}) \neq 0$, i.e. $x_{\nu-m+1+k} < t_{\nu-m+1+k}$. Altogether we have found the inclusions

$$x_{-m+1+k} \;\leq\; x_{\nu-m+1+k} \;<\; t_{\nu-m+1+k} \;<\; x_k \;<\; x_{k+1}\,, \tag{3.5}$$

which proves (3.4).

This remark implies that if some values of the k^{th} stage, say $T_\nu^k,\ldots,T_{\nu+\lambda}^k$ exist, then we already know that also the value $T_\nu^{k+\lambda}$ of stage $k+\lambda$ *and all intermediate* *values* exists and can be computed one after another by the recursive scheme (2.3). So, a first approach to a good *strategy* would be: Compute all values T_ν^0 of stage 0, which have non-zero denominators (note that these are just $1\times 1-$ "determinants"). Whenever two or more subsequent ones of the T_ν^0's exist, apply the iterative scheme (2.3) to get the corresponding values for $k=1,2,\ldots$. If some T_j^0 does not exist, then compute the related values of stage 1 directly, i.e. by evaluating the determinantal quotient (2.1). If some of the T_ν^1's still have zero denominators, compute the related T_ν^2 by means of (2.1), otherwise you may use (2.3), and so on.

If m is not too small compared with n, it turns out that rather soon all denominators become non-zero, such that we have from a certain low order stage on a completely iterative scheme for the computation of the desired value $a_{-m+1}=T_0^{m+n-2}(f)$.

For arbitrary values of μ, the situation is a little more complicated than in the above remark, since here we need an additional assumption on the non-vanishing of some B-spline values; however, note that our condition (3.6) is sufficient, but far from being necessary for the assertion of the following theorem. This is one reason for the "toward" in this paper's title.

Theorem 3.1: *We make the general assumption that*

$$B_{m,\mu+j}(t_{-m+1+j})\neq 0 \quad \textit{for } j=0,\ldots,n-1-\mu \tag{3.6}$$

(that is, the zero triangle in the upper right corner of the Schoenberg-Whitney matrix is not too large).

If, for some ν and k, the values $T_\nu^{k-1}(f)$ and $T_{\nu+1}^{k-1}(f)$ exist, then also $T_\nu^k(f)$ exists and can be computed by means of the recursive scheme (2.3).

Proof. At the beginning of this technical proof it is a good idea to visualize the Schoenberg-Whitney matrix as it looks like in our notation: We have

$$\begin{pmatrix} P_{n+m-2}(\tau_{n+m-2}) & \cdots & P_{n-\mu}(\tau_{n+m-2}) & P_0(\tau_{n+m-2}) & \cdots & P_{n-1-\mu}(\tau_{n+m-2}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{n+m-2}(\tau_{n-\mu}) & \cdots & P_{n-\mu}(\tau_{n-\mu}) & P_0(\tau_{n-\mu}) & \cdots & P_{n-1-\mu}(\tau_{n-\mu}) \\ P_{n+m-2}(\tau_0) & \cdots & P_{n-\mu}(\tau_0) & P_0(\tau_0) & \cdots & P_{n-1-\mu}(\tau_0) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{n+m-2}(\tau_{n-1-\mu}) & \cdots & P_{n-\mu}(\tau_{n-1-\mu}) & P_0(\tau_{n-1-\mu}) & \cdots & P_{n-1-\mu}(\tau_{n-1-\mu}) \end{pmatrix} \tag{3.7}$$

Note that in this matrix the B-splines P_j as well as the points τ_j appear in their "correct" ordering, i.e. Theorems 1.1 and 1.2 can be directly applied; it is regular if and only if

$$D\begin{pmatrix} P_0 & \cdots & \cdots & P_{n+m-2} \\ \tau_0 & \cdots & \cdots & \tau_{n+m-2} \end{pmatrix} \neq 0 \qquad (3.8)$$

holds, since (3.7) is constructed from the matrix in (3.8) only by permutations of lines and columns, which does not affect the regularity, and the same holds true for any submatrices of these two.

