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Categorical characterization of bisimulation

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Abstract

[AM89] and [JNW94] present abstract concepts of bisimulation in terms of category theory. This paper deals with the question how these approaches are related. Futheron it shows how different types of bisimulations on prime event structures can be modelled in terms of the abstract concepts.

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1 Introduction

Bisimulation was introduced by [Mil80] and [Par81] in order to identify processes that cannot be distinguished by an external agent. Since then various notions of "bisimulation" have been studied, e.g. on labelled transition systems by [DNMV90], [MS92], on event structures by [GG89], [GKP92], on petri nets by [GV87], [ABS91]. Recently attempts have been made to develop an abstract characterization of the various notions of bisimulation, see for example [DDNM93] and [Mal95]. We focus here on the work of [AM89] and [JNW94].

[AM89] characterize bisimulation (*AM-bisimulation*) as a coalgebra relative to a functor on the category **Class**. [JNW94] work with a general category $\mathbb{M}$ of models with a distinguished subcategory $\mathbb{P}$ of path objects. Two objects X_1 and X_2 are called *$\mathbb{P}$-bisimular* iff there is an object X in $\mathbb{M}$ together with so-called *$\mathbb{P}$-open morphisms* $f_i : X \to X_i$, $i = 1, 2$. We study here how AM-bisimulation and $\mathbb{P}$-bisimulation are related. To connect these concepts we use the formalism of *path-$\mathbb{P}$-bisimulation* of [JNW94].

Starting in a setting where one may speak about path-$\mathbb{P}$-bisimulation we prove that (strong) path-$\mathbb{P}$-bisimulation and (strong) AM-bisimulation are equivalent. As every $\mathbb{P}$-bisimulation induces a strong path-$\mathbb{P}$-bisimulation we get the result: If one can introduce the concept of $\mathbb{P}$-bisimulation in a category of models $\mathbb{M}$ this bisimulation induces a strong AM-bisimulation.

For the reverse direction – i.e. to characterize a given AM-bisimulation on a category $\mathbb{M}$ of models in terms of $\mathbb{P}$-bisimulation – numerous assumptions have to be made. In a first step we switch from AM-bisimulation to path-$\mathbb{P}$-bisimulation. Therefore we have to construct a suitable category $\mathbb{P}$ of path objects. Using a theorem of [JNW94] which characterizes the situations where strong path-$\mathbb{P}$-bisimulation coincides with $\mathbb{P}$-bisimulation one may conclude that AM-bisimulation is a more general concept than $\mathbb{P}$-bisimulation.

As an application we study AM-bisimulation and $\mathbb{P}$-bisimulation on labelled event structures where we consider the concepts of interleaving, backward-forward (bf), step, pomset, history-preserving and strong-history-preserving bisimulation.

2 Definition of the different bisimulation concepts

2.1 AM-bisimulation

A *coalgebra* for an endofunctor F on a category $\mathbb{C}$ is a pair (A, α) where A is an object of $\mathbb{C}$ and $\alpha : A \to FA$ a morphism. A morphism $\pi : A \to B$ in $\mathbb{C}$ is called a *homomorphism* between coalgebras (A, α) and (B, β) iff $\beta \circ \pi = (F\pi) \circ \alpha$ holds. The coalgebras and

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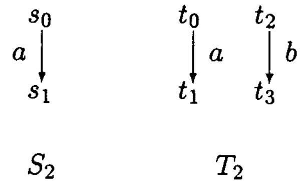


Figure 3: "Same behaviour" by choosing either s_0 and t_0 or s_1 and t_1 as initial states.

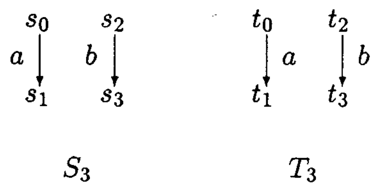


Figure 4: "Same behaviour" even in the not reachable parts.

2.1.1 An interpretation of AM-bisimulation

In order to relate two transition systems by AM-bisimulation it is not enough to require the existence of a coalgebra (R, γ) which makes the diagram of figure 1 commute. $(R, \gamma) = (\emptyset, \emptyset)$ is an AM-bisimulation between any two coalgebras. To model a particular kind of bisimulation as AM-bisimulation it is therefore necessary for the set R of the coalgebra (R, γ) to include a distinguished pair of states, e.g. the initial states of the transition systems. The transition systems of figure 2, 3 and 4 demonstrate, how different the "information content" of an AM-bisimulation may be:

$(R, \gamma) := (\{(s_1, t_0)\}, \emptyset)$ is an AM-bisimulation between the transition sytems S_1 and T_1 of figure 2. It says: "Both transition systems include final states."

For the transition systems S_2 and T_2 of figure 3 one may take as AM-bisimulation (R, γ) the set $R := (\{(s_0, t_0), (s_1, t_1)\}$ together with the map $\gamma(s_0, t_0) := \{(a, s_1, t_1)\}$. Here we get: "If one chooses s_0 and t_0 as initial states the behaviour of the transition systems S_2 and T_2 is identical. The same holds in the choice of s_1 and t_1 as initial states."

In case of the transition systems S_3 and T_3 of figure 4 it is possible to take (R, γ) as AM-bisimulation where $R := \{(s_0, t_0), (s_1, t_1), (s_2, t_2), (s_3, t_3)\}$ and $\gamma(s_0, t_0) := \{(a, s_1, t_1)\}$, $\gamma(s_2, t_2) := \{(b, s_3, t_3)\}$. This AM-bisimulation can be interpreted as: "Choose any state s of S_3 as initial state. Then there exists a state t of T_3 such that taking t as inital state of T_3 the behaviour of the transition systems S_3 and T_3 is identical. The same holds if one chooses first an initial state in T_3."

The transition systems of figures 2 and 3 show that an AM-bisimulation contains just a "subset" of those transitions which are "common" to both coalgebras. This view on AM-bisimulation justifies our observation that $(\emptyset, \emptyset)$ is an AM-bisimulation between any two coalgebras: The statement that all transitions which are contained in $\emptyset$ are "common" to any pair of coalgebras is true. As the discussion of the transition systems in figure 4 shows this "common part" is independent of a concept of "reachability".

The question whether two transition systems share a special part is equivalent to the question whether there exists an AM-bisimulation between these transition systems which "contains" this part. Such a part can be a whole transition system. We give just two examples:

1. Does for any state s of a transition system S a state t in a transition system T exist such that T started in t behaves like S started in s? This question may be coded as AM-bisimulation: Does an AM-bisimulation (R, γ) exist such that the projection π_1 on the coalgebra related to S is surjective?

2. Of course it is possible to reduce the first problem to that part of a transition system which is reachable from a state s_0. This leads to the question: Does for any state s of a transition system S which is reachable from s_0 a state t in a transition system T exist such that T started in t behaves like S started in s? Again this question may be coded as AM-bisimulation: Does an AM-bisimulation (R, γ) exist such that for the projection π_1 on the coalgebra related to S holds: $s_0 \in \pi_1(R)$?

These examples model "simulations": We asked whether T is able to show a part of the behaviour of S. To describe "bisimulations" we have to formulate the conditions in a symmetric manner. For the second example this could be: Assuming that both S and T have an intial state s_0 resp. t_0 we want to know whether there exists an AM-bisimulation (R, γ) with $s_0 \in \pi_1(R)$ and $t_0 \in \pi_2(R)$. Another – more discriminating – possibility would be: Is there an AM-bisimulation (R, γ) with $(s_0, t_0) \in R$? This last formulation will be the one we use to model different kinds of bisimulations as AM-bisimulation.

2.1.2 Some properties of AM-bisimulations

In case of AM-bisimulation the proof that for coalgebras (A, α), (B, β) and (R, γ) the diagram of figure 1 commutes consists of two parts for each square: For the left square for example one establishes first that $(\alpha \circ \pi_1)(x, y) \subseteq (F\pi_1 \circ \gamma)(x, y)$ for all $(x, y) \in R$, in a second step one proves the inclusion the other way round. The following lemma shows that in order to establish the property strong the second step is not necessary:

Lemma 2.1

Let (R,γ) be an AM-bisimulation between two coalgebras (A,α) and (B,β). Then for all $(x',y') \in R$ holds:

$$(F\pi_1 \circ \gamma^-)(x',y') \subseteq (\alpha^- \circ \pi_1)(x',y') \text{ and } (F\pi_2 \circ \gamma^-)(x',y') \subseteq (\beta^- \circ \pi_1)(x',y').$$

Proof: Let $(l,x) \in (F\pi_1 \circ \gamma^-)(x',y')$. Then there exists some $y \in B$ with $(l,x,y) \in \gamma^-(x',y')$. Therefore we have $(l,x',y') \in \gamma(x,y)$. This implies $(l,x') \in (F\pi_1 \circ \gamma)(x,y)$. As (R,γ) is an AM-bisimulation we get $(l,x') \in (\alpha \circ \pi_1)(x,y)$. Thus we may conclude that $(l,x') \in \alpha(x)$ which results in $(l,x) \in \alpha^-(x') = (\alpha^- \circ \pi_1)(x',y')$. ■

To translate the definition of a homomorphism between two coalgebras in terms of their related transition systems we cite a lemma from [MCR96]:

Lemma 2.2

A morphism $f : A \to B$ in **Set** *is a homomorphism between coalgebras (A,α) and (B,β) iff for the related transition systems $T_{(A,\alpha)}$ and $T_{(B,\beta)}$ holds:*

(i) if $x \xrightarrow{a} y$ in $T_{(A,\alpha)}$ then $f(x) \xrightarrow{a} f(y)$ in $T_{(B,\beta)}$ and

(ii) if $r \xrightarrow{a} s$ in $T_{(B,\beta)}$ and $r = f(x)$ then $s = f(y)$ for some $y \in A$ and $x \xrightarrow{a} y$ in $T_{(A,\alpha)}$.

Furtheron we provide a useful property of AM-bisimulation which helps us to translate different kinds of bisimulations into the setting of coalgebras.

Lemma 2.3

For an AM-bisimulation (R,γ) on coalgebras (A,α) and (B,β) holds:

- *if $x \xrightarrow{a} y$ in $T_{(A,\alpha)}$ where $x = \pi_1(r)$ for some $r \in R$ then there exists $s \in R$ such that $y = \pi_1(s)$ and $\pi_2(r) \xrightarrow{a} \pi_2(s)$ in $T_{(B,\beta)}$ and*
- *if $v \xrightarrow{a} w$ in $T_{(B,\beta)}$ where $v = \pi_2(r)$ for some $r \in R$ then there exists $s \in R$ such that $w = \pi_2(s)$ and $\pi_1(r) \xrightarrow{a} \pi_1(s)$ in $T_{(A,\alpha)}$.*

Proof: If $x \xrightarrow{a} y$ in $T_{(A,\alpha)}$ for some $x \in \pi_1(R)$ then by (ii) of lemma 2.2 there exists $s \in R$ with $r \xrightarrow{a} s$ in $T_{(R,\gamma)}$. This induces by (i) of lemma 2.2 $\pi_2(r) \xrightarrow{a} \pi_2(s)$ in $T_{(B,\beta)}$. ■

In this paper we are interested in AM-bisimulations, which contain a distinguished pair of elements. For these one obtains transitivity:

Lemma 2.4

Let (A_1, α_1), (A_2, α_2), (A_3, α_3) be coalgebras, let (R_1, γ_1) and (R_2, γ_2) be (strong) AM-bisimulations between (A_1, α_1) and (A_2, α_2) resp. (A_2, α_2) and (A_3, α_3) with $(x, y) \in R_1$ and $(y, z) \in R_2$. Then there exists a (strong) AM-bisimulation (R, γ) between (A_1, α_1) and (A_3, α_3) with $(x, z) \in R$.

Proof: Let $R := \{(r,t) \in A_1 \times A_3 \mid \exists s \in A_2 : (r,s) \in R_1, (s,t) \in R_2\}$. Let for all (r', t'), $(r, t) \in R$

$$(l, r', t') \in \gamma(r,t) :\Longleftrightarrow \exists s',\, s \in A_2 : (l, r', s') \in \gamma_1(r,s),\ (l, s', t') \in \gamma_2(s,t).$$

Obviously holds $(x, z) \in R$.

To prove that (R, γ) is an AM-bisimulation between (A_1, α_1) and (A_3, α_3) let $(l, r') \in (\alpha_1 \circ \pi_1)(r,t)$. As $(r,t) \in R$ there exists $s \in A_2$ such that $(r,s) \in R_1$ and $(s,t) \in R_2$. We get $(l, r') \in (\alpha_1 \circ \pi_1)(r,s)$ and as (R_1, γ_1) is an AM-bisimulation $(l, r') \in (F\pi_1 \circ \gamma_1)(r,s)$. Thus there exists $s' \in A_2$ with $(r', s') \in R_1$ and $(l, r', s') \in \gamma_1(r,s)$. This leads to $(l, s') \in (F\pi_2 \circ \gamma_1)(r,s)$ and – again as (R_1, γ_1) is an AM-bisimulation – $(l, s') \in (\alpha_2 \circ \pi_2)(r,s)$. Therfore $(l, s') \in (\alpha_2 \circ \pi_1)(s,t)$ and $(l, s') \in (F\pi_1 \circ \gamma_2)(s,t)$. Thus there exists $t' \in A_3$ with $(s', t') \in R_2$ and $(l, s', t') \in \gamma_2(s,t)$ and we may conclude that $(r', t') \in R$ and $(l, r', t') \in \gamma(r,t)$. Finally we get $(l, r') \in (F\pi_1 \circ \gamma)(r,t)$.

Now let $(l, r') \in (F\pi_1 \circ \gamma)(r,t)$. Then there exists $t' \in A_3$ such that $(l, r', t') \in \gamma(r,t)$. By the definition of γ there exist $s', s \in A_2$ such that $(l, r', s') \in \gamma_1(r,s)$. Thus $(l, r') \in (\alpha_1 \circ \pi_1)(r,s) = (\alpha_1 \circ \pi_1)(r,t)$.

To show that (R, γ) is strong if (R_1, γ_1) and (R_2, γ_2) are strong let $(l, r) \in (\alpha_1^- \circ \pi_1)(r', t')$. As $(r', t') \in R$ there exists $s' \in A_2$ such that $(r', s') \in R_1$ and $(s', t') \in R_2$. For (r', s') holds: $(l, r) \in (\alpha_1^- \circ \pi_1)(r', s')$. (R_1, γ_1) is strong. Therefore we know that $(l, r) \in (F\pi_1 \circ \gamma_1^-)(r', s')$. Thus there exists $s \in A_2$ with $(r, s) \in R_1$ and $(l, r, s) \in \gamma_1^-(r', s')$. This leads to $(l, s) \in (F\pi_2 \circ \gamma_1^-)(r', s')$. Using again the property "strong" of (R_1, γ_1) we get $(l, s) \in (\alpha_2^- \circ \pi_2)(r', s') = (\alpha_2^- \circ \pi_1)(s', t')$. As (R_2, γ_2) is strong this results in $(l, s) \in (F\pi_1 \circ \gamma_2^-)(s', t')$. Thus there exists $t \in A_2$ with $(s, t) \in R_2$ and $(l, s, t) \in \gamma_2^-(s', t')$. This leads to $(r, t) \in R$ and $(l, r, t) \in \gamma^-(r', t')$ and we may conlude that $(l, r) \in (F\pi_1 \circ \gamma^-)(r', t')$. Lemma 2.1 proves the other inclusion. ■

Remark 2.5

1. *Let (A, α) be a coalgebra, let $x \in A$. Then (R, γ) with $R := A \times A$ and $(l, a', a') \in \gamma(a, a) :\Longleftrightarrow (l, a') \in \alpha(a)$ is an AM-bisimulation between (A, α) and (A, α) with $(x, x) \in R$.* (reflexivity)

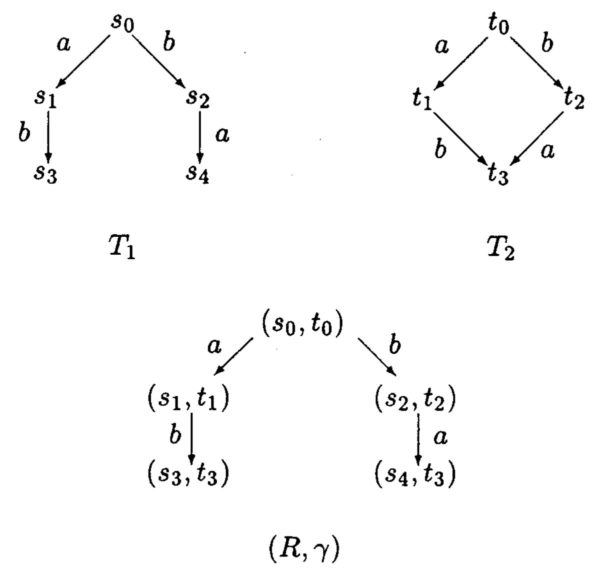


Figure 5: An AM-bisimulation (R, γ) between transition system T_1 and T_2.

2. *Let (R, γ) be an AM-bisimulation between coalgebras (A, α) and (B, β) with $(x, y) \in R$. Then $(\hat{R}, \hat{\gamma})$ with $\hat{R} := \{(b, a) \,|\, (a, b) \in R\}$ and $(l, b', a') \in \hat{\gamma}(b, a) :\iff (l, a', b') \in \gamma(a, b)$ is an AM-bisimulation between (B, β) and (A, α) with $(y, x) \in \hat{R}$.* (symmetry)

2.1.3 A note on strong AM-bisimulation

The concept of strong AM-bisimulation is new. Looking on transition systems with initial states example 2.6 shows that the strong and the non strong version of AM-bisimulation differ:

Example 2.6

Consider the transition systems T_1 and T_2 of figure 5. Let (A, α) and (B, β) be the related coalgebras of T_1 resp. T_2. Figure 5 shows an AM-bisimulation (R, γ) in **Set**$_F$ *between T_1 and T_2.*

But there is no strong AM-bisimimulation between $(\hat{R}, \hat{\gamma})$ between T_1 and T_2 which includes (s_0, t_0) as state. For any such bisimulation $(\hat{R}, \hat{\gamma})$ we get $(s_3, t_3) \in \hat{R}$: In T_1 we find the transition $s_0 \xrightarrow{a} s_1$, thus we need as state (s_1, t_1), furtheron we find in T_1 the transition $s_1 \xrightarrow{b} s_3$, therefore we have $(s_3, t_3) \in \hat{R}$. Assume that $(\hat{R}, \hat{\gamma})$ is strong. Then it has to include a transition $(a, x, t_2) \in \gamma^{-}(s_3, t_3)$ for some state $(x, t_2) \in \hat{R}$. This implies $(a, x) \in (F\pi_1 \circ \gamma^{-})(s_3, t_3)$ and as $(\hat{R}, \hat{\gamma})$ is strong $(a, x) \in (\alpha^{-} \circ \pi_1)(s_3, t_3)$. Therefore we have $(a, s_3) \in \alpha(x)$ - but there is no such transition for any state x in (A, α).

If one considers just trees any AM-bisimulation induces a strong one:

Theorem 2.7

Let S and T be trees with root s_0 resp. t_0. Let (R,γ) be an AM-bisimulation between S and T with $(s_0,t_0) \in R$. Then there exists a strong AM-bisimulation $(\hat{R},\hat{\gamma})$ between S and T with $(s_0,t_0) \in \hat{R}$.

Proof: In order to prove the theorem we define a new AM-bisimulation $(\hat{R},\hat{\gamma})$ from (R,γ). Let (A,α) denote the related coalgebra of the tree S, (B,β) the coalgebra related to T. Let

$$\begin{array}{lcl} R_0 & := & \{s_0, t_0\}, \\ R_{i+1} & := & \{(s',t') \in R \,|\, \exists (s,t) \in R_i, \exists l \in L: \; s \stackrel{l}{\longrightarrow} s',\, t \stackrel{l}{\longrightarrow} t'\}, \quad i \geq 0, \\ \hat{R} & := & \bigcup_{i\geq 0} R_i. \end{array}$$

Let for all (s',t'), $(s,t) \in \hat{R}$ and all labels l

$$(l,s',t') \in \hat{\gamma}(s,t) :\Longleftrightarrow s \stackrel{l}{\longrightarrow} s' \text{ in } (A,\alpha) \text{ and } t \stackrel{l}{\longrightarrow} t' \text{ in } (B,\beta).$$

Obviously $\hat{R} \subseteq R$.

We show first that $(\hat{R},\hat{\gamma})$ is an AM-bisimulation. Let $(l,s') \in (\alpha \circ \pi_1)(s,t)$. $(s,t) \in \hat{R}$ implies $(s,t) \in R$. As (R,γ) is an AM-bisimulation we get $(l,s') \in (F\pi_1 \circ \gamma)(s,t)$. Thus there exists some $t' \in B$ such that $(l,s',t') \in \gamma(s,t)$. This implies $t \stackrel{l}{\longrightarrow} t'$ in (B,β). Therefore we get $(s',t') \in \hat{R}$ and $(l,s',t') \in \hat{\gamma}(s,t)$ which induces $(l,s') \in (F\pi_1 \circ \hat{\gamma})(s,t)$. Now let $(l,s') \in (F\pi_1 \circ \hat{\gamma})(s,t)$. By the definition of $\hat{\gamma}$ we know that $(l,s') \in \alpha(s)$ and thus $(l,s') \in (\alpha \circ \pi_1)(s,t)$.

To prove that $(\hat{R},\hat{\gamma})$ is strong let $(l,s) \in (\alpha^- \circ \pi_1)(s',t')$. This implies that there exists a transition $s \stackrel{l}{\longrightarrow} s'$ in S. Thus $s' \neq s_0$ and therefore $t' \neq t_0$ by the definition of $\hat{R}$. As T is a tree we know that there exists exactly one transition which ends in state t'. Let $t \stackrel{l'}{\longrightarrow} t'$ be this transition. As S is a tree too $s \stackrel{l}{\longrightarrow} s'$ is only transition leading to s. On the other hand we have $(s',t') \in \hat{R}$. Thus by definition of $\hat{R}$ we get $l = l'$ and therfore $(l,s',t') \in \hat{\gamma}(s,t)$. This implies $(l,s) \in (F\pi_1 \circ \hat{\gamma}^-)(s',t')$. Lemma 2.1 etablishes the other inlusion. ■

It is important to note that strong AM-bisimulation is not an "abstraction" of the concept of back and forth bisimulation which [DNMV90] introduce on transition systems with initial state. If one does not consider the "silent action" τ [DNMV90] prove that the "usual" bisimulation on transition systems with initial state – which can be modelled as AM-bisimulation, see section 3.3 – and back and forth bisimulation are equivalent. Concerning strong AM-bisimulation example 2.6 showed that the strong and the non-strong version of AM-bisimulation differ on transition systems with initial state.

An instance of a strong AM-bisimulation can be found in [GKP92]. Among other kinds of bisimulations they introduce the concepts of forward bisimulation ($\sim_{f-b}$) and backward-forward bisimulation ($\sim_{bf-b}$) on prime event structures. For these bisimulations the following strict inclusions hold[1]:

- $\simeq \subset \sim_{bf-b}$,
- $\sim_{bf-b} \subset \sim_{wh-b} \subset \sim_{f-b}$ and
- $\sim_{bf-b} \subset \sim_{r-b} \subset \sim_{f-b}$,

where $\simeq$ stands for isomorphism, $\sim_{wh-b}$ for weak history-preserving bisimulation and $\sim_{r-b}$ for run bisimulation. The equivalences $\sim_{wh-b}$ and $\sim_{r-b}$ are not comparabel. [GKP92] characterize forward bisimulation and backward-forward-bisimulation by temporal logics: Two event structures are forward bisimilar iff their related models cannot be distinguished by formulas of the logic S4N; backward-forward bisimulation can be characterized by the logic POL – an extension of S4N by two modalities. In section 4.2.2 we show how one can model interleaving bisimulation (this is just another term for forward bisimulation) as AM-bisimulation. Backward-forward bisimulation (which we will call here bf-bisimulation) arises as the strong case of this AM-bisimulation.

