Reihe Informatik

2/1989

Towards a foundation for semantics in complete metric spaces

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keywords: semantics, fixed points, complete metric spaces

März 1989

This technical report is an extension of a previous report (1, 87) by the same authors.

0. Introduction

In the last years a series of papers appeared that deal with the semantics of those languages or systems that allow for some notion of concurrency [1, 3-9, 15-20, 24]. The approach of Francez et al. [8] e.g. is based on complete partial orders, the work of de Bakker and Zucker [7] is based on complete metric spaces, Plotkin presents an operational approach [19], axiomatic methods can be found in [1, 4, 11, 15, 17, 20]. The connection between some of the approaches has been investigated in [9, 13].

We are here presenting an investigation and foundation of the metric space approach of [7].

In order to do so we briefly sketch how semantics is defined in [7]. The basic concepts of [7] are the notion of a "process domain" and a "domain equation". Given a language L for which semantics is to be defined, the authors suggest to construct a suitable equation $P = \mathcal{F}(P)$, called domain equation, such that the solution of this equation (a complete metric space) provides a domain for the interpretation of programs, i.e. the meaning function maps programs to elements of this solution.

The authors demonstrate their ideas concerning the solution of such equations by considering the following four prototypes

$$P = \{p_0\} \cup (A \times P) \tag{1}$$

$$P = \{p_0\} \cup \wp_c(A \times P) \tag{2}$$

$$P = \{p_0\} \cup (A \to \wp_c(B \times P)) \tag{3}$$

$$P = \{p_0\} \cup (A \to \wp_c((B \times P) \cup (C \to P))) \tag{4}$$

where e.g. the Cartesian product is used to model the sequencing of actions, the powerset construction $\wp_c$ (see section I.) and the function space construction are used to model nondeterminism, concurrency and communication.

For each equation $P = \mathcal{F}_i(P)$, $i = 1,2,3$, the authors [7] construct a solution as follows (the last equation is left to the reader): A sequence $((P_n, d_n))$ of metric spaces is constructed by setting $P_0 = \{p_0\}$, $P_j = \mathcal{F}_i(P_{j-1})$, and P_ω is defined as $(\cup P_n, \cup d_n)$. It is then shown that the completion (P, d) of P_ω is a solution of the given equation. The thus constructed solutions serve as semantic domains for various sample languages.

When looking closer at the proposed handling of process domain equations, a number of questions arise immediately:

> *Is the thus constructed solution the only solution? If not, what features characterize the constructed solution? And most important, under what conditions is*

it possible to give a solution of an equation $P = \mathcal{F}(P)$ in such a way? What properties must the operator $\mathcal{F}$ have in order to guarantee the existence of a solution alltogether?

In this paper, which is based on a previous report [14], we are dealing with these questions. An independent investigation was developed in [2] and is discussed in section V. In particular, we establish a framework for discussing the existence of solutions of equations as discussed above. This is an important task, because, when we are trying to apply the techniques of [7] to some nontrivial language like CSP [9, 12, 13, 27] we have to have some criterion to decide if the respective equation does have a solution at all.

This problem already occurs with such simple-looking equations as equation (4), the solution of which is left to the reader in [7].

We will prove that this equation cannot be solved in the way claimed in [7].

This is interesting, as the associated operator $\mathcal{F}$ does not satisfy our conditions for existence of fixed points given in theorem 10 and theorem 12.

We finally make two observations. First, there is a strong analogy between the construction of a fixed point theory for the category CPO of complete partial orders from the theory of fixed points in complete partial orders on one side and our ideas on the other. Second, everyone who wants to use complete metric spaces to define the semantics of some language does not have to go into details about existence proofs of fixed points. One only has to ensure some contraction property of the operator involved according to theorem 10 or theorem 12. In this Lemma 9 is helpful.

The paper is divided into seven sections. Section I contains the definitions and elementary statements. In section II we establish conditions for existence and uniqueness of fixed points. Section III deals with the special role of the $\wp_c$-operator. Section IV deals with equation (4) from above and general considerations concerning the choice of the metric and section V creates the connection to related work. Section VI contains the conclusion, section VII an appendix.

I. Definitions and Elementary Properties

Definition 1

A *metric space* is a pair (M,d) with M a set and d a mapping, $d: M \times M \to [0,1]$ which satisfies [1]

(a) $\forall x,y \in M \quad (d(x,y) = 0 \Leftrightarrow x = y)$,

(b) $\forall x,y \in M \quad d(x,y) = d(y,x)$,

(c) $\forall x,y,z \in M \quad d(x,y) \leq d(x,z) + d(z,y)$.

A sequence (x_i) in a metric space (M,d) is a *Cauchy sequence*, whenever $\forall \epsilon > 0$ $\exists N \in I\!N \quad \forall n,m > N \quad d(x_n, x_m) < \epsilon$. The metric space (M,d) is called *complete* if every Cauchy sequence converges to an element of M. It is well known, that every metric space (M,d) can be embedded into a "unique" "minimal" complete metric space, called the *completion* of (M,d).

Let (N,d_N) and (M,d_M) be metric spaces. A function $f: N \to M$ is called a *weak contraction*, if $\forall\, x \in N \ \forall\, y \in N$

$$d_M(f(x), f(y)) \leq d_N(x,y).$$

Definition 2

Let (M,d_M) be a metric space $(d_M \leq 1)$, let $\wp_c(M)$ denote the collection of all closed nonempty subsets of M and let $\wp_c^\emptyset(M)$ denote $\wp_c(M) \cup \{\emptyset\}$. The *Hausdorff metric* on $\wp_c^\emptyset(M)$ is given by

$$d(X,Y) = max\{\sup_{x\in X} \inf_{y \in Y} d(x,y), \sup_{y \in Y} \inf_{x \in X} d(x,y)\}$$

for $X, Y \in \wp_c^\emptyset(M)$.

It has been shown by Hahn [10]:

Remark 1

If (M,d_M) is complete, so are $(\wp_c(M), d)$ and $(\wp_c^\emptyset(M), d)$.

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1) $0 \leq d(x,y) \leq 1$ can be always obtained for an arbitrary metric $\hat{d}: M \times M \to I\!R$ by substituting $\hat{d}(x,y)$ by $\frac{\hat{d}(x,y)}{\hat{d}(x,y)+1}$. d and $\hat{d}$ yield the same topology on M.

<u>Definition 3</u>

Let (N, d_N), (M, d_M) be metric spaces. A weak contraction $f : N \to M$ is called an *embedding*, if it preserves distances, i.e. if $d_M(f(x), f(y)) = d_N(x, y)\ \ \forall\, x, y \in N$. If the embedding f is onto, f is called an *isometry*.

<u>Remark 2</u>

Let (N, d_N), (M, d_M) be complete metric spaces. If $e : N \to M$ is an embedding then N can be identified with the closed subset $e(N)$ of M. Hence, we can talk about the distance of N and M (as elements of $\wp_c^{\emptyset}(M)$) with respect to the embedding e, denoted by $d_e(N, M) = d(e(N), M)$. The subscript e will often be omitted, if no ambiguity arises.

<u>Lemma 1</u>

Let N, M, Z be complete metric spaces. Let $e : N \to M$, $f : M \to Z$ be embeddings then

$$d_e(N, M) \le d_{e\circ f}(N, Z) \text{ [2] } \quad\text{and}\quad d_f(M, Z) \le d_{e\circ f}(N, Z)\,.$$

<u>Proof:</u>

We prove the first inequality

$$\begin{aligned} d_e(N, M) &= d(e(N), M) \\ &= \sup_{x\in M}\ \inf_{y\in e(N)} d(x, y) \\ &= \sup_{x\in M}\ \inf_{y\in e(N)} d(f(x), f(y)) \\ &= \sup_{x\in f(M)}\ \inf_{y\in f(e(N))} d(x, y) \\ &\le \sup_{x\in Z}\ \inf_{y\in f(e(N))} d(x, y) \\ &= d\left(f\left(e\left(N\right)\right), Z\right) \\ &= d_{e\circ f}(N, Z). \end{aligned}$$

Hence, if N can be embedded into M and M into Z we will write

$$d(N, M) \le d(N, Z)$$

bearing in mind that the assumed embedding of N into Z is the functional composition of the two given embeddings.

