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Abstract

In image sequence analysis, variational optical flow computations require the solution of a parameter dependent optimization problem with a data term and a regularizer. In this paper we study existence and uniqueness of the optimizers. Our studies rely on quasiconvex functionals on the spaces $W^{1,p}(\Omega, \mathbb{R}^d)$, with $p > 1$, $BV(\Omega, \mathbb{R}^d)$, $BD(\Omega)$. The methods that are covered by our results include several existing techniques. Experiments are presented that illustrate the behavior of these approaches.

Keywords: Optical flow, calculus of variations, quasiconvex functionals, functions of bounded variation and deformation

1 Introduction

Optical flow is the apparent 2D motion that needs to be recovered from a video sequence. 2D motions find diverse applications in *video processing* as well as in *computer vision.* In video compression knowledge of motion helps to remove temporal data redundancy which in turn is used to compress video sequences with high compression ratios.

For the detection of motion one typically uses the following model. Let $I(x,t)$ be the measured image intensity at position (x,t), with $x = (x_1, x_2)$ in $[0,1]^2 \subseteq \mathbb{R}^2$. Let $x(t)$

be the parametrisation of a path of constant image intensity, i.e., $I(x(t), t) = constant$ for $t \in [0, \infty)$, then

$$\frac{\partial I}{\partial t}(x(t), t) = 0 \,. \tag{1}$$

By applying the chain rule and assuming that structures do not change their intensities over time, (1) can be written as the *optical flow* equation

$$\begin{aligned} 0 &= I_{x_1}(x,t)u(x,t) + I_{x_2}(x,t)v(x,t) + I_t(x,t) \\ &= (\nabla I)^t(x,t)\vec{w}(x,t) + I_t(x,t)\,, \end{aligned} \tag{2}$$

where $\vec{w}(x,t) = (u,v)^t(x,t)$ denotes the optical flow field. We use the convention that subscripts denote partial derivatives and $\frac{\partial \cdot}{\partial t}$ denotes the *total* derivative.

In this paper we analyze models for recovering a motion field $\vec{w}$ in (2) from a sequence of image intensities.

2 Models for motion representation

The motion field $\vec{w} = (u, v)^t$ is not uniquely determined by (2), since it is *one* equation for *two* unknown functions u and v. Thus additional constraints have to imposed and there have been proposed several models in the literature.

Variational optical flow computations started with the pioneering work of Horn and Schunck [19] who proposed to calculate an approximate solution of (2) that minimizes the functional

$$J_{\text{HS}}(\vec{w}) = \frac{1}{2}\int_\Omega (|\nabla u(x)|^2 + |\nabla v(x)|^2)\, dx \,. \tag{3}$$

Recently there has been a trend to use more sophisticated constraints to preserve edges and corners in the motion field (see e.g. [23, 8, 27, 30, 2, 31]). This can be achieved by considering e.g. penalizing functionals like

$$J_{\text{NE}}(\vec{w}) = \int_\Omega \text{trace}\Big((\nabla\vec{w}(x))^t D_2(\nabla I)(x)(\nabla\vec{w}(x))\Big)\, dx\,, \tag{4}$$

with

$$D_2(\nabla I)(x) = \frac{1}{|\nabla I(x)|^2 + 2\lambda^2}\left\{ \begin{pmatrix} \frac{\partial I}{\partial x_2}(x) \\ -\frac{\partial I}{\partial x_1}(x) \end{pmatrix} \begin{pmatrix} \frac{\partial I}{\partial x_2}(x) \\ -\frac{\partial I}{\partial x_1}(x) \end{pmatrix}^t + \lambda^2 E \right\} .$$

Here and in the sequel of the paper $|.|$ denotes the Euclidean norm and E denotes the unitary matrix. The motivation for using such penalizing functionals comes from anisotropic diffusion filtering. For some background on this topic we refer to [29].

Another frequently used edge preserving technique is via BV penalizing functionals like [8]

$$J_{\text{BV}}(\vec{w}) = \int_\Omega (|\nabla u(x)| + |\nabla v(x)|)\, dx\,, \tag{5}$$

where both $\int_\Omega |\nabla u|\,dx$ and $\int_\Omega |\nabla v|\,dx$ are understood as the bounded variations semi-norms of u and v. For a definition of the space of functions of bounded variation and the semi-norm we refer to [13].

Following the standard way of solving a constraint optimization problems, we approximate minimizers of J_{HS}, J_{NE}, J_{BV} that satisfy the constraint (2) by solutions of the *unconstraint* optimization problem to minimize the functional

$$J_i^\lambda(\vec{w}) = \lambda \int_\Omega \phi(((\nabla I)^t \vec{w} + I_t)^2)\,dx + J_i(\vec{w})\ , \tag{6}$$

where λ is the positive *penalizing* parameter and $i = HS, NE, BV$, respectively. Here ϕ is a nonnegative function. Examples of frequently used functions ϕ in optical flow computations are $\phi(.) = |\cdot|^p$, $p = 1, 2$.

3 Lower semi-continuous functionals

For the analysis of optical flow problems we utilize classical result of *calculus of variations* and *nonlinear partial differential equation.* All the particular functionals for optical flow computations outlined in Section 2 fit into the class of problems of minimization of a functional

$$\vec{w} \to I[\vec{w}] := \int_\Omega f(x, \vec{w}, \nabla \vec{w})\,dx\ . \tag{7}$$

In this particular paper we consider minimization over Banach spaces $W^{1,p}(\Omega, \mathbb{R}^d)$ and $BV(\Omega, \mathbb{R}^d)$, $d \geq 1$. All along the remainder of this paper we assume that $\Omega \subseteq \mathbb{R}^n$ is bounded with piecewise smooth boundary.

In the first case we can utilize classical results on calculus of variations (see e.g., Morrey [21, 22], Ball [5], Dacorogna [10]) which guarantee lower semi-continuity of the functional $I[\vec{w}]$ in $W^{1,p}(\Omega, \mathbb{R}^d)$, $1 < p \leq \infty$. These abstract results will be applied afterwards to prove existence of minimizers of optical flow models. Weak lower semi–continuity of $I[\vec{w}]$ on $BV(\Omega, \mathbb{R}^d)$ is a rather challenging topic. There are several results in this direction (see e.g. [4, 15, 16, 14] to name but a few) which deal with semi-continuity of $I[\vec{w}]$ in a general setting. There are some easier results available if we take into account the special structure of penalized least squares functionals such as modeled in Section 2.

