On the Eigenvectors of a Finite-Difference Approximation to the Sturm-Liouville Eigenvalue Problem

by

Eckart Gekeler

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1. Introduction

The present contribution is concerned with the nonselfadjoint problem

$$(1)\qquad \begin{aligned} &-[a(x)u_x]_x - b(x)u_x + c(x)u = \lambda u, \qquad 0 < x < 1,\\ &u(0) = u(1) = 0 \end{aligned}$$

where $a(x) \geq \alpha > 0$, $c(x) \geq 0$, and a, b, c are all bounded and smooth functions. This problem has an infinite sequence of positive and distinct eigenvalues

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots$$

and a corresponding sequence of smooth eigenfunctions $u^1, u^2, u^3, \dots$ (see for instance Protter-Weinberger [10, p. 37] and Coddington-Levinson [4, p. 212]). Following Courant-Hilbert [5, p. 334] the eigenfunctions u^p are uniformly bounded in the supremum norm if they are normalized so that

$$\int_0^1 |u^p(x)|^2 dx = 1, \qquad p = 1, 2, 3, \dots .$$

Of course, by the well-known transformation

$$(2)\qquad u(x) = \exp\left(-\frac{1}{2} \int_0^x \frac{b(t)}{a(t)}\, dt\right) w(x)$$

(1) may be put in the selfadjoint form

$$\begin{aligned} &-[a(x)w_x]_x + \tilde{c}(x)w = \lambda w, \qquad 0 < x < 1,\\ &w(0) = w(1) = 0 \end{aligned}$$

where

$$\tilde{c}(x) = c(x) + \frac{1}{2} b_x(x) + \frac{1}{4} b^2(x)/a(x).$$

Here, in order to obtain $\tilde{c}(x) \geq 0$ we have to make a restricting assumption on b_x. Therefore we choose the direct approximation of (1) by means of the finite-difference equations

$$(3) \quad \begin{aligned} &- \frac{a_{k+1/2}(v_{k+1}-v_k) - a_{k-1/2}(v_k-v_{k-1})}{\Delta x^2} - b_k \frac{v_{k+1} - v_{k-1}}{2\Delta x} \\ &+ c_k v_k = \Lambda v_k, \qquad k = 1,\dots,M, \end{aligned}$$

$$v_o = v_{M+1} = 0$$

where $M \in \mathbb{N}$, $\Delta x = 1/(M+1)$, and $v_k = v(k\Delta x)$. Equivalently, we may write (3) in matrix-vector notation

$$(3') \qquad LV = \Lambda V$$

where $V = (v_1,\dots,v_M)^T$ and the matrix L may be easily derived from (3).

Let $|b(x)| \leq \beta$ and $0 < \Delta x < 2\alpha/\beta$. Then the matrix L is equivalent to a real symmetric matrix (see Carasso [2]). Using this fact and Theorem 1.8 of Varga [11] it can be shown that all eigenvalues Λ_p of (3) are real and positive,

$$0 < \Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \dots \leq \Lambda_M ,$$

and there exists a complete sequence of corresponding eigenvectors V^p . A result of Carasso [2, Corollary 1] says that there exist a constant K and an integer p_o, both independent of M, such that

$$|V^p|_\infty = \max_{1\le k\le M} |v_k^p| \le K\, p^{1/2}, \qquad p_o \le p \le M,$$

if

$$|V^p|_2^2 = \Delta x \sum_{k=1}^{M} |v_k^p|^2 = 1.$$

In the selfadjoint case this result goes back to Bückner [1]. In this paper we prove the following theorem:

Theorem. Let $a(x) \ge \alpha > 0$ and $c(x) \ge 0$, $0 < x < 1$. Assume that a, b, and c are differentiable bounded functions with bounded derivatives ; say $|b(x)| \le \beta$. Let $0 < \Delta x \le \alpha/\beta$ and let $\{V^p\}_{p=1}^M$ be the eigenvectors of (3) normalized so that $|V^p|_2 = 1$. Then

$$|V^p|_\infty \le \kappa, \qquad p = 1,\dots,M,$$

for some constant κ independent of M.

Remark 1. In the case of the equation $u_{xx} = \lambda u$ this result may be proved by explicit computation of the eigenvectors V^p (see Isaacson-Keller [9, 9.1.1]).

Applications of the Theorem to the theory of finite-difference approximations to parabolic and hyperbolic partial differential equations are given in [6, 7].

2. Proof of the Theorem

Instead of L we consider, as in [2], the eigenvectors $D^{-1}V^p$ of the similar matrix $D^{-1}LD$ defined below. But in contrast to Carasso [2], who uses a discrete maximum principle for his estimation, we transpose then the proof of Courant-Hilbert [5, p. 334] to the resulting discrete problem.

