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Abstract

The purpose of this paper is twofold: First we show in which way the initial solution of a domain equation for cpo's solved by the methods of [19] and the unique solution of a corresponding domain equation for metric spaces solved by the methods of [5, 14, 16] are related. Second we present a technique to lift a given domain equation for cpo's to a corresponding domain equation for metric spaces.

Contents

1 Introduction

The frameworks of cpo's and complete metric spaces have proved to be very useful for giving denotational semantics to concurrent programming languages. In various applications, e.g. [1, 3, 4, 8, 18], one has to solve recursive domain equations in order to obtain a suitable semantic domain. First [19] presented general techniques to solve recursive domain equations in a partial order setting, later [5, 10, 14, 16] considered a metric setting. [17] considered equations in a category of generalized ultrametric spaces, which summarizes in a sense previous work, as cpo's and ultrametric spaces are instances of generalized ultrametric spaces. In the same sense [20] includes previous work.

We are here interested in the following question: what is the effect of the choice of a mathematical discipline (i.e. cpo or metric) on the semantic domain obtained. Consider, e.g. the simple equation

$$X \simeq \{\perp\} \cup A \times X$$

(where A is a fixed set) which could arise in the construction of the semantics for a simple deterministic language with atomic actions $a \in A$. Introducing metric or partial order is a mathematical tool which helps solving such an equation. We consider equations that can be handled both in a metric and cpo setting and ask how the solutions are related. For this we introduce the notions of weak compatibility and of compatibility. The (initial) cpo solution and the (unique) metric solution of an equation are weakly compatible if the underlying sets coincide. Compatibility is motivated by semantic needs: the meaning of a recursive program is obtained as the least upper bound of a monotone approximation sequence in the partial order setting and is the limit of a Cauchy sequence in the metric setting. Hence two weakly compatible solutions are called compatible if for every monotone Cauchy sequence sequence (x_n): $\bigsqcup x_n = \lim x_n$. By this condition we guarantee that infinite behaviour of processes is treated alike in both approaches.

Some constructs of programming languages, e.g. concatenation of commands (;) are difficult to model in the partial order setting as they fail to exhibit the necessary monotonicity property. Other constructs, e.g. unguarded recursion, cause problems when metric is used as contractiveness cannot be guaranteed. Let us assume that we have an equation in the category of cpo's and that one language construct fails to have a monotone semantic description in the (initial) solution D. Then one might switch from order to metric. We investigate how this can be achieved.

The paper is organized as follows: Section 2 summarizes and unifies the various known results concerning the solutions of domain equations using metric resp. order. In section 3 we establish conditions on domain equations that ensure the (weak) compatibility of solutions. The passage from order to metric is stated in section 4. Examples satisfying the conditions of section 3 and 4 are given in section 5. Section 6 discusses further research and open problems. The appendix contains the formal definition of the functors used in section 5.

2 Solving domain equations in the partial order and metric approach

In this section we summarize and unify the results of [19] resp. [5, 14, 16] concerning the solutions of domain equations with partial order resp. metric.

2.1 Initial cones and initial fixed points

We recall the definitions and results of [19] concerning the connection between initial cones and initial fixed points of endofunctors in arbitrary categories.

Let *Cat* be a category. A tower in *Cat* is a sequence $(\mathcal{O}_n, e_n)_{n\geq 0}$ consisting of objects $\mathcal{O}_n$ of *Cat* and morphisms $e_n : \mathcal{O}_n \to \mathcal{O}_{n+1}$ in *Cat*. A cone of a tower $(\mathcal{O}_n, e_n)$ is a pair $(\mathcal{O}, (h_n)_{n\geq 0})$ (shortly $(\mathcal{O}, h_n)$) consisting of an object $\mathcal{O}$ of *Cat* and a family (h_n) of morphisms $h_n : \mathcal{O}_n \to \mathcal{O}$ in *Cat* such that $h_{n+1} \circ e_n = h_n$ for all $n \geq 0$. An initial cone of a tower $(\mathcal{O}_n, e_n)$ is a cone $(\mathcal{O}, h_n)$ of $(\mathcal{O}_n, e_n)$ such that for each cone $(\mathcal{U}, f_n)$ of $(\mathcal{O}_n, e_n)$ there exists a unique morphism $f : \mathcal{O} \to \mathcal{U}$ with $f \circ h_n = f_n$ for all $n \geq 0$.

Let $\mathcal{F} : Cat \to Cat$ be a functor. A fixed point of $\mathcal{F}$ is a pair $(\mathcal{O}, f)$ consisting of an object $\mathcal{O}$ in *Cat* and an isomorphism $f : \mathcal{O} \to \mathcal{F}(\mathcal{O})$. In this case $(\mathcal{O}, f)$ is called an initial fixed point of $\mathcal{F}$ iff for all fixed points $(\mathcal{U}, g)$ of $\mathcal{F}$ there exists a unique morphism $G : \mathcal{O} \to \mathcal{U}$ in *Cat* such that

$$\mathcal{F}(G) \circ f = g \circ G.$$

$(\mathcal{O}, f)$ is called the unique fixed point of $\mathcal{F}$ iff $(\mathcal{O}, f)$ is an initial fixed point of $\mathcal{F}$ such that for each fixed point of $(\mathcal{U}, g)$ of $\mathcal{F}$ the unique morphism $G : \mathcal{O} \to \mathcal{U}$ in *Cat* with $\mathcal{F}(G) \circ f = g \circ G$ is an isomorphism.

Lemma 2.1 *Let Cat be a category,* $\mathcal{F} : Cat \to Cat$ *a functor and* $\mathcal{O}_0$ *an initial object in Cat. Let* $(\mathcal{O}_n, e_n)$ *be given by:*

$$\mathcal{O}_{n+1} = \mathcal{F}(\mathcal{O}_n), \quad e_{n+1} = \mathcal{F}(e_n)$$

where $e_0 : \mathcal{O}_0 \to \mathcal{F}(\mathcal{O}_0)$ *is the unique arrow in Cat. Then we have:*

(a) If $(\mathcal{U}, f_n)$ *is a cone of* $(\mathcal{O}_n, e_n)$ *then also* $(\mathcal{F}(\mathcal{U}), f'_n)$ *is a cone of* $(\mathcal{O}_n, e_n)$ *where* $f'_0 : \mathcal{O}_0 \to \mathcal{F}(\mathcal{U})$ *is the unique arrow* $\mathcal{O}_0 \to \mathcal{F}(\mathcal{U})$ *in Cat and* $f'_{n+1} = \mathcal{F}(f_n)$*. If in addition* $(\mathcal{U}, f_n)$ *is an initial cone then there exists a unique morphism* $g : \mathcal{U} \to \mathcal{F}(\mathcal{U})$ *with* $g \circ f_n = \mathcal{F}(f_{n-1})$ *for all* $n \geq 1$.

(b) *If* $(\mathcal{U}, g)$ *is a fixed point of* $\mathcal{F}$ *then* $(\mathcal{U}, f_n)$ *is a cone of* $(\mathcal{O}_n, e_n)$ *where* $f_0 : \mathcal{O}_0 \to \mathcal{U}$ *denotes the unique arrow* $\mathcal{O}_0 \to \mathcal{U}$ *in* Cat *and* $f_{n+1} = g^{-1} \circ \mathcal{F}(f_n)$.

(c) *If* $(\mathcal{U}, f_n)$ *is an initial cone of* $(\mathcal{O}_n, e_n)$ *such that the unique arrow* $g : \mathcal{U} \to \mathcal{F}(\mathcal{U})$ *with* $g \circ f_n = \mathcal{F}(f_{n-1})$ *is an isomorphism then* $(\mathcal{U}, g)$ *is an initial fixed point of* $\mathcal{F}$. *In this case for each fixed point* $(\mathcal{U}', g')$ *of* $\mathcal{F}$ *we have: The unique morphism* $G : \mathcal{U} \to \mathcal{U}'$ *with* $g' \circ G = \mathcal{F}(G) \circ g$ *is the unique morphism* $\mathcal{U} \to \mathcal{U}'$ *with* $G \circ f_n = g_n$ *where* $(\mathcal{U}', g_n)$ *is the cone which is defined as in (b), i.e.* $g_{n+1} = g'^{-1} \circ \mathcal{F}(g_n)$.

In the proof of Theorem 5 we often use the following simple fact:

Lemma 2.2 *Let* Cat_1 *and* Cat_2 *be categories and* $\mathcal{I} : Cat_1 \to Cat_2$, $\mathcal{F}_1 : Cat_1 \to Cat_1$, $\mathcal{F}_2 : Cat_2 \to Cat_2$ *be functors such that* $\mathcal{F}_2 \circ \mathcal{I} = \mathcal{I} \circ \mathcal{F}_1$. *Let* $(\mathcal{O}, e)$ *be a fixed point of* $\mathcal{F}_1$. *Then* $(\mathcal{I}(\mathcal{O}), \mathcal{I}(e))$ *is a fixed point of* $\mathcal{F}_2$.

2.2 Categories of sets

By a pointed set we mean a pair (X, ξ) consisting of a set X and an element $\xi \in X$. ξ is called the basis point of (X, ξ). In the following we write X instead of (X, ξ). The basis point of X is denoted by ξ_X. A basis point preserving function $X \to Y$ is a function $f : X \to Y$ with $f(\xi_X) = \xi_Y$. An embedding projection pair $X \to Y$ is a pair $< e, c >$ consisting of functions $e : X \to Y$ and $c : Y \to X$ such that $c \circ e = id_X$.

Notation 2.3 *SET denotes category of sets and functions,* SET^* *the category of pointed sets and basis point preserving functions.* SET^E *resp.* SET^{*E} *denote the category of sets resp. pointed sets where the morphisms are embedding projection pairs resp. basis point preserving embedding projection pairs.*

In section 4 we need the following properties of endofunctors in SET^*:

Lemma 2.4 *Let* $\mathcal{K} : SET^* \to SET^*$ *be a functor and* $f : A \to B$, $g : A \to C$ *morphisms in* SET^*. *Then:*

(a) *If* f *is injective resp. surjective resp. bijective then* $\mathcal{K}(f)$ *is injective resp. surjective resp. bijective.*

(b) *If* $Kern(f) = Kern(g)$ *then* $Kern(\ \mathcal{K}(f)\ ) = Kern(\ \mathcal{K}(g)\ )$.

Here for each function $f : A \to B$: $Kern(f) = \{(\xi, \xi') \in A \times A : f(\xi) = f(\xi')\}$.

2.3 Initial solutions of domain equations for cpo's

We recall some basic notions of domain theory and fixed point theorems for endofunctors in order enriched categories. For further details see e.g. [2, 12, 13, 19].

A cpo (complete partial order) is a partially ordered set with a bottom element $\perp$ where each monotone sequence (x_n) has a least upper bound (which we denote by $\bigsqcup x_n$). If D,

D' are cpo's and $f : D \to D'$ is a function then f is called continuous iff f is monotone and $f(\bigsqcup x_n) = \bigsqcup f(x_n)$ for each monotone sequence (x_n) in D. f is called strict iff $f(\perp_D) = \perp_{D'}$. Let D, D' be cpo's. An embedding projection pair $D \to D'$ is a pair $< e, p >$ of continuous functions $e : D \to D'$ and $p : D' \to D$ such that $e \circ p \sqsubseteq id_{D'}$ and $p \circ e = id_D$.

Notation 2.5 *CPO denotes the category of cpo's and continuous functions.*

Let $\mathbf{D}$ *be a subcategory of CPO. Then* $\mathbf{D}_\perp$ *denotes the category whose objects are the objects of* $\mathbf{D}$ *and whose morphisms are strict* $\mathbf{D}$*-morphisms,* $\mathbf{D}^E$ *the category whose objects are the objects of* $\mathbf{D}$ *and whose morphisms are embedding projections pairs in* $\mathbf{D}$.

The forgetful functors $\mathbf{D}_\perp \to SET$, $\mathbf{D}_\perp \to SET^*$, $\mathbf{D}^E \to SET^{\mathrm{E}}$ *resp.* $\mathbf{D}^E \to SET^{*E}$ *are denoted by* $\mathcal{I}_{\mathrm{cpo}}$. *Here the basis point of a cpo is its bottom element.*

As shown in [19]: Each tower in CPO^{E} has an initial cone.

Notation 2.6 *In the following* $\mathbf{D}$ *denotes a subcategory of CPO such that:*

(i) The single element cpo $\{\perp\}$ *is an* $\mathbf{D}$*-object.*

(ii) For each tower $(D_n, \iota_n)_{n\geq 0}$ *in* $\mathbf{D}^E$ *the initial cone in* CPO^{E} *is the initial cone in* $\mathbf{D}^E$.

(iii) If D, D' *are* $\mathbf{D}$*-objects and* $f : D \to D'$ *is continuous then* f *is a* $\mathbf{D}$*-morphism.*

By (i) and (iii) $\{\perp\}$ is the initial object in $\mathbf{D}$. E.g. $\mathbf{D} = CPO$ or $\mathbf{D} = SFP$ (the category of SFP domains and continuous functions [15]) satisfy (i) - (iii). The categories $\mathbf{D}$, $\mathbf{D}_\perp$ and $\mathbf{D}^E$ are order enriched where the partial orders on the morphisms are defined as follows: If $f_i : D \to D'$ are $\mathbf{D}$-morphisms, $i = 1, 2$, then:

$$f_1 \sqsubseteq f_2 \quad :\iff \quad \forall\, x \in D \ : \ f_1(x) \sqsubseteq f_2(x).$$

If $< e_i, c_i >: D \to D'$, $i = 1, 2$, are $\mathbf{D}^E$-morphisms then:

$$< e_1, c_1 > \sqsubseteq < e_2, c_2 > \quad :\iff \quad (e_1 \sqsubseteq e_2) \wedge (c_1 \sqsubseteq c_2)$$

A functor $\mathcal{G} : \mathbf{D}_\perp \to \mathbf{D}_\perp$ resp. $\mathcal{G} : \mathbf{D}^E \to \mathbf{D}^E$ is called *locally continuous* if for all $\mathbf{D}$-objects D, D' the function $Mor(D, D') \to Mor(\mathcal{G}(D), \mathcal{G}(D'))$, $f \mapsto \mathcal{G}(f)$, is continuous. Here $Mor(D, D')$ means the set of morphisms $D \to D'$ in $\mathbf{D}_\perp$ resp. $\mathbf{D}^E$.

Lemma 2.7 *Each locally continuous functor* $\mathcal{G} : \mathbf{D}^E \to \mathbf{D}^E$ *or* $\mathcal{G} : \mathbf{D}_\perp \to \mathbf{D}_\perp$ *has an initial fixed point.*

The construction of the initial fixed point is as follows [19]: Let $\mathcal{G} : \mathbf{D}^E \to \mathbf{D}^E$ be a locally continuous functor and let (D_n, ι_n) be given by: $D_0 = \{\perp\}$, $D_{n+1} = \mathcal{G}(D_n)$. Let ι_0 be the unique arrow $D_0 \to D_1$ in $\mathbf{D}^E$ and $\iota_{n+1} = \mathcal{G}(\iota_n)$. Then the initial cone of (D_n, ι_n) is the initial fixed point of $\mathcal{G}$. More precisely: If (D, λ_n) is the initial cone of (D_n, ι_n) and $\lambda : D \to \mathcal{G}(D)$ is the unique arrow in $\mathbf{D}^E$ with $\lambda \circ \lambda_{n+1} = \mathcal{G}(\lambda_n)$ then λ is an isomorphism in $\mathbf{D}^E$ and (D, λ) is the initial fixed point of $\mathcal{G}$.

Notation 2.8 *If $\mathcal{G} : \mathbf{D}_\perp \to \mathbf{D}_\perp$ is a functor then $\mathcal{G}^E : \mathbf{D}^E \to \mathbf{D}^E$ is given by:*

$$\mathcal{G}^E(D) \;=\; \mathcal{G}(D), \quad \mathcal{G}^E(< e, c >) \;=< \mathcal{G}(e),\ \mathcal{G}(c) >$$

If $\mathcal{G} : \mathbf{D}_\perp \to \mathbf{D}_\perp$ is locally continuous then so is $\mathcal{G}^E$ and (D, h) is the initial fixed point of $\mathcal{G}$ where $(D, < h, h^{-1} >)$ is the initial fixed point of $\mathcal{G}^E$.

2.4 Unique solutions of domain equations for complete metric spaces

We recall the definitions and results of [5, 14, 16]. We always require that the underlying metric d_M of a metric space M satisfies $d_M \le 1$.

Notation 2.9 *$CMS_\emptyset$ denotes the category of (empty or nonempty) complete metric spaces and non-distance-increasing functions, CMS the subcategory of nonempty complete metric spaces and CUM the subcategory of nonempty complete ultrametric spaces.*

Notation 2.10 *Let $\mathbf{M}$ be a subcategory of $CMS_\emptyset$. Then $\mathbf{M}^*$ means the category whose objects are pointed $\mathbf{M}$-objects and whose morphisms are basis point preserving $\mathbf{M}$-morphisms. $\mathbf{M}_e$ resp. $\mathbf{M}_e^*$ means the subcategory of $\mathbf{M}$ resp. $\mathbf{M}^*$ where the morphisms are restricted to embeddings (i.e. distance preserving functions). The objects of $\mathbf{M}_e$ resp. $\mathbf{M}_e^*$ are the objects in $\mathbf{M}$ resp. $\mathbf{M}^*$. $\mathbf{M}^E$ resp. $\mathbf{M}^{*E}$ denotes the category whose objects are $\mathbf{M}$-objects resp. $\mathbf{M}^*$-objects and whose morphisms are embedding projection pairs in $\mathbf{M}$ resp. $\mathbf{M}^*$.*

The forgetful functors $\mathbf{M} \to SET$, $\mathbf{M}^* \to SET^*$, $\mathbf{M}^E \to SET^{\mathrm{E}}$ resp. $\mathbf{M}^{*E} \to SET^{*E}$ are denoted by $\mathcal{I}_{\mathrm{cms}}$.

Here an embedding projection pair $M \to M'$ in $\mathbf{M}$ resp. $\mathbf{M}^*$ is a pair $< e, c >$ consisting of $\mathbf{M}$-morphisms resp. $\mathbf{M}^*$-morphisms $e : M \to M'$ and $c : M' \to M$ such that $c \circ e \;=\; id_M$. By a pointed metric space we mean a pointed set which is endowed with a metric. $\emptyset$ is the initial object of $CMS_\emptyset$, the single element space the initial object of CMS^* resp. CMS^{*E}. CMS and CMS^E do not have initial objects.

Let M, N be metric spaces, $A \subseteq N$, $\eta \in N$, $< e, c >: M \to N$ an embedding projection pair. Then we put:

$$\delta_N(\eta, A) \;=\; \inf\ \{\ d_N(\eta, \zeta)\ :\ \zeta \in A\ \}$$

$$\delta(A, N) \;=\; \sup_{\eta \in N}\ \delta_N(\eta, A) \;=\; \sup\ \left\{\ \inf_{\zeta \in A}\ d_N(\eta, \zeta)\ :\ \eta \in N\ \right\}$$

$$\Delta_N(e, c) \;=\; \sup\ \{\ d_N(e(c(\eta)), \eta)\ :\ \eta \in N\ \}.$$

Here d_N denotes the underlying metric on N. If $\iota : M \to N$ is a morphism in CMS^E, $\iota \;=< e, c >$, then we put:

$$\Delta(\iota) \;=\; \Delta_N(e, c)$$

A tower (M_n, ι_n) in $\mathbf{M}^E$ is called *converging* iff for all $\varepsilon > 0$ there exists $n_0 \ge 0$ such that

$$\Delta(\ \iota_n \circ \iota_{n+1} \circ \ldots \circ \iota_m\ ) \;\le\; \varepsilon$$

for all $n > m \geq n_0$ or equivalently iff $\lim \Delta(\iota_n) = 0$.

As shown in [5]: Each converging tower in CMS^E resp. in CMS^{*E} has an initial cone. [14] gives a construction of an initial cone for towers (M_n, e_n) in $CMS_\emptyset$ where $e_n : M_n \to M_{n+1}$ are embeddings. It is easy to see that for $\mathbf{M}$ to be one of the categories CMS, CMS^*, CUM or CUM^* and each tower (M_n, e_n) in $\mathbf{M}$ where $e_n : M_n \to M_{n+1}$ are embeddings the initial cone in $CMS_\emptyset$ is at the same time the initial cone in $\mathbf{M}$ and in $\mathbf{M}_e$. The connection between the initial cones in CMS^E and CMS is the following:

Lemma 2.11 *Let (M_n, ι_n) be a converging tower in CMS^E, $\iota_n = < e_n, c_n >$, and let (M, λ_n) be the initial cone of (M_n, ι_n) in CMS^E where $\lambda_n = < h_n, b_n >$. Then (M, h_n) is the initial cone of the tower (M_n, e_n) in CMS.*

Let $\mathbf{M}$ be a subcategory of $CMS_\emptyset$. A functor $\mathcal{H} : \mathbf{M} \to \mathbf{M}$ or $\mathcal{H} : \mathbf{M}^* \to \mathbf{M}^*$ is called *contracting* iff $\mathcal{H}$ preserves embeddings and there exists a real number C with $0 \leq C < 1$ such that

$$\delta(\ \mathcal{H}(e)\,(\mathcal{H}(M))\,,\ \mathcal{H}(N)\ ) \ \leq\ C \cdot \delta(e(M), N)$$

for each embedding $e : M \to N$ in $\mathbf{M}$. $\mathcal{H}$ is called *cut-contracting* iff $\mathcal{H}$ preserves embeddings and there exists a real number C with $0 \leq C < 1$ such that for each embedding projection pair $< e, c >$ in $\mathbf{M}$:

$$\Delta_{\mathcal{H}(N)}(\mathcal{H}(e), \mathcal{H}(c)) \ \leq\ C \cdot \Delta_N(e, c).$$

$\mathcal{H}$ is called *locally contracting* iff there exists a real number C with $0 \leq C < 1$ such that

$$d_{\mathcal{H}(N)}(\mathcal{H}(f_1), \mathcal{H}(f_2)) \ \leq\ C \cdot\ d_N(f_1, f_2)$$

for all morphisms $f_i : M \to N$ in $\mathbf{M}$. I.e. the function $Mor(M, N) \to Mor(\mathcal{H}(M), \mathcal{H}(N))$, $f \mapsto \mathcal{H}(f)$, is contracting with contracting constant C. Here $Mor(M, N)$ means the set of morphisms $M \to N$ in $\mathbf{M}$ resp. $\mathbf{M}^*$ and

$$d_N(f_1, f_2) \ =\ \sup\ \{\ d_N(\ f_1(x),\ f_2(x)\ )\ :\ x \in M\ \}$$

for all $f_1,\ f_2\ \in\ Mor(M, N)$.