The following easy observation will be a fundamental tool in some places of the proof: If two numbers in a row of the matrix (3.7) are non-zero, then each number in between is also different from zero (follows exactly as in the proof of (3.4)), and the same is true for two non-zero elements in the columns (follows directly from the finite-support property of the B-splines).

We now prove the existence of $T_\nu^k(f)$, i.e. the non-vanishing of each diagonal element of the matrix

$$\begin{pmatrix} P_0 & \cdots & \cdots & P_k \\ \tau_\nu & \cdots & \cdots & \tau_{\nu+k} \end{pmatrix} , \qquad (3.9)$$

under the assumption that the diagonals of the matrices

$$\begin{pmatrix} P_0 & \cdots & \cdots & P_{k-1} \\ \tau_\nu & \cdots & \cdots & \tau_{\nu+k-1} \end{pmatrix} \text{ and } \begin{pmatrix} P_0 & \cdots & \cdots & P_{k-1} \\ \tau_{\nu+1} & \cdots & \cdots & \tau_{\nu+k} \end{pmatrix}$$

(the denominators of $T_\nu^{k-1}(f)$ and $T_{\nu+1}^{k-1}(f)$) do not vanish. We must distinguish three cases:

Case 1. It is $\nu + k \leq n - 1 - \mu$.

Then also $k \leq n - 1 - \mu$, and this means that we are in the lower right corner of the matrix (3.7). Therefore, the only new diagonal element in (3.9) is $P_k(\tau_{\nu+k})$, which is immediately seen to be non-zero exactly as in the above remark.

Case 2. It is $\nu + k > n - 1 - \mu$, but still $k \leq n - 1 - \mu$.

This means that we are in the right "half" of the matrix in (3.7). Let for a moment d_j^{k-1}, $j = 0, \ldots, k-1$ denote the diagonal element in the j^{th} column of $T_\nu^{k-1}(f)$ (which is known to be non-zero), and analogously let d_j^k , $j = 0, \ldots, k$ denote the diagonal element in the j^{th} column of $T_\nu^k(f)$.

Then we see that, for $j = 0, \ldots, k-1$, d_j^k lies in a column between

$$d_j^{k-1} \quad \text{and} \quad P_j(\tau_{n+m-2+j}) = B_{m,\mu+j}(t_{-m+1+j}) \neq 0 ,$$

and is therefore also different from zero. Finally, for $j = k$, the element d_k^k lies in a column between $P_k(\tau_k)$ and $P_k(\tau_{n+m-2+k}) = B_{m,\mu+k}(t_{-m+1+k}) \neq 0$, and the same implication follows.

Case 3. It is $k > n - 1 - \mu$.
This implies $\nu + k > n - 1 - \mu$, i.e. we have entered the left "half" of the matrix. In this case we have again only one new diagonal element, say $P_j(\tau_l)$. Since $\nu + k \geq k > n - 1 - \mu$, it is

$$l \geq j \geq n - \mu \,. \tag{3.10}$$

But since the main diagonal as well as the first line of the left "half" of the matrix (3.7) contain only non-zero numbers, a multiple application of our observation from above immediately implies that all elements $P_j(\tau_l)$ satisfying (3.10) are different from zero. This completes the proof of Theorem 3.1. □

Corollary 3.2: *Assume that (3.6) holds and that, for some k, ν and λ the values*

$$T_\nu^k(f)\,,\ \ldots\,,\ T_{\nu+\lambda}^k(f)$$

exist. Then also the value $T_\nu^{k+\lambda}(f)$ and all intermediate ones exist and can be computed one after another by the iterative procedure (2.3).

If assumption (3.6) is satisfied, we can take our strategy from above word-by-word for the computation of all a_μ's. In particular it follows from Corollary 3.2 that, if for some k all values $T_0^k(f), \ldots, T_{n+m-2-k}^k(f)$ exist, then we have a fully recursive scheme from this stage on.

