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On completions of semantic domains

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Abstract

This paper adds the technique of chain completion to the setting of [MCB94]. We develop the theory of chain completion $Ch(\mathcal{D})$ of a domain $\mathcal{D}$ and show how this completion relates to metric and ideal completion. Especially we study consistency results for denotational semantics on $\overline{D}$, $Ch(\mathcal{D})$ and $Idl(\mathcal{D})$.

Contents

Introduction

The aim of this paper is to add the technique of chain completion to the setting of [MCB94] which they describe as follows:

> *We assume that $\mathcal{D}$ is a semantic domain for non-recursive programs of a CCS-like language as finite strings on (labelled) trees of finite height. $\sqsubseteq$ is a partial order on $\mathcal{D}$ such that $\mathcal{D}$ has a bottom element $\perp$ which either can be the meaning of the* `nil` *program (the program which does not perform any action) or which represents a totally undefined process. If we have semantic operators on $\mathcal{D}$ which are monotene w.r.t. $\sqsubseteq$ then the ideal completion $Idl(\mathcal{D})$ can be used as semantic domain for a denotational cpo semantic which extends the semantics on $\mathcal{D}$ for recursive programs. On the other hand if $\mathcal{D}$ is endowed with a metric such that the semantic operators are non-distance-increasing resp. contracting we get a denotational semantics on the metric completion $\overline{\mathcal{D}}$. The question arises in which way the metric and ideal completion are related and how the denotational semantics on $Idl(\mathcal{D})$ resp. $\overline{\mathcal{D}}$ are connected. In this paper we answer this question under the assumption that $(\mathcal{D}, \sqsubseteq)$ can be endowed with a finite length. This length induces a metric on $\mathcal{D}$. By a finite length we mean a function which assigns the maximal number of atomic steps to each element x of $\mathcal{D}$ which are needed for the execution of x. Here the elements of $\mathcal{D}$ are considered as processes. E.g. the length of a finite string is its usual length, the length of a tree is its height. The distance $d(x, y)$ induced by a length counts the maximal number n of steps on which the execution of x and y coincide (and then $d(x, y) = 1/2^n$).*

This gives us our program: We have to develop the theory of the chain completion $Ch(\mathcal{D})$ of a domain $\mathcal{D}$ in a way that we are able to deal with semantics in the above described sense. Especially we are interested in a connection of the denotational semantics on $\overline{D}$, $Ch(\mathcal{D})$ and $Idl(\mathcal{D})$. To give an impression how similar or how different chain and ideal completion are we usually cite the corresponding results of [MCB94].

The paper is organized as follows: Chapter 1 gives some basic definitions. We introduce the technique of chain completion and relate it to the well known ideal completion. Furthermore we give a formal definition of the length ρ on a domain $\mathcal{D}$ and show how this induces a metric not only on $\mathcal{D}$ but also on the chain completion $Ch(\mathcal{D})$ and the ideal completion $Idl(\mathcal{D})$. Chapter 2 discusses the relations of the metric completions of $\mathcal{D}$, $Ch(\mathcal{D})$ and $Idl(\mathcal{D})$. In chapter 3 we present the application of the theory developed so far to denotational semantics.

1 Basic definitions

In this chapter we introduce different completions of a semantic domain $\mathcal{D}$ which is equipped with both a partial order $\sqsubseteq \subseteq \mathcal{D} \times \mathcal{D}$ and a length[1] $\rho : \mathcal{D} \to \mathbb{N}_0 \cup \{\infty\}$ which covers some information from $\sqsubseteq$. First we present some notions concerning partial orders. Then we discuss in some detail the chain completion of a partially ordered set and compare it with the well known ideal completion. Finally we show how one can define a metric on a partially ordered set with bottom using the length ρ.

1.1 Order theoretical notions

In order to compare completion techniques for semantic domains we need some order theoretical notions. Especially we introduce partially ordered sets, different completeness properties, corresponding stucture preserving functions and constructions to gain completeness.

A *partially ordered set* (*poset*) consists of a pair $(\mathcal{D}, \sqsubseteq)$ where $\mathcal{D}$ is a set and $\sqsubseteq \subseteq \mathcal{D} \times \mathcal{D}$ is a binary relation on $\mathcal{D}$ which is reflexive, antisymmetric and transitive. $(\mathcal{D}, \sqsubseteq)$ is *pointed*, iff it contains a least element. This element is called *bottom* and is denoted by $\perp$. If the relation $\sqsubseteq$ is only reflexive and transitive we call the pair $(\mathcal{D}, \sqsubseteq)$ *preorder*.

In a poset $(\mathcal{D}, \sqsubseteq)$ we write $x \sqcup y$ for the least upper bound of two elements $x, y \in \mathcal{D}$ and $\bigsqcup S$ for the least upper bound of a set $S \subseteq \mathcal{D}$ - if these bounds exist. Further we use the notions $\downarrow x := \{y \in \mathcal{D} \mid y \sqsubseteq x\}$ and $\downarrow S := \{y \in \mathcal{D} \mid \exists s \in S : \ y \sqsubseteq s\}$ for $x \in \mathcal{D}$, $S \subseteq \mathcal{D}$.