Let $\mathcal{H} : \mathbf{M}^E \to \mathbf{M}^E$ be a functor. $\mathcal{H}$ is called *e/p-contracting* iff there exists a constant C with $0 \leq C < 1$ such that

$$\Delta(\mathcal{H}(\iota)) \ \leq\ C \cdot \Delta(\iota)$$

for all morphisms $\iota : M \to N$ in $\mathbf{M}^E$. Note that [5] uses the notion 'contracting' instead of our notion 'e/p-contracting'. We decided to use the prefix 'e/p' (which stands for 'embedding/projection') to prevent a confusion with the notion 'contracting' of [14] which we use for endofunctors of $CMS_\emptyset$. $\mathcal{H}$ is called *hom-contracting* iff for all objects M, N in $\mathbf{M}$ the function $Mor(M, N) \to Mor(\mathcal{H}(M), \mathcal{H}(N))$, $\iota \mapsto \mathcal{H}(\iota)$, is contracting. Here $Mor(M, N)$ means the set of morphisms $M \to N$ in $\mathbf{M}^E$.

Lemma 2.12 *Each e/p-contracting functor $\mathcal{H} : CMS^E \to CMS^E$ has a fixed point. If in addition $\mathcal{H}$ is hom-contracting then $\mathcal{H}$ has a unique fixed point.*

Each e/p-contracting functor $\mathcal{H} : CMS^{*E} \to CMS^{*E}$ has a unique fixed point.

In both cases the fixed point of $\mathcal{H}$ can be constructed as follows [5]: Let M_0 be a metric space consisting of a single element and let ι_0 be a morphism $M_0 \to \mathcal{H}(M_0)$ in CMS^E and $M_{n+1} = \mathcal{H}(M_n)$, $\iota_{n+1} = \mathcal{H}(\iota_n)$. Then the tower (M_n, ι_n) is converging and has an initial cone (M, λ_n). (M, λ) is a fixed point of $\mathcal{H}$ where λ is the unique morphism $M \to \mathcal{H}(M)$ in CMS^E with $\lambda \circ \lambda_n = \mathcal{H}(\lambda_{n-1})$.

Notation 2.13 *Let* $\mathbf{M}$ *be a subcategory of* $CMS_\emptyset$ *and* $\mathcal{H} : \mathbf{M} \to \mathbf{M}$ *a functor. Then* $\mathcal{H}^E : \mathbf{M}^E \to \mathbf{M}^E$ *is given by:*

$$\mathcal{H}^E(M) = \mathcal{H}(M), \quad \mathcal{H}^E(< e, c >) = < \mathcal{H}(e), \ \mathcal{H}(c) > .$$

(M, h) is a fixed point of $\mathcal{H}$ if and only if $(M, < h, h^{-1} >)$ is a fixed point of $\mathcal{H}^E$. If $\mathcal{H}$ preserves embeddings then $\mathcal{H}$ is cut-contracting if and only if $\mathcal{H}^E$ is e/p-contracting.

Lemma 2.14 *If* $\mathcal{H} : CMS \to CMS$ *is locally contracting then* $\mathcal{H}$ *has a unique fixed point.*

The fixed point of a locally contracting functor $\mathcal{H}$ can be constructed as follows [16]: The induced functor $\mathcal{H}^E$ is e/p-contracting and hom-contracting. If (M, λ) is the unique fixed point of $\mathcal{H}^E$ then $\lambda = < h, h^{-1} >$ for some isometry $h : M \to \mathcal{H}(M)$. (M, h) is the unique fixed point of $\mathcal{H}$.

Lemma 2.15 *Let* $\mathcal{H} : CMS_\emptyset \to CMS_\emptyset$ *be a functor with* $\mathcal{H}(\emptyset) \neq \emptyset$. *If* $\mathcal{H}$ *is cut-contracting or contracting then* $\mathcal{H}$ *has a unique fixed point.*

The fixed point (M, e) of $\mathcal{H}$ can be constructed as follows [14]: Let $M_0 = \emptyset$ and e_0 the unique arrow $M_0 \to \mathcal{H}(M_0)$ in $CMS_\emptyset$ and $M_{n+1} = \mathcal{H}(M_n)$, $e_{n+1} = \mathcal{H}(e_n)$. Then (M_n, e_n) is a tower in $CMS_\emptyset$ such that $e_n : M_n \to M_{n+1}$ are embeddings. If (M, h_n) is the initial cone of (M_n, e_n) in $CMS_\emptyset$ then the unique morphism $e : M \to \mathcal{H}(M)$ with $\mathcal{H}(h_n) = e \circ h_{n+1}$ is an isometry and (M, e) the unique fixed point of $\mathcal{H}$.

It is easy to see that this result of [14] carries over to the pointed case. Here we have to deal with the tower $(\mathcal{H}^n(M_0), \mathcal{H}^n(e_0))$ where $M_0 = \{x_0\}$ is the initial object in CMS^* and e_0 the unique arrow $M_0 \to \mathcal{H}(M_0)$ in CMS^*. We obtain:

Lemma 2.16 *Each cut-contracting or contracting functor* $\mathcal{H} : CMS^* \to CMS^*$ *has a unique fixed point.*

The connection between the fixed point theorems of [5] and [14] is as follows: If $\mathcal{F}$ is a cut-contracting endofunctor of $CMS_\emptyset$ with $\mathcal{F}(\emptyset) \neq \emptyset$ then $\mathcal{F}(M) \neq \emptyset$ for all complete metric spaces M. Hence $\mathcal{F}$ can be restricted to an endofunctor $\mathcal{H}$ of CMS. The induced functor $\mathcal{H}^E$ is e/p-contracting. The result of [5] (Lemma 2.12) ensures the existence (but not the uniqueness) of cut-contracting endofunctors of $CMS_\emptyset$. Vice versa the result of [14] (Lemma 2.15) ensures the existence and uniqueness of the fixed point of e/p-contracting endofunctors of CMS^E which are induced by a cut-contracting endofunctor $\mathcal{F}$ of $CMS_\emptyset$ with $\mathcal{F}(\emptyset) \neq \emptyset$.

Notation 2.17 *In the following* $\mathbf{M}$ *denotes a subcategory of CMS such that:*

(i) Each metric space which consists of a single element is an object in $\mathbf{M}$.

(ii) *For each tower (M_n, e_n) in* $\mathbf{M}$ *where $e_n : M_n \to M_{n+1}$ are embeddings the initial cone in* $CMS_\emptyset$ *is the initial cone in* $\mathbf{M}$.

(iii) *Whenever M, M' are objects of* $\mathbf{M}$ *and $f : M \to M'$ is non-distance-increasing then f is a* $\mathbf{M}$*-morphism.*

E.g. $\mathbf{M} = CMS$ or $\mathbf{M} = CUM$ are categories that satisfy (i) - (iii). It is easy to see that the results of [5, 14, 16] remain true when one deals with endofunctors of $\mathbf{M}$. The proof of [14] for the existence and uniqueness of a fixed point of a contracting functor $\mathcal{H} : CMS_\emptyset \to CMS_\emptyset$ carries over to each contracting endofunctor of $\mathbf{M}^*_e$. Here one has to deal with the tower $(\mathcal{H}^n(M_0), \mathcal{H}^n(e_0))$ where $M_0 = \{x_0\}$ is the initial object in $\mathbf{M}^*_e$. We summarize:

Lemma 2.18

(a) *Each e/p-contracting functor $\mathcal{H} : \mathbf{M}^E \to \mathbf{M}^E$ has a fixed point. If in addition $\mathcal{H}$ is hom-contracting then $\mathcal{H}$ has a unique fixed point.*

(b) *Each e/p-contracting functor $\mathcal{H} : \mathbf{M}^{*E} \to \mathbf{M}^{*E}$ has a unique fixed point.*

(c) *Each cut-contracting functor $\mathcal{H} : \mathbf{M} \to \mathbf{M}$ has a fixed point.*

(d) *Each contracting or cut-contracting functor $\mathcal{H} : \mathbf{M}^* \to \mathbf{M}^*$ has a unique fixed point.*

(e) *Each contracting functor $\mathcal{H} : \mathbf{M}^*_e \to \mathbf{M}^*_e$ has a unique fixed point.*

(f) *Each locally contracting functor $\mathcal{H} : \mathbf{M} \to \mathbf{M}$ or $\mathcal{H} : \mathbf{M}^* \to \mathbf{M}^*$ has a unique fixed point.*

Notation 2.19 *Let $\mathcal{H}$ be a functor which satisfies one of the conditions (a) - (f) in Lemma 2.18. The fixed point of $\mathcal{H}$ which is constructed as the initial cone of the tower $(\mathcal{H}^n(M_0), \mathcal{H}^n(\iota_0))$ is called the* `canonical fixed point` *of $\mathcal{H}$ where M_0 consists of a single element and ι_0 is an arrow $M_0 \to \mathcal{H}(M_0)$ in the underlying category.*

In [16] it is shown that the initial resp. unique fixed point of a locally continuous resp. locally contracting functor $CPO_\perp \to CPO_\perp$ resp. $CMS \to CMS$ is also an initial algebra and a final coalgebra. In this paper we do not make use of this result.

3 (Weakly) compatible domain equations

In this section we show the relation between the initial solution of a domain equation for cpo's and the canonical solution of a corresponding domain equation for complete metric spaces (Theorem 1-4). In order to compare the solutions of a domain equation for cpo's and a domain equation for complete metric spaces we have to find a criterion which relates a cpo and a complete metric space. Second we have to say what we mean by 'corresponding' domain equations. In section 3.1 we introduce the notion of weakly compatible domains which means that the underlying sets are the same. Weak compatible domain equations are those which arise by lifting a domain equation for sets to a metric resp.

cpo equation. In section 3.2 we define compatible domains as weakly compatible domains which induce equivalent notions of approximability. Compatible domain equations are those which are given by functors that preserve compatibility.

In what follows $\mathbf{D}$ resp. $\mathbf{M}$ are subcategories of *CPO* or *CMS* that satisfy the conditions (i) - (iii) of section 2.3 resp. section 2.4.

3.1 Weakly compatible domain equations

We show that the canonical solutions D, M of domain equations for cpo's resp. complete metric spaces that arise by lifting a domain equation for sets have the same underlying set. This ensures that when D and M are used as semantic domain for denotational semantics (where the same semantic operators on the underlying set are used and display the necessary continuity and contractiveness properties) then the cpo and metric semantics coincide.

Definition 3.1 *Let D be a cpo and let M be a (pointed) complete metric space. Then D and M are called* `weakly compatible` *iff the underlying (pointed) sets are the same, i.e.* $\mathcal{I}_{\mathsf{cpo}}(D) = \mathcal{I}_{\mathsf{cms}}(M)$.

Definition 3.2 *Let $\mathcal{G} : \mathbf{D}_\perp \to \mathbf{D}_\perp$ and $\mathcal{H} : \mathbf{M} \to \mathbf{M}$ be functors. Then $\mathcal{G}$ and $\mathcal{H}$ are called* `weakly compatible` *iff there exists a functor $\mathcal{K} : SET \to SET$ such that*

$$\mathcal{K} \circ \mathcal{I}_{\mathsf{cpo}} = \mathcal{I}_{\mathsf{cpo}} \circ \mathcal{G} \quad \textit{and} \quad \mathcal{K} \circ \mathcal{I}_{\mathsf{cms}} = \mathcal{I}_{\mathsf{cms}} \circ \mathcal{H}.$$

I.e. the following diagram commutes:

$$\begin{tikzcd}
\mathbf{D}_\perp \arrow[r, "\mathcal{G}"] \arrow[d, "\mathcal{I}_{\mathsf{cpo}}"'] & \mathbf{D}_\perp \arrow[d, "\mathcal{I}_{\mathsf{cpo}}"] \\
SET \arrow[r, "\mathcal{K}"] & SET \\
\mathbf{M} \arrow[r, "\mathcal{H}"] \arrow[u, "\mathcal{I}_{\mathsf{cms}}"] & \mathbf{M} \arrow[u, "\mathcal{I}_{\mathsf{cms}}"']
\end{tikzcd}$$

Similary we define weak compatibility for functors $\mathcal{G} : \mathbf{D}^E \to \mathbf{D}^E$ and $\mathcal{H} : \mathbf{M}^E \to \mathbf{M}^E$ resp. $\mathcal{G} : \mathbf{D}^E \to \mathbf{D}^E$ and $\mathcal{H} : \mathbf{M}^{*E} \to \mathbf{M}^{*E}$ resp. $\mathcal{G} : \mathbf{D}_\perp \to \mathbf{D}_\perp$ and $\mathcal{H} : \mathbf{M}^* \to \mathbf{M}^*$.

We now show that weakly compatible domain equations have weakly compatible solutions.

Theorem 1

Let $\mathcal{G} : \mathbf{D}^E \to \mathbf{D}^E$ be locally continuous and $\mathcal{H} : \mathbf{M}^E \to \mathbf{M}^E$ (resp. $\mathcal{H} : \mathbf{M}^{*E} \to \mathbf{M}^{*E}$) e/p-contracting such that $\mathcal{G}$ and $\mathcal{H}$ are weakly compatible. Then the initial fixed point of $\mathcal{G}$ and the canonical (resp. unique) fixed point of $\mathcal{H}$ are weakly compatible.

Proof: We only consider the case that $\mathcal{H}$ is an endofunctor of $\mathbf{M}^E$. The argumentation for $\mathcal{H}$ to be an endofunctor of $\mathbf{M}^{*E}$ is similar.

We have to show that there exists a set A, a partial order $\sqsubseteq$ on A and a metric d on A such that (M, κ) is the canonical fixed point of $\mathcal{H}$ where $M = (A, d)$ and (D, λ) is the initial fixed point of $\mathcal{G}$ where $D = (A, \sqsubseteq)$. Let $\mathcal{K} : SET^{\mathrm{E}} \to SET^{\mathrm{E}}$ be a functor such that

$$\mathcal{K} \circ \mathcal{I}_{\mathrm{cpo}} = \mathcal{I}_{\mathrm{cpo}} \circ \mathcal{G} \text{ and } \mathcal{K} \circ \mathcal{I}_{\mathrm{cms}} = \mathcal{I}_{\mathrm{cms}} \circ \mathcal{H}.$$

Let $A_0 = M_0 = D_0 = \{\perp\}$ the set resp. complete metric space resp. cpo consisting of a single element and let ι_0 the unique arrow $D_0 \to \mathcal{G}(D_0)$ in $\mathbf{D}^E$. ι_0 can also be considered as an arrow $M_0 \to \mathcal{H}(M_0)$ in $\mathbf{M}^E$ and as an arrow $A_0 \to \mathcal{K}(A_0)$ in SET^{E}. We define:

$$A_{n+1} = \mathcal{K}(A_n), \; M_{n+1} = \mathcal{H}(M_n), \; D_{n+1} = \mathcal{G}(D_n), \; \iota_{n+1} = \mathcal{K}(\iota_n)$$

Then $\mathcal{I}_{\mathrm{cms}}(M_n) = \mathcal{I}_{\mathrm{cpo}}(D_n) = A_n$, $\iota_{n+1} = \mathcal{H}(\iota_n)$, $\iota_{n+1} = \mathcal{G}(\iota_n)$. Let $\iota_n = < e_n, c_n >$ and let

$$A = \{(\xi_n)_{n \geq 0} \in \Pi_{n \geq 0} A_n : c_n(\xi_{n+1}) = \xi_n \}.$$

By the results of [19] and [5]: A is the underlying set of the initial fixed point of $\mathcal{G}$ and of the canonical fixed point of $\mathcal{H}$. □

Theorem 2

Let $\mathcal{G} : \mathbf{D}_\perp \to \mathbf{D}_\perp$ be locally continuous and $\mathcal{H} : \mathbf{M} \to \mathbf{M}$ (resp. $\mathcal{H} : \mathbf{M}^* \to \mathbf{M}^*$) cut-contracting or locally contracting such that $\mathcal{G}$ and $\mathcal{H}$ are weakly compatible. Then the initial fixed point of $\mathcal{G}$ and the canonical (resp. unique) fixed point of $\mathcal{H}$ are weakly compatible.

Proof: We only consider the case that $\mathcal{H}$ is an endofunctor of $\mathbf{M}$.

Let $\mathcal{K} : SET \to SET$ with $\mathcal{K} \circ \mathcal{I}_{\mathrm{cpo}} = \mathcal{I}_{\mathrm{cpo}} \circ \mathcal{K}$ and $\mathcal{K} \circ \mathcal{I}_{\mathrm{cms}} = \mathcal{I}_{\mathrm{cms}} \circ \mathcal{H}$. It is clear that

$$\mathcal{K}^E : SET^{\mathrm{E}} \to SET^{\mathrm{E}}, \; \mathcal{K}^E(A) = \mathcal{K}(A), \; \mathcal{K}^E(< e, c >) = < \mathcal{K}(e), \mathcal{K}(c) >$$

is welldefined and $\mathcal{K}^E \circ \mathcal{I}_{\mathrm{cpo}} = \mathcal{I}_{\mathrm{cpo}} \circ \mathcal{G}^E$, $\mathcal{K}^E \circ \mathcal{I}_{\mathrm{cms}} = \mathcal{I}_{\mathrm{cms}} \circ \mathcal{H}^E$. Hence $\mathcal{G}^E$ and $\mathcal{H}^E$ are weakly compatible. Since $\mathcal{G}^E$ is locally continuous and since $\mathcal{H}^E$ is e/p-contracting the functors $\mathcal{G}^E$ and $\mathcal{H}^E$ satisfy the conditions of Theorem 1. The initial fixed point D of $\mathcal{G}$ is the initial fixed point of $\mathcal{G}^E$ and the canonical fixed point M of $\mathcal{H}$ is the canonical fixed point of $\mathcal{H}^E$. By Theorem 1 the underlying sets of D and M are the same. □

3.2 Compatible domain equations

In order to ensure that the partial order and the metric on weakly compatible domains induce 'compatible' notions of approximability we define compatible domains as weakly compatible domains where limits and least upper bounds of monotone Cauchy sequences coincide.

Definition 3.3 *Let D be a cpo and M a (pointed) complete metric space. Then D and M are called* `compatible` *iff the underlying (pointed) sets are the same and for each monotone Cauchy sequence (ξ_n):*

$$\lim_{n\to\infty} \xi_n = \bigsqcup_{n\geq 0} \xi_n.$$

The notion 'compatible' can be lifted to domain equations (or equivalenty to functors):

Definition 3.4 *Let $\mathcal{H}$ be an endofunctor $\mathbf{M}$ and $\mathcal{G}$ an endofunctor of $\mathbf{D}_\perp$. $\mathcal{H}$ and $\mathcal{G}$ are called* `compatible` *iff the following conditions (i) and (ii) are satisfied:*

(i) If M is a $\mathbf{M}$-object, D a $\mathbf{D}$-object such that M and D are compatible then $\mathcal{H}(M)$ and $\mathcal{G}(D)$ are compatible.

(ii) If $h : M \to M'$ is a $\mathbf{M}$-morphism, $g : D \to D'$ a $\mathbf{D}_\perp$-morphism such that M, D and M', D' are compatible and $\mathcal{I}_{\mathsf{cms}}(h) = \mathcal{I}_{\mathsf{cpo}}(g)$ then $\mathcal{I}_{\mathsf{cms}}(\mathcal{H}(h)) = \mathcal{I}_{\mathsf{cpo}}(\mathcal{G}(g))$.

Similary we define compatibility for functors $\mathcal{G} : \mathbf{D}^E \to \mathbf{D}^E$ and $\mathcal{H} : \mathbf{M}^E \to \mathbf{M}^E$ resp. $\mathcal{G} : \mathbf{D}^E \to \mathbf{D}^E$ and $\mathcal{H} : \mathbf{M}^{*E} \to \mathbf{M}^{*E}$ resp. $\mathcal{G} : \mathbf{D}_\perp \to \mathbf{D}_\perp$ and $\mathcal{H} : \mathbf{M}^* \to \mathbf{M}^*$.

Remark 3.5 Let $\mathcal{G} : \mathbf{D}_\perp \to \mathbf{D}_\perp$ and $\mathcal{H} : \mathbf{M} \to \mathbf{M}$ (resp. $\mathcal{H} : \mathbf{M}^* \to \mathbf{M}^*$) be functors such that $\mathcal{G}$ and $\mathcal{H}$ are compatible. Then $\mathcal{G}^E$ and $\mathcal{H}^E$ are compatible.

The following lemma shows that compatible domain equations are weakly compatible.

Lemma 3.6 *Assume that $\mathbf{M}$ contains all nonempty discrete metric spaces as objects. Then: If $\mathcal{H} : \mathbf{M} \to \mathbf{M}$ and $\mathcal{G} : \mathbf{D}_\perp \to \mathbf{D}_\perp$ are compatible then there exists a unique functor $\mathcal{K} : SET \to SET$ such that*

$$\mathcal{K} \circ \mathcal{I}_{\mathsf{cms}} = \mathcal{I}_{\mathsf{cms}} \circ \mathcal{H} \quad \text{and} \quad \mathcal{K} \circ \mathcal{I}_{\mathsf{cpo}} = \mathcal{I}_{\mathsf{cpo}} \circ \mathcal{G}.$$

In particular $\mathcal{H}$ and $\mathcal{G}$ are weakly compatible.

Similar results can be established for compatible endofunctors $\mathbf{D}_\perp \to \mathbf{D}_\perp$, $\mathbf{M}^* \to \mathbf{M}^*$ resp. $\mathbf{D}^E \to \mathbf{D}^E$, $\mathbf{M}^E \to \mathbf{M}^E$ resp. $\mathbf{D}^E \to \mathbf{D}^E$, $\mathbf{M}^{*E} \to \mathbf{M}^{*E}$.

