Measurability Theorems for Stochastic Extremals

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[42]

Summary: Measurability of the optimal value is proved for a rather general class of parametric optimization problems. The class considered includes in particular the stochastic convex programs. The measurability of the optimal solutions is discussed for a special case.

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In [1] a direct and elementary proof was given for the measurability of the optimal value of a stochastic linear program. It turns out that the same technique yields measurability statements for very general nonlinear optimization problems, too.

1. Let Ω be some measurable space, and let X be some subset of $\mathbb{R}^n$. Throughout we suppose that X contains a countable dense subset $\Xi = \{\xi_i\}_{i\in\mathbb{N}}$. Let the functions $F: X \times \Omega \to \mathbb{R}$ and $f: X \times \Omega \to \mathbb{R}$ be measurable on Ω for every $x\in X$. We are interested in the measurability of the optimal value

$$\Phi(\omega) = \begin{cases} \inf_x \{F(x,\omega)\,|\,x\in X, f(x,\omega) \le 0\} & \text{if } \{x\,|\,x\in X, f(x,\omega) \le 0\} \ne \emptyset, \\ +\infty & \text{else.} \end{cases}$$

Let us define in addition for $n\in\mathbb{N}$

$$\tau_n(\omega) = \begin{cases} \inf_x \{F(x,\omega)\,|\,x\in X, f(x,\omega) \le \frac{1}{n}\} & \text{if } \{x\,|\,x\in X, f(x,\omega) \le \frac{1}{n}\} \ne \emptyset, \\ +\infty & \text{else,} \end{cases}$$

and for all $n\in\mathbb{N}$, $i\in\mathbb{N}$

$$\Phi_{in}(\omega) = \begin{cases} F(\xi_i,\omega) & \text{if } f(\xi_i,\omega) \le \frac{1}{n}, \\ +\infty & \text{else.} \end{cases}$$

According to our measurability assumptions $\Phi_{in}(\omega)$ is an extended real valued measurable function for every $n\in\mathbb{N}$ and $i\in\mathbb{N}$.

<u>Lemma:</u> *Let* F *and* f *be upper semicontinuous on* X *for every* $\omega\in\Omega$, *and suppose that* $\sup_n \tau_n(\omega) \ge \Phi(\omega)$ *for all* $\omega\in\Omega$. *Then* $\Phi(\omega)$ *is measurable.*

Proof: For all $n\in\mathbb{N}$ and $i\in\mathbb{N}$ we have $\tau_n(\omega) \le \Phi_{in}(\omega)$, implying $\tau_n(\omega) \le \inf_i \Phi_{in}(\omega)$, and hence $\sup_n \tau_n(\omega) \le \sup_n \inf_i \Phi_{in}(\omega)$. By hypothesis then

(1) $$\Phi(\omega) \le \sup_n \inf_i \Phi_{in}(\omega).$$

To show the converse inequality we suppose first that $\Phi(\omega) < +\infty$. Then there exist points $x \in X$ satisfying $f(x,\omega) \le 0$, and - due to the upper semicontinuity of F and f - for every such x and for all $\varepsilon > 0$, $n \in \mathbb{N}$ there exists a $\xi_i \in \Xi$ such that

$$f(\xi_i,\omega) \le \frac{1}{n},\ F(\xi_i,\omega) \le F(x,\omega) + \varepsilon.$$

Hence for every $n \in \mathbb{N}$ we have $\inf_i \Phi_{in}(\omega) \le F(x,\omega) + \varepsilon$, and therefore $\sup_n \inf_i \Phi_{in}(\omega) \le F(x,\omega) + \varepsilon$. Since this inequality is true for all $x \in \{x \mid x \in X, f(x,\omega) \le 0\}$ and for every $\varepsilon > 0$, we have

(2) $$\sup_n \inf_i \Phi_{in}(\omega) \le \Phi(\omega).$$

Inequality (2) is trivially satisfied if $\Phi(\omega) = +\infty$. From (1) and (2) we obtain

(3) $$\Phi(\omega) = \sup_n \inf_i \Phi_{in}(\omega).$$

Since the infimum and supremum of countably many measurable functions is again measurable, the Lemma follows. q.e.d.