However, (3.6) is not a necessary condition for the existence of the higher order T_ν^k's, and therefore there is – depending on the special structure of the matrix under consideration – much freedom for creating better strategies in special cases. Also, there are other choices for the indexing of the τ_j's and P_j's, which might in some situations be better than ours; once again we have some reasons for the "toward" in our title.

If the condition given in (3.6) is not satisfied, it may happen that for some $l \in \mathbb{N}$ the values $T_\nu^k(f), \ldots, T_{\nu+l}^k(f)$ as well as $T_\nu^{k+l}(f)$ do exist, but the intermediate values do not. In this situation (2.3) cannot be applied, but it is possible to "jump" over this singularity by using the following multistep formula; the same is true if the denominator in (2.3) is small in absolute value. In this case a severe propagation of rounding errors can occur and thus using the following formula can improve the numerical stability of the algorithm considerably.

Theorem 3.3 (C. Brezinski & G. Walz [4]): *Let, for some $l \geq 1$, the determinant*

$$D\begin{pmatrix} T_\nu^k & T_{\nu+1}^k & \cdots & T_{\nu+l}^k \\ P_0 & P_{k+1} & \cdots & P_{k+l} \end{pmatrix} := \det \begin{pmatrix} T_\nu^k(P_0) & T_\nu^k(P_{k+1}) & \cdots & T_\nu^k(P_{k+l}) \\ T_{\nu+1}^k(P_0) & T_{\nu+1}^k(P_{k+1}) & \cdots & T_{\nu+1}^k(P_{k+l}) \\ \vdots & \vdots & \vdots & \vdots \\ T_{\nu+l}^k(P_0) & T_{\nu+l}^k(P_{k+1}) & \cdots & T_{\nu+l}^k(P_{k+l}) \end{pmatrix}$$

be different from zero. Then the value $T_\nu^{k+l}(f)$ can be expressed as linear combination of the values $T_\nu^k(f),\ldots,T_{\nu+l}^k(f)$. We have the relation

$$T_\nu^{k+l}(f) = \frac{D\begin{pmatrix} T_\nu^k & T_{\nu+1}^k & \cdots & T_{\nu+l}^k \\ f & P_{k+1} & \cdots & P_{k+l} \end{pmatrix}}{D\begin{pmatrix} T_\nu^k & T_{\nu+1}^k & \cdots & T_{\nu+l}^k \\ P_0 & P_{k+1} & \cdots & P_{k+l} \end{pmatrix}} .$$

Remark. For $k = 0$ we recover the determinantal formula (2.1) while, for $l = 1$, we obtain (2.3). For an arbitrary value of l this multistep formula can be used to compute directly $T_\nu^{k+l}(f)$ from the $T_\nu^k(f)$'s.

In general, we can say that our method works very efficiently (since it comes down to a very small size of the initial values), if the Schoenberg-Whitney matrix has not too many zero entries, i.e. if the band in the middle of this matrix is quite small. On the other hand, if we have a very sparse Schoenberg-Whitney matrix, then the explicit solution of the underlying linear system can be done quite easily.

4. A Newton-Type Interpolation Formula

In [8], G. Mühlbach presented a generalization of Newton's polynomial interpolation formula to the case of Haar spaces. We would like to conclude our paper with the presentation of a corresponding formula for spline interpolation, where – in contrast to the Haar case – additional attention to the location of the interpolation points t_j must be paid.

First we need some notation: For every $k \in \{-m+1,\ldots,n-1\}$, we define the *generalized divided difference* through

$$A_k(f) := \frac{D\begin{pmatrix} B_{m,-m+1} & \cdots & B_{m,k-1} & f \\ t_{-m+1} & \cdots & t_{k-1} & t_k \end{pmatrix}}{D\begin{pmatrix} B_{m,-m+1} & \cdots & B_{m,k-1} & B_{m,k} \\ t_{-m+1} & \cdots & t_{k-1} & t_k \end{pmatrix}} , \tag{4.1}$$

a special case of (2.1) (the name is inspired by the fact that replacing in (4.1) the B-splines by the monomials just gives the "ordinary" divided difference, see e.g. [5,7,9]).