Proof: If A is a set then we put: $M_A = (A, d)$ where d denotes the discret metric on A. By assumption M_A is a $\mathbf{M}$-object. Then $\mathcal{I}_{\mathsf{cms}}(M_A) = A$. If $f : A \to B$ is a morphism in SET then f can be considered as a morphism $M_A \to M_B$ in $\mathbf{M}$ (because of assumption (iii) about $\mathbf{M}$). Then f is the unique morphism $M_A \to M_B$ in $\mathbf{M}$ with $\mathcal{I}_{\mathsf{cms}}(f) = f$.

Uniqueness: If $\mathcal{K}$ is such a functor then for all objects A in SET:

$$\mathcal{K}(A) = \mathcal{K}(\mathcal{I}_{\mathsf{cms}}(M_A)) = \mathcal{I}_{\mathsf{cms}}(\mathcal{H}(M_A))$$

and for all morphisms $f : A \to B$ in SET:

$$\mathcal{K}(f) = \mathcal{K}(\mathcal{I}_{\mathsf{cms}}(f)) = \mathcal{I}_{\mathsf{cms}}(\mathcal{H}(f))$$

Existence: If A is an object in SET then we define $\mathcal{K}(A) = \mathcal{I}_{\rm cms}(\mathcal{H}(M_A))$. If $f : A \to B$ is a morphism in SET then f : is a **M**-morphism $M_A \to M_B$. We define:

$$\mathcal{K}(f) = \mathcal{I}_{\rm cms}(\mathcal{H}(f)) .$$

It is clear that $\mathcal{K} : SET \to SET$ is a welldefined functor with $\mathcal{K} \circ \mathcal{I}_{\rm cms} = \mathcal{I}_{\rm cms} \circ \mathcal{H}$. We show that $\mathcal{K} \circ \mathcal{I}_{\rm cpo} = \mathcal{I}_{\rm cpo} \circ \mathcal{G}$. If D is a **D**-object and $A = \mathcal{I}_{\rm cpo}(D)$ then D and M_A are compatible. Since $\mathcal{G}$ and $\mathcal{H}$ are compatible:

$$\mathcal{K}(\mathcal{I}_{\rm cpo}(D)) = \mathcal{K}(\mathcal{I}_{\rm cms}(M_A)) = \mathcal{I}_{\rm cms}(\mathcal{H}(M_A)) = \mathcal{I}_{\rm cpo}(\mathcal{G}(D)).$$

Let $g : D \to C$ be a morphism in $\mathbf{D}_\perp$. Let $A = \mathcal{I}_{\rm cpo}(D)$ and $B = \mathcal{I}_{\rm cpo}(C)$. g can be considered as a morphism $f : A \to B$ is a morphism in SET and also as a morphism h : $M_A \to M_B$ in **M**. I.e. $\mathcal{I}_{\rm cms}(h) = f = \mathcal{I}_{\rm cpo}(g)$. Since M_A, D and M_B, C are compatible (this is because Cauchy sequences in discret metric spaces are eventually constant) we get:

$$\mathcal{K}(\mathcal{I}_{\rm cpo}(g)) = \mathcal{K}(\mathcal{I}_{\rm cms}(h)) = \mathcal{I}_{\rm cms}(\mathcal{H}(h)) = \mathcal{I}_{\rm cpo}(\mathcal{G}(g)).$$

We conclude: $\mathcal{K} \circ \mathcal{I}_{\rm cpo} = \mathcal{I}_{\rm cpo} \circ \mathcal{G}$. $\square$

Next we show that compatible domain equations have compatible solutions. Moreover we show that the isomorphisms $M \to \mathcal{H}(M)$ and $D \to \mathcal{G}(D)$ are the same.

Theorem 3

Let $\mathcal{G} : \mathbf{D}^E \to \mathbf{D}^E$ be locally continuous and $\mathcal{H} : \mathbf{M}^E \to \mathbf{M}^E$ (resp. $\mathcal{H} : \mathbf{M}^{*E} \to \mathbf{M}^{*E}$) e/p-contracting such that $\mathcal{G}$ and $\mathcal{H}$ are compatible. Then the initial fixed point of $\mathcal{G}$ and the canonical (resp. unique) fixed point of $\mathcal{H}$ are compatible.

More precisely: If (D, λ) is the initial fixed point of $\mathcal{G}$, (M, κ) the canonical fixed point of $\mathcal{H}$ then D and M are compatible and $\mathcal{I}_{\rm cpo}(\lambda) = \mathcal{I}_{\rm cms}(\kappa)$.

Before we give the proof of Theorem 3 we present the following corollary:

Theorem 4

Let $\mathcal{G} : \mathbf{D}_\perp \to \mathbf{D}_\perp$ be locally continuous and $\mathcal{H} : \mathbf{M} \to \mathbf{M}$ (resp. $\mathcal{H} : \mathbf{M}^* \to \mathbf{M}^*$) cut-contracting or locally contracting. If $\mathcal{H}$ and $\mathcal{G}$ are compatible then the initial fixed point of $\mathcal{G}$ and the canonical (resp. unique) fixed point of $\mathcal{H}$ are compatible.

More precisely: If (D, h) is the initial fixed point of $\mathcal{G}$, (M, k) the canonical fixed point of $\mathcal{H}$ then D and M are compatible and $\mathcal{I}_{\rm cpo}(h) = \mathcal{I}_{\rm cms}(k)$.

Proof: Let $\mathcal{G} : \mathbf{D}_\perp \to \mathbf{D}_\perp$ be locally continuous and $\mathcal{H} : \mathbf{M} \to \mathbf{M}$ or $\mathcal{H} : \mathbf{M}^* \to \mathbf{M}^*$ cut-contracting or locally contracting such that $\mathcal{G}$ and $\mathcal{H}$ are compatible. Then the induced functors $\mathcal{G}^E$ and $\mathcal{H}^E$ satisfy the conditions of Theorem 3, i.e. $\mathcal{G}^E$ is locally continuous and $\mathcal{H}^E$ is e/p-contracting and $\mathcal{G}^E$, $\mathcal{H}^E$ are compatible. By Theorem 3 we get: If (D, λ) is the initial fixed point of $\mathcal{G}$ and (M, κ) the canonical fixed point of $\mathcal{H}$ then D and M are compatible and $\mathcal{I}_{\rm cpo}(\lambda) = \mathcal{I}_{\rm cms}(\kappa)$. Let $\lambda = < h, h^{-1} >$, $\kappa = < k, k^{-1} >$. Then (D, h) is the initial fixed point of $\mathcal{G}$, (M, k) the canonical fixed point of $\mathcal{H}$ and $\mathcal{I}_{\rm cpo}(h) = \mathcal{I}_{\rm cms}(k)$. $\square$

Now we give the proof of Theorem 3. We only consider the case that $\mathcal{H}$ is an endofunctor of $\mathbf{M}^E$. If $\mathcal{H}$ is an endofunctor of $\mathbf{M}^{*E}$ the argumentation is similar. The proof of Theorem 3 can be sketched as follows:

Step 1: If (D_n, ι_n) is a tower in CPO^{E} and (M_n, μ_n) a converging tower in $\mathbf{D}^E$ resp. $\mathbf{M}^E$ such that D_n, M_n are compatible and $\mathcal{I}_{\mathsf{cpo}}(\iota_n) = \mathcal{I}_{\mathsf{cms}}(\mu_n)$ for all $n \geq 0$ then the initial cones (D, λ_n) and (M, κ_n) are compatible and $\mathcal{I}_{\mathsf{cpo}}(\lambda_n) = \mathcal{I}_{\mathsf{cms}}(\kappa_n)$ for all $n \geq 0$.

Step 2: Let $\mathcal{G}$, $\mathcal{H}$ be as in Theorem 3 and

$$D_0 = M_0 = \{\perp\}, \quad D_{n+1} = \mathcal{G}(D_n), \quad M_{n+1} = \mathcal{H}(M_n).$$

Let ι_0 be the unique arrow $D_0 \to D_1$ in $\mathbf{D}^E$ and let μ_0 be the unique arrow $M_0 \to M_1$ in $\mathbf{M}^E$ which satisfies $\mathcal{I}_{\mathsf{cpo}}(\iota_0) = \mathcal{I}_{\mathsf{cms}}(\mu_0)$. Let $\iota_{n+1} = \mathcal{G}(\iota_n)$, $\mu_{n+1} = \mathcal{H}(\mu_n)$. Then the towers (D_n, ι_n) and (M_n, μ_n) satisfy the conditions of Step 1. Hence the initial/canonical fixed points of $\mathcal{G}$ and $\mathcal{H}$ which are the initial cones of (D_n, ι_n) resp. (M_n, μ_n) are compatible.

In order to show that the initial cones of 'compatible' towers in $\mathbf{D}^E$ and $\mathbf{M}^E$ are compatible (Step 1) we use a new category of 'complete metric partial orders' (i.e. sets which are endowed with a partial order and a compatible metric). We show that $\mathcal{G}$ and $\mathcal{H}$ induce an endofunctor $\mathcal{F}$ of the category of complete partial orders. $\mathcal{F}$ has an initial fixed point. This is the initial fixed point of $\mathcal{G}$ and the canonical fixed point of $\mathcal{H}$.

Definition 3.7 *A* `cmpo` *(complete metric partial order) is a tripel $(A, \sqsubseteq, d)$ consisting of a set A, a partial order $\sqsubseteq$ on A and a metric $d \leq 1$ on A such that:*

- *$(A, \sqsubseteq)$ is a cpo.*
- *(A, d) is a complete metric space.*
- *$(A, \sqsubseteq)$ and (A, d) are compatible.*

A `homomorphism` *from a cmpo $(A, \sqsubseteq_A, d_A)$ into a cmpo $(B, \sqsubseteq_B, d_B)$ is a function $f : A \to B$ which is non-distance-increasing w.r.t. d_A and d_B and strict and continuous w.r.t. $\sqsubseteq_A$ and $\sqsubseteq_B$. An* `embedding projection pair` *from $(A, \sqsubseteq_A, d_A)$ into $(B, \sqsubseteq_B, d_B)$ is a pair $< e, c >$ consisting of homomorphisms*

$$e \; : \; (A, \sqsubseteq_A, d_A) \;\to\; (B, \sqsubseteq_B, d_B)$$

$$c \; : \; (B, \sqsubseteq_B, d_B) \;\to\; (A, \sqsubseteq_A, d_A)$$

such that $c \circ e \; = \; id_A$ and $e \circ c \; \sqsubseteq_B \; id_B$.

Definition 3.8 $\mathbf{DM}$ *denotes the category whose objects are cmpo's $(A, \sqsubseteq, d)$ such that $(A, \sqsubseteq)$ is an $\mathbf{D}$-object and (A, d) an $\mathbf{M}$-object. The morphisms are embedding projection pairs between $\mathbf{DM}$-objects. The forgetful functors are denoted by:*

$$\mathcal{J}^{\mathsf{cmpo}}_{\mathsf{cpo}} : \mathbf{DM} \to \mathbf{D}^E, \quad \mathcal{J}^{\mathsf{cmpo}}_{\mathsf{cms}} : \mathbf{DM} \to \mathbf{M}^E.$$

Note that by the third assumption about **D** and **M** the morphisms in **DM** are those embedding projection pairs in SET^{E} which are at the same time morphisms in $\mathbf{D}^E$ and $\mathbf{M}^E$. It is clear that $\{\perp\}$ is the initial object of **DM**.

Lemma 3.9 *Let $\mathcal{G} : \mathbf{D}^E \to \mathbf{D}^E$ and $\mathcal{H} : \mathbf{M}^E \to \mathbf{M}^E$ be compatible functors. Then there exists a unique functor $\mathcal{F} : \mathbf{DM} \to \mathbf{DM}$ with*

$$\mathcal{J}^{\mathsf{cmpo}}_{\mathsf{cpo}} \circ \mathcal{G} \;=\; \mathcal{F} \circ \mathcal{J}^{\mathsf{cmpo}}_{\mathsf{cpo}},$$

$$\mathcal{J}^{\mathsf{cmpo}}_{\mathsf{cms}} \circ \mathcal{H} \;=\; \mathcal{F} \circ \mathcal{J}^{\mathsf{cmpo}}_{\mathsf{cms}}.$$

Proof: It is easy to see that the functor $\mathcal{F} : \mathbf{DM} \to \mathbf{DM}$ which is given by

- $\mathcal{F}(A, \sqsubseteq, d) \;=\; (A', \sqsubseteq', d')$ where $\mathcal{G}(A, \sqsubseteq) \;=\; (A', \sqsubseteq')$, $\mathcal{H}(A, d) \;=\; (A', d')$
- $\mathcal{F}(< e, c >) \;=\; \mathcal{G}^E(< e, c >) \;=\; \mathcal{H}^E(< e, c >)$

is the unique functor which satisfies $\mathcal{J}^{\mathsf{cmpo}}_{\mathsf{cpo}} \circ \mathcal{G} \;=\; \mathcal{F} \circ \mathcal{J}^{\mathsf{cmpo}}_{\mathsf{cpo}}$, $\mathcal{J}^{\mathsf{cmpo}}_{\mathsf{cms}} \circ \mathcal{H} \;=\; \mathcal{F} \circ \mathcal{J}^{\mathsf{cmpo}}_{\mathsf{cms}}$.
□

Definition 3.10 *A tower $(\mathcal{A}_n, \iota_n)$ in* **DM** *is called* converging *iff the tower (M_n, ι_n) in CMS^E is converging where $M_n \;=\; \mathcal{J}^{\mathsf{cmpo}}_{\mathsf{cms}}(\mathcal{A}_n)$.*

Lemma 3.11 *Each converging tower in* **DM** *has an initial cone.*

More precisely: Let $(\mathcal{A}_n, \iota_n)$ be a converging tower in **DM**, *where $\mathcal{A}_n \;=\; (A_n, \sqsubseteq_n, d_n)$ and $\iota_n \;=\; < e_n, c_n >$. Then the initial cone $(\mathcal{A}, \lambda_n)$ of $(\mathcal{A}_n, \iota_n)$ in* **DM** *satisfies:*

- *(D, λ_n) is the initial cone of (D_n, ι_n) in $\mathbf{D}^E$ where $D = \mathcal{J}^{\mathsf{cmpo}}_{\mathsf{cpo}}(\mathcal{A})$, $D_n = \mathcal{J}^{\mathsf{cmpo}}_{\mathsf{cpo}}(\mathcal{A}_n)$.*
- *(M, λ_n) is the initial cone of (M_n, ι_n) in $\mathbf{M}^E$ where $M = \mathcal{J}^{\mathsf{cmpo}}_{\mathsf{cms}}(\mathcal{A})$, $M_n = \mathcal{J}^{\mathsf{cmpo}}_{\mathsf{cms}}(\mathcal{A}_n)$.*

Proof: Let $\mathcal{A} \;=\; (A, \sqsubseteq, d)$ and $\lambda_n \;=\; < h_n, b_n >$ be given by:

$$A \;=\; \{\; (\xi_k)_{k \geq 0} \;\in\; \Pi_{k \geq 0}\, A_k \;:\; \xi_k = c_k(\xi_{k+1}) \;\}$$

$$(\xi_k) \;\sqsubseteq\; (y_k) \quad\Longleftrightarrow\quad \xi_k \;\sqsubseteq_k\; y_k \;\;\forall\, k \geq 0$$

$$d(\; (\xi_k)_{k \geq 0},\; (\eta_k)_{k \geq 0} \;) \;=\; \sup \;\{\; d_k(\xi_k, \eta_k) \;:\; k \geq 0 \;\}$$

$$h_n : A_n \to A, \quad h_n(\xi) \;=\; (\; e_{n,k}(\xi)\;)_{k \geq 0}, \quad b_n : A \to A_n, \quad b_n(\; (\xi_k)_{k \geq 0}\;) \;=\; \xi_n$$

Here $e_{n,k} : A_n \to A_k$ is given by

$$e_{n,k} \;=\; \begin{cases} e_{k-1} \circ e_{m-2} \circ \ldots \circ e_n & : \text{ if } k > n \\ id_{A_n} & : \text{ if } k = n \\ c_k \circ c_{m+1} \circ \ldots \circ c_{n-1} & : \text{ if } k < n. \end{cases}$$

$\sqsubseteq_k$ denotes the partial order on D_k and d_k the metric on M_k. By the results of [19] resp. [5]: (D, λ_n) is the initial cone of (D_n, ι_n) in $\mathbf{D}^E$ and (M, λ_n) is the initial cone of (M_n, ι_n) in $\mathbf{M}^E$. In particular: M is a complete metric space and D a cpo.

Claim 1: $\mathcal{A}$ is a cmpo and $\lambda_n : \mathcal{A}_n \to \mathcal{A}$ morphisms in **DM**.

Proof: Let $(\xi^{(n)})_{n\geq 0}$ be a monotone Cauchy sequence in A where $\xi^{(n)} = (\xi_m^{(n)})_{m\geq 0}$ then: $(\xi_m^{(n)})_{n\geq 0}$ are monotone Cauchy sequences in the cmpo's $\mathcal{A}_n$ and

$$\lim_{n\to\infty} \xi^{(n)} = \left(\lim_{n\to\infty} \xi_m^{(n)}\right)_{m\geq 0} = \left(\bigsqcup_{n\geq 0} \xi_m^{(n)}\right)_{m\geq 0} = \bigsqcup_{n\geq 0} \xi^{(n)}.$$

Since $\lambda_n : D_n \to D$ and $\lambda_n : M_n \to M$ are morphisms in $\mathbf{D}^E$ resp. $\mathbf{M}^E$ we get that λ_n are morphisms $\mathcal{A}_n \to \mathcal{A}$ in **DM**. In addition $\lambda_{n+1} \circ \iota_n = \lambda_n$.

Claim 2: $(\mathcal{A}, \lambda_n)$ is the initial cone of $(\mathcal{A}_n, \iota_n)$ in **DM**.

Proof: By Claim 1: $(\mathcal{A}, \lambda_n)$ is a cone of $(\mathcal{A}_n, \iota_n)$.

Let $(\mathcal{B}, \kappa_n)$ be a cone of $(\mathcal{A}_n, \iota_n)$, $\mathcal{B} = (B, \sqsubseteq_B, d_B)$, $\kappa_n = < g_n, a_n >$, $C = (B, \sqsubseteq_B)$, $N = (B, d_B)$. Then (C, κ_n) resp. (N, κ_n) is a cone of (D_n, ι_n) resp. (M_n, ι_n) in $\mathbf{D}^E$ resp. $\mathbf{M}^E$. Since (D, λ_n) is the initial cone of (D_n, ι_n) there exists a unique morphism

$$\kappa = < g, a >: D \to C$$

in $\mathbf{D}^E$ with $\kappa \circ \lambda_n = \kappa_n$. Hence $g \circ h_n = g_n$ and $b_n \circ a = a_n$.

If $\kappa' : \mathcal{A} \to \mathcal{B}$ is a morphism in **DM** with $\kappa' \circ \lambda_n = \kappa_n$ then $\kappa' : D \to C$ is a morphism in $\mathbf{D}^E$. By the uniqueness of κ as a morphism $D \to C$ in $\mathbf{D}^E$ with $\kappa \circ \lambda_n = \kappa_n$ we get $\kappa = \kappa'$.

Now we show that $\kappa : \mathcal{A} \to \mathcal{B}$ is a morphism in **DM**. We have:

$$g = \bigsqcup_{n\geq 0} g_n \circ b_n, \quad a = \bigsqcup_{n\geq 0} h_n \circ a_n.$$

Since (M, λ_n) is the initial cone of (M_n, ι_n) $\kappa' = < g', a' > : M \to N$ is a $\mathbf{M}^E$-morphism with $\kappa' \circ \lambda = \kappa_n$ where $g' = \lim g_n \circ b_n$, $a' = \lim h_n \circ a_n$. We conclude: For each $\xi \in A$ and $\zeta \in B$ the sequences $(g_n(b_n(\xi)))_{n\geq 0}$ resp. $(h_n(a_n(\zeta)))_{n\geq 0}$ are monotone Cauchy sequences in the cmpo's $\mathcal{B}$ resp. $\mathcal{A}$. Hence

$$g'(\xi) = \lim_{n\to\infty} g_n(b_n(\xi)) = \bigsqcup_{n\geq 0} g_n(b_n(\xi)) = g(\xi),$$

$$a'(\xi) = \lim_{n\to\infty} h_n(a_n(\xi)) = \bigsqcup_{n\geq 0} h_n(a_n(\xi)) = a(\xi).$$

I.e. $\kappa = < g, a > = < g', a' > = \kappa'$ is a morphism in $\mathbf{D}^E$ and in $\mathbf{M}^E$. Hence $\kappa : \mathcal{A} \to \mathcal{B}$ is a morphism in **DM**. $\Box$

We show the following stronger version of Theorem 3:

Lemma 3.12 *Let $\mathcal{G} : \mathbf{D}^E \to \mathbf{D}^E$ be locally continuous and let $\mathcal{H} : \mathbf{M}^E \to \mathbf{M}^E$ be e/p-contracting such that $\mathcal{G}$ and $\mathcal{H}$ are compatible. Let $\mathcal{F} : \mathbf{DM} \to \mathbf{DM}$ be the unique functor with*

$$\mathcal{J}_{\mathrm{cpo}}^{\mathrm{cmpo}} \circ \mathcal{G} = \mathcal{F} \circ \mathcal{J}_{\mathrm{cpo}}^{\mathrm{cmpo}} \quad \textit{and} \quad \mathcal{J}_{\mathrm{cms}}^{\mathrm{cmpo}} \circ \mathcal{H} = \mathcal{F} \circ \mathcal{J}_{\mathrm{cms}}^{\mathrm{cmpo}}$$

(see Lemma 3.9). Then $\mathcal{F}$ has an initial fixed point $(\mathcal{A}, \lambda)$ and:

- (M, λ) *is the canonical fixed point of* $\mathcal{H}$ *where* $M = \mathcal{J}_{\text{cms}}^{\text{cmpo}}(\mathcal{A})$,
- (D, λ) *is the initial fixed point of* $\mathcal{G}$ *where* $D = \mathcal{J}_{\text{cpo}}^{\text{cmpo}}(\mathcal{A})$.