The assumption $\sup_n \tau_n(\omega) \ge \Phi(\omega)$ may be replaced by the assumption that the Kuhn-Tucker condition holds for all ω with $\Phi(\omega) < +\infty$. More precisely we have

<u>Theorem 1:</u> *Let F and f be upper semicontinuous on X for every $\omega \in \Omega$. Suppose that for every $\omega \in \{\omega \mid \Phi(\omega) < +\infty\}$ there exists a real number $u(\omega) \ge 0$ such that*

$$\Phi(\omega) \le F(x,\omega)+u(\omega)\cdot f(x,\omega) \quad \forall x \in X \qquad \textit{(K.-T. condition)},$$

and suppose that for every $\omega \in \{\omega \mid \Phi(\omega) = +\infty\}$ we have $\sup_n \tau_n(\omega) = +\infty$. Then $\Phi(\omega)$ is measurable.

Proof: We have to show that $\sup_n \tau_n(\omega) \ge \Phi(\omega)$ for all ω satisfying $\Phi(\omega) < +\infty$. Then the result follows from the Lemma. According to the Kuhn-Tucker condition assumed, $F(x,\omega) \ge \Phi(\omega) - u(\omega)\cdot\frac{1}{n}$ for all $x \in X$ such that $f(x,\omega) \le \frac{1}{n}$. Hence $\tau_n(\omega) \ge \Phi(\omega) - u(\omega)\cdot\frac{1}{n}$, which implies

$$\sup_n \tau_n(\omega) \ge \Phi(\omega).$$ q.e.d.

<u>Corollary 1:</u> *If* $X = \mathbb{R}^n$, *if* F *is convex in* x *for every* $\omega\in\Omega$, *and if* $f(x,\omega) = \max_{1\le j\le m} \ell_j(x,\omega)$, *where the functions* ℓ_j *are linear-affine in* x, *then* $\Phi(\omega)$ *is measurable.*

Proof: F and f are continuous in x, since they are convex over all of $\mathbb{R}^n$. The Kuhn-Tucker condition is satisfied, since it always holds for convex programs with only linear constraints. If the linear system $\ell_j(x,\omega) \le 0$ (with $j=1,\dots,m$) has no solution, then it is a standard result of linear programming that the system $\ell_j(x,\omega) \le \frac{1}{n}$ $(j=1,\dots,m)$ also has no solution for all sufficiently large $n\in\mathbb{N}$. Thus $\Phi(\omega) = +\infty$ implies $\sup_n \tau_n(\omega) = +\infty$. The assumptions of Theorem 1 are therefore satisfied. q.e.d.

Corollary 1 implies in particular that the optimal value of a stochastic linear program is measurable.

2. The assumption, made in Theorem 1, that the Kuhn-Tucker condition be satisfied for all ω with $\Phi(\omega) < +\infty$ is very restrictive, since even for convex programs the Kuhn-Tucker condition generally holds only if $\inf_{x\in X} f(x,\omega) < 0$. It is for this reason that we introduce a modified optimal value, $\Psi(\omega)$, defined as

$$\Psi(\omega) = \begin{cases} \inf_x \{F(x,\omega) \mid x\in X, f(x,\omega) \le 0\} & \text{if } \inf_{x\in X} f(x,\omega) < 0, \\ \sup_n \tau_n(\omega) & \text{if } \inf_{x\in X} f(x,\omega) = 0, \\ +\infty & \text{if } \inf_{x\in X} f(x,\omega) > 0. \end{cases}$$

<u>Theorem 2:</u> *Let* F *and* f *be upper semicontinuous on* X *for every* $\omega\in\Omega$. *Suppose that for all* $\omega\in\{\omega \mid \inf_X f(x,\omega) < 0\}$ *there exists a real number* $u(\omega) \ge 0$ *such that*

$$\Psi(\omega) \le F(x,\omega)+u(\omega)\cdot f(x,\omega) \quad \forall x\in X \qquad \textit{(K.-T. condition).}$$