2.2 ℙ-bisimulation and Path-ℙ-bisimulation

To give an abstract characterization of bisimulation [JNW94] choose a category $\mathbb{M}$ of models and a subcategory $\mathbb{P}$ of $\mathbb{M}$ of "path objects". A *path* is a morphism $p : P \to X$ from an object P in $\mathbb{P}$ to an object X in $\mathbb{M}$. In $\mathbb{M}$ a morphism $f : X \to Y$ is called *ℙ-open,* iff whenever there are objects P, Q and a morphism $m : P \to Q$ in $\mathbb{P}$ and paths $p : P \to X$, $q : Q \to Y$, then there exists a path $r : Q \to X$ with $r \circ m = p$ and $f \circ r = q$. Figure 6 illustrates this "path lifting condition". ℙ-open morphisms include all the identity morphisms and are closed under composition. Two objects X_1 and X_2 of $\mathbb{M}$ are called *ℙ-bisimilar*, iff there exists an object X in $\mathbb{M}$ and ℙ-open morphisms $f_1 : X \to X_1$ and $f_2 : X \to X_2$.

To introduce the concept of path-ℙ-bisimulation [JNW94] assume that $\mathbb{P}$ is a small subcategory of $\mathbb{M}$ and that $\mathbb{P}$ and $\mathbb{M}$ have a common initial object I. Two objects X_1 and X_2 of $\mathbb{M}$ are called *path-ℙ-bisimilar* iff there is a set R of pairs of paths (p_1, p_2) with common domain P, so $p_1 : P \to X_1$ is a path in X_1 and $p_2 : P \to X_2$ is a path in X_2, such that

[1] [GKP92] consider just prime event structures without auto-concurrency where the related transition systems exhibit only finite branching.

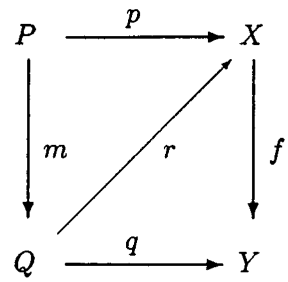


Figure 6: Path lifting condition

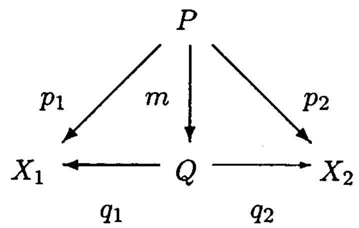


Figure 7: Path-$\mathbb{P}$-bisimulation, illustration for condition (i).

(o) $(\iota_1, \iota_2) \in R$, where $\iota_1 : I \to X_1$ and $\iota_2 : I \to X_2$ are the unique paths starting in the initial object,

and for all $(p_1, p_2) \in R$ and for all $m : P \to Q$, where m is in $\mathbb{P}$, holds

(i) if there exists $q_1 : Q \to X_1$ with $q_1 \circ m = p_1$ then there exists $q_2 : Q \to X_2$ with $q_2 \circ m = p_2$ and $(q_1, q_2) \in R$ (see figure 7) and

(ii) if there exists $q_2 : Q \to X_2$ with $q_2 \circ m = p_2$ then there exists $q_1 : Q \to X_1$ with $q_1 \circ m = p_1$ and $(q_1, q_2) \in R$.

Two objects X_1 and X_2 are *strong path-$\mathbb{P}$-bisimilar* iff they are path-$\mathbb{P}$-bisimilar and the set R further satisfies:

(iii) If $(q_1, q_2) \in R$, with $q_1 : Q \to X_1$ and $q_2 : Q \to X_2$ and $m : P \to Q$, then $(q_1 \circ m, q_2 \circ m) \in R$, see figure 8.

Sometimes we call the set R a (strong) path-$\mathbb{P}$-bisimulation between the objects X_1 and X_2.

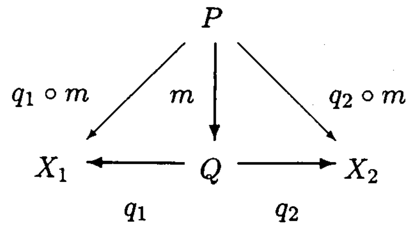


Figure 8: The new condition for strong Path-$\mathbb{P}$-bisimulation.

3 Relating the concepts

[JNW94] give the following relation between $\mathbb{P}$-bisimulation and strong path-$\mathbb{P}$-bisimulation:

Theorem 3.1

1. *Let $\mathbb{M}$ be a category of models, let $\mathbb{P}$ be a small subcategory of $\mathbb{M}$ of path objects, such that $\mathbb{P}$ and $\mathbb{M}$ have a common initial object I. If two objects X_1 and X_2 of $\mathbb{M}$ are $\mathbb{P}$-bisimilar then they are strong path-$\mathbb{P}$-bisimilar.*

2. *Let $\mathbb{M}$ be the subcategory of rooted presheaves in* $[\mathbb{P}^{op}, \mathbf{Set}]$. *Rooted presheaves X_1 and X_2 are strong path-$\mathbb{P}$-bisimilar iff they are $\mathbb{P}$-bisimilar.*

We show how the concept of (strong) AM-bisimulation fits into this picture. The first step – from a given path-$\mathbb{P}$-bisimulation to an AM-bisimulation – is easy, because we start with an "abstract" concept to get a more "concrete" one: Even if we don't know of what "type" the states and the labels of the transition system are, dealing with AM-bisimulation means speaking about transition systems. The second step – from a given AM-bisimulation to a path-$\mathbb{P}$-bisimulation – is more difficult: Here we have to introduce a concept just on categories with no knowledge about (for example) how to define a morphism.

3.1 From path-$\mathbb{P}$-bisimulation to AM-bisimulation

Let $\mathbb{M}$ be a category of models, let $\mathbb{P}$ be a small subcategory of $\mathbb{M}$ of path objects, such that $\mathbb{P}$ and $\mathbb{M}$ have a common initial object I. Then we define for each object X of $\mathbb{M}$ a labelled transition system $T_{path-\mathbb{P}}(X) = (S, \sigma)$ in $\mathbf{Set}_F$ over the set of labels $\bigcup_{P,Q \in \mathbb{P}} \{(m, P, Q) \mid m \in Mor(P, Q)\}$:

- $S := \{p : P \to X \mid P \in \mathbb{P}, p \in Mor(P, X)\}$.

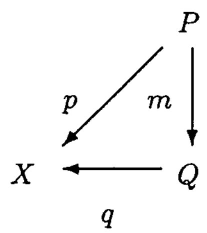


Figure 9: Defining the transitions of $T_{path-\mathbb{P}}$.

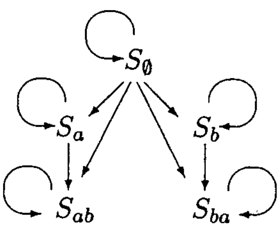


Figure 10: The structure of $T_{path-\mathbf{Bran}}(T_1)$.

- $(m, P, Q, q) \in \sigma(p) :\Longleftrightarrow q \circ m = p$, see figure 9.

Example 3.2

To illustrate the operator $T_{path-\mathbb{P}}$ we consider the transition system T_1 of figure 5 , where we take s_0 as initial state, as object of some category of transition systems **Tran**[2]. *A morphism σ in* **Tran** *between two transition systems $T_1 = (S_1, s_1, \longrightarrow_1)$ and $T_2 = (S_2, s_2, \longrightarrow_2)$ is a mapping $\sigma : S_1 \to S_2$ which satisfies: $\sigma(s_1) = s_2$ and if $x \stackrel{a}{\longrightarrow}_1 y$ then $\sigma(x) \stackrel{a}{\longrightarrow}_2 \sigma(y)$. For $\mathbb{P}$ take* **Bran** *the full subcategory of* **Tran** *whose objects are those acyclic transition systems which consist only of one finite branch.*

First we gather all path objects $P \in$ **Bran** *with $mor_{\mathbf{Tran}}(P, T_1) \neq \emptyset$ into sets: $S_\emptyset$ consists of all transition systems of* **Bran** *with one state but no transition, S_a consists of all transition systems of* **Bran** *with one transition labelled with a, S_{ab} consists of all transition systems of* **Bran** *with two consecutive transitions, the first labelled with a, the second labelled with b, the sets S_b and S_{ba} are defined similarly. The states of $T_{path-\mathbf{Bran}}(T_1)$ are the morphisms from an object $P \in (S_\emptyset \cup S_a \cup S_b \cup S_{ab} \cup S_{ba})$ to T_1.*

[2]The category **Tran** is described in detail in section 3.3.

Figure 10 shows the structure of the transitions in $T_{path-\mathbf{Bran}}(T_1)$. An arrow between two sets X and Y from $S_\emptyset$, S_a, S_b, S_{ab} and S_{ba} means that taking any object P of X and any object Q of Y there exists a transition $p \stackrel{(m,P,Q)}{\longrightarrow} q$ in $T_{path-\mathbf{Bran}}(T_1)$, where $p : P \to T_1$, $q : Q \to Q_1$ and $m : P \to Q$. As in this example all morphisms from a path object to T_1 are uniquely determined we obtain a very simple structure of the transitions in $T_{path-\mathbf{Bran}}(T_1)$. For the transition system T_2 of figure 5 holds: $T_{path-\mathbf{Bran}}(T_2) = T_{path-\mathbf{Bran}}(T_1)$.

Theorem 3.3

Let $\mathbb{M}$ be a category of models, let $\mathbb{P}$ be a small subcategory of $\mathbb{M}$ of path objects, such that $\mathbb{P}$ and $\mathbb{M}$ have a common initial object I. Then holds: Two objects X_1 and X_2 of $\mathbb{M}$ are (strong) path-$\mathbb{P}$-bisimilar iff there exists a (strong) AM-bisimulation (R, γ) between $(A, \alpha) := T_{path-\mathbb{P}}(X_1)$ and $(B, \beta) := T_{path-\mathbb{P}}(X_2)$ with $(\iota_1, \iota_2) \in R$, where $\iota_1 : I \to X_1$ and $\iota_2 : I \to X_2$ are the unique pathes from I to X_1 resp. X_2.

Proof: Let X_1 and X_2 be path-$\mathbb{P}$-bisimilar. Then there exists a set R consisting of pairs of paths (p_1, p_2) with common domain P. We define a map $\gamma : R \to FR$ and show that (R, γ) is an AM-bisimulation between (A, α) and (B, β). Let for all $(p_1, p_2), (q_1, q_2) \in R$, $p_i : P \to X_i$, $q_i : Q \to X_i$, $i = 1, 2$, $m \in Mor(P, Q)$

$$(m, P, Q, q_1, q_2) \in \gamma(p_1, p_2) :\Longleftrightarrow q_1 \circ m = p_1 \wedge q_2 \circ m = p_2.$$

Let $(m, P, Q, q_1) \in (\alpha \circ \pi_1)(p_1, p_2)$. Then $(m, P, Q, q_1) \in \alpha(p_1)$ and therefore $q_1 \circ m = p_1$. As $(p_1, p_2) \in R$ this implies by condition (i) of the definition of path-$\mathbb{P}$-bisimulation that there is some $q_2 : Q \to X_2$ with $q_2 \circ m = p_2$ and $(q_1, q_2) \in R$. Thus we have $(m, P, Q, q_1, q_2) \in \gamma(p_1, p_2)$ and hence $(m, P, Q, q_1) \in (F\pi_1 \circ \gamma)(p_1, p_2)$.

Let $(m, P, Q, q_1) \in (F\pi_1 \circ \gamma)(p_1, p_2)$. Then there exists some $q_2 : Q \to X_2$ such that $(m, P, Q, q_1, q_2) \in \gamma(p_1, p_2)$. By the above definition of γ this implies $q_1 \circ m = p_1$. By definition of $T_{path-\mathbb{P}}(X_1)$ we get $(m, P, Q, q_1) \in \alpha(p_1)$ and therefore $(m, P, Q, q_1) \in (\alpha \circ \pi_1)(p_1, p_2)$.

Assume furtheron that the set R is a strong path-$\mathbb{P}$-bisimulation between X_1 and X_2. In order to provide that in this case the constructed AM-bisimimulation (R, γ) is strong it is enough to show $(\alpha^- \circ \pi_1) \subseteq (F\pi_1 \circ \gamma^-)$ – see lemma 2.1.

Let $(m, P, Q, p_1) \in (\alpha^- \circ \pi_1)(q_1, q_2)$. Then we have $(m, P, Q, p_1) \in \alpha^-(q_1)$ and therefore $(m, P, Q, q_1) \in \alpha(p_1)$. Thus by definition of (A, α) we get the equation $q_1 \circ m = p_1$. As $(q_1, q_2) \in R$ we get by (iii) that $(q_1 \circ m, q_2 \circ m) \in R$. By definition of γ we obtain $(m, P, Q, q_1, q_2) \in \gamma(q_1 \circ m, q_2 \circ m)$. This implies $(m, P, Q, q_1 \circ m, q_2 \circ m) \in \gamma^-(q_1, q_2)$ and we get finally by the equation $q_1 \circ m = p_1$ that $(m, P, Q, p_1) \in (F\pi_1 \circ \gamma^-)(q_1, q_2)$.

Now let (R, γ) be an AM-bisimulation between (A, α) and (B, β), such that $(\iota_1, \iota_2) \in R$. As R may relate paths p_1 and p_2 with different domains we define a subset of R to establisch the path-$\mathbb{P}$-bisimilation:

$$R' := \{(p_1, p_2) \in R \mid \exists P \in \mathbb{P} : \; p_1 \in Mor(P, X_1), \, p_2 \in Mor(P, X_2)\}.$$

Obviously we have $(\iota_1, \iota_2) \in R'$. Now let $(p_1, p_2) \in R'$, $m \in Mor(P, Q)$ for some object Q in $\mathbb{P}$ and $q_1 : Q \to X_1$ a path, such that $q_1 \circ m = p_1$. This implies $(p_1, p_2) \in R$ and $(m, P, Q, q_1) \in (\alpha \circ \pi_1)(p_1, p_2)$. As (R, γ) is an AM-bisimulation there exists some $q_2 : Q \to X_2$ with $(m, P, Q, q_1, q_2) \in \gamma(p_1, p_2)$. Therefore we get $(m, P, Q, q_2) \in \beta(p_2)$ and thus by definition of (B, β) we have $q_2 \circ m = p_2$. As q_1 and q_2 have the same domain and $(q_1, q_2) \in R$ we conclude $(q_1, q_2) \in R'$ and thus R' fullfills condition (i).

Assume furtheron that the AM-bisimulation (R, γ) is strong. To show condition (iii) let $(q_1, q_2) \in R'$, i.e. q_1 and q_2 are paths with the same domain Q, let $m \in Mor(P, Q)$. Then $q_1 \circ m \in Mor(P, X_1)$. By definition of the operator $T_{path-\mathbb{P}}$ we get $(m, P, Q, q_1) \in \alpha(q_1 \circ m)$. This implies

$$(m, P, Q, q_1 \circ m) \in \alpha^-(q_1) = (\alpha^- \circ \pi_1)(q_1, q_2) = (F\pi_1 \circ \gamma^-)(q_1, q_2)$$

Thus there exists some $p_2 : P \to X_2$ such that $(m, P, Q, q_1 \circ m, p_2) \in \gamma^-(q_1, q_2)$. As R is a strong AM-bisimulation we get $(m, P, Q, p_2) \in \beta^-(q_2)$ and therefore $(m, P, Q, q_2) \in \beta(p_2)$. With the definition of $T_{path-\mathbb{P}}$ we conclude $q_2 \circ m = p_2$. Thus $(q_1 \circ m, q_2 \circ m) \in R'$. ■

3.2 From AM-bisimulation to path-$\mathbb{P}$-bisimulation

Let L be a set of labels, let $\mathbf{T}_L$ be the category of transition systems which consists of all objects (A, α) of $\mathbf{Set}_F$ which have an initial state $i_A \in A$ and all states $s \in A$ are reachable from i_A. Take as morphisms between two objects (A, α) and (B, β) of $\mathbf{T}_L$ the mappings $f : A \to B$ with $Ff \circ \alpha \subseteq \beta \circ f$ and $f(i_A) = i_B$, where i_A and i_B are the initial states of (A, α) resp. (B, β). A morphism $f : (A, \alpha) \to (B, \beta)$ in $\mathbf{Set}_F$ is a morphisms in $\mathbf{T}_L$ if (A, α) and (B, β) are objects of $\mathbf{T}_L$ and f preserves the initial state. But not all morphisms of $\mathbf{T}_L$ can be seen as morphisms of $\mathbf{Set}_F$. Reformulating lemma 2.2 for $\mathbf{T}_L$ leads to: A map $f : A \to B$ is a morphism between (A, α) and (B, β) in $\mathbf{T}_L$ iff $f(i_A) = i_B$ and whenever there is a transition $s \xrightarrow{a} s'$ in (A, α) then there is a transition $f(s) \xrightarrow{a} f(s')$ in (B, β).

We use the category $\mathbf{T}_L$ as a link between $\mathbf{Set}_F$ and $\mathbb{M}$ in the following sense: given a functor from $\mathbb{M}$ to $\mathbf{T}_L$ statisfying the conditions which we present in remark 3.5 we construct for a given path-$\mathbb{P}$-bisimulation in $\mathbb{M}$ an AM-bisimulation in $\mathbf{T}_L$ and vice versa. An AM-bisimulation in $\mathbf{T}_L$ is an AM-bisimulation in $\mathbf{Set}_F$ as the projections are morphism in both categories. If there is an AM-bisimulation (R,γ) between (A,α) and (B,β) in $\mathbf{Set}_F$ where (R,γ), (A,α) and (B,β) are objects in $\mathbf{T}_L$ and $(i_A, i_B) \in R$ then (R,γ) is an AM-bisimulation between (A,α) and (B,β) in $\mathbf{T}_L$. Call an AM-bisimulation (R,γ) between (A,α) and (B,β) strong in $\mathbf{T}_L$ iff (R,γ^-) is an AM-bisimulation between (A,α^-) and (B,β^-) in $\mathbf{Set}_F$.

In order to translate the concept of AM-bisimulation into path-$\mathbb{P}$-bisimulation we have to introduce an abstract formulation of AM-bisimulation:

Definition 3.4

Let $\mathbb{M}$ be a category of models. Let mt be an operator which turns an object X of $\mathbb{M}$ into a coalgebra $mt\,X$ in $\mathbf{T}_L$ - i.e. a transition system over a suitable set of labels L. Call two objects X_1 and X_2 of $\mathbb{M}$ (strong) AM-bisimilar relative to *mt iff $mt\,X_1$ and $mt\,X_2$ are (strong) AM-bisimilar in $\mathbf{T}_L$.*

Remark 3.5

As morphisms in $\mathbf{T}_L$ map initial states to inital states an AM-bisimulation (R,γ) in $\mathbf{T}_L$ between (A,α) and (B,β) includes the pair (i_A, i_B).

For the rest of this section we assume the following conditions:

1. The operator mt evolves into a functor from $\mathbb{M}$ to $\mathbf{T}_L$.

2. There is a small subcategory $\mathbb{P}$ in $\mathbb{M}$, such that $\mathbb{P}$ and $\mathbb{M}$ have a common initial object I.

3. For objects P in $\mathbb{P}$ holds: The transition system $mt\,P$ has a unique final state, which is reachable from all other states in $mt\,P$.

4. For any derivation $s_1 \xrightarrow{l_1} s_2 \xrightarrow{l_2} \dots \xrightarrow{l_{n-1}} s_n$ of a transition system in $\mathbf{T}_L$ exists an object P of $\mathbb{P}$ such that in $mt\,P$ exists a derivation $t_1 \xrightarrow{l_1} t_2 \xrightarrow{l_2} \dots \xrightarrow{l_{n-1}} t_n$, where t_1 is the initial state of $mt\,P$, t_n is the final state of $mt\,P$. For this object P holds furtheron: Whenever there exists an object X of $\mathbb{M}$ with $u_1 \xrightarrow{l_1} u_2 \xrightarrow{l_2} \dots \xrightarrow{l_{n-1}} u_n$ in $mt\,X$, where u_1 is the inital state of $mt\,X$, then there exists a morphism $p : P \to X$ in $\mathbb{M}$, such that $(mt\,p)(t_i) = u_i$, $i = 1, 2, \dots, n$.

5. For the transition system in $\mathbf{T}_L$ which consists of just one state and no transition the initial object I of $\mathbb{P}$ is one of the objects condition 4 speaks about.

6. Let P and Q be objects of $\mathbb{P}$, X be an object of $\mathbb{M}$, $p : P \to X$, $q : Q \to X$, $m : P \to Q$ morphims in $\mathbb{M}$ resp. $\mathbb{P}$. Let $t_1 \xrightarrow{l_1} t_2 \xrightarrow{l_2} \dots \xrightarrow{l_{n-1}} t_n$ be a derivation in $mt\,P$, where

t_1 is the initial and t_n the final state of $mt\,P$. Then we have:

$$q \circ m = p \iff \forall 1 \le i \le n : (mt\,q \circ mt\,m)(t_i) = (mt\,p)(t_i).$$

Theorem 3.6

Assume the above described conditions, let X_1 and X_2 be objects of $\mathbb{M}$. X_1 and X_1 are AM-bisimilar relative to mt iff X_1 and X_2 are path-$\mathbb{P}$-bisimular.

Proof: Let (R,γ) be an AM-bisimulation between $(A,\alpha) := mt\,X_1$ and $(B,\beta) := mt\,X_2$. For any element $(s,t) \in R$ exists a derivation from the initial state (s_1,t_1)

$$(s_1,t_1) \xrightarrow{l_1} (s_2,t_2) \xrightarrow{l_2} \dots \xrightarrow{l_{n-1}} (s_n,t_n) = (s,t)$$

in (R,γ). Condition 4 implies that there exists an object P of $\mathbb{P}$ such that $mt\,P$ includes a derivation $u_1 \xrightarrow{l_1} u_2 \xrightarrow{l_2} \dots \xrightarrow{l_{n-1}} u_n$. Using the projections π_1 and π_2 we know that $s_1 \xrightarrow{l_1} s_2 \xrightarrow{l_2} \dots \xrightarrow{l_{n-1}} s_n$ and $t_1 \xrightarrow{l_1} t_2 \xrightarrow{l_2} \dots \xrightarrow{l_{n-1}} t_n$ are derivations in $mt\,X_1$ resp. $mt\,X_2$. Thus there exist morphisms $p_i : P \to X_i$, $i = 1,2$, with $(mt\,p_1)(u_j) = s_j$ and $(mt\,p_2)(u_j) = t_j$, $j = 1,2,\dots,n$. Let $M(s,t)$ denote the set of all pairs of morphism (p_1,p_2), which can be obtained in this way from a state (s,t) of (R,γ), i.e. we look for any derivation which ends in (s,t), for any object P which corresponds to this derivation and for any pair of morphims (p_1,p_2) which embeds $mt\,P$ in $mt\,X_1$ resp. $mt\,X_2$ in the described way. We claim that

$$R' := \bigcup_{(s,t)\in R} M(s,t)$$

is a path-$\mathbb{P}$-bisimulation between X_1 and X_2.

Taking the initial state (s_1,t_1) of (R,γ) condition 5 ensures $(\iota_1,\iota_2) \in R'$, where $\iota_i : I \to X_i$, $i = 1,2$, are the uniquely determined morphisms from the initial object I to X_i.