In order to define completeness properties it is necessary to characterize some subsets of a poset. Let $(\mathcal{D}, \sqsubseteq)$ be a partially ordered set. A nonempty subset $S \subseteq \mathcal{D}$ is called

- *leftclosed,* iff $\forall x \in \mathcal{D},\ s \in S : \ x \sqsubseteq s \Rightarrow x \in S$.
- *directed,* iff $\forall x, y \in S\, \exists z \in S : \ x \sqsubseteq z \wedge y \sqsubseteq z$.
- *ideal,* iff S is directed and leftclosed.
- *bounded,* iff $\exists b \in \mathcal{D}\, \forall s \in S : \ s \sqsubseteq b$.
- *finite bounded,* iff S is a finite set and S is bounded.
- *chain,* iff S is totally ordered by $\sqsubseteq \cap (S \times S)$.
- *ω-chain,* iff S is countable and S is a chain. Sometimes we refer to the elements of an ω-chain S and denote it by $(c_i)_{i\in\mathbb{N}} \subseteq \mathcal{D}$. In this case holds $\forall i \in \mathbb{N} : c_i \sqsubseteq c_{i+1}$.

[1] See subsection 1.4 for details.

As a chain is totally ordered it is a special case of a directed set.

Completeness distinguishes posets by marking the existence of suprema for special kinds of subsets. We call a poset $(\mathcal{D}, \sqsubseteq)$

- *fb-cpo,* iff each finite bounded set $S \subseteq \mathcal{D}$ has a least upper bound in $\mathcal{D}$.
- *ωc-cpo,* iff each ω-chain $C \subseteq \mathcal{D}$ has a least upper bound in $\mathcal{D}$.
- *d-cpo,* iff each directed set $S \subseteq \mathcal{D}$ has a least upper bound in $\mathcal{D}$.
- *c-cpo,* iff each chain $C \subseteq \mathcal{D}$ has a least upper bound in $\mathcal{D}$.

Obviously every d-cpo is an ωc-cpo. [AJ92] mentions that the notions d-cpo and c-cpo are equivalent.

The concepts of a compact element[2] and of an algebraic poset are defined similarly in the setting of ωc-cpos and d-cpos. We present them in parallel, following the definitions of [WWT78][3]:

Let $(\mathcal{D}, \sqsubseteq)$ be an ωc-cpo. An element $d \in \mathcal{D}$ is called *ωc-compact* iff for all ω-chains $C \subseteq \mathcal{D}$ holds: $d \sqsubseteq \bigsqcup C \Rightarrow (\exists c \in C: \; d \sqsubseteq c.)$ We denote the set of all ωc-compact elements of $\mathcal{D}$ by $K_{\omega c}(\mathcal{D})$. $\mathcal{D}$ is called *ωc-algebraic*[4], iff for all elements $d \in \mathcal{D}$ there exists an ω-chain $C \subseteq K_{\omega c}(\mathcal{D})$ such that $d = \bigsqcup C$.

Substituting ω-chains by directed sets we get: Let $(\mathcal{D}, \sqsubseteq)$ be a d-cpo. An element $d \in \mathcal{D}$ is called *d-compact* iff for all directed sets $S \subseteq \mathcal{D}$ holds: $d \sqsubseteq \bigsqcup S \Rightarrow (\exists s \in S: \; d \sqsubseteq s)$. We denote the set of all d-compact elements of $\mathcal{D}$ by $K_d(\mathcal{D})$. $\mathcal{D}$ is called *d-algebraic,* iff for all elements $d \in \mathcal{D}$ there exists a directed set $S \subseteq K_d(\mathcal{D})$ such that $d = \bigsqcup S$.

Next we consider functions that preserve (some of) the structure in different posets:

- Let $(\mathcal{D}, \sqsubseteq)$ and $(\mathcal{E}, \sqsubseteq)$ be posets and $f : \mathcal{D} \to \mathcal{E}$. f is called *monotone* iff $\forall x, y \in \mathcal{D}: \; x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$.
- Let $(\mathcal{D}, \sqsubseteq)$ and $(\mathcal{E}, \sqsubseteq)$ be ωc-cpos and $f : \mathcal{D} \to \mathcal{E}$. f is called *ωc-continuous* iff f is monotone and for all ω-chains $C \subseteq \mathcal{D}$ holds: $f(\bigsqcup C) = \bigsqcup f(C)$.
- Let $(\mathcal{D}, \sqsubseteq)$ and $(\mathcal{E}, \sqsubseteq)$ be d-cpos and $f : \mathcal{D} \to \mathcal{E}$. f is called *d-continuous* iff f is monotone and for all directed sets $S \subseteq \mathcal{D}$ holds: $f(\bigsqcup S) = \bigsqcup f(S)$.

[2] Some authors call such an element "finite" or "isolated".

[3] [WWT78] uses the notion "core complete" instead of "algebraic".

[4] [AJ92] uses the term "ω-algebraic" to denote a d-cpo with a countable basis of compact elements. It should be noted that the property "ωc-algebraic" does not include any restriction in the number of ωc-compact elements.

