A-Convergence of Finite-Difference Approximations to Parabolic Initial Boundary Value Problems

by

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1. Introduction

The present contribution is concerned with the parabolic initial boundary problem

$$
(1)\quad
\begin{aligned}
&u_t - L(x,t;u) = d(x,t), \ (x,t) \in (0,1)\times(0,\infty),\\
&u(x,0) = f(x), \qquad 0 < x < 1,\\
&u(0,t) = g^-(t),\ u(1,t) = g^+(t), \quad 0 \leq t.
\end{aligned}
$$

where L is a linear or weakly nonlinear strongly elliptic differential operator which is more exactly defined below. For an arbitrary but fixed real T let $\Delta t = T/N$ and $\Delta x = 1/(M+1)$, $M,N \in \mathbb{N}$, be small increments of the variables t and x, and let

$$v_k^n = v(k\Delta x, n\Delta t),\quad V^n = (v_1^n,\dots,v_M^n)^T,$$

$$V = (V^2,\dots,V^N)^T.$$

We consider two well-known finite-difference approximations of (1), the first one is the Crank-Nicolson method

$$(2)\quad \frac{v_k^n - v_k^{n-1}}{2\Delta t} + \frac{1}{2}\{[l_\Delta v^n]_k + [l_\Delta v^{n-1}]_k\} = d_k^{n-1/2},$$

$$k = 1,\dots,M,\ n = 2,3,\dots ,$$

where l_Δ denotes the finite-difference operator corresponding to - L. The other method will be

$$(3)\quad \frac{3v_k^n - 4v_k^{n-1} + v_k^{n-2}}{2\Delta t} + [l_\Delta v^n]_k = d_k^n, \qquad k = 1,\dots,M,$$

$$n = 2,3,\dots .$$

This latter approximation constitutes an implicit method as well. By lack of better historical knowledge we call it in the following the Richtmyer-Morton method [13, p. 86, p. 190]. We denote by $U^n = (u_1^n, \dots u_M^n)^T$ the vector obtained from the solution u of the non-discrete problem (1) by mesh-point evaluation on the line $t = n\Delta t$ and V_Δ^n, $n = 1,2,\dots,$will be the solution of the discrete problem in consideration.

In this note estimations of the form

$$(4) \qquad |U^N - V_\Delta^N| \le \frac{\kappa}{\Delta t^\sigma}(\Delta t^2 + \Delta x^2)$$

with $\sigma = 0$ or $\sigma = 1/2$ are deduced by means of an expansion in the eigenvectors of the elliptical part of the corresponding finite-difference system and by the monotonicity principle of Minty [11, 12] . So, as the constant κ in (4) does not depend on $T = N\Delta t$, A-convergence of the above indicated methods is established. This means that the solution of the discrete analogues preserve the asymptotic behaviour of the solution of the analytical problem (1) (see Gustafsson [10]).

For some other, more sophisticated A-convergent difference approximations to parabolic problems we refer to Carasso [6].

2. Estimations by Means of Eigenvector Expansions

In this section we follow strictly the pattern of [7, 9]. Thus the presentation is allowed to be rather concised. Here we study the initial boundary problem (1) with the differential equation

(5) $u_t - [a(x)u_x]_x + b(x)u_x + c(x)u = d(x,t)$

under the following assumption

Assumption I. (i) Let $0 < \alpha_o \leq a(x)$, $0 \leq c(x)$.
(ii) Let $a,\dots,a_{xxx}$, $f,\dots, f_{xxxx}$, b, b_x, b_{xx}, c, c_x, c_{xx}, $d(\cdot,0)$, $d_x(\cdot,0)$, $d_{xx}(\cdot,0)$ be continuous and bounded in $G = \{x,\ 0 < x < 1\}$, say $|b(x)| \leq \beta$. Let g^-, g^+, d be continuous in $G\times(0,\infty)$ and d bounded in $G\times(0,T)$ for every $T > 0$.
(iii) Let $f(0) = g^-(0)$, $f(1) = g^+(0)$. Let all coefficients of the problem (1) be sufficiently smooth that a solution $u: (x,t) \longmapsto u(x,t)$ exists in the classical sense having in $G\times(0,\infty)$ three continuous time derivatives and four continuous space derivatives.