3.1 Quasiconvex functional on $W^{1,p}(\Omega, \mathbb{R}^d)$

In this section we recall the concept of quasiconvexity and summarize some results on (weak) lower semi-continuity of the functional (7) on the space $W^{1,p}(\Omega, \mathbb{R}^d)$. We associate with each space of vector-valued functions $X(\Omega, \mathbb{R}^d)$ the norm

$$\|\vec{f}\| := \sqrt{\sum_{i=1}^d \|f_i\|_X^2}\ .$$

Definition 1. *A function*

$$f : \mathbb{R}^{n\times d} \to \mathbb{R}$$
$$P \to f(P)$$

is called quasiconvex *if*

$$\int_\Omega f(P)\,dy \leq \int_\Omega f(P+\nabla\vec{v})\,dy \tag{8}$$

for each $P \in \mathbb{R}^{n\times d}$ and $\vec{v} \in C_c^\infty(\Omega;\mathbb{R}^d)$. Here $C_c^\infty(\Omega;\mathbb{R}^d)$ denotes the subspace of $C^\infty(\Omega;\mathbb{R}^d)$ with functions of compact support in Ω.

Note that any convex function is quasiconvex (see e.g [12]).
Sometimes instead of (8) an equivalent formulation of quasiconvexity is used

$$f(P) \leq \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega f(P+\nabla\vec{v})\,dy\,. \tag{9}$$

The following structural theorem holds for quasiconvex functionals (see e.g. [12]).

Theorem 2. *Let $1 < p < \infty$. Suppose f satisfies*

$$0 \leq f(P) \leq C(1+|P|^p) \qquad (P \in \mathbb{R}^{n\times d})\,, \tag{10}$$

for some constants C and $p > 1$; here $|P|$ denotes the Frobenius norm of the matrix P. Then the functional

$$G[\vec{w}] = \int_\Omega f(\nabla\vec{w})\,dx \tag{11}$$

is lower semi-continuous with respect to weak convergence in $W^{1,p}(\Omega,\mathbb{R}^d)$ if and only if f is quasiconvex.

Now we turn to the more general situation that f is of the general form (7). We can rely on a variety of results. A few of them are quoted here for the readers convenience.

One of the first results in this direction can be found in Morrey [21]. More recently Fonseca and Müller [15] proved the following result:

Theorem 3. (Fonseca and Müller) *Let f be continuous from $\Omega \times \mathbb{R}^d \times \mathbb{R}^{n\times d}$ into $[0,\infty)$ satisfying*

1. *$f(x,\vec{w},\cdot)$ is quasiconvex*
2. *There exits a nonnegative, bounded, continuous function $\overline{f} : \Omega \times \mathbb{R}^d \to [0,\infty)$, constants $\underline{\alpha}, \overline{\alpha} > 0$ such that*

$$\underline{\alpha}\overline{f}(x,\vec{w})|P| - \overline{\alpha} \;\leq\; f(x,\vec{w},P) \;\leq\; \overline{\alpha}\overline{f}(x,\vec{w})(1+|P|) \tag{12}$$

for all $(x,\vec{w},P) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{n\times d}$.

3. *For all* $(x_0, \vec{w}_0) \in \Omega \times \mathbb{R}^d$ *and for all* $\eta > 0$ *there exists* $\delta > 0$ *such that for* $|x - x_0| + |\vec{w} - \vec{w}_0| \leq \delta$ *implies that*

$$f(x_0, \vec{w}, P) - f(x_0, \vec{w}_0, P) \geq -\eta(1 + |P|)\,, \tag{13}$$

and

$$|f(x_0, \vec{w}, P) - f(x, \vec{w}, P)| \leq \eta(1 + |P|)\,. \tag{14}$$

Then if $\vec{w}_n, \vec{w} \in W^{1,1}(\Omega, \mathbb{R}^d)$ *and* $\vec{w}_n \to \vec{w}$ *in* $L^1(\Omega, \mathbb{R}^d)$*, then*

$$I[\vec{w}] \leq \liminf_{n\to\infty} I[\vec{w}_n]\,.$$

For our purposes Theorem 2 is not practicable since for optical flow simulations we require instead of (12) estimates of the form

$$\underline{\alpha}(1 + |\vec{w}| + |P|) \;\leq\; f(x, \vec{w}, P) \;\leq\; \overline{\alpha}(1 + |\vec{w}| + |P|) \tag{15}$$

on the space of functions of bounded variation (instead of $W^{1,1}(\Omega, \mathbb{R}^d)$). Modifications of this theorem which can be applied to the analysis of optical flow problems are given below.

In Dacorogna [10, p. 167] we find the following result:

Theorem 4. (see Dacorogna) *For* $1 < p < \infty$. *Let* f *be continuous from* $\Omega \times \mathbb{R}^d \times \mathbb{R}^{n\times d}$ *into* $[0, \infty)$ *satisfying*

1. $f(x, \vec{w}, \cdot)$ *is quasiconvex.*

2. *There exists a positive constant* $\overline{\alpha}$ *such that*

$$0 \leq f(x, \vec{w}, P) \leq \overline{\alpha}(1 + |\vec{w}|^p + |P|^p)\,.$$

3. *There exists a positive constant* $\beta > 0$ *such that*

$$|f(x, \vec{w}, P_1) - f(x, \vec{w}_0, P_2)| \leq \beta\Big(1 + |\vec{w}|^{p-1} + |\vec{w}_0|^{p-1} + |P_1|^{p-1} + |P_2|^{p-1}\Big) \cdot \Big(|\vec{w} - \vec{w}_0| + |P_1 - P_2|\Big)\,.$$

4. *There exists a continuous, increasing function* η *satisfying* $\eta(0) = 0$ *such that*

$$|f(x, \vec{w}, P) - f(y, \vec{w}, P)| \leq \eta(|x - y|)\Big(1 + |\vec{w}|^p + |P|^p\Big)\,.$$

Then $I[\vec{w}]$ *is weakly lower semi-continuous on* $W^{1,p}(\Omega; \mathbb{R}^d)$.

The following corollary can be proven similarly to Theorem 3 by taking into account the Sobolev embedding theorem.