Proof: Let $\mathcal{A}_0 = \{\perp\}$ be the initial object in **DM**, ι_0 the unique arrow $\mathcal{A}_0 \to \mathcal{F}(\mathcal{A}_0)$ in **DM** and $\mathcal{A}_{n+1} = \mathcal{F}(\mathcal{A}_n)$, $\iota_{n+1} = \mathcal{F}(\iota_n)$. Let

$$M_n = \mathcal{J}_{\text{cms}}^{\text{cmpo}}(\mathcal{A}_n), \quad D_n = \mathcal{J}_{\text{cpo}}^{\text{cmpo}}(\mathcal{A}_n).$$

Then $M_{n+1} = \mathcal{H}(M_n)$ and $D_{n+1} = \mathcal{G}(D_n)$. Since $\mathcal{H}$ is e/p-contracting (M_n, ι_n) is a converging tower in $\mathbf{M}^E$. Hence $(\mathcal{A}_n, \iota_n)$ is a converging tower in **DM**. By Lemma 3.11: The initial cone $(\mathcal{A}, \lambda_n)$ of $(\mathcal{A}_n, \iota_n)$ exists and (D, λ_n) resp. (M, λ_n) is the initial cone of (D_n, λ_n) resp. (M_n, λ_n) in $\mathbf{D}^E$ resp. $\mathbf{M}^E$ where

$$D = \mathcal{J}_{\text{cpo}}^{\text{cmpo}}(\mathcal{A}), \quad M = \mathcal{J}_{\text{cms}}^{\text{cmpo}}(\mathcal{A}).$$

Let $\lambda : \mathcal{A} \to \mathcal{F}(\mathcal{A})$ be the unique morphism in **DM** with $\lambda \circ \lambda_n = \mathcal{F}(\lambda_{n-1})$ (which exists by Lemma 2.1(a)). λ is a morphism $D \to \mathcal{G}(D)$ in $\mathbf{D}^E$ satisfying $\lambda \circ \lambda_n = \mathcal{G}(\lambda_{n-1})$. Hence λ is an isomorphism and (D, λ) the initial fixed point of $\mathcal{G}$. In particular λ as a morphism in **DM** is an isomorphism. By Lemma 2.1(c): $(\mathcal{A}, \lambda)$ is an initial fixed point of $\mathcal{F}$. λ as a morphism $M \to \mathcal{H}(M)$ in $\mathbf{M}^E$ satisfies $\lambda \circ \lambda_n = \mathcal{H}(\lambda_{n-1})$. Hence (M, λ) is the canonical fixed point of $\mathcal{H}$. $\square$

$$\begin{tikzcd}
\mathbf{D}^E \arrow[rr, "\mathcal{G}", "\text{locally continuous}"'] & & \mathbf{D}^E \\
\mathbf{DM} \arrow[u, "\mathcal{J}_{\text{cpo}}^{\text{cmpo}}"] \arrow[d, "\mathcal{J}_{\text{cms}}^{\text{cmpo}}"'] \arrow[rr, "\mathcal{F}"] & & \mathbf{DM} \arrow[u, "\mathcal{J}_{\text{cpo}}^{\text{cmpo}}"'] \arrow[d, "\mathcal{J}_{\text{cms}}^{\text{cmpo}}"] \\
\mathbf{M}^E \arrow[rr, "\mathcal{H}", "\text{e/p-contracting}"'] & & \mathbf{M}^E
\end{tikzcd}$$

4 Lifting of endofunctors of $CPO_\perp$ to endofunctors of CUM_e^*

In this section we show how a given domain equation for cpo's can be lifted to a 'corresponding' domain equation for pointed complete ultrametric spaces and in which way the solutions are connected. In general we cannot guarantee that there exists a corresponding domain equation in the whole category *CMS* but in the subcategory CUM_e^* (the category of pointed complete ultrametric spaces and embeddings). The induced domain equation in CUM_e^* is some new sense contracting which we call 'Γ-contracting'. We show that the induced metric equation has a unique solution which is compatible to the initial solution of the cpo equation.

Definition 4.1 *Let* $\gamma : [0,1] \to \{0, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots,\}$ *be given by:* $\gamma(0) = 0$ *and*

$$\frac{1}{2} \cdot \gamma(\xi) \ < \ \xi \ \leq \ \gamma(\xi)$$

for all $0 < \xi \leq 1$. *The functor* $\Gamma : CMS \to CMS$ *is given by:*

$$\Gamma(M, d) \ = \ (M, \gamma \circ d), \ \ \Gamma(f) \ = \ f.$$

In the following we write ΓM instead of $\Gamma(M, d)$. If $\mathbf{M}$ is a subcategory of CMS then $\Gamma\mathbf{M}$ denotes the subcategory of $\mathbf{M}$ whose objects are those objects M in $\mathbf{M}$ such that $M = \Gamma M$ and whose morphisms are $\mathbf{M}$-morphisms between $\Gamma\mathbf{M}$-objects.

If $\mathcal{H} : \mathbf{M} \to \mathbf{M}$ is a functor with $\Gamma \circ \mathcal{H} = \mathcal{H}$ then $\mathcal{H}(M)$ is an object in $\Gamma\mathbf{M}$ for all objects M in $\mathbf{M}$. In particular, if (M, e) is a fixed point of $\mathcal{H}$ then M is an object in $\Gamma\mathbf{M}$.

Definition 4.2 *Let* $\mathbf{M}$ *be a subcategory of CMS and* $\mathcal{H} : \mathbf{M} \to \mathbf{M}$ *a functor.* $\mathcal{H}$ *is called* Γ-`contracting` *iff* $\Gamma \circ \mathcal{H} \ = \ \mathcal{H}$ *and* $\mathcal{H}|\Gamma\mathbf{M} \to \Gamma\mathbf{M}$ *is contracting.*

In the following theorem we deal with Γ-contracting endofunctors of the category CUM_e^*. Since ΓCUM satisfies the conditions (i) - (iii) of section 2.4 we obtain by Lemma 2.18(e):

Lemma 4.3 *Each* Γ*-contracting functor* $\mathcal{H} : CUM_e^* \to CUM_e^*$ *has a unique fixed point.*

We now present the main result of this section which asserts that each domain equation for cpo's which is a lifting of a domain equation for pointed sets induces a weakly compatible domain equation for pointed complete ultrametric spaces such that the initial solution of the cpo equation and the unique solution of the metric equation are compatible.

Theorem 5

If $\mathcal{G} : \mathbf{D}_\perp \to \mathbf{D}_\perp$ *is a locally continuous functor and* $\mathcal{K} : SET^* \to SET^*$ *a functor with*

$$\mathcal{I}_{\mathrm{cpo}} \circ \mathcal{G} \ = \ \mathcal{K} \circ \mathcal{I}_{\mathrm{cpo}}$$

then there exists a Γ*-contracting functor*

$$\mathcal{H} : CUM_e^* \to CUM_e^*$$

with $\mathcal{I}_{\mathrm{cms}} \circ \mathcal{H} \ = \ \mathcal{K} \circ \mathcal{I}_{\mathrm{cms}}$ *and such that: If* (M, k) *is the unique fixed point of* $\mathcal{H}$ *and* (D, h) *the initial fixed point of* $\mathcal{G}$ *then* D, M *are compatible and* $\mathcal{I}_{\mathrm{cpo}}(h) \ = \ \mathcal{I}_{\mathrm{cms}}(k)$.

$$\begin{tikzcd}
\mathbf{D}_\perp \arrow[r, "\mathcal{G}", "\text{locally continuous}"'] \arrow[d, "\mathcal{I}_{\mathrm{cpo}}"'] & \mathbf{D}_\perp \arrow[d, "\mathcal{I}_{\mathrm{cpo}}"] \\
SET^* \arrow[r, "\mathcal{K}"] & SET^* \\
CUM_e^* \arrow[r, "\mathcal{H}", "\Gamma-\text{contracting}"'] \arrow[u, "\mathcal{I}_{\mathrm{cms}}"] & CUM_e^* \arrow[u, "\mathcal{I}_{\mathrm{cms}}"']
\end{tikzcd}$$

In the rest of this section we give the proof of Theorem 5. The main idea for the proof is the use of rank orderings as in [9] which are special kinds of ultrametric spaces. As in [7] the notion of a rank ordering is adapted to cpo's. The proof can be sketched as follows:

Step 1: If $\mathcal{G} : \mathbf{D}_\perp \to \mathbf{D}_\perp$ is locally continuous then $\mathcal{G}$ induces an endofunctor $\mathcal{G}_{\mathsf{rank}}$ of the category of rank ordered **D**-objects. The initial fixed point D of $\mathcal{G}$ can be endowed with a rank ordering such that D is the initial fixed point of $\mathcal{G}_{\mathsf{rank}}$.

Step 2: If $\mathcal{G} : \mathbf{D}_\perp \to \mathbf{D}_\perp$ is a locally continuous functor and $\mathcal{K} : SET^* \to SET^*$ and a functor such that $\mathcal{I}_{\mathsf{cpo}} \circ \mathcal{G} = \mathcal{K} \circ \mathcal{I}_{\mathsf{cpo}}$ then $\mathcal{K}$ induces an endofunctor $\mathcal{F}$ of complete rank ordered sets (i.e. rank ordered sets where the associated ultrametric space is complete).

Step 3: Let $\mathcal{G}$, $\mathcal{K}$, $\mathcal{F}$ be as in Step 2 and D as in Step 1. Then $\mathcal{F}$ induces a Γ-contracting functor $\mathcal{H} : CUM^*_e \to CUM^*_e$ with $\mathcal{I}_{\mathsf{cms}} \circ \mathcal{H} = \mathcal{K} \circ \mathcal{I}_{\mathsf{cms}}$. The rank ordering on D induces a metric on D that turns D into a complete ultrametric space which is the unique fixed point of $\mathcal{H}$.

In section 4.3, Lemma 4.17, we give the definition of the functor $\mathcal{G}_{\mathsf{rank}}$. Section 4.2, Lemma 4.9(c) and (d), shows how to define the functors $\mathcal{F}$ and $\mathcal{H}$. The complete proof of Theorem 5 (using Lemma 4.9 and Lemma 4.17) is given in section 4.4.

4.1 Rank ordered sets

Rank ordered sets are introduced in [9]. They are closely related to projection spaces [11]. Rank ordered sets are special kinds of pointed ultrametric spaces. In Lemma 4.7 we show the converse: pointed ultrametric spaces can be considered as rank ordered sets.

Definition 4.4 *Let M be a pointed set. A* `rank ordering` *on M is a family $\tilde{\pi} = (\pi_n)_{n\geq 0}$ of functions $\pi_n : M \to M$ such that*

(i) $\pi_n \circ \pi_m = \pi_m \circ \pi_n = \pi_n$ *for all* $0 \leq n \leq m$.

(ii) If $\xi, \eta \in M$ *and* $\pi_n(\xi) = \pi_n(\eta)$ *for all* $n \geq 0$ *then* $\xi = \eta$.

(iii) $\pi_0 = \lambda\xi.\xi_M$

Let $(M, \tilde{\pi})$ be a rank ordered set (i.e. M is a pointed set and $\tilde{\pi}$ a rank ordering on M). Then

$$d[\tilde{\pi}](\xi_1, \xi_2) = \inf \left\{ \frac{1}{2^n} : \pi_n(\xi_1) = \pi_n(\xi_2) \right\}$$

is an ultrametric on M and $d[\tilde{\pi}] = \gamma \circ d[\tilde{\pi}]$. We say that a rank ordered set $(M, \tilde{\pi})$ is complete iff the induced ultrametric space $(M, d[\tilde{\pi}])$ is complete. If d is metric on M then $\gamma \circ d = d[\tilde{\pi}]$ if and only if for all $\xi_1, \xi_2 \in M$:

$$d(\xi_1, \xi_2) \leq 1/2^n \iff \pi_n(\xi_1) = \pi_n(\xi_2)$$

In this case we say that $\tilde{\pi}$ is a suitable rank ordering for the pointed metric space (M, d).

Definition 4.5 *Let $(M, \tilde{\pi})$, $(N, \tilde{\mu})$ be rank ordered sets. A function $f : M \to N$ is called* `rank preserving` *iff $f \circ \pi_n = \mu_n \circ f$ for all $n \geq 0$. f is called a* `rank preserving embedding` *iff f is rank preserving and injective.*

Because of $f \circ \pi_0 = \mu_0 \circ f$ rank preserving functions always preserve the basis point.

Lemma 4.6 *Let $(M, \tilde{\pi})$ and $(N, \tilde{\mu})$ be rank ordered sets and $f : M \to N$ a function. Then: f is non-distance-increasing w.r.t. $d[\tilde{\pi}]$ resp. $d[\tilde{\mu}]$ if and only if*

$$\mu_n \circ f \circ \pi_n = \mu_n \circ f$$

for all $n \geq 0$. If f is rank preserving then f is non-distance-increasing. If f is a rank preserving embedding then f is an embedding of the metric space $(M, d[\tilde{\pi}])$ into the metric space $(N, d[\tilde{\mu}])$.

The next lemma shows that each pointed complete ultrametric space M can be endowed with a suitable rank ordering. In addition it shows that for any suitable rank ordering on a complete rank ordered set M and for any basis point preserving embedding $e : M \to N$ there exist a function $c : N \to M$ and a suitable rank ordering on N such that e and c are rank preserving. This result is needed for the lifting of a given endofunctor of SET^* to an endofunctor of CUM_e^* (Lemma 4.9).

Lemma 4.7 *For each pointed ultrametric space (M, d_M) there exists a rank ordering $\tilde{\pi}$ on M such that $d[\tilde{\pi}] = \gamma \circ d_M$.*

If (M, d_M) and (N, d_N) are pointed complete ultrametric spaces and $e : M \to N$ a basis point preserving embedding then we have: If $\tilde{\pi}$ is a rank ordering on M with $\gamma \circ d_M = d[\tilde{\pi}]$ then there exist a rank ordering $\tilde{\mu}$ on N with $\gamma \circ d_N = d[\tilde{\mu}]$ and a basis point preserving function $c : N \to M$ such that the following conditions are satisfied:

(i) e and c are rank preserving w.r.t. $\tilde{\pi}$ and $\tilde{\mu}$, i.e. $\mu_n \circ e = e \circ \pi_n$ and $\pi_n \circ c = c \circ \mu_n$ for all $n \geq 0$.

(ii) $c \circ e = id_M$ and $\delta(\, e(\Gamma M),\ \Gamma N\,) = \Delta_{\Gamma N}(e, c)$

Proof: If (M, d) is a pointed ultrametric space and $\tilde{\lambda} = (\lambda_n)_{n \geq 0}$ a family of functions $\lambda_n : M \to M$ then we say $\tilde{\lambda}$ a `pre-ranking` iff the following conditions are satisfied:

- $d(\xi, \lambda_n(\xi)) \leq 1/2^n$
- $\lambda_n(\xi) = \lambda_n(\eta) \iff d(\xi, \eta) \leq 1/2^n$
- λ_0 is constant with $\lambda_0(\xi) = \xi_M$ for all $\xi \in M$

$B(\xi, r)$ denotes the closed ball in M with center ξ and radius r. I.e.

$$B(\xi, r) = \{\eta \in M \ : \ d(\xi, \eta) \leq r\}.$$

Claim 1: For each pointed ultrametric space (M, d) with $d \leq 1$ there exists a pre-ranking.

Proof: Let $\mathcal{B} = \{B(\xi, 1/2^n) : \xi \in M, n \in \mathbb{N}_0\}$. By the axiom of choice there exists a function $\lambda : \mathcal{B} \to M$ with $\lambda(B) \in B$ for all $B \in \mathcal{B}$ and $\lambda(M) = \xi_M$. Then $\tilde{\lambda} = (\lambda_n)_{n \geq 0}$ is a pre-ranking on M where

$$\lambda_n(\xi) = \lambda\left(B(\xi, \frac{1}{2^n}) \right).$$

Claim 2: Let (M, d) be a pointed ultrametric space with $d \leq 1$. If $\tilde{\lambda}$ is a pre-ranking on M then $rank(\tilde{\lambda})$ is a rank ordering on M with $\gamma \circ d = d[rank(\tilde{\lambda})]$ where $rank(\tilde{\lambda}) = (\pi_n)_{n\geq 0}$ is defined as follows:

$$\pi_0 = \lambda_0, \quad \pi_{n+1}(\xi) = \begin{cases} \pi_n(\xi) & : \text{ if } d(\pi_n(\xi), \xi) \leq 1/2^{n+1} \\ \lambda_{n+1}(\xi) & : \text{ otherwise.} \end{cases}$$

Proof: Since $\pi_0 = \lambda_0$ we have $\pi_0(\xi) = \xi_M$ for all $\xi \in M$. We show by induction on n:

(i) $d(\pi_n(\xi), \xi) \leq 1/2^n$

(ii) $d(\xi, \eta) \leq 1/2^n \iff \pi_n(\xi) = \pi_n(\eta)$

(iii) $\pi_n \circ \pi_m = \pi_m \circ \pi_n = \pi_m$ for all $0 \leq m \leq n$

Then by (ii): If $\pi_n(\xi) = \pi_n(\eta)$ for all $n \geq 0$ then $d(\xi, \eta) = 0$ and hence $\xi = \eta$. By (iii) $(\pi_n)_{n\geq 0}$ is a rank ordering on M. (i) and (ii) imply that $\gamma \circ rank[\tilde{\lambda}]$.

The basis $n = 0$ is clear. Step of induction $n \Longrightarrow n + 1$:

(i) If $\xi \in M$ then $d(\pi_{n+1}(\xi), \xi) \leq 1/2^{n+1}$ since $d(\lambda_{n+1}(\xi), \xi) \leq 1/2^{n+1}$.

(ii) If $d(\xi, \eta) \leq 1/2^{n+1}$ then by induction hypothesis $\pi_n(\xi) = \pi_n(\eta)$. Hence (by the triangle inequality)

$$d(\pi_n(\xi), \xi) \leq \frac{1}{2^{n+1}} \iff d(\pi_n(\eta), \eta) \leq \frac{1}{2^{n+1}}$$

We get:

- If $d(\pi_n(\xi), \xi) \leq 1/2^{n+1}$ then $\pi_{n+1}(\xi) = \pi_n(\xi) = \pi_n(\eta) = \pi_{n+1}(\eta)$.
- If $d(\pi_n(\xi), \xi) > 1/2^{n+1}$ then $\pi_{n+1}(\xi) = \lambda_{n+1}(\xi) = \lambda_{n+1}(\eta) = \pi_{n+1}(\eta)$.

If $\pi_{n+1}(\xi) = \pi_{n+1}(\eta)$ then by (i) and the triangle inequality:

$$d(\xi, \eta) \leq \max \{ d(\xi, \pi_{n+1}(\xi)), d(\pi_{n+1}(\eta), \eta) \} \leq 1/2^{n+1}$$

(iii) If $0 \leq m \leq n$ then by induction hypothesis: $\pi_n(\pi_m(\xi)) = \pi_m(\xi)$. Hence

$$d(\pi_n(\pi_m(\xi)), \pi_m(\xi)) = 0 \leq \frac{1}{2^{n+1}}$$

and therefore $\pi_{n+1}(\pi_m(\xi)) = \pi_n(\pi_m(\xi)) = \pi_m(\xi)$.
If $0 \leq m \leq n + 1$ then (by (i)):

$$d(\pi_{n+1}(\xi), \xi) \leq 1/2^{n+1} \leq 1/2^m.$$

Then by (ii): $\pi_m(\pi_{n+1}(\xi)) = \pi_m(\xi)$.

Claim 3: Let (M, d_M), (N, d_N) be pointed ultrametric spaces, $e : M \to N$ a basis point preserving embedding and $\tilde{\pi} = (\pi_n)_{n \geq 0}$ a rank ordering on M with $d[\tilde{\pi}] = \gamma \circ d_M$. Let

$$\mathcal{B} = \{B(\eta, 1/2^n) : \eta \in N, n \geq 0\}.$$

Then:

(i) For all $n \geq 0$ and $\eta \in N$ we have: $\pi_n \mid e^{-1}(B(\eta, 1/2^n)) \to N$ is constant.

(ii) There exists a function $\lambda : \mathcal{B} \to N$ such that

- $\lambda(B) \in B$ for all $B \in \mathcal{B}$
- If $B = B(\eta, 1/2^n)$ then $\lambda(B) = e(\pi_n(\xi))$ for all $\xi \in e^{-1}(B)$.

(iii) $\tilde{\lambda} = (\lambda_n)_{n \geq 0}$ is a pre-ranking on N where $\lambda_n(\eta) = \lambda(\ B(\eta, 1/2^n)\ )$.

(iv) Let $\tilde{\mu} = rank(\tilde{\lambda})$ be the associated rank ordering (Claim 2) then $e \circ \pi_n = \mu_n \circ e$ for all $n \geq 0$. I.e. e is rank preserving w.r.t. $\tilde{\pi}$ and $\tilde{\mu}$.

Proof:

(i) If $B = B(\eta, 1/2^n) \in \mathcal{B}$ and $e(\xi)$, $e(\xi') \in B$ then

$$d_M(\xi, \xi') = d_N(e(\xi), e(\xi')) \leq \max\ \{d_N(e(\xi), \eta),\ d_N(\eta, e(\xi')\} \leq \frac{1}{2^n}.$$

Hence $\pi_n(\xi) = \pi_n(\xi')$. I.e. we get that $\pi_n \mid e^{-1}(B) \to M$ is constant.

(ii) If $B \in \mathcal{B}$, $e^{-1}(B) \neq \emptyset$ then let $\Lambda(B) \in M$ denote the value of the constant function $\pi_n|e^{-1}(B) \to M$. For each $B \in \mathcal{B}$ with $e^{-1}(B) = \emptyset$ let η_B be an arbitrary point in B (axiom of choice). Then $\lambda : \mathcal{B} \to N$,

$$\lambda(B) = \begin{cases} e(\Lambda(B)) & : \text{ if } e^{-1}(B) \neq \emptyset \\ \eta_B & : \text{ otherwise} \end{cases}$$

is a function which has the desired properties of (ii).

(iii) $\tilde{\lambda}$ is a pre-ranking:

For all $\xi \in N$: $\lambda_n(\xi) = \lambda(\ B(\xi, 1/2^n)\ ) \in B(\xi, 1/2^n)$. Hence

$$d_N(\xi, \lambda_n(\xi)) \leq \frac{1}{2^n}.$$

If $\lambda_n(\xi) = \lambda_n(\eta)$ then

$$d_N(\xi, \eta) \leq \max\ \{\ d_N(\xi, \lambda_n(\xi)),\ d_N(\lambda_n(\eta), \eta)\ \} \leq \frac{1}{2^n}.$$

On the other hand: If $d_N(\xi, \eta) \leq 1/2^n$ then $B(\xi, 1/2^n) = B(\eta, 1/2^n)$. Hence

$$\lambda_n(\xi) = \lambda(\ B(\xi, 1/2^n)\ ) = \lambda(\ B(\eta, 1/2^n)\ ) = \lambda_n(\eta).$$

Let $\xi \in N$. Then $B(\xi, 1) = N$ (because of the assumption $d_N \leq 1$). By definition $\Lambda(N)$ is the value of the constant function π_0, i.e. $\Lambda(N) = \xi_M$. Hence

$$\lambda_0(\xi) = \lambda(N) = e(\ \Lambda(N)\ ) = e(\xi_M).$$

Since e is basis point preserving: $\lambda_0(\xi) = e(\xi_M) = \xi_N$.

