Calculating p–adic orbital integrals on GSp(4) via a family of special subgroups

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Particularly in applications of the trace formula intriguing problems are already posed by the local orbital integrals

$$O_s^G(f) = \int_{G_s(F)\backslash G(F)} f(g^{-1}sg)\, d\dot{g}.$$

For $G = GSp(4)$ and a semisimple element s in $G(F)$ which is regular we propose in this paper a two–step method for calculating them explicitly. It is based on subgroups H_C of G depending only on the stable conjugacy class C of s, and possessing a hypothetical codiscreteness property which we prove and apply.

0. Notation: Unless otherwise specified F is a nonarchimedean local field with uniformizing element π, ring of integers $\mathcal{O}_F$ and residue field $k(F)$. We write $U(F)$ for the set of units in $\mathcal{O}_F$. The order on F is normalized such that $\operatorname{ord}\pi = 1$, and $|\pi| = 1/\#k(F) = q^{-1}$.
Let I be the involution on $M(2n, F)$, the $2n$ by $2n$ matrices with coefficients in F, defined by

$$I(g) = J^{-1}\cdot {}^tg \cdot J \qquad \text{with} \qquad J = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}.$$

The group $GSp(2n, F)$ of symplectic similitudes is the set of all g in $M(2n, F)$ such that ${}^tg\cdot J\cdot g = \mu(g)\cdot J$ or equivalently such that $I(g)\cdot g = \mu(g)\cdot E_{2n}$.

1. The groups H_C: Let F be any perfect field. A method to classify the stable conjugacy classes C of maximal F–tori in $GSp(2n)$, by which H_C are parametrized, dates back to A. Weil. Let T be a torus in C and s in $T(F)$ regular. The centralizer $C(s)$ of s in $M(2n, F)$ is isomorphic to the algebra $F[s]$. Now $F[s]$ is isomorphic to the direct sum $\mathcal{E}_C = \oplus\{E : E \in \mathcal{F}\}$ of the extension fields E of F defined by the irreducible factors of the characteristic polynomial of s. The image τ_C of $T(F)$ in $\mathcal{E}_C$ consists of all x in $\mathcal{E}_C$ with $I(x)x = \mu(x)1$. One is thus led to study the action of I on the elements of $\mathcal{F}$.

By [S, Kapitel 5] each E in $\mathcal{F}$ either belongs to a pair of factors (E, E') which I interchanges, i.e. with $I(E) = E'$, or I restricts to a nontrivial involution σ_E on E. The first case gives rise to tori in Levi factors coming from general linear groups. In the second case we obtain tori of unitary groups. The group H_C can be thought of as intersection with $GSp(2n)$ of the smallest product of general linear groups which contains representatives of all conjugacy classes in C. Thus it decomposes in two factors uniquely determined by C only: One factor is constructed from the unitary data of the second case, the other from the Levi factors of the first case above. No new insights are required for the part of general linear type.

We thus assume that each factor in $\mathcal{F}$ is of unitary type: $\mathcal{F} = \mathcal{F}_{\text{unitary}}$. For any E in $\mathcal{F}$ let E^+ and E^- be the $(+1)$– and (-1)–eigenspaces of σ_E respectively. Decompose $\mathcal{E}_C = \mathcal{E}_C^+ \oplus \mathcal{E}_C^-$ accordingly. For any invertible $a = (a_E)$ in $\mathcal{E}_C^-$ define the symplectic form $B_C(a)$ on the F–space $\mathcal{E}_C$ by

(1)
$$B_C(a)\Big((x_E),(y_E)\Big) = \sum\nolimits_{E\in\mathcal{F}} \operatorname{tr}_{E/K}\Big(x_E\cdot a_E\cdot\sigma_E(y_E)\Big).$$

Conjugating by $\operatorname{diag}(\ldots,\operatorname{diag}(1,b_E),\ldots)$ with $b = (b_E)$ invertible in $\mathcal{E}_C^+$ transforms $GSp(\mathcal{E}_C, B(a))$ into $GSp(\mathcal{E}_C, B(ab))$. Therefore

(2)
$$\begin{aligned} H_C(F) &= GSp\big(F_C, B_C(a)\big)\bigcap\bigoplus\Big\{GL_{E^+}(E) : E\in\mathcal{F}\Big\} \\ &= \Big\{(\phi_E)\in\bigoplus\big\{GL_{E^+}(E) : E\in\mathcal{F}\big\} : (\ldots,\det\phi_E,\ldots)\in K^*\cdot 1_{F_C}\Big\} \end{aligned}$$

is independent of the form $B(a)$ and defines the algebraic F–subgroup H_C of $GSp(2n)$. We embed $\mathcal{E}_C$ in $End_F(\mathcal{E}_C)$ mapping a to the multiplication $\ell(a)$ by a on the left. The intersection of $\ell(\mathcal{E}_C^*)$

with each group $GSp(\mathcal{E}_C, B(a))$ is $\ell(\tau_C)$, so that $\ell(\tau_C)$ defines a subtorus T_C of H_C in the stable conjugacy class C. For all $b = (b_E)$ invertible in F_C^+ we let

$$T_C(b) = \left(\text{Int diag}\left(\ldots, \begin{pmatrix} 1 & 0 \\ 0 & b_E \end{pmatrix}, \ldots\right)\right)\left(T_C\right). \tag{3}$$

The $T_C(b)$ account for all conjugacy classes of tori in C. Furthermore, T_C is conjugate to a $T_C(b)$ in $GSp(2n, F)$ if and only if T_C is conjugate to $T_C(b)$ in $H_C(F)$.

