On ω-admissible vector space topologies on C(X)

by

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A convergence structure Λ on $C(X)$, the $\mathbb{R}$-algebra of all real-valued continuous functions on a completely regular topological space X, is called *ω-admissible* if the evaluation map

$$\omega : C_{\Lambda}(X) \times X \longrightarrow \mathbb{R} ,$$

which sends each $(f,x) \in C(X) \times X$ to $f(x)$, is continuous. In this paper, we determine whether there exist coarsest, or indeed any ω-admissible vector space topologies or pseudo-topological structures on $C(X)$. (A convergence structure Λ on a vector space L over $\mathbb{R}$ is said to be *pseudo-topological* if L_{Λ} is an inductive limit of topological vector spaces in the category of convergence spaces.) We will describe that class of spaces X allowing an ω-admissible vector space topology on $C(X)$. It will be established that there exists a coarsest ω-admissible pseudo-topological structure on $C(X)$ for any space X. Furthermore, we characterize that class of spaces X, for which the continuous convergence structure (i.e., the coarsest ω-admissible convergence structure, see [1]) on $C(X)$ is a pseudo-topological structure. We conclude the note by demonstrating that this class contains non-locally compact spaces.

We first prove a lemma that will be extremely useful throughout this paper. The symbol X will always denote a completely regular topological space, and βX (respectively υX) its Stone-Čech compactification (respectively, Hewitt realcompactification). We regard X and υX as subspaces of βX. As usual, we identify eauch function $f \in C(T)$, where $X \subset T \subset \beta X$, with its restriction to X , and denote the evaluation map from $C(T) \times T$ into $\mathbb{R}$ also by the symbol ω . We write $C_c(X)$ for the algebra $C(X)$ endowed with the continuous convergence structure. By a topological vector space, we mean a Hausdorff topological vector space over the reals.

Lemma 1. *Let* α *be a continuous linear map from a topological vector space* E *into* $C_c(X)$. *Then there exists a compact subset* $K \subset \beta X \setminus X$ *with the property that* $\alpha(E) \subset C(\beta X \setminus K)$ *and the map* $\alpha: E \longrightarrow C_c(\beta X \setminus K)$ *is continuous.*

Proof: Let $\mathcal{U}$ be the neighborhood filter of zero in E . Since $\alpha(\mathcal{U})$ converges to zero in $C_c(X)$, there exists for each point $x \in X$ a neighborhood V_x of x in X and an element F_x in $\mathcal{U}$ with the property that

$$\omega(\alpha(F_x) \times V_x) \subset [-1,1] .$$

The closure of V_x in βX is a neighborhood of x in βX , and therefore we can choose an open neighborhood U_x of x in βX so that

$$\omega(\alpha(F_x) \times U_x) \subset [-1,1] .$$

For $U = \bigcup_{x \in X} U_x$, the set $K = \beta X \setminus U$ is certainly a compact subset of $\beta X \setminus X$ and $\alpha(E) \subset C(U)$. To see that $\alpha(\mathcal{U})$ converges to zero in $C_c(\beta X \setminus K)$, choose $r > 0$ and $y \in U$. Obviously, y is an element of an open set U_x for some $x \in X$ and

$$\omega(r \cdot \alpha(F_x) \times V_x) \subset [-r,r] .$$

Since $r \cdot F_x \in \mathcal{U}$, the proof is complete.

Given an ω-admissible vector space topology τ on $C(X)$, the identity map from $C_\tau(X)$ into $C_c(X)$ is continuous. Our lemma implies that there exists a compact subset $K \subset \beta X \setminus X$ such that

$$C(X) \subset C(\beta X \setminus K) ,$$

and thus $\beta X \setminus K$ is contained in υX . This means υX is a neigborhood of X in βX . We note that since $\beta X \setminus K$ is locally compact for any compact $K \subset \beta X \setminus X$, the continuous convergence structure on $C(\beta X \setminus K)$ coincides with the compact-open topology. We have now proved

Theorem 1. *For a completely regular topological space* X , *there exists an* ω-*admissible vector space topology on* $C(X)$ *if and only if* υX *is a neighborhood of* X *in* βX .

As an immediate consequence of theorem 1, we can state

<u>Corollary</u>. *For a realcompact space* X *, there exists an* ω*-admissible vector space topology on* $C(X)$ *if and only if* X *is locally compact.*

With the help of lemma 1, we provide an alternative proof for the following known result (see [2]). Recall that an ideal $I \subset C(X)$ is called fixed if there is a point p in X, at which all functions of I vanish.