A d-continuous function f is always ωc-continuous.

For a given poset $(\mathcal{D}, \sqsubseteq)$ it is possible to construct a d-cpo and an ωc-cpo which both "contain" $\mathcal{D}$. We discuss the second concept in the following subsection. In theorem 1.1 we compile some properties which can be found for example in [AJ92]:

Theorem 1.1 (Basic properties of the ideal completion)
Let $(\mathcal{D}, \sqsubseteq)$ be a poset. We consider the set $Idl(\mathcal{D}) := \{I \subseteq \mathcal{D} \mid I \text{ ideal}\}$. Ordered by inclusion it is a d-algebraic d-cpo. We call $(Idl(\mathcal{D}), \subseteq)$ the ideal completion *of $(\mathcal{D}, \sqsubseteq)$. The least upper bound of a directed set $S \subseteq Idl(\mathcal{D})$ is computed as $\bigsqcup S = \bigcup_{I \in S} I$. The function $\iota_d : \mathcal{D} \to Idl(\mathcal{D})$, $d \mapsto \downarrow d$, is monotone and injective, $K_d(Idl(\mathcal{D})) = \iota_d(\mathcal{D})$.*

In order to compare metric and poset completions it is useful to introduce another d-complete poset: Given a poset $(\mathcal{D}, \sqsubseteq)$ the set $P_\downarrow(\mathcal{D}) := \{X \subset \mathcal{D} \mid X \text{ leftclosed}\}$ ordered by inclusion is a d-cpo. The least upper bound of a directed set $S \subseteq P_\downarrow(\mathcal{D})$ is computed as $\bigsqcup S = \bigcup_{X \in S} X$. The function $\iota_P : (\mathcal{D}, \sqsubseteq) \to (P_\downarrow(\mathcal{D}), \subseteq)$, $d \mapsto \downarrow d$, is monotone and injective. For the ideal completion of D holds $Idl(\mathcal{D}) \subseteq P_\downarrow(\mathcal{D})$. If the poset $(\mathcal{D}, \sqsubseteq)$ is pointed then $(Idl(\mathcal{D}), \subseteq)$ and $(P_\downarrow(\mathcal{D}), \subseteq)$ are just so.

1.2 The chain completion of a poset $(\mathcal{D}, \sqsubseteq)$

The chain completion of a poset $(\mathcal{D}, \sqsubseteq)$ is defined as the set of all ω-chains in $\mathcal{D}$ equipped with some suitable equivalence relation. This technique works similar to the metric completion.

[WWT78] presents two general approaches – including the one we will use – to construct from a "suitable" poset $\mathcal{D}$ a cpo $\mathcal{I}(D)$ with, in effect, $\mathcal{D}$ as its compact elements. The completness property of $\mathcal{I}(D)$ is used as a parameter. It may be chosen widely, including finite-bounded, directed, ω-chain and chain.

In spite of the elegant theory presented in [WWT78] we will do the completion "by hand" and prove their properties in our own way. The reason is that this process sheds some light on the nature of the chain completion.

Let $(\mathcal{D}, \sqsubseteq)$ be a poset. Construct a preorderd set $(\mathcal{D}', \sqsubseteq_{\mathcal{D}'})$ by

- $\mathcal{D}' := \{(c_i)_{i \in \mathbb{N}} \subseteq \mathcal{D} \mid \forall i \in \mathbb{N} : c_i \sqsubseteq c_{i+1}\}$ and
- $(a_i)_{i \in \mathbb{N}} \sqsubseteq_{\mathcal{D}'} (b_i)_{i \in \mathbb{N}} :\Longleftrightarrow \forall i \in \mathbb{N}\, \exists j \in \mathbb{N} : \ a_i \sqsubseteq b_j$.

In order to obain a poset we define an equivalence relation $\equiv \subseteq \mathcal{D}' \times \mathcal{D}'$ by

$$(a_i)_{i \in \mathbb{N}} \equiv (b_i)_{i \in \mathbb{N}} :\Longleftrightarrow (a_i)_{i \in \mathbb{N}} \sqsubseteq_{\mathcal{D}'} (b_i)_{i \in \mathbb{N}} \ \wedge \ (b_i)_{i \in \mathbb{N}} \sqsubseteq_{\mathcal{D}'} (a_i)_{i \in \mathbb{N}}.$$

We write $[(a_i)]$ or – if it is useful to refer to the index of the ω-chain – $[(a_i)_{i\in\mathbb{N}}]$ for the induced equivalence classes on $\mathcal{D}'$. The equivalence relation $\equiv$ induces furthermore a partial order $\sqsubseteq_c \subseteq \mathcal{D}'_{/\equiv} \times \mathcal{D}'_{/\equiv}$. For all $[(a_i)], [(b_i)] \in \mathcal{D}'_{/\equiv}$ we define:

$$[(a_i)] \sqsubseteq_c [(b_i)] \;:\Longleftrightarrow\; (a_i)_{i\in\mathbb{N}} \sqsubseteq_{\mathcal{D}'} (b_i)_{i\in\mathbb{N}}.$$