Corollary 5. *For $1 < p < \infty$ and q satisfying*

$$\begin{array}{ccc} 1 \le q < \frac{np}{n-p} & \text{if} & 1 < p < n \\ & \text{or} & \\ 1 \le q < \infty & \text{if} & p \ge n \end{array} \tag{16}$$

set $\overline{s} := \max\{q,p\}$. Moreover, let f be continuous from $\Omega \times \mathbb{R}^d \times \mathbb{R}^{n\times d}$ into $[0,\infty)$ satisfying

1. *$f(x,\vec{w},\cdot)$ is quasiconvex.*
2. *There exists a positive constant $\overline{\alpha}$ such that*
$$0 \le f(x,\vec{w},P) \le \overline{\alpha}(1+|\vec{w}|^{\overline{s}}+|P|^p) .$$
3. *There exists a positive constant $\beta > 0$ such that*
$$|f(x,\vec{w},P_1) - f(x,\vec{w}_0,P_2)| \le \beta\Big(1+|\vec{w}|^{\overline{s}-1}+|\vec{w}_0|^{\overline{s}-1}+|P_1|^{p-1}+|P_2|^{p-1}\Big) \cdot \Big(|\vec{w}-\vec{w}_0|+|P_1-P_2|\Big) .$$
4. *There exists a continuous, increasing function η satisfying $\eta(0) = 0$ such that*
$$|f(x,\vec{w},P) - f(y,\vec{w},P)| \le \eta(|x-y|)\Big(1+|\vec{w}|^{\overline{s}}+|P|^p\Big) .$$

Then $I[\vec{w}]$ is weakly lower semi-continuous on $W^{1,p}(\Omega;\mathbb{R}^d)$.

3.2 Quasiconvex functional on $BV(\Omega,\mathbb{R}^d)$

If $f(x,\vec{w},\cdot)$ is quasiconvex then the functional (7) defined on $BV(\Omega,\mathbb{R}^d)$ is implicitly defined via the following limiting procedure (*relaxation*)

$$I[\vec{w}] := \inf_{\{\vec{w}_n\}}\Big\{\liminf_{n\to\infty}\int_\Omega f(x,\vec{w}_n(x),\nabla\vec{w}_n(x))\,dx : \vec{w}_n \in W^{1,1}(\Omega,\mathbb{R}^d) \text{ and } \vec{w}_n \to \vec{w} \in L^1(\Omega,\mathbb{R}^d)\Big\} .$$

If f is quasiconvex and satisfies some growth properties then there exists an integral representation for $I[\vec{w}]$ consisting of three integrals (see e.g. [15, 16]); the first integral takes care of the regular parts of $\nabla\vec{w}$ and the second and third parts take care of the singular parts of the measure $\nabla\vec{w}$.

A few results are available in the literature on weak lower semi-continuity of $I[\vec{w}]$ on $BV(\Omega,\mathbb{R}^d)$. One result showing semi-continuity of this functional has been given in Aviles and Giga [4]. Their result is technically complicated and we confine our considerations to a subclass of functionals (7) on the space of functions of bounded variation for which an easier analysis is accessible. Let us assume bounded variation penalization models of the form

$$f(x, \vec{w}, \nabla \vec{w}) = \phi(x, \vec{w}) + |\nabla \vec{w}|_1 , \tag{17}$$

where for $\vec{w} = (w^1, \cdots, w^d)^t$

$$|\nabla \vec{w}|_1 = \sum_{i=1}^{d} |\nabla w^i| .$$

That is we assume that the functional f consists of two sums, where only one is dependent on the gradient. In this situation lower semi-continuity of the functional $I[w]$ is easier tractable. At the current status of research in optical flow models on the space of functions of bounded variation it is sufficient to consider such models; all (numerically) investigated models are of such a form. The technical advantage of the term $|\nabla \vec{w}|_1$ is that the relaxed functional $\int_\Omega |\nabla \vec{w}|_1 \, dx$ is the bounded variation semi-norm of all components for which an integral decomposition in regular and singular parts of the measure is well–understood (see e.g. [13]).

In the following we utilize a weak continuity result on $L^p(\Omega, \mathbb{R}^d)$ similar to one stated in Dacorogna [9]. Surprisingly we were not able to find the particular result in the literature on calculus of variations. The difference to the result stated in [9] is that the function ϕ is also dependent on the space variable x. Thus for the sake of completeness of the paper we include a proof although it is a straight forward modification of Theorem 1.1 in [9].

Lemma 6. *Let ϕ be uniformly Lipschitz continuous in Ω with respect to $\vec{w}$, i.e.,*

$$|\phi(x, \vec{w}) - \phi(x, \vec{w}_0)| \leq L|\vec{w} - \vec{w}_0| .$$

Then the functional

$$H(\vec{w}) := \int_\Omega \phi(x, \vec{w}) \, dx ,$$

is weakly lower semi-continuous on $L^p(\Omega, \mathbb{R}^d)$ for any $1 \leq p < \infty$ if $\phi(x, \cdot)$ is convex for all $x \in \Omega$.

Proof. Let $\vec{w}_n$ be weakly convergent to $\vec{\overline{w}}$ and $L := \liminf_{n \to \infty} H(\vec{w}_n)$. We want to show that $L \geq H(\vec{\overline{w}})$. Without loss of generality we assume that the sequence $\{\vec{w}_n\}$ satisfies

$$L = \lim_{n \to \infty} H(\vec{w}_n) .$$

From Mazur's lemma we get (see e.g. [11]) that there exists a sequence of convex combinations $\{\vec{v}_n\}$ such that

$$\vec{v}_n = \sum_{k=n}^{N} \alpha_k \vec{w}_k , \text{ where } \sum_{k=n}^{N} \alpha_k = 1 \text{ and } \alpha_k \geq 0 \text{ and } n \leq k \leq N$$

which converges to $\vec{\vec{w}}$ in $L^p(\Omega, \mathbb{R}^d)$.

Using the Hölder-inequality and the Lipschitz continuity of ϕ we find

$$\begin{aligned} |H(\vec{\vec{w}}) - H(\vec{v}_n)| &\le \int_\Omega |\phi(x, \vec{\vec{w}}) - \phi(x, \vec{v}_n)| \, dx \\ &\le L \int_\Omega \left|\vec{\vec{w}} - \vec{v}_n\right| \, dx \\ &\le L(\text{meas}(\Omega))^{(p-1)/p} \left(\int_\Omega |\vec{\vec{w}} - \vec{v}_n|^p \, dx \right)^{1/p} , \end{aligned} \tag{18}$$

where we set $(\text{meas}(\Omega))^{(p-1)/p} = 1$ if $p = 1$.