<u>Proposition</u> 1. *There exists an* ω*-admissible algebra topology* τ *on* $C(X)$ *with the property that every closed maximal ideal in* $C_\tau(X)$ *is fixed, if and only if* X *is locally compact.*

<u>Proof</u>. Let τ be an ω-admissible algebra topology on $C(X)$. By lemma 1, we have the following diagram of continuous maps for some compact $K \subset \beta X \setminus X$:

$$\begin{array}{ccc} C_\tau(X) & \longrightarrow & C_c(X) \\ & \searrow \quad \nearrow & \\ & C_c(\beta X \setminus K) & \end{array} .$$

For a completely regular topological space Y and an ω-admissible algebra topology τ on $C(Y)$, there is a natural injective map $i_Y \colon Y \longrightarrow HomC_\tau(Y)$, where $HomC_\tau(Y)$ denotes the set of all the continuous $\mathbb{R}$-algebra homomorphisms from $C_\tau(Y)$ onto $\mathbb{R}$. Each point $y \in Y$ is sent under i_Y to the point evaluation by y (i.e., $i_Y(y)(f) = f(y)$ for every $f \in C(Y)$). Therefore, we have the following commutative diagram of injective maps:

$$\begin{array}{ccc} \mathit{Hom}C_c(\beta X\setminus K) & \xrightarrow{\quad id^* \quad} & \mathit{Hom}C_\tau(X) \\ \uparrow{\scriptstyle i_{\beta X\setminus K}} & & \uparrow{\scriptstyle i_X} \\ \beta X\setminus K & \longleftarrow & X \end{array},$$

where id^* is the map induced from $id\colon C_\tau(X) \longrightarrow C_c(\beta X\setminus K)$. If each maximal closed ideal in $C_\tau(X)$ is fixed, then i_X is surjective. Thus the inclusion map from X into $\beta X\setminus K$ is a bijection, which means that X is locally compact. Conversely, if X is locally compact, $C_c(X)$ carries a topology with the claimed properties.

The rest of our topological questions is answered by the following well-known result (see [4], p. 329).

<u>Theorem</u> 2. *For a completely regular topological space* X , *the following three statements are equivalent:*

(a). X *is locally compact*

(b). $C_c(X)$ *is topological*

(c). *There exists a coarsest* ω*-admissible vector space topology on* $C(X)$.

"(a) implies (b)" follows from a standard calculation and (b) clearly implies (c) . By applying lemma 1, we provide a quick proof of "(c) implies (a)" . Assume there exists a coarsest ω-admissible topology τ on $C(X)$. It follows from lemma 1 that the subalgebra $C(\beta X \setminus K)$, for some compact $K \subset \beta X \setminus X$, is all of $C(X)$ and

$$\mathrm{id}: C_\tau(X) \longrightarrow C_c(\beta X \setminus K)$$

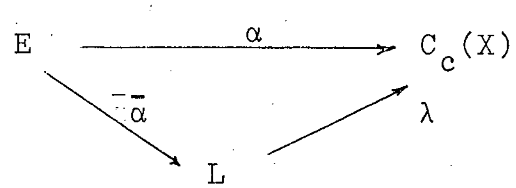
is continuous. If there existed a compact $K' \subset \beta X \setminus X$ which strictly contained K , then the topology of $C_c(\beta X \setminus K')$ would be strictly coarser than τ . Hence X must be locally compact.

Since we have shown that there do not, in general, exist ω-admissible vector space topologies on $C(X)$, we will discuss ω-admissible pseudo-topological structures, whose existence has been established in [3]. There, the convergence space $C_I(X)$ is defined as the inductive limit of the family

$$\{C_c(\beta X \setminus K) : K \text{ a compact subset of } \beta X \setminus X\}$$

together with the inclusion maps. The convergence algebra $C_I(X)$ carries an ω-admissible pseudo-topological structure with the further property that each closed maximal ideal is fixed (see [3]). We shall now work towards an universal characterization of $C_I(X)$, starting with a helpful definition:

An inductive limit L of topological vector spaces (taken in the category of convergence spaces) is called a c-inductive limit for X if there is a continuous linear map $\lambda: L \longrightarrow C_c(X)$ such that, given any topological vector space E, each continuous linear map $\alpha: E \longrightarrow C_c(X)$ factors uniquely through λ (that is, there is a unique continuous linear map $\bar{\alpha}$ making the diagram