(iv) Let $\tilde{\mu} = rank(\tilde{\lambda})$ be defined as in Claim 2. We show by induction on n that

$$e \circ \pi_n = \mu_n \circ e.$$

Basis of induction $n = 0$: Since e is basis point preserving and $\pi_0(\xi) = \xi_M$:

$$\mu_0(e(\xi)) = \lambda_0(\ e(\xi)\ ) = \xi_N = e(\xi_M) = e(\pi_0(\xi)).$$

Step of induction $n \Longrightarrow n+1$:

- If $d_M(\pi_n(\xi), \xi) \leq 1/2^{n+1}$ then $\pi_{n+1}(\xi) = \pi_{n+1}(\pi_n(\xi)) = \pi_n(\xi)$. By induction hypothesis and since e is an embedding:

$$d_N(\ \mu_n(e(\xi)),\ e(\xi)\ ) = d_N(\ e(\pi_n(\xi)),\ e(\xi)\ ) = d_M(\pi_n(\xi), \xi) \leq \frac{1}{2^{n+1}}.$$

Therefore $\mu_{n+1}(e(\xi)) = \mu_{n+1}(\mu_n(e(\xi))) = \mu_n(e(\xi))$. Again by induction hypothesis:

$$\mu_{n+1}(e(\xi)) = \mu_n(e(\xi)) = e(\pi_n(\xi)) = e(\pi_{n+1}(\xi)).$$

- If $d_M(\pi_n(\xi), \xi) > 1/2^{n+1}$ then by induction hypothesis

$$d_N(\ \mu_n(e(\xi)),\ e(\xi)\ ) = d_N(\ e(\pi_n(\xi)),\ e(\xi)\ ) = d_M(\pi_n(\xi), \xi) > \frac{1}{2^{n+1}}.$$

Hence by definition of μ_{n+1}, λ_{n+1} and by (ii):

$$\mu_{n+1}(e(\xi)) = \lambda_{n+1}(e(\xi)) = \lambda(B(e(\xi), 1/2^{n+1})) = e(\pi_{n+1}(\xi)).$$

Claim 4: Let (M, d_M) and (N, d_N) be pointed complete ultrametric spaces, $\tilde{\pi}$ resp. $\tilde{\mu}$ rank orderings on M resp. N with $\gamma \circ d_M = d[\tilde{\pi}]$ resp. $\gamma \circ d_N = d[\tilde{\mu}]$ and $e : M \to N$ a rank preserving embedding. Then there exists a function $c : N \to M$ such that

- $c \circ e = id_M$
- $d_{\Gamma N}(e(c(\eta)), \eta) = \delta_{\Gamma N}(\eta, e(M))$
- c is rank preserving.

In particular: c is basis point preserving and $\Delta_{\Gamma N}(e, c) = \delta(e(\Gamma M), \Gamma N)$.

Proof: W.l.o.g. $\Gamma M = M, \Gamma N = N$. Then $d_M = d[\tilde{\pi}]$, $d_N = d[\tilde{\mu}]$. Let $\Xi : N \to I\!N_0 \cup \{\infty\}$ be given by

$$\Xi(\eta) = \sup \left\{ n \geq 0 \ : \ e^{-1}\left(B(\eta, \frac{1}{2^n})\right) \neq \emptyset \right\}$$

and let

$$B(\eta) = \begin{cases} B(\eta, 1/2^{\Xi(\eta)}) & : \text{ if } \Xi(\eta) \neq \infty \\ \{\eta\} & : \text{ otherwise.} \end{cases}$$

Then for all η, $\eta' \in N$:

(I) If $d_N(\eta,\eta') \;\leq\; 1/2^k$ then

- either $\Xi(\eta) = \Xi(\eta') \leq k,\;\; B(\eta) = B(\eta')$
- or $\Xi(\eta), \Xi(\eta') \;\geq\; k,\;\; B(\eta) \cap B(\eta') \;\subseteq\; B(\eta, 1/2^k) \;=\; B(\eta', 1/2^k)$

(II) $\Xi(\eta) = \infty \qquad \Longleftrightarrow \qquad \eta \in e(M)$

(I) and (II) are easy verifications. In (II) we need the completeness of M. (II) implies that for all $\eta \in N$:

$$e^{-1}(\;B(\eta)\;) \quad \neq \emptyset$$

If $\Xi(\eta) = \infty$ then $e^{-1}(B(\eta))$ consists of a single element (since e is injective). Otherwise for all ξ, $\xi' \in e^{-1}(B(\eta))$: Since $e(\xi)$, $e(\xi') \in B(\eta)$:

$$d_M(\xi,\xi') \;=\; d_N(e(\xi), e(\xi')) \;\leq\; \frac{1}{2^{\Xi(\eta)}}$$

and therefore $\pi_{\Xi(\eta)}(\xi) \;=\; \pi_{\Xi(\eta)}(\xi')$. Hence we may define $c : N \to M$ as follows:

$$c(\eta) \;=\; \pi_{\Xi(\eta)}(\xi) \;\text{ where } \xi \in e^{-1}(B(\eta)) \text{ and } \pi_\infty(\xi) \;=\; \xi.$$

Now we show:

(A) $c \circ e \;=\; id_M$

(B) $d_N(e(c(\eta)), \eta) \;=\; \delta_N(\eta, e(M) \;\;(= \;\inf\;\{d_N(e(\xi), \eta) : \xi \in M\}\;)$

(C) $c \circ \mu_n \;=\; \pi_n \circ c$

ad (A): Since $\Xi(e(\xi)) \;=\; \infty$ and $B(e(\xi)) \;=\; \{e(\xi)\}$ we get $c(e(\xi)) \;=\; \xi$.

ad (B): If $\delta_N(\eta, e(M)) = 1/2^m$ then $B(\eta, 1/2^n) \cap e(M) \neq \emptyset$ for all $n \leq m$ and

$$B(\eta, 1/2^{m+1}) \cap e(M) = \emptyset.$$

I.e. $\Xi(\eta) \;=\; m$ and $e(c(\eta)) \;\in\; B(\eta) \;=\; B(\eta, 1/2^m)$. Hence

$$d_N(\;e(c(\eta)),\;\eta\;) \;\leq\; \frac{1}{2^m} \;=\; \delta_N(\eta, e(M)).$$

On the other hand $\delta_N(\eta, e(M)) \;\leq\; d_N(e(c(\eta)), \eta)$.

ad (C): Let $\eta \in N$, $\Xi(\eta) = m$, $c(\eta) \;=\; \pi_m(\xi)$ where $\xi \in B(\eta)$.

<u>Case 1</u>: $0 \leq n \leq m$. Since $d_N(\mu_n(\eta), \eta) \;\leq\; 1/2^n$:

$$\xi \in\; e^{-1}(\;B(\eta, 1/2^n)\;) \;=\; e^{-1}\left(\;B\left(\mu_n(\eta), \frac{1}{2^n}\right)\;\right).$$

Therefore

$$\Xi(\mu_n(\eta)) \;\geq\; n, \quad c(\mu_n(\eta)) \;=\; \pi_k(\xi')$$

where $k \;=\; \Xi(\mu_n(\eta))$ and $\xi' \in\; e^{-1}\left(\;B(\mu_n(\eta), 1/2^k)\;\right)$. Then $k \geq n$ and

$$\pi_n(c(\eta)) \;=\; \pi_n(\pi_m(\xi)) \;=\; \pi_n(\xi).$$

We have to show that $\pi_n(\xi) = c(\mu_n(\eta))$. Therefore we have to show that $\pi_n(\xi) = \pi_k(\xi')$: Since $k \geq n$ and $d_N(\mu_n(\eta), e(\xi')) \leq 1/2^k$ we get:

$$\mu_n(\eta) = \mu_k(\mu_n(\eta)) = \mu_k(e(\xi'))$$

Therefore

$$\mu_n(e(\xi')) = \mu_n(\mu_k(e(\xi'))) = \mu_n(\mu_n(\eta)) = \mu_n(\eta).$$

Since e is a rank preserving embedding:

$$d_M(\pi_n(\xi'), \xi') = d_N(e(\pi_n(\xi')), e(\xi')) = d_N(\mu_n(e(\xi')), e(\xi'))$$

$$= d_N(\mu_n(\eta), e(\xi')) \leq \frac{1}{2^k}.$$

Hence $\pi_k(\pi_n(\xi')) = \pi_k(\xi')$. Since $k \geq n$: $\pi_n(\xi') = \pi_k(\pi_n(\xi'))$. Hence

$$\pi_k(\xi') = \pi_n(\xi').$$

Since $\xi \in e^{-1}(B(\eta, 1/2^n))$:

$$d_N(e(\xi), \eta) \leq \frac{1}{2^n}$$

Since $\mu_n(\eta) = \mu_n(e(\xi'))$:

$$d_N(\mu_n(\eta), e(\xi')) \leq \frac{1}{2^n}$$

We get:

$$d_N(e(\xi), e(\xi')) \leq \max\{\, d_N(e(\xi), \eta), d_N(\eta, \mu_n(\eta)), d_N(\mu_n(\eta), e(\xi')) \,\} \leq \frac{1}{2^n}$$

Since e is distance preserving:

$$d_M(\xi, \xi') = d_N(e(\xi), e(\xi')) \leq \frac{1}{2^n}$$

We conclude: $\pi_k(\xi') = \pi_n(\xi') = \pi_n(\xi)$.

Case 2: $n > m$. Then by (I): $\Xi(\mu_n(\eta)) = \Xi(\eta) = m$ and $B(\, \mu_n(\eta)\,) = B(\eta)$. Hence by definition of c:

$$c(\mu_n(\eta)) = c(\eta) = \pi_m(\xi)$$

where $\xi \in e^{-1}(B(\eta))$. Therefore $\pi_n(c(\eta)) = \pi_n(\pi_m(\xi)) = \pi_m(\xi) = c(\mu_n(\eta))$. □

4.2 Lifting of endofunctors in SET^* to endofunctors in CUM_e^*

We present a technique to lift endofunctors of SET^* to Γ-contracting endofunctors of CUM_e^*. Given an endofunctor of SET^* we define an endofunctor of the category of complete rank ordered sets and then an endofunctor of CUM_e^*.

Notation 4.8 *CRankSET* denotes the category whose objects are complete rank ordered sets and whose morphisms are rank preserving functions. CRankSET$^*_{\mathrm{e}}$ denotes the subcategory of complete rank ordered sets and rank preserving embeddings as morphisms.*

$\mathcal{I}_{\mathrm{rank}}$ denotes the forgetful functor CRankSET* $\to$ SET*. $\mathcal{M}$: CRankSET$^*_{\mathrm{e}}$ $\to$ CUM$^*_{\mathrm{e}}$ is given by:

$$\mathcal{M}(M, \tilde{\pi}) \;=\; (M, d[\tilde{\pi}]), \quad \mathcal{M}(e) = e.$$

Note that by Lemma 4.6 rank preserving embeddings (which are by definition rank preserving injections) are embeddings of the underlying metric spaces. Hence $\mathcal{M}$ is welldefined.

Given an endofunctor $\mathcal{K}$ of *SET** we present conditions that allow the definition of a functor $\mathcal{F}$: *CRankSET** $\to$ *CRankSET**. These conditions are satisfied when $\mathcal{K}$ has a lifting $\mathcal{G}$ on the category $\mathbf{D}_\perp$ (as it is the case in Theorem 5, see Claim 1 in section 4.4). We show that this functor $\mathcal{F}$ preserves embeddings and hence can be considered as endofunctor of *CRankSET$^*_{\mathrm{e}}$*. Then Lemma 4.7 ensures that $\mathcal{F}$ (as an endofunctor of *CRankSET$^*_{\mathrm{e}}$*) can be lifted to a Γ-contracting endofunctor $\mathcal{H}$ of *CUM$^*_{\mathrm{e}}$*.

$$\begin{tikzcd}
SET^* \arrow[r, "\mathcal{K}"] & SET^* \\
CRankSET^*_{\mathrm{e}} \arrow[u, "\mathcal{I}_{\mathrm{rank}}"] \arrow[r, "\mathcal{F}"] \arrow[d, "\mathcal{M}"'] & CRankSET^*_{\mathrm{e}} \arrow[u, "\mathcal{I}_{\mathrm{rank}}"'] \arrow[d, "\mathcal{M}"] \\
CUM^*_{\mathrm{e}} \arrow[r, "\mathcal{H}"] & CUM^*_{\mathrm{e}}
\end{tikzcd}$$

Lemma 4.9 *Let $\mathcal{K}$: SET* $\to$ SET* be a functor. Then for each rank ordered set $(M, \tilde{\pi})$:*

$$\mathcal{K}(\tilde{\pi}) \;=\; (\pi'_n)_{n\geq 0}, \quad \pi'_0 \;=\; \lambda\eta.\text{basis point of } \mathcal{K}(M), \quad \pi'_{n+1} \;=\; \mathcal{K}(\pi_n)$$

satisfies the conditions (i) and (iii) of rank orderings (Definition 4.4).

Now we assume that $\mathcal{K}$ satisfies the following two conditions: For each complete rank ordered set $(M, \tilde{\pi})$:

(I) If ξ, $\xi' \in \mathcal{K}(M)$ with $\mathcal{K}(\pi_n)(\xi) \;=\; \mathcal{K}(\pi_n)(\xi')$ for all $n \geq 0$ then $\xi = \xi'$.

(II) Whenever (η_n) is a sequence in $\mathcal{K}(M)$ such that $\mathcal{K}(\pi_n)(\eta_n) \;=\; \mathcal{K}(\pi_n)(\eta_m)$ for all $m \geq n \geq 0$ then there exists $\eta \in \mathcal{K}(M)$ such that for all $n \geq 0$:

$$\mathcal{K}(\pi_n)(\eta) \;=\; \mathcal{K}(\pi_n)(\eta_n)$$

Then we have:

(a) For all complete rank ordered sets $(M, \tilde{\pi})$: $\mathcal{K}(\tilde{\pi})$ is a rank ordering on $\mathcal{K}(M)$ and the rank ordered set $(\mathcal{K}(M), \mathcal{K}(\tilde{\pi}))$ is complete.

(b) If $\tilde{\pi}$, $\tilde{\mu}$ are complete rank orderings on a pointed set M with $\mathcal{M}(M,\tilde{\pi}) = \mathcal{M}(M,\tilde{\mu})$ then

$$\mathcal{M}(\mathcal{K}(M),\mathcal{K}(\tilde{\pi})) = \mathcal{M}(\mathcal{K}(M),\mathcal{K}(\tilde{\mu})).$$

(c) The functor $\mathcal{F}$: CRankSET* $\to$ CRankSET* which is given by

$$\mathcal{F}(M,\tilde{\pi}) = (\mathcal{K}(M),\mathcal{K}(\tilde{\pi})), \quad \mathcal{F}(f) = \mathcal{K}(f)$$

is welldefined, preserves rank preserving embeddings and $\mathcal{I}_{\mathsf{rank}} \circ \mathcal{F} = \mathcal{K} \circ \mathcal{I}_{\mathsf{rank}}$.

(d) The functor $\mathcal{H}$: CUM$^*_{\mathsf{e}}$ $\to$ CUM$^*_{\mathsf{e}}$ which is given by

$$\mathcal{H}(M) = \mathcal{M}(\ \mathcal{F}(M,\tilde{\pi}^M)\ ), \quad \mathcal{H}(e) = \mathcal{K}(e)$$

is welldefined, Γ-contracting and satisfies

$$\mathcal{M} \circ \mathcal{F} = \mathcal{H} \circ \mathcal{M}, \quad \mathcal{I}_{\mathsf{cms}} \circ \mathcal{H} = \mathcal{K} \circ \mathcal{I}_{\mathsf{cms}}.$$

Here $\mathcal{F}$ is considered as functor CRankSET$^*_{\mathsf{e}}$ $\to$ CRankSET$^*_{\mathsf{e}}$ and $\tilde{\pi}^M$ is a fixed rank ordering on M with $\gamma \circ d_M = d[\tilde{\pi}^M]$ (Lemma 4.7 and axiom of choice for classes).

Proof: By definition π'_0 is constant and its value is the basis point of $\mathcal{K}(M)$. Since $\mathcal{K}(\pi_{n-1}) = \pi'_n$ is basis point preserving (by definition of morphisms in SET^*) we get:

$$\pi'_0 \circ \pi'_n = \pi'_n \circ \pi'_0 = \pi'_0$$

If n, $m \geq 1$, $k = \min\{n,m\}$ then

$$\pi'_n \circ \pi'_m = \mathcal{K}(\pi_{n-1} \circ \pi_{m-1}) = \mathcal{K}(\pi_{k-1}) = \pi'_k.$$

Hence by assumption (I) $\mathcal{K}(\tilde{\pi})$ is a rank ordering on $\mathcal{K}(M)$. It is easy to see that assumption (II) ensures the completeness of the rank ordered set $(\mathcal{K}(M),\mathcal{K}(\tilde{\pi}))$.

If $\mathcal{M}(M,\tilde{\pi}) = \mathcal{M}(M,\tilde{\mu})$ then we have:

$$\pi_n(\xi) = \pi_n(\xi') \iff d_M(\xi,\xi') \leq \frac{1}{2^n} \iff \mu_n(\xi) = \mu_n(\xi').$$

Therefore by Lemma 2.4(b):

$$\pi'_n(\eta) = \pi'_n(\eta') \iff \mu'_n(\eta) = \mu'_n(\eta').$$

I.e. $d[\mathcal{K}(\tilde{\pi})] = d[\mathcal{K}(\tilde{\mu})]$. Hence $\mathcal{M}(\mathcal{K}(M),\mathcal{K}(\tilde{\pi})) = \mathcal{M}(\mathcal{K}(M),\mathcal{K}(\tilde{\mu}))$.

For each rank preserving function $f : (M,\tilde{\pi}) \to (N,\tilde{\mu})$ we have:

$$\mathcal{K}(f) \circ \pi'_n = \mathcal{K}(f) \circ \mathcal{K}(\pi_{n-1}) = \mathcal{K}(f \circ \pi_{n-1})$$

$$= \mathcal{K}(\mu_{n-1} \circ f) = \mathcal{K}(\mu_{n-1}) \circ \mathcal{K}(f) = \mu'_n \circ \mathcal{K}(f).$$

I.e. $\mathcal{K}(f)$ is rank preserving w.r.t. $\mathcal{K}(\tilde{\pi})$ and $\mathcal{K}(\tilde{\mu})$. Hence $\mathcal{F}$ is welldefined (by (a)). It is clear that $\mathcal{I}_{\mathsf{rank}} \circ \mathcal{F} = \mathcal{K} \circ \mathcal{I}_{\mathsf{rank}}$.

If f is a rank preserving embedding then f is injective. By Lemma 2.4(a) $\mathcal{K}(f)$ is injective. I.e. $\mathcal{K}(f) = \mathcal{F}(f)$ is a rank preserving embedding. Hence $\mathcal{F}$ preserves rank preserving embeddings.

Let M, N be objects in CUM_c^* and let $e : M \to N$ be an embedding. First we show that $\mathcal{K}(e)$ is an embedding

$$\mathcal{M}(\mathcal{F}(M, \tilde{\pi}^M)) \;\to\; \mathcal{M}(\mathcal{F}(N, \tilde{\pi}^N)).$$

By Lemma 4.7 there exist a rank ordering $\tilde{\mu}$ on N with

$$\gamma \circ d_N \;=\; d[\tilde{\mu}]$$

and a function $c : N \to M$ with $c \circ e \;=\; id_M$ such that e and c are rank preserving w.r.t. $\tilde{\pi}^M$ and $\tilde{\mu}$ and

$$\delta(\, e(\Gamma M), \Gamma N) \;=\; \Delta_{\Gamma N}(e, c).$$

By (c) we get that $\mathcal{K}(e)$ and $\mathcal{K}(c)$ are rank preserving w.r.t. $\mathcal{K}(\tilde{\pi}^M)$ and $\mathcal{K}(\tilde{\mu})$. Hence $\mathcal{K}(e)$ and $\mathcal{K}(c)$ are non-distance-increasing w.r.t. $d[\mathcal{K}(\tilde{\pi})]$ and $d[\mathcal{K}(\tilde{\mu})]$ (Lemma 4.6). Since

$$\mathcal{K}(c) \circ \mathcal{K}(e) \;=\; \mathcal{K}(c \circ e) \;=\; \mathcal{K}(id_M) \;=\; id_{\mathcal{K}(M)}$$

we get that $\mathcal{K}(e)$ is an embedding $\mathcal{M}(\mathcal{K}(M), \mathcal{K}(\tilde{\pi}^M)) \;\to\; \mathcal{M}(\mathcal{K}(N), \mathcal{K}(\tilde{\mu}))$. By definition of $\tilde{\pi}^N$ we have: $d[\tilde{\pi}^N] \;=\; d[\tilde{\mu}]$. Hence

$$\mathcal{M}(N, \tilde{\mu}) \;=\; \mathcal{M}(N, \tilde{\pi}^N).$$

Then by (b):

$$\mathcal{M}\left(\, \mathcal{K}(N),\; \mathcal{K}(\tilde{\pi}^N) \,\right) \;=\; \mathcal{M}\,(\, \mathcal{K}(M),\; \mathcal{K}(\tilde{\mu})\,).$$

Hence c is an embedding $\mathcal{M}(\mathcal{F}(M, \tilde{\pi}^M)) \;\to\; \mathcal{M}(\mathcal{F}(N, \tilde{\pi}^N))$. I.e. $\mathcal{H}$ is welldefined. It is clear that $\mathcal{H}$ satisfies: $\mathcal{M} \circ \mathcal{F} \;=\; \mathcal{H} \circ \mathcal{M}$ and $\mathcal{I}_{\text{cms}} \circ \mathcal{H} \;=\; \mathcal{K} \circ \mathcal{I}_{\text{cms}}$.