Thus for every $\varepsilon > 0$, a sufficiently large n exists such that

$$H(\vec{\vec{w}}) \le H(\vec{v}_n) + \varepsilon \, . \tag{19}$$

Using the fact that ϕ is convex with respect to the second component we find for sufficiently large N

$$\begin{aligned} \int_\Omega \phi(x, \vec{\vec{w}}) \, dx &\le \int_\Omega \phi\left(x, \sum_{k=n}^N \alpha_k \vec{w}_k\right) \, dx + \varepsilon \\ &\le \sum_{k=n}^N \alpha_k \int_\Omega \phi(x, \vec{w}_k) \, dx + \varepsilon \\ &\le L + 2\varepsilon \, . \end{aligned}$$

Since ε is arbitrary the assertion follows. □

The assumption on the Lipschitz continuity in Lemma 6 can be modified: let us denote by $L_{x,\vec{w},\vec{w}_0}$ the Lipschitz constant of ϕ at x, $\vec{w}$, and $\vec{w}_0$, i.e.,

$$|\phi(x, \vec{w}) - \phi(x, \vec{w}_0)| \le L_{x,\vec{w},\vec{w}_0} |\vec{w} - \vec{w}_0| \, . \tag{20}$$

If

$$L_{x,\vec{w},\vec{w}_0} \le C(1 + \max\{|\vec{w}|^s, |\vec{w}_0|^s\}) \, , \tag{21}$$

with $0 \le s \le p - 1$, holds, then the conclusion of (18), i.e., (19), remains valid.

Using Lemma 6 we are able to prove a result on lower semi-continuity on $BV(\Omega)$.

Theorem 7. *Let $\phi(x, \vec{w})$ be convex with respect to $\vec{w}$. Moreover, let ϕ satisfy (21). Then for any uniformly bounded sequence $\{\vec{w}_n\}$ in $BV(\Omega, \mathbb{R}^d)$ with weak limit $\vec{w}$ in $L^1(\Omega)$ we have $\vec{w} \in BV(\Omega, \mathbb{R}^d)$ and*

$$I[\vec{w}] \le \liminf_{n \to \infty} I[\vec{w}_n] \, .$$

Proof. The bounded variation semi-norm is weakly lower semi-continuous on $L^1(\Omega, \mathbb{R}^d)$, and thus $\vec{w} \in BV(\Omega, \mathbb{R}^d)$ satisfies

$$\int_\Omega |\nabla \vec{w}|_1 \, dx \leq \liminf_{n\to\infty} \int_\Omega |\nabla \vec{w}_n|_1 \, dx < \infty \, .$$

This result follows from the fact that the dual of L^1 and L^∞ are isomorphic, and therefore for each $\vec{v} \in C_0^1(\Omega, \mathbb{R}^n)$ and $i = 1, \cdots, d$ we have

$$\int_\Omega w^i \nabla . \vec{v} \, dx = \lim_{n\to\infty} \int_\Omega w_n^i \nabla . \vec{v} \, dx \, .$$

Consequently,

$$\int_\Omega |\nabla w^i| \, dx = \sup_{\{\vec{v} \in C_0^\infty(\Omega, \mathbb{R}^n), |\vec{v}|_{L^\infty(\Omega, \mathbb{R}^n)} \leq 1\}} \int_\Omega w^i \nabla . \vec{v} \, dx \leq \liminf_{n\to\infty} \int_\Omega |\nabla w_n^i| \, dx \, ,$$

which shows the assertion.

The functional $H(\vec{w})$ is weakly lower semi-continuous on $L^p(\Omega, \mathbb{R}^d)$ for any $p \geq 1$ by Lemma 6. Thus I is weakly lower semi-continuous. □

Under some additional assumptions we are even able to prove a lower semi-continuity result for nonconvex functionals which satisfy a growth condition.

Theorem 8. *For $n = 2, 3$ let $\phi(x, \vec{w})$ be continuous and satisfy*

$$|\phi(x, \vec{w}_n) - \phi(x, \vec{w})| \leq L_{x, \vec{w}, \vec{w}_n} |\vec{w}_n - \vec{w}|$$

with

$$L_{x, \vec{w}, \vec{w}_n} \leq C(1 + |\vec{w}|^q + |\vec{w}_n|^q),$$

where $0 \leq q < 1$ [1] *for $n = 2$ and $0 \leq q < 1/2$ for $n = 3$. Then for any uniformly bounded sequence $\{\vec{w}_n\}$ in $BV(\Omega, \mathbb{R}^d)$ with weak limit $\vec{w}$ in $L^1(\Omega)$ we have*

$$I[w] \leq \liminf_{n\to\infty} I[w_n] \, .$$

Proof. By means of the weak lower semi-continuity of the BV-seminorm we conclude that $\vec{w} \in BV(\Omega, \mathbb{R}^d)$ (see e.g. [17]).

- For $n = 2$ each subsequence of $\{\vec{w}_n\}$ has a strongly convergent subsequence to $\vec{w}$ in $L^s(\Omega, \mathbb{R}^d)$ with $1 \leq s < 2$. This follows from the compact Sobolev embedding theorem [20, Theorem 3.5.2 and Section 6.1.2]. Thus $\{\vec{w}_n\}$ is itself strongly convergent in $L^s(\Omega, \mathbb{R}^d)$. Moreover, the embedding of $BV(\Omega, \mathbb{R}^d)$ into $L^2(\Omega, \mathbb{R}^d)$

[1] All along this paper we use the convention $x^0 = 1$ for $x \geq 0$

is bounded. Then by the Cauchy-Hölder inequality we get with the setting $t > 2$ and $qt = 2$ and $1/s = 1 - 1/t$ (for $q = 0$ set formally $t = \infty$ and use $x^0 = 1$)

$$\begin{aligned} &\left| \int_\Omega \phi(x, \vec{w}_n) - \phi(x, \vec{w})\, dx \right| \\ &\leq C \int_\Omega \max\{1 + |\vec{w}|^q, 1 + |\vec{w}_n|^q\} |\vec{w} - \vec{w}_n|\, dx \\ &\leq C \left(\max\left\{ \int_\Omega (1 + |\vec{w}_n|^q)^t\, dx, \int_\Omega (1 + |\vec{w}|^q)^t\, dx \right\} \right)^{1/t} \\ &\quad \left(\int_\Omega |\vec{w} - \vec{w}_n|^s\, dx \right)^{1/s} . \end{aligned}$$

The last term tends to zero by the Sobolev embedding theorem.