Let P and Q be objects from $\mathbb{P}$, $p_i : P \to X_i$, $i = 1,2$, $q_1 : Q \to X_1$ and $m : P \to Q$ morphisms with $q_1 \circ m = p_1$ and $(p_1,p_2) \in R'$. As $(p_1,p_2) \in R'$ we know that there exists a derivation $(s_1,t_1) \xrightarrow{l_1} (s_2,t_2) \xrightarrow{l_2} \dots \xrightarrow{l_{n-1}} (s_n,t_n)$ in (R,γ), derivations $s_1 \xrightarrow{l_1} s_2 \xrightarrow{l_2} \dots \xrightarrow{l_{n-1}} s_n$ and $t_1 \xrightarrow{l_1} t_2 \xrightarrow{l_2} \dots \xrightarrow{l_{n-1}} t_n$ in (A,α) resp. (B,β), a derivation $u_1 \xrightarrow{l_1} u_2 \xrightarrow{l_2} \dots \xrightarrow{l_{n-1}} u_n$ in $mt\,P$. By the construction of R' holds: $(mt\,p_1)(u_j) = s_j$, $j = 1,2,\dots,n$, and $(mt\,p_2)(u_j) = t_j$, $j = 1,2,\dots,n$. As $mt\,m$ is a morphims in $\mathbf{T}_L$ we find a derivation $(mt\,m)(u_1) \xrightarrow{l_1} (mt\,m)(u_2) \xrightarrow{l_2} \dots \xrightarrow{l_{n-1}} (mt\,m)(u_n)$ in $mt\,Q$. Let f be the final state of $mt\,Q$. By condition 2 we know that there exists a derivation $(mt\,m)(u_n) \xrightarrow{l_n} v_{n+1} \xrightarrow{l_{n+1}} \dots \xrightarrow{l_{n+k-1}} v_{n+k} = f$ in $mt\,Q$. Combining both derivations we get $s_1 \xrightarrow{l_1} s_2 \xrightarrow{l_2} \dots \xrightarrow{l_{n-1}} s_n \xrightarrow{l_n} (mt\,q_1)(v_{n+1}) \xrightarrow{l_{n+1}} \dots \xrightarrow{l_{n+k-1}} (mt\,q_1)(v_{n+k})$ in (A,α). As (R,γ) is an AM-bisimulation we find derivation $t_1 \xrightarrow{l_1} t_2 \xrightarrow{l_2} \dots \xrightarrow{l_{n-1}} t_n \xrightarrow{l_n} t_{n+1} \xrightarrow{l_{n+1}} \dots \xrightarrow{l_{n+k-1}} t_{n+k}$ in (B,β) for some states $t_{n+1}, \dots, t_{n+k} \in B$. Thus by condition 4 there exists a morphism $q_2 : Q \to X_2$ with $(mt\, q_2)((mt\, m)(u_j)) = t_j$, $j = 1, 2, \dots, n$ and $(mt\, q_2)(v_{n+j}) = t_{n+j}$, $j = 1, 2, \dots, k$. Condition 6 ensures $q_2 \circ m = p_2$ and by construction $(q_1, q_2) \in R'$.

Let R' be a path-ℙ-bisimulation between X_1 and X_2, let $(A, \alpha) := mt\, X_1$ and $(B, \beta) := mt\, X_2$. Let P be an object from ℙ, f the final state of $mt\, P$, X an object from 𝕄 and $p : P \to X$ a morphism. $\mathit{final}(p, P, X) := (mt\, p)(f)$ denotes the image of the final state of $mt\, P$ under $mt\, p$. Let

$$\begin{aligned} \hat{R} := \{(s,t) \mid\ & \exists P \in \mathbb{P}, (p_1, p_2) \in R' : \\ & p_1 : P \to X_1,\ p_2 : P \to X_2, \\ & s = \mathit{final}(p_1, P, X_1),\ t = \mathit{final}(p_2, P, X_2)\}. \end{aligned}$$

Let $(s_1, t_1), (s_2, t_2) \in \hat{R}$, let P, Q be objects of ℙ, $(p_1, p_2), (q_1, q_2) \in R'$ such that $s_1 = \mathit{final}(p_1, P, X_1)$, $t_1 = \mathit{final}(p_2, P, X_2)$, $s_2 = \mathit{final}(q_1, Q, X_1)$, $t_2 = \mathit{final}(q_2, Q, X_2)$. Define

$$(a, s_2, t_2) \in \hat{\gamma}(s_1, t_1)$$

iff there exists a morphism $m : P \to Q$ such that

- $p_1 = q_1 \circ m$,
- $p_2 = q_2 \circ m$ and
- $(mt\, m)(f) \xrightarrow{a} g$ is a derivation in $mt\, Q$, where f is the final state of $mt\, P$ and g the final state of $mt\, Q$.

As R' is a path-ℙ-bisimulation we have $(\iota_1, \iota_2) \in R'$, where I is the initial object of 𝕄 and ℙ and $\iota_i : I \to X_i$, $i = 1, 2$, are the uniquely determined morphisms. Therefore we get $(\mathit{final}(\iota_1, I, X_1), \mathit{final}(\iota_2, I, X_2)) \in \hat{R}$, where $\mathit{final}(\iota_i, I, X_i)$ are the initial states of $mt\, X_i$, $i = 1, 2$. Let

- $R := \{(s,t) \in \hat{R} \mid (s,t)\ \textit{is reachable in}\ (\hat{R}, \hat{\gamma})\ \textit{from}\ (\mathit{final}(\iota_1, I, X_1), \mathit{final}(\iota_2, I, X_2))\}$,
- $\gamma := \begin{cases} R & \to\ \mathcal{P}(L \times R) \\ (s,t) & \mapsto\ \{(a, s', t') \in \hat{\gamma}(s,t) \mid (s', t') \in R\}. \end{cases}$

Then (R, γ) is an object of $\mathbf{T}_L$.

Let $(a, s_2) \in (\alpha \circ \pi_1)(s_1, t_1)$. As $(s_1, t_1) \in R$ there exists an object $P \in$ ℙ and morphisms $p_1 : P \to X_1$, $p_2 : P \to X_2$ such that $s_1 = \mathit{final}(p_1, P, X_1)$, $t_1 = \mathit{final}(p_2, P, X_2)$ and $(p_1, p_2) \in R'$. Let $u_1 \xrightarrow{l_1} u_2 \xrightarrow{l_2} \dots \xrightarrow{l_{n-1}} u_n$ be a derivation from the intial state u_1 to the final state u_n in $mt\, P$. Then $(mt\, p_1)(u_1) \xrightarrow{l_1} (mt\, p_1)(u_2) \xrightarrow{l_2} \dots \xrightarrow{l_{n-1}} (mt\, p_1)(u_n) = s_1$

is a derivation for s_1 in (A,α). Combining it with $(a,s_2) \in \alpha(s_1)$ we get $(mt\,p_1)(u_1) \xrightarrow{l_1} (mt\,p_1)(u_2) \xrightarrow{l_2} \dots \xrightarrow{l_{n-1}} s_1 \xrightarrow{a} s_2$. By condition 4 there exists an object Q in $\mathbb{P}$ such that $v_1 \xrightarrow{l_1} v_2 \xrightarrow{l_2} \dots \xrightarrow{l_{n-1}} v_n \xrightarrow{a} v_{n+1}$ is a derivation in $mt\,Q$, where v_1 is the initial and v_{n+1} is the final state. Further there exist a morphism $m : P \to Q$ with $(mt\,m)(u_j) = v_j$, $j = 1, 2, \dots, n$, and a morphism $q_1 : Q \to X_1$ with $(mt\,q_1)(v_j) = (mt\,p_1)(u_j)$, $j = 1, 2, \dots, n$ and $(mt\,q_1)(v_{n+1}) = s_2$. By condition 6 this implies $q_1 \circ m = p_1$. As R' is a path-$\mathbb{P}$-bisimulation there exists a morphism $q_2 : Q \to X_2$ with $q_2 \circ m = p_2$ and $(q_1, q_2) \in R'$. Therefore we have $(\mathit{final}(q_1, P, X_1), \mathit{final}(q_2, Q, X_2)) \in \hat{R}$, where $s_2 = \mathit{final}(q_1, Q, X_1)$, and $(a, s_2, \mathit{final}(q_2, Q, X_2)) \in \hat{\gamma}(s_1, t_1)$. As (s_1, t_1) is reachable in $(\hat{R}, \hat{\gamma})$ so is $(\mathit{final}(q_1, P, X_1), \mathit{final}(q_2, Q, X_2))$. Thus we get $(a, s_2) \in (F\pi_1 \circ \gamma)(s_1, s_2)$.

Let $(a, s_2) \in (F\pi_1 \circ \gamma)(s_1, t_1)$. Then there exists some $t_2 \in B$ such that $(a, s_2, t_2) \in \gamma(s_1, t_1)$. By the definition of R and γ we get: there exist objects P and Q of $\mathbb{P}$, morphisms $p_i : P \to X_1$, $q_i : Q \to X_i$, $i = 1, 2$, and a morphism $m : P \to Q$, which fullfill the above described conditions. Especially we have $(mt\,m)(f) \xrightarrow{a} g$ in $mt\,Q$, where f is the final state of $mt\,P$ and g the final state of $mt\,Q$. This implies $s_1 = (mt\,p_1)(f) = (mt\,q_1)((mt\,m)(f)) \xrightarrow{a} (mt\,q_1)(g) = s_2$ in (A, α) and therefore $(a, s_2) \in (\alpha \circ \pi_1)(s_1, t_1)$. ∎

Corollary 3.7
Let $\mathbb{M}$ be a category of models, let mt be an operator such that conditions 1 to 6 hold, let $\mathbb{P}$ be the subcategory of $\mathbb{M}$ from condition 2, let X_1 and X_2 be objects of $\mathbb{M}$. X_1 and X_2 are AM-bisimilar relative to mt iff X_1 and X_2 are AM-bisimilar relative to $T_{path-\mathbb{P}}$.

3.3 A first application: transition systems

For traditional models of concurrency like transition systems, event structures and petri nets various notions of bisimulation have been studied. It is an interesting question to what extent the frameworks of [AM89] and [JNW94] are capable of modelling these various notions. As a first application we consider transition systems.

Let *States* be a "universal" set of states. Take as category $\mathbb{M}$ the category **Tran** which has as objects transition systems $(S, s, \longrightarrow)$ over some set of labels L, where $S \subseteq \mathit{States}$ is a set of states, $s \in S$ is the initial state and $\longrightarrow \subseteq S \times L \times S$ is the transition relation. The existence of an initial state implies $S \neq \emptyset$. A morphism σ in **Tran** between two transition systems $T_1 = (S_1, s_1, \longrightarrow_1)$ and $T_2 = (S_2, s_2, \longrightarrow_2)$ is a mapping $\sigma : S_1 \to S_2$ which satisfies: $\sigma(s_1) = s_2$ and if $x \xrightarrow{a}_1 y$ then $\sigma(x) \xrightarrow{a}_2 \sigma(y)$. For $\mathbb{P}$ take **Bran** the full subcategory of **Tran** whose objects are those acyclic transition systems which consist only of one finite branch. Take as functor mt from **Tran** to $\mathbf{T}_L$ the map which restricts a transition system to its reachable states. In this setting all six conditions hold and therefore theorem 3.6 translates AM-bisimulation into path-**Bran**-bisimulation.

Corollary 3.8

Let $T_1 = (S_1, s_1, \longrightarrow_1)$ and $T_2 = (S_2, s_2, \longrightarrow_2)$ be transition systems in **Tran**. *The following are equivalent:*

1. *Between T_1 and T_2 exists an AM-bisimulation (R, γ) in $\mathbf{Set}_F$ with $(s_1, s_2) \in R$.*

2. *Between $T_{path-\mathbf{Bran}}(T_1)$ and $T_{path-\mathbf{Bran}}(T_2)$ exists an AM-bisimulation (R, γ) with $(\iota_1, \iota_2) \in R$.*

3. *Between $T_{path-\mathbf{Bran}}(T_1)$ and $T_{path-\mathbf{Bran}}(T_2)$ exists a strong AM-bisimulation (R, γ) with $(\iota_1, \iota_2) \in R$.*

4. *T_1 and T_2 are path-***Bran***-bisimilar.*

5. *T_1 and T_2 are strong path-***Bran***-bisimilar.*

6. *T_1 and T_2 are* **Bran***-bisimilar.*

Proof: Theorem 3.6 proves the equivalence of 1. and 4. In [MCR96] we showed 1. $\Leftrightarrow$ 6. Furtheron theorem 3.1 (which we cite after [JNW94]) yields 6. implies 5. The equivalences between 2 and 4, 3 and 5 are due to theorem 3.3. ∎

Remark 3.9

Obviously Milner's notion of bisimulation on transition systems coincides with AM-bisimulation. Hence this concept can be modelled by both approaches and yields equivalent results.

Theorem 3.6 only provides an equivalence between AM-bisimulation and path-$\mathbb{P}$-bisimulation. Concerning the strong variants of those bisimulation concepts example 3.10 shows, that in general strong path-$\mathbb{P}$-bisimulation does not imply strong AM-bisimulation:

Example 3.10

*Consider again the transition systems T_1 and T_2 from figure 5. Let (A, α) and (B, β) be the related coalgebras of T_1 resp. T_2. Figure 5 shows an AM-bisimulation (R, γ) in $\mathbf{Set}_F$ between T_1 and T_2. Thus with corollary 3.8 we may conclude that T_1 and T_2 are strong path-***Bran***-bisimilar. But as we have seen in example 2.6 there is no strong AM-bisimimulation $(\hat{R}, \hat{\gamma})$ between T_1 and T_2 which includes (s_0, t_0) as state.*

4 Illustrating the concepts in terms of event structures

Let *Act* be a set of actions. A *(prime) event structure* $\mathcal{E} = (E, \leq, \sharp, l)$ over the set of actions *Act* consists of a set of events E, a causal dependency relation $\leq \subseteq E \times E$, which is a partial order, an irreflexive and symmetric conflict relation $\sharp \subseteq E \times E$ and a labelling function $l : E \to Act$, which together satisfy:

1. For all $e \in E$ the set $\downarrow(e) := \{e' \in E \mid e' \leq e\}$ is finite and

2. for all $d, e, f \in E$ holds: if $d \leq e$ and $d \sharp f$ then $e \sharp f$.

We call a set $X \subseteq E$ a *configuration* of $\mathcal{E}$ iff X is a finite set, leftclosed in E and for all $e, f \in X$ holds: $\neg e \sharp f$. Sometimes we look at a configuration X not just as a set but as a labelled poset. In this case X inherits the causal dependency relation and the labelling function from $\mathcal{E}$: $X = (X, \leq \cap (X \times X), \emptyset, l_{|X})$. $Conf(\mathcal{E})$ denotes the set of all configurations of an event structure $\mathcal{E}$. We call two events $e_1, e_2 \in E$ *concurrent*, $e_1 \, co \, e_2$, iff they are not related by $\leq$ or $\sharp$.

For a configuration X of an event structure $\mathcal{E} = (E, \leq, \sharp, l)$ the set $\sharp_{\mathcal{E}}(X) := \{f \in E \mid \exists e \in X : e \sharp f\}$ includes all events of $\mathcal{E}$ which are in conflict with an element of X. Let $E' := E \backslash (X \cup \sharp_{\mathcal{E}}(X))$. $\mathcal{E} \backslash X := (E', \leq \cap (E' \times E'), \sharp \cap (E' \times E'), l_{|E'})$ denotes the "sub-event structure" of $\mathcal{E}$ including all events which may be added to X in order to get a larger configuration.

The category $\mathbf{E}_{Act}$ has as objects the prime event structures $\mathcal{E} = (E, \leq, \sharp, l)$ over a fixed set of actions *Act*, where $E \subseteq Ev$ for some "universal" set of events Ev. This condition ensures that $\mathbf{E}_{Act}$ is small and therefore all subcategories $\mathbb{P}$ of $\mathbf{E}_{Act}$ which we introduce to define some kind of path-$\mathbb{P}$-bisimulaton are small. Let $\mathcal{E} = (E, \leq_E, \sharp_E, l_E)$ and $\mathcal{F} = (F, \leq_F, \sharp_F, l_F)$ be objects of $\mathbf{E}_{Act}$. A total map $\eta : E \to F$ is a morphism from $\mathcal{E}$ to $\mathcal{F}$ iff

- for all $e \in E : \; l_E(e) = l_F(\eta(e))$,

- $\forall X \in Conf(\mathcal{E}) : \eta(X) \in Conf(\mathcal{F})$ and

- $\forall X \in Conf(\mathcal{E}) \, \forall e, e' \in X : \eta(e) = \eta(e') \Rightarrow e = e'$.

Lin denotes the full subcategory of $\mathbf{E}_{Act}$ which consists of conflict free event structures $(E, \leq, \emptyset, l)$, where E is a finite set and the dependency relation is a total order.

Let $\mathcal{E} = (E, \leq_E, \emptyset, l_E)$, $\mathcal{M} = (M, \leq_M, \emptyset, l_M)$ be finite event structures with $E \cap M = \emptyset$ and $\leq_M = \{(m, m) \mid m \in M\}$. Then $\mathcal{F} := \mathcal{E}; \mathcal{M}$ denotes the event structere $(E \cup M, \leq_F, \emptyset, l_E \cup l_M)$, where $e \leq_F f$ iff $e = f$ or ($e \in E$ and $f \in M$) or $e \leq_E f$.

Call an event structure

$$\mathcal{S} := \mathcal{M}_1; \mathcal{M}_2; \ldots; \mathcal{M}_n, \; n \geq 0,$$

a *step*, where $\mathcal{M}_i = (M_i, \leq_{M_i}, \emptyset, l_i)$ are event structures, M_i are finite sets, M_i are pairwise disjoint and $\leq_{M_i} = \{(m,m) \mid m \in M_i\}$. For an event e of an event structure $\mathcal{E}$ let

$$depth_{\mathcal{E}}(e) := \begin{cases} 1 & \downarrow\{e\} = \{e\} \\ 1 + \max\{depth_{\mathcal{E}}(f) \mid f \in \downarrow\{e\},\, f \neq e\} & \textit{otherwise.} \end{cases}$$

Let $\mathcal{S} := \mathcal{M}_1; \mathcal{M}_2; \ldots; \mathcal{M}_n$, be a step, where all $\mathcal{M}_i$ are different from the empty event structure, let e be an event of $\mathcal{S}$. Then $e \in \mathcal{M}_i \iff depth_{\mathcal{S}}(e) = i,\ i \in \{1, 2, \ldots, n\}$. Thus the representation of a step by nonempty event structures $\mathcal{M}_i$ is uniquely determined. **Step** denotes the full subcategory of $\mathbf{E}_{Act}$ which consists of steps as objects.

Call **Pom** the full subcategory of $\mathbf{E}_{Act}$ which has as objects those conflict free event structures $(E, \leq, \emptyset, l)$ where E is a finite set.

A pomset $[\mathcal{E}]$ is the equivalence class of an event structure $\mathcal{E}$ from **Pom** where we take isomorphism as equivalence relation. **P** denotes the set of all pomsets which can be derived from $\mathbf{E}_{Act}$. A pomset $[\mathcal{E} = (E, \leq_E, \emptyset, l_E)]$ is less sequentiell than a pomset $[\mathcal{F} = (F, \leq_F, \emptyset, l_F)]$, $[\mathcal{E}] \leq [\mathcal{F}]$, iff there exists a map $f : E \to F$ which is bijective and a morphims in $\mathbf{E}_{Act}$.

4.1 Notions of bisimulation on event structures

In order to introduce some notions of bisimulation on event structures we define different transition relations on the configurations of an event structure. Let $\mathcal{E} = (E, \leq, \sharp, l)$ be an event structure over Act, let $X, X' \in Conf(\mathcal{E})$ be configurations of $\mathcal{E}$.

- $X \xrightarrow{a} X'$, iff $a \in Act$, $X \subseteq X'$, $X' \backslash X = \{e\}$.
- $X \xrightarrow{M} X'$ iff $M \in N_0^{Act}$, $X \subseteq X'$, $\forall e, f \in X' \backslash X : e \neq f \Rightarrow e\, co\, f$ and $\forall a \in Act : M(a) = |\{e \in X' \backslash X \mid l(e) = a\}|$.
- $X \xrightarrow{p} X'$ iff $p \in \mathbf{P}$, $X \subseteq X'$ and $p = [X' \backslash X]$.

Definition 4.1
Let $\mathcal{E}_1$, $\mathcal{E}_2$ be event structures.

1. *A relation $R \subseteq Conf(\mathcal{E}_1) \times Conf(\mathcal{E}_2)$ with $(\emptyset, \emptyset) \in R$ is called*

   interleaving bisimulation *iff for all $(X, Y) \in R$, $a \in Act$ holds:*

- *if* $X \xrightarrow{a} X'$ *in* $Conf(\mathcal{E}_1)$ *then* $Y \xrightarrow{a} Y'$ *in* $Conf(\mathcal{E}_2)$ *for some* $Y' \in Conf(\mathcal{E}_2)$ *with* $(X', Y') \in R$ *and*
- *if* $Y \xrightarrow{a} Y'$ *in* $Conf(\mathcal{E}_2)$ *then* $X \xrightarrow{a} X'$ *in* $Conf(\mathcal{E}_1)$ *for some* $X' \in Conf(\mathcal{E}_1)$ *with* $(X', Y') \in R$.

bf-bisimulation *(this definition is due to [GKP92], they call this relation backward-forward bisimulation) iff it is an interleaving bisimulation and for all* $(X', Y') \in R$, $a \in Act$ *holds:*

- *if* $X \xrightarrow{a} X'$ *in* $Conf(\mathcal{E}_1)$ *then* $Y \xrightarrow{a} Y'$ *in* $Conf(\mathcal{E}_2)$ *for some* $Y' \in Conf(\mathcal{E}_2)$ *with* $(X, Y) \in R$ *and*
- *if* $Y \xrightarrow{a} Y'$ *in* $Conf(\mathcal{E}_2)$ *then* $X \xrightarrow{a} X'$ *in* $Conf(\mathcal{E}_1)$ *for some* $X' \in Conf(\mathcal{E}_1)$ *with* $(X, Y) \in R$.

step bisimulation *iff for all* $(X, Y) \in R$, $M \in \mathbb{N}_0^{Act}$ *holds:*

- *if* $X \xrightarrow{M} X'$ *in* $Conf(\mathcal{E}_1)$ *then* $Y \xrightarrow{M} Y'$ *in* $Conf(\mathcal{E}_2)$ *for some* $Y' \in Conf(\mathcal{E}_2)$ *with* $(X', Y') \in R$ *and*
- *if* $Y \xrightarrow{M} Y'$ *in* $Conf(\mathcal{E}_2)$ *then* $X \xrightarrow{M} X'$ *in* $Conf(\mathcal{E}_1)$ *for some* $X' \in Conf(\mathcal{E}_1)$ *with* $(X', Y') \in R$.

pomset bisimulation *iff for all* $(X, Y) \in R$, $p \in \mathbf{P}$ *holds:*

- *if* $X \xrightarrow{p} X'$ *in* $Conf(\mathcal{E}_1)$ *then* $Y \xrightarrow{p} Y'$ *in* $Conf(\mathcal{E}_2)$ *for some* $Y' \in Conf(\mathcal{E}_2)$ *with* $(X', Y') \in R$ *and*
- *if* $Y \xrightarrow{p} Y'$ *in* $Conf(\mathcal{E}_2)$ *then* $X \xrightarrow{p} X'$ *in* $Conf(\mathcal{E}_1)$ *for some* $X' \in Conf(\mathcal{E}_1)$ *with* $(X', Y') \in R$.

2. *A set* R *of triples* (X, Y, f) *where* $X \in Conf(\mathcal{E}_1)$, $Y \in Conf(\mathcal{E}_2)$ *and* $f : X \to Y$ *is an isomorphism in* **Pom** *is called*

history preserving bisimulation *iff for all* $(X, Y, f) \in R$, $p \in \mathbf{P}$ *holds:*

- *if* $X \xrightarrow{p} X'$ *in* $Conf(\mathcal{E}_1)$ *then* $Y \xrightarrow{p} Y'$ *in* $Conf(\mathcal{E}_2)$ *for some* $Y' \in Conf(\mathcal{E}_2)$, $f' : E_1 \to E_2$ *with* $(X', Y', f') \in R$, $f'_{|X} = f$ *and*
- *if* $Y \xrightarrow{p} Y'$ *in* $Conf(\mathcal{E}_2)$ *then* $X \xrightarrow{p} X'$ *in* $Conf(\mathcal{E}_1)$ *for some* $X' \in Conf(\mathcal{E}_1)$, $f' : E_1 \to E_2$ *with* $(X', Y', f') \in R$, $f'_{|X} = f$.

strong history preserving bisimulation *iff* R *is a history preserving bisimulation and satisfies further*

- $(X', Y', f') \in R$ *and* $X \subseteq X'$ *for some configuration* $X \in Conf(\mathcal{E}_1)$ *implies* $(X, Y, f) \in R$ *for some* $Y \subseteq Y'$ *and* $f = f'_{|X}$ *and*

- *$(X', Y', f') \in R$ and $Y \subseteq Y'$ for some configuration $Y \in Conf(\mathcal{E}_1)$ implies $(X, Y, f) \in R$ for some $X \subseteq X'$ and $f = f'_{|X}$.*

For these bisimulations the following strict inclusions hold:

$$\simeq \subset \sim_{bf-b} \subset \sim_{shp-b} \subset \sim_{hp-b} \subset \sim_{pom-b} \subset \sim_{step-b} \subset \sim_{int-b},$$

where $\simeq$ stands for isomorphism, $\sim_{bf-b}$ for bf-bisimulation, $\sim_{shp-b}$ for strong history preserving bisimulation, $\sim_{hp-b}$ for history preserving bisimulation, $\sim_{pom-b}$ for pomset bisimulation, $\sim_{step-b}$ for step bisimulation and $\sim_{int-b}$ for interleaving bisimulation[3].