Definition 1.2 (Chain completion)
Let $(\mathcal{D},\sqsubseteq)$ be a poset. We call $Chain(\mathcal{D}) := \mathcal{D}'_{/\equiv}$ ordered by $\sqsubseteq_c$ the chain completion *of $\mathcal{D}$.*

Definition 1.3 (Stationary ω-chains)
Let $(\mathcal{D},\sqsubseteq)$ be a poset. We call an ω-chain $(c_i)_{i\in\mathbb{N}} \subseteq \mathcal{D}$ stationary *iff $\exists k \in \mathbb{N}\, \forall l \geq k : c_k = c_l$.*

It is easy to proof that all ω-chains $(a_i) \in [(c_i)] \in Chain(\mathcal{D})$ are stationary iff (c_i) is stationary. The chain completion of a poset has the desired properties:

Theorem 1.4 (Basic properties of the chain completion)
Let $(\mathcal{D},\sqsubseteq)$ be a poset. The chain completion $(Chain(\mathcal{D}),\sqsubseteq_c)$ is an ωc-algebraic ωc-cpo. The function $\iota_c : (\mathcal{D},\sqsubseteq) \to (Chain(\mathcal{D}),\sqsubseteq_c)$, $d \mapsto [(d_i)]$ with $\forall i \in \mathbb{N} : d_i = d$, is monotone and injective, $K_{\omega c}(Chain(\mathcal{D})) = \iota_c(\mathcal{D})$.

Proof: We prove first that $(Chain(\mathcal{D}),\sqsubseteq_c)$ is an ωc-cpo. Consider an ω-chain $(C_i)_{i\in\mathbb{N}} \subseteq Chain(\mathcal{D})$ with $C_i = [(c_{i,j})_{j\in\mathbb{N}}]$. We define by induction on i and j a new represantant $(a_{i,j})_{j\in\mathbb{N}} \subseteq \mathcal{D}$ for each C_i. Let

- for $i = 1$: $\forall j \in \mathbb{N} : a_{1,j} := c_{1,j}$ and
- for $i > 1$: $\forall j \in \mathbb{N} : a_{i,j} := c_{i,k}$, where $k := \min\{l \in \mathbb{N} \mid l \geq j \wedge a_{i-1,j} \sqsubseteq c_{i,l}\}$. Such a k exists for every j because (C_i) is an ω-chain in $Chain(\mathcal{D})$.

By induction on i one can prove that we gained indeed the property $\forall j \in \mathbb{N} : a_{i,j} \sqsubseteq a_{i,j+1}$. The construction ensures $c_{i,j} \sqsubseteq a_{i,j}$ for all $i, j \in \mathbb{N}$. The other way round for each $a_{i,j}$ exists $k \in \mathbb{N}$ such that $a_{i,j} = c_{i,k}$. Therefore $[(a_{i,j})_{j\in\mathbb{N}}] = [(c_{i,j})_{j\in\mathbb{N}}]$ for all $i \in \mathbb{N}$.

We claim that $A := [(a_{i,i})_{i\in\mathbb{N}}] \in Chain(\mathcal{D})$ is the least upper bound of (C_i). By construction we have for all $i \in \mathbb{N} : a_{i,i} \sqsubseteq a_{i+1,i} \sqsubseteq a_{i+1,i+1}$, i.e. $(a_{i,i})_{i\in\mathbb{N}}$ is an ω-chain in $\mathcal{D}$. For $a_{i,j}$ choose $k := \max\{i,j\}$. This results in $a_{i,j} \sqsubseteq a_{k,k}$. Therefore A is an upper bound for (C_i). Let $B := (b_i) \in Chain(\mathcal{D})$ be an upper bound. Then $\forall i,j\, \exists k : a_{i,j} \sqsubseteq b_k$. This holds especially in the case $i = j$ and we get $A \sqsubseteq_c B$.

Next we show that the elements of $\iota_c(\mathcal{D}) \subseteq Chain(\mathcal{D})$ are ωc-compact. Let for $a \in \mathcal{D}$ $A := \iota_c(a) \in Chain(\mathcal{D})$, $(C_i)_{i\in\mathbb{N}} \subseteq Chain(\mathcal{D})$ an ω-chain with $C_i = [(c_{i,j})_{j\in\mathbb{N}}]$ and $A \sqsubseteq_c \bigsqcup C_i =: C$. W.l.o.g. we may assume $C = [(c_{i,i})_{i\in\mathbb{N}}]$. As $A \sqsubseteq_c C$ there exists $k \in \mathbb{N}$ such that $a \sqsubseteq c_{k,k}$. Therefore we get $\iota_c(a) \sqsubseteq_c C_k$.

To prove that $Chain(\mathcal{D})$ is ωc-algebraic let $C := [(c_i)] \in Chain(\mathcal{D})$. Define an ω-chain $(A_i)_{i\in\mathbb{N}} \subseteq Chain(\mathcal{D})$ by $A_i := \iota_c(c_i)$. Obviously $\forall i \in \mathbb{N} : A_i \in \iota_c(\mathcal{D}) \subseteq K_{\omega c}(Chain(\mathcal{D}))$ and $\bigsqcup A_i = C$.