- For $n = 3$ each subsequence of $\{\vec{w}_n\}$ has a strongly convergent subsequence to $\vec{w}$ in $L^s(\Omega, \mathbb{R}^d)$ with $1 \leq s < 3/2$. Thus it follows from the Cauchy-Hölder inequality with the setting $t > 3/2$ and $qt = 3/2$ and $1/s = 1 - 1/t$ and the growth property of ϕ that

$$\begin{aligned} &\left| \int_\Omega (\phi(x, \vec{w}_n) - \phi(x, \vec{w}))\, dx \right| \\ &\leq C \int_\Omega \max\{(1 + |\vec{w}|^q), (1 + |\vec{w}_n|^q)\} |\vec{w} - \vec{w}_n|\, dx \\ &\leq C \left(\int_\Omega \max\{(1 + |\vec{w}|^q)^t, (1 + |\vec{w}_n|^q)^t\}\, dx \right)^{1/t} \\ &\quad \left(\int_\Omega |\vec{w} - \vec{w}_n|^s\, dx \right)^{1/s} . \end{aligned}$$

Again from the Sobolev embedding theorem the assertion follows.

□

In particular the above proof reveals that the operator H is even continuous on $BV(\Omega, \mathbb{R}^d)$.

3.3 Quasiconvex functionals on $BD(\Omega)$

Let $\Omega \subseteq \mathbb{R}^n$ be the space $BD(\Omega)$ of *vector fields with bounded deformation* (see e.g. [28]). This is the space of all vector fields $\vec{w} \in L^1(\Omega, \mathbb{R}^n)$ satisfying

$$j(\vec{w}) = \sum_{i,j=1}^{n} \int_\Omega |D_{ij}\vec{w}|\, dx < \infty ; \tag{22}$$

here

$$D_{ij}(\vec{w}) = \frac{\partial w^i}{\partial x_j} + \frac{\partial w^j}{\partial x_i}$$

has to be understood as a Radon measure.

We remark that analogously to BV-functions it can be shown that $j(\cdot)$ is weakly lower semi-continuous on $L^p(\Omega, \mathbb{R}^d)$, $p \geq 1$.

Lemma 9. *1. Let $\{\vec{w}_n\}$ a sequence of functions in $BD(\Omega)$ which converges weakly in $(L^p(\Omega))^n$, $1 \leq p < \infty$ to $\vec{w}$, then*

$$j(\vec{w}) \leq j(\vec{w}_n) .$$

Due to similar difficulties as for the functions space $BV(\Omega, \mathbb{R}^d)$ we restrict our attention to functionals of the form

$$J[w] := \int_\Omega f(x, \vec{w}, D_{ij}\vec{w})\, dx \tag{23}$$

with

$$f(x, \vec{w}, D_{ij}(\vec{w})) = \phi(x, \vec{w}) + \sum_{i,j=1}^{d} |D_{ij}(\vec{w})| . \tag{24}$$

Theorem 10. *Let $\phi(x, \vec{w})$ satisfy the assumptions of Theorem 7. Then for any uniformly bounded sequence $\{\vec{w}_m\}$ in $BD(\Omega)$ with weak limit $\vec{w}$ in $L^1(\Omega)$ we have $\vec{w} \in BD(\Omega)$ and*

$$I[\vec{w}] \leq \liminf_{m\to\infty} I[\vec{w}_m] .$$

Proof. Temam and Strang [28] showed that the embedding

$$i : BD(\Omega) \to \left(L^{n/(n-1)}(\Omega)\right)^n \tag{25}$$

is continuous. Thus any bounded subsequence $\{\vec{w}_m\}$ in $BD(\Omega)$ has a weakly convergent subsequence in $\left(L^{n/(n-1)}(\Omega)\right)^n$ and the weak limit $\vec{w}$ is in $BD(\Omega)$ due to the weak lower semi-continuity of the BD-norm. This proves the assertion. □

4 Existence of minimizers of quasiconvex functionals

The functional $G[\vec{w}]$ as introduced in (11) attains a minimum on any closed ball of $W^{1,p}(\Omega, \mathbb{R}^d)$, $1 < p < \infty$:

Theorem 11. *Let f satisfy (10) and be quasiconvex. Then $G[\vec{w}]$ attains a minimum on any closed ball of $W^{1,p}(\Omega, \mathbb{R}^d)$, $1 < p < \infty$.*

Proof. Let $\{\vec{w}_n\}$ be a sequence in a closed ball Θ of $W^{1,p}(\Omega, \mathbb{R}^d)$. Suppose that $G[\vec{w}_n]$ converges to z, the global minimum of $G[\vec{w}]$ in Θ. By the theorem of Alaoglu-Bourbaki-Kakutani, since $W^{1,p}(\Omega, \mathbb{R}^d)$ is reflexive, each ball is weakly compact and we can select a subsequence that is weakly convergent to $\vec{w} \in \Theta$ such that $G[\vec{w}_n] \to z$. Since G is weakly lower semi-continuous the assertion follows. □

For the functional $I[\vec{w}]$ defined in (7) we have the following results on existence of minimizers:

Theorem 12. *Let $f : \overline{\Omega} \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ be continuous and $f(x, \vec{w}, \cdot)$ quasiconvex for all $(x, \vec{w}) \in \overline{\Omega} \times \mathbb{R}^d$.*

1. *Let f additionally satisfy the assumptions item 2- 4 in Theorem 4 and*
$$\underline{f}(x) + \underline{\alpha}(|\vec{w}|^p + |P|^p) \leq f(x, \vec{w}, P) \tag{26}$$
with $\underline{\alpha} > 0$ and $\underline{f} \in L^1(\Omega)$, then $I[\vec{w}]$ attains a minimum on $W^{1,p}(\Omega, \mathbb{R}^d)$, $1 < p < \infty$.

2. *Let f additionally satisfy assumptions item 2- 4in Corollary 5 and*
$$\underline{f}(x) + \underline{\alpha}(|\vec{w}|^p + |P|^p) \leq f(x, \vec{w}, P)\,, \tag{27}$$
with
$$0 < \underline{\alpha}\,, \qquad \underline{f} \in L^1(\Omega)\,.$$
Then $I[\vec{w}]$ attains a minimum on $W^{1,p}(\Omega, \mathbb{R}^d)$.

3. *Let f satisfy (17) and*
$$\underline{f}(x) + \underline{\alpha}|\vec{w}| \leq \phi(x, \vec{w})\,, \tag{28}$$
where $\underline{f} \in L^1(\Omega)$. If ϕ satisfies either the assumptions of Theorem 7 or Theorem 8, then the functional $I[\vec{w}]$ attains a minimum in $BV(\Omega, \mathbb{R}^d)$.