2. The $GSp(4)$ results: The stable conjugacy classes of maximal F–tori in $GSp(4)$ with unitary parts were classified in [**S**, Kapitel 6]. They are the tori of type T_{3A} and the basic tori which, in fact, both have no parts of general linear type. This is the situation where we expect our codiscreteness results to hold in general.

(2.1) Tori of type T_{3A}: Such a torus is characterized by a pair $\xi = (E, L)$ of quadratic extensions of F and by definition is isomorphic to the F–subtorus

$$\tau_{E,L} = \left\{(x, y) \in E^* \times L^* : N_{E/F}(x) = N_{L/F}(y)\right\} \tag{1}$$

of the F–torus $E^* \times L^*$. Fix normalized primitive elements $\sqrt{A}$ of E and $\sqrt{B}$ of L over F, so that A and B are representatives in F of $F^*/(F^*)^2$ which both have orders 0 or 1. We take 1_E, 1_L, $\sqrt{A}$, $\sqrt{B}$ as symplectic orthonormal basis so that $H(F) = H_\xi(F)$ consists of all matrices

$$[h, h'] = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix} \quad \text{with} \quad h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ h' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \tag{2}$$

in $GL(E) \times GL(L)$ which satisfy the symplecticity condition $\det h = \det h'$. We will prove

(2.2) Theorem: *The set $H(F)\backslash GSp(4, F)/GSp(4, \mathcal{O}_F)$ is discrete, representatives are*

$$g(\gamma) = \begin{pmatrix} E_2 & \gamma W \\ 0 & E_2 \end{pmatrix}, \qquad W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with either $\gamma = 0$ or $\gamma = \pi^{-\ell}$ and $\ell > 0$ any natural number.

(2.3) Basic tori: Such a torus is characterized by a pair $\xi = (E, \sigma_E)$ consisting of an extension E over F of degree four which has a nontrivial involution σ_E. Let $\sqrt{A}$ be a normalized primitive element over F of the fixed field E^+ of σ_E. Then $H_\xi \cong \{g \in GL(2, E^+) : \det g \in F^*\}$. By [**S**, A.19.8] it can be realized as the subgroup of $GSp(4, F)$ of all matrices

$$\begin{pmatrix} a_1 & a_2 A^{-1} & b_1 & b_2 \\ a_2 & a_1 & b_2 & b_1 A \\ c_1 & c_2 A^{-1} & d_1 & d_2 \\ c_2 A^{-1} & c_1 A^{-1} & d_2 A^{-1} & d_1 \end{pmatrix} \tag{3}$$

Actually, this is the embedding used by Prof. Weissauer in his proof of the fundamental lemma. By definition, the basic torus to ξ is $\tau_b(\xi) = \{x \in E^* : N_{E/E^+}(x) \in F^*\}$. We will prove

(2.4) Theorem: *The set $\mathcal{H}_\xi = H_\xi(F)\backslash GSp(4, F)/GSp(4, \mathcal{O}_F)$ is discrete. For all $\ell \geq 1$ let*

$$g(\ell) = \begin{pmatrix} E_2 & S(\ell) \\ 0 & E_2 \end{pmatrix} \qquad \text{with} \qquad S(\ell) = \begin{pmatrix} \pi^{-\ell} & 0 \\ 0 & 0 \end{pmatrix}.$$

If E^+ is unramified over F, representatives of $\mathcal{H}_\xi$ are E_4 and $g(\ell)$ with $\ell \geq 1$. If E^+ is ramified over F, representatives of $\mathcal{H}_\xi$ are $g(\ell)$ with $\ell \geq 1$.

3. A technique for calculating orbital integrals: Let s be a regular, F-rational element of the torus $T_C(b)$. Let $K = GSp(2n, \mathcal{O}_F)$. Assume $H_C\backslash GSp(2n,F)/K$ discrete, thus countable. For any Hecke operator f on $GSp(2n,F)$ one has by [**Wa** I, p.477, A 1.2] and [**KoGL**, p.361f]

$$
\text{(1)}\qquad \begin{aligned} O_s^{GSp(2n)}(f) &\overset{\text{def}}{=} \int_{T_C(b)\backslash GSp(2n)} f(g^{-1}sg)d\dot{g} \\ &= \sum\nolimits_{x\in H_C\backslash GSp(2n)/K} \frac{\mathrm{vol}_{GSp(2n)}(K)}{\mathrm{vol}_{H_C}(H_C\cap xKx^{-1})} \int_{T_C(b)\backslash H_C} (f\circ \mathrm{Ad}\,x^{-1})(h^{-1}sh)\,d\dot{h} \end{aligned}
$$

where we identified the groups with their F-rational points and measures are suitably normalized. The support of $f\circ\mathrm{Ad}\,x^{-1}$ is $\mathrm{supp}(f\circ\mathrm{Ad}\,x^{-1}) = H_C(F)\cap(\mathrm{Ad}x)(\mathrm{supp}(f)) = H_C(F)\cap x\cdot\mathrm{supp}(f)\cdot x^{-1}$. We want to show next that this is a tool for calculating orbital integrals for the group $GSp(4)$.