To finish the prove of $K_{\omega c}(Chain(\mathcal{D})) = \iota_c(\mathcal{D})$ we have to show that elements $C \in Chain(\mathcal{D})\backslash\iota_c(\mathcal{D})$ are not ωc-compact. Here we use an easy to proof characteristic of $\iota_c(\mathcal{D})$: $C = [(c_i)] \in \iota_c(\mathcal{D})$ iff (c_i) is a stationary ω-chain in $\mathcal{D}$. Let $C := [(c_i)] \in Chain(\mathcal{D})$, $(c_i) \subseteq \mathcal{D}$ a non-stationary ω-chain. We construct an ω-chain $A_i \subseteq Chain(\mathcal{D})$ by $A_i := \iota_c(c_i)$ for all $i \in \mathbb{N}$. Then we get: $C \sqsubseteq \bigsqcup A_i$, but there is no $k \in \mathbb{N}$ such that $[(c_i)] \sqsubseteq_c A_k$. ■

If the poset $(\mathcal{D}, \sqsubseteq)$ is pointed then $(Chain(\mathcal{D}), \sqsubseteq_c)$ is just so. We make the two central constructions of the above proof explicit and present them as results of their own kind. Finally we give a simple but useful lemma.

Corollary 1.5 (Computing least upper bounds of ω-chains in $Chain(\mathcal{D})$)
Let $(\mathcal{D}, \sqsubseteq)$ be a poset, $(C_i)_{i\in\mathbb{N}} \subseteq Chain(\mathcal{D})$ an ω-chain, $C_i = [(c_{i,j})_{j\in\mathbb{N}}]$.

1. *Then there exist elements $a_{i,j} \in \mathcal{D}$ such that the following conditions hold:*

   i.) $\forall i \in \mathbb{N} : [(a_{i,j})_{j\in\mathbb{N}}] = [(c_{i,j})_{j\in\mathbb{N}}]$,

   ii.) $\forall i, j \in \mathbb{N}\, \exists l \in \mathbb{N} : \ a_{i,j} = c_{i,l}$,

   iii.) $\forall i, j \in \mathbb{N} : a_{i,j} \sqsubseteq a_{i,j+1} \wedge a_{i,j} \sqsubseteq a_{i+1,j}$.

2. *If the ω-chains $(c_{i,j})$ fulfill condition iii.) then the supremum may be computed by $\bigsqcup C_i = [(c_{i,i})_{i\in\mathbb{N}}]$.*

Corollary 1.6 (Construction of an ω-chain with least upper bound C)
Let $(\mathcal{D}, \sqsubseteq)$ be a poset, $C = [(c_i)] \in Chain(\mathcal{D})$. Then $\iota_c(c_i)$ is an ω-chain in $K_{\omega c}(Chain(\mathcal{D}))$ and $\bigsqcup \iota_c(c_i) = [(c_i)] = C$.

Let $(\mathcal{D}, \sqsubseteq)$ be a poset. The function ι_c is monotone, i.e. $\forall x, y \in \mathcal{D} : x \sqsubseteq y \Rightarrow \iota_c(x) \sqsubseteq_c \iota_c(y)$. For elements of $\iota_c(\mathcal{D})$ the other direction holds as well:

Lemma 1.7 (Connection between $\sqsubseteq_c$ and $\sqsubseteq$ on $K_{\omega c}(Chain(\mathcal{D}))$)
Let $(\mathcal{D}, \sqsubseteq)$ be a poset, $x, y \in \mathcal{D}$. Then $\iota_c(x) \sqsubseteq_c \iota_c(y)$ implies $x \sqsubseteq y$.

1.3 Chain completion versus ideal completion

In order to relate chain and ideal completion of a poset $\mathcal{D}$ we define an injective and ωc-continuous function $f : (Chain(\mathcal{D}), \sqsubseteq_c) \to (Idl(\mathcal{D}), \subseteq)$. In the case that $\mathcal{D}$ is countable we prove $(Chain(\mathcal{D}), \sqsubseteq_c) \simeq (Idl(\mathcal{D}), \subseteq)$. Finally we give two examples. The first demonstrates that ideal completion and chain completion are different concepts. In the second $Chain(\mathcal{D})$ and $Idl(\mathcal{D})$ coincide even though $\mathcal{D}$ is not countable.

Let $(\mathcal{D}, \sqsubseteq)$ be a poset. Using corollary 1.6 we define a function

$$f : \begin{cases} (Chain(\mathcal{D}), \sqsubseteq_c) & \rightarrow \quad (Idl(\mathcal{D}), \subseteq) \\ C = [(c_i)] & \mapsto \quad \bigsqcup_{i \in \mathbb{N}} \iota_d(c_i). \end{cases}$$

As (c_i) is an ω-chain in $\mathcal{D}$ and ι_d is monotone $(\iota_d(c_i))$ is an ω-chain in $Idl(\mathcal{D})$ and therefore has a least upper bound in $Idl(\mathcal{D})$. Let $[(c_i)], [(d_i)] \in Chain(\mathcal{D})$ with $(c_i) \equiv (d_i)$. This implies $\forall i \in \mathbb{N}\, \exists k : c_i \sqsubseteq d_k$ and $\forall j \in \mathbb{N}\, \exists l : d_j \sqsubseteq c_l$. Therefore $\bigsqcup \iota_d(d_i)$ is an upper bound for $\iota_d(c_i)$ and $\bigsqcup \iota_d(c_i)$ is an upper bound for $\iota_d(d_i)$ for all $i \in \mathbb{N}$. This results in $\bigsqcup \iota_d(c_i) = \bigsqcup \iota_d(d_i)$. Thus the function f is indeed well defined.