[2] $e \circ f$ denotes the composition of e and f such that first e is applied and then f.

Definition 4

A sequence $((M_i, d_i))_{i\geq 0}$ of metric spaces together with a sequence of embeddings $(e_i)_{i\geq 0}$, $e_i : M_i \to M_{i+1}$, is called an *embedding sequence*.

Definition 5

Let (N, d_N), (M, d_M) be metric spaces, $e : N \to M$ an embedding. A weak contraction $c : M \to N$ is called a *μ-cut for e* if,

$$i)\ \forall\, x \in N \quad c(e(x)) = x$$

$$ii)\ \forall\, x \in M \quad d_M\left(x, e\left(c\left(x\right)\right)\right) \leq \mu.$$

Let $((M_i, d_i))_{i\geq 0}$ with $(e_i)_{i\geq 0}$ be an embedding sequence with associated μ_i-cuts c_i then

$$c_{mn} : M_m \to M_n$$

is defined by

$$c_{mn} = \begin{cases} id, & \text{if } m = n \\ c_{m-1} \circ \cdots \circ c_n, & \text{if } m > n \\ e_m \circ \cdots \circ e_{n-1}, & \text{if } m < n. \end{cases}$$

Remark 3

Let (N, d_N), (M, d_M) be metric spaces, $e : N \to M$ an embedding, $c : M \to N$ a weak contraction such that (i) holds. One may interprete $c(x)$ as an approximation of x in N. Then (ii) implies that the approximation is at least as good as μ.

Lemma 2

Let (N, d_N), (M, d_M) be complete metric spaces, $e : N \to M$ an embedding with μ-cut c then

$$d_e(N, M) \leq \mu.$$

Proof:

By remark 2 and definition 2.

In order to be able to formulate the fixed point problem we have to define a suitable category in which the equations have to be solved.

Definition 6

The category *MS* is defined as follows: the objects of MS are the metric spaces $(d \leq 1)$, the morphisms are the weak contractions. The category *CMS* has as objects complete metric spaces, the morphisms are the weak contractions.

Remark 4

In MS and CMS the empty set is initial.

Remark 5

Let $((M_i, d_i))$ with (e_i) be an embedding sequence in MS. Then the direct limit of (M_i) in MS with respect to (e_i) exists and is denoted by $(\cup M_i, \cup d_i)$.

Lemma 3

Let $((M_i, d_i))$ with (e_i) be an embedding sequence in CMS, $M_i \neq \emptyset$. Let M denote the completion of the direct limit $(\cup M_i, \cup d_i)$ of $((M_i, d_i))$ in MS. If there exists a $0 \leq k < 1$ such that $d(M_i, M_{i+1}) \leq k \cdot d(M_{i-1}, M_i)$ for all i then $M = \lim M_i$ in $\wp_c(M)$.

Proof:

Obviously each M_i can be embedded into M, hence $(M_i)_{i\geq 0}$ is a Cauchy sequence in $(\wp_c(M), d)$. By Hahn's Theorem [10] one concludes that its limit N equals $\{x : x = \lim x_n, (x_n) \text{ Cauchy sequence}, x_n \in M_n\}$ and hence $N \subseteq M$. Let now $x \in M$, $x = \lim x_n$, (x_n) Cauchy sequence in $\cup M_n$. If $x \in M_n$ for some n nothing has to be shown. Let us consider the case $x \notin M_n$ for all n. We claim that there is a subsequence (y_n) of (x_n) with $y_n \in M_{k_n}$ and $k_{n+1} > k_n$: let $y_1 = x_1$ and $x_1 \in M_{k_1}$. Choose now n with $x_n \notin M_{k_1}$, $x_n \in M_{k_2}$. Such n exists, otherwise, as M_{k_1} is closed in M, $x \in M_{k_1}$, which yields a contradiction. Thus $k_2 > k_1$ because otherwise $M_{k_2} \subseteq M_{k_1}$, yielding a contradiction to $x_n \notin M_{k_1}$. We now choose $y_2 = x_n$. We continue this construction for the remaining y_i. It is easy to complete the sequence (y_n) to yield a sequence (z_n) with $z_n \in M_n$ and $\lim z_n = x$.

Lemma 4

Let $((M_i, d_i))_{i\geq 0}$ with $(e_i)_{i\geq 0}$ be an embedding sequence in CMS. The completion M of $(\cup M_i, \cup d_i)$ is the direct limit of (M_i) in CMS.

Proof:

Let N be an object in CMS and $g_n : M_n \to N$ with $g_i = e_i \circ g_{i+1}$ morphisms. The g_n determine a unique weak contraction $g : \cup M_n \to N$ such that we may first embed M_i into $\cup M_n$ and then apply g or immediately apply g_i. From the universal property of the completion M we can uniquely extend g to yield a continuous $g' : M \to N$. We have to show that g' is a weak contraction. Let $x, y \in M$, $x = \lim x_n$, $y = \lim y_n$, w.l.o.g. $M_n \neq \emptyset$ and $x_n, y_n \in M_n$.

$$d_N(g'(x), g'(y)) = d_N(\lim g_n(x_n), \lim g_n(y_n))$$

$$\le \lim d_M(x_n, y_n)$$
$$= d_M(x, y) \quad .$$

In the following we will be interested in such solutions of equations that are complete metric spaces as in [7]. The reason why fixed points that are not complete metric spaces are not interesting for the semantic specification of programming languages is easily understood by the following example.

Example 1 (see [7]):

Let (X, d) be a metric space, $d \le 1$, p_0 a distinguished element, A an arbitrary set. Consider $Y = \{p_0\} \cup A \times X$ together with the metric

$$d(p_0, p_0) = 0,$$
$$d(p_0, y) = d(y, p_0) = 1 \quad \text{for} \quad y \ne p_0,$$
$$d((a, x), (a', x')) = \begin{cases} 1, & \text{if } a \ne a' \\ \frac{1}{2} d(x, x'), & \text{if } a = a'. \end{cases}$$

Let $\mathcal{F}$ be the functor in MS that maps X to $\{p_0\} \cup A \times X$. For a morphism $f : X \to Y$ we define $\mathcal{F}(f) : \{p_0\} \cup A \times X \to \{p_0\} \cup A \times Y$, $\mathcal{F}(f)(p_0) = p_0$ and if $X \ne \emptyset$ $\mathcal{F}(f)(a, x) = (a, f(x))$. Define

$$P_0 = \{p_0\}, \quad P_{i+1} = \mathcal{F}(P_i), \ i \ge 0$$

and $P_\omega = \bigcup_{i \ge 0} P_i$ with the inherited metric then clearly there is an isometry between P_ω and $\mathcal{F}(P_\omega)$, hence P_ω is a fixed point of $\mathcal{F}$. If, however, P_ω is to be used as a semantic domain for the interpretation of programs, the problem arises that nonterminating program executions cannot be handled. This can be achieved by taking the completion of P_ω as a semantic domain. A nonterminating computation can then be modelled by the limit of the Cauchy sequence of its finite approximations.