4.2 Modelling with the abstract concepts

We are again considering the suitability of the two frameworks for handling the various notions of bisimulation.

In a first approach we might – for a given bisimulation type – attempt to formulate a suitable subcategory $\mathbb{P}$ of $\mathbf{E}_{Act}$ and model this bisimulation as path-$\mathbb{P}$- or as $\mathbb{P}$-bisimulation. If we succeed we might proceed and apply theorem 3.3 to obtain transition systems $T_{path-\mathbb{P}}(\mathcal{E})$ and an AM-bisimulation. This approach has two drawbacks: First it might be impossible to find an adequate path-$\mathbb{P}$-modelling, see section 4.3, second, even if we found a suitable modelling in the framework of [JNW94], the highly abstract transition systems and the AM-bisimulation provided by theorem 3.3 are probably not the ones we are aiming for.

Most notions of bisimulation induce in a natural way transition systems. Hence one might in a second attempt model this "natural way" by an operator mt and invoke theorem 3.6. As it turns out, however, there are interesting types of bisimulation for which mt does not extend to a functor, so this approach will not work either.

In the following we first study two different strategies to associate a transition system with an event structure. In section 4.2.2 we model interleaving, bf, step and pomset bisimulation as AM-bisimulations. We also show that bf-bisimulation can only be modelled by one of the two strategies. In section 4.2.3 we use our results on the relations between the abstract bisimulation concepts to model interleaving, step and pomset bisimulation in the framework of [JNW94]. (Strong) history preserving bisimulation is considered in section 4.2.4.

[3]The relation $\sim_{bf-b} \subset \sim_{shp-b}$ is due to [GKP92], who consider just prime event structures without auto-concurrency where the related transition systems exhibit only finite branching.

4.2.1 Defining transition systems related to an event structure

We will discuss two types of transition systems which one can associate with an event structure: The first one takes the configurations of the event structure as the states of a transition systems. This approach corresponds to the definition of occurence transition systems (OTS) in [GKP92]. We will define three operators T_{int}, T_{step} and T_{pom} where each of them makes of an event structure a transition systems over a particular set of labels. Another possibility to translate an event structure into a transition system can be found in [LG91] or [BMC94]: Here one takes event structures as states. Again we define three operators, this time TE_{int}, TE_{step} and TE_{pom}. The two types of transition systems represent different views on event structures:

- Taking configurations as states means to distinguish between "situations" in an event structure which arise from different histories: Two states, i.e. two configurations of an event structure, are different, iff they differ in at least one event. The computational possibilities of a state, i.e. its future, are not considered. Therefore a transition system based on configurations may have more than one final state.

- Taking event structures as states means to collect in a state all "situations" in an event structure which have the same future: Two configurations, i.e. histories, lead to the same state, iff the same sets of events may be added to them in order to obtain a larger configuration, i.e. both configurations have the same "computational possibilities". The different histories which may lead to one state are not considered. Thus a transition system of this type has at most one final state.

Common to both approaches is that they respect different names of events, i.e. isomorphic histories resp. computational possibilities are not identified.

Comparing both types of transition systems we show that for any prime event structure $\mathcal{E}$ there exist AM-bisimulations (R, γ_*) between $T_*(\mathcal{E})$ and $TE_*(\mathcal{E})$ with $(\emptyset, \mathcal{E}) \in R$, where $* \in \{int, step, pom\}$.

For an event structure $\mathcal{E} = (E, \leq, \sharp, l)$ of $\mathbf{E}_{Act}$ we construct different coalgebras, i.e. transition systems, $T_{int}(\mathcal{E}) = (Conf(\mathcal{E}), \alpha_{int})$, $T_{step}(\mathcal{E}) = (Conf(\mathcal{E}), \alpha_{step})$ and $T_{pom}(\mathcal{E}) = (Conf(\mathcal{E}), \alpha_{pom})$ in $\mathbf{Set}_F$, where the functor $F := \mathcal{P}(L \times \_)$. They consist all of the same set $Conf(\mathcal{E})$ in the first component but are defined over different sets of labels L. In case of

$T_{int}(\mathcal{E})$ we choose $L := Act$ and define $(a, X') \in \alpha_{int}(X)$ iff
$X \subseteq X'$, $X' \backslash X = \{e\}$ and $l(e) = a$.

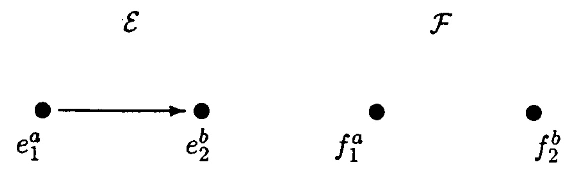


Figure 11: T_{pom} is not a functor to $\mathbf{T_P}$.

$T_{step}(\mathcal{E})$ we choose $L := \mathbb{N}_0^{Act}$ and define $(M, X') \in \alpha_{step}(X)$ iff
$X \subseteq X'$, $\forall e, f \in X' \backslash X : e \neq f \Rightarrow e \, co \, f$ and
$\forall a \in Act : M(a) = |\{e \in X' \backslash X \mid l(e) = a\}|$.

$T_{pom}(\mathcal{E})$ we choose $L := \mathbf{P}$ and define $(p, X') \in \alpha_{pom}(X)$ iff
$X \subseteq X'$ and $p = [X' \backslash X]$.

Thus for the above defined maps α_* holds: $(l, X') \in \alpha_*(X) \iff X \xrightarrow{l} X'$, where $* \in \{int, step, pom\}$ and l is a label of appropriate type. Neglecting the different types of the labels L, i.e. looking at an action a as a multiset with $M(a) = 1$ and for $x \neq a : M(x) = 0$ resp. on a multiset M as a pomset $[\mathcal{E} = (E, \leq_E, \emptyset, l_E)]$, where $\forall a \in Act : M(a) = |\{e \in E \mid l(e) = a\}|$, $\leq_E = \{(e, e) \mid e \in E\}$, we get the following relations between the above introduced transition systems: $\forall X \in Conf(\mathcal{E}) : \alpha_{int}(X) \subseteq \alpha_{step}(X) \subseteq \alpha_{pom}(X)$.

In section 4.2.2 we will show that the operators T_* are suitable to model some of the above defined bisimulations on event structures as AM-bisimulation. Thus we are interested to invoke theorem 3.6 in order to obtain from an AM-bisimulation a path-$\mathbb{P}$-bisimulation. One condition of theorem 3.6 is that a choosen operator evolves into functor.

Let $\mathcal{E}$ and $\mathcal{F}$ be prime event structures, $\eta : \mathcal{E} \to \mathcal{F}$ a morphism in $\mathbf{E}_{Act}$. To obtain from η a morphism in $\mathbf{T}_{Act}$ define $T_{int}(\eta)(X) := \eta(X)$ for configurations $X \in Conf(\mathcal{E})$. It is easy to see that with this definition the operator T_{int} evolves into a functor from $\mathbf{E}_{Act}$ to $\mathbf{T}_{Act}$. Concerning the operator T_{step} we take again $T_{step}(\eta)(X) := \eta(X)$ for configurations $X \in Conf(\mathcal{E})$. With this definition T_{step} evolves into a functor from $\mathbf{E}_{Act}$ to $\mathbf{T}_{\mathbb{N}_0^{Act}}$ – for a proof see corollary 4.14. The operator T_{pom} is not a functor from $\mathbf{E}_{Act}$ to $\mathbf{T_P}$, see example 4.2. The reason is that an event structure morphism may map causal dependent events on concurrent events. But the operator T_{pom} is able to distinguish between these different concepts: As we will see in section 4.2.3 pomset bisimulation can be modelled using the operator T_{pom}.

Example 4.2

Let $\mathcal{E}$ and $\mathcal{F}$ be the event structures from figure 11. In $T_{pom}(\mathcal{E})$ we find the transition $\emptyset \xrightarrow{[\mathcal{E}]} \{e_1, e_2\}$ while there is no transition labelled with $[\mathcal{E}]$ in $T_{pom}(\mathcal{F})$. Thus in $\mathbf{T_P}$ exists no morphism from $T_{pom}(\mathcal{E})$ to $T_{pom}(\mathcal{F})$. On the other hand $\eta : \mathcal{E} \to \mathcal{F}$ with $\eta(e_1) = f_1$ and $\eta(e_2) = f_2$ is a morphims between $\mathcal{E}$ and $\mathcal{F}$ in $\mathbf{E}_{Act}$. Therefore the operator T_{pom} cannot evolve into a functor.

In order to introduce the operators TE_{int}, TE_{step} and TE_{pom} we define transition relations $\xrightarrow{l}_*$ between the objects of $\mathbf{E}_{Act}$, where $* \in \{int, step, pom\}$ and l is a label of appropriate type. Let $\mathcal{E} = (E, \leq_E, \sharp_E, l_E)$ and $\mathcal{F} = (F, \leq_F, \sharp_F, l_F)$ be prime event structures.

$\mathcal{E} \xrightarrow{a}_{int} \mathcal{F}$ iff $\exists e \in E : \downarrow(e) = \{e\}$, $l(e) = a$, $\mathcal{F} = \mathcal{E} \setminus \{e\}$.

$\mathcal{E} \xrightarrow{M}_{step} \mathcal{F}$ iff there exists a configuration $X \in Conf(\mathcal{E})$ such that

- $\forall e, f \in X : \ e \ co \ f$,
- $M \in \mathbb{N}_0^{Act}$ with $\forall a \in Act : \ M(a) = |\{e \in X \mid l_E(e) = a\}|$ and
- $\mathcal{F} = \mathcal{E} \setminus X$.

$\mathcal{E} \xrightarrow{p}_{pom} \mathcal{F}$ iff there exists a configuration $X \in Conf(\mathcal{E})$ such that $p = [X]$ and $\mathcal{F} = \mathcal{E} \setminus X$.

The set

$$\begin{aligned} Reach_*(\mathcal{E}) \ := \ \{\mathcal{F} \in \mathbf{E}_{Act} \mid \ & \exists\, k \geq 0, \exists\, \mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_k \in \mathbf{E}_{Act} : \\ & \mathcal{E}_0 = \mathcal{E},\, \mathcal{E}_k = \mathcal{F},\, \mathcal{E}_i \xrightarrow{l}_* \mathcal{E}_{i+1} \text{ for } i < k\} \end{aligned}$$

includes all event structures which can be derived from $\mathcal{E}$ by a finite number of steps with $\xrightarrow{l}_*$, where $* \in \{int, step, pom\}$ and l is an element of Act, $\mathbb{N}_0^{Act}$ resp. $\mathbf{P}$.

Lemma 4.3

Let $\mathcal{E} = (E, \leq_E, \sharp_E, l_E)$ be a prime event structure.

1. *Let $\mathcal{E}' := \mathcal{E} \setminus X$ for some configuration $X \in Conf(\mathcal{E})$, $\mathcal{E}'' := \mathcal{E}' \setminus X'$ for some configuration $X' \in Conf(\mathcal{E}')$. Then $X \cup X'$ is a configuration of $\mathcal{E}$ and $\mathcal{E}'' = \mathcal{E} \setminus (X \cup X')$.*
2. *For all event structures $\mathcal{E}' \in Reach_{int}(\mathcal{E})$ holds: There exists a configuration $X \in Conf(\mathcal{E})$ such that $\mathcal{E}' = \mathcal{E} \setminus X$.*
3. *For all $X \in Conf(\mathcal{E})$ holds: $\mathcal{E} \setminus X \in Reach_{int}(\mathcal{E})$.*
4. *$Reach_{int}(\mathcal{E}) = Reach_{step}(\mathcal{E}) = Reach_{pom}(\mathcal{E})$.*

5. *Let $\mathcal{E}'$, $\mathcal{E}'' \in Reach_{int}(\mathcal{E})$. For all labels a, M, p holds:*

   (a) $\mathcal{E}' \xrightarrow{a}_{int} \mathcal{E}'' \Longrightarrow$
   $\exists\, X', X'' \in Conf(\mathcal{E}):\ \mathcal{E}' = \mathcal{E}\backslash X',\ \mathcal{E}'' = \mathcal{E}\backslash X'',\ X' \xrightarrow{a}_{int} X''.$

   (b) $\mathcal{E}' \xrightarrow{M}_{step} \mathcal{E}'' \Longrightarrow$
   $\exists\, X', X'' \in Conf(\mathcal{E}):\ \mathcal{E}' = \mathcal{E}\backslash X',\ \mathcal{E}'' = \mathcal{E}\backslash X'',\ X' \xrightarrow{M}_{step} X''.$

   (c) $\mathcal{E}' \xrightarrow{p}_{pom} \mathcal{E}'' \Longrightarrow$
   $\exists\, X', X'' \in Conf(\mathcal{E}):\ \mathcal{E}' = \mathcal{E}\backslash X',\ \mathcal{E}'' = \mathcal{E}\backslash X'',\ X' \xrightarrow{p}_{pom} X''.$

6. *Let X', $X'' \in Conf(\mathcal{E})$ with $X' \subseteq X''$. Define $\mathcal{E}' := \mathcal{E}\backslash X'$, $\mathcal{E}'' := \mathcal{E}\backslash X''$ and $X := X''\backslash X'$. Then X is a configuration of $\mathcal{E}'$ and $\mathcal{E}'' = \mathcal{E}'\backslash X$.*

7. *Let X', $X'' \in Conf(\mathcal{E}.)$ For all labels a, M, p holds:*

   (a) $X' \xrightarrow{a}_{int} X'' \Longrightarrow \mathcal{E}\backslash X' \xrightarrow{a}_{int} \mathcal{E}\backslash X''$

   (b) $X' \xrightarrow{M}_{step} X'' \Longrightarrow \mathcal{E}\backslash X' \xrightarrow{M}_{step} \mathcal{E}\backslash X''$

   (c) $X' \xrightarrow{p}_{pom} X'' \Longrightarrow \mathcal{E}\backslash X' \xrightarrow{p}_{pom} \mathcal{E}\backslash X''$

Proof:

1. Let $X \in Conf(\mathcal{E})$, $\mathcal{E}' := \mathcal{E}\backslash X = (E', \leq', \sharp', l')$, let $X' \in Conf(\mathcal{E}')$. We first prove that $X \cup X' \in Conf(\mathcal{E})$. As X and X' are finite sets so is $X \cup X'$. As X is a configuration of $\mathcal{E}$ it contains for all $e \in X$ their predecessors. Consider now the set $\{\hat{e} \leq_E e \,|\, \hat{e} \in E,\ e \in X'\}$ of all predecessors which the events from X' have in E. We have to show that this set is contained in $X \cup X'$. Let $\hat{e} \leq_E e$ for some event $e \in X'$, where $\hat{e} \in E$. If $\hat{e} \in E'$ we know $\hat{e} \in X'$ because X' is leftclosed. If $\hat{e} \notin E'$ either $\hat{e} \in X$ or $\hat{e} \in \sharp_{\mathcal{E}}(X)$. In the first case we are done again. In the second case exists some event $f \in X$ with $\hat{e}\sharp_{\mathcal{E}} f$. This implies $f \sharp_{\mathcal{E}} e$ and thus $e \notin E'$ – contradiction. To show that $X \cup X'$ are conflictfree with respect to $\sharp_E$ let e_1, $e_2 \in X \cup X'$. If both events are in X they are not in conflict as X is a configuration of $\mathcal{E}$. Let $e_1 \in X$ and $e_2 \in X'$, assume $e_1 \sharp_E e_2$. Then $e_2 \in \sharp_{\mathcal{E}}(X)$ and therefore $e_2 \notin X'$ – contradiction. Finally let e_1, $e_2 \in X'$, assume $e_1 \sharp_E e_2$. As $\mathcal{E}'$ inherits its conflict relation from $\mathcal{E}$ this leads to $e_1 \sharp' e_2$ – contradiction to $X' \in Conf(E')$.

   Next we show $\sharp_{\mathcal{E}}(X) \cup \sharp_{\mathcal{E}'}(X') = \sharp_{\mathcal{E}}(X \cup X')$. As any conflict in $\mathcal{E}'$ is inherited from $\mathcal{E}$ the inclusion "$\subseteq$" holds. To prove the other direction let $f \in \sharp_{\mathcal{E}}(X \cup X')$. Then there exists some event $e \in X \cup X'$ such that $e \sharp_E f$. If $e \in X$ we are done. If $e \in X'$ and $f \in E'$ we get $f \in \sharp_{\mathcal{E}'}(X')$. Now consider the situation $f \notin E'$: Then either $f \in X$ of

$f \in \sharp_{\mathcal{E}}(X)$. In the second case we are done again. In the first we get $e \in \sharp_{\mathcal{E}}(X)$ and therefore $e \notin X'$ – contradiction. With this equation we get

$$\begin{aligned} E'' &= E' \setminus (X' \cup \sharp_{\mathcal{E}'}(X')) \\ &= (E \setminus (X \cup \sharp_{\mathcal{E}}(X))) \setminus (X' \cup \sharp_{\mathcal{E}'}(X')) \\ &= E \setminus (X \cup X' \cup \sharp_{\mathcal{E}}(X) \cup \sharp_{\mathcal{E}'}(X')) \\ &= E \setminus ((X \cup X') \cup \sharp_{\mathcal{E}}(X \cup X')) \end{aligned}$$

and may conclude $\mathcal{E}'' = \mathcal{E} \setminus (X \cup X')$.

2. Let $\mathcal{E}' \in \mathit{Reach}_{int}(\mathcal{E})$. We show the existence of a configuration $X \in \mathit{Conf}(\mathcal{E})$ with $\mathcal{E}' = \mathcal{E} \setminus X$ by induction on the length n of a shortest derivation from $\mathcal{E}$ to $\mathcal{E}'$.

   If $n = 0$ take $X := \emptyset$ as configuration: $\mathcal{E} = \mathcal{E} \setminus X = \mathcal{E}'$. Now let $\mathcal{E}'$ be an event structure in $\mathit{Reach}_{int}(\mathcal{E})$ with a shortest derivation from $\mathcal{E}$ of length $n+1$. Then there exists an event structure $\mathcal{E}'' \in \mathit{Reach}_{int}(\mathcal{E})$ such that $\mathcal{E}'' \xrightarrow{a}_{int} \mathcal{E}'$ and $\mathcal{E}''$ can be derived from $\mathcal{E}$ in n steps. By the induction hypothesis there exists a configuration $X \subseteq E$ with $\mathcal{E}'' = \mathcal{E} \setminus X$. Let e be the event with $\mathcal{E}'' = \mathcal{E}' \setminus \{e\}$. From part 1 of this lemma we know that $X \cup \{e\}$ is a configuration of $\mathcal{E}$ and $\mathcal{E}' = \mathcal{E} \setminus (X \cup \{e\})$.

3. Let X be a configuration of $\mathcal{E}$. If $X = \emptyset$ then $\mathcal{E} \setminus X = \mathcal{E} \setminus \emptyset = \mathcal{E} \in \mathit{Reach}_{int}(\mathcal{E})$. If $X \neq \emptyset$ we consider the configuration X as lposet. As X is finite there exists a total order $\leq_t \subseteq (X \times X)$ such that $e \leq_E f \Rightarrow e \leq_t f$ for all $e, f \in X$. Let $e_1 \leq_t e_2 \leq_t \ldots \leq_t e_n$ be the order on the elements of X, $a_i = l_E(e_i)$, $1 \leq i \leq n$. Let $\mathcal{E}_0 = \mathcal{E}$ and for $1 \leq i \leq n:\ \mathcal{E}_i := \mathcal{E} \setminus \{e_1, e_2, \ldots, e_i\}$. For $\mathcal{E}_i$ and $\mathcal{E}_{i+1}$, $0 \leq i < n$, holds: $\mathcal{E}_{i+1} = \mathcal{E}_i \setminus \{e_i\}$. Therefore $\mathcal{E} = \mathcal{E}_0 \xrightarrow{a_1}_{int} \mathcal{E}_1 \xrightarrow{a_2}_{int} \mathcal{E}_2 \xrightarrow{a_2}_{int} \cdots \xrightarrow{e_n}_{int} \mathcal{E}_n = \mathcal{E} \setminus X$ is a derivation of $\mathcal{E} \setminus X$ and $\mathcal{E} \setminus X \in \mathit{Reach}_{int}(\mathcal{E})$.

4. Obviously $\mathit{Reach}_{int}(\mathcal{E}) \subseteq \mathit{Reach}_{step}(\mathcal{E}) \subseteq \mathit{Reach}_{pom}(\mathcal{E})$: Any single action can be viewed as multiset, any multiset is a special case of a pomset.

   As a partial order on a finite set can always be linearized, it is possible to "simulate" any transition $\mathcal{E}' \xrightarrow{p}_{pom} \mathcal{E}''$ between event structures $\mathcal{E}'$ and $\mathcal{E}''$ of $\mathbf{E}_{Act}$ by a finite number of transitions $\mathcal{E}' \xrightarrow{a_1}_{int} \mathcal{E}_1 \xrightarrow{a_2}_{int} \cdots \xrightarrow{a_n}_{int} \mathcal{E}_n \xrightarrow{a_{n+1}}_{int} \mathcal{E}''$. Therefore $\mathit{Reach}_{pom}(\mathcal{E}) \subseteq \mathit{Reach}_{int}(\mathcal{E})$.

5. Let $\mathcal{E}' = (E', \leq_{E'}, \sharp_{E'}, l_{E'})$ and $\mathcal{E}'' = (E'', \leq_{E''}, \sharp_{E''}, l_{E''}) \in \mathit{Reach}_{int}(\mathcal{E})$.

   (a) If $\mathcal{E}' \xrightarrow{a}_{int} \mathcal{E}''$ we know that $\mathcal{E}'' = \mathcal{E}' \setminus \{e\}$ for some $e \in E$ with $l'_E(e) = a$. As $\mathcal{E}' \in \mathit{Reach}_{int}(\mathcal{E})$ there exists a configuration $X' \in \mathit{Conf}(\mathcal{E})$ such that $\mathcal{E}' = \mathcal{E} \setminus X'$. Let $X'' := X' \cup \{e\}$. From the part 1 of this lemma we know that X'' is a configuration of $\mathcal{E}$ and we get: $\mathcal{E}'' = (\mathcal{E}\backslash X')\backslash\{e\} = \mathcal{E}\backslash(X' \cup \{e\}) = \mathcal{E}\backslash X''$ and $X' \xrightarrow{a}_{int} X''$.

(b) and (c) Let $\mathcal{E}' \xrightarrow{M}_{step} \mathcal{E}''$ resp. $\mathcal{E}' \xrightarrow{p}_{pom} \mathcal{E}''$. In both cases we know from part 2 of this lemma that there exists a configuration $X \in Conf(\mathcal{E})$ such that $\mathcal{E}' = \mathcal{E}\backslash X$. By definition of both transition relations there exists a configuration $X' \in Conf(\mathcal{E}')$ with $\mathcal{E}'' = \mathcal{E}'\backslash X'$. Let $X'' := X \cup X'$. X'' is a configuration of $\mathcal{E}$, $\mathcal{E}\backslash X'' = \mathcal{E}''$. If we start with a multistep the elements of $X' = X''\backslash X$ are concurrent in $\mathcal{E}$, if we start with a pomset X' is a lposet.

6. Let X', $X'' \in Conf(\mathcal{E})$ with $X' \subseteq X''$. Let $\mathcal{E}' := \mathcal{E}\backslash X' = (E', \leq_{E'}, \sharp_{E'}, l_{E'})$ and $\mathcal{E}'' := \mathcal{E}\backslash X'' = (E'', \leq_{E''}, \sharp_{E''}, l_{E''})$. Let $X := X''\backslash X'$. If $X = \emptyset$ obviously $X \in Conf(\mathcal{E}')$ and $\mathcal{E}'' = \mathcal{E}'$.

   Now let $X \neq \emptyset$. First we show that X is a configuration of $\mathcal{E}'$. Let $e \in X$. Then $e \in X''$ and therefore for all $f \in X''$ holds $\neg(e \sharp_E f)$. Thus $e \in E\backslash(X' \cup \sharp_{\mathcal{E}} X')$ and we get $X \subseteq E'$. X is finite as it is a subset of X''. In order to prove that X is leftclosed in $\mathcal{E}'$ let $e \in X$. Consider an $f \in E$ with $f \leq_E e$. As X'' is a configuration of $\mathcal{E}$ we get $f \in X''$. If $f \in E'$ then $f \notin X'$ and thus $f \in X = X''\backslash X'$. X is conflict-free because X'' is.

