Nr. 179/94

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Abstract

In this paper we study a class of parabolic equations with a non-linear gradient term. The system is disturbed by white noise in time. We show that the solution of this problem can be represented as the Wick product between a normalized random variable of exponential form and the solution of a nonlinear parabolic equation. We allow random initial data which might be anticipating. A relation between the Wick product with a normalized exponential and translation is proved in order to establish our results.

1 Introduction

In recent years there has been a growing interest in stochastic partial differential equations. Within the White Noise Analysis, many authors have been studying several kinds of partial differential equations involving noise. We mention here just a few works: Potthoff [P1,P2], Lindstrøm et. al. [LØU1,LØU2], Gjessing [G] and Benth [B2]. The main advantage of the white noise framework, are the $\mathcal{S}$-transform and the Hermite-transform. The philosophy behind these two transformations is to map stochastic problems into deterministc problems, where classical theory can be invoked.

In this paper we investigate a nonlinear parabolic problem of the form

$$\phi_t(t,x,\omega) + \left(f(t,x,\phi(t,x,\omega))\right)_x = \nu\phi_{xx}(t,x,\omega) + \sigma(t)\phi(t,x,\omega)\cdot W_t(\omega) \quad (1)$$

Here, $W_t(\omega)$ is white noise in time. We are going to precise the interpretation of the noise term in section 2 and 4. Time will be assumed to run over a finite intervall $[0,T]$, and the space variable $x \in \mathbb{R}$. We recognize the equation as a

Burgers equation with noise in the case when $f(t,x,u) = \frac{1}{2}u^2$. However, in this paper we will restrict ourselves to functions f which are of uniform Lipschitz type. The usual technique within the white noise framework is to transform the stochastic problem into a deterministic one, using either the $\mathcal{S}$-transform or the Hermite-transform. However, due to the nonlinear gradient term, this transformation technique seems to fail.

To solve equation (1), we will use a reduction method introduced by Gjessing in [G] for ordinary stochastic differential equations. To motivate our method, consider the stochastic process

$$u(t,x,\omega) = v(t,x,\omega) \diamond X(t,\omega)$$

where X and v are solutions of the problems

$$X(t,\omega) = 1 + \int_0^t X(s,\omega)dB_s(\omega)$$

and

$$v_t(t,x,\omega) = v_{xx}(t,x,\omega)$$
$$v(0,x,\omega) = v_0(x,\omega)$$

For a definition of the Wick product, see section 2 and Hida et. al. [HKPS]. Observe that $u(0,x,\omega) = v_0(x,\omega)$. Moreover, differentiation of $u(t,x,\omega)$ with respect to t (on a formal level) gives:

$$u_t(t,x,\omega) = v_t(t,x,\omega) \diamond X(t,\omega) + v(t,x,\omega) \diamond \frac{d}{dt}X(t,\omega)$$

$$= v_{xx}(t,x,\omega) \diamond X(t,\omega) + v(t,x,\omega) \diamond X(t,\omega) \cdot \frac{d}{dt}B_t(\omega)$$

$$= u_{xx}(t,x,\omega) + u(t,x,\omega) \cdot \frac{d}{dt}B_t(\omega)$$

Hence, the solution of the problem (written on Ito form)

$$du(t,x,\omega) = u_{xx}(t,x,\omega)dt + u(t,x,\omega)dB_t(\omega)$$

$$u(0,x,\omega) = v_0(x,\omega)$$

can be split into the solution of a (deterministic) heat problem and an ordinary stochastic differential equation. There are reasons to expect that the solution of (1) also can be represented in a similar way. However, the heat equation part of the solution will now include a nonlinear term which is a modification of the f in (1). The reduced equation has the form

$$\psi_t(t,x,\omega) + X(t,\omega+\sigma_t)^{-1}\left(f(t,x,X(t,\omega+\sigma_t)\cdot\psi(t,x,\omega))\right)_x = \nu\psi_{xx}(t,x,\omega) \tag{2}$$

σ_t is the function

$$\sigma_t(\cdot) = \mathbf{1}_{[0,t)}(\cdot)\sigma(\cdot)$$

In section 4 we will study equation (2), and show that it has a solution with the correct regularity properties.

To be able to reduce problem (1), we will use a formula telling us the relation between the Wick product with a normalized exponential, and translation. Such a formula is worked out in section 3. If we denote $\mathrm{Exp}W_\eta = e^{<\omega,\eta>-\frac{1}{2}|\eta|^2}$, the normalized exponential, where $\eta \in L^2(I\!R)$, we show that

$$\mathrm{Exp}W_\eta \diamond \Phi = T_{-\eta}\Phi \cdot \mathrm{Exp}W_\eta$$

Φ is a generalized random variable, and T denotes (generalized) translation. The product on the right hand side makes sense, since the exponential will be a smooth random variable in our setting. In section 3 we define translation of generalized random variables.

In connection to the stochastic partial differential equations which we are going to study, η will naturally be an element of $L^2(I\!R)$, and not a Schwartz function. This means that we cannot work with the usual white noise spaces of Hida test functions and Hida distributions since normalized exponentials will not be test functions when $\eta \in L^2(I\!R)$. Using a dual pair of spaces studied by Potthoff and Timpel in [PT], we can allow $\eta \in L^2(I\!R)$.