Definition 7

Let $n \ge 1$ and let

$$\mathcal{F} : \underbrace{MS \times \cdots \times MS}_{n \text{ times}} \to MS$$

be a functor. $\mathcal{F}$ *preserves completeness*, if for $M_1, \ldots, M_n$ in CMS, $\mathcal{F}(M_1, \ldots, M_n)$ is an object in CMS. $\mathcal{F}$ *preserves embeddings* if, given embeddings $e_i : N_i \to M_i$, $i = 1, \ldots, n$, $\mathcal{F}(e_1, \ldots, e_n)$ is an embedding from $\mathcal{F}(N_1, \ldots, N_n)$ to $\mathcal{F}(M_1, \ldots, M_n)$. If $\mathcal{F}$ preserves embeddings we say that $\mathcal{F}$ *preserves μ-cuts* if, given embeddings e_i with μ-cuts c_i, $i = 1, \ldots, n$, then $\mathcal{F}(c_1, \ldots, c_n)$ is a μ-cut for $\mathcal{F}(e_1, \ldots, e_n)$.

Let $\mathcal{F} : MS \to MS$ be a functor that preserves completeness and embeddings. We define an embedding sequence as follows: let $M_0 = \emptyset$ { the empty space },

$$M_i = \mathcal{F}(M_{i-1}), \quad i \geq 1$$

and let $e_0 : M_0 \to M_1$ be the unique embedding and $e_i = \mathcal{F}(e_{i-1})$, $i \geq 1$. Clearly, the M_i are complete. Let M denote the completion of the direct limit of this embedding sequence in MS.

<u>Lemma 5</u>

Let $\mathcal{F} : MS \to MS$ be a functor that preserves completeness and embeddings. Let M be given as above. Then there is an embedding $e : M \to \mathcal{F}(M)$.

<u>Proof:</u>

Let $h_i : M_i \to M$ be the canonical embeddings, $i \geq 0$. As

$$h_i = e_i \circ h_{i+1}$$

we obtain

$$\alpha_{i+1} := \mathcal{F}h_i = e_{i+1} \circ \mathcal{F}h_{i+1}, \quad i \geq 0$$

where $\mathcal{F}h_i : M_{i+1} \to \mathcal{F}(M)$. M_0 can be trivially embedded into $\mathcal{F}(M)$, say by α_0, and $e_0 \circ \mathcal{F}h_0 = \alpha_0$ by the initiality of M_0. As by Lemma 4 M is the direct limit of the embedding sequence in CMS and $\mathcal{F}(M)$ is complete we conclude the existence of a weak contraction $e : M \to \mathcal{F}(M)$ such that $h_0 \circ e = \alpha_0$ and $\mathcal{F}h_i = h_{i+1} \circ e$. By the construction of e, see lemma 4, it is clear that e is an embedding.

<u>Definition 8</u>

i) Let A be a set, (X, d) a metric space. Define a metric on $A \times X$ by

$$d((a,x),(a',x')) = \begin{cases} 1 & \text{if } a \neq a' \\ \frac{1}{2}d(x,x') & \text{else.} \end{cases}$$

ii) Let A be a set, (X, d) a metric space. $A \to X$ is the set of functions from A to X. Define a metric on $A \to X$ by

$$d(f,g) = \sup_{a \in A} d(f(a), g(a)).$$

iii) Let (X_1, d_1), (X_2, d_2) be metric spaces. Define a metric on $M_1 \times M_2$ by

$$d((x_1,x_2),(y_1,y_2)) = \max\{d_1(x_1,y_1), d_2(x_2,y_2)\}.$$

<u>Lemma 6</u>

Let the endofunctor $\mathcal{F}$ in MS be defined by

$$\mathcal{F}(X) = A \times X$$
$$\mathcal{F}(f) = \lambda(a,x)(a,f(x))$$

then $\mathcal{F}$ preserves completeness, embeddings und μ-cuts.

<u>Proof:</u>

Let $\{y_n\}$ be a Cauchy sequence in $A \times X$; from the definition of the metric it follows that there is $n_0 \in I\!N$ and $a \in A$ such that $y_n = (a, x_n)$ for $n > n_0$ and (x_n) is a Cauchy sequence in X. Hence $\{y_n\}$ converges to $(a, \lim x_n)$. Let $e : X \to Y$ be an embedding then

$$\begin{aligned} d\left(\mathcal{F}(e)\left((a,x)\right), \mathcal{F}(e)\left((a',x')\right)\right) &= d\left((a, e(x)), (a', e(x'))\right) \\ &= \begin{cases} 1 & \text{if } a \neq a' \\ \frac{1}{2} d((a,x),(a',x')) & \text{else} \end{cases} \\ &= d((a,x),(a',x')). \end{aligned}$$

Let c be a μ-cut for e, i.e.

$$c(e(x)) = x \quad \forall\, x \in X$$
$$d(y, e(c(y))) \leq \mu \quad \forall\, y \in Y.$$

Let $z \in \mathcal{F}(X) = A \times X$, $z = (a,x)$, then

$$\begin{aligned} \mathcal{F}(c)(\mathcal{F}(e)(z)) &= (a, c(e(x))) \\ &= (a,x) \\ &= z. \end{aligned}$$

and for $z \in \mathcal{F}(Y) = A \times Y$, $z = (a,y)$

$$\begin{aligned} d(z, \mathcal{F}(e)\mathcal{F}(c)(z)) &= d\left((a,y), (a, e(c(y)))\right) \\ &= \frac{1}{2} d(y, e(c(y))) \\ &\leq \mu. \end{aligned}$$

<u>Lemma 7</u>

The endofunctor $\mathcal{F} : MS \to MS$

$$\begin{aligned} \mathcal{F}(X) &= A \to X \\ \mathcal{F}(f) &= \lambda g\, \lambda a\, f(g(a)) \end{aligned}$$

preserves completeness, embeddings and μ-cuts.

Proof:

in analogy to Lemma 6.

Lemma 8

The functor $\mathcal{F} : MS \times MS \to MS$

$$\mathcal{F}(X_1, X_2) = X_1 \times X_2$$

$$\mathcal{F}(f_1, f_2) = \lambda(x, y)\,(f_1(x), f_2(y))$$

preserves completeness, embeddings and μ-cuts.

Proof:

in analogy to Lemma 6.

By now, we have treated some examples of functors that are relevant for the definition of the semantics of programming languages. One functor of interest in this context, the functor $\wp_c$, is given special treatment in section III.

II. The Existence and Uniqueness of Fixed Points

In this section we are going to derive conditions for the existence of fixed points.

In analogy to the classical case of fixed points in complete metric spaces we establish conditions that guarantee that

i) a sequence $\{M_i\}$ of metric spaces generated by iteration as in Lemma 5 is a "Cauchy sequence"

and

ii) its "limit" is a fixed point.

The first criterion is derived from the fixed point theorem by Banach–Cacciopoli.

<u>Definition 9</u>

Let $\mathcal{F} : MS \to MS$ be a functor that preserves completeness and embeddings. $\mathcal{F}$ is called a *<u>contraction functor</u>*, if there exists a k, $0 \leq k < 1$, such that for all N, M in CMS and all embeddings $e : N \to M$ with $N \neq \emptyset$

$$d_{\mathcal{F}(e)}(\mathcal{F}(N), \mathcal{F}(M)) \leq k \cdot d_e(N, M)$$

holds.

<u>Definition 10</u>

Let $\mathcal{F} : MS \to MS$ be a functor that preserves completeness, embeddings and μ-cuts. $\mathcal{F}$ is called *<u>cut-contractive</u>*, if there is a k, $0 \leq k < 1$, such that for every embedding e with μ-cut c, $\mathcal{F}(c)$ is a $(k \cdot \mu)$-cut for $\mathcal{F}(e)$.