   In order to prove $\mathcal{E}'' = \mathcal{E}'\backslash X$ we first establish $\sharp_{\mathcal{E}}(X'') = \sharp_{\mathcal{E}}(X') \cup \sharp_{\mathcal{E}'}(X)$. Let $f \in \sharp_{\mathcal{E}}(X'')$. Then we find an element $e \in X''$ with $f \sharp_E e$. As $X'' = X' \cup X$ we get: $e \in X'$ or $e \in X$. The first case implies $f \in \sharp_{\mathcal{E}}(X')$. In the second case we know that $f \notin X''$ and thus $f \notin X'$. This leads to $f \in E'$ and $f \sharp_{E'} e$ – which means: $f \in \sharp_{\mathcal{E}'}(X)$ – or to $f \notin E'$ which implies $f \in \sharp_{\mathcal{E}}(X')$. To prove the reverse direction let first $f \in \sharp_{\mathcal{E}}(X')$. Then obviously $f \in \sharp_{\mathcal{E}}(X'')$. Now consider $f \in \sharp_{\mathcal{E}'}(X)$. Then $f \in \sharp_{\mathcal{E}}(X)$ and thus $f \in \sharp_{\mathcal{E}}(X'')$.

   With this equation we get:

$$\begin{aligned} E'' &= E\backslash(X'' \cup \sharp_{\mathcal{E}}(X'')) \\ &= E\backslash(X'' \cup \sharp_{\mathcal{E}}(X') \cup \sharp_{\mathcal{E}'}(X)) \\ &= (((E\backslash X') \setminus \sharp_{\mathcal{E}}(X')) \backslash X) \setminus \sharp_{\mathcal{E}'}(X) \\ &= E'\backslash(X \cup \sharp'_{\mathcal{E}}(X)). \end{aligned}$$

   As the causality relation, conflict relation and label function are inherited from $\mathcal{E}$ these equations on sets of events lead to $\mathcal{E}'' = \mathcal{E}'\backslash X$.

7. Let finally X', $X'' \in Conf(\mathcal{E})$. If

   (a) $X' \xrightarrow{a}_{int} X''$,

(b) $X' \xrightarrow{M}_{step} X''$ or

(c) $X' \xrightarrow{p}_{pom} X''$

we get $X' \subseteq X''$ and we may conclude with part 6 of this lemma: $X := X''\backslash X'$ is a configuration of $\mathcal{E}' := \mathcal{E}\backslash X'$ and $\mathcal{E}'' := \mathcal{E}\backslash X'' = \mathcal{E}'\backslash X$. In the first case the configuration X consists of one element labelled with a, in the second X corresponds to a multiset M and in the third to an object p of $\mathbf{P}$. ■

With lemma 4.1 we can define the operators TE_* on an event structure $\mathcal{E}$ as follows: $TE_{int}(\mathcal{E}) := (Reach_{int}(\mathcal{E}), \hat{\alpha}_{int})$, $TE_{step}(\mathcal{E}) := (Reach_{int}(\mathcal{E}), \hat{\alpha}_{step})$ and $TE_{pom}(\mathcal{E}) := (Reach_{int}(\mathcal{E}), \hat{\alpha}_{pom})$, where

- $(a, \mathcal{E}'') \in \hat{\alpha}_{int}(\mathcal{E}')$ iff $\mathcal{E}' \xrightarrow{a}_{int} \mathcal{E}''$,
- $(M, \mathcal{E}'') \in \hat{\alpha}_{step}(\mathcal{E}')$ iff $\mathcal{E}' \xrightarrow{M}_{step} \mathcal{E}''$ and
- $(p, \mathcal{E}'') \in \hat{\alpha}_{pom}(\mathcal{E}')$ iff $\mathcal{E}' \xrightarrow{p}_{pom} \mathcal{E}''$.

As for the operators T_* one obtains: $\hat{\alpha}_{int}(X) \subseteq \hat{\alpha}_{step}(X) \subseteq \hat{\alpha}_{pom}(X)$.

As example 4.4 shows the operators TE_{int}, TE_{step} and TE_{pom} fail to evolve into functors from $\mathbf{E}_{Act}$ to $\mathbf{T}_L$, where L is choosen from $\{Act, \mathbb{N}_0^{Act}, \mathbf{P}\}$. The reason is that an event stucture morphism $\eta : \mathcal{E} \rightarrow \mathcal{F}$ cannot "control" all computing capabilities of $\mathcal{F}$: The execution of events e_1, e_2 in $\mathcal{E}$ may lead to the same computational capabities in $\mathcal{E}$, while the execution of the event $\eta(e_1)$ in $\mathcal{F}$ leads to a "future" different from the one after the execution of $\eta(e_2)$ in $\mathcal{F}$. An event $f \notin \eta(E)$ may be in conflict with $\eta(e_1)$ but not with $\eta(e_2)$. This "splits" one state in $TE_*(\mathcal{E})$ into two states in $TE_*(\mathcal{F})$.

Example 4.4

Consider the event structures $\mathcal{E}$ and $\mathcal{F}$ in the initial nodes of the transition systems $TE_{int}(\mathcal{E})$ and $TE_{int}(\mathcal{F})$ of figure 12. Obviously the map $\eta : \{e_1, e_2\} \rightarrow \{f_1, f_2, f_3\}$ with $\eta(e_1) = f_1$ and $\eta(e_2) = f_2$ is an event structure morphism. But there is no morphism in $\mathbf{T}_{Act}$ between $TE_{int}(\mathcal{E})$ and $TE_{int}(\mathcal{F})$. As $\longrightarrow_{int} \subseteq \longrightarrow_{step} \subseteq \longrightarrow_{pom}$ there are no morphisms in $\mathbf{T}_{\mathbb{N}_0^{Act}}$ resp. $\mathbf{T}_{\mathbf{P}}$ between $TE_*(\mathcal{E})$ and $TE_*(\mathcal{F})$ for $* \in \{step, pom\}$.

Example 4.5

The transition systems $T_{int}(\mathcal{E})$ and $TE_{int}(\mathcal{E})$ of an event structure $\mathcal{E}$ are in general not isomorphic. Consider the event structure of figure 13. Figure 14 shows both transition systems. While $T_{int}(\mathcal{E})$ has two final states $TE_{int}(\mathcal{E})$ has just one final state. As the operators T_{step}, T_{pom}, TE_{step} and TE_{pom} lead to the same final states as T_{int} resp. TE_{int}

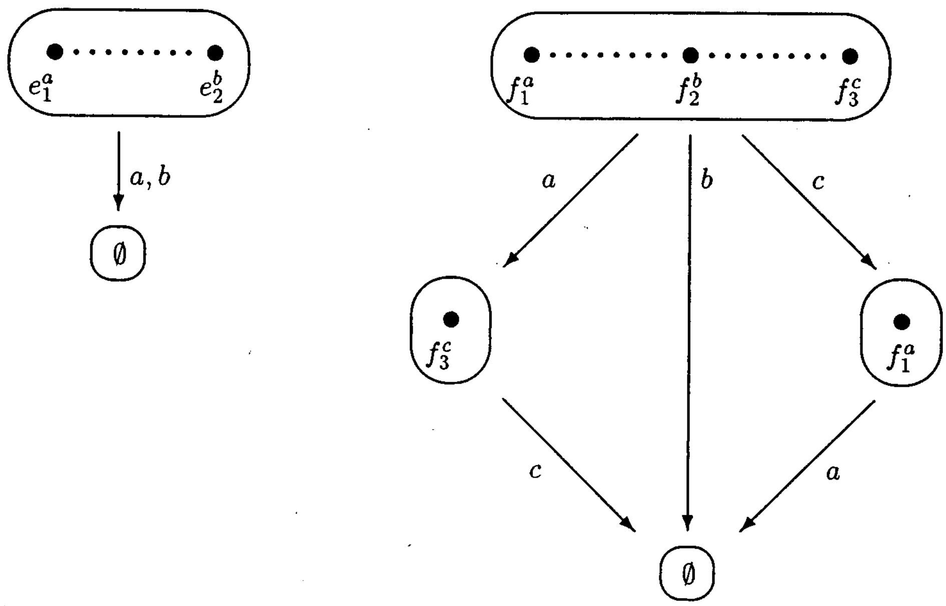


Figure 12: No morphism between $TE_{int}(\mathcal{E})$ and $TE_{int}(\mathcal{F})$.

we may conclude that in general transition systems of type T and of type TE are not isomorphic.

Theorem 4.6

Let $\mathcal{E}$ be an event structure. Then for the transition systems obtained by the operators of type T resp. TE holds: There exists an AM-bisimulation (R, γ) with $(\emptyset, \mathcal{E}) \in R$ between $T_*(\mathcal{E})$ and $TE_*(\mathcal{E})$, where $* \in \{int, step, pom\}$.

Proof: Let $\mathcal{E}$ be a prime event structure, $(A, \alpha) := T_*(\mathcal{E})$ and $(B, \beta) := TE_*(\mathcal{E})$, where $* \in \{int,\ step,\ pom\}$. Let $R := \{(X, \mathcal{E}') \mid X \in Conf(\mathcal{E}),\ \mathcal{E}' = \mathcal{E} \backslash X\}$, let for all $(Y, \mathcal{E}''), (X, \mathcal{E}') \in R$

$$(l, Y, \mathcal{E}'') \in \gamma(X, \mathcal{E}') :\Longleftrightarrow (l, Y) \in \alpha(X),\ (l, \mathcal{E}'') \in \beta(\mathcal{E}'),$$

where l is a label of appropriate type, i.e. an action, a multiset or a pomset. We claim that (R, γ) is an AM-bisimulation between (A, α) and (B, β). As the transition relations of (A, α) and (B, β) are defined in a different way, this time we have to prove

(i) $(\alpha \circ \pi_1) \subseteq (F\pi_1 \circ \gamma)$,

(ii) $(F\pi_1 \circ \gamma) \subseteq (\alpha \circ \pi_1)$,

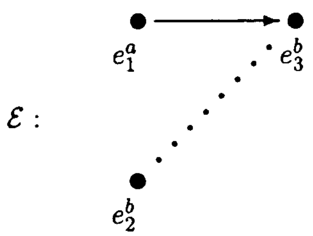


Figure 13: The event structure $\mathcal{E}$.

(iii) $(\beta \circ \pi_2) \subseteq (F\pi_2 \circ \gamma)$ and

(iv) $(F\pi_2 \circ \gamma) \subseteq (\beta \circ \pi_2)$.

Let $(l, Y) \in (\alpha \circ \pi_1)(X, \mathcal{E}')$. Then $(l, Y) \in \alpha(X)$ and $\mathcal{E}' = \mathcal{E}\backslash X$. Define $\mathcal{E}'' := \mathcal{E}\backslash Y$. Part 7 of lemma 4.1 provides $(l, \mathcal{E}'') \in \beta(\mathcal{E}')$. This results in $(l, Y, \mathcal{E}'') \in \gamma(X, \mathcal{E}')$ and we may conclude $(l, Y) \in (F\pi_1 \circ \gamma)(X, \mathcal{E}')$.

Let $(l, Y) \in (F\pi_1 \circ \gamma)(X, \mathcal{E}')$. Then there exists some event structure $\mathcal{E}''$ such that $(l, Y, \mathcal{E}'') \in \gamma(X, \mathcal{E}')$. By definition of γ we may conclude $(l, Y) \in \alpha(X)$ and get: $(l, Y) \in (\alpha \circ \pi_1)(X, \mathcal{E}')$.

Let $(l, \mathcal{E}'') \in (\beta \circ \pi_2)(X, \mathcal{E}')$. Then we get $(l, \mathcal{E}'') \in \beta(\mathcal{E}')$, i.e. there exists some configuration $Y \in Conf(\mathcal{E}')$ such that $\mathcal{E}'' = \mathcal{E}'\backslash Y$. With lemma 4.1, part 1, we know that $X \cup Y \in Conf(\mathcal{E})$ and $\mathcal{E}'' = \mathcal{E}\backslash(X \cup Y)$. Thus we obain $(X \cup Y, \mathcal{E}'') \in R$. By definition of the operators T_* we get $(l, X \cup Y) \in \alpha(X)$ and therefore $(l, X \cup Y, \mathcal{E}'') \in \gamma(X, \mathcal{E}')$. Thus we have: $(l, \mathcal{E}'') \in (F\pi_2 \circ \gamma)(X, \mathcal{E}')$.

Let $(l, \mathcal{E}'') \in (F\pi_2 \circ \gamma)(X, \mathcal{E}')$. Then there exists some configuration $Y \in Conf(\mathcal{E})$ such that $(l, Y, \mathcal{E}'') \in \gamma(X, \mathcal{E}')$. This implies $(l, \mathcal{E}'') \in \beta(\mathcal{E}')$ and we get: $(l, \mathcal{E}'') \in (\beta \circ \pi_2)(X, \mathcal{E}')$.

We finally remark that by definition holds: $(\emptyset, \mathcal{E}) \in R$. ■

Remark 4.7

The projections $\pi_1 : R \to A$ and $\pi_2 : R \to B$ in the proof of theorem 4.3 are both surjective. π_1 is even injective: If $\pi_1(X, \mathcal{E}\backslash X) = \pi_2(Y, \mathcal{E}\backslash Y)$ then $X = Y$, therefore $\mathcal{E}\backslash X = \mathcal{E}\backslash Y$ and thus $(X, \mathcal{E}\backslash X) = (Y, \mathcal{E}\backslash Y)$.

Example 4.8

As the event structure from figure 13 shows there is in general no strong AM-bisimulation with $(\emptyset, \mathcal{E}) \in R$ between $T_*(\mathcal{E})$ and $TE_*(\mathcal{E})$, see figure 14: Any AM-bisimulation between $T_{int}(\mathcal{E})$ and $TE_{int}(\mathcal{E})$ with $(\emptyset, \mathcal{E}) \in R$ has to contain $(\{e_1, e_3\}, \emptyset\}$. In $TE_{int}(\mathcal{E})$ there are two

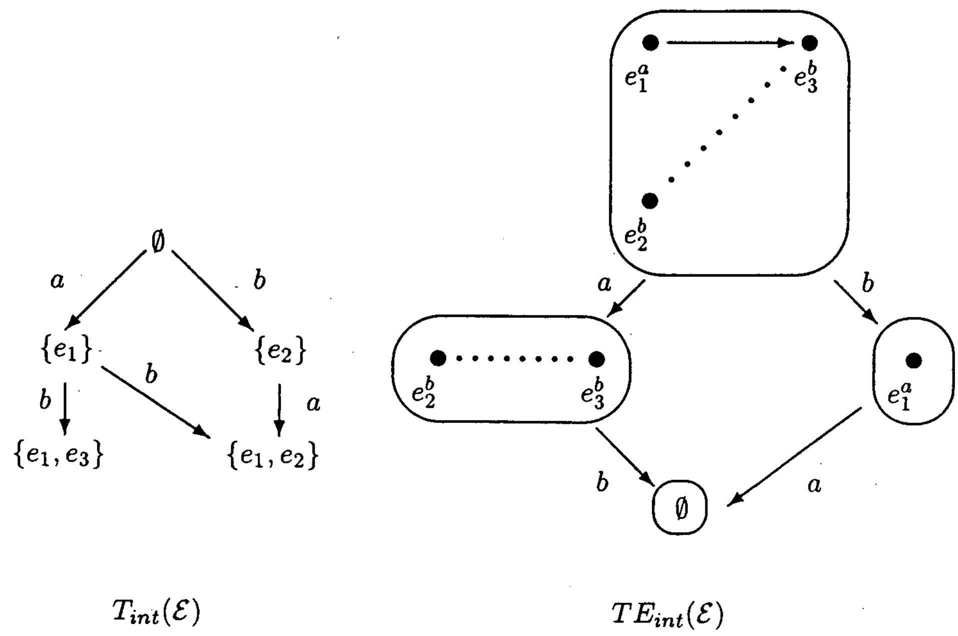


Figure 14: $T_{int}(\mathcal{E})$ and $TE_{int}(\mathcal{E})$ for the event structure from figure 13.

arcs leading to the state $\emptyset$ *: one labelled with "a", the other labelled with "b". But the state* $\{e_1, e_3\}$ *of* $T_{int}(\mathcal{E})$ *is reachable only with a transition labelled "b".*

In order to model some kind of bisimulation on event structures as AM-bisimulation theorem 4.6 together with the transitivity result of lemma 2.4 says that it is enough to study just one type of transition systems related to prime event structures: Let $\mathcal{E}$ and $\mathcal{F}$ be event structures, let (R, γ) be an AM-bisimulation between $T_*(\mathcal{E})$ and $T_*(\mathcal{F})$ with $(\emptyset, \emptyset) \in R$, where $* \in \{\text{int, step, pom}\}$. By theorem 4.6 we know that there exists AM-bisimulations (S_1, σ_1) and (S_2, σ_2) between $T_*(\mathcal{E})$ and $TE_*(\mathcal{E})$ with $(\emptyset, \mathcal{E}) \in S_1$ resp. between $T_*(\mathcal{F})$ and $TE_*(\mathcal{F})$ with $(\emptyset, \mathcal{F}) \in S_2$. Applying lemma 2.4 on these three AM-bisimulations we get an AM-bisimulation $(\hat{R}, \hat{\gamma})$ between $TE_*(\mathcal{E})$ and $TE_*(\mathcal{F})$ with $(\mathcal{E}, \mathcal{F}) \in \hat{R}$. The same holds for the other direction. But as example 4.8 shows it is necessary to study both types of operators, if we deal with strong AM-bisimulation.

4.2.2 Modelling interleaving, bf, step and pomset bisimulation as AM-bisimulation

With lemma 2.3 we can translate the definition of interleaving, step, pomset and bf-bisimu-lation directly into AM-bisimulations:

Theorem 4.9

For event strucures $\mathcal{E}$, $\mathcal{F}$ holds:

1. *$\mathcal{E}$ and $\mathcal{F}$ are interleaving bisimilar iff there exists an AM-bisimulation (R, γ) between $T_{int}(\mathcal{E})$ and $T_{int}(\mathcal{F})$ with $(\emptyset, \emptyset) \in R$.*

2. *$\mathcal{E}$ and $\mathcal{F}$ are bf-bisimilar iff there exists a strong AM-bisimulation (R, γ) between $T_{int}(\mathcal{E})$ and $T_{int}(\mathcal{F})$ with $(\emptyset, \emptyset) \in R$.*

3. *$\mathcal{E}$ and $\mathcal{F}$ are step bisimilar iff there exists an AM-bisimulation (R, γ) between $T_{step}(\mathcal{E})$ and $T_{step}(\mathcal{F})$ with $(\emptyset, \emptyset) \in R$.*

4. *$\mathcal{E}$ and $\mathcal{F}$ are pomset bisimilar iff there exists an AM-bisimulation (R, γ) between $T_{pom}(\mathcal{E})$ and $T_{pom}(\mathcal{F})$ with $(\emptyset, \emptyset) \in R$.*

Proof: We prove the theorem only for interleaving and bf-bisimulation.

Let $\mathcal{E}$ and $\mathcal{F}$ be interleaving bisimilar. Then there exists an interleaving bisimulation $R \subseteq Conf(\mathcal{E}) \times Conf(\mathcal{F})$. Let $T_{int}(\mathcal{E}) = (Conf(\mathcal{E}), \alpha)$ and $T_{int}(\mathcal{F}) = (Conf(\mathcal{F}), \beta)$ be the related coalgebras. Let for all (X, Y), $(X', Y') \in R$

$$(a, X', Y') \in \gamma(X, Y) :\Longleftrightarrow (a, X') \in \alpha(X),\ (a, Y') \in \beta(Y).$$

Let $(a, X') \in (F\pi_1 \circ \gamma)(X, Y)$. Then there exists Y' such that $(a, X', Y') \in \gamma(X, Y)$. By the definition of γ this implies $(a, X') \in \alpha(X)$ and hence $(a, X') \in (\alpha \circ \pi_1)(X, Y)$. To prove the inclusion the other way round let $(a, X') \in (\alpha \circ \pi_1)(X, Y)$. This implies $(a, X') \in \alpha(X)$ and as R is an interleaving bisimulation there exists some Y' such that $(a, Y') \in \beta(Y)$ and $(X', Y') \in R$. Therefore we get $(a, X', Y') \in \gamma(X, Y)$ and finally $(a, X') \in (F\pi_1 \circ \gamma)(X, Y)$. Lemma 2.3 proves the other implication.

Now let R be an bf-bisimulation between $\mathcal{E}$ and $\mathcal{F}$. Then R is especially an interleaving bisimulation and from the prove above we know that there exists a map γ such that (R, γ) is an AM-bisimulation. We claim that (R, γ) is strong. Due to lemma 2.1 we have only to prove that $(\alpha^- \circ \pi_1) \subseteq (F\pi_1 \circ \gamma^-)$.

Let $(X', Y') \in R$, let $(a, X) \in (\alpha^- \circ \pi_1)(X', Y')$. Then $(a, X) \in \alpha^-(X')$ and thus $(a, X') \in \alpha(X)$. As R is a bf-bisimulation this implies that there exists $Y \in Conf(\mathcal{F})$ with $(a, Y') \in \beta(Y)$ and $(X, Y) \in R$. By definition of γ we get $(a, X', Y') \in \gamma(X, Y)$, therefore $(a, X, Y) \in \gamma^-(X', Y')$ and finally $(a, X) \in (F\pi_1 \circ \gamma^-)(X', Y')$.

Let (R, γ) be a strong AM-bisimulation between $\mathcal{E}$ and $\mathcal{F}$ with $(\emptyset, \emptyset) \in R$. Then we know that R is an interleaving bisimulation between $\mathcal{E}$ and $\mathcal{F}$. In order to prove that R is a bf-bisimulation consider a transition $(a, X') \in \alpha(X)$ in $T_{int}(\mathcal{E})$, where $(X', Y') \in R$. We

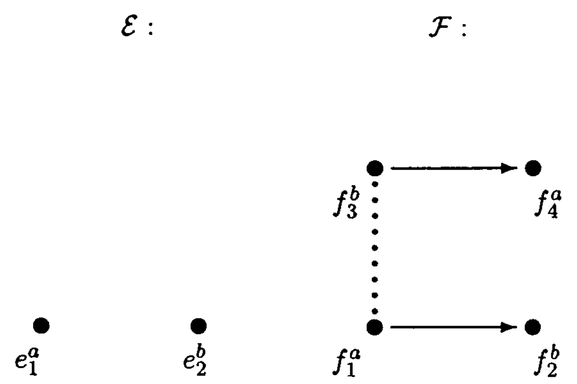


Figure 15: Two interleaving bisimilar event structures.