Now we show that $\mathcal{H}$ is Γ-contracting. It is clear that $\Gamma \circ \mathcal{H} \;=\; \mathcal{H}$. Let $e : M \to N$ be an embedding, $\Gamma M = M$ and $\Gamma N = N$. Let c be as above. Then

$$\delta(e(M), N) \;=\; \Delta_N(e, c).$$

We assume that $\Delta_N(e, c) \;=\; 1/2^n$ for some $n \geq 0$. Then

$$d_N(\, e(c(\eta)),\; \eta\,) \;\leq\; \frac{1}{2^n}$$

for all $\eta \in N$. Hence by Lemma 4.6: $\pi_n^N \circ e \circ c \;=\; \pi_n^N$ for all $n \geq 0$. Then

$$\mathcal{K}(\pi_n^N) \circ \mathcal{K}(e) \circ \mathcal{K}(c) \;=\; \mathcal{K}(\pi_n^N)$$

for all $n \geq 0$. Let d denote the metric on $\mathcal{H}(N)$. Then by definition of $\mathcal{H}$:

$$d(\xi, \xi') \;=\; \inf \left\{ \frac{1}{2^{n+1}} \;:\; \mathcal{K}(\pi_n^N)(\xi) \;=\; \mathcal{K}(\pi_n^N)(\xi') \right\}$$

where $\inf \emptyset \;=\; 1$. Hence for all $\eta \in \mathcal{H}(N)$:

$$d\,(\, \mathcal{K}(e)\, (\mathcal{K}(c)(\eta))\, , \eta) \;\leq\; \frac{1}{2^{n+1}}$$

Since $\mathcal{H}(e) = \mathcal{K}(e)$ (or more precisely $\mathcal{I}_{\text{cms}}(\mathcal{H}(e)) = \mathcal{K}(e)$):

$$\Delta_{\mathcal{H}(N)}(\, \mathcal{H}(e),\ \mathcal{H}(c)\,) \ \leq\ \frac{1}{2^{n+1}} \ =\ \frac{1}{2} \cdot \delta(\, e(M),\ N\,)$$

We conclude:

$$\delta(\, \mathcal{H}(e)\,(\mathcal{H}(M)),\ \mathcal{H}(N)\,) \ \leq\ \Delta_{\mathcal{H}(N)}(\, \mathcal{H}(e), \mathcal{H}(c)) \ \leq\ \frac{1}{2} \cdot \delta(e(M), N).$$

□

Remark 4.10 There exist endofunctors $\mathcal{K}$ of SET^* and rank ordered sets $(M, \tilde{\pi})$ such that $\mathcal{K}(\tilde{\pi})$ is not a rank ordering on $\mathcal{K}(M)$. For instance consider the functor $M \mapsto \mathcal{P}_{\text{n}}(M)$, $f \mapsto \lambda A.f(A)$. Here $\mathcal{P}_{\text{n}}(\cdot)$ means the collection of all nonempty subsets of $(\cdot)$. Let X be a nonempty set. X^∞ (the collection of all finite or infinite sequences over X) with the empty sequence as basis point can be endowed with the rank ordering (π_n) where $\pi_n(\omega)$ is the n-th prefix of ω. Then

$$\mathcal{P}_{\text{n}}(\pi_n)(X^*) \ =\ \mathcal{P}_{\text{n}}(\pi_n)(X^\infty) \ \ \forall\, n \geq 0$$

where X^* means the set of all finite sequences over X. I.e. $\mathcal{P}_{\text{n}}(\tilde{\pi})$ is not a rank ordering on $\mathcal{P}_{\text{n}}(X^\infty)$.

4.3 Rank ordered cpo's

As in [7] we introduce the notion of a rank ordered cpo which means a cpo with a suitable rank ordering. In Lemma 4.17 we show that each locally continuous endofunctor $\mathcal{G}$ of $\mathbf{D}_\perp$ induces an endofunctor $\mathcal{G}_{\text{rank}}$ of rank ordered $\mathbf{D}$-objects such that the initial fixed point of D endowed with a suitable rank ordering is the initial fixed point of $\mathcal{G}_{\text{rank}}$.

Definition 4.11 *Let $(D, \sqsubseteq)$ be a cpo. A* `rank ordering` *on $(D, \sqsubseteq)$ is a rank ordering $\tilde{\pi} = (\pi_n)_{n\geq 0}$ on D (where the basis point of D is $\perp_D$) such that π_n is continuous and $\pi_n \sqsubseteq id_D$ for all $n \geq 0$. A* `rank ordered cpo` *is a tripel $(D, \sqsubseteq, \tilde{\pi})$ consisting of a cpo $(D, \sqsubseteq)$ and a rank ordering $\tilde{\pi}$ on $(D, \sqsubseteq)$.*

The following lemma shows that a rank orderings on cpo D induces a complete rank ordered set such that D and the associated metric space are compatible. The proof is omitted. It can be found in [7].

Lemma 4.12 *Let $(D, \sqsubseteq, \tilde{\pi})$ be a rank ordered cpo. Then $(\pi_n)_{n\geq 0}$ is monotone and*

$$\bigsqcup_{n\geq 0} \pi_n \ =\ id_D.$$

$(D, \tilde{\pi})$ is a complete rank ordered set and $(D, \sqsubseteq)$, $(D, d[\tilde{\pi}])$ are compatible.

Lemma 4.13 shows the converse of Lemma 4.12. It asserts that each complete rank ordered set induces a compatible rank ordered cpo.

Lemma 4.13 *Let $(M, \tilde{\pi})$ be a complete rank ordered set. Then*

$$\mathcal{D}(M, \tilde{\pi}) \;=\; (M, \sqsubseteq)$$

is a cpo and $\tilde{\pi}$ a rank ordering on $\mathcal{D}(M, \tilde{\pi})$ where the partial order $\sqsubseteq$ is given by:

$$\xi \sqsubseteq \eta \quad \Longleftrightarrow \quad \xi = \eta \;\vee\; \exists n \in \mathbb{N}_0 \; \xi = \pi_n(\eta)$$

$\mathcal{D}(M, \tilde{\pi})$ and $(M, d[\tilde{\pi}])$ are compatible.

If $(M, \tilde{\pi})$ and $(N, \tilde{\mu})$ are complete rank ordered sets and $f : M \to N$ is rank preserving function then f as a function $\mathcal{D}(M, \tilde{\pi}) \to \mathcal{D}(N, \tilde{\mu})$ is strict and continuous.

Proof: Let $D = \mathcal{D}(M, \tilde{\pi})$. It is easy to see that $\sqsubseteq$ is a partial order. Since

$$\xi_M \;=\; \pi_0(\xi) \;\sqsubseteq\; \xi$$

the basis point of M is the bottom element of D. We put

$$\pi_\infty(\xi) \;=\; \xi$$

for all $\xi \in M$. Then $\xi \sqsubseteq \xi'$ implies $\xi = \pi_m(\xi')$ for some $m \in \mathbb{N}_0 \cup \{\infty\}$. If $\xi \sqsubseteq \xi'$ and $\xi' \sqsubseteq \xi''$ then $\xi = \pi_k(\xi')$, $\xi' = \pi_l(\xi'')$ for some k and l. If $k > l$ then

$$\xi \;=\; \pi_k(\pi_l(\xi'')) \;=\; \pi_l(\xi'') \;=\; \xi' \;=\; \pi_l(\xi').$$

Hence whenever $(\xi_n)_{n\geq 0}$ is a monotone sequence in D then there exists a monotone sequence $(m_n)_{n\geq 0}$ in $\mathbb{N}_0 \cup \{\infty\}$ such that $\xi_n = \pi_{m_n}(\xi_{n+1})$.

<u>Claim 1</u>: Whenever (ξ_n) is a monotone sequence in D then (ξ_n) is a Cauchy sequence in M and the limit of (ξ_n) in M is the least upper bound in D.

<u>Proof</u>: Let (ξ_n) be a monotone sequence in D and (m_n) a monotone sequence in $\mathbb{N}_0 \cup \{\infty\}$ such that $\xi_n = \pi_{m_n}(\xi_{n+1}$. Then for all $l \geq n \geq 0$:

$$\xi_n \;=\; \pi_{m_n}(\xi_l)$$

We first consider the case that there is some subsequence $(m_{n_k})_{k\geq 0}$ such that

$$m_{n_0} \;<\; m_{n_1} \;<\; m_{n_2} \;<\; \ldots$$

Let K be a natural number. There is some $N_K \geq 0$ such that $m_n \geq K$ for all $n \geq N_K$. Therefore for all $l > n \geq N_K$:

$$\pi_K(\xi_n) \;=\; \pi_K(\;\pi_{m_n}(\xi_l)\;) \;=\; \pi_K(\xi_l)$$

and hence $d[\tilde{\pi}](\xi_n, \xi_l) \;\leq\; 1/2^K$ for all $l > n \geq N_K$. Hence (ξ_n) is a Cauchy sequence in M. Let $\xi = \lim \xi_n$. Then:

$$d[\tilde{\pi}](\xi_n, \xi) \;=\; \lim_{l\to\infty} d[\tilde{\pi}](\xi_n, \xi_l) \;\leq\; \frac{1}{2^K}$$

I.e. $\pi_K(\xi) = \pi_K(\xi_n)$ for all $n \geq N_K$. We show that ξ is the least upper bound of (ξ_n): If n is an arbitrary natural number and $K = m_n$, $l >$ min $\{N_K, n\}$ then

$$\xi_n = \pi_{m_n}(\xi_l) = \pi_K(\xi_l) = \pi_K(\xi) \sqsubseteq \xi.$$

If ξ' is also an upper bound of (ξ_n) then $\xi_n = \pi_{m'_n}(\xi')$ for some m'_n. As mentioned above we may assume that

$$m'_0 \leq m'_1 \leq m'_2 \leq \ldots$$

If there is some $m' \in I\!N_0 \cup \{\infty\}$ such that $m'_n = m'$ for almost all $n \geq 0$ then

$$\xi_n = \pi_{m'}(\xi')$$

for almost all n. Hence

$$\xi = \lim_{n\to\infty} \xi_n = \pi_{m'}(\xi') \sqsubseteq \xi'.$$

Otherwise for each natural number K there is some $N'_K \geq 0$ such that $m'_n \geq K$ for all $n \geq N'_K$. Then for all $n \geq$ max $\{N_K, N'_K\}$:

$$\pi_K(\xi) = \pi_K(\xi_n) = \pi_K(\ \pi_{m'_n}(\xi')\ ) = \pi_K(\xi').$$

Hence $d[\tilde{\pi}](\xi, \xi') = 0$ and therefore $\xi = \xi'$.

Now we consider the case that there exists some $m \in I\!N_0 \cup \{\infty\}$ such that $m_n = m$ for almost all $n \geq 0$. Then there exists some $n_0 \geq 0$ such that $m_n = m$ for all $n \geq n_0$. Then

$$\xi_n = \pi_m(\xi_{n+1}) = \pi_m(\pi_m(\xi_{n+2})) = \pi_m(\xi_{n+2}) = \xi_{n+1}$$

for all $n \geq n_0$. Hence $\xi_{n_0} = \xi_{n_0+1} = \xi_{n_0+2} = \ldots$ is the limit and least upper bound of (ξ_n).

Claim 2: D is a cpo and D, M are compatible.

Proof: follows immediately by Claim 1.

Claim 3: $\tilde{\pi}$ is a rank ordering on the cpo D.

Proof: Since $\tilde{\pi}$ is a rank ordering on M and since $\pi_0(\xi) = \xi_M = \perp$ and $\pi_n \sqsubseteq id_D$ we only have to show that the functions π_k are continuous. Let $k \geq 0$ and let (ξ_n) be a monotone sequence in D and $\xi = \bigsqcup \xi_n$. By Claim 1:

$$\lim_{n\to\infty} \xi_n = \xi.$$

Hence for each $k \geq 0$ there is some $n_0 \geq 0$ with $d[\tilde{\pi}](\xi, \xi_n) \leq 1/2^k$ for all $n \geq n_0$. Then $\pi_k(\xi) = \pi_k(\xi_n)$ for all $n \geq n_0$ and therefore

$$\pi_k(\xi) = \bigsqcup_{n\geq 0} \pi_k(\xi_n).$$

Claim 4: If $(M, \tilde{\pi})$ and $(N, \tilde{\mu})$ are complete rank ordered sets and $f : M \to N$ is rank preserving function then f as a function $\mathcal{D}(M, \tilde{\pi}) \to \mathcal{D}(N, \tilde{\mu})$ is strict and continuous.

Proof: f is strict since $f \circ \pi_0 = \mu_0 \circ f$. We show that f is monotone: If $\xi \sqsubseteq \xi'$, $\xi \neq \xi'$ then $\xi = \pi_n(\xi')$ for some $n \geq 0$. Hence

$$f(\xi) = f(\pi_n(\xi')) = \mu_n(f(\xi')) \sqsubseteq f(\xi').$$

Now we show that f is continuous: Let (ξ_n) be a monotone sequence in $\mathcal{D}(M, \tilde{\pi})$ and $\xi = \bigsqcup \xi_n$. Since f is monotone the sequence $(f(\xi_n))$ is monotone. By Claim 1:

$$\lim_{n \to \infty} \xi_n = \xi, \quad \lim_{n \to \infty} f(\xi_n) = \bigsqcup_{n \geq 0} f(\xi_n)$$

Since f is non-distance-increasing (Lemma 4.6) we have:

$$f(\xi) = \lim_{n \to \infty} f(\xi_n) = \bigsqcup_{n \geq 0} f(\xi_n)$$

Hence f is continuous. □

Definition 4.14 *RankD denotes the category of rank ordered* **D***-objects. The morphism in RankD are rank preserving* **D***-morphisms.*

$$\mathcal{J}^{\text{rank}}_{\text{cpo}} \; : \; \mathit{Rank}\mathbf{D} \to \mathit{CPO}_{\perp}$$

$$\mathcal{J}^{\text{rank}}_{\text{cro}} \; : \; \mathit{Rank}\mathbf{D} \to \mathit{CRankSET}^{*}$$

denote the forgetful functors. $\mathit{Rank}\mathbf{D}^E$ *denotes the category of rank ordered* **D***-objects and* $\mathbf{D}^E$*-morphisms* $< e, c >$ *such* e *and* c *are rank preserving.*

It is clear that $\{\perp\}$ (considered as a rank ordered cpo) is the initial object in $\mathit{Rank}\mathbf{D}$ and $\mathit{Rank}\mathbf{D}^E$.

Lemma 4.15 *Each tower in* $\mathit{Rank}\mathbf{D}^E$ *has an initial cone.*

More precisely: If $((D_n, \tilde{\pi}^{(n)}), \iota_n)$ *is a tower in* $\mathit{Rank}\mathbf{D}^E$ *and* (D, λ_n) *the initial cone of the tower* (D_n, ι_n) *in* $\mathbf{D}^E$ *then there exists a rank ordering* $\tilde{\pi}$ *on* D *such that* $((D, \tilde{\pi}), \lambda_n)$ *is the initial cone of* $((D_n, \tilde{\pi}^{(n)}), \iota_n)$ *in* $\mathit{Rank}\mathbf{D}^E$.

Proof: Let $((D_n, \tilde{\pi}^{(n)}), \iota_n)$ be a tower in $\mathit{Rank}\mathbf{D}^E$ and $\iota_n = < e_n, c_n >$. Let (D, λ_n), $\lambda_n = < h_n, b_n >$, be the initial cone of the tower (D_n, ι_n) in $\mathbf{D}^E$. Let $e_{n,k} : D_n \to D_k$ be defined as in the proof of Lemma 3.11. $\sqsubseteq_n$ denotes the underlying partial order on D_n. We use the following properties of the initial cone (D, λ_n):

(I) $b_k \circ h_n = e_{n,k}$ for all n, $k \geq 0$.

(II) If C is a cpo and $f : D \to C$ a function then f is continuous if and only if for all $n \geq 0$ the function $f \circ h_n : D_n \to C$ is continuous.

(III) $\bigsqcup (h_k \circ b_k) = id_D$.

<u>Claim 0</u>: $\pi^{(j)}_n \circ e_{i,j} = e_{i,j} \circ \pi^{(i)}_n$ for all i, j, $n \geq 0$.

<u>Proof</u>: easy verification, uses the fact that e_k and c_k are rank preserving.

<u>Claim 1</u>: For each $k \geq 0$ the sequence $(h_n \circ \pi^{(n)}_k \circ b_n)_{n \geq 0}$ is monotone.

Proof: For all $n \geq 0$ we have: $h_n = h_{n+1} \circ e_n$ and $b_n = c_n \circ b_{n+1}$. Since c_n is rank preserving we have:

$$\pi_k^{(n)} \circ c_n = c_n \circ \pi_k^{(n+1)}$$

Therefore

$$h_n \circ \pi_k^{(n)} \circ b_n = h_{n+1} \circ e_n \circ \pi_k^{(n)} \circ c_n \circ b_{n+1} = h_{n+1} \circ e_n \circ c_n \circ \pi_k^{(n+1)} \circ b_{n+1}.$$

Since $e_n \circ c_n \sqsubseteq_{n+1} id_{D_{n+1}}$ we get: $h_n \circ \pi_k^{(n)} \circ b_n \sqsubseteq h_{n+1} \circ \pi_k^{(n+1)} \circ b_{n+1}$.

Definition: Let $\tilde{\pi} = (\pi_k)_{k \geq 0}$ be given by:

$$\pi_k : D \to D, \quad \pi_k = \bigsqcup_{n \geq 0} h_n \circ \pi_k^{(n)} \circ b_n$$

Claim 2: $h_j \circ \pi_k^{(j)} = \pi_k \circ h_j$ for all $j, k \geq 0$

Proof: Let $j \geq 0$. By Claim 0 and (I) we get for all $n \geq 0$:

$$h_n \circ \pi_k^{(n)} \circ b_n \circ h_j = h_n \circ \pi_k^{(n)} \circ e_{j,n} = h_n \circ e_{j,n} \circ \pi_k^{(j)} = (h_n \circ b_n) \circ (h_j \circ \pi_k^{(j)}).$$

By (III) we get:

$$\pi_k \circ h_j = \bigsqcup_{n \geq 0} h_n \circ \pi_k^{(n)} \circ b_n \circ h_j = \bigsqcup_{n \geq 0} (h_n \circ b_n) \circ (h_j \circ \pi_k^{(j)}) = h_j \circ \pi_k^{(j)}.$$

Claim 3: π_k is continuous for all $k \geq 0$.

Proof: Since h_j and $\pi_k^{(j)}$ are continuous the functions $\pi_k \circ h_j = h_j \circ \pi_k^{(j)}$ are continuous for all $j \geq 0$ (Claim 2). Therefore π_k is continuous (by (II)).

Claim 4: $\pi_n \circ \pi_m = \pi_m \circ \pi_n = \pi_k$ where $k = \min\{n, m\}$.

Proof: By Claim 0 and (I) we get:

$$h_i \circ \pi_n^{(i)} \circ b_i \circ h_j \circ \pi_m^{(j)} \circ b_j = h_i \circ \pi_n^{(i)} \circ e_{j,i} \circ \pi_m^{(j)} \circ b_j$$

$$= h_i \circ e_{j,i} \circ \pi_n^{(j)} \circ \pi_m^{(j)} \circ b_j = (h_i \circ b_i) \circ (h_j \circ \pi_k^{(j)} \circ b_j)$$

Since $h_i \circ \pi_n^{(i)} \circ b_i$ and $h_i \circ b_i$ are continuous we get:

$$h_i \circ \pi_n^{(i)} \circ b_i \circ \pi_m = (h_i \circ \pi_n^{(i)} \circ b_i) \circ \left(\bigsqcup_{j \geq 0} h_j \circ \pi_m^{(j)} \circ b_j \right)$$

$$= \bigsqcup_{j \geq 0} \left(h_i \circ \pi_n^{(i)} \circ b_i \circ h_j \circ \pi_m^{(j)} \circ b_j \right) = \bigsqcup_{j \geq 0} (h_i \circ b_i) \circ (h_j \circ \pi_k^{(j)} \circ b_j)$$

$$= (h_i \circ b_i) \circ \left(\bigsqcup_{j \geq 0} h_j \circ \pi_k^{(j)} \circ b_j \right) = (h_i \circ b_i) \circ \pi_k.$$

Since $\bigsqcup(h_i \circ b_i) = id_D$ (see (III)) we get:

$$\pi_n \circ \pi_m = \bigsqcup_{i \geq 0} h_i \circ \pi_n^{(i)} \circ b_i \circ \pi_m = \bigsqcup_{i \geq 0} (h_i \circ b_i) \circ \pi_k = \pi_k.$$

Claim 5: $\bigsqcup \pi_k = id_D$

Proof: In each cpo we have

$$\bigsqcup_{k \geq 0} \bigsqcup_{n \geq 0} a_{k,n} = \bigsqcup_{n \geq 0} \bigsqcup_{k \geq 0} a_{k,n}$$

where $(a_{k,n})_{k \geq 0}$ and $(a_{k,n})_{n \geq 0}$ are monotone. Since $(\tilde{\pi}^{(n)})$ is a rank ordering on D_n we have:

$$\bigsqcup_{k \geq 0} \pi_k^{(n)} \circ b_n = b_n$$

Since h_n is continuous we get by (III):

$$\bigsqcup_{k \geq 0} \pi_k = \bigsqcup_{k \geq 0} \bigsqcup_{n \geq 0} h_n \circ \pi_k^{(n)} \circ b_n = \bigsqcup_{n \geq 0} \bigsqcup_{k \geq 0} h_n \circ \pi_k^{(n)} \circ b_n$$

$$= \bigsqcup_{n \geq 0} h_n \circ \left(\bigsqcup_{k \geq 0} \pi_k^{(n)} \circ b_n \right) = \bigsqcup_{n \geq 0} h_n \circ b_n = id_D$$

Claim 6: $b_n \circ \pi_m = \pi_m^{(n)} \circ b_n$ for all $m, n \geq 0$.

