Interpolation by Bivariate Splines on Crosscut Partitions

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Abstract. We give a survey of recent methods to construct Lagrange interpolation points for splines of arbitrary smoothness r and degree q on general crosscut partitions in $\mathbb{R}^2$. For certain regular types of partitions, also results on Hermite interpolation sets and on the approximation order of the corresponding interpolating splines are given.

1. Introduction

We consider bivariate spline spaces of the following type. Let a convex compact region $\Omega \in \mathbb{R}^2$ be given, which is subdivided by a finite number of straight lines (crosscuts) into convex subregions $\{T\}$, called a *partition* Δ of Ω. The space of all functions $s \in C^r(\Omega)$, whose restriction to each T is a bivariate polynomial of degree q, is denoted as the spline space $S_q^r(\Delta)$.

In Section 2, a construction method for point sets $\{z_1, \ldots, z_{\dim S_q^r(\Delta)}\}$ on a general partition Ω, which admit unique Lagrange interpolation w.r.t. $S_q^r(\Delta)$, is described.

The construction of Hermite interpolation sets (which can be considered as limits of Lagrange interpolation sets) for certain rectangular types of partitions Δ, denoted as Δ^1 resp. Δ^2 partitions, is given in Section 3. Here, Δ^1 denotes the partitions where to each subrectangle the same diagonal is added, while Δ^2 means that both diagonals are added.

We also give results on the approximation order of the interpolating spline function. The approximation order for $S_q^r(\Delta^1)$ equals $q+1$ (which is optimal), if $q \geq 3.5r+1$, $r \geq 1$, and q, if $r = 1$ and $q = 4$. For $S_q^1(\Delta^2)$, we get the optimal order $q+1$, if $q \geq 4$, and the order q, if $q = 2, 3$.

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2. Construction of Lagrange Interpolation Sets for Crosscut Partitions

Let $\Omega \subset \mathbb{R}^2$ be a convex compact domain. Any finite set of straight lines (called *crosscuts*) $l_1, \ldots, l_M$ having nonempty intersections with the interior of Ω, produces a *partition* Δ of Ω into convex compact subregions (called *cells*) with pairwise disjoint interiors. We denote by $\mathcal{T}$ the set of all cells. The straight line boundaries of each cell are called *edges* and their endpoints *vertices*. Let $\{v_1, \ldots, v_L\}$ be the set of all *interior vertices* of the partition Δ. Thus, each v_i is an intersection point of two or more crosscuts which lies in the interior of Ω.

The space of polynomial splines of degree q and smoothness r, $0 \le r < q$, with respect to the partition Δ is defined by

$$S_q^r(\Delta) := \{s \in C^r(\Omega) : \ s|_T \in \tilde{\Pi}_q \,,\ T \in \mathcal{T}\}\,,$$

where

$$\tilde{\Pi}_q := \operatorname{span}\{x^i y^j : \ i \ge 0,\ j \ge 0,\ i+j \le q\}$$

is the space of bivariate polynomials of total degree q.

The dimension of $S_q^r(\Delta)$ was determined by Chui & Wang [6].

We consider the following problem. Determine points $z_1, \ldots, z_N$ in Ω, where $N = \dim S_q^r(\Delta)$, such that for any $f \in C(\Omega)$, the Lagrange interpolation problem $s(z_i) = f(z_i)$, $i = 1, \ldots, N$, has a unique solution s in $S_q^r(\Delta)$. Such sets $\{z_1, \ldots, z_N\}$ are called *Lagrange interpolation sets for* $S_q^r(\Delta)$. (Sometimes we say that $z_1, \ldots, z_N$ are *Lagrange interpolation points for* $S_q^r(\Delta)$.)

First we describe bases and interpolation points for three types of subspaces of $S_q^r(\Delta)$ and then show how these points can be combined into an interpolation set for the whole space.

A. The space of bivariate polynomials $\tilde{\Pi}_q$. It is well known that $\dim \tilde{\Pi}_q = \binom{q+2}{2}$, and a basis of $\tilde{\Pi}_q$ is given by

$$\{x^i y^j : \ i \ge 0,\ j \ge 0,\ i+j \le q\}\,.$$

Interpolation sets for $\tilde{\Pi}_q$ can be obtained in the following way (see, for example, Nürnberger [11]). Let $\gamma_0, \ldots, \gamma_q$ be distinct parallel lines and $z_{i,0}, \ldots, z_{i,q-i}$ distinct points on γ_i, $i = 0, \ldots, q$ (see Fig. 2.1). Then

$$\{z_{0,0}, \ldots, z_{0,q}, \ldots, z_{q-1,0}, z_{q-1,1}, z_{q,0}\}$$

is a Lagrange interpolation set for $\tilde{\Pi}_q$, and the points are called *interpolation points of type A*.

B. The space of truncated power functions. Let l be any crosscut dividing Ω into two subdomains Ω_0 and Ω_1. We consider the following space:

$$T_q^r := \{s \in C^r(\Omega) : \ s|_{\Omega_0} \equiv 0\,,\ s|_{\Omega_1} \in \tilde{\Pi}_q\}\,.$$

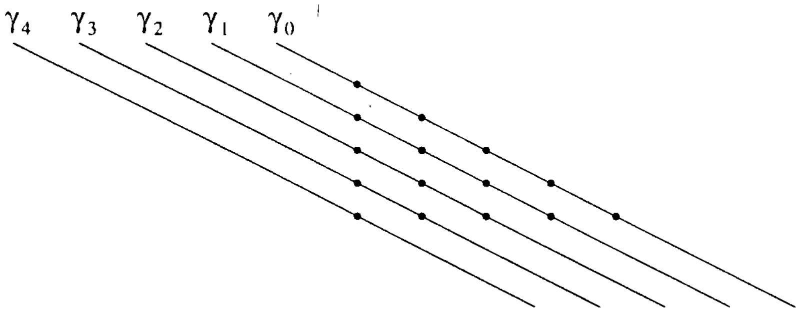

Fig. 2.1. Interpolation points of type A.