For practical purposes there is an easy way to determine these properties for a given functor:

<u>Lemma 9</u>

Let $\mathcal{F} = \mathcal{F}_1 \circ \mathcal{F}_2$ or $\mathcal{F} = \mathcal{F}_2 \circ \mathcal{F}_1$ where $\mathcal{F}_i$ is an endofunctor in MS, $i = 1, 2$, that preserves embeddings and completeness.

a) If $\mathcal{F}_1$ is a contraction functor and $\mathcal{F}_2$ satisfies a weak contraction property; i.e. $d_{\mathcal{F}_2(e)}(\mathcal{F}_2(N), \mathcal{F}_2(M)) \leq d_e(N, M)$ for every embedding e, $e : N \to M$, $N \neq \emptyset$, then $\mathcal{F}$ is a contraction functor.

b) If $\mathcal{F}_1$ is cut-contractive and $\mathcal{F}_2$ preserves μ-cuts then $\mathcal{F}$ is cut-contractive.

Proof:

a) Let $\mathcal{F} = \mathcal{F}_1 \circ \mathcal{F}_2$, $e : N \to M$ an embedding, $N \neq \emptyset$,

$$\begin{aligned} d_{\mathcal{F}(e)}(\mathcal{F}(N), \mathcal{F}(M)) &= d_{\mathcal{F}_2(\mathcal{F}_1(e))}\Big(\mathcal{F}_2(\mathcal{F}_1(N)), \mathcal{F}_2(\mathcal{F}_1(M))\Big) \\ &\leq d_{\mathcal{F}_1(e)}(\mathcal{F}_1(N), \mathcal{F}_1(M)) \\ &\leq k \cdot d_e(N,M). \end{aligned}$$

b) Let $\mathcal{F} = \mathcal{F}_1 \circ \mathcal{F}_2$, $e : N \to M$ an embedding with μ-cut $c : M \to N$. We have to show that there is a k such that $\mathcal{F}(c)$ is a $(k \cdot \mu)$-cut. Clearly $\mathcal{F}(c)(\mathcal{F}(e)(x)) = x$. Consider

$$\begin{aligned} d\Big(\mathcal{F}(e)(\mathcal{F}(c)(x)), x\Big) &= d\Big(\mathcal{F}_2(\mathcal{F}_1(e))\Big(\mathcal{F}_2(\mathcal{F}_1(c))(x)\Big), x\Big) \\ &\leq k \cdot \mu \end{aligned}$$

as $\mathcal{F}_1(c)$ is a $(k \cdot \mu)$-cut for $\mathcal{F}_1(e)$ and $\mathcal{F}_2$ preserves this property.

Theorem 10

Let $\mathcal{F} : MS \to MS$ be a contraction functor then $\mathcal{F}$ has a fixed point in CMS. If in addition $\mathcal{F}\emptyset \neq \emptyset$, this fixed point is unique up to isometry among the objects of CMS. In other words, $\mathcal{F}$ considered as functor from CMS to CMS has a unique fixed point.

Proof:

If $\mathcal{F}\emptyset = \emptyset$ the statement is trivial. Let now $\mathcal{F}\emptyset \neq \emptyset$. As a first step we construct an embedding sequence (M_i) as in Lemma 5 by choosing M_0 as the empty space and $M_i = \mathcal{F}(M_{i-1})$, $i \geq 1$; each M_i is complete and can be identified with an element of $\wp_c^{\emptyset}(M)$, where M is the completion of $\cup M_i$. We already know by Lemma 5 that there is an embedding $e : M \to \mathcal{F}(M)$ with $\mathcal{F}h_i = h_{i+1} \circ e$, hence

$$\begin{aligned} d_{h_{n+1}}(M_{n+1}, M) &= d_{h_{n+1}}(\mathcal{F}(M_n), M) \\ &\leq d_{h_{n+1} \circ e}(\mathcal{F}(M_n), \mathcal{F}(M)) \quad \text{by Lemma 1} \\ &= d_{\mathcal{F}(h_n)}(\mathcal{F}(M_n), \mathcal{F}(M)) \\ &\leq k \cdot d_{h_n}(M_n, M), \end{aligned}$$

where the h_i are the canonical embeddings. Continuing this argument we get

$$d(M_{n+1}, M) \leq k^n\, d(M_1, M),$$

hence M is the limit of the embedding sequence (M_i) with (e_i) in $\wp_c^{\emptyset}(M)$.

On the other hand

$$\begin{aligned} d(M_{n+1}, \mathcal{F}(M)) &= d_{\mathcal{F}(h_n)}(\mathcal{F}(M_n), \mathcal{F}(M)) \\ &\leq k \cdot d_{h_n}(M_n, M), \end{aligned}$$

hence $\mathcal{F}(M)$ is the limit of the embedding sequence (M_i), (e_i) in $\wp_c^e(\mathcal{F}(M))$ from which we conclude that M is a fixed point.

Let N be another fixed point of $\mathcal{F}$ in CMS, hence there is an isometry

$$h : \mathcal{F}(N) \to N .$$

As M_0 is initial we have a unique embedding $g_0 : M_0 \to N$ and $g_0 = e_0 \circ \mathcal{F} g_0 \circ h$ where $e_0 : M_0 \to \mathcal{F}(M_0) = M_1$. Let for $i \geq 1$ embeddings g_i be defined by

$$g_i = \mathcal{F} g_{i-1} \circ h$$
$$g_i : M_i \to N$$

then $g_i = e_i \circ g_{i+1}$ for $i \geq 0$, i.e. the M_i can be embedded into N in a way that is <u>compatible</u> with the embeddings e_i. Hence there is an embedding f,

$$f : M \to N ,$$

such that $h_i \circ f = g_i$, $i \geq 0$. In addition for $i \geq 1$

$$\begin{aligned} d_{g_{i+1}}(M_{i+1}, N) &= d_{g_{i+1}}(\mathcal{F}(M_i), N) \\ &\leq d_{g_{i+1} \circ h^{-1}}(\mathcal{F}(M_i), \mathcal{F}(N)) \\ &= d_{\mathcal{F} g_i}(\mathcal{F}(M_i), \mathcal{F}(N)) \\ &\leq k \cdot d_{g_i}(M_i, N) \end{aligned}$$

hence the M_i converge towards N from where we conclude $M = N$.

<u>Remark 6</u>

Obviously definitions 9 and 10, as well as lemma 9 and theorem 10 can be adapted to n-ary functors.

<u>Example 2</u>

The functor $\mathcal{F}$ given in Example 1 satisfies the conditions of Theorem 10 with $k = \frac{1}{2}$ as contraction constant.

<u>Example 3</u>

The functor $\mathcal{G} : MS \to MS$ given by

$$\mathcal{G}(X) = \{p_0\} \cup A \times (X \cup (B \times X))$$

and suitably defined for morphisms satisfies the conditions of theorem 10.

By applying lemma 9 various functors can be shown to satisfy the conditions of Theorem 10. There are, however, interesting cases for which the conditions of Theorem 10 are too strong, e.g. functors that are built with the $\wp_c$-functor as $\mathcal{F}(X) = \{p_0\} \cup \wp_c(A \times X)$. For these cases we use the concept cut-contractive.

Lemma 11

Let $\mathcal{F} : MS \to MS$ be a cut-contractive functor. Let M_0 be a complete metric space such that there is an embedding $e_0 : M_0 \to \mathcal{F}(M_0)$ with μ-cut $c_0 : \mathcal{F}(M_0) \to M_0$. Let $M_i = \mathcal{F}(M_{i-1})$, $i \geq 1$, $e_i = \mathcal{F}(e_{i-1})$, $c_i = \mathcal{F}(c_{i-1})$, $i \geq 1$. Let M be the completion of $\cup M_i$ and let $h_i : M_i \to M$ be the canonical embedding. Then there is a μ_i-cut $l_i : M \to M_i$ for h_i with $\lim_{i \to \infty} \mu_i = 0$.