Proof: It is easy to see that $b_n \circ h_k \circ b_k = b_n$ for all $k \geq n$. Hence by Claim 0:

$$b_n \circ h_k \circ \pi_m^{(k)} \circ b_k = \pi_m^{(n)} \circ b_n$$

Therefore:

$$b_n \circ \pi_m = b_n \circ \left(\bigsqcup_{k \geq 0} h_k \circ \pi_m^{(k)} \circ b_k \right)$$

$$= \bigsqcup_{k \geq 0} b_n \circ h_k \circ \pi_m^{(k)} \circ b_k = \pi_m^{(n)} \circ b_n.$$

Claim 7: $\tilde{\pi}$ is a rank ordering on D and h_j, b_j are rank preserving.

Proof: Since $\tilde{\pi}^{(n)}$ are rank orderings $\pi_0^{(n)} = \lambda\xi.\bot_{D_n}$. Since h_n is strict we get:

$$\pi_0(\xi) = \bigsqcup_{n \geq 0} h_n \left(\pi_0^{(n)}(b_n(\xi)) \right) = \bigsqcup_{n \geq 0} h_n(\bot_{D_n}) = \bot_D.$$

By Claim 3-5 we get that $\tilde{\pi}$ is a rank ordering on D. By Claim 2 and 6 h_j and b_j rank preserving.

Claim 8: Let (C, κ_n) be a cone of (D_n, ι_n), $\kappa_n = \langle g_n, a_n \rangle$, and let $\tilde{\mu} = (\mu_k)_{k \geq 0}$ be a rank ordering on C such that g_n and a_n are rank preserving w.r.t. $\tilde{\pi}^{(n)}$ and $\tilde{\mu}$. Let $\Phi = \langle g, a \rangle$ be the unique morphism in $\mathbf{D}^E$ with $\Phi \circ \lambda_n = \kappa_n$. Then g and a are rank preserving w.r.t. $\tilde{\pi}$ and $\tilde{\mu}$.

Proof: We have to show that for all $k \geq 0$:

$$g \circ \pi_k = \mu_k \circ g, \quad a \circ \mu_k = \pi_k \circ a$$

Since $g_n = g \circ h_n$ is rank preserving we get:

$$g \circ h_n \circ \pi_k^{(n)} \circ b_n = g_n \circ \pi_k^{(n)} \circ b_n = \mu_k \circ g_n \circ b_n = \mu_k \circ g \circ h_n \circ b_n$$

Since $\mu_k \circ g$ is continuous we get by (III):

$$\bigsqcup_{n \geq 0} \mu_k \circ g \circ h_n \circ b_n = (\mu_k \circ g) \circ \left(\bigsqcup_{n \geq 0} h_n \circ b_n \right) = \mu_k \circ g$$

Since g is continuous we conclude:

$$g \circ \pi_k = g \circ \left(\bigsqcup_{n \geq 0} h_n \circ \pi_k^{(n)} \circ b_n \right) = \bigsqcup_{n \geq 0} g \circ h_n \circ \pi_k^{(n)} \circ b_n$$

$$= \bigsqcup_{n \geq 0} \mu_k \circ g \circ h_n \circ b_n = \mu_k \circ g$$

Since $b_n \circ a = a_n$ and since a_n is rank preserving we have:

$$\pi_k \circ a = \bigsqcup_{n \geq 0} h_n \circ \pi_k^{(n)} \circ b_n \circ a$$

$$= \bigsqcup_{n \geq 0} h_n \circ a_n \circ \mu_k = a \circ \mu_k.$$

Claim 9: $((D, \tilde{\pi}), \lambda_n)$ is the initial cone of the tower $((D_n, \tilde{\pi}^{(n)}), \iota_n)$ in $Rank\mathbf{D}^E$.

Proof: By Claim 7 $((D, \tilde{\pi}), \lambda_n)$ is a cone. If $((C, \tilde{\mu}), \kappa_n)$ is also a cone then (C, κ_n) is a cone of (D_n, ι_n) in $\mathbf{D}^E$. By Claim 8 the unique arrow $\Phi : D \to C$ in $\mathbf{D}^E$ with $\Phi \circ \lambda_n = \kappa_n$ is an arrow $(D, \tilde{\pi}) \to (C, \tilde{\mu})$ in $Rank\mathbf{D}^E$.

If $\Phi' : (D, \tilde{\pi}) \to (C, \tilde{\mu})$ is also an arrow in $Rank\mathbf{D}^E$ with $\Phi' \circ \lambda_n = \kappa_n$ then $\Phi' : D \to C$ is an arrow in $\mathbf{D}^E$. By the uniqueness of Φ as an arrow in $\mathbf{D}^E$ with $\Phi \circ \lambda_n = \kappa_n$ we conclude: $\Phi = \Phi'$. □

Lemma 4.16 *Let $\mathcal{G} : \mathbf{D}_\perp \to \mathbf{D}_\perp$ be a locally continuous functor. Then:*

(a) If $(D, \tilde{\pi})$ is a rank ordered cpo then $(\mathcal{G}(D), \mathcal{G}(\tilde{\pi}))$ is a rank ordered cpo where

$$\mathcal{G}(\tilde{\pi}) = (\pi'_n)_{n \geq 0}, \quad \pi'_0 = \lambda \xi . \perp_{\mathcal{G}(D)}, \quad \pi'_{n+1} = \mathcal{G}(\pi_n).$$

(b) If $(D, \tilde{\pi})$, $(E, \tilde{\mu})$ are rank ordered cpo's and $f : D \to E$ is continuous and rank preserving then $\mathcal{G}(f) : \mathcal{G}(D) \to \mathcal{G}(E)$ is rank preserving w.r.t. $\mathcal{G}(\tilde{\pi})$ and $\mathcal{G}(\tilde{\mu})$.

Proof: Since π_n are $\mathbf{D}_\perp$-morphisms π'_n is strict and continuous. Since $\mathcal{G}$ is locally continuous:

$$\bigsqcup_{n \geq 0} \pi'_n = \bigsqcup_{n \geq 1} \mathcal{G}(\pi_n) = \mathcal{G}\left(\bigsqcup_{n \geq 1} \pi_n \right) = id_{\mathcal{G}(D)}.$$

If $\xi, \eta \in \mathcal{G}(D)$ and $\pi'_n(\xi) = \pi'_n(\eta)$ for all $n \geq 0$ then

$$\xi = \bigsqcup_{n\geq 0} \pi'_n(\xi) = \bigsqcup_{n\geq 0} \pi'_n(\eta) = \eta.$$

As in the proof of Lemma 4.9 it can be shown that $\mathcal{G}(\tilde{\pi})$ is a rank ordering on $\mathcal{G}(D)$ and that for each rank preserving **D**-morphism f the function $\mathcal{G}(f)$ is rank preserving w.r.t. $\mathcal{G}(\tilde{\pi})$ and $\mathcal{G}(\tilde{\mu})$. □

Lemma 4.17 *Let $\mathcal{G} : \mathbf{D}_\perp \to \mathbf{D}_\perp$ be a locally continuous functor. Then*

$$\mathcal{G}_{\mathsf{rank}} : Rank\mathbf{D} \to Rank\mathbf{D}, \quad \mathcal{G}_{\mathsf{rank}}(D, \tilde{\pi}) = (\mathcal{G}(D), \mathcal{G}(\tilde{\pi})), \quad \mathcal{G}_{\mathsf{rank}}(f) = \mathcal{G}(f),$$

is a welldefined functor and

$$\mathcal{J}^{\mathsf{rank}}_{\mathsf{cpo}} \circ \mathcal{G}_{\mathsf{rank}} = \mathcal{G} \circ \mathcal{J}^{\mathsf{rank}}_{\mathsf{cpo}}.$$

Let (D, h) the initial fixed point of $\mathcal{G}$. Then there exists a rank ordering $\tilde{\pi}$ on D such that $(\ (D, \tilde{\pi}),\ h\ )$ is the initial fixed point of $\mathcal{G}_{\mathsf{rank}}$.

Proof: $\mathcal{G}_{\mathsf{rank}}$ is welldefined by Lemma 4.16. It is clear that $\mathcal{J}^{\mathsf{rank}}_{\mathsf{cpo}} \circ \mathcal{G}_{\mathsf{rank}} = \mathcal{G} \circ \mathcal{J}^{\mathsf{rank}}_{\mathsf{cpo}}$. Let $\mathcal{G}^E_{\mathsf{rank}} : Rank\mathbf{D}^E \to Rank\mathbf{D}^E$ be given by:

$$\mathcal{G}^E_{\mathsf{rank}}(D, \tilde{\pi}) = \mathcal{G}_{\mathsf{rank}}(D, \tilde{\pi}), \quad \mathcal{G}^E_{\mathsf{rank}}(< e, c >) = < \mathcal{G}_{\mathsf{rank}}(e),\ \mathcal{G}_{\mathsf{rank}}(c) >$$

Using Lemma 2.1 and Lemma 4.15 it is easy to show that the initial fixed point (D, λ) of $\mathcal{G}^E$ can be endowed with a rank ordering $\tilde{\pi}$ such that $((D, \tilde{\pi}), \lambda)$ is the initial fixed point of $\mathcal{G}^E_{\mathsf{rank}}$. Then $\lambda = < h, h^{-1} >$ for some isomorphism $h : (D, \tilde{\pi}) \to \mathcal{G}_{\mathsf{rank}}(D, \tilde{\pi})$ in $Rank\mathbf{D}^E$. Then (D, h) is the initial fixed point of $\mathcal{G}$, $((D, \tilde{\pi}), h)$ a fixed point of $\mathcal{G}_{\mathsf{rank}}$.

We show that $((D, \tilde{\pi}), h)$ is the initial fixed point of $\mathcal{G}_{\mathsf{rank}}$. Let $((C, \tilde{\mu}), i)$ be a fixed point of $\mathcal{G}_{\mathsf{rank}}$. Then (C, i) is a fixed point of $\mathcal{G}$. Hence whenever $F : (D, \tilde{\pi}) \to (C, \tilde{\mu})$ is an arrow in $Rank\mathbf{D}$ such that $i \circ F = \mathcal{G}_{\mathsf{rank}}(F) \circ h$ then F is the unique arrow $D \to C$ in $\mathbf{D}_\perp$ with $i \circ F = \mathcal{G}(F) \circ h$. Hence there exists at most one arrow $F : (D, \tilde{\pi}) \to (C, \tilde{\mu})$ with $i \circ F = \mathcal{G}_{\mathsf{rank}}(F) \circ h$.

Let $\kappa = < i, i^{-1} >$. Then $((C, \tilde{\mu}), \kappa)$ is a fixed point of $\mathcal{G}^E_{\mathsf{rank}}$. Hence there exists a unique morphism $\Phi : (D, \tilde{\pi}) \to (C, \tilde{\mu})$ in $Rank\mathbf{D}^E$ satisfying

$$\kappa \circ \Phi = \mathcal{G}^E_{\mathsf{rank}}(\Phi) \circ \lambda.$$

Let F be the first component of Φ. Then $F : (D, \tilde{\pi}) \to (C, \tilde{\mu})$ is a morphism in $Rank\mathbf{D}$ satisfying $i \circ F = \mathcal{G}_{\mathsf{rank}}(F) \circ h$. □

4.4 Proof of Theorem 5

We suppose that $\mathcal{G} : \mathbf{D}_\perp \to \mathbf{D}_\perp$ is locally continuous and $\mathcal{K} : SET^* \to SET^*$ is a functor with $\mathcal{I}_{\mathsf{cpo}} \circ \mathcal{G} = \mathcal{K} \circ \mathcal{I}_{\mathsf{cpo}}$. Let (D, h) be the initial fixed point of $\mathcal{G}$.

<u>Claim 1</u>: $\mathcal{K}$ satisfies the conditions of Lemma 4.9, i.e.: if $(M, \tilde{\pi})$ is a complete rank ordered set then:

(I) $\mathcal{K}(\pi_n)(\xi) = \mathcal{K}(\pi_n)(\eta)$ for all $n \geq 0$ implies $\xi = \eta$.

(II) Whenever (η_n) is a sequence in $\mathcal{K}(M)$ such that $\mathcal{K}(\pi_n)(\eta_n) = \mathcal{K}(\pi_n)(\eta_m)$ for all $m \geq n \geq 0$ then there exists $\eta \in \mathcal{K}(M)$ such that for all $n \geq 0$:

$$\mathcal{K}(\pi_n)(\eta) = \mathcal{K}(\pi_n)(\eta_n)$$

Proof: Let $(M, \tilde{\pi})$ be a complete rank ordered set. By Lemma 4.13: $\tilde{\pi}$ is a rank ordering on the cpo $\mathcal{D}(M, \tilde{\pi})$. Then by Lemma 4.16: $\mathcal{K}(\tilde{\pi}) = \mathcal{G}(\tilde{\pi})$ is a rank ordering on the cpo $\mathcal{G}(\ \mathcal{D}(M, \tilde{\pi})\ )$. In particular $\mathcal{K}(\tilde{\pi})$ is a rank ordering on the pointed set

$$\mathcal{I}_{\mathsf{cpo}}\left(\mathcal{G}(\ \mathcal{D}(M, \tilde{\pi})\ )\right) = \mathcal{K}\left(\mathcal{I}_{\mathsf{cpo}}(\ \mathcal{D}(M, \tilde{\pi})\ )\right) = \mathcal{K}(M)$$

and hence satisfies (I). If (η_n) is a sequence in $\mathcal{K}(M)$ with $\mathcal{K}(\pi_n)(\eta_n) = \mathcal{K}(\pi_n)(\eta_m)$ for all $m \geq n \geq 0$ then the sequence $(\ \mathcal{K}(\pi_m)(\eta_m)\ )_{m \geq 0}$ is monotone in $\mathcal{G}(\ \mathcal{D}(M, \tilde{\pi})\ )$. Let

$$\eta = \bigsqcup_{m \geq 0} \mathcal{K}(\pi_m)(\eta_m).$$

Since $\mathcal{K}(\pi_n) = \mathcal{I}_{\mathsf{cpo}}(\ \mathcal{G}(\pi_n)\ )$ and $\mathcal{G}(\pi_n)\ :\ \mathcal{G}(\mathcal{D}(M, \tilde{\pi})) \to \mathcal{G}(\mathcal{D}(M, \tilde{\pi})$ is continuous and since

$$\mathcal{K}(\pi_n) \circ \mathcal{K}(\pi_m) = \mathcal{K}(\pi_n)$$

for all $m \geq n$ we have:

$$\mathcal{K}(\pi_n)(\eta) = \bigsqcup_{m \geq 0} \mathcal{K}(\pi_n)(\ \mathcal{K}(\pi_m)(\eta_m)\ ) = \mathcal{K}(\pi_n)(\eta_n)$$

$\square$

Because of Claim 1 we may apply Lemma 4.9: Let

$$\mathcal{F} : \mathit{CRankSET}^{*} \to \mathit{CRankSET}^{*}, \quad \mathcal{H} : \mathit{CUM}_{\mathsf{e}}^{*} \to \mathit{CUM}_{\mathsf{e}}^{*}$$

be the liftings of $\mathcal{K}$ as in Lemma 4.9 with

$$\mathcal{I}_{\mathsf{rank}} \circ \mathcal{F} = \mathcal{K} \circ \mathcal{I}_{\mathsf{rank}}, \quad \mathcal{I}_{\mathsf{cms}} \circ \mathcal{H} = \mathcal{K} \circ \mathcal{I}_{\mathsf{cms}}.$$

$\mathcal{H}$ is Γ-contracting and hence has an unique fixed point (Lemma 4.3). $\mathcal{F}$ preserves embeddings and can be considered as a functor $\mathit{CRankSET}_{\mathsf{e}}^{*} \to \mathit{CRankSET}_{\mathsf{e}}^{*}$. Then

$$\mathcal{M} \circ \mathcal{F} = \mathcal{H} \circ \mathcal{M}.$$

Claim 2: Let $M = (A, d[\tilde{\pi}])$ where $A = \mathcal{I}_{\mathsf{cpo}}(D)$. Then:

(a) $((A, \tilde{\pi}), h)$ is a fixed point of $\mathcal{F}$.

(b) (M, h) is the unique fixed point of $\mathcal{H}$.

(c) For each monotone Cauchy sequence (ξ_n) in A: $\lim \xi_n = \bigsqcup \xi_n$

Proof: Let $\mathcal{G}_{\mathsf{rank}} : \mathit{Rank}\mathbf{D} \to \mathit{Rank}\mathbf{D}$ be as in Lemma 4.17 and let $\tilde{\pi}$ be a rank ordering on D such that $((D, \tilde{\pi}), h)$ is the initial fixed point of $\mathcal{G}_{\mathsf{rank}}$ (Lemma 4.17).

(a) It is easy to see that $\mathcal{J}^{\mathsf{rank}}_{\mathsf{cro}} \circ \mathcal{G}_{\mathsf{rank}} = \mathcal{F} \circ \mathcal{J}^{\mathsf{rank}}_{\mathsf{cro}}$. Since

$$(A, \tilde{\pi}) = \mathcal{J}^{\mathsf{rank}}_{\mathsf{cro}}(D, \tilde{\pi})$$

we get by Lemma 2.2: $((A, \tilde{\pi}), h)$ is a fixed point of $\mathcal{F}$.

(b) Since $M = (A, d[\tilde{\pi}]) = \mathcal{M}(A, \tilde{\pi})$ and since $\mathcal{M} \circ \mathcal{F} = \mathcal{H} \circ \mathcal{M}$ we get by Lemma 2.2 that (M, h) is a fixed point of $\mathcal{H}$. Since $\mathcal{H}$ is Γ-contracting (M, h) is the unique fixed point of $\mathcal{H}$ (Lemma 4.3).

(c) follows immediately by Lemma 4.12. □

$$\begin{tikzcd}
\mathit{Rank}\mathbf{D} \arrow[rrr, "\mathcal{G}_{\mathsf{rank}}"] \arrow[dr, "\mathcal{J}^{\mathsf{rank}}_{\mathsf{cpo}}"] \arrow[dddd, "\mathcal{J}^{\mathsf{rank}}_{\mathsf{cro}}"'] & & & \mathit{Rank}\mathbf{D} \arrow[dl, "\mathcal{J}^{\mathsf{rank}}_{\mathsf{cpo}}"'] \arrow[dddd, "\mathcal{J}^{\mathsf{rank}}_{\mathsf{cro}}"] \\
& \mathbf{D}_\perp \arrow[r, "\mathcal{G}", "\text{locally continuous}"'] \arrow[d, "\mathcal{I}_{\mathsf{cpo}}"'] & \mathbf{D}_\perp \arrow[d, "\mathcal{I}_{\mathsf{cpo}}"] & \\
& \mathit{SET}^* \arrow[r, "\mathcal{K}"] & \mathit{SET}^* & \\
& \mathit{CUM}^*_{\mathsf{e}} \arrow[r, "\mathcal{H}", "\Gamma-\text{contracting}"'] \arrow[u, "\mathcal{I}_{\mathsf{cms}}"] & \mathit{CUM}^*_{\mathsf{e}} \arrow[u, "\mathcal{I}_{\mathsf{cms}}"'] & \\
\mathit{CRankSET}^* \arrow[rrr, "\mathcal{F}"] \arrow[ur, "\mathcal{M}"'] & & & \mathit{CRankSET}^* \arrow[ul, "\mathcal{M}"]
\end{tikzcd}$$

5 Examples

We give examples for functors which satisfy the conditions of Theorem 1-5. These functors are built from production systems that contain:

- identity functors *id* and in the metric case identity functors id_c which multiply the metric by a positive factor $c < 1$
- constant functors $const_A$ that assign to each object a fixed object A
- function space functors $X \rightarrow \mathcal{F}$ where all functions from the set X into the underlying set of $\mathcal{F}(\cdot)$ are considered
- functors of the form $X \otimes \mathcal{F}$ which assign to each object A an object of the form

$$\{\perp\} \cup X \times \mathcal{F}(A)$$

- product functors $\mathcal{F}_1 \times \mathcal{F}_2$
- functors of the form $\mathcal{F}_1 \oplus \mathcal{F}_2$ which assign to each A an object of the form

$$\{\perp\} \uplus \mathcal{F}_1(A) \uplus \mathcal{F}_2(A)$$

  where $\uplus$ means disjoint union.

In the cases where the morphisms of the underlying categories are embedding projection pairs we also use function space functors of the form

$$\mathcal{F}_1 \ \rightarrow^E \ \mathcal{F}_2$$

which assign to each object A an object of the form $\mathcal{F}_1(A) \rightarrow \mathcal{F}_2(A)$ (i.e. an object which consists of all functions from the underlying set of $\mathcal{F}_1(A)$ into the underlying set of $\mathcal{F}_2(A)$). The exact definitions of these functors are given in the appendix.

Lemma 5.1 *$id : CPO_\perp \rightarrow CPO_\perp$ and $id_c : CMS \rightarrow CMS$ (or $id_c : CMS^* \rightarrow CMS^*$) are compatible. If D is a cpo and M a (pointed) complete metric space such that D and M are compatible then $const_D : CPO_\perp \rightarrow CPO_\perp$ and $const_M : CMS \rightarrow CMS$ (resp. $const_M : CMS^* \rightarrow CMS^*$) are compatible.*

If $\mathcal{G}$, $\mathcal{G}_i : CPO_\perp \rightarrow CPO_\perp$ and $\mathcal{H}$, $\mathcal{H}_i : CMS \rightarrow CMS$ (or $\mathcal{H}$, $\mathcal{H}_i : CMS^* \rightarrow CMS^*$) are functors such that $\mathcal{G}$, $\mathcal{H}$ resp. $\mathcal{G}_i$, $\mathcal{H}_i$ are compatible then also

- *$X \rightarrow \mathcal{G}$ and $X \rightarrow \mathcal{H}$*
- *$X \otimes \mathcal{G}$ and $X \otimes \mathcal{H}$*
- *$\mathcal{G}_1 \times \mathcal{G}_2$ and $\mathcal{H}_1 \times \mathcal{H}_2$*
- *$\mathcal{G}_1 \oplus \mathcal{G}_2$ and $\mathcal{H}_1 \oplus \mathcal{H}_2$*
- *$\mathcal{G}^E$ and $\mathcal{H}^E$*

are compatible. If $\mathcal{G}_i : CPO^{\mathrm{E}} \rightarrow CPO^{\mathrm{E}}$, $\mathcal{H}_i : CMS^E \rightarrow CMS^E$ (or $\mathcal{H}_i : CMS^{*E} \rightarrow CMS^{*E}$) are compatible, $i = 1, 2$, then also $\mathcal{G}_1 \rightarrow^E \mathcal{G}_2$ and $\mathcal{H}_1 \rightarrow^E \mathcal{H}_2$ are compatible.