4. Calculating the $GSp(4)$-orbital integral $O_s(T(\pi))$ for s in a torus of type $\mathrm{T_{3A}}$: Let F be a local field with odd residue characteristic and $G = GSp(4)$. Let $T(\pi)$ be the Hecke operator on $G(F)$ defined as characteristic function of the double coset $K\cdot\mathrm{diag}(E_2,\pi E_2)\cdot K$ with $K = GSp(4,\mathcal{O}_F)$. This operator is closely connected with the problem of counting points mod p of the Shimura variety to G by the trace formula.

(4.1) Embeddings of tori of type $\mathrm{T_{3A}}$: In the local case the first cohomology group of the torus $\tau_{E/L}$ of type $\mathrm{T_{3A}}$ is by class field theory trivial for $E\not\cong L$ and cyclic of order two for $E=L$. Representatives of the conjugacy classes of F-embeddings of $\tau_{E,L}$ into $GSp(4)$ taking their values in H were determined explicitly in [S, §11B]. We fix the F-rational, semisimple element s in the image T of $\tau_{E,L}$ under any of these, so that

$$
\text{(1)}\qquad s = [s_E, s_L] = \left[\begin{pmatrix} a & bD^{-1}A \\ bD & a\end{pmatrix}, \begin{pmatrix} a' & b'B \\ b' & a'\end{pmatrix}\right] \sim_{\overline{F}} \left[\begin{pmatrix} \lambda & \\ & \mu\end{pmatrix}, \begin{pmatrix} \lambda' & \\ & \mu'\end{pmatrix}\right]
$$

where $D=1$ in the stable case $E\not\cong L$ and where $D\in\{1,\Theta\}\cong F^*/N_{E/F}(E^*)$ in the instable case $E=L$. Our aim is to prove

(4.2) Theorem: *Let F be a local field with odd residue characteristic and let s be regular.*

The $GSp(4)$-orbital integral $O_s(T(\pi))$ is nonzero only if the similarity factor $\mu(s)$ of s has order one in F and E, L are both ramified over F.

Then in the stable case $E\not\cong L$

$$
O_s(T(\pi)) \overset{\text{def}}{=} \int_{T\backslash GSp(4)} T(\pi)(g^{-1}sg)d\left(\frac{\mu_{GSp(4)}}{\mu_T}\right)(\dot{g}) = 2\cdot\frac{\mathrm{vol}_G(GSp(4,\mathcal{O}_F))}{\mathrm{vol}_T(T(\mathcal{O}_F))}.
$$

In the instable case $E=L$ let $\delta_D(s)=1$ if $-bD/b'$ is a quadratic residue modulo $\pi\mathcal{O}_F$ and $\delta_D(s)=0$ otherwise. Then

$$
O_s(T(\pi)) = \frac{\mathrm{vol}_G(GSp(4,\mathcal{O}_F))}{\mathrm{vol}_T(T(\mathcal{O}_F))}\left(1+\frac{2\cdot\delta_D(s)}{|(\lambda+\mu)-(\lambda'+\mu')|}\cdot\frac{\xi_F(1)}{\xi_F(N)}\right)
$$

where $N = \mathrm{ord}_F\big((\lambda+\mu)-(\lambda'+\mu')\big)$ and $\xi_F(\ell) = 1/(1-q^{-\ell})$ is the zeta function of F evaluated at ℓ.

Remarks: In the instable case $E=L$ our calculation identifies the instable contribution from the regular elements in tori of type $\mathrm{T_{3A}}$ to the trace formula "evaluated for $T(\pi)$".
The κ-orbital integral to s on $GSp(4)$ is in this case up to a sign the difference of the orbital integrals to the two conjugates of s. Thus we get $\Delta(s)\cdot O_s^\kappa(T(\pi)) = O_s^{\mathrm{st}}(T(0,\pi))$ with transfer factor $\Delta(s) = \pm|\lambda\mu|^{1/2}\cdot|(\lambda/\lambda'-1)\cdot(\lambda'/\lambda-1)\cdot(\mu/\lambda'-1)\cdot(\lambda'/\mu-1)|^{1/2}\cdot\xi_F(N)/\xi_F(1) = \pm|(\lambda+\mu)-(\lambda'+\mu')|\cdot\xi_F(N)/\xi_F(1)$.