Proof:

From the properties of $\mathcal{F}$ it is clear that c_i is a $(\mu \cdot k^i)$-cut for e_i. For fixed n we consider the family of morphisms $(c_{mn})_{m \geq 0}$ as given in Definition 5, $c_{mn} : M_m \to M_n$. As M is the direct limit of the M_i according to Lemma 4 there is a uniquely determined contraction $l_n : M \to M_n$ such that

$$c_{mn} = h_m \circ l_n \quad m \geq 0.$$

From this we immediately get that

$$l_n h_n(x) = x \quad \forall\, x \in M_n.$$

It remains to evaluate $d(x, h_n l_n(x))$ for $x \in M$. For this let $n \geq 0$ and $x \in M_{n+2}$. As c_n is a $(\mu \cdot k^n)$-cut

$$d\Big(c_{n+1}(x), e_n\big(c_n(c_{n+1}(x))\big)\Big) \leq \mu \cdot k^n$$

hence

$$d\Big(e_{n+1}(c_{n+1}(x)), e_{n+1}\Big(e_n\big(c_n(c_{n+1}(x))\big)\Big)\Big) \leq \mu \cdot k^n$$

implying

$$\begin{aligned} d\Big(x, e_{n+1}\Big(e_n\big(c_n(c_{n+1}(x))\big)\Big)\Big) &\leq d\big(x, e_{n+1}(c_{n+1}(x))\big) \\ &\quad + d\Big(e_{n+1}(c_{n+1}(x)), e_{n+1}\Big(e_n\big(c_n(c_{n+1}(x))\big)\Big)\Big) \\ &\leq \mu \cdot k^{n+1} + \mu \cdot k^n \\ &= \mu \cdot (k^n + k^{n+1}) \end{aligned}$$

thus

$$d(x, c_{n+2,n}(x)) \leq \mu \cdot (k^n + k^{n+1})$$

and in general

$$d(x, c_{mn}(x)) \leq \mu \cdot (k^n + k^{n-1} + \cdots + k^{m-1})$$

for all $m \geq n$. Put

$$\begin{aligned} \mu_n &= \mu \cdot \sum_{m \geq n} k^m \\ &= \mu \cdot \left(\frac{k^n}{1-k}\right) \end{aligned}$$

Let now $x \in M$, $x = \lim x_m$, $x_m \in M_m$.

$$\begin{aligned} d(x, l_n(x)) &= d(\lim_m x_m, \lim_m c_{mn}(x_m)) \\ &= \lim_m d(x_m, c_{mn}(x_m)) \\ &\leq \mu_n, \end{aligned}$$

omitting the explicit notation of the canonical embeddings.

In the following we present an existence and uniqueness result for cut–contractive functors. The existence part has been independently found in a similar form by [2]. See also section V for detailed discussion.

<u>Theorem 12</u>

Let $\mathcal{F} : MS \to MS$ be a cut-contractive functor. Then $\mathcal{F}$ has a fixed point in CMS. If in addition $\mathcal{F}\emptyset \neq \emptyset$ then $\mathcal{F}$ has a fixed point that is unique up to isometry among the objects of CMS.

<u>Proof</u>

If $\mathcal{F}\emptyset = \emptyset$ the statement is trivial. Let now $\mathcal{F}\emptyset \neq \emptyset$. Choose a one-element space $S_0 = \{x_0\}$ and let $S_i = \mathcal{F}S_{i-1}$, $i \geq 1$. Clearly S_i is a complete metric space. As before let $M_0 = \emptyset$, $M_i = \mathcal{F}M_{i-1}$, $i \geq 1$, and

$$e_0 : M_0 \to M_1$$

the unique embedding and

$$e_i = \mathcal{F}e_{i-1}, \quad i \geq 1.$$

There is a unique embedding

$$i_0 : M_0 \to S_0 .$$

We choose in addition an embedding

$$\lambda_0 : S_0 \to M_1$$

which is possible as $S_0 = \{x_0\}$ and $M_1 = \mathcal{F}\emptyset \neq \emptyset$ by assumption. From the initiality of M_0 we obtain

$$e_0 = i_0 \circ \lambda_0 . \tag{I}$$

We now put $\sigma_0 = \lambda_0 \circ \mathcal{F} i_0$

$$\sigma_0 : S_0 \to S_1$$

and $\sigma_i = \mathcal{F}\sigma_{i-1}$, $i \geq 1$, having thus turned the sequence S_i into an embedding sequence (with embeddings σ_i).

Let S denote the completion of the $\cup S_i$ and let $k_i : S_i \to S$ be the canonical embeddings, $i \geq 0$.

We first observe that

$$r_{i+1} = \mathcal{F} k_i : S_{i+1} \to \mathcal{F}S , \quad i \geq 0$$

is an embedding and

$$r_{i+1} = \sigma_{i+1} \circ r_{i+2} , \quad i \geq 0 . \tag{II}$$

We put $r_0 = \sigma_0 \circ \mathcal{F} k_0$ and get

$$r_0 = \sigma_0 \circ r_1 . \tag{III}$$

As S is the direct limit of the S_i in CMS we conclude the existence of an embedding

$$e : S \to \mathcal{F}S \tag{IV}$$

such that

$$r_i = k_i \circ e . \tag{V}$$

Let us define

$$c_0 : S_1 \to S_0$$

$$c_0 = \lambda x . x_0$$

then according to lemma 11 there are μ_i -cuts $l_i : S \to S_i$ for k_i with $\lim \mu_i = 0$. As

$$d_{k_i}(S_i, S) \leq \mu_i$$

by lemma 2, we conclude that $S = \lim S_i$ in $\wp_c(S)$. On the other hand

$$d_{r_{i+1}}(S_{i+1}, \mathcal{F}S) = d_{\mathcal{F}k_i}(S_{i+1}, \mathcal{F}S) \leq k \cdot \mu_i$$

and hence $\mathcal{F}S = \lim S_i$ in $\wp_c(\mathcal{F}S)$. By (II) , (III) , (IV) , (V) we conclude that $\mathcal{F}S$ and S coincide up to isomorphism.

Let now N be another fixed point. Hence there is an isometry

$$h : \mathcal{F}N \to N .$$

Let $j_0 : \emptyset \to N$ be the unique morphism then by the initiality of M_0 we have

$$j_0 = e_0 \circ \mathcal{F} j_0 \circ h \, . \qquad (VI)$$

We define now $\tau_0 : S_0 \to N$

$$\tau_0 = \lambda_0 \circ \mathcal{F} j_0 \circ h$$

and set

$$\tau_i = \mathcal{F} \tau_{i-1} \circ h$$

and get

$$\begin{aligned}
\sigma_0 \circ \tau_1 &= \sigma_0 \circ (\mathcal{F} \tau_0 \circ h) \quad \text{by Def. of } \tau_1 \\
&= (\lambda_0 \circ \mathcal{F} i_0) \circ (\mathcal{F} \tau_0 \circ h) \quad \text{by Def. of } \sigma_0 \\
&= \lambda_0 \circ \mathcal{F}(i_0 \circ \tau_0) \circ h \\
&= \lambda_0 \circ \mathcal{F}(i_0 \circ \lambda_0 \circ \mathcal{F} j_0 \circ h) \circ h \quad \text{by Def. of } \tau_0 \\
&= \lambda_0 \circ \mathcal{F}(\epsilon_0 \circ \mathcal{F} j_0 \circ h) \circ h \quad \text{by } (I) \\
&= \lambda_0 \circ \mathcal{F}(j_0) \circ h \quad \text{by } (VI) \\
&= \tau_0 \quad \text{by Def. of } \tau_0 \qquad (VII)
\end{aligned}$$

hence τ_0 is an embedding such that the following diagram

$$\begin{tikzcd}
\{x_0\} \arrow[rr, "\sigma_0"] \arrow[dr, "\tau_0"'] & & \mathcal{F}\{x_0\} \arrow[dl, "\mathcal{F}\tau_0 \circ h"] \\
& N &
\end{tikzcd}$$

commutes. Consequently

$$\tau_i = \sigma_i \circ \tau_{i+1}$$

by induction. Hence, as S is the direct limit of the S_i with respect to the σ_i we conclude that there is a unique embedding

$$f : S \to N$$

with $\tau_i = k_i \circ f$, $i \geq 0$.