Theorem 1 and 3 can be applied to all functors $\mathcal{G} : CPO^{\mathrm{E}} \rightarrow CPO^{\mathrm{E}}$, $\mathcal{H} : CMS^E \rightarrow CMS^E$ or $\mathcal{H} : CMS^{*E} \rightarrow CMS^{*E}$ which are given by the following production system and which satisfy $c(\mathcal{H}) < 1$.

$$\begin{aligned}
(\mathcal{G}, \mathcal{H}) \ ::= \quad & (id, id_c) \ \mid \ (const_D, const_M) \ \mid \\
& (X \rightarrow \mathcal{G}, X \rightarrow \mathcal{H}) \ \mid \ (X \otimes \mathcal{G}, X \otimes \mathcal{H}) \ \mid \\
& (\mathcal{G}_1 \times \mathcal{G}_2, \mathcal{H}_1 \times \mathcal{H}_2) \ \mid \ (\mathcal{G}_1 \oplus \mathcal{G}_2, \mathcal{H}_1 \oplus \mathcal{H}_2) \ \mid \\
& (\mathcal{G}_1 \rightarrow^E \mathcal{G}_2, \mathcal{H}_1 \rightarrow^E \mathcal{H}_2)
\end{aligned}$$

Here D and M are compatible. $c(\mathcal{H})$ is defined as in [5]:

$$c(id_c) = c, \quad c(const_M) \;=\; 0,$$

$$c(X \to \mathcal{H}) \;=\; c(X \otimes \mathcal{H}) \;=\; c(\mathcal{H}),$$

$$c(\mathcal{H}_1 \times \mathcal{H}_2) \;=\; c(\mathcal{H}_1 \oplus \mathcal{H}_2) \;=\; \max\{\, c(\mathcal{H}_1),\ c(\mathcal{H}_2) \,\},$$

$$c(\mathcal{H}_1 \to^E \mathcal{H}_2) \;=\; \max\{\, \infty \cdot c(\mathcal{H}_1),\ c(\mathcal{H}_2) \,\}.$$

with $\infty \cdot 0 \;=\; 0$ and $\infty \cdot c \;=\; \infty$ if $c > 0$. Removing the function space functor $\to^E$ from the production system of above we get examples for functors satisfying the conditions of Theorem 2 and 4.

Each functor $\mathcal{G} : CPO_\perp \to CPO_\perp$ which is given by the following production system satisfies the conditions of Theorem 5:

$$\mathcal{G} \;::=\; id \mid const_{D_0} \mid X \to \mathcal{G} \mid X \otimes \mathcal{G} \mid \mathcal{G}_1 \times \mathcal{G}_2 \mid \mathcal{G}_1 \oplus \mathcal{G}_2$$

The induced functors $\mathcal{H} : CUM_e^* \to CUM_e^*$ are given by:

$$\mathcal{H} \;::=\; id_{\frac{1}{2}} \circ \Gamma \mid const_{M_0} \mid X \to \mathcal{H} \mid X \otimes \mathcal{H} \mid \mathcal{H}_1 \times \mathcal{H}_2 \mid \mathcal{H}_1 \oplus \mathcal{H}_2$$

Here $M_0 \;=\; (A_0, d)$ where A_0 is the underlying set of D_0 and d is the discrete metric on A_0.

Example 5.2 $\{\perp\}$ is the initial fixed point of $\mathcal{G} = id$ as endofunctor of $CPO_\perp$ and the unique fixed point of the associated functor

$$\mathcal{H} \;=\; id_{\frac{1}{2}} \circ \Gamma : CUM_e^* \to CUM_e^*,\ \mathcal{H}(M,d) \;=\; \left(M, \frac{1}{2} \cdot \gamma \circ d\right).$$

$\{\perp\}$ is also the initial fixed point of the functor

$$\mathcal{G} = (X \to id) : CPO_\perp \to CPO_\perp$$

and the unique fixed point of the associated functor

$$\mathcal{H} : CUM_e^* \to CUM_e^*, \quad \mathcal{H}(M) \;=\; X \to id_{\frac{1}{2}}(\Gamma M)$$

The functor $\mathcal{G} \;=\; X \otimes id \;: CPO_\perp \to CPO_\perp$ induces the functor $\mathcal{H} : CUM_e^* \to CUM_e^*$ which is given by:

$$\mathcal{H}(M) \;=\; \{\perp\} \uplus X \times id_{\frac{1}{2}}(\Gamma M)$$

X^∞ (the collection of all finite or infinite sequences over X) endowed with the prefix ordering is the initial fixed point of $\mathcal{G}$. X^∞ endowed with the distance

$$d(\omega_1, \omega_2) \;=\; \inf \left\{ \frac{1}{2^n} \;:\; \omega_1[n] = \omega_2[n] \right\}$$

is the unique fixed point of $\mathcal{H}$. Here $\omega[n]$ is the n-th prefix of ω. $\square$

6 Conclusion

In Theorem 1-4 we have shown that domain equations which can be considered in the metric and in the cpo setting and which can be solved by the methods of [19] resp. [5, 14, 16] have (weakly) compatible solutions. In Theorem 5 we presented a method to define a corresponding metric domain equation for a giving cpo domain equation such that the solutions are compatible. The question arises whether in analogy to Theorem 5 domain equations in the metric setting can be lifted to domain equations for cpo's. Given functors

$$\mathcal{H} : CUM^* \to CUM^*, \quad \mathcal{K} : SET^* \to SET^*$$

such that $\mathcal{H}$ is locally contracting and $\mathcal{I}_{\mathsf{cms}} \circ \mathcal{H} = \mathcal{K} \circ \mathcal{I}_{\mathsf{cms}}$ a locally continuous functor

$$\mathcal{R} : RankCPO \to RankCPO$$

can be derived such that $\mathcal{R}$ has an initial fixed point $(D, \tilde{\pi})$ where D and the unique fixed point of $\mathcal{H}$ are compatible. The construction of $\mathcal{R}$ is as follows: $\mathcal{H}$ and $\mathcal{K}$ induces a functor

$$\mathcal{F} : CRankSET^* \to CRankSET^*$$

which is given by: $\mathcal{F}(M, \tilde{\pi}) = (\ \mathcal{K}(M),\ \mathcal{K}(\tilde{\pi})\ ),\ \mathcal{F}(f) = \mathcal{K}(f)$. In order to show that $\mathcal{K}(\tilde{\pi})$ is a rank ordering on $\mathcal{K}(M)$ we use the first part of Lemma 4.9 and the fact that (because $\mathcal{H}$ is locally contracting):

$$d\Big(\ id_{\mathcal{K}(M)},\ \mathcal{K}(\pi_n)\ \Big) \le d(\ id_M,\ \pi_n\ ) \le \frac{1}{2^n}$$

Hence whenever $\mathcal{K}(\pi_n)(\xi) = \mathcal{K}(\pi_n)(\eta)$ for all $n \ge 0$ then

$$d(\xi, \eta) \le \max\ \{\ d(\xi, \mathcal{K}(\pi_n)(\xi)),\ d(\mathcal{K}(\pi_n)(\eta), \eta)\ \} \le \frac{1}{2^n}$$

and therefore $\xi = \eta$.

$\mathcal{F}$ induces the functor $\mathcal{R}$ as follows: Using Lemma 4.13

$$\mathcal{D} : CRankSET^* \to RankCPO, \quad (M, \tilde{\pi}) \mapsto \mathcal{D}(M, \tilde{\pi}), \quad f \mapsto f$$

is a welldefined functor and hence we may put $\mathcal{R} = \mathcal{D} \circ \mathcal{F} \circ \mathcal{J}^{\mathsf{rank}}_{\mathsf{cpo}}$. Similar to the construction of the initial fixed point for locally continuous endofunctors in $CPO_\perp$ an initial fixed point $(D, \tilde{\pi})$ of $\mathcal{R}$ can be constructed as the initial cone of the tower $(\mathcal{R}^n(\{\perp\}))$ in $RankCPO^E$. Here we use Lemma 2.1 and Lemma 4.15. Using Lemma 3.11 it can be shown that D and the unique fixed point of $\mathcal{H}$ are compatible.

In general for this functor $\mathcal{R}$ there does not exist a functor $\mathcal{G} : CPO_\perp \to CPO_\perp$ such that $\mathcal{R} = \mathcal{G}_{\mathsf{rank}}$. This can be seen as follows: Let $\mathcal{H} = id_{\frac{1}{2}}$ and $\mathcal{K} = id$ then $\mathcal{F}(M, \tilde{\pi}) = (M, \tilde{\pi}')$ where $\pi'_n = \pi_{n+1}$. Therefore

$$\mathcal{R}(D, \tilde{\pi}) = \mathcal{D}(\ \mathcal{I}_{\mathsf{cpo}}(D),\ \tilde{\pi}\ ).$$

When we assume that there is a functor $\mathcal{G} : CPO_\perp \to CPO_\perp$ with $\mathcal{R} = \mathcal{G}_{\mathsf{rank}}$ then for each cpo D and each two rank orderings $\tilde{\pi}$, $\tilde{\mu}$ on D:

$$\mathcal{J}^{\mathsf{cmpo}}_{\mathsf{cpo}}(\ \mathcal{R}(D, \tilde{\pi})\ ) = \mathcal{J}^{\mathsf{cmpo}}_{\mathsf{cpo}}(\ \mathcal{R}(D, \tilde{\mu})\ )$$

Consider the cpo $D = \{\perp, \xi_1, \xi_2, \xi\}$ where $\perp \sqsubseteq \xi_1$, $\xi_2 \sqsubseteq \xi$ and $\xi_1 \not\sqsubseteq \xi_2$, $\xi_2 \not\sqsubseteq \xi_1$. Then (π_n), (μ_n) are rank orderings on D where $\pi_n = \mu_n = id_D$ for $n \geq 2$ and

$$\pi_1(\eta) = \begin{cases} \perp & : \eta = \perp \\ \xi_2 & : \eta = \xi_2 \\ \xi_1 & : \eta = \xi_1 \text{ or } \eta = \xi \end{cases} \qquad \mu_1(\eta) = \begin{cases} \perp & : \eta = \perp \\ \xi_1 & : \eta = \xi_1 \\ \xi_2 & : \eta = \xi_2 \text{ or } \eta = \xi \end{cases}$$

Then ξ_2 and ξ are incomparible in $\mathcal{R}(D, \tilde{\pi})$ but $\xi_2 \sqsubseteq \xi$ in $\mathcal{R}(D, \tilde{\mu})$. Hence

$$\mathcal{J}^{\text{rank}}_{\text{cpo}}(\mathcal{R}(D, \tilde{\pi})) \neq \mathcal{J}^{\text{rank}}_{\text{cpo}}(\mathcal{R}(D, \tilde{\mu})).$$

Contradiction.

Hence using the idea to go the opposite way in the proof of Theorem 4, i.e. going from complete ultrametric spaces to complete rank orderings to rank ordered cpo's and then to cpo's, fails. At this moment we have no idea how a given metric domain equation can be lifted to a corresponding domain equation for cpo's.

In Theorem 1-5 we cannot deal with powerdomain constructions like $\mathcal{P}_{\text{Plotkin}}(\cdot)$ or $\mathcal{P}_{\text{closed}}(\cdot)$ (where $\mathcal{P}_{\text{Plotkin}}(\cdot)$ denotes the Plotkin powerdomain of $(\cdot)$ and $\mathcal{P}_{\text{closed}}(M)$ the collection of closed subsets of a metric space M) or with function space functors like

$$\mathcal{H}(M_1, M_2) = M_1 \longrightarrow^{\text{ndi}} M_2$$

(which means the collection of all non-distance-increasing functions from the metric space M_1 into the metric space M_2) or

$$\mathcal{G}(D_1, D_2) = D_1 \longrightarrow^{\text{cont}} D_2$$

(which means the collection of all continuous functions from the cpo D_1 into the cpo D_2). The reason is that the underlying set of the image $\mathcal{F}(A)$ of an object A under such functors $\mathcal{F}$ depends on the underlying partial order resp. metric and not only on the underlying set $\mathcal{I}(A)$. This implies that there does not exist an endofunctor $\mathcal{K}$ of a suitable category of sets such that $\mathcal{K} \circ \mathcal{I} = \mathcal{I} \circ \mathcal{F}$ (where $\mathcal{I}$ means the forgetful functor). I.e. domain equations envolving powerdomain or function space constructions like $\mathcal{P}_{\text{Plotkin}}$, $\mathcal{P}_{\text{closed}}$, $\rightarrow^{\text{ndi}}$, $\rightarrow^{\text{cont}}$ violate our assumption that the domain equations under consideration arise by lifting a domain equation for sets. It is an open question whether a suitable notion of the correspondence for domain equations of the form

$$D \simeq \mathcal{P}_{\text{cpo}}(\mathcal{G}(D)), \quad M \simeq \mathcal{P}_{\text{cms}}(\mathcal{H}(M))$$

resp.

$$D \simeq \mathcal{G}_1(D) \longrightarrow^{\text{cpo}} \mathcal{G}_2(D), \quad M \simeq \mathcal{H}_1(M) \longrightarrow^{\text{cms}} \mathcal{H}_2(M)$$

can be found where $\mathcal{G}$, $\mathcal{H}$ resp. $\mathcal{G}_i$, $\mathcal{H}_i$ induces compatible domain equations (in our or in some other sense). Here $\mathcal{P}_{\text{cpo}}(\cdot)$ resp. $\mathcal{P}_{\text{cms}}(\cdot)$ means a suitable powerdomain construction in the partial order rsp. metric setting, $\rightarrow^{\text{cpo}}$ resp. $\rightarrow^{\text{cms}}$ a suitable function space construction for cpo's resp. complete metric spaces.

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A Formal definitions of the functors in section 5

In what follows $\mathcal{C}$ is one of the categories $CPO_\perp$, CMS, CMS^*, SET or SET^*, $\mathcal{O}$ and $\mathcal{U}$ $\mathcal{C}$-objects and $f : \mathcal{O} \to \mathcal{U}$ a $\mathcal{C}$-morphism. id means the identity on $\mathcal{C}$, i.e. $id(\mathcal{O}) = \mathcal{O}$ and $id(f) = f$. If $\mathbf{M}$ is one of the categories CMS or CUM_e^* and c a real number with $0 < c \leq 1$ then $id_c : \mathbf{M} \to \mathbf{M}$ is given by: $id_c(M,d) = (M, c \cdot d)$, $id_c(f) = f$. If A is a fixed object in $\mathcal{C}$ then $const_A(\mathcal{O}) = A$, $const_A(f) = id_A$. If X is a set and $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ a functor then: $X \to \mathcal{F} = \mathcal{E}_1 \circ \mathcal{F}$ where $\mathcal{E}_1 : \mathcal{C} \to \mathcal{C}$ is given by:

$$\mathcal{E}_1(\mathcal{O}) = X \to \mathcal{O}, \quad \mathcal{E}_1(f) = \lambda\varphi.(f \circ \varphi)$$

Here $X \to \mathcal{O}$ means the set of functions from X into the underlying set of $\mathcal{O}$. In the pointed case the function $\lambda\xi.\,\xi_0$ is the basis point of $X \to \mathcal{O}$ where ξ_0 is the basis point of $\mathcal{O}$. The partial order $\sqsubseteq'$ on $X \to (D, \sqsubseteq)$ is given by:

$$\varphi \sqsubseteq' \psi \iff \varphi(\xi) \sqsubseteq \psi(\xi) \ \forall \xi \in D.$$

The metric d' on $X \to (M,d)$ is given by:

$$d(\varphi, \psi) = \sup\{ d(\varphi(\xi), \psi(\xi)) : \xi \in X \}.$$

If X is a set and $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ a functor then: $X \otimes \mathcal{F} = \mathcal{E}_2 \circ \mathcal{F}$ where $\mathcal{E}_2 : \mathcal{C} \to \mathcal{C}$ is given by:

$$\mathcal{E}_2(\mathcal{O}) = \{\perp\} \uplus X \times \mathcal{O}, \quad \mathcal{E}_2(f) = f_X$$

where $\uplus$ means disjoint union. In the pointed case $\perp$ is the basis point. f_X is given by:

$$f_X : (\{\perp\} \uplus X \times \mathcal{O}) \to (\{\perp\} \uplus X \times \mathcal{U})$$

where

$$f_X(\eta) = \begin{cases} \perp & : \text{ if } \eta = \perp \\ <\alpha, f(\xi)> & : \text{ if } \eta = <\alpha, \xi>. \end{cases}$$

The partial order $\sqsubseteq'$ on $\mathcal{E}_2(D, \sqsubseteq)$ is:

$$p \sqsubseteq' p' \iff (p = \perp) \vee (p = <\alpha, \xi> \wedge p' = <\alpha, \xi'> \wedge \xi \sqsubseteq \xi').$$

The metric d' on $\mathcal{E}_2(M,d)$ is: $d'(\perp, \perp) = 0$ and $d'(\perp, p) = d'(p, \perp) = 1$ if $p \neq \perp$ and

$$d'(<\alpha, \xi>, <\beta, q>) = \begin{cases} d(\xi, \eta) & : \text{ if } \alpha = \beta \\ 1 & : \text{ otherwise.} \end{cases}$$

Let $\mathcal{F}_1, \mathcal{F}_2 : \mathcal{C} \to \mathcal{C}$ be functors. Then: $\mathcal{F}_1 \times \mathcal{F}_2 \;=\; \mathcal{E}_3 \circ (\mathcal{F}_1, \mathcal{F}_2)$ where $\mathcal{E}_3 : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is given by:

$$\mathcal{E}_3(\mathcal{O}_1, \mathcal{O}_2) \;=\; \mathcal{O}_1 \times \mathcal{O}_2$$

$$\mathcal{E}_3(f_1, f_2) \;=\; f_1 \times f_2 \;=\; \lambda(\xi, \eta).(f_1(\xi), f_2(\eta))$$

In the pointed case $(\xi_{1,0}, \xi_{2,0})$ is the basis point of $\mathcal{O}_1 \times \mathcal{O}_2$ where $\xi_{i,0}$ is the basis point of $\mathcal{O}_i$. The partial order on $(D_1, \sqsubseteq_1) \times (D_2, \sqsubseteq_2)$ is:

$$(\xi, \eta) \sqsubseteq (\xi', \eta') \iff \xi \sqsubseteq_1 \xi' \;\wedge\; \eta \sqsubseteq_2 \eta'$$

The metric d on $(M_1, d_1) \times (M_2, d_2)$ is:

$$d(\;(\xi, \eta),\; (\xi', \eta')\;) \;=\; \max\;\{\; d_1(\xi, \xi'),\; d_2(\eta, \eta')\;\}.$$

Let $\mathcal{F}_1, \mathcal{F}_2 : \mathcal{C} \to \mathcal{C}$ be functors. Then: $\mathcal{F}_1 \oplus \mathcal{F}_2 \;=\; \mathcal{E}_4 \circ (\mathcal{F}_1, \mathcal{F}_2)$ where $\mathcal{E}_4 : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is given by:

$$\mathcal{E}_4(\mathcal{O}_1, \mathcal{O}_2) \;=\; \mathcal{O}_1 \oplus \mathcal{O}_2 \;=\; \{\bot\} \uplus \mathcal{O}_1 \uplus \mathcal{O}_2,$$

$$\mathcal{E}_4(f_1, f_2) \;=\; f_1 \oplus f_2$$

where $\uplus$ means disjoint union and $\bot$ is the basis point. If $f_i : \mathcal{O}_i \to \mathcal{U}_i$ are morphisms in $\mathcal{C}$ then

$$f_1 \oplus f_2 \;:\; (\mathcal{O}_1 \oplus \mathcal{O}_2) \;\to\; (\mathcal{U}_1 \oplus \mathcal{U}_2)$$

is defined by:

$$(f_1 \oplus f_2)(\xi) \;=\; \begin{cases} \bot & : \text{ if } \xi = \bot \\ f_i(\xi) & : \text{ if } \xi \in D_i. \end{cases}$$

The partial order on $(D_1, \sqsubseteq_1) \;\oplus\; (D_2, \sqsubseteq_2)$ is:

$$\xi \sqsubseteq \eta \iff \xi = \bot \;\vee\; (\xi, \eta \in D_i \;\wedge\; \xi \sqsubseteq_i \eta).$$

The metric on $(M_1, d_1) \;\oplus\; (M_2, d_2)$:

$$d(\xi, \eta) \;=\; \begin{cases} d_i(\xi, \eta) & : \text{ if } \xi, \eta \in M_i \\ 0 & : \text{ if } \xi = \eta = \bot \\ 1 & : \text{ otherwise.} \end{cases}$$

The same notations are used to denote the induced functors

$$\mathcal{F}^E : \mathcal{C}^E \to \mathcal{C}^E,\;\; \mathcal{F}^E(\mathcal{O}) \;=\; \mathcal{F}(\mathcal{O}),\;\; \mathcal{F}^E(< e, c >) \;=\; < \mathcal{F}(e), \mathcal{F}(c) >$$

E.g. $X \otimes \mathcal{F}^E \;=\; (X \otimes \mathcal{F})^E$. If $\mathcal{F}_i : \mathcal{C}^E \to \mathcal{C}^E$ are functors then $\mathcal{F}_1 \to^E \mathcal{F}_2 : \mathcal{C}^E \to \mathcal{C}^E$ is given by:

$$\mathcal{F}_1 \to^E \mathcal{F}_2 \;=\; \mathcal{E}_5 \circ (\mathcal{F}_1, \mathcal{F}_2)$$

where $\mathcal{E}_5 : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is given by:

$$\mathcal{E}_5(\mathcal{O}_1, \mathcal{O}_2) \;=\; \mathcal{O}_1 \to \mathcal{O}_2$$

$$\mathcal{E}_5(\; < e_1, c_1 >,\; < e_2, c_2 > \;) \;=\; < \;\lambda\varphi.(e_2 \circ \varphi \circ c_1),\; \lambda\psi.(c_2 \circ \psi \circ e_1)\; >$$

Here $\mathcal{O}_1 \to \mathcal{O}_2$ means the set of functions $A_1 \to A_2$ where A_i is the underlying set of $\mathcal{O}_i$. In the pointed case $< \lambda\varphi.\lambda\xi.\xi_2, \lambda\psi.\lambda\eta.\xi_1 >$ is the basis point where ξ_i is the basis point of $\mathcal{O}_i$.