One of the advantages with our method, is that the initial condition of (1), $\phi_0(x,\omega)$, does not have to be independent of the driving white noise. We can allow for anticipating initial conditions, and still obtain existence results. This opens for initial conditions which are functions of the Brownian motion, for instance. We remark that Buckdahn [Bu1-5] has done similar work within the Malliavin Calculus. We refer to Gjessing [G] for further comments and references.

In section 5 we state the existence and regularity results for problem (1). We finally look at a concrete example.

2 Some Mathematical Preliminaries

We start with some preliminaries from the white noise analysis. For a complete account on this theory, we refer to the excellent book by Hida et. al. [HKPS]. See also Kondratiev et. al. [KLPSW] and Potthoff and Timpel [PT].

Let

$$(\mathcal{S}'(I\!R), \mathcal{B}, \mu)$$

be our probability space. We denote $L^p(\mathcal{S}'(I\!R), \mathcal{B}, \mu)$ by (L^p). Now consider the Ornstein-Uhlenbeck or the number operator N. Denote by $\mathcal{P}$ the algebra of polynomials in (L^2). For an element $\varphi \in \mathcal{P}$, we have the chaos expansion

$$\varphi = \sum_{n=1}^{\infty} \varphi^{(n)}$$

for finitely many non-zero $\varphi^{(n)}$. Hence $e^{\lambda N}\varphi \in (L^2)$ for each $\lambda \in I\!R$, and we denote by $\mathcal{G}_\lambda$ the completion of $\mathcal{P}$ under the norm

$$\|\varphi\|_\lambda := \|e^{\lambda N}\varphi\|_{(L^2)}$$

By usual construction, $\mathcal{G}$ is the projective limit of $\mathcal{G}_\lambda$, and hence becomes a countably Hilbert space (see [GV]). $\mathcal{G}^*$ is the dual of $\mathcal{G}$, and equals

$$\mathcal{G}^* = \bigcup_{\lambda \in I\!R} \mathcal{G}_\lambda$$

Hence, we have the triplet

$$\mathcal{G} \subset (L^2) \subset \mathcal{G}^*$$

For more information about these spaces, the interested reader should confer [PT].

We define the *Wick product*: Let Φ, Ψ be two elements of $\mathcal{G}^*$ with chaos decompositions $\{f^{(n)}, n \in I\!N_0\}, \{g^{(n)}, n \in I\!N_0\}$ respectively. Then the Wick product between Φ, Ψ, denoted $\Phi \diamond \Psi$, has the chaos decomposition $\{h^{(n)}, n \in I\!N_0\}$, where

$$h^{(n)} = \sum_{m=0}^{n} f^{(n-m)} \widehat{\otimes} g^{(m)}$$

It can be shown (cf. [PT]) that both $\mathcal{G}$ and $\mathcal{G}^*$ are closed under the Wick product.

Later in this section we will need the $\mathcal{S}$-transform, (see [HKPS]). Since $\mathcal{G}^*$ is a subspace of $(\mathcal{S})^*$, the $\mathcal{S}$-transform is well defined on $\mathcal{G}^*$. It is defined as follows: Let $\Phi \in \mathcal{G}^*$. Then for $\xi \in \mathcal{S}(I\!R)$

$$\mathcal{S}\Phi(\xi) = \langle \Phi, \mathrm{Exp} W_\xi \rangle$$

Here, $\langle \cdot, \cdot \rangle$ is the dual pairing between $\mathcal{G}$ and $\mathcal{G}^*$, and $\mathrm{Exp}W_\xi$ is called the Wick exponential of the coordinate process $W_\xi(\omega) = <\omega, \xi>$. We have

$$\mathrm{Exp} W_\xi = e^{<\omega,\xi> - \frac{1}{2}|\xi|^2}$$

It is known that the $\mathcal{S}$-transform transforms Wick products into ordinary products, (see [HKPS]): If $\Phi, \Psi \in \mathcal{G}^*$, then

$$\mathcal{S}(\Phi \diamond \Psi)(\xi) = \mathcal{S}\Phi(\xi) \cdot \mathcal{S}\Psi(\xi)$$

There exists a natural definition of a Brownian motion in $\mathcal{G}$: Define

$$B_t(\omega) = \langle \omega, \mathbf{1}_{[0,t)} \rangle$$

There exists a continuous version of B_t, which will be a standard Brownian motion. *(We denote both processes by B_t).* One can show that for $\gamma \in L^2(I\!R)$

$$\int_{I\!R} \gamma(s) dB_s(\omega) = \langle \omega, \gamma \rangle = W_\gamma(\omega)$$

We define integration of parametrized $\mathcal{G}^*$ elements, *(see [HKPS,PT])*: Let $\Phi(x) \in \mathcal{G}^*$ for each $x \in \Lambda$, where (Λ, λ) is a measure space. If

$$\langle \Phi(x), \phi \rangle \in L^1(\Lambda, \lambda)$$

for all $\phi \in \mathcal{G}$, we define the (Pettis) integral of $\Phi(x)$, $\int_\Lambda \Phi(x) d\lambda(x)$, as the unique $\mathcal{G}^*$-element

$$\langle \int_\Lambda \Phi(x) d\lambda(x), \phi \rangle = \int_\Lambda \langle \phi(x), \phi \rangle d\lambda(x)$$