It remains to show that the embedding sequence S_i (with respect to σ_i) converges to

N. For this we define

$$g_0 : N \to S_0$$

$$g_0 = \lambda x . x_0$$

and

$$g_i = h^{-1} \circ \mathcal{F} g_{i-1}\,, \quad i \geq 1$$

$$g_i : N \to S_i$$

Clearly g_0 is a 1-cut for τ_0. By induction

$$\tau_i \circ g_i = 1\,, \quad i \geq 0\,.$$

By induction g_i is a k^i-cut for τ_i, as for all $x \in N$

$$\begin{aligned} d_N\Big(x, \tau_{i+1}(g_{i+1}(x))\Big) &= d\Big(x, h\big(\mathcal{F}\tau_i(\mathcal{F}g_i(h^{-1}(x)))\big)\Big) \\ &= d_N\Big(h(y), h\big(\mathcal{F}\tau_i(\mathcal{F}g_i(y))\big)\Big) \\ &= d_{\mathcal{F}(N)}\Big(y, \mathcal{F}\tau_i\big(\mathcal{F}g_i(y)\big)\Big) \\ &\leq k \cdot k^i\,. \end{aligned}$$

Hence $d_{\tau_i}(S_i, N) \leq k^i$ hence N and S coincide up to isometry.

<u>Remark 7</u>

As $\wp_c$ preserves μ-cuts (see section III) Theorem 12 together with Lemma 9 allow us to handle a variety of interesting functors.

<u>Example 4</u>

The functor $\mathcal{F}(X) = \{p_0\} \cup \Big(A \to \wp_c\big(B \times (X \cup (C \to X))\big)\Big)$ satisfies the conditions of Theorem 12.

<u>Example 5</u>

In [13] a detailed semantic definition of Hoare's communicating sequential processes [12] is given using the metric space approach. The equation, that is the basis for this definition is described by the functor:

$$\mathcal{F}(X) = \{p_0\} \cup \Big(A \to \wp_c\big(\{f, \delta, \bot\} \cup (A \cup C \cup \wp(I)) \times (X \cup (V \times X) \cup (V \to X))\big)\Big)$$

which satisfies the conditions of Theorem 12. The details of this semantic description are too lengthy to be presented here.

III. The Functor $\wp_c$

In this section we deal with the operator $\wp_c$ that deserves some special consideration because it cannot be simply considered as an endofunctor in MS, as, in general, an arbitrary morphism in MS

$$f : N \to M$$

will not yield a morphism from $\wp_c(N)$ to $\wp_c(M)$ via $\lambda U\, f(U)$. So $\wp_c$ has to be restricted to those morphisms $f : N \to M$ that are closed, i.e. that they map closed subsets of N to closed subsets of M. If we denote by MS_c the subcategory of MS that has the same objects as MS and closed morphisms as morphisms then $\wp_c$ is a functor from MS_c to MS.

Clearly all the definitions of prefixed point, completeness preserving etc. can be easily adapted to the case of such a "partial" functor.

<u>Lemma 13:</u>

The functor $\wp_c : MS_c \to MS$ preserves completeness, embeddings and μ-cuts.

<u>Proof:</u>

$\wp_c$ preserves completeness according to Remark 1. Preservation of embeddings is trivial, preservation of μ-cuts follows from the definition of the Hausdorff metric.

For functors $\mathcal{F}$ that arise from combination of $\wp_c$ with other functors it has to be ensured that the construction of fixed points by iteratively defining an embedding sequence (S_i) with respective μ-cuts is not affected.

We have to establish that starting with

$$\sigma_0 : S_0 \to \mathcal{F}(S_0) \qquad S_0 = \{x_0\}$$

$$c_0 : \mathcal{F}(S_0) \to S_0 \qquad c_0 = \lambda x x_0$$

we can always apply $\mathcal{F}$ iteratively to get

$$\sigma_i = \mathcal{F}^i(\sigma_0)$$

$$c_i = \mathcal{F}^i(c_0).$$

<u>Definition 11:</u>

A metric space (X, d) has the <u>*minimum distance property*</u>, if there exists $\delta \in \mathbb{R}$, $\delta \geq 0$, such that for all $x, y \in X$, $x \neq y$, $d(x, y) \geq \delta$.

Remark 8:

The topology of a metric space with the minimum distance property is the discrete topology, as every one-element set is open.

Lemma 14:

The functors $\mathcal{F}_1(X) = A \times X$, $\mathcal{F}_2(X) = A \to X$, $\mathcal{F}_3(X) = \wp_c(X)$, $\mathcal{F}_4(X_1, X_2) = X_1 \uplus X_2$, $\mathcal{F}_5(X_1, X_2) = X_1 \times X_2$ preserve the minimum distance property, i.e. if the arguments of $\mathcal{F}_i$ inhibit the minimum distance property, so does the resulting metric space.

Proof:

As an example we treat the case of $\mathcal{F}_2$. Let (X, d) be a metric space and $\delta \in \mathbb{R}$, $\delta > 0$ such that $d(x, y) \geq \delta \;\; \forall\, x, y \in X$. Let $f, g \in \mathcal{F}_2(X)$

$$
\begin{aligned}
d(f, g) &= \sup_{a \in A} d(f(a), g(a)) \\
&\geq \delta.
\end{aligned}
$$

Lemma 15:

Let $\mathcal{F}$ be a functor that is composed of functors in $\{\mathcal{F}_1, \ldots, \mathcal{F}_5\}$ (see Lemma 14). Let N be a metric space that has the minimum distance property and $g : M \to N$ a contraction, then $\mathcal{F}$ is defined for g.

Proof:

For ease of notation we only treat unary functors in $\{\mathcal{F}_1, \ldots, \mathcal{F}_5\}$. Let hence $\mathcal{F} = \mathcal{G}_1 \circ \mathcal{G}_2 \circ \cdots \circ \mathcal{G}_k$ with $\mathcal{G}_i$ (unary) in $\{\mathcal{F}_1, \ldots, \mathcal{F}_5\}$, $1 \leq i \leq k$. As N has the minimum distance property so does $\mathcal{G}_1(N)$, $\mathcal{G}_2(\mathcal{G}_1(N))$, etc. and finally $\mathcal{F}(N)$ by Lemma 16. Hence the topology of $\mathcal{G}_1(N)$, $\mathcal{G}_2(\mathcal{G}_1(N))$ etc. is the discrete topology by Remark 11. As N has the discrete topology we conclude that g is a closed morphism, hence $\mathcal{G}_1$ is defined for g, $\mathcal{G}_1(g) : \mathcal{G}_1(M) \to \mathcal{G}_1(N)$. Similarly $\mathcal{G}_1(N)$ has the discrete topology, hence $\mathcal{G}_2$ is defined for $\mathcal{G}_1(g)$ and so on.

Corollary 16:

Let $\mathcal{F}$ be as in Lemma 15, N a metric space that has the minimum distance property, $g : M \to N$ a morphism. Then $\mathcal{F}^n$ is defined for g for all $n \geq 1$.

The above observations guarantee that our results also hold for functors that are composed from $\wp_c$ and others. Obviously the above results can be extended to any other functors that preserve the minimum distance property.