The following norms are used

$$|a|_\infty = \sup_{0<x<1} |a(x)| \qquad |V^n|_\infty = \max_{1\leq k\leq M} \{|v_k^n|\},$$

$$|V^n|_2^2 = \Delta x \sum_{k=1}^{M} |v_k^n|^2, \quad |V|_\infty = \max_{2\leq n\leq N} \{|V^n|_\infty\},$$

$$|V|_2^2 = \Delta t \sum_{n=2}^{N} |V^n|_2^2 .$$

At first we consider the Crank-Nicolson approximation to the problem (1) with the differential equation (5), i.e. the finite difference equation (2) where

$$[l_\Delta v]_k =$$

$$-\frac{a_{k+1/2}(v_{k+1}-v_k)-a_{k-1/2}(v_k-v_{k-1})}{\Delta x^2}+b_k\frac{v_{k+1}-v_{k-1}}{2\Delta x}+c_kv_k,$$

$$k = 1,\ldots,M.$$

The discrete boundary conditions are

$$\text{(6)}\qquad v_o^n = (g^-)^n,\quad v_{M+1}^n = (g^+)^n,\qquad n = 0,1,\ldots ,$$

and as initial conditions we choose

$$\text{(7)}\qquad \begin{aligned} v_k^o &= f_k, \\ v_k^1 &= f_k + \Delta t\{[(af_x)_x]_k-b_k[f_x]_k-c_kf_k+d_k^o\} + \frac{\Delta t^2}{2}[u_{tt}]_k^o \end{aligned}$$

where for $[u_{tt}]_k^o$ the following quantity is to be inserted

$$[u_{tt}]_k^o = a_k\{[(af_x)_{xxx}]_k-[(bf_x)_{xx}]_k-[(cf)_{xx}]_k+[d_{xx}]_k^o\}$$

$$+\{[a_x]_k-b_k\}\{[(af_x)_{xx}]_k-[(bf_x)_x]_k-[(cf)_x]_k+[d_x]_k^o\}$$

$$-c_k\{[(af_x)_x]_k - b_k[f_x]_k - c_kf_k + d_k^o\}$$

(then $U^1 - V^1 = \mathcal{O}(\Delta t^3)$). Thus, computing at first V^o and V^1 by means of (7) then inserting the quantities (6) and (7) as far as possible in the equations (2) we can obtain the vectors V^n, $n = 2,3,\ldots$, by an implicit difference approximation which is consistent of order two. Equivalently we may write the conditional equations (2) for the components v_k^n of the vector V in matrix-vector notation

(8) $$P_1 V := \frac{1}{\Delta t}(I - S_1)V + \frac{1}{2}\,T(I + S_1)V = H_1 .$$

Here S_1 is the block matrix

(9) $$S_1 = \begin{pmatrix} 0 & & & \\ I & 0 & & \mathbf{0} \\ & \ddots & \ddots & \\ \mathbf{0} & & I & 0 \end{pmatrix}$$

(I denoting in S_1 the identity matrix of dimension M, in equation (8) the identity matrix of dimension $M\times(N-1)$).
As the coefficients of the differential equation (5) do not depend on t the block diagonal matrix T consists of identical submatrices

(10) $$T = (\tilde{T},\ldots,\tilde{T}).$$

The right side H_1 of (8) contains only known quantities; the exact notation may be suppressed here. Instead the $M\times M$-matrix $\tilde{T}$ may be displayed in detail, since some facts on its eigenvalues and eigenvectors are needed below

$\tilde{T} = A + B + C$, $A = (a_{ik})_{i,k=1,\ldots,M}$ etc.,

$$a_{ik} = \begin{cases} (a_{k+1/2}+a_{k-1/2})/\Delta x^2 \\ -\,a_{k+1/2}/\Delta x^2 \\ -\,a_{i+1/2}/\Delta x^2 \\ 0 \end{cases}, \quad b_{ik} = \begin{cases} 0 & i = k \\ b_i/2\Delta x & i = k+1 \\ -b_i/2\Delta x & k = i+1 \\ 0 & \text{otherwise} \end{cases},$$

$$c_{ik} = c_k \delta_{ik}$$

(δ_{ik} denoting the Kronecker symbol).