In the last part of this section we look at some relations for stochastic integration in $\mathcal{G}^*$: In Hida et. al. [HKPS], Lindstrøm et. al. [LØU2] and Benth [B1] the following relation between the Skorohod integral and the white noise integral is discussed:

$$\int_0^t X_s \delta B_s = \int_0^t X_s \diamond W_s ds \tag{3}$$

The right hand side is to be understood as the Hitsuda-Skorohod integral *(see [HKPS])*. For a discussion of Skorohod integration, see Nualart and Zakai [NZ]. W_t is the singular white noise defined by the $\mathcal{S}$-transform

$$\mathcal{S}W_t(\xi) = \xi(t)$$

We remark that W_t is an element of $(\mathcal{S})^*$. In Potthoff and Timpel [PT] it is proved that the Hitsuda-Skorohod integral generalizes Skorohod integration for $\mathcal{G}^*$ elements.

We show an integration by parts formula for Pettis integrals in $\mathcal{G}^*$ which will be useful later. We remark that this formula is given in [G]. However, we will here prove it in a slightly different way:

Proposition 1 *Assume Φ_s, Ψ_s, $\int_0^s \Psi_u du \diamond \Phi_s$ and $\int_0^s \Phi_u du \diamond \Psi_s$ are Pettis integrable elements of $\mathcal{G}^*$ on $[0,t]$. Then*

$$\int_0^t \Phi_s ds \diamond \int_0^t \Psi_s ds = \int_0^t \Phi_s \diamond (\int_0^s \Psi_u du) ds + \int_0^t \Psi_s \diamond (\int_0^s \Phi_u du) ds$$

Proof: The proposition follows by the $\mathcal{S}$-transform together with the classical integration by parts formula:

$$\mathcal{S}(\int_0^t \Phi_s ds \diamond \int_0^t \Psi_s ds)(\xi) = \int_0^t \mathcal{S}\Phi_s(\xi) ds \int_0^t \mathcal{S}\Psi_s(\xi) ds$$

$$= \int_0^t \mathcal{S}\Phi_s(\xi) \left(\int_0^s \mathcal{S}\Psi_u(\xi) du \right) ds + \int_0^t \mathcal{S}\Psi_s(\xi) \left(\int_0^s \mathcal{S}\Phi_u(\xi) du \right) ds$$

$$= \int_0^t \mathcal{S} \left(\Phi_s \diamond (\int_0^s \Psi_s du) \right) (\xi) ds + \int_0^t \mathcal{S} \left(\Psi_s \diamond (\int_0^s \Phi_u du) \right) (\xi) ds$$

$$= \mathcal{S}(\int_0^t \Phi_s \diamond (\int_0^s \Psi_u du)ds)(\xi) + \mathcal{S}(\int_0^t \Psi_s \diamond (\int_0^s \Phi_u du)ds)(\xi)$$

■

We end this discussion of integration with a lemma describing the linearity of the Pettis integral under the Wick product:

Lemma 2 *Assume $X \in \mathcal{G}^*$, and $\Phi_s, X \diamond \Phi_s$ are Pettis integrable on $[0,t]$. Then*

$$X \diamond \int_0^t \Phi_s ds = \int_0^t X \diamond \Phi_s ds$$

Proof: The $\mathcal{S}$-transform yields

$$\mathcal{S}X(\xi) \int_0^t \mathcal{S}\Phi_s(\xi) ds = \int_0^t \mathcal{S}X(\xi)\mathcal{S}\Phi_s(\xi) ds$$

■

This last lemma has an interesting consequence for the Skorohod integral. It is well-known that the Skorohod integral is not linear under ordinary product *(see Nualart and Zakai [NZ])*. But, by the above lemma, we see that under the Wick product it is linear: Assume $X \in (L^2)$ and $\Phi_s, X \diamond \Phi_s$ Skorohod integrable on $[0,t]$. Then

$$X \diamond \int_0^t \Phi_s \delta B_s = \int_0^t X \diamond \Phi_s \delta B_s$$

3 Translation and the Wick Product with a Normalized Exponential

To produce our reduction formula for stochastic partial differential equations, we will need a connection between the Wick product and the ordinary product when Wick exponentials are involved. We now investigate this topic in more detail:

Define for each $\eta \in L^2(\mathbb{R})$ the translation operator

$$\tau_\eta : \mathcal{G} \to \mathcal{G}$$

by

$$\tau_\eta \phi(\omega) = \phi(\omega + \eta)$$

From [PT] we know that this is a linear and continuous operator on $\mathcal{G}$. We define the adjoint of the translation operator:

$$\tau_\eta^* : \mathcal{G}^* \to \mathcal{G}^*$$

by

$$\langle \tau_\eta^* \Phi, \phi \rangle = \langle \Phi, \tau_\eta \phi \rangle$$

The adjoint of the translation operator has an explicit representation:

Lemma 3 *For each $\eta \in L^2(I\!R)$*

$$\tau_\eta^* \Phi = ExpW_\eta \diamond \Phi$$

Proof: The S-transform and prop.2.3 in [HKPS] give

$$\mathcal{S}(\tau_\eta^*\Phi)(\xi) = \langle \tau_\eta^*\Phi, \mathrm{Exp}W_\xi \rangle = \langle \Phi, \tau_\eta \mathrm{Exp}W_\xi \rangle$$

$$\langle \Phi, e^{\langle \eta, \xi \rangle} \mathrm{Exp}W_\xi \rangle = e^{\langle \eta, \xi \rangle} \mathcal{S}\Phi(\xi) = \mathcal{S}(\mathrm{Exp}W_\eta)(\xi)\mathcal{S}\Phi(\xi)$$

■

This lemma immediately implies a Cameron-Martin-Girsanov type of result.