There is an alternative approach to treat the powerset construction which has been recently proposed by [2]. There, the authors define for a complete metric space M, $M \neq \emptyset$,

$$\tilde{\wp}_c(M) = \{U \subset M : U \text{ closed}, U \neq \emptyset\}$$

and for $f : N \to M$

$$\tilde{\wp}_c(f) = \lambda U.\, \mathbf{cl}(f(U))$$

where $\mathbf{cl}(X)$ of a subset X of M stands for the closure of M. It probably depends on the particular application which of the two different powerset functors is adequate.

IV. The Equation $P = \{p_0\} \cup (A \rightarrow \wp_c((B \times P) \cup (C \rightarrow P)))$ and General Considerations on the Choice of Metric

We claimed in the introduction that the above equation in [7], the solution of which is left to the reader, cannot be solved as proposed by [7], namely by putting

$$Y_0 = \{p_0\}$$
$$Y_i = \{p_0\} \cup \Big(A \rightarrow \wp\big((B \times Y_i) \cup (C \rightarrow Y_i)\big)\Big)$$

and showing that the completion Y of $\cup Y_i$ is a solution of the above equation by establishing an isometry between Y and $\mathcal{F}(Y)$.

We do not claim that the equation does not have a solution at all. We do claim that Y cannot be one.

Let us consider the functor $\mathcal{F}(X) = \{p_0\} \cup \Big(A \rightarrow \wp_c\big((B \times X) \cup (C \rightarrow X)\big)\Big)$ in more detail. $\mathcal{F}$ clearly preserves completeness and embeddings and, according to Lemma 5, there is an embedding $\Phi : Y \rightarrow \mathcal{F}(Y)$. In order to establish that Φ is an isometry, we have to show that Φ is onto.

We claim that this cannot be the case. Let us for simplicity only consider the case where A, B and C are finite sets.

We define the infinite set

$$S_\infty = \{\lambda c p_0, \lambda c \lambda a \lambda c p_0, \lambda c \lambda a \lambda c \lambda a \lambda c p_0, \ldots\}$$

and observe

i) $S_\infty \subset (C \rightarrow Y)$,

ii) S_∞ is closed, as there do not exist any nontrivial convergent sequences in S_∞, i.e. $S_\infty \in \wp_c((B \times Y) \cup (C \rightarrow Y))$,

iii) S_∞ has non countably many infinite subsets T_∞, each of which is closed, as there are no nontrivial convergent sequences.

To see this, remember that the metric on Y_{n+1} is given by

$$d_{n+1}(p, p_0) = d(p, p_0) = 1 \quad p \neq p_0$$
$$d_{n+1}(p, p') = \sup_{a \in A} \{d(p'(a), p(a))\}$$

and for $x, y \in (B \times Y_n) \cup (C \rightarrow Y_n)$

$$d(x.y) = \begin{cases} 1, & \text{if } x \in B \times Y_n,\ y \in C \rightarrow Y_n \text{ or viceversa} \\ 1, & \text{if } x, y \in B \times Y_n,\ x = (a, x'),\ y = (b, y'),\ a \neq b \\ \frac{1}{2} d(x', y'), & \text{if } x = (a, x'),\ y = (a, y') \\ \sup_{c \in C} (x(c), y(c)), & \text{if } x, y \in C \rightarrow Y_n. \end{cases}$$

Let us assume that there is an isometry $\Phi : Y \to \mathcal{F}(Y)$. We consider the family of functions

$$g = \lambda a S_\infty$$

$$g_{T_\infty} = \lambda a T_\infty$$

where T_∞ is an infinite subset of S_∞. Clearly g and all g_T are elements of $\mathcal{F}(Y)$. If $\Phi : Y \to \mathcal{F}(Y)$ is onto there must be an $f \in Y$ such that $\Phi(f) = g$. $f \in Y$ implies that either

$$f \in \bigcup_{i\geq 0} Y_i$$

or

$$f = \lim f_n \quad f_n \in Y_n.$$

Assume that $f = \lim f_n$ and $f \notin \cup Y_i$ then we get

$$\begin{aligned} 0 &= \lim d(\Phi(f), \Phi(f_n)) \\ &= \lim d(g, \Phi(f_n)) \end{aligned}$$

yielding $\Phi(f_n) \to g$ and hence a contradiction, because only a trivial (finally constant) sequence can converge towards g. On the other hand Φ is one-to-one and $\{f_n\}$ cannot be trivial because $f \notin \cup Y_i$ was assumed.

So we conclude that no element in $Y \setminus \bigcup_{i\geq 0} Y_i$ can be mapped to g or analogously to any g_{T_∞}, thus only remain the elements of $\cup Y_i$ as candidates. But from the definition of the functor it is clear that $\cup Y_i$ has only countably many elements. Hence there cannot exist an isometry. It is easy to see that the functor

$$\mathcal{F}(X) = \{p_0\} \cup \Big(A \to \wp_c\big((B \times X) \cup (C \to X)\big)\Big)$$

is not cut-contractive. As by the above the standard construction does not work to construct a fixed point the condition "cut-contractive" seems to be quite narrow a criterion for the existence of fixed points.

Let us now consider this matter a little further. We slightly modify the functor $\mathcal{G}(X) = C \to X$ where the metric on $C \to X$ is given in Definition 8 by $d(f,g) = \sup_{a\in C} d_X(f(a), g(a))$ and put $\mathcal{G}'(X) = C \to X$ where the metric on $C \to X$ is now given by $d'(f,g) = \frac{1}{2}\sup_{a\in C} d_X(f(a), g(a))$. What happens now is, that if we use the modified definition $\mathcal{G}'$ instead of the original one, then the resulting functor $\mathcal{F}'$ is cut-contractive, $\mathcal{F}'\emptyset \neq \emptyset$ and hence $\mathcal{F}'$ has a unique fixed point. In general, it is true that if we have a functor $\mathcal{G} : CMS \to CMS$ that preserves μ-cuts then we get a cut-contractive functor $\mathcal{G}'$ by proceeding as follows: let (X, d_X) be a complete metric space and let $\mathcal{G}((X,d_X)) = (Y,d_Y)$, then $\mathcal{G}'((X,d_X)) = (Y,kd_Y)$, where k is a fixed constant, $0 < k < 1$.

Here, immediately the question arises which definition is adequate for our original purposes, i.e. the semantic definition of programming languages. Given a language $\mathcal{L}$, one constructs a suitable domain equation $P = \mathcal{F}_{\mathcal{L}} P$ such that the solution S of this equation, if any, is the range of the meaning function

$$Me : \mathbf{Programs} \to S \ .$$

Very roughly speaking the functor $\mathcal{F}_{\mathcal{L}}$ reflects the kind of operations that can be performed in the language $\mathcal{L}$. So e.g. a functor that maps (X,d_X) to $Y = A \times X$ together with some metric d_Y serves to describe the "sequencing" of actions. There is, however, some freedom with respect to the choice of the metric d_Y. In the original paper of [7] d_Y is chosen to be

$$d_Y((a,x),(a',y)) = \begin{cases} 1 & a \neq a' \\ \frac{1}{2}d(x,y) & a = a' \end{cases} \ .$$

(Clearly, any $0 < k < 1$ instead would serve the same purpose). One might interprete this choice of d_Y as follows: the semantics of languages $\mathcal{L}$ treated by [7] is operational in flavour, i.e. for the case of a sequential program p, its meaning in this approach is basicly the "sequence" of the meaning of its actions. Under the above choice of d_Y two programs p_1 and p_2 that coincide in their first l actions are regarded to be "closer" than two programs p_1' and p_2' that coincide only on some h actions, $h < l$. Hence this choice of d_Y by [7] is very intuitive having the above interpretation in mind. We find it very hard, however, to justify – on the grounds of relevance for programming language semantics – the choice of $d'(f,g) = \frac{1}{2}\sup_{a\in C} d_X(f(a),g(a))$ for the set $C \to X$ instead of the original d. We cannot find an intuitive explanation for this change in metric. Consider equations (3) and (4) from the introduction, both of which are the basis for semantic specification of certain programming languages in [7]. Whereas (3) can be solved if the "old" metric d on the function space is used, equation (4) is only solved by the standard approach if the "new" metric d' is introduced.