The following basic results are needed.

Lemma 1 (Carasso [2, Lemma 1, 3, Lemma 3.1]). Let $D = (d_1, \ldots, d_M)$ be the diagonal matrix with

$$d_1 = 1, \quad d_i = + \left[\prod_{k=1}^{i-1} \frac{a_{k+1/2} - b_{k+1}\Delta x/2}{a_{k+1/2} + b_k \Delta x/2} \right]^{1/2}, \quad i = 2, \ldots, M .$$

For $0 < \Delta x < 2\alpha/\beta$ we have $d_i > 0$ and

$$|D|_\infty \leq \kappa_1, \quad |D^{-1}|_\infty \leq \kappa_2$$

for constants κ_1, κ_2 independent of M. Furthermore

$$D^{-1}L\,D = (P + Q)/\Delta x^2$$

where $P = (p_{ik})_{i,k=1,\ldots,M}$

$$p_{ik} = \begin{cases} (p_{k+1/2} + p_{k-1/2}) & i = k \\ -\,p_{k+1/2} & i = k+1 \\ -\,p_{i+1/2} & k = i+1 \\ 0 & \text{otherwise} \end{cases},$$

$$p_{k+1/2} = (a_{k+1/2} - b_{k+1}\Delta x/2)^{1/2} (a_{k+1/2} + b_k \Delta x/2)^{1/2} ,$$

and $Q = (q_1, \ldots, q_M)$ is the diagonal matrix with

$$q_k = (a_{k+1/2} + a_{k-1/2}) - (p_{k+1/2} + p_{k-1/2}) + \Delta x^2 c_k .$$

Remark 2. The change of variables $V = DW$ is a discrete analog to (2) [2].

Lemma 2 (Carasso [2, Theorem 1]). Let Λ_p, V^p be the characteristic pairs of the matrix L with $|V^p|_2 = 1$. Let u^p be an eigenfunction of (1) corresponding to λ_p and let U^p be the vector of dimension M obtained from u^p by mesh-point evaluation. Assume u^p normalized so that $|D^{-1}U^p|_2 = |D^{-1}V^p|_2$ then as $\Delta x \longrightarrow 0$, we have

$$|\lambda_p - \Lambda_p| \leq \kappa_3(p)\Delta x^2,$$

$$|U^p - V^p|_2 \leq \kappa_4(p)\Delta x^2 \tag{4}$$

where κ_3, κ_4 are positive constants depending only on p.

In the selfadjoint case Lemma 2 was proved by Gary [6].

Remark 3. The estimation (4) implies

$$|U^p - V^p|_\infty \leq \kappa_4(p)\Delta x^{3/2} .$$

Lemma 3. Let

$$C_l(W) = \sum_{k=1}^{l} [p_{k+1/2}(w_k-w_{k+1}) - p_{k-1/2}(w_k-w_{k-1})]q_k w_k/\Delta x^2 .$$

Then, under the assumptions of the Theorem ,

$$C_l(W^p) = - q_l p_{l+1/2} w^p_{l+1} w^p_l + \mathcal{O}(1),\ l = 1,\ldots,M,$$

where $\mathcal{O}(1)$ denotes a function which has a bound independent of M.

Proof. We show at first that $|q_k/\Delta x^2| \leq \kappa_5$ independently of M. To this end it suffices to consider $a_{k+1/2} - p_{k+1/2}$.

By means of the binomial theorem we obtain

$$a_{k+1/2} - p_{k+1/2}$$

$$= a_{k+1/2} - a_{k+1/2}\left(1 - \frac{b_{k+1}\Delta x}{4a_{k+1/2}} + \mathcal{O}(\Delta x^2)\right)\left(1 + \frac{b_k \Delta x}{4a_{k+1/2}} + \mathcal{O}(\Delta x^2)\right).$$

Inserting $b_{k+1} = b_k + \mathcal{O}(\Delta x)$ we find that

$$(5) \qquad a_{k+1/2} - p_{k+1/2} = \mathcal{O}(\Delta x^2).$$

Now, since $w_o^p = 0$,

$$C_l(W^p) = - q_l p_{l+1/2} w_{l+1}^p w_l^p/\Delta x^2$$
$$\sum_{k=1}^{l} \frac{q_k}{\Delta x^2}(p_{k+1/2} - p_{k-1/2}) w_k^p w_k^p + \sum_{k=1}^{l-1} \frac{q_{k+1} - q_k}{\Delta x^2} p_{k+1/2} w_{k+1}^p w_k^p .$$

But by the mean value theorem we have $p_{k+1} - p_k = \mathcal{O}(\Delta x)$ and $(q_{k+1} - q_k)/\Delta x^2 = \mathcal{O}(\Delta x)$. Hence, using Schwarz's inequality and $|W^p|_2 \leq \kappa_6$ we obtain the desired result.