Theorem 1.8 (Properties of $f : (Chain(\mathcal{D}), \sqsubseteq_c) \rightarrow (Idl(\mathcal{D}), \subseteq)$**)**
Let $(\mathcal{D}, \sqsubseteq)$ be a poset. For the above defined function $f : (Chain(\mathcal{D}), \sqsubseteq_c) \rightarrow (Idl(\mathcal{D}), \subseteq)$ holds:

- $\iota_d = f \circ \iota_c$,
- *f is injective,*
- *f is ωc-continuous and*
- $Im(f) = \{\iota_d(d) \mid d \in \mathcal{D}\} \,\dot{\cup}\, \{\bigcup_{i \in \mathbb{N}} \iota_d(c_i) \mid (c_i) \subseteq \mathcal{D}$ *non-stationary* ω-*chain*$\} \subseteq Idl(\mathcal{D})$.

Proof: The first and the last property of f are obvious.

To prove that f is injective let $A = [(a_i)]$, $B = [(b_j)] \in Chain(\mathcal{D})$ with $f(A) = f(B)$. Using theorem 1.1 we may compute $f(A) = \bigsqcup \iota_d(a_i) = \bigcup \iota_d(a_i)$ and $f(B) = \bigsqcup \iota_d(b_j) = \bigcup \iota_d(b_j)$. As for all i we have $a_i \in \iota_d(a_i)$ this implies $\forall i \in \mathbb{N} : a_i \in \bigcup \iota_d(b_j)$. Therefore we get $\forall i \in \mathbb{N}\, \exists j \in \mathbb{N} : a_i \in \iota_d(b_j)$ and hence $a_i \sqsubseteq b_j$. Thus we can conclude $(a_i) \sqsubseteq_{\mathcal{D}'} (b_j)$. The prove of $(b_j) \sqsubseteq_{\mathcal{D}'} (a_i)$ uses the same argument – with the roles of (a_i) and (b_j) exchanged. This implies $(a_i) \equiv (b_j)$.

Next we verify that f is monotone. Let $A = [(a_i)]$, $B = [(b_i)] \in Chain(\mathcal{D})$ with $A \sqsubseteq_c B$. Then for all $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $a_i \sqsubseteq b_j$. This implies $\iota_d(a_i) \subseteq \iota_d(b_j)$ and hence $f(A) \subseteq f(B)$.

Now we deal with the continuity of f. Let $(C_i) \subseteq Chain(\mathcal{D})$ be an ω-chain with $\bigsqcup C_i =: C \in Chain(\mathcal{D})$. As $Chain(\mathcal{D})$ is ω-algebraic there exists an ω-chain $(A_r)_{r \in \mathbb{N}} \subseteq \iota_c(\mathcal{D})$ with $\bigsqcup A_r = C$. Further for all $i \in \mathbb{N}$ exist ω-chains $(B_{i,s})_{s \in \mathbb{N}} \subseteq \iota_c(\mathcal{D})$ such that $\bigsqcup_{s \in \mathbb{N}} B_{i,s} = C_i$. As $A_r, B_{i,s} \in \iota_c(\mathcal{D})$ there exist elements $a_i, b_{i,s} \in \mathcal{D}$ such that $A_r = \iota_c(a_r)$, and $B_{i,s} = \iota_c(b_{i,s})$ for all r, i, $s \in \mathbb{N}$. Due to the special form of the ω-chains (A_r) and $(B_{i,s})_{s \in \mathbb{N}}$ we may compute their least upper bounds by $\bigsqcup A_r = [(a_r)]$ and $\bigsqcup_{s\in\mathbb{N}} B_{i,s} = [(b_{i,s})_{s\in\mathbb{N}}]$. We already know that f is monotone. Thus to finish the proof for the continuity of f we have to establish $f(\bigsqcup C_i) = \bigsqcup f(C_i)$. The lefthandside evolves to

$$f(\bigsqcup C_i) = f(C) = f([(a_r)]) = \bigsqcup \iota_d(a_r)$$

while we get for the righthandside

$$\bigsqcup f(C_i) = \bigsqcup_{i\in\mathbb{N}} f([(b_{i,s})_{s\in\mathbb{N}}]) = \bigsqcup_{i\in\mathbb{N}} \bigsqcup_{s\in\mathbb{N}} \iota_d(b_{i,s}).$$