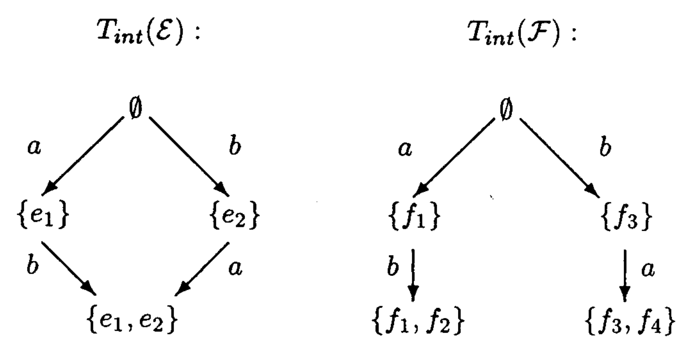


Figure 16: The transition systems $T_{int}(\mathcal{E})$ and $T_{int}(\mathcal{F})$.

get $(a, X) \in \alpha^{-}(X')$ and $(a, X) \in (\alpha^{-} \circ \pi_1)(X', Y')$. As (R, γ) is strong we may conclude: $(a, X) \in (F\pi_1 \circ \gamma^{-})(X', Y')$. Thus there exists some configuration $Y \in Conf(\mathcal{F})$ such that $(a, X, Y) \in \gamma^{-}(X', Y')$. This implies $(X, Y) \in R$. Using again the property strong of (R, γ) we get $(a, Y) \in \beta^{-}(Y')$ and finally $(a, Y') \in \beta(Y)$. ■

Remark 4.10

It is important to note that bf-bisimulation cannot be modelled as AM-bisimulation using the operator TE_{int} :

Consider the event structures $\mathcal{E}$ and $\mathcal{F}$ of figure 15. Figure 16 shows the transition systems which one obtains by applying the operator T_{int}. They are isomorphic to the transition systems we discussed in example 2.6 to show that strong and non strong AM-bisimulation

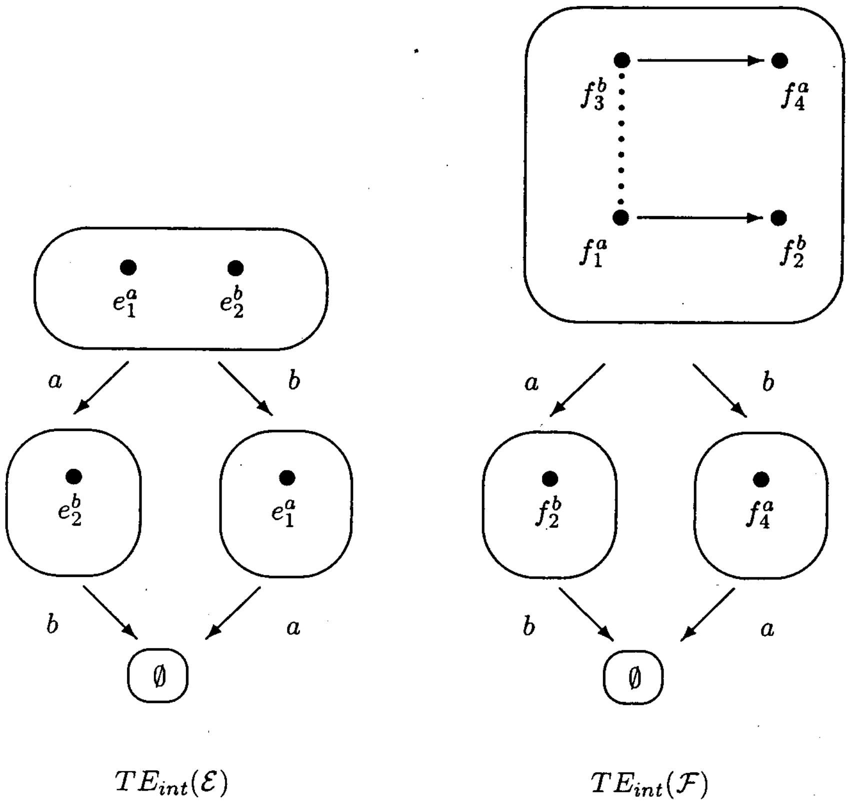


Figure 17: $TE_{int}(\mathcal{E})$ and $TE_{int}(\mathcal{F})$ for the event structures from figure 15.

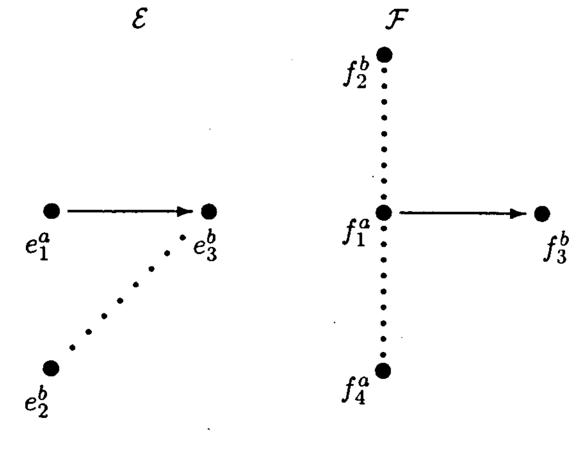


Figure 18: Pomset-bisimilar event structures.

are different concepts. Therefore $\mathcal{E}$ and $\mathcal{F}$ are interleaving bisimilar but not bf-bisimilar. Now consider the transition systems $TE_{int}(\mathcal{E})$ and $TE_{int}(\mathcal{F})$. As figure 17 shows they are isomorphic and thus there exists a strong AM-bisimulation between them.

Two event structures, which are interleaving, step or pomset bisimilar, are in general not strong AM-bisimilar:

Example 4.11

Take as event structeres $\mathcal{E}$ and $\mathcal{F}$ from figure 18. The set

$$R := \{ (\emptyset,\emptyset), (\{e_1\},\{f_1\}), (\{e_1\},\{f_4\}), (\{e_2\},\{f_2\}), \\ (\{e_1,e_3\},\{f_1,f_3\}), (\{e_1,e_2\},\{f_1,f_3\}), (\{e_1,e_3\},\{f_2,f_4\}), (\{e_1,e_2\},\{f_2,f_4\})\}$$

is a pomset bisimulation between $\mathcal{E}$ and $\mathcal{F}$ and therefore equally a step and an interleaving bisimulation (this bisimulation-example is due to [Vog93]).

Let $(A,\alpha) = T_*(\mathcal{E})$, $(B,\beta) = T_*(\mathcal{F})$, where $* \in \{int, step, pom\}$. Let (R,γ) be some AM-bisimulation between (A,α) and (B,β). As we find the transition $\emptyset \xrightarrow{a} \{f_1\}$ in (B,β), we get $(\{e_1\},\{f_1\}) \in R$. In (A,α) we find the transition $\{e_1\} \xrightarrow{b} \{e_1,e_2\}$. Therefore R has to contain $(\{e_1,e_2\},\{f_1,f_3\})$.

Assume that (R,γ) is strong. As in (A,α) we find the transition $\{e_2\} \xrightarrow{a} \{e_1,e_2\}$ we get by $(\alpha^- \circ \pi_1)(\{e_1,e_2\},\{f_1,f_3\}) = (F\pi_1 \circ \gamma^-)(\{e_1,e_2\},\{f_1,f_3\})$ a transition $(a,\{e_2\},Y) \in \gamma^-(\{e_1,e_2\},\{f_1,f_3\})$ and thus a transition $Y \xrightarrow{a} \{f_1,f_3\}$ in (B,β). But there is no such configuration $Y \in B$. Therefore no AM-bisimulation between $\mathcal{E}$ and $\mathcal{F}$ can be strong.

4.2.3 Modelling interleaving, step and pomset bisimulation in the framework of [JNW94]

[JNW94] studied history preserving and strong history preserving bisimulation on event structures to show the suitability of their framework. We show in this section that interleaving bisimulation and step bisimulation can also be modelled by choosing suitable subcategories $\mathbb{P}$. To obtain the result of step bisimulation we use theorem 4.9 and theorem 3.6. We also discuss pomset bisimulation.

Theorem 4.12

Two event structures $\mathcal{E}_1$ and $\mathcal{E}_2$ are **Lin**-*bisimilar iff they are interleaving bisimilar.*

Proof: Let $\mathcal{E}_1 = (E_1, \leq_1, \sharp_1, l_1)$ and $\mathcal{E}_2 = (E_2, \leq_2, \sharp_2, l_2)$ be **Lin**-bisimilar, i.e. there exists an event structure $\mathcal{E} = (E, \leq, \sharp, l)$ and **Lin**-open maps $p_i : \mathcal{E} \to \mathcal{E}_i$, $i = 1, 2$. We claim that

$$R := \{(p_1(X), p_2(X)) \mid X \in Conf(\mathcal{E})\}$$

is an interleaving bisimulation between $\mathcal{E}_1$ and $\mathcal{E}_2$. As $\emptyset \in Conf(\mathcal{E})$ we have $(\emptyset, \emptyset) \in R$.

Consider the element $(p_1(X), p_2(X))$ of R for some configuration $X \in Conf(\mathcal{E})$. Let $p_1(X) \stackrel{a}{\longrightarrow} Y'$ be a transition in $T_{int}(\mathcal{E}_1)$.

Make from the configuration $p_1(X) \in Conf(\mathcal{E}_1)$ an event structure $\mathcal{P} = (P, \leq_P, \sharp_P, l_P)$ as follows: $P := X$, $\leq_P$ is a linearization of $\leq_1 \cap (X \times X)$, $\sharp_P := \emptyset$ and $l_p := l_{1|X}$. Let $\hat{e}$ be the event in which $p_1(X)$ and Y' differ, i.e. $\{\hat{e}\} = Y' \backslash p_1(X)$. Let $\mathcal{Q} := (Q, \leq_Q, \emptyset, l_Q)$, where $Q := P \cup \{\hat{e}\}$, let $\forall e \in Q: \ e \leq_Q \hat{e}$ and $\forall e, f \in P : e \leq_Q f :\iff e \leq_P f$, $\sharp_Q := \emptyset$ and $\forall e \in P : l_Q(e) := l_P(e)$ and $l_Q(\hat{e}) := a$. Obviously $\mathcal{P}$ and $\mathcal{Q}$ are objects of **Lin**.

We define morphism $p : \mathcal{P} \to \mathcal{E}$, $m : \mathcal{P} \to \mathcal{Q}$ and $q : \mathcal{Q} \to \mathcal{E}_1$ by:

- $\forall e \in P: \ p(e) := e$,
- $\forall e \in P: \ m(e) := e$ and
- $\forall e \in P: \ q(e) := p_1(e), \ q(\hat{e}) = \hat{e}$.

Obviously we have: $p_1 \circ p = q \circ m$. As p_1 is **Lin**-open, there exists a morphim $r : \mathcal{Q} \to \mathcal{E}$ such that $r \circ m = p$ and $p_1 \circ r = q$. Therefore $Y := r(Q) = X \cup \{r(\hat{e})\} \in Conf(\mathcal{E})$, $p_1(Y) = Y'$ and $X \stackrel{a}{\longrightarrow} Y$ is a transition of $T_{int}(\mathcal{E})$. Therefore $p_2(X) \stackrel{a}{\longrightarrow} p_2(Y)$ is a transition of $T_{int}(\mathcal{E}_2)$. Furtheron by definition of R we have $(p_1(Y), p_2(Y)) = (Y', p_2(Y)) \in R$.

Now let $\mathcal{E}_1$ and $\mathcal{E}_2$ be interleaving bisimilar. In theorem 4.9 we constructed an AM-bisimulation (R, γ) between $T_{int}(\mathcal{E}_1)$ and $T_{int}(\mathcal{E}_2)$ with $(\emptyset, \emptyset) \in R$. We claim that unfolding this coalgebra (R, γ) to a tree S and constructing from S an event structure $\mathcal{E}$ with morphism $p_i : \mathcal{E} \to \mathcal{E}_i$, $i = 1, 2$, makes a $\mathcal{E}_1$ and $\mathcal{E}_2$ **Lin**-bisimilar.

First unfold (R,γ) interpreted as transition system with initial state $(\emptyset,\emptyset)$ to a tree $S=(S_1,i_1,Tran_1)$: Take as states all nonempty, finite sequences of elements of R, i.e. the one element sequence $\langle(\emptyset,\emptyset)\rangle$ is the initial state i_1 of S, a sequence

$$\langle(X_1,Y_1),(X_2,Y_2),\ldots,(X_n,Y_n)\rangle$$

is an element of S_1 iff $(X_1,Y_1)\xrightarrow{a_1}(X_2,Y_2)\xrightarrow{a_2}\ldots\xrightarrow{a_{n-1}}(X_n,Y_n)$ is a derivation in (R,γ) and $(X_1,Y_1)=(\emptyset,\emptyset)$. There is a transition

$$\langle(X_1,Y_1),(X_2,Y_2),\ldots,(X_n,Y_n)\rangle\xrightarrow{a}\langle(X_1,Y_1),(X_2,Y_2),\ldots,(X_n,Y_n),(X_{n+1},Y_{n+1})\rangle$$

in $Tran_1$ between two states of S iff $(X_n,Y_n)\xrightarrow{a}(X_{n+1},Y_{n+1})$ in (R,γ).

Now construct from $S=(S_1,i_1,Tran_1)$ an event structure $\mathcal{E}=(E,\leq,\sharp,l)$. Define

- $E := S_1\backslash\{i_1\}$,
- $e\leq f :\Longleftrightarrow (e,f)\in Tran_1^*$, where $Tran_1^*$ denotes the reflexive transitive closure of $\{(e,f)\,|\,(e,a,f)\in Tran_1 \text{ for some label } a\}$,
- $e\sharp f :\Longleftrightarrow \neg(e\leq f\vee f\leq e)$ and
- $l(e)=a :\Longleftrightarrow e=\langle(X_1,Y_1),(X_2,Y_2),\ldots,(X_n,Y_n)\rangle\wedge(X_{n-1},Y_{n-1})\xrightarrow{a}(X_n,Y_n)$.

Define maps $p_1:\mathcal{E}\to\mathcal{E}_1$, $p_2:\mathcal{E}\to\mathcal{E}_2$ by

- $p_1(\,\langle(X_1,Y_1),(X_2,Y_2),\ldots,(X_n,Y_n)\rangle\,) := e$ iff $\{e\}=X_n\backslash X_{n-1}$ and
- $p_2(\,\langle(X_1,Y_1),(X_2,Y_2),\ldots,(X_n,Y_n)\rangle\,) := e$ iff $\{e\}=Y_n\backslash Y_{n-1}$.

We claim that p_1 and p_2 are **Lin**-open morphisms.

By the construction of (R,γ) we get: $(X,Y)\xrightarrow{a}(X',Y')$ implies $X\xrightarrow{a}X'$ and $Y\xrightarrow{a}Y'$. Therefore p_1 and p_2 preserve labels. As all events of $\mathcal{E}$ are in conflict iff they are not related by $\leq_E$ a configuration C with $n\geq 1$ elements of $\mathcal{E}$ is a set

$$\begin{aligned}C=\{\ &\langle(\emptyset,\emptyset),(X_2,Y_2)\rangle,\\ &\langle(\emptyset,\emptyset),(X_2,Y_2),(X_3,Y_3)\rangle,\\ &\ldots\\ &\langle(\emptyset,\emptyset),(X_2,Y_2),(X_3,Y_3),\ldots,(X_{n+1},Y_{n+1})\rangle\}.\end{aligned}$$

Applying p_1 on such a configuration $C\in Conf(\mathcal{E})$ results in:

$$p_1(C)=\bigcup_{i=2}^{n+1}X_i\backslash X_{i-1}=X_{n+1}\in Conf(\mathcal{E}_1).$$

Let e, e' be events of a configuration $C \in Conf(\mathcal{E})$ with $p_1(e) = p_2(e')$. Then there exist configurations $X_1 \subsetneq X_2$ and $X_3 \subsetneq X_4$ of $\mathcal{E}_1$ with $p_1(e) = p_1(e') = \hat{e}$, where $X_2 \backslash X_1 = X_4 \backslash X_3 = \{\hat{e}\}$. Let w.o.l.g. $e \leq_E e'$. Assume $e \neq e'$. Then $X_2 \subseteq X_3$ and $p_1(e) = p_1(e') \in X_3 \subseteq X_4$ – contradiction to $p_1(e') \in X_4 \backslash X_3$. Thus $e = e'$ and we may conclude that p_1 and p_2 are morphisms.

Finally we prove that p_1 is **Lin**-open. Let $\mathcal{P} = (P, \leq_P, \emptyset, l_P)$ and $\mathcal{Q} = (Q, \leq_Q, \emptyset, l_Q)$ be objects of **Lin**, let $p : \mathcal{P} \to \mathcal{E}$, $m : \mathcal{P} \to \mathcal{Q}$, $q : \mathcal{Q} \to \mathcal{E}_1$ be morphism with $q \circ m = p_1 \circ p$. In case of $P = Q = \emptyset$ there exists obviously a morphism $r : \mathcal{Q} \to \mathcal{E}$ with $p = r \circ m$ and $q = p_1 \circ r$. Thus we assume $P, Q \neq \emptyset$. We prove the existence of the morphism $r : \mathcal{Q} \to \mathcal{E}$ by induction on the difference $n := |Q| - |P|$.

If $n = 0$ the morphism m is bijective: $P \in Conf(\mathcal{P})$, m restricted to configurations is an injection, and as $|P| = |Q|$ the morphism m is also surjective. As the map m^{-1} preserves labels, maps configurations of $\mathcal{Q}$ on configurations of $\mathcal{P}$ and is injective on Q it is especially a morphism in $\mathbf{E}_{Act}$. Thus we may define $r := p \circ m^{-1}$ and get: $r \circ m = p \circ m^{-1} \circ m = p$ and $p_1 \circ r = p_1 \circ p \circ m^{-1} = q$, as $q \circ m = p_1 \circ p$.

Now let $|Q| - |P| = n + 1$. Let $\hat{e}$ be the largest event of Q. Let $\mathcal{Q}' := (Q', \leq', \emptyset, l')$ with $Q' := Q \backslash \{\hat{e}\}$, $\leq' := \leq_Q \cap (Q' \times Q')$, $l' := l_{Q|Q'}$. Let $m' : \mathcal{P} \to \mathcal{Q}'$ the morphism with $m'(e) := m(e)$ for all $e \in P$ and $q' : \mathcal{Q}' \to \mathcal{E}_1$ be the morphims with $q'(e) := q(e)$ for all $e \in Q'$. Then obviously $q' \circ m' = p_1 \circ p$ and thus by induction hypothesis there exists a morphism $r' : \mathcal{Q}' \to \mathcal{E}$ with $p = r' \circ m'$ and $q' = p_1 \circ r'$.

Consider the image of Q' under the morphism r' : This is a configuration C of $\mathcal{E}$ and has therefore the form

$$\begin{aligned} C = \{ \ & \langle(\emptyset,\emptyset),(X_2,Y_2)\rangle, \\ & \langle(\emptyset,\emptyset),(X_2,Y_2),(X_3,Y_3)\rangle, \\ & \ldots \\ & \langle(\emptyset,\emptyset),(X_2,Y_2),(X_3,Y_3),\ldots,(X_{k+1},Y_{k+1})\rangle\}, \end{aligned}$$

where $k = |Q'|$, $p_1(C) = X_{k+1}$ and $q'(Q) = p_1(r'(Q)) = X_{k+1}$. The transition $Q' \xrightarrow{a} Q$ in $T_{int}(\mathcal{Q})$ implies that there is a transition $q(Q') = X_{k+1} \xrightarrow{a} q(Q)$ in $T_{int}(E_1)$. As R is an interleaving bisimulation and $(X_{k+1}, Y_{k+1}) \in R$ there exists a configuration $Y' \in C(\mathcal{E}_2)$ such that $(q(Q), Y') \in R$ and $Y_{k+1} \xrightarrow{a} Y'$ is a transition in $T_{int}(\mathcal{E}_2)$. Thus by definiton of γ we get the transition $(X_{k+1}, Y_{k+1}) \xrightarrow{a} (q(Q), Y')$ in (R, γ) and therefore an event

$$f := \langle(\emptyset,\emptyset),(X_2,Y_2),(X_3,Y_3),\ldots,(X_{k+1},Y_{k+1}),((q(Q),Y')\rangle$$

in the event structure $\mathcal{E}$. Define $\forall e \in Q' : r(e) := r'(e)$ and $r(\hat{e}) := f$. This map r is the desired morphism. ■

Categorical characterization of bisimulation

Corollary 4.13

Let $\mathcal{E}_1$, $\mathcal{E}_2$ be event structures in $\mathbf{E}_{Act}$. The following are equivalent:

1. *$\mathcal{E}_1$ and $\mathcal{E}_2$ are interleaving-bisimilar.*
2. *There exists an AM-bisimulation (R,γ) between $T_{int}(\mathcal{E}_1)$ and $T_{int}(\mathcal{E}_2)$ with $(\emptyset,\emptyset) \in R$.*
3. *There exists an AM-bisimulation (R,γ) between $T_{path-\mathbf{Lin}}(\mathcal{E}_1)$ and $T_{path-\mathbf{Lin}}(\mathcal{E}_2)$ with $(\iota_1,\iota_2) \in R$.*
4. *There exists a strong AM-bisimulation (R,γ) with $(\iota_1,\iota_2) \in R$ between $T_{path-\mathbf{Lin}}(\mathcal{E}_1)$ and $T_{path-\mathbf{Lin}}(\mathcal{E}_2)$.*
5. *$\mathcal{E}_1$ and $\mathcal{E}_2$ are path-**Lin**-bisimilar.*
6. *$\mathcal{E}_1$ and $\mathcal{E}_2$ are strong path-**Lin**-bisimilar.*
7. *$\mathcal{E}_1$ and $\mathcal{E}_2$ are **Lin**-bisimilar.*

Proof: Theorem 4.9 proves the equivalence of 1 and 2, applying theorem 3.6 results in the equivalence of 2 and 5. In theorem 4.12 we find the equivalence of 1 and 7, with theorem 3.1 we may conclude that 7 implies 6. Thus 5 and 6 are equivalent. Theorem 3.3 gives us the equivalences between 3 and 5 resp. 4 and 6. ■

One should note that the strong AM-bisimulation of corollary 4.13 is between transition systems of type $T_{path-\mathbf{Lin}}$ and *not* of type T_{int}.

Looking on step bisimulation we get the following characterization in terms of path-$\mathbb{P}$-bisimulation:

Corollary 4.14

*Two event structures of $\mathbf{E}_{Act}$ are step bisimilar iff they are path-**Step**-bisimilar.*

Proof: We prove the equivalence using the characterization of step bisimulation in terms of AM-bisimulation in theorem 4.9. On this AM-bisimulation we apply theorem 3.6 in order to translate it into a path-**Step**-bisimulation. Therefore we have only to show that all six conditions are fullfilled. We choose $\mathbf{Set}_F$ with $F = \mathcal{P}(\mathbb{N}_0^{Act} \times \_)$, $\mathbf{T}_{\mathbb{N}_0^{Act}}$ as the above described "link"-category, $\mathbb{M}=\mathbf{E}_{Act}$ and $\mathbb{P}=\mathbf{Step}$.

Condition 1: Let $mt\,\mathcal{E} := T_{step}(\mathcal{E})$. Let $f : \mathcal{E} \to \mathcal{F}$ be a morphism between event structures. Define $mt\,f : mt\,\mathcal{E} \to mt\,\mathcal{F}$, $X \mapsto f(X)$, where $X \in Conf(\mathcal{E})$. Let $X \xrightarrow{M} X'$ be a transition in $T_{step}(\mathcal{E})$. As f is an event structure morphism, $f(X)$, $f(X') \in Conf(\mathcal{F})$ and f is injective on X'. Therefore we find a transition $f(X) \xrightarrow{M} f(X')$ in $T_{step}(\mathcal{F})$. As furtheron $(mt\,f)(\emptyset) = \emptyset$ we may conclude that $mt\,f$ is a morphims in $\mathbf{T}_{\mathbb{N}_0^{Act}}$.

Obviously we have $mt\,id_{\mathcal{E}} = id_{mt\,\mathcal{E}}$, where $\mathcal{E}$ is an event structure from $\mathbf{E}_{Act}$, and $mt(f \circ g) = (mt\,f) \circ (mt\,g)$ for all morphism $f : \mathcal{E}_2 \to \mathcal{E}_3$, $g : \mathcal{E}_1 \to E_2$ of $\mathbf{E}_{Act}$.

Condition 2: Take as initial object I the empty event structure. As $\mathbf{E}_{Act}$ is small so is **Step**.

Condition 3: Let $\mathcal{S} = (S, \leq, \sharp, l) = \mathcal{M}_1; \mathcal{M}_2; \ldots; \mathcal{M}_n$, $n \geq 0$, be a step, where $\mathcal{M}_i = (M_i, \leq_{M_i}, \emptyset, l_i)$. S is the final state of $T_{step}(\mathcal{S})$. Let X be a configuration of $\mathcal{S}$. Then holds: $S \backslash X = R \cup \bigcup_{i=k+1}^{n} M_i$, $R \subseteq M_k$, for some $k \in \{1, 2, \ldots, n\}$. Let $A(a) := |\{e \in R \,|\, l(e) = a\}|$, $A_i(a) := |\{e \in M_{i+1} \,|\, l(e) = a\}|$, $i = k, k+1, \ldots, n-1$, $a \in Act$, be multisets over Act. Then $X \xrightarrow{A} \bigcup_{i=1}^{k} M_i \xrightarrow{A_k} \bigcup_{i=1}^{k+1} M_i \xrightarrow{A_k} \ldots \xrightarrow{A_n} S$ is a derivation from X to S in $T_{step}(\mathcal{S})$.