(4.3) The operators $T(\ell,\pi)$ and the groups $H(\ell)$: Define for all integers $\ell \geq 0$

$$(2)\qquad z(0)=E_4,\qquad z(\ell)=\begin{pmatrix}0 & \pi^{-\ell}E_2\\ E_2 & 0\end{pmatrix} g(\pi^{-\ell})=\begin{pmatrix}0 & \pi^{-\ell}E_2\\ E_2 & \pi^{-\ell}W\end{pmatrix}\qquad W=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \ell\geq 1,$$

$$(2)\qquad H(\ell)=H(F)\bigcap z(\ell)\cdot K\cdot z(\ell)^{-1},$$

$$(4)\qquad T(\ell,\pi)=T(\pi)\circ\left(\operatorname{Ad} z(\ell)^{-1}\right)\big|_{H(F)}\in\mathcal{H}\big(H(F),H(\ell)\big).$$

The family of all $z(\ell)$ is again a system of representatives for $H(F)\backslash GSp(4,F)/K$. We claim

(4.3.1) Proposition: *For any $\ell\geq 0$ the support of $T(\ell,\pi)$ is $H(\ell)\cdot[\operatorname{diag}(1,\pi),\operatorname{diag}(\pi,1)]\cdot H(\ell)$ and $\operatorname{supp}(T(\ell,\pi))\subseteq\operatorname{supp}(T(\ell',\pi))\subseteq\operatorname{supp}(T(0,\pi))$ for all $\ell>\ell'$.*

(4.3.2) Lemma: *Let pr be the projection of $H(F)$ on its first $GL(2)$-factor. Then for all $\ell\geq 0$ the sequence $1\longrightarrow\{E_2\}\times\Gamma(\ell)\longrightarrow H(\ell)\xrightarrow{\text{pr}}GL(2,\mathcal{O}_F)\longrightarrow 1$ is exact, where $\Gamma(\ell)$ is the principal congruence subgroup of $SL(2,\mathcal{O}_F)$ of level π^ℓ.*
Furthermore $H(\ell)=\{[X,Y]\in H(\mathcal{O}_F): X\equiv Y^w(\operatorname{mod}\gamma^{-1}\mathcal{O}_F)\}$ where $Y^w=(\operatorname{Int}w)(Y)$ and $w=E_{12}+E_{21}$.

(4.3.3) Symmetrization by the automorphism $1\times\Phi$ of H: For $W=E_{12}+E_{21}$ let

$$(5)\qquad (1\times\Phi)([h,h'])=\left(\operatorname{Ad}([E_2,W])\right)([h,h'])=\left[h,(\operatorname{Ad}W)(h')\right]=\left[h,\Phi(h')\right].$$

Then $H_\Phi(\ell)=(1\times\Phi)(H(\ell))=\{[X,Y]\in H(\mathcal{O}_F): X\equiv Y(\operatorname{mod}\gamma^{-1}\mathcal{O}_F)\}$ and the support of the pullback of $T(\ell,\pi)$ by $1\times\Phi$ is $H_\Phi(\ell)\cdot[\operatorname{diag}(1,\pi),\operatorname{diag}(1,\pi)]\cdot H_\Phi(\ell)$.

A straightforward calculation shows (4.3.2). To prove (4.3.1) we first indicate a general strategy to determine the $H(\ell)$-double cosets ξ in the support of the pullback $f\circ(\operatorname{Ad}z(\ell)^{-1})$ to $H(F)$ of a Hecke operator f. Choose a representative of ξ whose first $GL(2)$-component is a diagonal matrix $\operatorname{diag}(a_1,d_1)$ with pure π-powers a_1, d_1 and $\operatorname{ord}a_1\leq\operatorname{ord}d_1$. We have to decide when $Y_{\ell,S}=z(\ell)^{-1}\cdot[\operatorname{diag}(a_1,d_1),\operatorname{diag}(d_1,a_1)\cdot S]\cdot z(\ell)$ is in the support of f, where S is chosen in $SL(2,F)/\Gamma(\ell)$.
Using the filtration $\Gamma(\ell+1)\subseteq\Gamma(\ell)\subseteq\Gamma(0)$ one should deal with this problem iteratively, starting with $\ell=0$. For ℓ fixed, one decides in a first step for which parameters $Y_{\ell,S}$ has entries in $\mathcal{O}_F$. Only for these one determines then in a second step the elementary divisors.
Let $f=T(\pi)$. Then $a_1=1$, $d_1=\pi$ imply that $Y_{\ell,S}$ has entries in $\mathcal{O}_F$ only if S is in $\Gamma(\ell)$.

(4.4) Necessary conditions on s: The similarity factor $\mu(s)=a^2-b^2A=(a')^2-(b')^2B$ of s has by Hensel's lemma the same order one as $\mu(T(\pi))$ only if $\operatorname{ord}A=\operatorname{ord}B=1$, if $\operatorname{ord}a$, $\operatorname{ord}a'\geq 1$, and if b, b' are both units. Especially then, E and L are ramified over F.