Now, for $0 < \Delta x \le \alpha_0/\beta$ the matrix $\tilde{T}$ has real positive eigenvalues λ_p which are bounded away from zero (see

Carasso [2] and [9]). Moreover, following [2, Lemma 1] there exists a real diagonal matrix $\tilde{D}$, which is together with its inverse bounded in the maximum norm independently of M, such that $\tilde{D}^{-1}\tilde{T}\,\tilde{D}$ is a symmetric matrix. Consequently $\tilde{D}^{-1}\tilde{T}\,\tilde{D}$ has a complete system of orthogonal eigenvectors W_p, $p = 1,\ldots,M$. By a result of [8] these eigenvectors are bounded in the maximum norm independently of M if they are normalized so that $|W_p|_2^2 = 1$. Thus, by a result of Carasso [2, Lemma 3], it can be shown that

$$(11)\qquad \sum_{p=1}^{M} \frac{|W_p|_\infty}{\lambda_p^{(1+\varepsilon)/2}} \le \kappa_1 \sum_{p=1}^{M} p^{-(1+\varepsilon)/2}$$

for $0 < \Delta x \le \alpha_0/\beta$ where κ_1 denotes a constant independent of M (see also [9]). Let D, Λ, and W be the block diagonal matrices of order N-1 with the diagonal submatrices $\tilde{D}$, $\tilde{\Lambda} = (\lambda_1,\ldots,\lambda_M)$, respectively with the matrix $\tilde{W} = (W_1,\ldots,W_M)$. Further, define the block vector Φ_1 and the block matrix Q_1 by the relations

$$D^{-1}(P_1U - H_1) = W\Phi_1, \quad D^{-1}P_1D = WQ_1W^{-1}.$$

Then, following the pattern of [9] (see also Carasso-Parter [4] and [7]) we may write

$$(12)\qquad \begin{aligned} |U^N - V_\Delta^N|_\infty &= |DD^{-1}P_1^{-1}(P_1U - H_1)|_\infty \le |D|_\infty |W\Lambda^{-1}\Lambda Q_1^{-1}\Phi_1|_\infty \\ &\le |D|_\infty \sum_{p=1}^{M} \frac{|W_p|_\infty}{\lambda_p^{(1+\varepsilon)/2}} \left| \frac{1}{\lambda_p^{(1-\varepsilon)/2}} \hat{P}_1(\lambda_p \Delta t)^{-1} \right|_\infty |\Phi_{1,p}|_\infty \end{aligned}$$

where $\Phi_p = (\phi_p^2,\ldots,\phi_p^N)^T$ and $\hat{P}_1(\lambda_p\Delta t)$ is the $(N-1)\times(N-1)$-matrix

(13) $$\hat{P}_1(\lambda_p \Delta t) = \frac{1}{\lambda_p \Delta t}(I - S_1^*) + \frac{1}{2}(I + S_1^*) \ .$$

Here I denotes the identity matrix of dimension N-1 and S_1^* is a matrix of the form (9) but with unities in place of the matrices I (see [7]). We remark that $\lambda_p \Delta x^2$ has a bound independently of M (see [2, Lemma 3]), i.e. $\lambda_p^{\varepsilon/2} \leq \kappa_2 \Delta x^{-\varepsilon}$. Hence, observing inequality (11) and the consistence of the Crank-Nicolson method, i.e. $|\Phi_{1,p}|_\infty = \mathcal{O}(\Delta t^2 + \Delta x^2)$, we find that an estimation of $|U^N - V_\Delta^N|_\infty$ depends on a bound for the values

(14) $$\left|\frac{1}{\lambda_p^{1/2}} \hat{P}_1(\lambda_p \Delta t)^{-1}\right|_\infty, \qquad p = 1,\dots,M.$$