V. Related Work

Recursive specification of "domains" plays a crucial role in the denotational semantics based on metric spaces [7] as well as in the denotational semantics as developed by Scott and Strachey. First approaches of Scott to solve recursive equations were his inverse limit construction [22], which were later substituted by using a universal domain and a fixed point construction [23].

The categorical aspects of these approaches were studied e.g. by Reynolds [21] and Wand [26]. These investigations typically stuck to one fixed category, e.g. the category CPO of complete partial orders with strict continuous functions or the category of countably based continuous lattices and continuous functions, and are at the same level of abstraction as our work presented here.

In [25] and [26] a further abstraction step is initiated to develop a theory of solving recursive equations for general categories. For this [25] elaborate a basic lemma:

Basic lemma [25]

"Let k be a category with initial object $\perp_k$ and let $\mathcal{F} : k \to k$ be a functor. Define the ω-chain Δ to be $(\mathcal{F}^n(\perp_k), \mathcal{F}^n(\perp_{F\perp}))$. Suppose that both $\mu : \Delta \to A$ and $\mathcal{F}\mu : \mathcal{F}\Delta \to \mathcal{F}A$ are colimiting cones then the initial fixed point exist."

In the sequel [25] discuss how the conditions of the lemma can be satisfied for the class of O-categories, i.e. categories that exhibit certain order structures in their hom-sets.

If we compare our procedere with that implied by the basic lemma, then obviously choosing $k = CMS$ our M (the completion of $\cup M_i$ in theorem 10) plays the role of A and we know that M is the direct limit of (M_i) in CMS. In order to prove the fixed point property, however, we do not show that $\mathcal{F}(M)$ is direct limit of $\mathcal{F}(M_i)$, but rather show that the distance between $\mathcal{F}(M_i) = M_{i+1}$ and $\mathcal{F}(M)$ (understood as elements in $\wp_c(\mathcal{F}(M))$) tends to zero as $i \to \infty$. Having then established the fixed point property of M we get as a trivial conclusion that $\mathcal{F}(M)$ is the direct limit of $\mathcal{F}(M_i)$. So, M is a fixed point if and only if $\mathcal{F}(M)$ is direct limit of (M_i). In addition, in CMS besides existence the uniqueness of fixed points is guaranteed for functors with a contraction property.

While this present paper was being refereed we learned about the recent and independent work of [2]. Let us relate our work to [2], in which also a problem of the present paper is tackled, i.e. the question of the solution of equations $P = \mathcal{F}P$. In [2] the authors first establish a criterion, that ensures the existence of solutions of equations of the form $P = \mathcal{F}P$ in a category of complete metric spaces. In a second step they develop a criterion that ensures uniqueness in a slightly modified category of complete metric spaces by adding base points.

Where do our results and those by [2] coincide, where do they differ? We show in the appendix that the notion of "contracting" functor of [2] (not to be confused with our definition of contracting) is about the same as our concept of μ-contractive functor (modulo slight changes in categories). So the result concerning the existence part of our theorem 12 is about the same as the result of [2] concerning the existence of fixed points. In contrast to [2] we show, however, that a cut-contractive $\mathcal{F}$ already has a <u>unique</u> fixed point, unless $\mathcal{F}\emptyset = \emptyset$. By this we also answer an open question of [2], namely to exhibit a contracting functor (in their terminology) that has nonisometric fixed points: There <u>does not</u> exist such a functor $\mathcal{F}$ with $\mathcal{F}\emptyset \neq \emptyset$.

Note at this point that all functors considered in [7] fulfill the condition $\mathcal{F}\emptyset \neq \emptyset$ because of the so-called "nil" process p_0. One could argue that our approach gives a "reason" why this nil process is introduced. It guarantees that $\mathcal{F}\emptyset \neq \emptyset$.

In addition to theorem 12 we derived in theorem 10 another criterion for existence and uniqueness that is unrelated to theorem 12 and the results of [2] as it does not make use of cuts.

VI. Conclusion

We have proposed a rigorous framework within which the problem of solving recursive equations such that the solution constitutes a complete metric space can be formulated and discussed. We established conditions, under which the (unique) existence of a solution is guaranteed. For example, all equations in [7] - except for equation (4) from our introduction - satisfy either the conditions of Theorem 10 or Theorem 12. We have also given special attention to the functor φ_c because of its partiality and we pointed out some connection to related work. Equation (4) has been investigated and it has been shown that the methods of [7] do not apply to it. The question if this equation does have a solution at all is open. Moreover we discussed to some extent the problem of choice of metric. Relation to other work is discussed in detail.

VII. Appendix

We briefly introduce the concepts of [2] in order to be able to relate them to the ones used in this paper.

Let **C** be the category with complete, nonempty metric spaces as objects and pairs $\iota = (i, j)$ as arrows where i is an embedding

$$i : M_1 \to M_2$$

and j is a weak contraction

$$j : M_2 \to M_1$$

such that $i \circ j = \mathrm{id}_{M_1}$.

A functor $\mathcal{F} : \mathbf{C} \to \mathbf{C}$ is called *contracting* in [2] (which we will call *2-contracting* for distinction) if there is ϵ, $0 \le \epsilon < 1$, such that

$$\delta(\mathcal{F}\iota) \le \epsilon \cdot \delta(\iota)$$

where $\delta(\iota) = \sup_{x \in M_2} \left\{ d_{M_2}(x, i(j(x))) \right\}$.

A functor that is 2-contracting is shown in [2] to have a fixed point. For uniqueness an additional property has to be satisfied in [2].

Our criterion of cut-contractiveness was formulated for functors in the slightly different category CMS (which was defined differently just in order to be able to include the empty space) and amounts to

$$\exists k,\ 0 \le k < 1 : \forall \text{ embeddings } i \text{ with cuts j}$$

$$\Big(\forall x \in M_2\ d(x, i(j(x))) \le \mu \Rightarrow \forall y \in \mathcal{F}M_2\ d(y, \mathcal{F}i(\mathcal{F}j(y))) \le k \cdot \mu \Big). \qquad (*)$$

Let now $\iota = (i, j)$ be an arrow in **C** and $\mathcal{F}$ 2-contracting, i.e. $\delta(\mathcal{F}\iota) \le \epsilon \cdot \delta(\iota)$ then $(*)$ is satisfied: let

$$d(x, i(j(x))) \le \mu\ \forall\, x \in M_2$$

then $\delta(\iota) \le \mu$ and $\mathcal{F}$ 2-contracting yields

$$\delta(\mathcal{F}\iota) \le \epsilon \cdot \delta(\iota) \le \epsilon \cdot \mu$$

hence $d(y, \mathcal{F}i(\mathcal{F}j(y))) \le \epsilon \cdot \mu \ \ \forall\, y \in \mathcal{F}M_2$ by definition of δ.

Let conversely $\mathcal{F}$ satisfy $(*)$ then clearly

$$d(x, i(j(x))) \le \delta(\iota)\ \ \forall\, x \in M_2$$

hence $d(y, \mathcal{F}i(\mathcal{F}j(y))) \le k \cdot \delta(\iota)\ \ \forall y \in \mathcal{F}M_2$ hence $\delta(\mathcal{F}\iota) \le k \cdot \delta(\iota)$. Hence, neglecting the slight differences in categories, the notion of a cut–contractive functor and a 2–contracting functor is the same.

VIII. References

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