We use an iterative procedure based on the support filtration (4.3.1) to calculate the orbital integrals $O_s(T(\ell,\pi))$, the key step being (4.6).

(4.5) The parameter set $\mathcal{N}(E,L)$: We have $E^*/\operatorname{pr}(T(F))\cong\mathcal{N}(E,L)$, where

$$\mathcal{N}(E,L)=\begin{cases}\{E_4\} & E=L\\ \{E_4,x\} & E\neq L\end{cases}\qquad\text{with}\qquad x=[x_E,x_L]=\left[\begin{pmatrix}0 & D^{-1}A\\ D & 0\end{pmatrix},\begin{pmatrix}0 & A\\ 1 & 0\end{pmatrix}\right],$$

since by construction $\mu(T(F))=N_{E/F}(E^*)\cap N_{L/F}(L^*)$.

(4.6) Proposition: *In the case $E=L$ the support of $O_s^H(T(0,\pi))$ is*

$$\left\{h\in H(F):T(0,\pi)(h^{-1}sh)\neq 0\right\}=T(\mathcal{O}_F)\backslash H(\mathcal{O}_F)=\left\{T(\mathcal{O}_F)\cdot h:h\in H(\mathcal{O}_F)\right\}.$$

For $E\neq L$ the support $\{h\in H(F):T(0,\pi)(h^{-1}sh)\neq 0\}$ is the disjoint union of the two sets

$$T(\mathcal{O}_F)\backslash H(\mathcal{O}_F)=\left\{T(\mathcal{O}_F)\cdot g:g\in H(\mathcal{O}_F)\right\},$$

$$x^{-1}\cdot T(\mathcal{O}_F)\cdot x\backslash H(\mathcal{O}_F)\cong T(F)\backslash T(F)\cdot x\cdot H(\mathcal{O}_F)=\left\{T(F)\cdot x\cdot h:h\in H(\mathcal{O}_F)\right\}.$$

(4.7) Proposition: *For $\alpha = [\alpha_E, \alpha_L]$ in $\mathcal{N}(E,L)$ and $\ell \geq 1$ let $R(\alpha,\ell,s)$ be the number of y in $SL(2,\mathcal{O}_F/\pi^\ell\mathcal{O}_F)$ such that $[s_E, y^{-1}(\alpha_L^{-1}s_L\alpha_L)y]$ is in the support of $T(\ell,\pi)$. Then*

$$O_s(T(\ell,\pi)) = \frac{\mathrm{vol}_H(H(\ell))}{\mathrm{vol}_T(T(\mathcal{O}_F))}\sum\nolimits_{\alpha\in\mathcal{N}(E,L)} R(\alpha,\ell,s),$$

$$O_s(T(\pi)) = \frac{\mathrm{vol}_G(GSp(4,\mathcal{O}_F))}{\mathrm{vol}_T(T(\mathcal{O}_F))}\left([EL:E] + \sum\nolimits_{\alpha\in\mathcal{N}(E,L)}\sum\nolimits_{\ell>0} R(\alpha,\ell,s)\right).$$

The elements $[1,y]$ with y in $SL(2,\mathcal{O}_F/\pi^\ell\mathcal{O}_F)$, which we consider as section in $M(2,\mathcal{O}_F)$ for $SL(2,\mathcal{O}_F)/\Gamma(\ell)$, are representatives for $H(\mathcal{O}_F)/H(\ell)$. So (4.7) follows by (4.6) and §3(1).

Proof of (4.6): Our argument is based on the fact that $g_n = \mathrm{diag}(1,\pi^n)$ with $n \geq 0$ are representatives of $\tau(F)\backslash GL(2,F)/GL(2,\mathcal{O}_F)$, where τ is any of the tori E^*, L^* or $x_L^{-1}L^*x_L$.
Fix $[h,h']$ in the support set. By (4.3.1) – (4.3.2) each factor of $[h^{-1}s_Eh, (h')^{-1}s_Lh']$ is in $\xi = GL(2,\mathcal{O}_F)\mathrm{diag}(1,\pi)GL(2,\mathcal{O}_F)$. Writing $h = t_0g_nk$ with t_0 in E^* and k in $GL(2,\mathcal{O}_F)$ it follows that $g_n^{-1}s_Eg_n$ is in ξ. Since b is a unit an explicit calculation shows that this implies $n=0$.
Decompose t_0 in the form $t_0 = t_E\alpha_E$ where α_E is in $\mathcal{N}(E,L)$. Choose t_L in L^* such that $\det t_E = \det t_L$. By construction $y = (t_L\alpha_L)^{-1}h'k^{-1}$ is in $SL(2,F)$ and $T(0,\pi)([h,h']^{-1}\cdot s\cdot[h,h']) = T(0,\pi)([s_E, y^{-1}(\alpha_L^{-1}s_L\alpha_L)y])$. As above, $y = (\alpha_L^{-1}t'\alpha_L)k'$ with t' in L^* and k' in $GL(2,\mathcal{O}_F)$. Then t' is a unit, so that the entries of $\alpha_L^{-1}t'\alpha_L$ are in $\mathcal{O}_F$. Hence y is in $SL(2,\mathcal{O}_F)$. The reverse inclusions are easily checked.