Suppose

$$l = \{(x, y) \in \mathbb{R}^2 : \; ax + by + c = 0\}\,,$$

such that

$$ax + by + c > 0\,, \qquad (x, y) \in \operatorname{int} \Omega_1\,.$$

A basis for T_q^r is given by truncated powers multiplied with polynomials,

$$\{(ax + by + c)_+^{r+1} x^i y^j : \; i \geq 0,\; j \geq 0,\; i + j \leq q - r - 1\}\,,$$

where

$$w_+^k := \begin{cases} w^k\,, & w \geq 0\,, \\ 0\,, & w < 0\,. \end{cases}$$

Therefore,

$$\dim T_q^r = \binom{q - r + 1}{2}\,.$$

In order to determine interpolation sets for T_q^r, we choose $q - r$ parallel lines $\gamma_0, \ldots, \gamma_{q-r-1}$ intersecting l and put $q - r - i$ distinct points $z_{i,0}, \ldots, z_{i,q-r-1-i}$ on $\gamma_i \cap (\Omega_1 \setminus l)$, $i = 0, \ldots, q - r - 1$ (see Fig. 2.2). Then

$$\{z_{0,0}, \ldots, z_{0,q-r-1}, \ldots, z_{q-r-2,0}, z_{q-r-2,1}, z_{q-r-1,0}\}$$

is a Lagrange interpolation set for T_q^r, and the points are called *interpolation points of type B*.

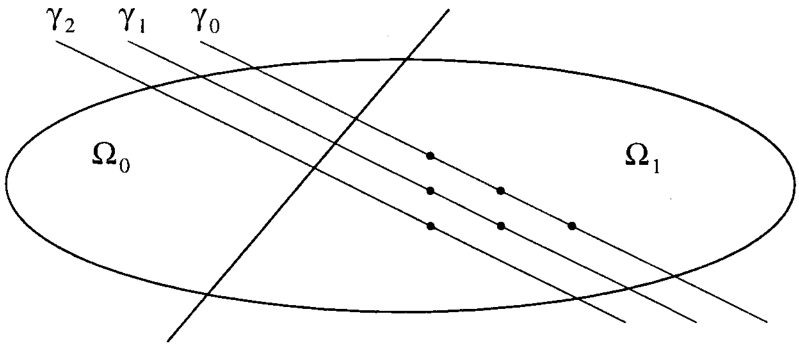

Fig. 2.2. Interpolation points of type B.

C. The space of cone splines. Let v be any interior vertex such that m crosscuts $l_1, \ldots, l_m$ intersect at v. Thus, $2m$ rays originate at v. We take m consecutive rays $r_1, \ldots, r_m$ (so that all m crosscuts are involved) and divide Ω into m subdomains $\Omega_0, \Omega_1, \ldots, \Omega_{m-1}$ as in Fig. 2.3.

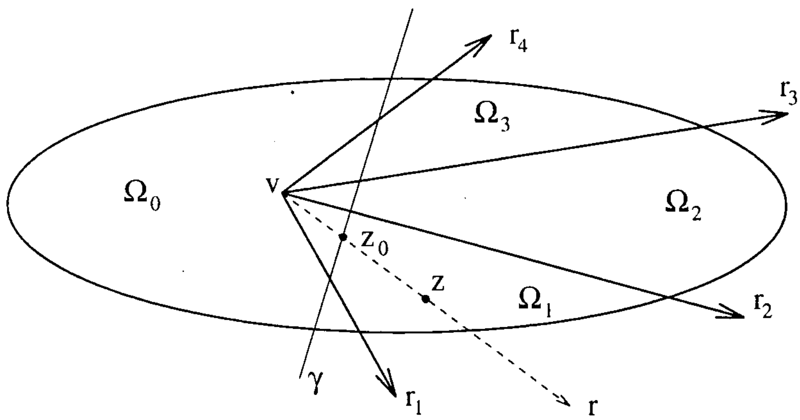

Fig. 2.3.

The space of cone splines K_q^r is defined as follows.

$$K_q^r := \{s \in C^r(\Omega) : \; s|_{\Omega_0} \equiv 0 \,,\; s|_{\Omega_i} \in \tilde{\Pi}_q \,,\; i = 1, \ldots, m-1\}\,.$$

Thus, all splines in K_q^r are zero outside the cone $\bigcup_{i=1}^{m-1} \Omega_i$. We now define a basis for K_q^r. We choose a line γ which intersects all the rays $r_1, \ldots, r_m$, with $v \notin \gamma$ (see Fig. 2.3). For $n = q, q-1, \ldots$, we consider the univariate spline spaces $K_n^r|_\gamma$ on γ such that $\dim K_n^r|_\gamma > 0$. Then we extend each univariate B-spline B in $K_n^r|_\gamma$ to a function in K_n^r as follows. Let $B(z) \equiv 0$, $z \in \Omega_0$. For each ray r in $\Omega \setminus \Omega_0$ passing through v, we define B to be the univariate truncated power function t_+^n multiplied with an appropriate constant. More precisely, let $z \in \Omega \setminus \Omega_0$ and let z_0 be the intersection point of γ and the line through v and z (see Fig. 2.3). Then $B(z) := B(z_0)|z - v|^n / |z_0 - v|^n$. All the bivariate splines in K_n^r, $n = q, q-1, \ldots$, obtained in this way (called *cone splines*), form a basis of K_q^r (see Chui & Wang [6] and Dahmen & Micchelli [7]). Therefore,