Condition 4: Let $s_1 \xrightarrow{A_1} s_2 \xrightarrow{A_2} \ldots \xrightarrow{A_{n-1}} s_n$ be a derivation in some transition system in $\mathbf{T}$. Let $\mathcal{S} = (S, \leq, \sharp, l) = \mathcal{M}_1; \mathcal{M}_2; \ldots; \mathcal{M}_{n-1}$, $n \geq 1$, be a step, where $\mathcal{M}_i = (M_i, \leq_{M_i}, \emptyset, l_i)$, $\leq_{M_i} = \{(m, m) \,|\, m \in M_i\}$, M_i pairwise disjoint, $\forall a \in Act : A_i(a) = |\{e \in M_i \,|\, l_i(e) = a\}|$. In $T_{step}(\mathcal{S})$ we find the derivation $\emptyset \xrightarrow{A_1} M_1 \xrightarrow{A_2} M_1 \cup M_2 \xrightarrow{A_3} \ldots \xrightarrow{A_{n-1}} S$, where $\emptyset$ is the initial and S the final state of $mt\,\mathcal{S}$.

Let $\mathcal{E} = (E, \leq_E, \sharp_E, l_E)$ be an event structure from $\mathbf{E}_{Act}$ with a derivation $X_1 = \emptyset \xrightarrow{A_1} X_2 \xrightarrow{A_2} X_3 \xrightarrow{A_3} \ldots \xrightarrow{A_{n-1}} X_n$ in $mt\,\mathcal{E}$. For the above introduced M_i and $X_{i+1} \backslash X_i$ holds: $\forall a \in Act : A_i(a) = |\{e \in M_i \,|\, l_i(e) = a\}| = |\{e \in X_{i+1} \backslash X_i \,|\, l_E(e) = a\}|$. Thus there exist bijective mappings $p_i : M_i \to X_{i+1} \backslash X_i$, $i = 1, 2, \ldots, n-1$, with $l_E(p_i(e)) = l_i(e)$ for all $e \in M_i$. We claim that $p := \bigcup_{i=1}^{n-1} p_i$ is a morphism between $\mathcal{S}$ and $\mathcal{E}$. Obviously p fullfills the label condition and is injective on every configuration Y of $\mathcal{S}$. As X_n is conflictfree $p(Y) \subseteq X_n$ is conflictfree for all $Y \in Conf(\mathcal{S})$. Thus it remains to prove that the image of a configuration $Y \in Conf(\mathcal{S})$ is leftclosed in E. Let $e \in p(Y)$ for some configuration $Y \in Conf(\mathcal{S})$. Let $e' \leq_E e$. As X_n is leftclosed, we have $e' \in X_n$. $e' \leq_E e$ implies that for some $j \in \{1, 2, \ldots, n-1\}$ we have $e' \in X_j, e \notin X_j$. Thus for the elements $f, f' \in S$ with $p(f) = e$, $p(f') = e'$ holds: $f' \leq_S f$. As Y is a configuration we get $f' \in Y$ and therefore $p(f') = e' \in p(Y)$. Obviously holds: $\forall 1 \leq i \leq n : (mt\,p)(\bigcup_{j<i} M_i) = X_{i+1}$.

Condition 5: $mt\,I$ is the transition system with $\emptyset$ as its only state and no transition.

Condition 6: Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be steps, let $\mathcal{E}$ be an event structure, let $m : \mathcal{S}_1 \to \mathcal{S}_2$, $p : \mathcal{S}_1 \to \mathcal{E}$, and $q : \mathcal{S}_2 \to \mathcal{E}$ be morphisms. Let $\emptyset = X_0 \xrightarrow{A_1} X_1 \xrightarrow{A_2} X_2 \xrightarrow{A_3} \ldots \xrightarrow{X_{n-1}} X_n$ be a derivation in $\mathcal{S}_1$, where X_n is the final state of $\mathcal{S}_1$. Let $\forall 0 \leq i \leq n : (mt\,q \circ$

$mt\, m)(X_i) = (mt\, p)(X_i)$. This implies especially $(mt\, q \circ mt\, m)(X_n) = (mt\, p)(X_n)$ and therefore for all $e \in X_n$ we get: $(q \circ m)(e) = p(e)$. Thus $q \circ m = p$. ■

Example 4.15

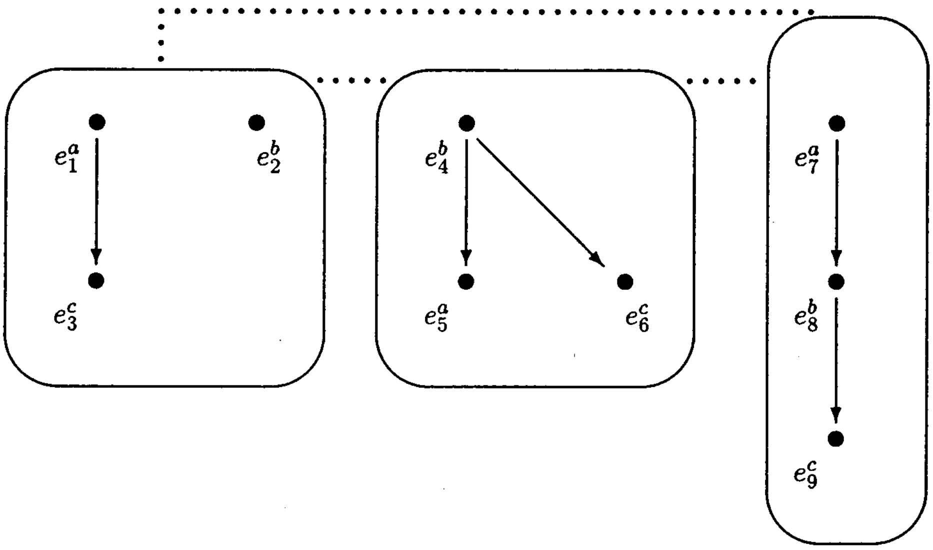
While for the path category **Lin** *the strong and the non strong version of path-$\mathbb{P}$-bisimulation coincide this does not hold for path-***Step***-bisimulation and strong path-***Step***-bisimulation: Consider the event structures $\mathcal{E}$ and $\mathcal{F}$ from figure 19. The dotted lines between the circles around the events mean that all events inside one circle are in conflict with all event inside the other circle. For example the events e_1, e_2 and e_3 are in conflict with all events e_i, where $i \geq 4$.*

*Figure 20 shows the transition systems $T_{step}(\mathcal{E})$ and $T_{step}(\mathcal{F})$. A label "a" - which we find for example in the transition system $T_{step}(\mathcal{E})$ on the arc from $\emptyset$ to $\{e_1\}$ - stands for the multiset M with $M(a) = 1$ and $M(x) = 0$ for all $x \in Act$ with $x \neq a$. Similarly a label "ab" - which we find for example in the transition system $T_{step}(\mathcal{E})$ on the arc from $\emptyset$ to $\{e_1, e_2\}$ - stands for the multiset M with $M(a) = 1$, $M(b) = 1$ and $M(x) = 0$ for all $x \in Act$ with $x \notin \{a, b\}$. Figure 21 shows an AM-bisimulation (R, γ) between $T_{step}(\mathcal{E})$ and $T_{step}(\mathcal{F})$ with $(\emptyset, \emptyset) \in R$. Thus $\mathcal{E}$ and $\mathcal{F}$ are step-bisimilar and by theorem 4.9 and corollary 4.14 we know that they are path-***Step***-bisimilar.*

*Assume that there exists a strong path-***Step*** bisimulation R between $\mathcal{E}$ and $\mathcal{F}$. Consider an event structure $O := (\{g_1, g_2\}, \emptyset, \emptyset, l_O)$ with two concurrent events g_1 and g_2, where $l_O(g_1) := a$, $l_O(g_2) := b$. Obviously O is a step. The maps $o_1 : O \to \mathcal{E}$ with $o_1(g_1) := e_1, o_2(g_2) := e_2$ and $o_2 : O \to \mathcal{F}$ with $o_1(g_1) := f_1, o_2(g_2) := f_2$ are morphisms in $\mathbf{E}_{Act}$. Thus $(o_1, o_2) \in R$ for any path-***Step***-bisimulation R. Let $P := (\{g'\}, \emptyset, \emptyset, l_P(g') := a)$ be an event structure with just one event labelled with a. Define a morphism $m_1 : P \to O$ with $m_1(g') := g_1$. As R is strong we get $(o_1 \circ m_1, o_2 \circ m_2) \in R$. Let $Q := (\{g_1'', g_2''\}, \leq_Q, \emptyset, l_Q)$ be an event structure with two events g_1'' and g_2'', where $l_Q(g_1'') := a$, $l_Q(g_2'') := c$ and $g_1'' \leq_Q g_2''$. Let finally $m_2 : P \to Q$ the morphism which maps the event g' on g_1''. For the morphism $q_1 : Q \to \mathcal{E}$ with $q_1(g_1'') := e_1$ and $q_1(g_2'') := e_3$ holds: $q_1 \circ m_2 = (o_1 \circ m_1)$, but there exists no morphism $q_2 : Q \to \mathcal{F}$ with $q_2(g_1'') = f_1$ - contradiction to the closure property (i) of path-$\mathbb{P}$-bisimulation.*

Concerning pomset bisimulation it is not possible to apply theorem 3.6 on the equivalent characterization as AM-bisimulation from theorem 4.9: The operator $T_{pom}(\mathcal{E})$ fails to evolve to a functor as one can see in example 4.2. This coincides with a result of [JNW94], which we will present in theorem 4.16: Path-**Pom**-bisimulation is equivalent with history preserving bisimulation.

The event structure $\mathcal{E}$:



The event structure $\mathcal{F}$:

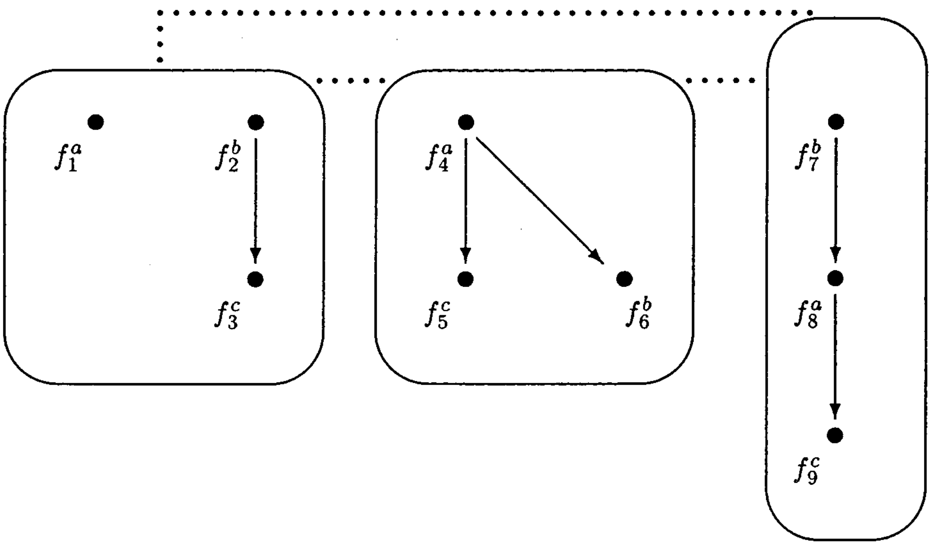


Figure 19: Step-bisimilar event structures $\mathcal{E}$ and $\mathcal{F}$.

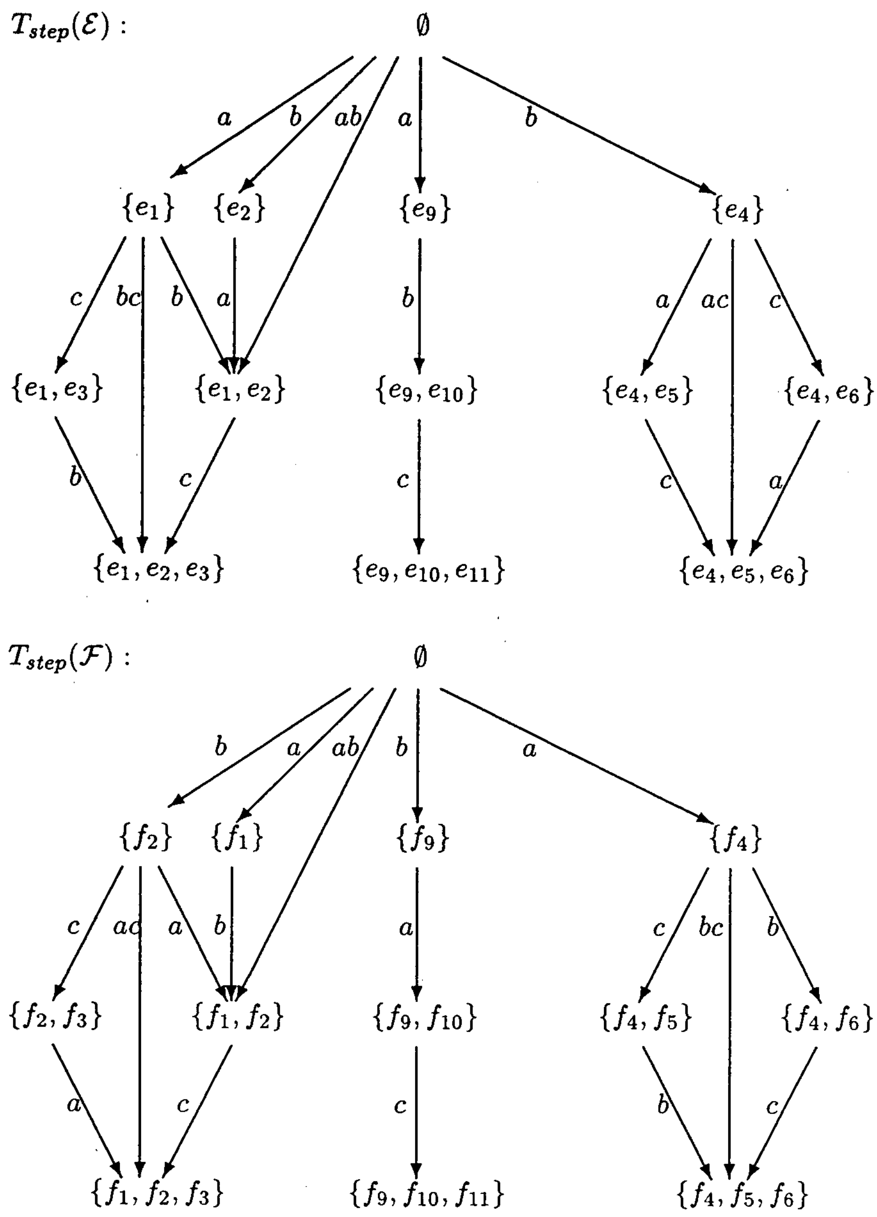


Figure 20: The transition systems $T_{step}(\mathcal{E})$ and $T_{step}(\mathcal{F})$.

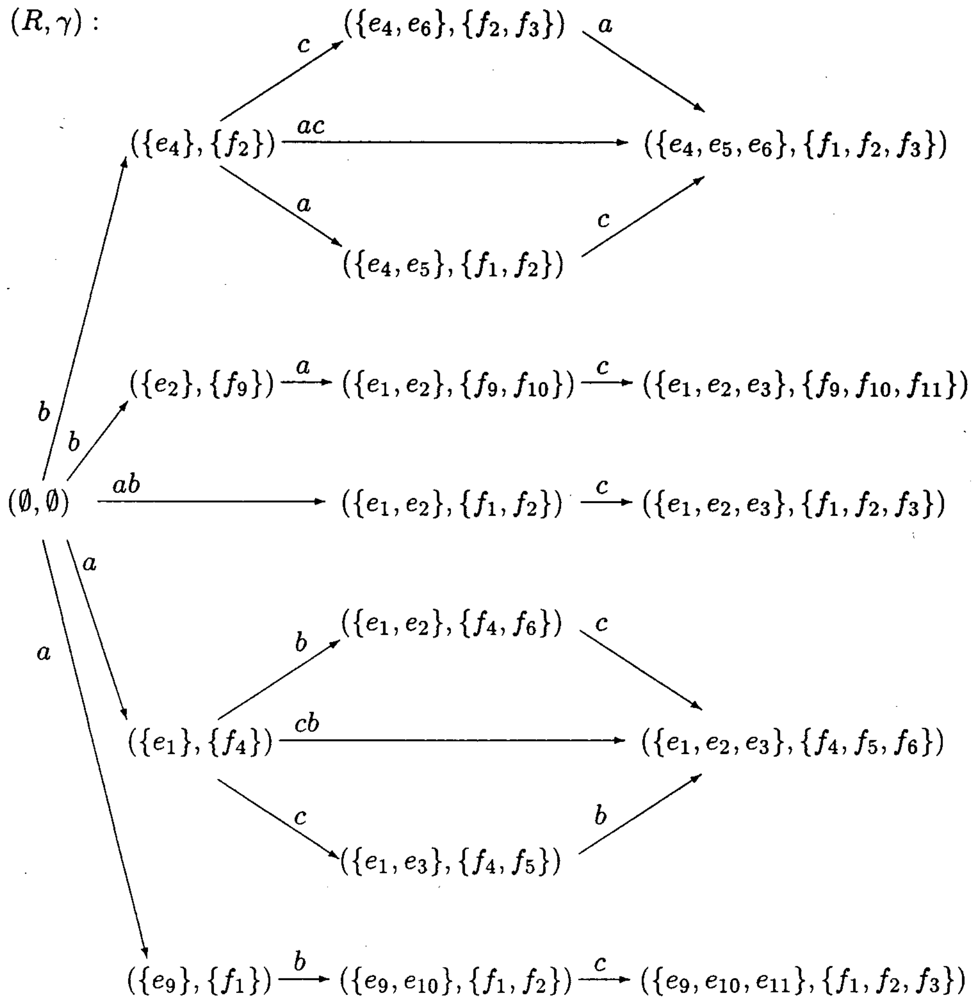


Figure 21: An AM-bisimimulation (R, γ) between $T_{step}(\mathcal{E})$ and $T_{step}(\mathcal{F})$.

4.2.4 Modelling (strong) history preserving bisimulation

[JNW94] give the following characterizations of (strong) history preserving bisimulation on event structures with consistency relation:

Theorem 4.16
Two event structeres $\mathcal{E}_1$ and $\mathcal{E}_2$ are

- *strong history preserving bisimular iff they are* **Pom**-*bisimilar.*
- *(strong) history preserving bisimilar iff they are (strong) path-***Pom**-*bisimilar.*

Applying theorem 3.3 on the second result we get a characterization of (strong) history preserving bisimulation in terms of AM-bisimulation on event structures with consistency relation as well as on prime event structures.

Corollary 4.17
Event structures $\mathcal{E}_1$ and $\mathcal{E}_2$ are (strong) history preserving bisimilar iff there exists a (strong) AM-bisimulation (R,γ) between $T_{path-\mathbf{Pom}}(\mathcal{E}_1)$ and $T_{path-\mathbf{Pom}}(\mathcal{E}_2)$ with $(\iota_1,\iota_2) \in R$.

In corollary 4.17 we used transition systems of kind $T_{path-\mathbb{P}}(\mathcal{E})$. Choosing the operator T_{pom} leads to an alternative characterization:

Theorem 4.18
Let $\mathcal{E} = (E, \leq_E, \sharp_E, l_E)$, $\mathcal{F} = (F, \leq_F, \sharp_F, l_F)$ be event structures, $T_{pom}(\mathcal{E}) = (Conf(\mathcal{E}), \alpha)$ and $T_{pom}(\mathcal{F}) = (Conf(\mathcal{F}), \beta)$ be their related coalgebras. Let

$$\begin{aligned} M := \{f \in mor(\mathcal{X},\mathcal{Y}) \mid \ & X \in Conf(\mathcal{E}), Y \in Conf(\mathcal{F}),\\ & \mathcal{X} = (X, \leq_E \cap (X \times X), l_{E|X}),\\ & \mathcal{Y} = (Y, \leq_F \cap (Y \times Y), l_{E|Y})\}. \end{aligned}$$

Let $\mathcal{P}(M)$ be the powerset of M.

1. *$\mathcal{E}$ and $\mathcal{F}$ are history preserving bisimular iff there exists an AM-bisimulation (R,γ) between $T_{pom}(\mathcal{E})$ and $T_{pom}(\mathcal{F})$ with $(\emptyset,\emptyset) \in R$, such that there exists a mapping $Isom : R \to \mathcal{P}(M)$ with*

   (o) $Isom(X,Y) \neq \emptyset$, all $f \in Isom(X,Y)$ are isomorphisms,

   (i) if $(p, X', Y') \in \gamma(X,Y)$, $(p, X') \in \alpha(X)$ and $f \in Isom(X,Y)$ then there exists $Y'' \in Conf(\mathcal{F})$, $f' \in Isom(X', Y'')$ with $f'_X = f$, $(p, X', Y'') \in \gamma(X,Y)$ and

   (ii) if $(p, X', Y') \in \gamma(X,Y)$, $(p, Y') \in \beta(Y)$ and $f \in Isom(X,Y)$ then there exists $X'' \in Conf(\mathcal{E})$, $f' \in Isom(X'', Y')$ with $f'_{|X} = f$, $(p, X'', Y') \in \gamma(X,Y)$.

2. The event structures $\mathcal{E}$ and $\mathcal{F}$ are strong history preserving bisimular iff there exists a strong AM-bisimulation (R, γ) between $T_{pom}(\mathcal{E})$ and $T_{pom}(\mathcal{F})$ with $(\emptyset, \emptyset) \in R$ such that there exists a mapping Isom $: R \to \mathcal{P}(M)$ which satisfies (o), (i) and (ii) and for which furtheron holds:

(iii) if $(p, X, Y) \in \gamma^-(X', Y')$, $(p, X) \in \alpha^-(X')$ and $f' \in Isom(X', Y')$ then there exists $Y'' \in Conf(\mathcal{F})$, $f \in Isom(X, Y'')$ with $f'_X = f$, $(p, X, Y'') \in \gamma^-(X', Y')$ and

(iv) if $(p, X, Y) \in \gamma^-(X', Y')$, $(p, Y) \in \beta^-(Y')$ and $f' \in Isom(X', Y')$ then there exists $X'' \in Conf(\mathcal{E})$, $f \in Isom(X'', Y)$ with $f'_X = f$, $(p, X'', Y) \in \gamma^-(X', Y')$.

Proof: Let $\mathcal{E}$, $\mathcal{F}$ be history preserving bisimular. Then there exists a history preserving bisimulation R' which is a set of triples (X, Y, f) where $X \in Conf(\mathcal{E})$, $Y \in Conf(\mathcal{F})$ and $f : X \to Y$ is an isomorphism in **Pom**. Let

- $R := \{(X, Y) \in Conf(\mathcal{E}) \times Conf(\mathcal{F}) \mid (X, Y, f) \in R'\}$ and
- $Isom(X, Y) := \{f \mid (X, Y, f) \in R'\}$.

Let for all (X, Y), $(X', Y') \in R'$

$$(p, X', Y') \in \gamma(X, Y) \;:\Longleftrightarrow\; (p, X') \in \alpha(X),\ (p, Y') \in \beta(Y),\ \exists f \in Isom(X, Y),\ f' \in Isom(X', Y') : f'_{|X} = f.$$

The definitions of *Isom* and γ imply $(\emptyset, \emptyset) \in R$ and (o). Thus it remains to prove that (R, γ) is indeed an AM-bisimulation and that *Isom* fullfills (i) and (ii).

Let $(p, X') \in (\alpha \circ \pi_1)(X, Y)$. This implies $(p, X') \in \alpha(X)$. By definition of R there exists an $f : E_1 \to E_2$ such that $(X, Y, f) \in R'$. As R' is a history preserving bisimulation we get $(p, Y') \in \beta(Y)$, $(X', Y', f') \in R'$ and $f'_{|X} = f$ for some $Y' \in Conf(E_2)$ and some $f' : E_1 \to E_2$. Thus $(p, X', Y') \in \gamma(X, Y)$ and hence $(p, X') \in (F\pi_1 \circ \gamma)(X, Y)$.

Let $(p, X') \in (F\pi_1 \circ \gamma)(X, Y)$. Then for some $Y' \in Conf(\mathcal{F})$ we have $(p, X', Y') \in \gamma(X, Y)$. This implies especially $(p, X') \in \alpha(X)$ and therefore we get $(p, X') \in (\alpha \circ \pi_1)(X, Y)$.