(4.8) Characterizing $R(\alpha,\ell,s)$ by congruence conditions: Let $P = \mathrm{diag}(1,\pi)$, so that $s_E = h_EP$ for h_E in $GL(2,\mathcal{O}_F)$. Let y be in $SL(2,\mathcal{O}_F/\pi^\ell\mathcal{O}_F)$, taken as set of representatives for $SL(2,\mathcal{O}_F)/\Gamma(\ell)$ in $M(2,\mathcal{O}_F)$. Then $[s_E, y^{-1}(\alpha_L^{-1}s_L\alpha_L)y]$ is in the support of $T(\ell,\pi)$ if and only if

$$(6)\qquad \Phi\Big(y^{-1}(\alpha_L^{-1}s_L\alpha_L)y\Big) \in \Gamma(\ell)\cdot s_E\cdot\Gamma(\ell) \subseteq \begin{pmatrix}\pi\mathcal{O}_F & \pi U(F)\\ U(F) & \pi\mathcal{O}_F\end{pmatrix}.$$

The order conditions hold only for the elements

$$(7)\qquad Y(\omega,\beta,\delta) = \begin{pmatrix}\omega & \gamma\\ \delta & \beta\end{pmatrix}\quad\text{with}\quad \gamma = \frac{\omega\beta-1}{\delta}\quad\text{and}\quad \begin{array}{l}\omega\in\langle\pi,\ldots,\pi^{\ell-1}\rangle,\\ \beta\in\langle 1,\pi,\ldots,\pi^{\ell-1}\rangle,\\ \delta\in k(F)^*\oplus\langle\pi,\ldots,\pi^{\ell-1}\rangle.\end{array}$$

They are defined by the intersection of three quadrics in the affine space of dimension four over $\mathcal{O}_F/\pi^\ell\mathcal{O}_F$: In the case $\alpha = E_4$ multiply Y, with entries denoted as in (7), by $\omega - \delta\sqrt{B}$ on the left and let $\Delta_1(Y) = \omega^2 - \delta^2B$, $z_1(Y) = \omega\gamma - \delta\beta B$. For $\alpha = x$ multiply by $x_L^{-1}(\omega - \delta(A/B)\sqrt{B})x_L$ and let $\Delta_x(Y) = \omega^2 - \delta^2(A/B)\cdot A$, $z_x(Y) = \omega\gamma - \delta\beta(A/B)\cdot A$. Then

$$(8-1)\qquad \Phi\Big(Y^{-1}s_LY\Big) = \begin{pmatrix} a' + z_1b' & b'\Delta_1\\ \dfrac{b'(B-z_1^2)}{\Delta_1} & a' - z_1b'\end{pmatrix},$$

$$(8-2)\qquad \Phi\Big(Y^{-1}(x_L^{-1}s_Lx_L)Y\Big) = \begin{pmatrix} a' + z_x\frac{B}{A}b' & b'\frac{B}{A}\Delta_x\\ \dfrac{b'(A - A^{-1}Bz_x^2)}{\Delta_x} & a' - z_x\frac{B}{A}b'\end{pmatrix}.$$

Their entries have orders as in (6) only if $\mathrm{ord}\,\Delta_\alpha = 1$ and $\mathrm{ord}\,z_\alpha \geq 1$. These conditions characterize the elements $Y(\omega,\beta,\delta)$.
By construction $\Phi(Y^{-1}(\alpha_L^{-1}s_L\alpha_L)Y)P^{-1}$ has entries in $\mathcal{O}_F$ for all $Y = Y(\omega,\beta,\delta)$. Thus the following easily proved criterion applies to determine for which (Δ, z) the $\Gamma(\ell)$–double cosets of $\Phi(Y^{-1}(\alpha_L^{-1}s_L\alpha_L)Y)$ and $s_E = h_EP$ are equal.

(4.8.1) Lemma: *Let g', g'' be in $GL(2,\mathcal{O}_F)$. Then $\Gamma(\ell)\cdot g'P\cdot\Gamma(\ell) = \Gamma(\ell)\cdot g''P\cdot\Gamma(\ell)$ if and only if there is β in $\pi^{\ell-1}k(F)$ such that*

$$g' \equiv g''\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}(\mathrm{mod}\,\Gamma(\ell)).$$

For $\alpha = E_4$ these congruences are equivalent to $a \equiv a' (\mathrm{mod}\, \pi^\ell \mathcal{O}_F)$, $0 \equiv z (\mathrm{mod}\, \pi^\ell \mathcal{O}_F)$ and

$$\Delta_0 = \frac{\Delta}{\pi} \equiv \frac{A}{\pi D} \frac{b}{b'} (\mathrm{mod}\, \pi^\ell \mathcal{O}_F), \qquad \left(\frac{b}{b'}\right)^2 \frac{A}{\pi} \equiv \frac{B - z^2}{\pi} (\mathrm{mod}\, \pi^\ell \mathcal{O}_F).$$

Especially then, the values $\mathrm{mod}\, \pi^\ell \mathcal{O}_F$ of Δ_0, z are completely determined. Since $\mathrm{ord}\, z \geq \ell$ we obtain $(b/b')^2 \equiv B/A (\mathrm{mod}\, \pi^\ell)$. So $A = B$ by Hensel's lemma and since A and B are normalized. Consequently $(b/b')^2 \equiv 1 (\mathrm{mod}\, \pi^\ell)$. In the case $\alpha = x$ we obtain similar congruences. They, too, hold only for $A = B$, contradicting the assumption $A \neq B$.