Then $\Psi(\omega)$ *is measurable.*

Proof: As in the proof of the Lemma we have for all $\omega\in\Omega$ that

$$\sup_n \tau_n(\omega) \le \sup_n \inf_i \Phi_{in}(\omega).$$

If $\inf_X f(x,\omega) < 0$ we conclude from the Kuhn-Tucker condition, as in the proof of Theorem 1, that

$$\Psi(\omega) \leq \sup_n \tau_n(\omega).$$

This is also true if $\inf_X f(x,\omega) = 0$, from the definition of Ψ. If $\inf_X f(x,\omega) > 0$, then there is a real number $\rho(\omega) > 0$ such that $f(x,\omega) \geq \rho(\omega)$ for all $x \in X$, implying $\tau_n(\omega) = +\infty$ for all $n > \frac{1}{\rho(\omega)}$, and thereby $\Psi(\omega) = \sup_n \tau_n(\omega) = +\infty$. Hence we have for all $\omega \in \Omega$

(4) $$\Psi(\omega) \leq \sup_n \inf_i \Phi_{in}(\omega).$$

On the other hand for all ω satisfying $\inf_X f(x,\omega) \neq 0$ the relation

(5) $$\sup_n \inf_i \Phi_{in}(\omega) \leq \Psi(\omega)$$

follows from the upper semicontinuity of F and f, as in the proof of the Lemma. Let now $\inf_X f(x,\omega) = 0$. Choose $n \in \mathbb{N}$ and $\varepsilon > 0$ arbitrarily. Then for every $x \in X$ satisfying $f(x,\omega) \leq \frac{1}{2n}$ there exists, according to the upper semicontinuity of F and f, an element $\xi_i \in \Xi$ such that

$$f(\xi_i,\omega) \leq f(x,\omega) + \frac{1}{2n} \leq \frac{1}{n}, \; F(\xi_i,\omega) \leq F(x,\omega) + \varepsilon.$$

Hence $\Phi_{in}(\omega) \leq F(x,\omega) + \varepsilon$ and $\inf_i \Phi_{in}(\omega) \leq \tau_{2n} + \varepsilon$. Since ε was arbitrary we get

$$\sup_n \inf_i \Phi_{in}(\omega) \leq \sup_n \tau_{2n}(\omega) \leq \sup_n \tau_n(\omega).$$

Since $\sup_n \tau_n(\omega) = \Psi(\omega)$ in the case under consideration, (5) again holds. In conclusion we have from (4), (5)

$$\Psi(\omega) = \sup_n \inf_i \Phi_{in}(\omega),$$

which proves the measurability of $\Psi(\omega)$. q.e.d.

Corollary 2: *Let* X *be a convex set, and let* F *and* f *be convex functions in* x *for every* $\omega \in \Omega$. *Then* $\Psi(\omega)$ *is measurable.*

Proof: The Kuhn-Tucker condition, as required in Theorem 2, is satisfied, since $\inf_X f(x,\omega) < 0$ is the well-known Slater-condition, the latter implying in the convex case the validity of the Kuhn-Tucker condition. The requirement of upper semicontinuity may be dropped in the convex case. Indeed, the upper semicontinuity of F (resp. f) was used only to conclude that for every $x \in X$ and $\varepsilon > 0$ there exists $\xi_i \in \Xi$ satisfying

(6) $$F(\xi_i,\omega) \leq F(x,\omega) + \varepsilon.$$

In the convex case the same conclusion may be reached as follows: Let z be an arbitrary point in the relative interior of X. Then $x_\lambda = x+\lambda(z-x)$ with $0 < \lambda \leq 1$ is also in the relative interior of X. Since F is convex in x we have

$$F(x_\lambda,\omega) \leq F(x,\omega) + \lambda\{F(z,\omega) - F(x,\omega)\}.$$

Choose $\lambda > 0$ so small that

(7) $$F(x_\lambda,\omega) \leq F(x,\omega) + \frac{\varepsilon}{2}.$$

Since x_λ is in the relative interior of X, since F – as a convex function – is continuous in the relative interior of its domain X, and since Ξ is dense in X, there exists $\xi_i \in \Xi$ such that

(8) $$|F(\xi_i,\omega) - F(x_\lambda,\omega)| \leq \frac{\varepsilon}{2}.$$

From (7) and (8) follows (6). q.e.d.