$$\dim K_q^r = \sum_{n \le q} \dim K_n^r|_\gamma \,.$$

We now describe interpolation points for K_q^r. Let $\gamma_0, \ldots, \gamma_p$ be parallel lines which intersect the rays $r_1, \ldots, r_m$ and do not pass through the vertex v, where p is determined by the conditions $\dim K_{q-p}^r|_\gamma > 0$ and $\dim K_{q-p-1}^r|_\gamma = 0$. We choose $\dim K_{q-i}^r|_\gamma$ points on $\gamma_i \cap (\Omega \setminus \Omega_0)$, $i = 0, \ldots, p$, so that these points satisfy Schoenberg-Whitney condition for the univariate spline space $K_{q-i}^r|_{\gamma_i}$ (see Fig. 2.4, a)).

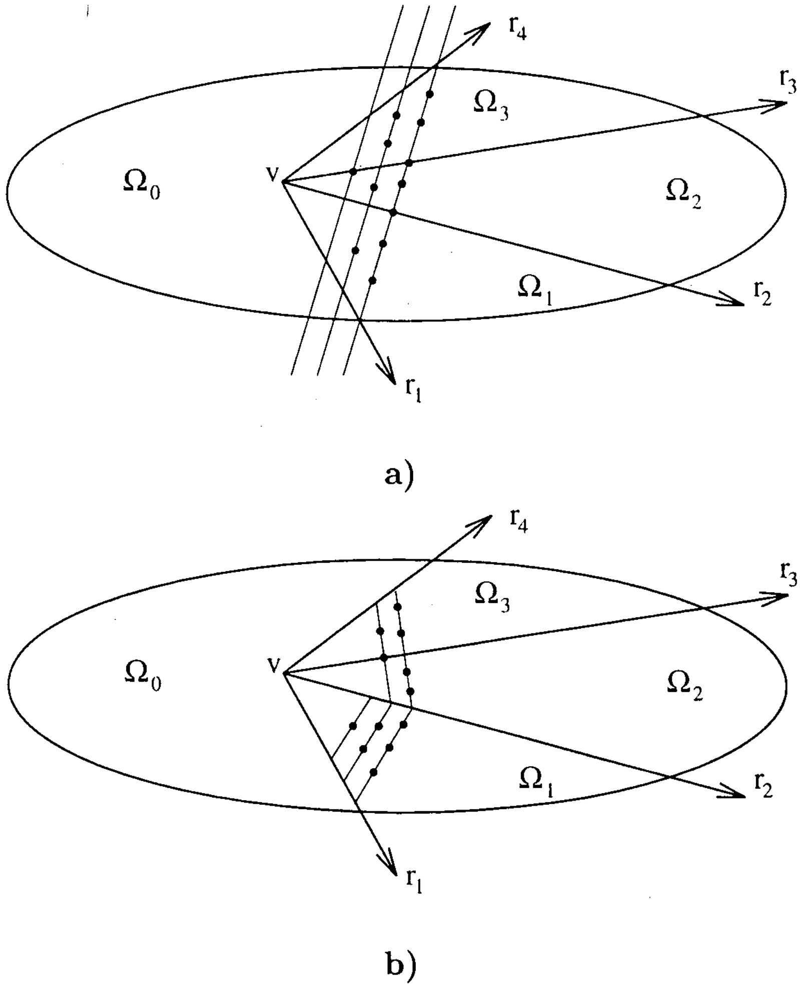

Fig. 2.4. Interpolation points of type C.

In other words, we take a point in a support of each univariate B-spline in $K^r_{q-i}|_{\gamma_i}$. It was shown by Nürnberger & Riessinger [14] that each set of points obtained in this way admits unique Lagrange interpolation from K^r_q. We call them *interpolation points of type C.*

A more general scheme of constructing interpolation points of type C (see Fig. 2.4, b)) was also proposed in [14]. The cone formed by r_1 and r_m can be divided into several subcones. Then the points are chosen on line segments inside the subcones according to some Schoenberg-Whitney type conditions for the spaces K^r_n restricted to the line segments as in Fig. 2.4, b). (For details see [14].) Another configuration of interpolation points of type C was given by Adam [1], where the points are lying on certain rays.

Now we are able to describe a basis for $S^r_q(\Delta)$ which is due to Chui & Wang [6] and Dahmen & Micchelli [7].

Theorem 2.1. *A basis of* $S^r_q(\Delta)$ *is given by the following functions:*

A. *Polynomial basis functions* $x^i y^j$, $i \geq 0$, $j \geq 0$, $i + j \leq q$.

B. *Truncated power functions, for each crosscut* l_i, $i = 1, \ldots, M$.

C. *Cone splines, as described above, for each interior vertex* v_i, $i = 1, \ldots, L$.

We note that there is some freedom in choosing basis functions in B. and C. of Theorem 2.1 since there are two possible spaces of truncated powers T_q^r with respect to a given crosscut, and also the rays which define cone splines can be choosen differently. On the other hand, the interpolation points of type A, B and C cannot be freely combined to obtain a Lagrange interpolation set for $S_q^r(\Delta)$. A method to assign a type A, B or C to each cell of the partition so that the combination of corresponding interpolation points on the cells is an interpolation set for the whole spline space was proposed by Nürnberger & Riessinger [13,14] for rectangular partitions with diagonals and extended to arbitrary crosscut partitions by Adam [1]. Their construction depends upon an order of the cells which is the natural ordering with respect to rows and columns in the case of rectangular partitions. We now describe this order in the general case of crosscut partitions (see Adam [1]).