Corollary 4 *If $\Phi \in \mathcal{G}^*$ and $\eta \in L^2(I\!R)$, then*

$$\langle ExpW_\eta \diamond \Phi, \phi \rangle = \langle \Phi, \tau_\eta \phi \rangle$$

If $\Phi \in \mathcal{G}$, then

$$\int_{S'(I\!R)} \phi(\omega)(ExpW_\eta \diamond \Phi)(\omega) d\mu(\omega) = \int_{S'(I\!R)} \Phi(\omega)\phi(\omega + \eta) d\mu(\omega)$$

Proof: The proposition follows from

$$\langle \mathrm{Exp}W_\eta \diamond \Phi, \phi \rangle = \langle \tau_\eta^* \Phi, \phi \rangle = \langle \Phi, \tau_\eta \phi \rangle$$

■

Observe that when $\Phi \equiv 1$, we recover the Cameron-Martin-Girsanov formula

$$\int_{S'(I\!R)} \phi(\omega) \mathrm{Exp}W_\eta(\omega) d\mu(\omega) = \int_{S'(I\!R)} \phi(\omega + \eta) d\mu(\omega)$$

The adjoint translation operator has the following property:

Corollary 5 *For $\eta, \sigma \in L^2(I\!R)$, we have*

$$\tau_{\eta+\sigma}^* = \tau_\eta^* \tau_\sigma^* = \tau_\sigma^* \tau_\eta^*$$

Proof: This follows immediately from lemma 3, and the identity

$$\mathrm{Exp}W_{\eta+\sigma} = \mathrm{Exp}(W_\eta + W_\sigma) = \mathrm{Exp}W_\eta \diamond \mathrm{Exp}W_\sigma$$

■

Lemma 6 *Assume $\Phi \in \mathcal{G}$. Then*

$$\tau_{-\eta}\Phi(\omega) \cdot ExpW_\eta(\omega) = \Phi(\omega) \diamond ExpW_\eta(\omega) \tag{4}$$

Proof: Direct calculation gives

$$\int_{S'(\mathbb{R})} \phi(\omega)(\mathrm{Exp}W_\eta \diamond \Phi)(\omega)d\mu(\omega) = \int_{S'(\mathbb{R})} \Phi(\omega)\phi(\omega+\eta)d\mu(\omega)$$

$$= \int_{S'(\mathbb{R})} \Phi(\omega-\eta+\eta)\phi(\omega+\eta)d\mu(\omega) = \int_{S'(\mathbb{R})} \Phi(\omega-\eta)\phi(\omega)\mathrm{Exp}W_\eta(\omega)d\mu(\omega)$$

We have used the Cameron-Martin-Girsanov theorem in the last equality. Since this is valid for all $\phi \in \mathcal{G}$, (4) follows. ■

We generalize formula (4) to $\mathcal{G}^*$: Note that for $\eta \in L^2(\mathbb{R})$, $\mathrm{Exp}W_\eta \in \mathcal{G}$. Observe that since $\mathcal{G}$ is closed under ordinary product (see [PT]), τ_η^* is a continuous linear operator from $\mathcal{G}$ into $\mathcal{G}$. We then define generalized translation T_η of $\mathcal{G}^*$-elements as follows: Let $\Phi \in \mathcal{G}^*$. Then

$$\langle T_\eta \Phi, \varphi\rangle = \langle \Phi, \tau_\eta^* \varphi\rangle$$

Observe that $T_\eta = \tau_\eta$ on $\mathcal{G}$. Since $\mathcal{G}$ is sequentially dense in $\mathcal{G}^*$, it is easily seen that (4) extends to $\mathcal{G}^*$, i.e. for $\eta \in L^2(\mathbb{R})$ and $\Phi \in \mathcal{G}^*$ we have

$$T_{-\eta}\Phi \cdot \mathrm{Exp}W_\eta = \Phi \diamond \mathrm{Exp}W_\eta \tag{5}$$

We understand the product on the left hand side of (5) as follows: If $\Psi \in \mathcal{G}^*, \psi \in \mathcal{G}$, define $\Psi \cdot \psi \in \mathcal{G}^*$ by

$$\langle \Psi \cdot \psi, \phi\rangle = \langle \Psi, \psi \cdot \phi\rangle$$

see [HKPS].

From corollary 5, we observe that

$$T_{\eta+\sigma} = T_\eta T_\sigma = T_\sigma T_\eta$$

when $\eta, \sigma \in L^2(\mathbb{R})$.