But, as a consequence of [7, proof of Theorem 5 and Remark 4] and of the uniform boundedness of the values $|W_p|_\infty$ [8] it is easily deduced that the quantities (14) are uniformly bounded independent of $\lambda \Delta t \geq 0$ and N. Summarizing the above estimations we obtain the following theorem

Theorem 1. Suppose that the parabolic initial boundary problem (1) with the differential equation (5) satisfies Assumption I. Let $0 < \Delta x = 1/(M+1) \leq \alpha_0/\beta$, $\Delta t = T/N$, $M,N \in \mathbb{N}$, and let $\Delta t/\Delta x$ be arbitrary but fixed as $\Delta x \longrightarrow 0$. Denote by U^N the vector of dimension M obtained from the solution of the problem (1), (5) by mesh-point evaluation on the line $t = N\Delta t = T$. Let V_Δ^n, $n = 1,\dots,N$, be the solution of the system (8) with the initial values (7). Then

$$|U^N - V_\Delta^N|_\infty \leq \frac{\kappa_3}{\Delta x^\varepsilon} (K_1(T)\Delta t^2 + K_3(T)\Delta x^2) \sum_{p=1}^{M} p^{-(1+\varepsilon)}$$

where κ_3 is a constant independent of T, Δt, and Δx. The constants K_1, K_3 are uniformly bounded quantities independent of T if the solution u of the problem (1), (5) together with its derivatives up to order four is bounded in $(0,1)\times(0,\infty)$.

Remark 1. Note that

$$K_1(T) = \max_{0\leq t\leq T} \{\tfrac{1}{24}|u_{ttt}(\cdot,t)|_\infty\} \; ;$$

an estimation for $K_3(T)$ is given in [9].

Next we apply the Richtmyer-Morton method to the problem (1) with the equation (5). Again we start with (7) and insert the quantities (6), (7) in the conditional equations (3). We follow the above considerations exactly and obtain again an inequality of the form (12):

$$(15) \quad |U^N - V_\Delta^N|_\infty \leq |D|_\infty \sum_{p=1}^{M} \frac{|W_p|_\infty}{\lambda_p} |\hat{P}_2(\lambda_p\Delta t)^{-1}|_\infty |\Phi_{2,p}|_\infty \; ,$$

where now the components of V_Δ are computed by (7) and (3). $\hat{P}_2(\lambda\Delta t)$ is the following $(N-1)\times(N-1)$-matrix (see [7])

$$(16)\quad \hat{P}_2(\lambda\Delta t) = I + \frac{1}{2\lambda\Delta t}\begin{pmatrix} 3 & & & & \\ -4 & 3 & & & \\ 1 & -4 & 3 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 3 \end{pmatrix} =: I + \frac{1}{2\lambda\Delta t} S_2^* .$$

Accordingly, in order to establish the result of Theorem 1 with $\varepsilon = 0$ for the Richtmyer-Morton method we have to show that $|\hat{P}_2(\lambda\Delta t)^{-1}|_\infty$ has a bound independent of $\lambda\Delta t \geq 0$ and N. This result is a consequence of the proof of Theorem 6 in [7]. But in the present case it can be proved in a much simplier way using some of the ideas in [7]. For simplicity let $\sigma = 1/2\lambda\Delta t$ then $0 < \sigma < \infty$ as λ denotes an arbitrary eigenvalue of T. Further assume that

$$\tilde{L} = \begin{pmatrix} 1+3\sigma & 0 \\ -4\sigma & 1+3\sigma \end{pmatrix}, \quad \tilde{R} = \begin{pmatrix} \sigma & -4\sigma \\ 0 & \sigma \end{pmatrix}.$$