4. *Let f satisfy (24) and (28). If ϕ satisfies the assumptions of Theorem 7, then the functional $J[\vec{w}]$ (defined in (23)) attains a minimum in $BD(\Omega)$.*

Proof. 1.-2. The proof of the first and second item is very similar to the proof of Theorem 2.9 in [10, p.180].

3.-4. The proofs are similar to the proof of the first item by taking into account the special structure of the functionals $I[\vec{w}]$ and $J[\vec{w}]$.

□

5 Analysis of optical flow models

The optical flow models considered in (2) reveal a special structure which is inherent in many variational problems in *nonlinear elasticity* (see e.g. [5]). The most commonly used model are of the form

$$f(x, \vec{w}, \nabla\vec{w}) = \mathcal{W}(x, \nabla\vec{w}) + \phi(x, \vec{w})\,.$$

In nonlinear elasticity $\mathcal{W}(x, \nabla\vec{w})$ is the stored-energy function and ϕ is a body force potential. The obvious coherence between nonlinear elasticity and optical flow models allows us to give physical interpretations in terms of nonlinear elastic models.

In the following we apply the general results of Sections 3 and 4 to the models outlined in Section 2.

All along this Section we restrict our attention to $\Omega = [0,1]^2$ and assume that the image data I is in $C^2([0,1]^2)$, and we denote

$$C := \max\{|I_{x_1}(x)|, |I_{x_2}(x)|, |I_t(x)|, |I_{x_i x_j}(x)|, |I_{x_1,t}| : 1 \le i,j \le 2, x \in \overline{\Omega}\} .$$

Let ρ be a continuous non-negative function from $\mathbb{R}$ into $[0,\infty)$, then

$$\phi(x,\vec{w}) := \rho\Big((\nabla I)^t(x)\vec{w} + I_t(x) \Big) + \underline{\alpha}|\vec{w}|^p \tag{29}$$

is continuous with respect to $(x,\vec{w})$. The term $\underline{\alpha}|\vec{w}|^p$, with $\underline{\alpha} > 0$, is used to guarantee ellipticity in the space $W^{1,p}(\Omega,\mathbb{R}^d)$ with $p > 1$ or in $BV(\Omega,\mathbb{R}^d)$ with $1 \le p \le 2$. In most numerical simulations this term can be neglected since already the properties of I guarantee that

$$\rho\Big((\nabla I)^t(x)\vec{w} + I_t(x) \Big) \ge \underline{\alpha}|\vec{w}|^p . \tag{30}$$

However, in general it is not possible to derive this estimate as one sees from the trivial example $I = 0$. In order to guarantee (30) one would have to impose technical assumptions on I which we want to avoid.

In the following lemma we summarize a basic result on Lipschitz continuity of ϕ.

Lemma 13. *Let ρ be uniformly Lipschitz continuous on every closed ball $\overline{B(r)} \subseteq \mathbb{R}$ with radius r, with Lipschitz constant L_r, i.e., for all $s_1, s_2 \in \overline{B(r)}$ we have*

$$|\rho(s_1) - \rho(s_2)| \le L_r |s_1 - s_2| .$$

Let $\hat{r} = C(1+\sqrt{2}|\vec{w}|)$, then

$$|\phi(x,\vec{w}) - \phi(x_0,\vec{w})| \le L_{\hat{r}} C(\sqrt{2} + 2|\vec{w}|)|x - x_0| . \tag{31}$$

Moreover, with $\hat{r} = C(1 + \sqrt{2}\max\{|\vec{w}|, |\vec{w}_0|\})$ we have

$$|\phi(x_0,\vec{w}) - \phi(x_0,\vec{w}_0)| \le \left(\sqrt{2}L_{\hat{r}}C + \underline{\alpha}p\max\{|\vec{w}|^{p-1}, |\vec{w}_0|^{p-1}\}\right)|\vec{w} - \vec{w}_0| . \tag{32}$$

The following lemma relates convexity of ρ and ϕ:

Lemma 14. *Let $p \ge 1$ and ρ convex. Then $\phi(x,\vec{w})$ is convex with respect to $\vec{w}$.*

Proof. For fixed $x \in \Omega$ the function

$$h(\vec{w}) = (\nabla I)^t(x)\vec{w} + I_t(x)$$

is affine linear and thus the superposition

$$\phi(x, \vec{w}) = \rho \circ h(\vec{w})$$

is convex (with respect to $\vec{w}$), too. □

Now we are able to formulate an existence result in $BV(\Omega, \mathbb{R}^2)$:

Theorem 15. • *Let* $1 \leq p < 2$ *in (29). If* ρ *satisfies the growth condition*

$$L_{\hat{r}} \leq C_L(1 + \hat{r}^s)\,, \tag{33}$$

with $0 \leq s < 1$*, then*

$$I[\vec{w}] = \int \phi(x, \vec{w})\, dx + \alpha \int_\Omega |\nabla \vec{w}|_1\, dx$$

attains a minimum on $BV(\Omega, \mathbb{R}^2)$.

- *For* $1 \leq p \leq 2$ *in (29). If* ρ *is convex with respect to* $\vec{w}$*, and satisfies (33) with* $0 \leq s \leq 1$*, then* $I[\vec{w}]$ *attains a minimum on* $BV(\Omega, \mathbb{R}^2)$.

Proof. Let $q := \max\{s, p-1\}$, then from (32) we get

$$|\phi(x, \vec{w}) - \phi(x, \vec{w}_0)| \leq \tilde{C}(1 + \max\{|w|^q, |w_0|^q\})|\vec{w} - \vec{w}_0|\,.$$

1. In the first case the function ϕ satisfies the general assumption of Theorem 8. For $p = 1$, (28) is trivially satisfied with $0 = \underline{c} = \underline{f}$. For $p > 1$,

$$\underline{\alpha}|\vec{w}|^p = \underline{\alpha}(1 + |\vec{w}|^p) - \underline{\alpha} \geq \underline{\alpha}|\vec{w}| - \underline{\alpha}.$$

   Thus, (28) holds with $\underline{f} = \underline{c} = -\underline{\alpha}$. From Theorem 12 the assertion follows.