The *lexicographical order* of the points in $\mathbb{R}^2$ is defined as follows. Given two points $z' = (x', y')$, $z'' = (x'', y'') \in \mathbb{R}^2$ we say that $z' \leq z''$ if

$$x' < x'' \quad \text{or} \quad (x' = x'' \text{ and } y' \leq y'') .$$

As usual, $z' < z''$ if $z' \leq z''$ and $z' \neq z''$. For any compact set $K \subset \mathbb{R}^2$, $m(K) \in \mathbb{R}^2$ denotes the minimal point of K with respect to the lexicographical order.

The total order of cells $T \in \mathcal{T}$ of the partition Δ is defined as follows. In the case when $m(T') < m(T'')$ (in lexicographical order), $T', T'' \in \mathcal{T}$, we set $T' < T''$. In the case when several cells have the same minimal point m, they are situated to the right of m and separated from each other by ray segments originating at m. Therefore, they can be ordered either clockwise or counterclockwise. We choose every time one of these two orders according to the following rule:

Case 1. $m = m(\Omega)$ or $m \in \operatorname{int} \Omega$. The cells $\{T \in \mathcal{T} : m(T) = m\}$ are ordered *counterclockwise* (see Fig. 2.5).

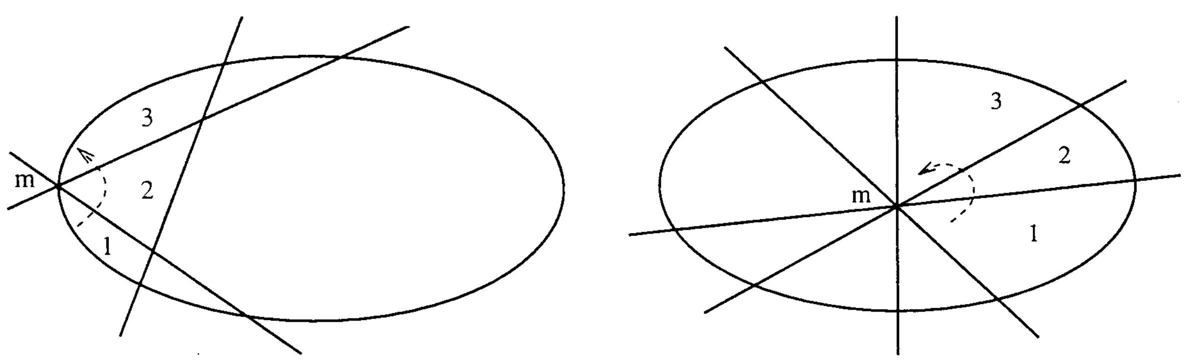

Fig. 2.5. $m = m(\Omega)$ or $m \in \operatorname{int} \Omega$.

Case 2. $m \in \partial\Omega \setminus \{m(\Omega)\}$. The cells $\{T \in \mathcal{T} : \; m(T) = m\}$ are ordered *clockwise* if $m = \min(\Omega \cap \xi_m)$ and *counterclockwise* if $m = \max(\Omega \cap \xi_m)$, where ξ_m is the vertical line through m (see Fig. 2.6).

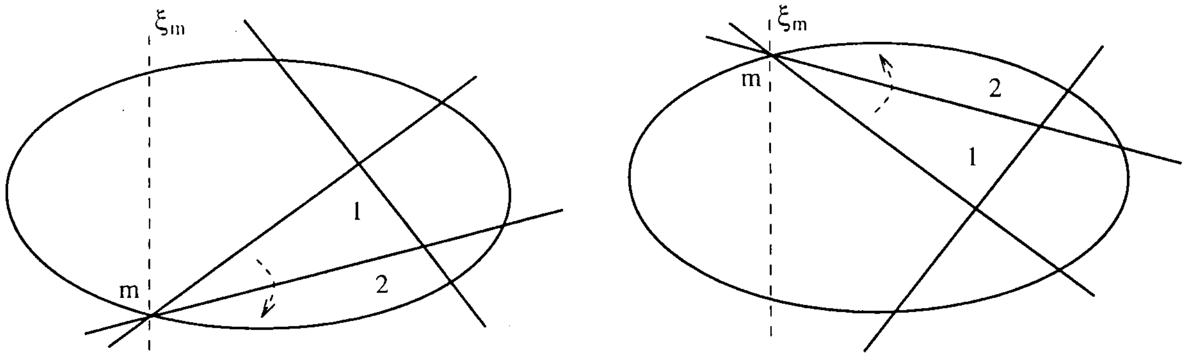

Fig. 2.6. $m \in \partial\Omega \setminus \{m(\Omega)\}$.

Thus, a total order for the elements of $\mathcal{T}$ has been defined, and we can write

$$\mathcal{T} = \{T_1, \dots, T_n\}, \quad \text{where} \quad T_i < T_{i+1}, \; i = 1, \dots, n-1.$$

We now assign a type A, B or C to each cell T_i and then choose interpolation points on T_i's according to their types. Namely, T_1 is the only *cell of type A* (since $m(T_1) = m(\Omega)$), T_i is a *cell of type B* if $m(T_i) \in \partial\Omega \setminus \{m(\Omega)\}$, and a *cell of type C* if $m(T_i) \in \operatorname{int}\Omega$. We put interpolation points of type A (as in Fig. 2.1) on T_1. It is easy to see that if T_i is of type B, then

$$I_i := \partial T_i \cap \partial \left(\bigcup_{j=1}^{i-1} T_j \right)$$

is a nondegenerate line segment with an endpoint at $m(T_i)$. We choose interpolation points of type B on T_i, as in Fig. 2.2, where $T_i \subset \Omega_1$, $\bigcup_{j=1}^{i-1} T_j \subset \Omega_0$, $I_i \subset l$.