Then

$$\hat{P}_2(\lambda\Delta t) = L + RS_1^* = L(I + L^{-1}RS_1^*)$$

where L and R are block diagonal matrices with the identical submatrices $\tilde{L}$ and $\tilde{R}$, and S_1 is the matrix of equation (13). However, the eigenvalues of the matrix $\tilde{L}^{-1}\tilde{R}$ are

$$\eta_{1,2} = (1 + 3\sigma)^{-2}(-5\sigma^2 + \sigma \pm 4\sigma(\sigma^2 - \sigma)^{1/2}),$$

i.e. $|\eta_{1,2}| < 1$ for $\sigma \geq 0$. The corresponding matrix $\tilde{X}$ of the eigenvectors reads in case $\sigma \neq 1$

$$\tilde{X} = \begin{pmatrix} \frac{2\sigma+(\sigma^2-\sigma)^{1/2}}{\sigma} & \frac{2\sigma-(\sigma^2-\sigma)^{1/2}}{\sigma} \\ 1 & 1 \end{pmatrix} = 2(\sigma^2-\sigma)^{1/2} \begin{pmatrix} \sigma & -2\sigma+(\sigma^2-\sigma)^{1/2} \\ -\sigma & 2\sigma+(\sigma^2-\sigma)^{1/2} \end{pmatrix}^{-1}$$

(see [7]). In case $\sigma = 1$ we obtain the following Jordan canonical form

$$\tilde{L}^{-1}\tilde{R} = \tilde{Y}\tilde{\Omega}\tilde{Y}^{-1} = \begin{pmatrix} 4 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1/4 & 0 \\ 1 & -1/4 \end{pmatrix} \begin{pmatrix} 1/4 & -1/2 \\ 0 & 1 \end{pmatrix} .$$

Therefore, denoting by X, Y, Ω, and Θ the block diagonal matrices with the submatrices $\tilde{X}$, $\tilde{Y}$, $\tilde{\Omega}$, and $\tilde{\Theta} = (\eta_1, \eta_2)$ respectively, we have to show that

$$|\hat{P}_2(1/2)^{-1}|_\infty = |Y(I + \Omega S_1^*)^{-1} Y^{-1} L^{-1}|_\infty$$

has a bound independent of N, and that

$$|\hat{P}_2(1/2\sigma)^{-1}|_\infty = |X(I + \Theta S_1^*)^{-1} X^{-1} L^{-1}|_\infty$$

has a bound independent of N if $\sigma \geq 0$ remains bounded away from one. Then the proof is finished since $\hat{P}_2(\lambda\Delta t)$ has an inverse for $\sigma = 1/2\lambda\Delta t \geq 0$ and depends continuously on $\sigma \geq 0$.

In the case $\sigma = 1$ it suffices to show that $|(I + \Omega S_1^*)^{-1}|_\infty$ has a bound independent of N. This is true since

$$\tilde{\Omega}^n = \begin{pmatrix} (-1/4)^n & 0 \\ n(-1/4)^{n-1} & (-1/4)^n \end{pmatrix}$$

and therefore

$$|(I + \Omega S_1^*)^{-1}|_\infty \leq \sum_{n=1}^{\infty} [(1/4)^n + n(1/4)^{n-1}] \leq \kappa_4 .$$

If $0 < \sigma < 1 - \delta$ ($\delta > 0$) we find in the same way that $|\hat{P}_2(1/2\delta)^{-1}|_\infty$ has a bound which depends only on δ estimating each of the values $|L^{-1}|_\infty$, $|X|_\infty$, $|X^{-1}|_\infty$ for its own and observing that

$$|(I + \Theta S_1^*)^{-1}|_\infty \leq (1 - \max\{|\eta_1|, |\eta_2|\})^{-1}$$

$$\leq \frac{(1+3\sigma)^2}{(1+3\sigma)^2 - [(\sigma-5\sigma^2)^2+16\sigma^2(\sigma-\sigma^2)]^{1/2}} \leq \kappa_5 .$$

In case $1 + \delta < \sigma$ ($\delta > 0$) we deduce easily that $|X|_\infty$, $|X^{-1}|_\infty$, and $|(1 + 3\sigma)L^{-1}|_\infty$ have bounds depending only on δ. Moreover

$$|(1+3\sigma)^{-1}(I+\Theta S_1^*)^{-1}|_\infty \leq \frac{1+3\sigma}{(1+3\sigma)^2-(5\sigma^2-\sigma+4\sigma(\sigma^2-\sigma)^{1/2})}$$

$$= \frac{1+3\sigma}{1+4\sigma^2+7\sigma-4\sigma(\sigma^2-\sigma)^{1/2}} \leq \kappa_6 .$$