3. We would like to point out that Corollary 1 could also be derived from Theorem 2 instead of from Theorem 1, since it may be shown under the assumptions of Corollary 1 that Φ and Ψ coincide. Under the weaker assumptions of Corollary 2, however, Φ and Ψ may differ, as the following examples show: Choose

$$X = \{(x_1,x_2)\,|\,x_1 \geq 1\} \subset \mathbb{R}^2,\ F(x_1,x_2,\omega) = x_2, f(x_1,x_2,\omega) = (x_2)^2/x_1.$$

Then $\Phi(\omega) = 0$, but $\Psi(\omega) = \sup_n \tau_n(\omega) = -\infty$. To take another example, let

$$X = [0,1] \subset \mathbb{R}^1,\ F(x,\omega) \equiv 0,\ f(0,\omega) = 1,\ f(x,\omega) = x^2 \ \text{ for } x > 0.$$

Then $\Phi(\omega) = +\infty$, but $\Psi(\omega) = 0$. A further comparison of Φ and Ψ therefore seems appropriate.

<u>Theorem 3:</u> *If X is compact, and F and f are lower semicontinuous in x for every $\omega \in \Omega$, then $\Phi(\omega) = \Psi(\omega)$.*

Proof: We have to show that if $\inf_X f(x,\omega) = 0$ then $\sup_n \tau_n(\omega) = \Phi(\omega)$. Obviously $\sup_n \tau_n(\omega) \le \Phi(\omega)$. On the other hand, by lower semicontinuity and compactness, for every $n \in \mathbb{N}$ there exists $x_n \in X$ satisfying

$$f(x_n,\omega) \le \frac{1}{n}, \quad F(x_n,\omega) = \tau_n(\omega).$$

Let $\{x_{n_j}\}$ be a subsequence converging to some $x_o \in X$. Then, by lower semicontinuity and by the monotonicity of $\tau_n(\omega)$,

$$f(x_o,\omega) \le \liminf_{j\to\infty} f(x_{n_j},\omega) \le 0, \quad F(x_o,\omega) \le \liminf_{j\to\infty} F(x_{n_j},\omega) \le \sup_n \tau_n(\omega),$$

implying $\sup_n \tau_n(\omega) \ge \Phi(\omega)$. q.e.d.

Combining Theorems 2 and 3 one can derive measurability statements for $\Phi(\omega)$. In particular we obtain very easily the following result which is contained in [2, Corollary 4.3].

Corollary 3: *Let* X *be a closed convex set, and let* F *and* f *be lower semicontinuous convex functions on* X *for every* $\omega \in \Omega$. *Then* $\Phi(\omega)$ *is measurable.*

Proof: For all $k \in \mathbb{N}$ denote by $\Phi_k(\omega)$ [resp. $\Psi_k(\omega)$] the functions which are obtained if in the definition of Φ [resp. Ψ] we replace X by the compact subset $X_k \equiv \{x \mid x \in X, \|x\| \le k\}$. By Corollary 2 $\Psi_k(\omega)$ is measurable. By theorem 3 Φ_k equals Ψ_k, hence is measurable. The measurability of Φ follows since obviously

$$\Phi(\omega) = \inf_k \Phi_k(\omega).$$ q.e.d.

4. In this section we want to discuss briefly the measurability of the solution mapping for the case of set-valued constraints.

Let X be a *nonvoid, compact, convex* subset of $\mathbb{R}^n$. Let Γ be the family of all nonvoid, closed, convex subsets of X. For arbitrary $C \in \Gamma$ define $U(C,\varepsilon) = \{x \in X \mid \exists \xi \in C: \|x - \xi\| \le \varepsilon\}$. Obviously $U(C,\varepsilon)$ is also in Γ. We make Γ a metric space by introducing the Hausdorff-distance

$$d(C_1,C_2) = \min\{\vartheta \mid \theta \in \mathbb{R},\ C_1 \subset U(C_2,\theta),\ C_2 \subset U(C_1,\theta)\}.$$