Let now $T_i, T_{i+1}, \dots, T_k$ be a family of cells of type C with the same minimal point $m \in \operatorname{int}\Omega$, so that $m(T_{i-1}) < m$, $m(T_i) = m(T_{i+1}) = \cdots = m(T_k) = m$, $m(T_{k+1}) > m$. Then $\bigcup_{j=1}^{k} T_j$ is a subset of a cone, and we choose interpolation points of type C (as in Fig. 2.4, a) or b)) on it.

Theorem 2.2. [1] *The set of all points chosen on the cells T_i, $i = 1, \dots, n$, as described above, is a Lagrange interpolation set for $S_q^r(\Delta)$.*

We briefly describe the idea of the proof of Theorem 2.2. Let $z_1, \dots, z_N$ be all the points chosen on the cells in accordance with the above procedure. Then it follows from Theorem 2.1 that $N = \dim S_q^r(\Delta)$. Therefore, in order to prove that $\{z_1, \dots, z_N\}$ is a Lagrange interpolation set for $S_q^r(\Delta)$ it is sufficient to check that for any $s \in S_q^r(\Delta)$,

$$s(z_1) = 0, \dots, s(z_N) = 0, \tag{2.1}$$

implies

$$s(z) = 0\,, \quad \text{for any} \quad z \in \Omega\,. \tag{2.2}$$

To this end we start from T_1 and see that (2.1) implies that

$$s(z) = 0\,, \quad \text{for any} \quad z \in T_1\,,$$

since $\{z_1, \ldots, z_N\} \cap T_1$ is a Lagrange interpolation set for $\widetilde{\Pi}_q$. Then we pass from T_2 to T_n and see that for any $i \in \{2, \ldots, n\}$ such that T_i is a cell of type B,

$$s(z) = 0\,, \quad \text{for any} \quad z \in \bigcup_{j=1}^{i-1} T_j \quad \text{and} \quad z \in \{z_1, \ldots, z_N\} \cap T_i\,,$$

implies

$$s(z) = 0\,, \quad \text{for any} \quad z \in T_i\,,$$

since $\{z_1, \ldots, z_N\} \cap T_i$ is a Lagrange interpolation set for the corresponding space of truncated power functions T_q^r. Similarly, if $T_i, T_{i+1}, \ldots, T_k$ is a family of cells of type C with the same minimal point $m \in \operatorname{int}\Omega$, so that $m(T_{i-1}) < m$, $m(T_i) = m(T_{i+1}) = \cdots = m(T_k) = m$, $m(T_{k+1}) > m$, then

$$s(z) = 0\,, \quad \text{for any} \quad z \in \bigcup_{j=1}^{i-1} T_j \quad \text{and} \quad z \in \{z_1, \ldots, z_N\} \cap \bigcup_{j=i}^{k} T_j\,,$$

implies

$$s(z) = 0\,, \quad \text{for any} \quad z \in \bigcup_{j=i}^{k} T_j\,,$$

since $\{z_1, \ldots, z_N\} \cap \bigcup_{j=i}^{k} T_j$ is a Lagrange interpolation set for the corresponding space of cone splines K_q^r. Thus, (2.2) follows by induction.

3. Approximation Order of Bivariate Spline Interpolation

In this section we consider spaces of bivariate splines with respect to special crosscut partitions Δ^1 and Δ^2. Let a rectangle $\Omega = [a, b] \times [c, d]$ and points $a = x_0 < x_1 < \cdots < x_{n_1} = b$, $c = y_0 < y_1 < \cdots < y_{n_2} = d$ such that $x_i - x_{i-1} = h_1$, $i = 1, \ldots, n_1$; $y_j - y_{j-1} = h_2$, $j = 1, \ldots, n_2$, be given. We set $h = \max\{h_1, h_2\}$. By defining $R_{i,j} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, $i = 1, \ldots, n_1$; $j = 1, \ldots, n_2$, we obtain a partition of Ω into subrectangles $R_{i,j}$. If the diagonal from the lower left to the upper right vertex is added to each subrectangle $R_{i,j}$, then we denote the resulting partition by Δ^1. If we add both diagonals to each subrectangle, then the resulting partition is denoted by Δ^2.

Since both Δ^1 and Δ^2 are crosscut partitions, a basis for the spline spaces $S_q^r(\Delta^\mu)$, $\mu = 1, 2$, is given in Theorem 2.1. Similarly, the application of Theorem 2.2 to $S_q^r(\Delta^\mu)$, $\mu = 1, 2$, yields Lagrange interpolation sets of Nürnberger &

Riessinger [13,14]. We first describe these Lagrange interpolation sets for $S_q^r(\Delta^1)$. Then, by "taking limits", some Hermite interpolation sets are obtained, such that interpolation at them yields (nearly) optimal approximation order, under some restrictions on r and q.

For constructing Lagrange interpolation sets for $S_q^r(\Delta^1)$, $q \geq 4$, we describe four basic steps. For an arbitrary subtriangle T of the partition Δ^1, one of the following steps will be applied to T. (If the number of lines in Step C or D below is non-positive, then no points are chosen.)