An important question is on which (L^p)-spaces do generalized translation T_η and classical translation τ_η coincide? To help us answering this question, consider the following lemma:

Lemma 7 *Let $p \in [1, \infty]$. Assume $\Phi \in (L^{3p})$. Then $\tau_\eta \Phi \in (L^{3p/2})$.*

Proof: Observe that $|\tau_\eta \Phi|^{3p/2} \in (L^2)$. Hence, by the translation formula for Gaussian measures, we have

$$\|\tau_\eta \Phi\|_{3p/2}^{3p/2} = \int_{S'(\mathbb{R})} |\Phi(\omega+\eta)|^{3p/2} d\mu(\omega) = \int_{S'(\mathbb{R})} |\phi|^{3p/2}\mathrm{Exp}W_\eta(\omega)d\mu(\omega)$$

Cauchy-Schwarz implies

$$\leq \left(\int_{S'(\mathbb{R})} |\Phi(\omega)|^{3p} d\mu(\omega)\right)^{\frac{1}{3}} \left(\int_{S'(\mathbb{R})} (\mathrm{Exp}W_\eta(\omega))^2 d\mu(\omega)\right)^{\frac{1}{2}} = \|\Phi\|_{3p}^{3p/2} \|\mathrm{Exp}W_\eta\|_2$$

which proves the lemma. ■

From Corollary 4.14 in [HKPS], we have that $(L^p) \subset \mathcal{G}^*$ for $p \in (1, \infty]$. Note that $\Phi \in (L^1)$ can not be considered as a generalized functional. Consider $\Phi \in (L^{3p})$ for $p \in [1, \infty]$: We can find a sequence $(\phi_n)_{n \in \mathbb{N}} \subset \mathcal{G}$ such that $\phi_n \to \Phi$ in (L^{3p}). Recall that the translation operator is a mapping from $\mathcal{G}$ into $\mathcal{G}$. Hence, $(\tau_\eta \phi_n)_{n \in \mathbb{N}} \subset \mathcal{G}$. By the translation formula for Gaussian measures, it is easy to show that $\tau_\eta \phi_n \hookrightarrow \tau_\eta \Phi$ in $(L^{3p/2})$. Moreover, using that $T_\eta = \tau_\eta$ on $\mathcal{G}$:

$$\langle T_\eta \Phi, \psi \rangle = \langle \Phi, \tau_\eta^* \psi \rangle = \lim_n \langle \phi_n, \tau_\eta^* \psi \rangle$$

$$= \lim_n \langle \tau_\eta \phi_n, \psi \rangle = \langle \tau_\eta \Phi, \psi \rangle$$

Therefore, we have that $T_\eta = \tau_\eta$ on (L^p), for $p \in [3, \infty]$.

In the last proposition, we study how regularity is preserved under Wick product with a normalized exponential. We will need this result in the last section, in order to assure that our solution is a stochastic variable, and not a *generalized* one.

Proposition 8 *Put*

$$X = ExpW_\eta$$

for a $\eta \in L^2(\mathbb{R})$. *If* $p \in [1, \infty)$ *and* $Y \in (L^{3p})$, *we have*

$$X \diamond Y \in (L^p)$$

Moreover

$$\|X \diamond Y\|_p \leq e^{K(p)|\eta|_2^2} \|Y\|_{3p} \tag{6}$$

where $K(p) = \frac{1}{2}(3p-1) + 1/3p$.

If $Y \in (L^\infty)$, *then*

$$X \diamond Y \in (L^q)$$

for all $q \in [1, \infty)$. *Moreover, (6) holds for* $X \diamond Y$ *for all such* q.

Proof: Recall that $X \in (L^q)$ for all $q \geq 1$. If $Y \in (L^{3p})$ for $p \in [1, \infty)$, estimation using Cauchy-Schwarz gives

$$E[|X \diamond Y|^p] = E[|X|^p \cdot \tau_{-\eta}|Y|^p] \leq E[|X|^{3p}]^{1/3} \cdot E[\tau_{-\eta}|Y|^{3p/2}]^{2/3}$$

Since $Y^{3p/2} \in (L^2)$, we can use the translation rule for Gaussian measures:

$$E[|X \diamond Y|^p] \leq \|X\|_{3p}^p \cdot E[\mathrm{Exp}W_{-\eta} \cdot |Y|^{3p/2}]^{2/3}$$

$$\leq \|X\|_{3p}^p \|\text{Exp}W_{-\eta}\|_2^{2/3} \cdot (E[|Y|^{3p}]^{1/2})^{2/3} = \|X\|_{3p}^p \|\text{Exp}W_{-\eta}\|_2^{2/3} \|Y\|_{3p}^p$$

To find the $\|X\|_q$ and $\|\text{Exp}W_{-\eta}\|_2$, we use the identity

$$X = \text{Exp}W_\eta = e^{\langle \omega, \eta \rangle - \frac{1}{2}|\eta|_2^2}$$

The rest is then standard calculation.

For the case $Y \in (L^\infty)$, we have that $Y \in (L^q)$ for all $q \in [1, \infty]$. Hence, using the above calculations, the second conclusion in the proposition follows.

■

4 The Parabolic Problem and its Reduced Version

This section is split into two. We start with a precision of problem (1), and state the conditions which we need. Then we look at the so-called reduced version of our parabolic problem. Existence and uniqueness of (L^p)-solutions will be proved using a fixed point technique.