Theorem 2. Suppose that the assumptions of Theorem 1 are satisfied. Let V_Δ^n, $n = 1,\dots,N$, be the solution of the equations (3) with the initial values (7). Then

$$|U^N - V_\Delta^N|_\infty \leq \kappa_7(K_2(T)\Delta t^2 + K_3(T)\Delta x^2)$$

where κ_7 is a constant independent of T, Δt, and Δx. The constants K_2, K_3 are uniformly bounded quantities independent of T if the solution u of the problem (1), (5) together with its derivatives up to order four is bounded in $(0,1)\times(0,\infty)$.

Remark 2. To be completely we note that

$$K_2(T) = \max_{0 \le t \le T} \{|u_{ttt}(\cdot,t)|_\infty\}$$

while $K_3(T)$ is the same quantity as in Theorem 1.

3. The Richtmyer-Morton Method and Weakly Nonlinear Problems

In this section we apply the Richtmyer-Morton method to the problem (1) with the following differential equation

$$(17)\qquad u_t - [a(x,t)u_x]_x + b(x,t)u_x + c(x,t,u) = 0.$$

Then, of course, the initial values (8) must be modified:

$$(18)\qquad \begin{aligned} v_k^0 &= f_k, \\ v_k^1 &= f_k + \Delta t\{[(af_x)_x]_k^0 - b_k^0[f_x]_k - c_k^0(f_k)\} + \frac{\Delta t^2}{2}[u_{tt}]_k^0, \end{aligned}$$

where now

$$\begin{aligned} [u_{tt}]_k^0 &= a_k^0\{[(af_x)_{xxx}]_k^0 - [(bf_x)_{xx}]_k^0 - [c(x,t,f(x))_{xx}]_k^0\} \\ &+ \{[a_x]_k^0 - b_k^0\}\{[(af_x)_{xx}]_k^0 - [(bf_x)_x]_k^0 - [c(x,t,f(x)_x]_k^0\} \\ &+ [a_{xt}f_x + a_t f_{xx} - b_t f_x - c_t(x,t,f(x))]_k^0 \end{aligned}$$

is to be inserted. In the same way as above we obtain the following system of equations

$$(19)\qquad P_3(V) := \frac{1}{2\Delta t} S_2 V + T(V) = H_3$$

where

$$S_2 = \begin{pmatrix} 3I & & & & \\ -4I & 3I & & \text{\Large 0} & \\ I & -4I & \ddots & & \\ & \ddots & \ddots & \ddots & \\ \text{\Large 0} & & I & -4I & 3I \end{pmatrix}$$

(I identity matrix of dimension M),

$$T(V) = (T^2(V^2),\dots,T^N(V^N))^T,\quad T^n(V^n) = (A^n + B^n)V^n + C^n(V^n),$$

$A^n = (a_{ik}^n)_{i,k=1,\dots,M}$ etc.,

$$a_{ik}^n = \begin{cases} (a_{k+1/2}^n + a_{k-1/2}^n)/\Delta x^2 \\ -\ a_{k+1/2}^n/\Delta x^2 \\ -\ a_{i+1/2}^n/\Delta x^2 \\ 0 \end{cases}, \quad b_{ik}^n = \begin{cases} 0 & i = k \\ b_i^n/2\Delta x & i = k+1 \\ -b_i^n/2\Delta x & k = i+1 \\ 0 & \text{otherwise} \end{cases},$$

and

$$C^n(V^n) = (c_1^n(v_1^n),\ldots,c_M^n(v_M^n))^T, \qquad n = 2,3,\ldots,N.$$

Assumption II. Let $0 < \alpha_o \leq a(x,t)$ and

$$(c(x,t,v) - c(x,t,w))(v - w) \geq 0, \qquad (x,t) \in G \times (0,\infty); v,w \in \mathbb{R}.$$