Step A. (Starting step) Choose $q+1$ disjoint line segments $a_1, \ldots, a_{q+1}$ in T. For $i = 1, \ldots, q+1$, choose $q+2-i$ distinct points on a_i.

Step B. Choose $q-r$ disjoint line segments $b_1, \ldots, b_{q-r}$ in T. For $i = 1, \ldots, q-r$, choose $q+1-r-i$ distinct points on b_i.

Step C. Choose $q-2r+[\frac{r}{2}]$ disjoint line segments $c_1, \ldots, c_{q-2r+[\frac{r}{2}]}$ in T. For $i = 1, \ldots, q-2r$, choose $q+1-r-i$ distinct points on c_i and for $i = q-2r+1, \ldots, q-2r+[\frac{r}{2}]$ choose $2(q-i)-3r+1$ distinct points on c_i. (Here $[b] := \max\{a \in \mathbb{Z} : a \leq b\}$.)

Step D. Choose $q-2r-1$ disjoint line segments $d_1, \ldots, d_{q-2r-1}$ in T. For $i = 1, \ldots, q-2r-1$, choose $q-2r-i$ distinct points on d_i.

Given a partition Δ^1, we apply the above steps to the subtriangles of Δ^1 as indicated in Fig. 3.2, where we choose horizontal, vertical and diagonal line segments as indicated in Fig. 3.1.

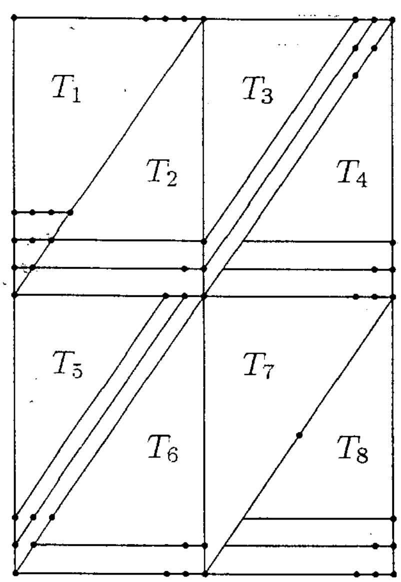

Fig. 3.1.

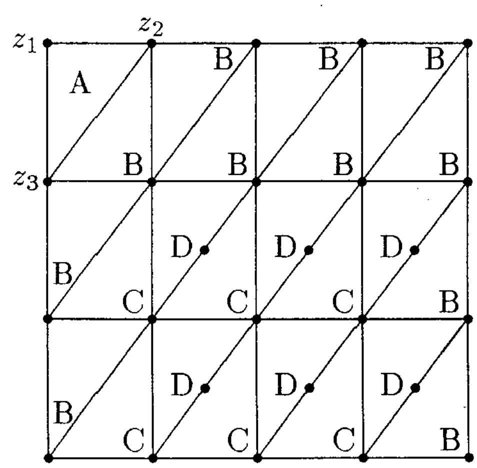

Fig. 3.2.

The following construction of Hermite interpolation sets for $S_q^r(\Delta^1)$, which yields (nearly) optimal order approximation, was given by Nürnberger [12] for $r = 1$ and by Davydov, Nürnberger & Zeilfelder [8] for $r \geq 2$.

Let a sufficiently differentiable function $f \in C(\Omega)$ be given. In order to define Hermite interpolation conditions for a spline $s \in S_q^r(\Delta^1)$, where $q \geq 4$ if $r = 1$, and $q \geq 3.5r + 1$ if $r \geq 2$, we describe four basic conditions. Let T be an arbitrary subtriangle of the partition Δ^1. If T is not the first from the left triangle in the top row, then $\widetilde{T}$ denotes the adjacent subtriangle left of T in the same row if it exists, and up of T otherwise. We impose one of the following four conditions on the polynomial $p = s|_T \in \widetilde{\Pi}_q$.

Condition A. (Starting condition) $D^\omega p(z) = D^\omega f(z)$, $\omega = 0, \ldots, q$, where z is a vertex of T.

Condition B. $D^\omega p(z) = D^\omega f(z)$, $\omega = 0, \ldots, q - r - 1$, where z is the vertex of T not belonging to $\widetilde{T}$.

Condition C. $p_{x^\alpha y^\beta}(z) = f_{x^\alpha y^\beta}(z)$, $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta \leq q - r - 1$, $\alpha + 2\beta \leq 2q - 3r - 2$, where z is the vertex of T not belonging to $\widetilde{T}$.

Condition D. $D^\omega p(z) = D^\omega f(z)$, $\omega = 0, \ldots, q - 2r - 2$, where z is the midpoint of the diagonal of T.

Note that while Conditions A, B and D are symmetric with respect to x and y, this is not the fact for Condition C. Fig. 3.3 presents the domain in which all integer points (α, β) should be taken in order to define Condition C.

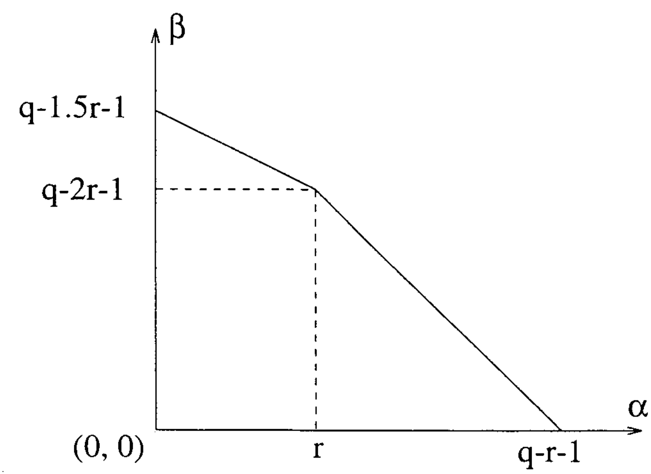

Fig. 3.3. Condition C.