We interpret (1) in weak integral form

$$\phi(t,x,\omega) = (G_t * \phi_0(\cdot,\omega))(x) + \int_0^t (G'_{t-s} * f(s,\cdot,\phi(s,\cdot,\omega)))(x)ds$$

$$+ \int_0^t \sigma(s)(G_{t-s} * \phi(s,\cdot,\omega))(x) \diamond W_s ds \tag{7}$$

$G_t(x)$ denotes the heat kernel associated to the differential operator $\nu\triangle$, and $G'_t(x)$ its derivative with respect to x. By $*$ we understand the convolution product with respect to x. Recall that when Φ_s is Skorohod integrable, we have

$$\int_0^t \Phi_s \delta B_s = \int_0^t \Phi_s \diamond W_s ds$$

This means that the last integral term in (7) really is a (generalized) Skorohod integral.

We make the following assumptions:

a: f is measurable in (t,x), and Lipschitz in the following sense: There exists a positive function $C(t,x)$ such that

$$|f(t,x,u) - f(t,x,v)| \leq C(t,x)|u-v|$$

and

$$\sup_{t,x} |C(t,x)| < \infty$$

b: f is zero in zero, i. e.

$$f(t,x,0)=0$$

for all (t,x)

c: ϕ_0 is measurable in x and ω, and $\sigma(t)$ is bounded on $[0,T]$

In the next section, we will prove that the solution of (7) can be written as

$$\phi(t,x,\omega)=\psi(t,x,\omega)\diamond X(t,\omega)$$

where

$$X(t,\omega)=\mathrm{Exp}(\int_0^t\sigma(s)dB_s(\omega))$$

and ψ satisfies a reduced version of (7):

$$\psi(t,x,\omega)=(G_t*\phi_0(\cdot,\omega))(x)$$

$$+\int_0^t(X(s,\omega+\sigma_s))^{-1}\left(G'_{t-s}*f(s,\cdot,X(s,\omega+\sigma_s)\cdot\psi(s,x,\omega))\right)(x)ds \quad (8)$$

This rest of this section is devoted to the study of the reduced equation (8). Moreover, we will generalize this problem, and study the equation:

$$u(t,x,\omega)=(G_t*u_0(\cdot,\omega))(x)+\int_0^t\left(G'_{t-s}*g(s,\cdot,\omega,u(s,\cdot,\omega))\right)(x)ds \quad (9)$$

We make the following assumptions for a given (in advance) $p\in[1,\infty]$:

1: g is measurable in (t,x,ω), and Lipschitz in the following sense: There exists a positive $K(t,x,\omega)$ such that

$$|g(t,x,\omega,u)-g(t,x,\omega,v)|\leq K(t,x,\omega)|u-v|$$

and

$$K:=\sup_{t,x}(\|K(t,x,\cdot)\|_\infty)<\infty$$

2: g is zero in zero, i. e.

$$g(t,x,\omega,0)=0$$

for all t,x,ω.

3: u_0 is measurable in x and ω, and

$$\sup_x\|u_0(x,\cdot)\|_p<\infty$$

Above, $\|\cdot\|_p$ denotes the (L^p)-norm. Using these assumptions, we show that (9) has a unique solution in (L^p):

We first prove that (9) attains a local solution, i.e., that we can find a t_0 for which our problem has a unique solution. This will be done by using a fixed point principle. Define the space

$$B_p = \{u : C([0,t_0]; C_{\infty,p}) | C_{\infty,p} \text{ is the bounded and}$$
$$\text{uniformly continuous functions from } \mathbb{R} \text{ into } (L^p)\}$$

with norm

$$\|\cdot\|_{B_p} := \sup_{(t,x)\in[0,t_0]\times\mathbb{R}} \|\cdot\|_p$$

B_p is a Banach space. Define the (contraction) map

$$\mathcal{C}u(t,x,\omega) = (G_t * u_0)(x) + \int_0^t \left(G'_{t-s} * g(s,\cdot,\omega,u(s,\cdot,\omega))\right)(x)ds$$

We show that $\mathcal{C}$ is a mapping from B_p into itself: Note that by condition 3 we have that $G_t * u_0$ is bounded uniformly continuous. Moreover, if u is bounded uniformly continuous in x, then $g(u)$ is bounded and uniformly continuous by conditions 1 and 2. Hence, $\mathcal{C}u$ is bounded uniformly continuous. We need to estimate the B_p-norm of $\mathcal{C}u$: Let K_1 be the constant such that

$$\int_{\mathbb{R}} |G'_{t-s}(x)|dx \le K_1(t-s)^{-\frac{1}{2}}$$

If $u \in B_p$, we get by standard estimation using the Lipschitz property of g

$$|\mathcal{C}u(t,x,\omega)| \le (G_t * |u_0|)(x) + \int_0^t |G'_{t-s}| * (|u(s,\cdot,\omega)|K(s,\cdot,\omega))(x)ds$$