Furthermore, for $k \in \{-m+1,\ldots,n-1\}$, define the spline space

$$\mathcal{B}_k := \mathrm{span}\{B_{m,-m+1},\cdots,B_{m,k}\}|_{[x_0,x_n]} . \tag{4.2}$$

Then, according to Theorem 1.1, for each k there is one and only one spline function $s_k \in \mathcal{B}_k$ satisfying

$$s_k(t_j) = f(t_j) \quad \text{for } j = -m+1, \ldots, k \,. \tag{4.3}$$

Moreover, we consider for $k \in \{-m+1, \ldots, n-1\}$ the error function

$$\begin{aligned} r_k(f)(x) &:= \frac{D\begin{pmatrix} B_{m,-m+1} & \cdots & \cdots & B_{m,k} & f \\ t_{-m+1} & \cdots & \cdots & t_k & x \end{pmatrix}}{D\begin{pmatrix} B_{m,-m+1} & \cdots & \cdots & B_{m,k} \\ t_{-m+1} & \cdots & \cdots & t_k \end{pmatrix}} \\ &= s_k(x) - f(x) \,, \end{aligned} \tag{4.4}$$

and set for completeness $r_{-m}(f)(x) := f(x)$. Then the following generalization of Newton's interpolation formula holds:

Theorem 4.1: *a) For each $k \in \{-m+1, \ldots, n-1\}$ the interpolating spline s_k can be written as*

$$s_k(x) = \sum_{j=-m+1}^{k} A_j(f) \cdot r_{j-1}(B_{m,j})(x) \,. \tag{4.5}$$

In particular, for $k = n-1$ we obtain a new representation for the spline function $s = s_{n-1} \in S_m(K_n)$ which satisfies our initial interpolation problem (1.4).

b) The determinantal quotients defining A_k and $r_k(f)$ can be computed by means of the iterative procedure (2.3) resp. by the strategies suggested in section 3.

Proof. It is important to emphasize that for all values of k the denominator in (4.1) resp. (4.4) does not vanish, due to the Schoenberg-Whitney position of the interpolation points $t_{-m+1}, \ldots, t_k$ for all k, and therefore the coefficients $A_k(f)$ and the functions $r_k(f)$ are well-defined.

We only have to prove (4.5), and do this by induction with respect to k. For $k = -m+1$, the assertion is obviously true, so let now $k > -m+1$, and assume that (4.5) holds for $k-1$. We consider the spline function

$$s^* := s_k - s_{k-1} \in \mathcal{B}_k \,,$$

and we have to show that

$$s^* = A_k(f) \cdot r_{k-1}(B_{m,k}) \,. \tag{4.6}$$

To do this, we first note that s^* has the representation

$$s^*(x) = \sum_{j=-m+1}^{k-1} \beta_j B_{m,j}(x) + A_k(f)\, B_{m,k}(x) \tag{4.7}$$

with $A_k(f)$ from (4.1), according to Cramer's rule. On the other hand, $r_{k-1}(B_k)(x)$ is also a function from the space $\mathcal{B}_k$, the coefficient of $B_{m,k}$ in its basis representation equals 1.

Combining this with (4.7), we obtain the following result: The spline function

$$e(x) := s^*(x) - A_k(f) \cdot r_{k-1}(B_k)(x)$$

lies in the space $\mathcal{B}_{k-1}$ and vanishes at the points $t_{-m+1}, \ldots, t_{k-1}$. Since these points are for each k in Schoenberg-Whitney position, we must have $e(x) \equiv 0$, which yields (4.6) and finally proves the assertion. □

Remark. Obviously, the result of Theorem 4.1 also holds for any re-ordering of the B-splines $\{B_{m,-m+1}, \ldots, B_{m,n-1}\}$.

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