Given a partition Δ^1, the distribution of the Hermite interpolation conditions to the subtriangles is the same as for Lagrange interpolation and is indicated in Fig. 3.2.

In Theorems 3.1, 3.2 and 3.4 below, the norm denotes the maximum of the uniform norm over all subtriangles of the partition (w.r.t. the polynomial pieces).

By using Bernstein-Bezier techniques, several authors proved results similar to those of Theorems 3.1 and 3.4 below for the special spline spaces $S_3^1(\Delta^1)$ (Sha [16]) and $S_2^1(\Delta^2)$ (Chui & He [4], Sha [17], Zedek [18]). Moreover, Jeeawock-Zedek [9] proved that interpolation by $S_3^1(\Delta^2)$ yields approximation order two.

Theorem 3.1. [12] *For each function $f \in C^{q+1}(\Omega)$, there exists a constant $K > 0$ such that for the unique spline $s \in S_q^1(\Delta^1)$ which satisfies the above Hermite interpolation conditions, the following statements hold: For all $i \in \{0, \dots, \rho - 1\}$, $\|D^i(f - s)\| \leq Kh^{\rho - i}$, where $\rho = 4$ if $q = 4$, and $\rho = q + 1$ if $q \geq 5$. (The constant $K > 0$ depends on $\|D^{q+1}f\|$ and is independent of h.)*

Theorem 3.2. [8] *Let integers $r \geq 2$ and $q \geq 3.5r + 1$ be given. For each function $f \in C^{q+1}(\Omega)$, there exists a constant $K > 0$ such that for the unique spline $s \in S_q^r(\Delta^1)$ which satisfies the above Hermite interpolation conditions,*

$$\|D^i(f - s)\| \leq Kh^{q+1-i}, \quad i = 0, \dots, q.$$

(The constant $K > 0$ depends on $\|D^{q+1}f\|$ and is independent of h.)

In view of Theorem 3.2 it is interesting to note that the approximation order of the spline space $S_q^r(\Delta)$ is optimal (i.e., $q + 1$), if $q \geq 3r + 2$ (see de Boor & Höllig [2], Chui, Hong & Jia [5], and Lai & Schumaker [10]). On the other hand, it was proved by de Boor & Jia [3] that this is not true, if $q < 3r + 2$, even for the Δ^1-partition.

Remark 3.3. The method of proof in [12] can be applied to spline spaces $S_q^1(\widetilde{\Delta}^1)$, $q \geq 5$, where $\widetilde{\Delta}^1$ is a "deformation" of partition Δ^1, by which we mean an arbitrary rectilinear embedding of the same triangulation in $\mathbb{R}^2$, as in Fig. 3.4. Under some restrictions on the angles between adjacent edges, corresponding Hermite interpolation scheme possesses optimal approximation order h^{q+1}, where h is the maximal sidelength of the subtriangles.

We now describe in a similar way the construction of Lagrange and Hermite interpolation sets for $S_q^1(\Delta^2)$ (Results for $S_q^r(\Delta^2)$, $r \geq 2$, are not yet available). As above, it turns out that interpolation at these points yields (nearly) optimal approximation order.

For constructing Lagrange interpolation sets for $S_q^1(\Delta^2)$, $q \geq 2$, we again describe four basic steps.

Step A. (Starting step) Choose $q + 1$ disjoint line segments $a_1, \dots, a_{q+1}$ in T. For $i = 1, \dots, q + 1$, choose $q + 2 - i$ distinct points on a_i.

Step B. Choose $q - 1$ disjoint line segments $b_1, \dots, b_{q-1}$ in T. For $i = 1, \dots, q - 1$, choose $q - i$ distinct points on b_i.

Step C. Choose $q - 3$ disjoint line segments $c_1, \dots, c_{q-3}$ in T. For $i = 1, \dots, q - 3$, choose $q - 2 - i$ distinct points on c_i.

Step D. Choose $q - 2$ disjoint line segments $d_1, \dots, d_{q-2}$ in T. For $i = 1, \dots, q - 2$, choose $q - i$ distinct points on d_i.

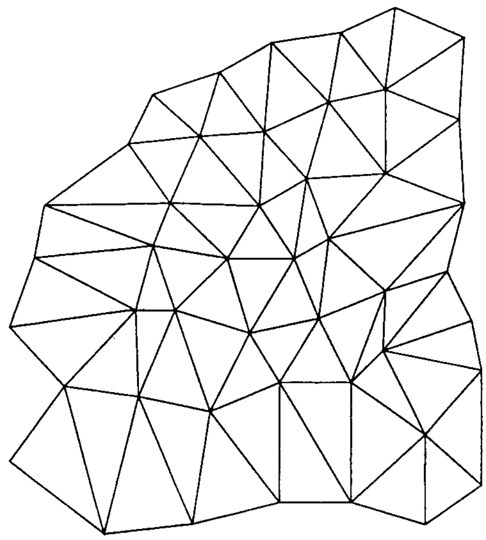

Fig. 3.4. An example of partition $\widetilde{\Delta}^1$.

Given a partition Δ^2, we apply the above steps to the subtriangles of Δ^2 as indicated in Fig. 3.6, where we choose horizontal, vertical and diagonal line segments as indicated in Fig. 3.5.