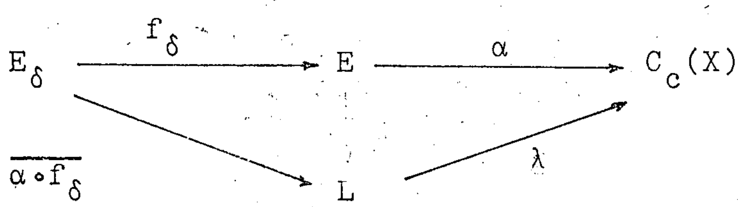


commutative).

Merely by looking at lemma 1, one sees that $C_I(X)$ is a c-inductive limit for X, by means of $id: C_I(X) \longrightarrow C_c(X)$. The next lemma forms the backbone of our characterization of $C_I(X)$:

Lemma 2. *Let* L *be a c-inductive limit for* X, *and* $\alpha: E \longrightarrow C_c(X)$ *a continuous linear map, where* E *is an inductive limit of topological vector spaces. Then* α *factors uniquely through* λ.

Proof: Suppose E to be the inductive limit of the inductive system $\{E_\delta : \delta \in \Delta\}$ of topological vector spaces, and $f_\delta: E_\delta \longrightarrow E$ the canonical mapping, for each $\delta \in \Delta$. Then the maps $\alpha \circ f_\delta$ all factor uniquely through λ, giving the following commutative diagram, for each $\delta \in \Delta$:



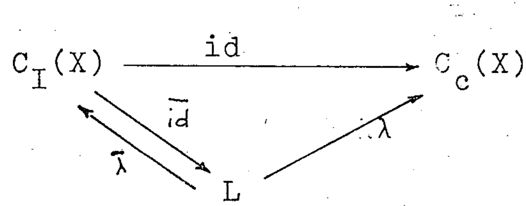
Now, by the universal property of inductive limits, the family $\{\alpha \circ f_\delta : \delta \in \Delta\}$ of continuous linear mappings induces an unique map $\bar{\alpha}: E \longrightarrow L$, which is actually continuous. It is easy to verify that $\alpha = \lambda \circ \bar{\alpha}$, and that $\bar{\alpha}$ is indeed unique with this property.

<u>Theorem</u> 3. *Let* X *be a completely regular topological space. Then*

i). $C_I(X)$ *is linearly homeomorphic to each* c-*inductive limit for* X , *and*

ii). *the convergence structure on* $C_I(X)$ *is the coarsest* ω-*admissible pseudo-topological structure on* C(X) .

<u>Proof</u>. Part ii) follows immediately from the preceding lemma, when one recalls that $C_I(X)$ is a c-inductive limit for X , and part i) from the observation that, if L is any c-inductive limit, in the diagram



the map id factors uniquely through $\bar{\lambda}$, and the map λ in turn factors uniquely through $\overline{id}$. By the usual category-theoretic argument, the reader can himself show that $\lambda = \bar{\lambda}$ is the claimed linear homeomorphism.

Now we wish to determine when $C_c(X)$ is an inductive limit of topological vector spaces. To simplify the notation, we say that a filter Φ <u>admits countable intersections</u> if the intersection of every countable collection of elements in Φ is again an element of Φ.

<u>Theorem</u> 4. *For a completely regular topological space* X, *the following three statements are equivalent:*

(a). $C_c(X)$ *is an inductive limit of topological vector spaces in the category of convergence spaces.*

(b). The identity map from $C_I(X)$ *onto* $C_c(X)$ *is a homeomorphism.*

(c). The space X *has both properties*

(i). The neighborhood filter of X *in* βX *admits countable intersections.*

(ii). The set $\tilde{X}$, *consisting of all points in* X *having no compact neighborhood in* υX, *is a compact subspace of* X.

<u>Proof</u>. The equivalence of statements (a) and (b) follows directly from theorem 3.

Throughout this proof, neighborhoods and closed sets are taken in βX. Assume that the conditions in statement (c) are satisfied. Let Θ be a filter convergent to zero in $C_c(X)$.