As the A_r are ωc-compact we may conclude:

$$\begin{aligned} & \forall r \in \mathbb{N} : \ A_r \sqsubseteq_c C = \bigsqcup C_i \\ \Longrightarrow\ & \forall r \in \mathbb{N}\, \exists i \in \mathbb{N} : \ A_r \sqsubseteq_c C_i = \textstyle\bigsqcup_{s\in\mathbb{N}} B_{i,s} \\ \Longrightarrow\ & \forall r \in \mathbb{N}\, \exists i,\ s \in \mathbb{N} : \ A_r \sqsubseteq_c B_{i,s}. \end{aligned}$$

Starting with $\forall i, s \in \mathbb{N} : B_{i,s} \sqsubseteq_c C_i \sqsubseteq_c C \sqsubseteq_c \bigsqcup A_r$ we establish $\forall i, s \in \mathbb{N}\, \exists r \in \mathbb{N} : \ B_{i,s} \sqsubseteq_c A_r$ using the same argument as above. This results with lemma 1.7 in $\forall r \in \mathbb{N}\, \exists i,\ s \in \mathbb{N} : \ a_r \sqsubseteq b_{i,s}$ and $\forall i, s \in \mathbb{N}\, \exists r \in \mathbb{N} : \ b_{i,s} \sqsubseteq a_r$. Now we can compute

$$\begin{aligned} & \forall r \in \mathbb{N}\, \exists i,\ s \in \mathbb{N} : \ a_r \sqsubseteq b_{i,s} \\ \Longrightarrow\ & \forall r \in \mathbb{N} \ : \ \iota_d(a_r) \subseteq \textstyle\bigsqcup_{i\in\mathbb{N}} \bigsqcup_{s\in\mathbb{N}} \iota_d(b_{i,s}) \\ \Longrightarrow\ & \textstyle\bigsqcup \iota_d(a_r) \subseteq \bigsqcup_{i\in\mathbb{N}} \bigsqcup_{s\in\mathbb{N}} \iota_d(b_{i,s}) \\ \Longrightarrow\ & f(\textstyle\bigsqcup C_i) \subseteq \bigsqcup f(C_i). \end{aligned}$$

The other direction is computed in the same way:

$$\begin{aligned} & \forall i, s \in \mathbb{N}\, \exists r \in \mathbb{N} : \ b_{i,s} \sqsubseteq a_r \\ \Longrightarrow\ & \forall i, s \in \mathbb{N} \ : \ \iota_d(b_{i,s}) \subseteq \textstyle\bigsqcup \iota_d(a_r) \\ \Longrightarrow\ & \textstyle\bigsqcup_{i\in\mathbb{N}} \bigsqcup_{s\in\mathbb{N}} \iota_d(b_{i,s}) \subseteq \bigsqcup \iota_d(a_r) \\ \Longrightarrow\ & \textstyle\bigsqcup f(C_i) \subseteq f(\bigsqcup C_i). \end{aligned}$$

Thus we have $f(\bigsqcup C_i) = \bigsqcup f(C_i)$.

■

For the purpose to relate metric completion with chain and ideal completion it is easier to use the image $f(Chain(\mathcal{D}))$ instead of $Chain(\mathcal{D})$ itself. Therefore we have to establish that $(Chain(\mathcal{D}), \sqsubseteq_c) \simeq (f(Chain(\mathcal{D})), \subseteq)$. Figure 1 illustrates the situation: We use the notion $Ch(\mathcal{D})$ instead of $Im(f)$ and introduce an ωc-continuous function g that is an inverse to f.

$$\begin{tikzcd}
 & (Chain(\mathcal{D}),\sqsubseteq_c) \arrow[dd, shift right=2, "f"'] & & \\
(\mathcal{D},\sqsubseteq) \arrow[ur, "\iota_c"] \arrow[dr, "\iota_d"'] & & & \\
 & (Ch(\mathcal{D}),\subseteq) \arrow[uu, shift right=2, "g"'] & \subseteq & (Idl(\mathcal{D}),\subseteq)
\end{tikzcd}$$

Figure 1: $(Chain(\mathcal{D}),\sqsubseteq_c)$ and $(Ch(\mathcal{D}),\subseteq)$.

Theorem 1.9 (The chain completion as subset of the ideal completion)
Let $(\mathcal{D},\sqsubseteq)$ be a poset. $Ch(\mathcal{D}) := Im(f)$ ordered by inclusion is an ωc-algebraic ωc-cpo which is isomorphic to $(Chain(\mathcal{D}),\sqsubseteq_c)$. The least upper bound of an ω-chain $(I_i)_{i\in\mathbb{N}} \subseteq Ch(\mathcal{D})$ is computed as $\bigsqcup I_i = \bigcup I_i$. The function $\iota_d : (\mathcal{D},\sqsubseteq) \to (Ch(\mathcal{D}),\subseteq), d \mapsto \downarrow d$, is injective and monotone, $K_{\omega c}(Ch(\mathcal{D})) = \iota_d(\mathcal{D})$.