We note that $a \equiv a' (\mathrm{mod}\, \pi^\ell \mathcal{O}_F)$ implies $b^2 \equiv (b')^2 (\mathrm{mod}\, \pi^\ell \mathcal{O}_F)$ and thus $b \equiv \varepsilon b' (\mathrm{mod}\, \pi^\ell \mathcal{O}_F)$ because of (4.4).

For a smooth affine variety V over $\mathcal{O}_F$ the fibres of $V(\mathcal{O}_F/\pi^{i+1}\mathcal{O}_F) \longrightarrow V(\mathcal{O}_F/\pi^i \mathcal{O}_F)$ have cardinality $\# k(F)^{\dim V}$ for all $i \geq 1$. The following result, proved by showing that the Jacobian has full rank, now completes the proof of (4.2)

(4.8.2) Lemma: *For z in $\pi\mathcal{O}_F$, Υ in $\pi U(F)$ and Δ_0 in $U(F)$ let Q_Υ be the zero set of $\Delta_0 = \pi^{-1}(\omega^2 - \delta^2 \Upsilon)$, $z = \omega\gamma - \delta\beta\Upsilon$, $1 = \omega\beta - \delta\gamma$ in the fourdimensional affine space. Then Q_Υ is a smooth variety over $\mathcal{O}_F$ and*

$$\#\left\{(\omega, \beta, \delta, \gamma) \in Q_\Upsilon(\mathcal{O}_F/\pi^\ell \mathcal{O}_F) : \begin{matrix} \omega \equiv 0 (\mathrm{mod}\, \pi\mathcal{O}_F), \\ \delta \not\equiv 0 (\mathrm{mod}\, \pi\mathcal{O}_F) \end{matrix}\right\} = \begin{cases} 2 \cdot (\# k(F))^\ell & -\frac{\pi}{\Upsilon}\Delta_0 \in U(F)^2 \\ 0 & \textit{otherwise.} \end{cases}$$

5. Proof of Theorem (2.2): Using the Iwasawa decomposition of $GSp(4, F)$ one starts with representatives in the Borel subgroup of upper triangular matrices in $GSp(4, F)$. Multiplying on the left by suitable upper triangular matrices in $H(F)$ they can be modified to representatives of the form $h(a, b, 0)$ in the Heisenberg subgroup of $GSp(4, F)$ consisting of all matrices

$$h(a, b, c) = \begin{pmatrix} 1 & a & c & b \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -a & 1 \end{pmatrix}$$

with a, b, c in F. Since $h(0, 0, c)$ is in $H(F)$ for all c in F, the relations

$$h(a, b, 0) \cdot h(a', b', 0) = h(a + a', b + b', ab' - a'b), \qquad h(0, 0, -c) \cdot h(a, b, c) = h(a, b, 0)$$

show that one may in fact choose representatives $g(a, b) = h(a, b, 0)$ with a, b in the $k(F)$–space $\langle \ldots, \pi^{-2}, \pi^{-1} \rangle$. Here we follow a suggestion of Prof. Weissauer for simplifying our original proof.

We reduce to pure π–powers a and b. Let $\alpha = ua$ and $\beta = wb$ for units u und w in $U(F)$. Then

$$g^{-1}(\alpha, \beta) \left[\begin{pmatrix} uw & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} w & 0 \\ 0 & u \end{pmatrix} \right] g(a, b) = \mathrm{diag}(uw, w, 1, u)$$

is in $GSp(4, \mathcal{O}_F)$ so that $g(\alpha, \beta)$ and $g(a, b)$ are in the same coset of $H(F) \backslash GSp(4, F)/GSp(4, \mathcal{O}_F)$.

By the same reasoning we reduce further to representatives $g(\gamma) = g(0, \gamma)$. In the case $\mathrm{ord}\, b \leq \mathrm{ord}\, a$

$$g^{-1}(0, b) \left[\begin{pmatrix} 0 & b \\ b^{-1} & 0 \end{pmatrix}, \begin{pmatrix} a & b \\ b^{-1} & 0 \end{pmatrix} \right] g(a, b) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ b^{-1} & ab^{-1} & 0 & 1 \\ 0 & b^{-1} & 1 & 0 \end{pmatrix}$$

is an element of $GSp(4, \mathcal{O}_F)$. For $\mathrm{ord}\, a < \mathrm{ord}\, b$

$$g^{-1}(0, a) \left[\begin{pmatrix} 0 & -a \\ a^{-1} & 0 \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & -a^{-1} \end{pmatrix} \right] g(a, b) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ a^{-1} & 1 & 0 & a^{-1}b \\ 0 & 0 & -1 & -a^{-1} \end{pmatrix}$$

is in $GSp(4, \mathcal{O}_F)$.