We will show that θ converges to zero in $C_I(X)$, finding first an open neighbourhood V of X, such that θ has a base on $C(V)$:

Since θ is convergent with respect to the continuous convergence structure, for each $x \in X$ and $n \in \mathbb{N}$ there is an open neighbourhood $U_{x,n}$ of x and an element $F_{x,n}$ in θ so that

$$\omega(F_{x,n} \times U_{x,n}) \subseteq \left[\frac{-1}{n}, \frac{1}{n}\right]$$

(by the same argument used in lemma 1). The collection $\{ U_{x,1} : x \in X \}$ is an open covering of the compact set $\tilde{X}$. Thus there are points $x_1, x_2, \dots , x_k$ in X with

$$\tilde{X} \subseteq U_{x_1,1} \cup \dots \cup U_{x_k,1} \ .$$

This implies that the set

$$V = ((\upsilon X)_\ell \cup U_{x_1,1} \cup \dots \cup U_{x_k,1})$$

contains X, where $(\upsilon X)_\ell$ denotes that subspace of all points in υX possessing a compact neighbourhood in υX. Obviously V is open (since $(\upsilon X)_\ell$ is open), and further,

$$F_{x_1,1} \cap \dots \cap F_{x_k,1} \subseteq C(V).$$

Next we construct an open neighbourhood W of X (with $W \subseteq V$, so that θ still has a base on $C(W)$) such that the filter $\{ F \cap C(W) : F \in \theta \}$ converges to zero in $C_c(W)$:

For each $n \in \mathbb{N}$, the set

$$U_n = \bigcup_{x \in X} U_{x,n}$$

is open. By assumption (condition ii), the set

$$V \cap \bigcap_{n=1}^{\infty} U_n$$

is a neighbourhood of X, and so we can find an open neighbourhood W of X contained within it. One can readily verify that $C(W)$ has the desired properties.

To complete the proof, we show " (b) implies (c) ". To this end, assume statement (c) is not satisfied, meaning that $\tilde{X}$ is not compact or the neighbourhood filter of X does not admit countable intersections. In both cases, we construct filters converging to zero in $C_c(X)$ but not convergent in $C_I(X)$.

To begin with, let $\tilde{X}$ be not compact. Then there is a family $\{ U_x : x \in \tilde{X} \}$ with the property that each U_x is a closed neighbourhood of x and no finite subfamily covers $\tilde{X}$. For each point $x \in X \setminus \tilde{X}$, we choose a closed neighbourhood U_x of x contained in $(\upsilon X)_{\ell}$. Now for each point $x \in X$, let

$$F_x = \{ f \in C(X) : f(U_x) = \{0\} \}.$$

Clearly the family of all F_x for $x \in X$ generates a filter θ convergent to zero in $C_c(X)$. We claim that θ does not converge in $C_I(X)$. Assume that θ has a basis in $C(\beta X \setminus K)$, for some compact subset K of $\beta X \setminus X$. This means there are points $x_1, x_2, \ldots, x_n$ in X with

$$F_{x_1} \cap \ldots \cap F_{x_n} \subseteq C(\beta X \setminus K) .$$

By construction, there is no finite subcollection of $\{ U_x : x \in X \}$ covering X, and hence we can find a point p in the set $\tilde{X} \setminus (U_{x_1} \cup \ldots \cup U_{x_n})$. Furthermore, we pick a closed neighbourhood V of p disjoint from $K \cup U_{x_1} \cup \ldots \cup U_{x_n}$. Since $p \notin (\upsilon X)_\ell$, we know that

$$V \cap \beta X \setminus X \neq \emptyset ,$$

and hence there is a function

$$f \in F_{x_1} \cap \ldots \cap F_{x_n}$$

which does not belong to $C(\beta X \setminus K)$ - see [5], section 7.9 . This contradicts our assumption.

On the other hand, let $\{ U_n : n \in \mathbb{N} \}$ be a sequence of neighbourhoods of X whose intersection fails to be a neighbourhood of X. Without loss of generality, we assume that each U_n contains U_{n+1}. For $x \in X$ and $n \in \mathbb{N}$, choose a closed neighbourhood $U_{x,n}$ of x contained in U_n, and put

$$F_{x,n} = \{ f \in C(\beta X) : f(U_{x,n}) \subseteq \left[\frac{-1}{n}, \frac{1}{n}\right] \} .$$