To prove (i) let $(p, X', Y') \in \gamma(X, Y)$, $(p, X') \in \alpha(X)$ and $f \in Isom(X, Y)$. By definition of R and *Isom* this implies $(X, Y, f) \in R'$. As R' is a history preserving bisimulation there exists $(p, Y'') \in \beta(Y)$ such that $(X', Y'', f') \in R'$ with $f'_{|X} = f$. Therefore we have $f' \in Isom(X', Y'')$ and hence $(p, X', Y'') \in \gamma(X, Y)$.

Now let (R, γ) be an AM-bisimulation with $(\emptyset, \emptyset) \in R$ between $(Conf(\mathcal{E}), \alpha)$ and $(Conf(\mathcal{F}), \beta)$ which fullfills (o),(i) and (ii). Let

$$R' := \{(X, Y, f) \mid (X, Y) \in R,\ f \in Isom(X, Y)\}.$$

Let $(X, Y, f) \in R'$, let $(p, X') \in \alpha(X)$. Then $(X, Y) \in R$, $(p, X') \in (\alpha \circ \pi_1)(X, Y)$ and therefore $(p, X') \in (F\pi_1 \circ \gamma)(X, Y)$. Thus there exists some $Y' \in \mathit{Conf}(E_2)$ such that $(p, X', Y') \in \gamma(X, Y)$. By condition (i) we find some $Y'' \in \mathit{Conf}(\mathcal{F})$ and some $f' \in \mathit{Isom}(X', Y'')$ such that $f'_{|X} = f$ and $(p, X', Y'') \in \gamma(X, Y)$. Thus we have $(X', Y'', f') \in R'$. As (R, γ) is a bisimulation we get finally $(p, Y'') \in \beta(Y)$.

Now let $\mathcal{E}$, $\mathcal{F}$ be strong history preserving bisimular. Then there exists a strong history preserving bisimulation R'. Choose the coalgebra (R, γ) and the mapping *Isom* as above.

In order to show that (R, γ) is strong let $(p, X) \in (\alpha^- \circ \pi_1)(X', Y')$. Then we get $(p, X) \in \alpha^-(X')$ and from the definition of R follows: There exists f' such that $(X', Y', f') \in R'$. As R' is a strong history preserving bisimulation there exists $(X, Y, f) \in R'$ such that $Y \subseteq Y'$ and $f = f'_{|X}$. As X and Y are isomorphic, X' and Y' are isomorphic and for the isomorphisms holds $f = f'_{|X}$ we may conclude that $p = [X' \backslash X] = [Y' \backslash Y]$ and therefore $(p, Y) \in \beta^-(Y')$. By definition of γ we get $(p, X, Y) \in \gamma^-(X', Y')$ which results in $(p, X) \in (F\pi_1 \circ \gamma^-)(X', Y')$.

To prove (iii) let $(p, X, Y) \in \gamma^-(X', Y')$, $(p, X) \in \alpha^-(X')$ and $f' \in \mathit{Isom}(X', Y')$. By definition of *Isom* this implies $(X', Y', f') \in R'$. As R' is a strong history preserving bisimulation there exist $Y'' \in \mathit{Conf}(\mathcal{F})$ and an isomorphism $f : X \to Y''$ such that $(p, Y'') \in \beta^-(Y')$, $(X, Y'', f) \in R'$ and $f = f'_{|X}$. Thus we get $(X, Y'') \in R$, $f \in \mathit{Isom}(X, Y'')$ and $(p, X, Y'') \in \gamma^-(X', Y')$.

Let finally (R, γ) be a strong AM-bisimulation together with a mapping *Isom* such that the conditions (o) – (iv) are fullfilled. We know that conditions (o), (i) and (ii) ensure that the above constructed set R' is a history preserving bisimulation. We claim that R' is strong. Let $(X', Y', f') \in R'$, let $X \subseteq X'$ for some configuration $X \in \mathit{Conf}(\mathcal{E})$, i.e. $(p, X) \in \alpha^-(X')$ for $p = [X' \backslash X]$. By construction of R' we get $(X', Y') \in R$, $f' \in \mathit{Isom}(X', Y')$ and therefore $(p, X) \in (\alpha^- \circ \pi_1)(X', Y')$. As (R, γ) is strong this implies $(p, X) \in (F\pi_1 \circ \gamma^-)(X', Y')$. Thus there exists some configuration $Y \in \mathit{Conf}(\mathcal{F})$ with $(p, X, Y) \in \gamma^-(X', Y')$. With condition (iii) we may conclude that there exist $Y'' \in \mathit{Conf}(\mathcal{F})$ and $f \in \mathit{Isom}(X, Y'')$ such that $f = f'_{|X}$ and $(p, X, Y'') \in \gamma^-(X', Y')$. Using again the property strong of (R, γ) we get $(p, Y'') \in \beta^-(Y')$ and therefore $Y'' \subseteq Y'$. ■

4.3 Synopsis

Figure 22 summerizes the above results on modelling different types of bisimulation on prime event structures with (strong) AM-bisimulation, (strong) path-ℙ-bisimulation and ℙ-bisimulation.

Column “AM I” refers on modelling with AM-bisimulation where the transition systems

bisimulation	AM I	AM II	path-$\mathbb{P}$	$\mathbb{P}$
interleaving	T_{int}	$T_{path-\mathbf{Lin}}$	**Lin**	**Lin**
step	T_{step}	$T_{path-\mathbf{Step}}$	**Step**	
pomset	T_{pom}			
history preserving	$T_{pom} + C$	$T_{path-\mathbf{Pom}}$	**Pom**	
strong history preserving	$T_{pom} + C + (s)$	$T_{path-\mathbf{Pom}} + (s)$	**Pom** + (s)	[**Pom**]

Figure 22: Modelling bisimulations on event structures.

are different from $T_{path-\mathbb{P}}$. Theorem 4.9 gives the first three rows: Choosing the transition system of type T_{int}, T_{step} resp. T_{pom} AM-bisimulation is equivalent to interleaving, step resp. pomset bisimulation. As example 4.11 showed these bisimulations are not equivalent with the respective strong variants of AM-bisimulation. Theorem 4.18 leads to the last two rows: History preserving bisimulation is equivalent to AM-bisimulation on T_{pom} which further fullfills certain conditions "C"; in case of strong history preserving bisimulation it is necessary that the AM-bisimulation is furtheron strong which we denote by "(s)" in figure 22.

Column "$\mathbb{P}$" presents the results concerning $\mathbb{P}$-bisimulation: Theorem 4.12 shows that interleaving bisimulation and **Lin**-bisimulation are equivalent. For the category of event structures with consistency relation theorem 4.16 provides the equivalence between strong history preserving bisimulation and **Pom**-bisimulation. As it is an open problem whether this results holds too in the category of prime event structures we write [**Pom**] in figure 22. Concerning the flexibility of $\mathbb{P}$-bisimulation on event structures [JNW94] write:

> *It might be thought that strong history-preserving bisimulation, presented as* **Pom**-*bisimilarity, is affected by restricting the category* **Pom** *to a smaller class of objects. However, no matter how much the objects in the path category* **Pom** *are restricted, provided they include all pomsets of the "stick" and "lollipop" forms in the proof of Proposition 7, then the relation of bisimulation that results will coincide with strong history-preserving bisimulation.*

Thus one does not expect that step, pomset or history preserving bisimulation can be modelled as $\mathbb{P}$-bisimulation and only interleaving and strong history preserving bisimulation fit in the concept of $\mathbb{P}$-bisimulation.

Column "path-$\mathbb{P}$" shows what kind of bisimulation on event structures we modelled by path-$\mathbb{P}$-bisimulation: Corollary 4.13 contains the equivalence of interleaving bisimulation, path-**Lin**-bisimulation and strong path-**Lin**-bisimulation. Corollary 4.14 provides the equivalence of step-bisimulation and path-**Step**-bisimulation. History preserving bisimulation is equivalent to path-**Pom**-bisimulation; in case of strong history preserving bisimulation it is necessary that the path-**Pom**-bisimulation is furtheron strong which we denote by "(s)" in figure 22. For event structures with consistency relation these two results can be found in theorem 4.16. It is easy to see that they hold too for prime event structures.

Column "AM II" results from column "path-$\mathbb{P}$": Using theorem 3.3 we can translate any (strong) path-$\mathbb{P}$-bisimulation into a (strong) AM-bisimulation.

Thus we can model all the mentioned types of bisimulation on event structures with (strong) AM-bisimulation in a unifying way: Two event structures $\mathcal{E}$ and $\mathcal{F}$ are $*$-bisimilar, iff there exists a (strong) AM-bisimulation (R, γ) with $(i_1, i_2) \in R$ between $T(\mathcal{E})$ and $T(\mathcal{F})$, where $* \in \{$*interleaving, bf, step, pomset, history-preserving, strong-history-preserving*$\}$, (i_1, i_2) is a distinguished pair of states and T is an operator which maps an event structure on a suitable coalgebra. In case of interleaving, step and pomset bisimulation we choose $(i_1, i_2) = (\emptyset, \emptyset)$ and one of the operators T_{int}, T_{step} resp. T_{pom}. bf-bisimulation is modelled as strong AM-bisimulation with $(i_1, i_2) = (\emptyset, \emptyset)$ and $T = T_{int}$. For history preserving bisimulation we take $(i_1, i_2) = (\iota_1, \iota_2)$ and the operator $T_{path-\mathbf{Pom}}$, for strong history preserving bisimulation we make the same choice but take this time the strong version of AM-bisimulation.

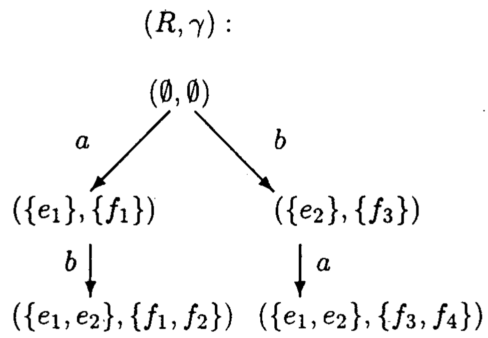


Figure 23: An AM-bisimulation between $T_{int}(\mathcal{E})$ and $T_{int}(\mathcal{F})$ from figure 16.

5 Different answers: what is a bisimulation?

In order to discuss the differences between the abstract concepts (strong) AM-bisimulation, $\mathbb{P}$-bisimulation and (strong) path-$\mathbb{P}$-bisimulation consider the interleaving bisimilar event structures $\mathcal{E}$ and $\mathcal{F}$ from figure 15. As we know from corollary 4.13 this type of bisimulation can be modelled by any of the three abstract concepts.

To establish an AM-bisimulation between $\mathcal{E}$ and $\mathcal{F}$ we first have to transform the event structures into transition systems. Therefore we use the operator T_{int}. The resulting transitions systems are shown in figure 16. Figure 23 shows an AM-bisimulation (R, γ) between $T_{int}(\mathcal{E})$ and $T_{int}(\mathcal{F})$ with $(\emptyset, \emptyset) \in R$. It relates those states from $T_{int}(\mathcal{E})$ and $T_{int}(\mathcal{F})$ which show the same behaviour.

In order to show that $\mathcal{E}$ and $\mathcal{F}$ are $\mathbb{P}$-bisimilar we choose as path category the category **Lin**. As object X we take the event structure $\mathcal{F}$, and define the morphism $g_1 : \mathcal{F} \to \mathcal{E}$ by $g_1(f_1) = g_1(f_4) := e_1$ and $g_1(f_2) = g_1(f_3) := e_2$ and the morphism $g_2 : \mathcal{F} \to \mathcal{F}$ as identity on $\mathcal{F}$. As we know from [JNW94] the morphism g_2 is **Lin**-open. In order to show that g_1 is **Lin**-open too we consider the commuting square from figure 6, where $X = \mathcal{F}$, $Y = \mathcal{E}$ and P and Q are objects of **Lin**. As all configurations of $\mathcal{E}$ or $\mathcal{F}$ have no more than two elements, the path objects P and Q may consist of maximal two events. We consider just one case: Let P be an event structure with one event e labelled with a. Then $p(e) = f_1$. In order to make the square of figure 6 commute there are only two types of event structures possible for Q. The first is that Q consists of one event $\hat{e}$ labelled with a. In this case $m(e) = \hat{e}$, $q(e) = e_1$ and we can define $r(\hat{e}) := f_1$ in order to obtain the desired morphism r. The second type of Q consists of two events $\hat{e}_1$ labelled with a and $\hat{e}_2$ labelled with b, where

$\hat{e}_1 \leq_Q \hat{e}_2$. This time we get $m(e) = \hat{e}_1$, $q(\hat{e}_1) = e_1$ and $q(\hat{e}_2) = e_2$. Defining $r(\hat{e}_1) := f_1$ and $r(\hat{e}_2) := f_2$ we get a morphism with $r \circ m = p$ and $g_1 \circ r = q$.

The event structure $X = \mathcal{F}$ from the $\mathbb{P}$-bisimulation is isomorphic to the event structure which we obtain from the AM-bisimulation (R, γ) of figure 23 if we first unfold (R, γ) into a synchronisation tree and then transform the result into an event structure. We formalized this technique to obtain a **Lin**-bisimulation from an AM-bisimulation between transition systems of type T_{int} in theorem 4.12.

In case of path-$\mathbb{P}$-bisimulation we choose again the category **Lin** as path category and take the empty event structure $(\emptyset, \emptyset, \emptyset, \emptyset)$ as the common initial object I of **Lin** and $\mathbf{E}_{Act}$. We define sets of event structures

- $\mathcal{S}_a$: The set of all event structures of $\mathbf{E}_{Act}$ which consist of just one event labelled with a,
- $\mathcal{S}_b$: The set of all event structures of $\mathbf{E}_{Act}$ which constist of just one event labelled with b,
- $\mathcal{S}_{ab}$: The set of all event structures of $\mathbf{E}_{Act}$ which constist of two events, one event labelled with a, the other labelled with b, where the event labelled with a is predecessor of the event labelled with b and
- $\mathcal{S}_{ba}$: The set of all event structures of $\mathbf{E}_{Act}$ which constist of two events, one event labelled with b, the other labelled with a, where the event labelled with b is predecessor of the event labelled with a.

With these sets we define a path-**Lin**-bisimulation $\hat{R}$ between $\mathcal{E}$ and $\mathcal{F}$:

$$
\begin{aligned}
\hat{R} \;:=\; & \{(p_1, p_2) \mid p_1 \in mor(I, \mathcal{E}), p_2 \in mor(I, \mathcal{F})\} \cup \\
& \{(p_1, p_2) \mid p_1 \in mor(X, \mathcal{E}), p_2 \in mor(X, \mathcal{F}), X \in \mathcal{S}_a\} \cup \\
& \{(p_1, p_2) \mid p_1 \in mor(X, \mathcal{E}), p_2 \in mor(X, \mathcal{F}), X \in \mathcal{S}_b\} \cup \\
& \{(p_1, p_2) \mid p_1 \in mor(X, \mathcal{E}), p_2 \in mor(X, \mathcal{F}), X \in \mathcal{S}_{ab}\} \cup \\
& \{(p_1, p_2) \mid p_1 \in mor(X, \mathcal{E}), p_2 \in mor(X, \mathcal{F}), X \in \mathcal{S}_{ba}\} \cup
\end{aligned}
$$

The argument why $\hat{R}$ is a path-**Lin**-bisimulation is similar to the proof that the morphism g_1 is **Lin**-open: Again we argue that any path object P with $mor(P, \mathcal{E}) \neq \emptyset$ and $mor(P, \mathcal{F}) \neq \emptyset$ is either the initial object I or is element of one of the sets S_a, S_b, S_{ab} or S_{ba}. We consider just one case: Let P be an event structure with one event e labelled a. Then $p_1(e) = e_1$ and $p_2(e) = f_1$. By definition of $\hat{R}$ we know that $(p_1, p_2) \in \hat{R}$. If there is an event structure $Q \in$ **Lin** such that there exist morphism $m : P \to Q$ and $q_1 : Q \to \mathcal{E}$ with $q_1 \circ m = p_1$, see

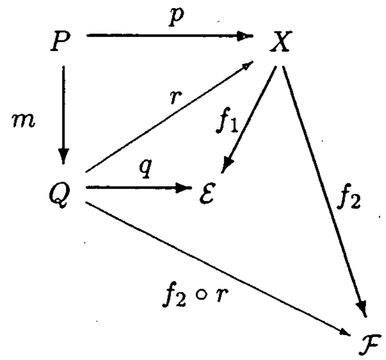


Figure 24: Path augementation in case of ℙ-bisimulation.

figure 7, then Q has to be an element of S_a or of S_{ab}. In the first case Q consists of one event $\hat{e}$ labelled with a and $m(e) = \hat{e}$. We define $q_2(\hat{e}) := f_1$ and obtain $q_2 \circ m = p_2$ and $(q_1, q_2) \in \hat{R}$. Now let $Q \in S_{ab}$. Then Q consist of two events, one event $\hat{e}_1$ labelled with a and one event $\hat{e}_2$ labelled with b, where $\hat{e}_1 \leq_Q \hat{e}_2$. For m we get: $m(e) = \hat{e}_1$. We define $q_2(\hat{e}_1) := f_1$, $q_2(\hat{e}_2) := f_2$ and obtain again: $q_2 \circ m = p_2$ and $(q_1, q_2) \in \hat{R}$.

One should note that even in this simple example the set $\hat{R}$ includes infinitly many pairs of morphisms, but that $\hat{R}$ is the finite union of morphism sets of the same type: Any of the above mentioned subsets of $\hat{R}$ is parametrized by an isomorphism class of event structures.

With corollary 4.13 we may conclude from any of the three above mentioned bisimulations that the event structures $\mathcal{E}$ and $\mathcal{F}$ are interleaving bisimilar.

The three abstract concepts give different answers on the question: "What is a (interleaving) bisimulation between event structures $\mathcal{E}$ and $\mathcal{F}$?"

AM-bisimulation: In order to establish an AM-bisimulation we first have to construct some kind of transition systems related to $\mathcal{E}$ and $\mathcal{F}$. For the event structures from figure 15 we chose $T_{int}(\mathcal{E}_1)$ and $T_{int}(\mathcal{E}_2)$. These transition systems make the "dynamics" of the event structures explicit. A bisimulation is then a *transition system* (R, γ) with a "behaviour" common to $T_{int}(\mathcal{E}_1)$ and $T_{int}(\mathcal{E}_2)$. The condition $(\emptyset, \emptyset) \in R$ ensures that the whole transition sytems are taken into account.

ℙ-bisimulation: Here bisimulation is expressed as an *event structure* X together with ℙ-open morphisms $f_1 : X \to \mathcal{E}$ and $f_2 : X \to \mathcal{F}$. This construction ensures (see figure 24): If a morphism p from a path object P into X can be augmented to a

$\mathbb{P}$-bisimulation		
$\Updownarrow$ (theorem 3.1)		
strong path-$\mathbb{P}$-bisimulation	$\Rightarrow$	path-$\mathbb{P}$-bisimulation
$\Updownarrow$ (theorem 3.3)		$\Updownarrow$ (theorem 3.3)
strong AM-bisimulation on $T_{path-\mathbb{P}}$	$\Rightarrow$	AM-bisimulation on $T_{path-\mathbb{P}}$
$\not\Downarrow$ (example 3.10)		$\Updownarrow$ (theorem 3.6)
strong AM-bisimulation on $T \neq T_{path-\mathbb{P}}$	$\Rightarrow$	AM-bisimulation on $T \neq T_{path-\mathbb{P}}$

Figure 25: Relations between the different bisimulation concepts.

morphism q from a path object Q into $\mathcal{E}$, i.e. there exists a morphism $m : P \to Q$ such that $q \circ m = f_1 \circ p$, then this augmentation is possible with the same morphism m for the event structure $\mathcal{F}$: As f_1 is $\mathbb{P}$-open, there exists a morphism r such that $r \circ m = p$ and $q = f_1 \circ r$. With r we get a morphism $f_2 \circ r : Q \to F$ such that $(f_2 \circ r) \circ m = f_2 \circ p$. The same holds symmetrically for $\mathcal{F}$. The event structure X from the $\mathbb{P}$-bisimulation can be interpreted as the "image" of all path objects "common" to $\mathcal{E}$ and $\mathcal{F}$. $\mathbb{P}$-openess of f_1 and f_2 guarantees that for any path augmentation of $\mathcal{E}$ one can find a corresponding one of $\mathcal{F}$ and vice versa.

path-$\mathbb{P}$-bisimulation: A bisimulation is a *set of morphism pairs* $\hat{R}$ which fullfills some closure properties. The existence of an initial object together with the closure properties ensure that $\hat{R}$ includes all pairs (p_1, p_2), where P is a path object and $p_1 : P \to \mathcal{E}$ and $p_2 : P \to \mathcal{F}$ are morphism. Furtheron this type of bisimulation guarantees some kind of "path augmentation" (see figure 7): If there are morphisms p_1 and p_2 from a path object P to $\mathcal{E}$ resp. $\mathcal{F}$, the path p_1 can be augmented to a path $q_1 : Q \to \mathcal{E}$, i.e. there exists a morphism $m : P \to Q$ with $q_1 \circ m = p_1$, then there exists a path $q_2 : Q \to \mathcal{F}$ with $q_2 \circ m = p_2$. The same holds symmetrically for $\mathcal{F}$. Thus a path-$\mathbb{P}$-bisimulation $\hat{R}$ includes all pathes which can be derived by path augmentation (beginning with the initial pathes) for both event structure $\mathcal{E}$ and $\mathcal{F}$.

6 Conclusion

Figure 25 summarizes the general relations between (strong) AM-bisimulation, (strong) path-$\mathbb{P}$-bisimulation and $\mathbb{P}$-bisimulation. For simplicity we do *not* mention the conditions which are (sometimes) necessary to establish an equivalence.

On top of figure 25 we find $\mathbb{P}$-bisimulation. Theorem 3.1 gives an equivalence with strong path-$\mathbb{P}$-bisimulation. Of course strong path-$\mathbb{P}$-bisimulation implies path-$\mathbb{P}$-bisimulation. Theorem 3.3 shows that choosing suitable transition systems with the operator $T_{path-\mathbb{P}}$ (strong) path-$\mathbb{P}$-bisimulation and (strong) AM-bisimulation are equivalent. Concerning (strong) AM-bisimulation on arbitrary transition systems which are different from those we get by the operator $T_{path-\mathbb{P}}$ the situation is more complicated: Theorem 3.6 provides an equivalence between path-$\mathbb{P}$-bisimulation and AM-bisimulation; for the strong variants example 3.10 showed that in general strong path-$\mathbb{P}$-bisimulation does not imply strong AM-bisimulation.

Combining theorem 3.1 and theorem 3.3 we can conclude: If a bisimulation between objects of a category of models $\mathbb{M}$ can be modelled as $\mathbb{P}$-bisimulation for a suitable subcategory $\mathbb{P}$ of $\mathbb{M}$ and the assumptions of theorem 3.1 are fullfilled, then this bisimulation can also be modelled as strong AM-bisimulation where we choose the operator $T_{path-\mathbb{P}}$ to get a transition system. Applications of this combination are interleaving and strong history preserving bisimulation on event structures, see section 4.2.3 resp. 4.2.4, and **Bran**-bisimulation on transition systems which we discussed in section 3.3.

For the converse direction one obtains: If a bisimulation can be modelled as AM-bisimulation, the assumptions of theorem 3.6 are fullfilled, the AM-bisimulations on $T_{path-\mathbb{P}}$ are always strong, the assumptions of theorem 3.1 are fullfilled, then this bisimulation can also be modelled as $\mathbb{P}$-bisimulation. Applications of this equivalence are interleaving bisimulation on event structures and AM-bisimulation on transition systems.

Applying these results to concrete models we showed: For transition systems the concepts of AM-bisimulation, **Bran**-bisimulation and (strong) path-**Bran**-bisimulation coincide (corollary 3.8). Differences arise for the more complex model of event structures: Looking for an approach which is able to model various types of bisimulations on event structures AM-bisimulation turned out to be the most flexible of the three concepts.

It is left as open question how strong AM-bisimulation and strong path-$\mathbb{P}$-bisimulation are related. In order to get more insight into the "nature" of bisimulation other types of bisimulation on event strucures, bisimulations on other models of concurrency and other abstract characterizations of bisimulation should be studied.

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