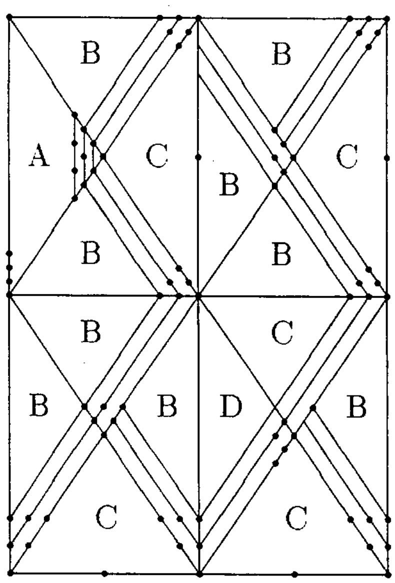

Fig. 3.5.

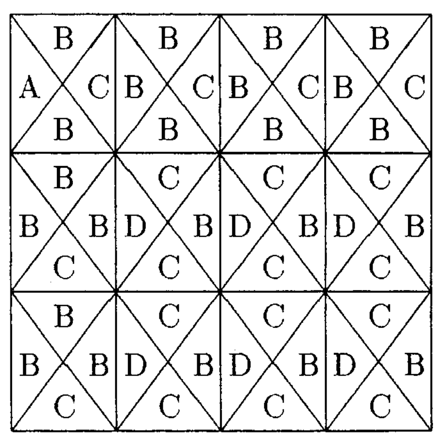

Fig. 3.6.

The following construction of Hermite interpolation sets for $S_q^1(\Delta^2)$, which differs in some parts from the one described above for Δ^1, was given by Nürnberger & Walz [15].

Let a sufficiently differentiable function $f \in C(\Omega)$ be given. We again have to describe four basic conditions. Let T be an arbitrary subtriangle of the partition Δ^2. We impose one of the following four conditions on the polynomial $p = s|_T \in \widetilde{\Pi}_q$, where z is a vertex resp. a midpoint of an edge of T as described below (cf. also Fig. 3.7).

Condition A. (Starting condition) $D^\omega p(z) = D^\omega f(z)$, $\omega = 0, \ldots, q$, where z is a vertex of T.

Condition B. $D^\omega p(z) = D^\omega f(z)$, $\omega = 0, \ldots, q-2$, where z is a vertex of T not adjacent to the subtriangles already considered, e.g. z_4 or z_6.

Condition C. $D^\omega p(z) = D^\omega f(z)$, $\omega = 0, \ldots, q-4$, where z is the midpoint of the edge of T which is not adjacent to the subtriangles already considered, e.g. z_5.

Condition D. $p_{\varrho^\alpha \overline{\varrho}^\beta}(z) = f_{\varrho^\alpha \overline{\varrho}^\beta}(z)$, $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta \leq q-2$, $\beta \neq q-2$, where $\varrho = (\varrho_1, \varrho_2)$ is the unit vector in direction of the diagonal of Δ^1, and $\overline{\varrho} = (-\varrho_1, \varrho_2)$, and where z is the crossing point of the two diagonals in one subrectangle, e.g. z_7.

Given a partition Δ^2, we apply the above steps to the subtriangles of Δ^2 as indicated in Fig. 3.7. Note that the configuration of Hermite conditions is different from that of Lagrange conditions (cp. Fig. 3.6 and Fig. 3.7).

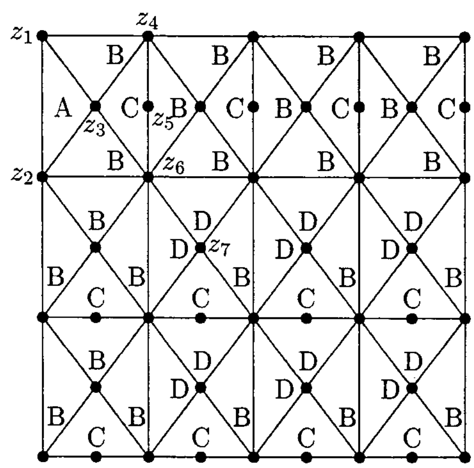

Fig. 3.7.

Theorem 3.4. [15] *For each function $f \in C^{q+1}(\Omega)$, there exists a constant $K > 0$ such that for the unique spline $s \in S_q^1(\Delta^2)$ which satisfies the above Hermite interpolation conditions, the following statements hold: For all $i \in \{0, \ldots, \rho - 1\}$, $\|D^i(f - s)\| \leq K h^{\rho - i}$, where $\rho = q$ if $q \in \{2, 3\}$, and $\rho = q + 1$ if $q \geq 4$. (The constant $K > 0$ depends on $\|D^{q+1} f\|$ and is independent of h.)*

We briefly mention that these results can also be used for fitting of scattered data by using a two-step method, originally developed in [12]. The method is as follows: Let a (possibly non-rectangular) domain Ω, points $w_i \in \Omega$ and corresponding data f_i be given. In the first step, we approximate the data f_i by any local method, e.g. interpolation by a piecewise polynomial $\widetilde{s}$ of degree q such that $\|f - \widetilde{s}\| = O(h^{q+1})$ if $f_i = f(w_i)$ and $f \in C^{q+1}(\Omega)$. In general, piecewise polynomial interpolation is a simpler problem than spline interpolation and in any case, this is always possible if the data is regularly distributed over Ω. In the second step, we interpolate the resulting function $\widetilde{s}$ (which may not even be continuous) by a smooth spline s as described in this paper. As in Theorems 3.1, 3.2 and 3.4, it can be shown that $|f_i - s(w_i)| = O(h^q)$ or $O(h^{q+1})$. More details on this and several numerical examples can be found in [12] and [15].

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