Now according to Lemma 1 it suffices to prove the Theorem for the eigenvectors $W^p = D^{-1}V^p$ of the matrix $(P+Q)/\Delta x^2$ which has the eigenvalues Λ_p too. We multiply the k-th row of

$$\frac{1}{\Delta x^2}(P + Q)W^p = \Lambda_p W^p$$

by

$$p_{k+1/2}(w_k^p - w_{k+1}^p) - p_{k-1/2}(w_k^p - w_{k-1}^p)$$

and obtain by adding all rows from $k = 1$ until $k = l$

$$(6) \qquad \left[\frac{p_{l+1/2}(w_l^p - w_{l+1}^p)}{\Delta x}\right]^2 + C_l(W^p)$$
$$+ \Lambda_p p_{l+1/2} w_{l+1}^p w_l^p - \Lambda_p \Delta x \sum_{k=1}^{l} p_x(\xi_k) w_k^p w_k^p = \left[\frac{p_{1/2}(w_1^p - w_o^p)}{\Delta x}\right]^2$$

where $(k-1/2)\Delta x < \xi_k < (k+1/2)\Delta x$. In order to eliminate the term on the right side of (6) we sum up the equations (6) for $l = 1$ to $l = M$, add $(p_{1/2}(w_1^p - w_o^p)/\Delta x)^2$ to both sides, and divide by $M+1 = 1/\Delta x$. Then

$$\left[\frac{p_{1/2}(w_1^p - w_o^p)}{\Delta x}\right]^2 = \Delta x \sum_{l=o}^{M} \left[\frac{p_{l+1/2}(w_{l+1}^p - w_l^p)}{\Delta x}\right]^2 + \Delta x \sum_{l=1}^{M} C_l(W^p)$$

$$(7) \qquad + \Lambda_p \Delta x \sum_{l=1}^{M} p_{l+1/2} w_{l+1}^p w_l^p - \Lambda_p \Delta x^2 \sum_{l=1}^{M} \sum_{k=1}^{l} p_x(\xi_k) w_k^p w_k^p .$$

But

$$(8) \qquad 0 < \alpha/2 < p_{l+1/2} < \kappa_7$$

if $0 < \Delta x \leq \alpha/\beta$. Thus, using the fundamental relation

$$\sum_{l=o}^{M} p_{l+1/2}(w_{l+1} - w_l)^2 = W^T P W$$

and Schwarz's inequality we derive

$$\Delta x \sum_{l=o}^{M} \left[\frac{p_{l+1/2}(w_{l+1}^p - w_l^p)}{\Delta x}\right]^2 + \Delta x \kappa_7 \sum_{l=1}^{M} \frac{q_l}{\Delta x^2} w_l^p w_l^p - \Delta x \kappa_7 \sum_{l=1}^{M} \frac{q_l}{\Delta x^2} w_l^p w_l^p$$

$$\leq \kappa_7 \Lambda_p + \kappa_5 \kappa_7$$

since $|q_l/\Delta x^2| \leq \kappa_5$ independently of M. Hence, applying Schwarz's inequality once more we find from (7) by means of the Assumption and Lemma 3 that

$$\left[\frac{p_{1/2}(w_1^p - w_o^p)}{\Delta x}\right]^2 \leq \kappa_8 \Lambda_p + \kappa_9 .$$

From this estimation, equation (6), and Lemma 3 we deduce that

$$(\Lambda_p - \frac{q_l}{\Delta x^2}) p_{l+1/2} w_{l+1}^p w_l^p \leq \kappa_{10} \Lambda_p + \kappa_{11} .$$

Consequently, observing (8) we obtain, in case $\Lambda_p > \kappa_5$, that

(9) $$w^p_{l+1} w^p_l \leq \kappa_{12}, \qquad l = 1,\ldots,M,$$

for some constant κ_{12} independent of M. For $\Lambda_p \leq \kappa_5$ the assertion of the Theorem follows by Lemma 2.

Finally, we return once more to equation (6). The above estimations yield

$$\left[\frac{p_{l+1/2}(w^p_{l+1} - w^p_l)}{\Delta x}\right]^2 \leq \kappa_{13}\Lambda_p + \kappa_{14}, \quad l = 1,\ldots,M,$$

or, using (9),

$$\max_{1\leq l\leq M} \{w^p_{l+1}, w^p_l\} \leq \Delta x^2(\kappa_{15}\Lambda_p + \kappa_{16}) + \kappa_{17} \leq \kappa$$

because $\Delta x^2 \Lambda_p$ is bounded independently of M.

Dr. E. Gekeler
Universität (WH)
6800 Mannheim
Postfach 2428
(Germany)

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