2. The embedding of $BV(\Omega, \mathbb{R}^d)$ in $L^2(\Omega, \mathbb{R}^d)$ is bounded. Let $I[w_n]$ converge to its infimum, then due to the boundedness of $\{w_n\}$ in $BV(\Omega, \mathbb{R}^d)$ it has a subsequence which is weakly convergent in $L^2(\Omega, \mathbb{R}^d)$. From Lemma 6 it follows (taking into account the remark after this lemma) that $\int_\Omega \phi(x, \vec{w})\, dx$ is weakly lower semi-continuous on $L^2(\Omega, \mathbb{R}^d)$. Since the bounded variation seminorm is weakly lower semi-continuous on $BV(\Omega, \mathbb{R}^d)$ as well, the assertion follows.

□

Now we turn to existence results on $W^{1,p}(\Omega,\mathbb{R}^d)$. For $1 < p < 2$, let $1 \leq q \leq \frac{2p}{2-p}$, for $p \geq 2$ let $1 \leq q < \infty$. Set $\overline{s} = max\{p,q\}$ and let ρ be nonnegative with $\rho(0) = 0$, satisfying

$$L_{\hat{r}} \leq C_L(1 + \hat{r}^{\overline{s}-1}) \,. \tag{34}$$

Then we have

$$\begin{aligned} 0 &\leq \phi(x,\vec{w}) \\ &= \rho\Big((\nabla I)^t(x)\vec{w} + I_t(x)\Big) - \rho(0) + \underline{\alpha}|\vec{w}|^p \\ &\leq \tilde{C}(1 + |\vec{w}|^{\overline{s}}) \,, \end{aligned}$$

with a generic constant $\tilde{C}$. This in particular shows that item 2 in Corollary 5 holds with f replaced by $\phi + \mathcal{W}$ if $\mathcal{W}(x, \nabla\vec{w})$ quasiconvex, satisfying

1. there exists a positive constant $\beta > 0$ such that

$$|\mathcal{W}(x,P_1) - \mathcal{W}(x,P_2)| \leq \beta(1 + |P_1|^{p-1} + |P_2|^{p-1})|P_1 - P_2| \,, \tag{35}$$

2. there exists a continuous, increasing function η satisfying $\eta(0) = 0$ such that

$$|\mathcal{W}(x,P) - \mathcal{W}(y,P)| \leq \eta(|x-y|)(1 + |P|^p) \,, \tag{36}$$

3. there exist constants $\underline{\alpha} > 0, \overline{\alpha} > 0$, and $\underline{c} \in \mathbb{R}$, such that

$$\underline{c} + \underline{\alpha}(1 + |P|^p) \leq \mathcal{W}(x,P) \leq \overline{\alpha}(1 + |P|^p) \,. \tag{37}$$

From (31) and (32) we see that items 3 and 4 in Corollary 5 hold with f replaced by $\phi + \mathcal{W}$. Moreover, (27) holds by the imposed assumptions on $\mathcal{W}$ and (29). Thus according to Theorem 12 the functional $I[\vec{w}]$ with ρ satisfying (34) attains a minimum in $W^{1,p}(\Omega,\mathbb{R}^2)$.

5.1 Examples

In the following we show particular examples of weakly lower semi-continuous optical flow models.

1. Let $1 \leq s+1 \leq 2$. The function $\rho(\cdot) = |\cdot|^{s+1}$ is convex and Lipschitz continuous on bounded sets and satisfies

$$L_{\hat{r}} \leq p\hat{r}^s \,.$$

   Thus according to Lemma 14 and Theorem 15 the functional

$$I[\vec{w}] := \int_\Omega |(\nabla I)^t\vec{w} + I_t|^{s+1}\,dx + \alpha\int_\Omega (\underline{\alpha}|\vec{w}| + |\nabla\vec{w}|_1)\,dx,$$

   attains a minimum on $BV(\Omega,\mathbb{R}^d)$.

The case $s = 0$ has been studied in [3].

2. Let us consider the specific example

$$\int_\Omega |(\nabla I)^t(x)\vec{w} + I_t(x)|^q \, dx + \alpha \|\vec{w}\|^p_{W^{1,p}(\Omega,\mathbb{R}^d)} ,$$

with $1 < p$ and q satisfying (16) (with $n = 2$).

The according function $\rho(t) = t^q$ is Lipschitz continuous with Lipschitz constant $L_{\hat{r}} = C(1 + \hat{r}^{q-1})$. Thus from Theorem 12 and Corollary 5 it follows that $I[\vec{w}]$ attains a minimum on $W^{1,p}(\Omega, \mathbb{R}^2)$.

The case $q = 1$, $p = 2$ has been studied in [18].

The case $q = 2$, $p = 2$ goes back to [19] and has been analyzed in [26].

As long as $\mathcal{W}(P)$ is quasiconvex, satisfies some growth rate and is elliptic, Theorem 12 and Corollary 5 are valid and guarantee weak lower semi-continuity of $I[\vec{w}]$ on $W^{1,p}(\Omega, \mathbb{R}^d)$ and existence of a minimizer.

In particular the general results are applicable in the following situations:

Let $p = 2$, $1 \leq q$ and ρ satisfy

$$L_{\hat{r}} \leq \hat{c}(1 + \hat{r}^{q-1}) . \tag{38}$$

- With

$$\mathcal{W}(P) = |P_1|^2 + |P_2|^2$$

the functional $I[\vec{w}]$ is weakly lower semi-continuous on $W^{1,2}(\Omega, \mathbb{R}^2)$ and attains a minimum.

- For the integrand in the anisotropic diffusion penalizing functional (4) there exist constants c, C such that

$$c(|\nabla P_1|^2 + |\nabla P_2|^2) \;\leq\; \mathcal{W}(x, P) \;\leq\; C(|\nabla P_1|^2 + |\nabla P_2|^2)$$

for all $P \in \mathbb{R}^{2\times 2}$. Investigating the eigenvalues of $D_2(\nabla I)$ shows that one may choose

$$\begin{aligned} c &:= \frac{\lambda^2}{\sup\limits_x |\nabla I(x)|^2 + 2\lambda^2} \\ C &:= 1 \end{aligned}$$

This shows (37). The functional $\mathcal{W}(x, \cdot)$ is quasiconvex since it is a sum of convex operations. The estimates (35) and (36) follow from elementary calculus.

- Let $\tau = |P_1|^2 + |P_2|^2$, then the function

$$\phi_i(\tau) = \epsilon\tau + \lambda_i^2\sqrt{1+\tau/\lambda_i^2} - \lambda_i^2 \tag{39}$$

is convex with respect to (P_1, P_2) and satisfies (35)-(37); here $\epsilon > 0$ is a positive parameter and $\lambda_i > 0$ denotes a contrast parameter.