By th. 6.19 in [F], and Cauchy-Schwarz, we have

$$\|\mathcal{C}u(t,x,\cdot)\|_p \le G_t * \|u_0(x,\cdot)\|_p(x) + \int_0^t \sup_{t,x} \|K(t,x,\cdot)\|_\infty |G'_{t-s}| * \|u(s,\cdot,\cdot)\|_p(x)ds$$

$$\le \sup_x \|u_0\|_p + K \cdot K_1 \|u\|_{B_p} \int_0^t |t-s|^{-\frac{1}{2}} ds$$

$$\le \sup_x \|u_0\|_p + K \cdot K_1 \|u\|_{B_p} 2t_0^{\frac{1}{2}}$$

Hence, $\mathcal{C}$ is a mapping from B_p into itself. Choose

$$t_0 = \frac{1}{(4KK_1)^2}$$

With this t_0, we show that $\mathcal{C}$ is a contraction: By similar arguments as above, we estimate with $u, v \in B_p$

$$\|\mathcal{C}u - \mathcal{C}v\|_p \le 2KK_1\|u - v\|_{B_p} \cdot t_0^{\frac{1}{2}}$$

and hence

$$\|\mathcal{C}u - \mathcal{C}v\|_{B_p} \le \frac{1}{2}\|u - v\|_{B_p}$$

By Banachs fixed point theorem, (9) has a unique solution in B_p. The solution can now be constructed on $[0, T]$ by standard techniques, since t_0 is chosen as a constant. Using $u(t_0)$ as an initial condition, we obtain a solution on $[t_0, 2t_0]$. Continuing this process, we will obtain a unique solution on the whole time interval after a finite number of steps. Hence, we have the following existence and uniqueness theorem:

Theorem 9 *Assume conditions 1-3 for g and u_0. There exists a unique solution $u(t, x, \omega)$ of (9) which is continuous in t, and bounded and uniformly continuous in x. Moreover, for each (t, x) we have*

$$u(t, x, \cdot) \in (L^p)$$

■

5 Existence of a Solution

We have the following result, which is a generalization of the theorem in Gjessing, [G]:

Theorem 10 *Assume the conditions a-c in section 4 for f, σ and ϕ_0. If, for $p \in [1, \infty)$,*

$$\sup_x \|\phi_0\|_{3p} < \infty$$

then there exists a solution $\phi(t, x, \cdot) \in (L^p)$ of (7) represented as

$$\phi(t, x, \omega) = \psi(t, x, \omega) \diamond Exp(\int_0^t \sigma(s) dB_s(\omega))$$

where $\psi(t, x, \omega)$ satisfies (8). Moreover,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \|\phi(t, x, \cdot)\|_p \le e^{K(p)\int_0^T \sigma^2(s)ds} \cdot \|\psi\|_{B_{3p}} \tag{10}$$

If

$$\sup_x \|\phi_0\|_\infty < \infty$$

then we have a solution $\phi(t,x,\cdot) \in (L^q)$ for all $q \in [1,\infty)$. Moreover, the estimate (10) is valid for all $q \in [1,\infty)$.
$K(p)$ is given in proposition 8.

Proof: In the proof, we have defined

$$\sigma_t(s) := \mathbf{1}_{[0,t)}(s)\sigma(s)$$

and

$$X_t(\omega) := \operatorname{Exp}(\int_0^t \sigma(s)dB_s(\omega))$$

Let $p \in [1,\infty)$. Put

$$g(t,x,\omega,\psi) = (\tau_{\sigma_t}X_t)^{-1} f(t,x,(\tau_{\sigma_t}X_t)\psi)$$

Then we see that

$$|g(t,x,\omega,u) - g(t,x,\omega,v)| \leq C(t,x)|u-v|$$

where $C(t,x)$ is the Lipschitz constant to f *(recall condition a in section 4)*. Hence, theorem 9 gives a solution ψ of the reduced equation belonging to (L^{3p}). Since $p \geq 1$, we are garanteed that $\phi = \psi \diamond X$ are well-defined and belongs to $\mathcal{G}^*$. Moreover, by the proposition 8, we get that $\phi(t,x,\omega) \in (L^p)$, and satisfies (10). For the case $p = \infty$, the same argument applies for each $q \geq 1$.

We prove that $\phi(t,x,\omega)$ solves (7): The proof follows the idea of Gjessing, [G]: Note that $X_t(\omega)$ solves the following stochastic differential equation:

$$X_t(\omega) = 1 + \int_0^t \sigma(s)X_s(\omega)dB_s(\omega)$$

Put

$$\Psi_{t,s} = (X_s(\omega+\sigma_s))^{-1}\left(G'_{t-s} * f(s,x,X_s(\omega+\sigma_s)\cdot\psi)\right)(x)$$

Since the Wick product obeys the standard calculus rules, we get

$$\psi \diamond X_t = (G_t * \phi_0)(x) + \int_0^t \Psi_{t,s}ds$$

$$+ \int_0^t (G_t * \phi_0)(x) \diamond \sigma(s)X_s \diamond W_s ds + \int_0^t \Psi_{t,s}ds \diamond \int_0^t \sigma(s)X_s \diamond W_s ds$$