(ii) Let $a,\ldots,a_{xxx},f,\ldots,f_{xxxx},b,\ b_x,b_{xx},c,c_x,c_{xx},c_t,$ c_u,c_{uu} be continuous and bounded in $G \times \{0\}$. Let $a,\ldots,a_{xxx},$ $b,\ g^-,g^+$ be continuous in $\bar{G} \times (0,\infty)$ and bounded in $G \times (0,T)$ for every $T > 0$. Let $|b_x(x,t)| \leq \mu$ and c be continuous in $G \times (0,\infty)$.

(iii) as Assumption I (iii).

Theorem 3. Suppose that the parabolic initial boundary problem (1) with the differential equation (17) satisfies Assumption II. Assume that there exists an $\varepsilon > 0$ such that $\alpha_o \pi^2 - \mu/2 - \varepsilon > 0$. Let $\Delta x = 1/(M+1)$, $\Delta t = T/N$, $M,N \in \mathbb{N}$, and let V_Δ^n, $n = 1,\ldots,N$ be the solution of the system (19) with the initial values (18). If $0 < \Delta x \leq (12\varepsilon/\alpha_o \pi^4)^{1/2}$ and $\Delta t/\Delta x$ arbitrary but fixed as $\Delta x \to 0$ then

$$|U - V_\Delta|_2 \leq \kappa_8(K_2(T)\Delta t^2 + K_3(T)\Delta x^2)$$

where κ_8 denotes a constant independent of T, Δt, and Δx, and K_2, K_3 are the values of Theorem 2.

Proof. The $(N-1)\times(N-1)$-matrix

$$S_2^* = \begin{pmatrix} 3 & & & & & \\ -4 & 3 & & & 0 & \\ 1 & -4 & 3 & & & \\ & \ddots & \ddots & \ddots & & \\ & 0 & & 1 & -4 & 3 \end{pmatrix}$$

is monotone, i.e.

$$Y^T(S_2^* + (S_2^*)^T)Y \geq 0 \ , \qquad \forall\, Y \in \mathbb{R}^{N-1} \ ,$$

since

$$\sum_{n=1}^{N-1} y_n(y_{n-2}-4y_{n-1}+6y_n-4y_{n+1}+y_{n+2}) = \sum_{n=o}^{N} (y_{n-1}-2y_n+y_{n+1})^2$$

$(y_{-1} = y_o = y_N = y_{N+1} = 0)$. Consequently. using a result of Carasso [5, p.310, 3, p. 789], we find that P_3 is strictly monotone:

$$(P_3(Y) - P_3(Z),\ Y - Z) \geq \alpha_\varepsilon |Y - Z|_2^2, \forall\, Y,Z \in \mathbb{R}^{M\times(N-1)},$$

where $\alpha_\varepsilon = \alpha_o\pi^2 - \mu/2 - \varepsilon > 0$ independent of T, Δt, and Δx. Hence $P_3^{-1}\colon \mathbb{R}^{M\times(N-1)} \longrightarrow \mathbb{R}^{M\times(N-1)}$ exists and it is continuous (see Browder [1] and Minty [11, 12]); moreover

$$|U - V_\Delta|_2 = |P_3^{-1}(P_3(U)) - P_3^{-1}(P_3(V_\Delta))|_2 \leq \alpha_\varepsilon^{-1}|P_3(U) - H_3|_2 \ .$$

Now the assertion follows since the Richtmyer-Morton method with the initial values (18) is consistent of second order.

As a consequence of the inequality

$$\Delta t |U^N - V_\Delta^N|_2^2 \leq |U - V_\Delta|_2^2$$

we obtain from Theorem 3 immediately the following

Corollary

Corollary. Under the assumptions of Theorem 3 we have

$$|U^N - V_\Delta^N|_2 \leq \frac{\kappa_8}{\Delta t^{1/2}} (K_2(T)\Delta t^2 + K_3(T)\Delta x^2).$$

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