3. Nonconvex growth functions may be considered as well:

$$\int_\Omega \log(1 + |(\nabla I)^t(x)\vec{w} + I_t(x)|^2)\,dx + \alpha\|\vec{w}\|^p_{W^{1,p}(\Omega,\mathbb{R}^d)} ,$$

with $1 < p < \infty$ attains a minimum on $W^{1,p}(\Omega, \mathbb{R}^d)$.

$$\int_\Omega \log(1 + |(\nabla I)^t(x)\vec{w} + I_t(x)|^2)\,dx + \alpha\|\vec{w}\|_{BV(\Omega,\mathbb{R}^d)} ,$$

attains a minimum on $BV(\Omega, \mathbb{R}^d)$.

4. $|\cdot|$ is convex and thus from Lemma 14 we see that $\phi(x,\cdot)$ is convex, too. Thus from Theorem 12 it follows that

$$\int_\Omega |(\nabla I)^t(x)\vec{w} + I_t(x)|\,dx + \alpha j(\vec{w}) + \underline{\alpha}\int_\Omega |\vec{w}|_{L^1(\Omega)}\,dx$$

attains a minimum on $BD(\Omega)$.

6 Numerical Experiments

For our numerical experiments we consider the non-quadratic functional

$$E(\vec{w}) = \int_\Omega (\phi_1(|(\nabla I)^t(x)\vec{w} + I_t(x)|^2) + \alpha\,\phi_2(|\nabla\vec{w}|^2)\,dx \tag{40}$$

where the growth function ϕ_i was specified in (39). In order to show the influence of the growth function on the data term we compare this functional with [30]

$$E(\vec{w}) = \int_\Omega (|(\nabla I)^t(x)\vec{w} + I_t(x)|^2 + \alpha\,\phi_2(|\nabla\vec{w}|^2)\,dx, \tag{41}$$

For both convex functionals, the steepest descent equations have been discretized with a simple explicit finite difference scheme.

The results are depicted in Figure 1. It shows a well-known test sequence with a taxi scene. Using functional (41) leads to a relatively noisy optical flow field. With functional (40), noise is successfully removed and the flow field of the taxi is more homogeneous and realistic. For a suitable parameter choice, it is even possible to focus on the taxi movement by smoothing away the flow fields of the two faster vehicles. Since they were faster, they are treated as outliers in the data term. As a result, they are significantly less penalized than in the functional (41).

This shows that from a practical viewpoint, it may be interesting to consider non-quadratic growth functions not only in the regularizer, but also in the data term. They may lead to increased robustness and give additional degrees of freedom. Experiments by Black and Ananden [6, 7] point in the same direction. In their articles, nonconvex growth functionals have been motivated from robust statistics. However, our numerical experiments indicate that a similar effect might be achieved using convex non-quadratic growth functionals (40) which are more convenient from a numerical point of view.

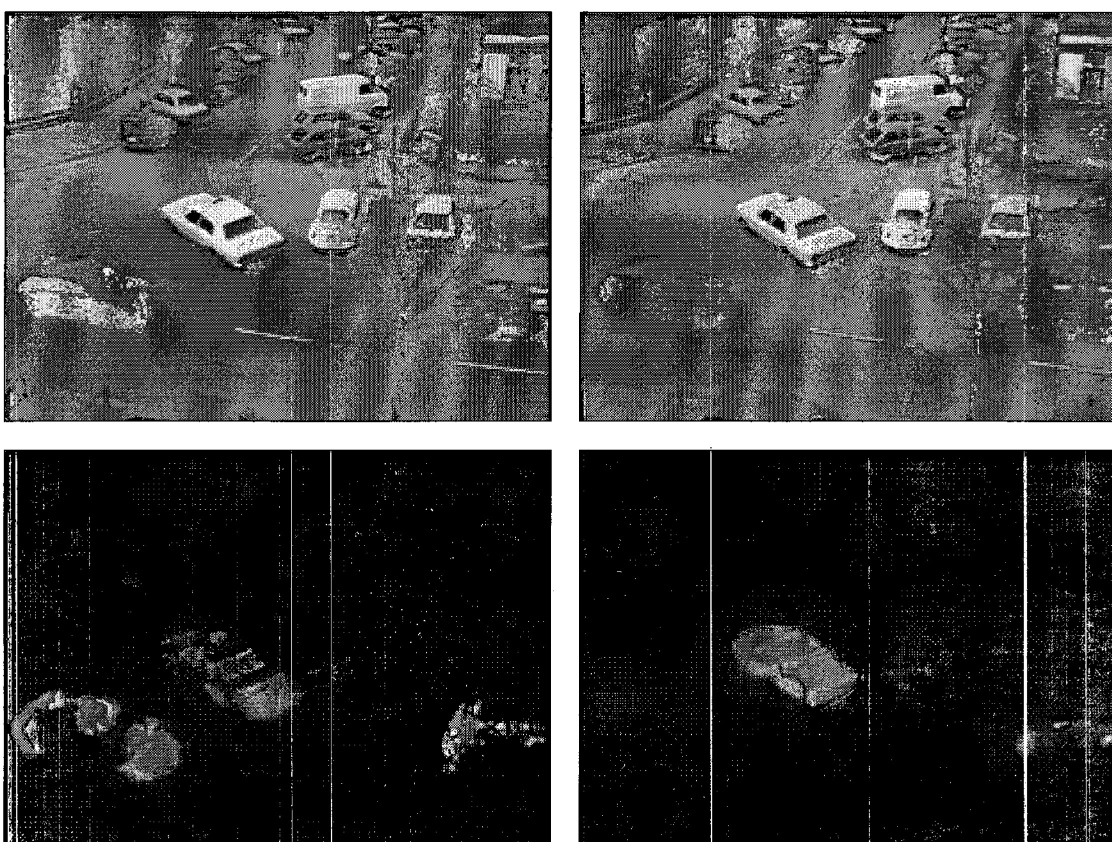

Figure 1: (a) TOP LEFT: Frame 10 of the Hamburg taxi sequence, $\Omega = (0,256) \times (0,190)$. (b) TOP RIGHT: Frame 11. The car in the lower left corner moved to the right, the taxi turned around the street corner, and the bus at the lower right corner moved to the left. (c) BOTTOM LEFT: Computed optic flow magnitude using the functional (41) with the parameters $\alpha = 1000$ and $\lambda_2 = 0.01$. (d) BOTTOM RIGHT: Result using (40) with $\alpha = 1000$, $\lambda_1 = 0.000001$, and $\lambda_2 = 0.01$.

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