By the integration by parts formula we obtain

$$= (G_t * \phi_0)(x) + \int_0^t \Psi_{t,s}ds + \int_0^t \sigma(s)(G_t * \phi_0)(x) \diamond X_s \diamond W_s ds$$

$$+\int_0^t \Psi_{t,s} \diamond \left(\int_0^s \sigma(u) X_u \diamond W_u du\right) ds + \int_0^t \sigma(s) X_s \diamond \left(\int_0^s \Psi_{t,u} du\right) ds$$

$$= (G_t * \phi_0)(x) + \int_0^t X_s \diamond \Psi_{t,s} ds + \int_0^t \sigma(s) X_s \diamond \left((G_t * \phi_0)(x) + \int_0^s \Psi_{t,u} du\right) \diamond W_s ds$$

We consider the last two integrals: Recall that $T_\eta = \tau_\eta$ on $\mathcal{G}$, and that $T_{\eta_1+\eta_2} = T_{\eta_1} T_{\eta_2}$. Using the relation (5) between the Wick exponential and translation, we obtain the following:

$$X_s \diamond \Psi_{t,s} = X_s T_{-\sigma_s} \Psi_{t,s} = X_s T_{-\sigma_s} \left\{ (\tau_{\sigma_s} X_s)^{-1} \left(G'_{t-s} * f(s, \cdot, (\tau_{\sigma_s} X_s)\psi) \right)(x) \right\}$$

$$= X_s X_s^{-1} \left(G'_{t-s} * f(s, \cdot, X_s \cdot T_{-\sigma_s} \psi) \right)(x) = G'_{t-s} * f(s, \cdot, X_s \diamond \psi)(x)$$

An easy calculation shows the following relation for the derivative of the heat kernel:

$$G_t * G'_s = G'_{t+s}$$

Hence, we have

$$G_{t-s} * \Psi_{s,u} = G_{t-s} * \left\{ (\tau_{\sigma_u} X_u)^{-1} \left(G'_{s-u} * f(s, \cdot, (\tau_{\sigma_u} X_u)\psi(u)) \right)(x) \right\}$$

$$= (\tau_{\sigma_u} X_u)^{-1} \left(G'_{t-u} * f(s, \cdot, (\tau_{\sigma_u} X_u)\psi(u)) \right)(x) = \Psi_{t,u}$$

which implies,

$$G_{t-s} * \psi(s) = G_t * \phi_0 + \int_0^s (G_{t-s} * \Psi_{s,u}) du = G_t * \phi_0 + \int_0^s \Psi_{t,u} du$$

This proves that

$$\phi(t,x,\omega) = \psi(t,x,\omega) \diamond X_t(\omega)$$

is a solution of problem (7). ■

Remark: We stress that in theorem 10 we have not assumed any nonanticipation conditions on the initial condition $\phi_0(x,\omega)$.

We have an equivalent representation of the solution:

Corollary 11 *Under the conditions in theorem 10, we can write the solution $\phi(t,x,\omega)$ of (7) on product form:*

$$\phi(t,x,\omega) = \psi(t,x,\omega-\sigma_t) \cdot \exp\left(\int_0^t \sigma(s) dB_s(\omega) - \frac{1}{2}\int_0^t \sigma^2(s) ds\right) \tag{11}$$

where $\psi(t,x,\omega)$ solves (8).

Proof: By formula (5) and theorem 10, we see that the solution can be written

$$\phi(t,x,\omega) = T_{-\sigma_t}\psi(t,x,\omega)\cdot \mathrm{Exp}(\int_0^t \sigma(s)dB_s(\omega)$$

Since $\psi(t,x,\cdot) \in (L^p)$ for $p \geq 3$, generalized translation and ordinary translation coincide. Recall also from section 2 that the Wick exponential has the representation

$$\mathrm{Exp}W_\gamma = \exp(W_\gamma - \frac{1}{2}|\gamma|_2^2)$$

The corollary follows. ■

We end this paper with an example:
Example: A flux function which is often used in model studies, is

$$f(u) = \frac{u^2}{u^2 + (1-u)^2}$$

(see for instance the numerical example by Holden and Risebro in [HR]). This function is uniformly Lipschitz. Moreover,

$$|f(u) - f(v)| \leq 2|u - v|$$

Hence, the stochastic initial value problem

$$u_t + \left(\frac{u^2}{u^2 + (1-u)^2}\right)_x = \nu u_{xx} + u \cdot W_t$$

has a solution $u(t,x,\omega)$ in (L^p), when the initial condition u_0 is in (L^{3p}). We can write the solution as

$$u(t,x,\omega) = \psi(t,x,\omega - \mathbf{1}_{[0,t)}) \cdot \exp(B_t(\omega) - \frac{1}{2}t)$$

where $\psi(t,x,\omega)$ is the solution of the problem

$$\psi_t + \exp(B_t(\omega) + \frac{1}{2}t)\left(\frac{\psi^2}{\psi^2 + (\exp(B_t(\omega) + \frac{1}{2}t) - \psi)^2}\right)_x = \nu\psi_{xx}$$

This problem can be looked upon as a deterministic parabolic equation, considering the equation for each path of the Brownian motion.

Acknowledgements: This work has been supported by the Norwegian Research Council (NFR), under grants 100549/410 (Benth) and 412.90/028 (Gjessing). The authors would like to thank Bruce Driver, Bernt Øksendal and Jürgen Potthoff for inspiring discussions and fruitful suggestions.

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