The collection of all $F_{x,n}$ for $x \in X$ and $n \in \mathbb{N}$ again generates a filter θ converging to zero in $C_c(X)$. Given any arbitrary compact subset K of $\beta X \setminus X$, we shall show that θ does not even converge pointwise, when regarded as a filter on $C_c(\beta X \setminus K)$. Since $\beta X \setminus K$ is a neighbourhood of X, our assumption implies the existence of a point q in $\beta X \setminus K$ but not in $\bigcap_{n=1}^{\infty} U_n$, and thus not in U_n, for some $n \in \mathbb{N}$. Suppose there are points $x_1, x_2, \ldots, x_k$ of X and positive integers $n_1, \ldots, n_k$ such that

$$(F_{x_1,n_1} \cap \ldots \cap F_{x_k,n_k})(q) \subseteq \left[\frac{-1}{n+1}, \frac{1}{n+1}\right].$$

Clearly we can assume also that

$$n_1 \geq n_2 \geq \ldots \geq n_r \geq n \geq n_{r+1} \geq \ldots \geq n_k .$$

Since $q \notin U_{x_1,n_1} \cup \ldots \cup U_{x_r,n_r}$, there is a function $f \in C(X)$, with

$$f(U_{x_1,n_1} \cup \ldots \cup U_{x_r,n_r}) = \{0\},$$

$f(q) = \frac{1}{n}$, and $f(X) \subseteq \left[\frac{-1}{n}, \frac{1}{n}\right]$. Now f belongs to $F_{x_1,n_1} \cap \ldots \cap F_{x_k,n_k}$ but $f(q) \notin \left[\frac{-1}{n+1}, \frac{1}{n+1}\right]$. With this contradiction the theorem is established.

Corollary 1. *If the subspace* $X_{n\ell}$ *of all points of* X *without compact neighbourhoods in* X *is compact and the neighbourhood filter of* $X_{n\ell}$ *in* X *admits countable intersections, then* $C_I(X)$ *and* $C_c(X)$ *coincide.*

Proof. Under these conditions on X, we show that requirement (c) of theorem 4 is fulfilled. Let $\{U_n : n \in \mathbb{N}\}$ be a sequence of neighbourhoods of X in βX, and thus also neighbourhoods of

$X_{n\ell}$ in βX. As $X_{n\ell}$ is compact, we can choose for each $n \in \mathbb{N}$ a closed neighbourhood V_n in βX of $X_{n\ell}$, which lies in U_n. Every $V_n \cap X$ is a neighbourhood of $X_{n\ell}$ in X, and hence the set

$$D = \bigcap_{n=1}^{\infty} (V_n \cap X)$$

is a neighbourhood of $X_{n\ell}$ in X. Thus the closure $\bar{D}$ of D in βX is a neighbourhood of $X_{n\ell}$ in βX, and of course

$$\bar{D} \subseteq \bigcap_{n=1}^{\infty} V_n \subseteq \bigcap_{n=1}^{\infty} U_n.$$

Now, since X itself is a neighbourhood of $X \setminus X_{n\ell}$ in βX, it follows that $\bigcap_{n=1}^{\infty} U_n$ is a neighbourhood of X in βX. To complete the proof, we need only see that the set $\tilde{X} = (\upsilon X)_{n\ell} \cap X$ is a closed subset of the compact set $X_{n\ell}$, and thus itself compact.

We provide next a short proof of the following result, which appears in [3].

Corollary 2. *Let* p *be a point in* X *with a countable neighbourhood base in* X, *but no compact neighbourhoods in* X. *Then* $C_c(X)$ *does not carry a pseudo-topological structure.*

Proof. By assumption, we can find a sequence $(x_n)_{n \in \mathbb{N}}$ of points of $\beta X \setminus X$ converging to the point p in βX. Clearly the set

$$\bigcap_{n=1}^{\infty} (\beta X \setminus \{x_n\}) .$$

is not a neighbourhood of X in βX, despite being a countable intersection of open neighbourhoods of X.

When X is a locally compact topological space, $C_c(X)$ is a topological vector space, in particular, carrying a pseudo-topological structure. However, locally compact spaces are not characterized by this latter fact, as the following example shows:

Under the interval topology, the set $[0,\Omega]$ of all ordinal numbers less than or equal to Ω, the first uncountable ordinal, becomes a compact topological space. Hence Y, that subspace of $[0,\Omega]$ obtained by deleting all countable limit ordinals, is completely regular. Since there is but one point, namely Ω, of Y without compact neighbourhoods, and since the neighbourhood filter of Ω in Y admits countable intersections, corollary 1 to theorem 4 implies that $C_c(Y)$ and $C_I(Y)$ coincide

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