Let Ω be a measurable space. Let $c: \Omega \ni \omega \to c(\omega) \in \Gamma$ be a *measurable* mapping, and let $F: X \to \mathbb{R}$ be a *continuous, convex* function. The function $\Phi(\omega): \Omega \to \mathbb{R}$ is defined as

$$\Phi(\omega) = \min \{F(x) | x \in c(\omega)\}.$$

Under the assumptions made we may associate with each $\omega \in \Omega$ a set $\hat{c}(\omega) \in \Gamma$ - the solution set - according to

$$\hat{c}(\omega) = \{x \in c(\omega) | F(x) = \Phi(\omega)\}.$$

The measurability of $\hat{c}(\omega)$, considered as a multivalued mapping from Ω into $\mathbb{R}^n$, has been discussed in [2]. We do not know about practical conditions which ensure the measurability of $\hat{c}(\omega)$ if considered as a singlevalued mapping from Ω into Γ. But instead we can show measurability of a multivalued mapping γ closely related to $\hat{c}$. Recall that a multivalued mapping

$$\gamma: \Omega \ni \omega \to \gamma(\omega) \subset \Gamma$$

is called measurable if the set

$$\gamma^{-1}(H) \equiv \{\omega \in \Omega | \gamma(\omega) \cap H \neq \emptyset\}$$

is measurable in Ω for any closed subset $H \subset \Gamma$. Now take as γ the multivalued mapping which assigns to each $\omega \in \Omega$ the family of all nonvoid, closed, convex subsets of $\hat{c}(\omega)$. Then we have

Theorem 4: *The multivalued mapping* $\gamma(\omega)$ *is measurable.*

Proof: It is easy to see that for arbitrary $C \in \Gamma$ the requirement $C \in \gamma(\omega)$ is equivalent with the two requirements

(9) $$C \subset c(\omega),\ \min \{F(x) | x \in c(\omega)\} \geq \max \{F(x) | x \in C\}.$$

The function $F(x)$ is uniformly continuous on the compact set X. This implies that the two functions

$$m(C) = \min \{F(x) | x \in C\},\ M(C) = \max \{F(x) | x \in C\}$$

are continuous on Γ. Now let $H \subset \Gamma$ be closed in Γ. We have to show that

$\gamma^{-1}(H)$ is measurable in Ω. Because of (9) we have

$$\gamma^{-1}(H) = \{\omega\in\Omega \mid \exists C\in H\colon C \subset c(\omega) \;\&\; m(c(\omega)) \geq M(C)\}.$$

i.e., $\gamma^{-1}(H)$ is the inverse image with regard to the measurable mapping $c(\omega)$ of the set

$$K = \{D\in\Gamma \mid \exists C\in H\colon C \subset D \;\&\; m(D) \geq M(C)\}.$$

We show that K is closed in Γ. Let $\{D_n\}_{n\in\mathbb{N}}$ be a sequence from K, converging to some element $D\in\Gamma$. By the definition of K there exists for all $n\in\mathbb{N}$ a set $C_n\in H$ such that

$$C_n \subset D_n \;\&\; m(D_n) \geq M(C_n).$$

In view of our convexity assumptions Blaschke's selection theorem [3] ensures that Γ is sequentially compact. Hence H is also sequentially compact, and there exists a convergent subsequence $C_{n_j} \to C\in H$. The continuity of m and M implies in the limit

$$m(D) \geq M(C).$$

From $C \subset U\left(C_n, d(C_n,C)\right)$, $C_n \subset D_n$, $D_n \subset U\left(D, d(D,D_n)\right)$ it follows that

$$C \subset U\left(D, d(C_n,C) + d\,(D_n,D)\right).$$

Since $d(C_{n_j},C) \to 0$ and $d(D_n,D) \to 0$ we have in the limit

$$C \subset D.$$

Therefore $D\in K$, and K is closed. Since $\gamma^{-1}(H) = c^{-1}(K)$ with c measurable and K closed, the measurability of the set $\gamma^{-1}(H)$ in Ω follows. q.e.d.

Note that the function $\Phi(\omega)$ is measurable, being the composition of the measurable mapping $c(\omega)$ and the continuous function $m(C)$.

References

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