To prove independence, assume that $g(\alpha)$ and $g(\beta)$ are in the same coset. Equivalently, there is h in $H(F)$ such that $hg(\alpha)\cdot GSp(4,\mathcal{O}_F) = g(\beta)\cdot GSp(4,\mathcal{O}_F)$. Taking images of this $GSp(4,\mathcal{O}_F)$–coset under each element of a dual basis one obtains four equalities of ideals in $\mathcal{O}_F$. They translate into four conditions on the orders of the entries of $hg(\alpha)$ and $g(\beta)$. Distinguishing the cases $\alpha\beta = 0$ and $\alpha\beta \neq 0$ it follows easily that α and β have the same orders and thus are in fact equal.

6. Proof of Theorem (2.4): One starts again with representatives in the Borel subgroup of upper triangular matrices in $GSp(4,F)$. Their components in the Levi factor $\{\mathrm{diag}(A,\lambda^t A) : A \in GL(2,F),\ \lambda \in F^*\}$ can be reduced to matrices $\mathrm{diag}(g_\ell, g_\ell^{-1})$, where $g_\ell = \mathrm{diag}(1,\pi^\ell)$ with $\ell \geq 0$ are representatives of $(E^+)^*\backslash GL(2,F)/GL(2,\mathcal{O}_F)$. Because of

$$\begin{pmatrix} 1 & 0 & b_1 & b_2 \\ 0 & 1 & b_2 & b_1 A \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & x & y \\ 0 & \pi^\ell & \pi^\ell y & \pi^\ell z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \pi^{-\ell} \end{pmatrix} = \begin{pmatrix} 1 & 0 & a+b_1 & \pi^{-\ell}(b_2+\pi^\ell y) \\ 0 & \pi^\ell & b_2+\pi^\ell y & \pi^\ell z + \pi^{-\ell} b_1 A \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \pi^{-\ell} \end{pmatrix}$$

we can choose representatives with $y = z = 0$. After multiplying from the right by a suitable unipotent matrix in $GSp(4,\mathcal{O}_F)$ we can assume x in $\langle \ldots, \pi^{-2}, \pi^{-1}\rangle$ and obtain the matrices $g(\ell, x)$.

We now show that we can achieve $\ell = 0$. For $\ell \geq 1$

$$g(0,\pi^{-\ell})^{-1}\begin{pmatrix} \pi^{-\ell} & 0 & 0 & 0 \\ 0 & \pi^{-\ell} & 0 & 0 \\ 1 & 0 & \pi^\ell & 0 \\ 0 & A^{-1} & 0 & \pi^\ell \end{pmatrix} g(\ell,0) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & \pi^\ell & 0 \\ 0 & A^{-1}\pi^\ell & 0 & 1 \end{pmatrix}$$

is in $GSp(4,\mathcal{O}_F)$. For $x \neq 0$ let $z = -(1+\pi^\ell)x^{-1}$. Then

$$g(0,-x\pi^{-\ell})^{-1}\begin{pmatrix} \pi^{-\ell} & 0 & 0 & 0 \\ 0 & \pi^{-\ell} & 0 & 0 \\ z & 0 & \pi^\ell & 0 \\ 0 & zA^{-1} & 0 & \pi^\ell \end{pmatrix} g(\ell,x) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ z & 0 & -1 & 0 \\ 0 & zA^{-1} & 0 & 1 \end{pmatrix}$$

is in $GSp(4,\mathcal{O}_F)$. We reduce to pure π–powers by the calculation

$$g(0,x)^{-1}\begin{pmatrix} E_2 & 0 \\ 0 & uE_2 \end{pmatrix} g(0,y) = \begin{pmatrix} E_2 & \begin{matrix} y-ux & 0 \\ 0 & 0 \end{matrix} \\ 0 & uE_2 \end{pmatrix}.$$

$$\begin{pmatrix} E_2 & \begin{matrix} -y & 0 \\ 0 & -yA \end{matrix} \\ 0 & E_2 \end{pmatrix} g(0,y) = \begin{pmatrix} E_2 & \begin{matrix} 0 & 0 \\ 0 & -yA \end{matrix} \\ 0 & E_2 \end{pmatrix}$$

eventually shows that representatives of $\mathcal{H}_\xi$ are of the form $E_4 = g(0,0)$ and $g(\ell) = g(0,\pi^{-\ell})$ with $\ell \geq 1$ for ord $A = 0$, i.e. for E^+ unramified over F, and $g(\ell)$ with $\ell \geq 1$ for ord $A = 1$.
To check their independence is tedious, but straightforward given the method indicated in §5.

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8. References

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