Proof: As $Ch(\mathcal{D})$ is ordered by inclusion it is a poset. Let I be an element of $Ch(\mathcal{D}) = Im(f)$. Then there exists an element $[(c_i)] \in Chain(\mathcal{D})$ with $f([(c_i)] = \bigsqcup \iota_d(c_i) = I$. Using the ω-chain $(c_i) \subseteq \mathcal{D}$ we define a function

$$g : \begin{cases} (Ch(\mathcal{D}),\subseteq) & \to \quad (Chain(\mathcal{D}),\sqsubseteq_c) \\ I = f([(c_i)]) & \mapsto \quad \bigsqcup \iota_c(c_i). \end{cases}$$

We claim that

1. g is well defined,
2. g is ωc-continuous,
3. $f \circ g = id_{Ch(\mathcal{D})}$ and $g \circ f = id_{Chain(\mathcal{D})}$.

To prove that g is well defined let $I \in Ch(\mathcal{D})$, $[(c_i)], [(d_i)] \in Chain(\mathcal{D})$ such that $f([(c_i)]) = f([(d_i)]) = I$. For all $k \in \mathbb{N}$ we have $\iota_d(c_k) \subseteq f([(c_i)]) = f([(d_i)]) = \bigcup \iota_d(d_i)$ and hence there exists $l \in \mathbb{N} : c_k \sqsubseteq d_l$. The other way round we get $\forall m \in \mathbb{N}\, \exists n \in \mathbb{N} : d_m \sqsubseteq c_n$. Therefore holds $(c_i) \equiv (d_i)$.

Next we verity that g is monotone. Let $I = f([(c_i)])$, $J = f([(d_i)]) \in Ch(\mathcal{D})$ with $I \subseteq J$. Using the same argument as above we gain $\forall k \in \mathbb{N}\, \exists l \in \mathbb{N} : c_l \sqsubseteq d_k$. This implies $[(c_i)] \sqsubseteq_c [(d_i)]$.

The proof that g is an inverse to f is straight forward. With this knowledge we can establish that $(Ch(\mathcal{D}),\subseteq)$ is an ωc-cpo and that the least upper bound of an ω-chain

$(I_i) \subseteq Ch(\mathcal{D})$ may be computed as $\bigsqcup I_i = \bigcup I_i$. Let $(I_i) \subseteq Ch(\mathcal{D})$ be an ω-chain. As g is monotone $(g(I_i))$ is an ω-chain in $Chain(\mathcal{D})$ and $C := \bigsqcup g(I_i)$ exists in $Chain(\mathcal{D})$. We compute $f(C) = f(\bigsqcup g(I_i)) = \bigsqcup (f \circ g)(I_i) = \bigsqcup I_i$ which is an element of $Ch(\mathcal{D}) = Im(f)$. Using corollary 1.5 we may assume w.o.l.g. that

- $\forall i \in \mathbb{N} : \ g(I_i) = [(a_{i,j})]$,
- $\forall i, j \in \mathbb{N} : a_{i,j} \sqsubseteq a_{i,j+1} \wedge \ a_{i,j} \sqsubseteq a_{i+1,j}$ and
- $C = [(a_{i,i})]$.

With this notions we have $f(C) = \bigcup_{i \in \mathbb{N}} \iota_d(a_{i,i})$ and $I_i = f(g(I_i)) = f([(a_{i,j})]) = \bigcup_{j \in \mathbb{N}} \iota_d(a_{i,j})$. Using the ordering property of the $a_{i,j}$ we get

$$\bigsqcup I_i = f(C) = \bigcup_{i \in \mathbb{N}} \iota_d(a_{i,i}) = \bigcup_{i \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} \iota_d(a_{i,j}) = \bigcup I_i.$$

Every d-compact element of $Idl(\mathcal{D})$ is ωc-compact in $Idl(\mathcal{D})$. As $\iota_d(\mathcal{D}) \subseteq Ch(\mathcal{D})$ we have $K_d(Idl(\mathcal{D})) \subseteq K_{\omega c}(Ch(\mathcal{D}))$. To get the inclusion the other way round we consider an element $I \in K_{\omega c}(Ch(\mathcal{D}))$. Let $g(I) \sqsubseteq_c \bigsqcup C_i$ for an ω-chain $(C_i) \subseteq Chain(\mathcal{D})$. Then we have $(f \circ g)(I) = I \subseteq f(\bigsqcup C_i) = \bigsqcup f(C_i)$. As I is ωc-compact there exists $k \in \mathbb{N}$ such that $I \subseteq f(C_k)$. Therefore $g(I) \sqsubseteq_c C_k$ and thus $g(I) \in K_{\omega c}(Chain(\mathcal{D})) = \iota_c(\mathcal{D})$. This implies $I \in \iota_d(\mathcal{D})$.

We already know that g is monotone. Thus to finish the proof for the continuity of g we have to establish $g(\bigsqcup I_i) = \bigsqcup g(I_i)$ for an ω-chain $(I_i) \subseteq Ch(\mathcal{D})$. This can be done in the same way as for f in the proof of theorem 1.8. $Ch(\mathcal{D})$ is ωc-algebraic, therefore the construction of the desired ω-chains is possible. As g is built like f the evaluation of $g(\bigsqcup I_i)$ and $\bigsqcup g(I_i)$ results in similar terms as for f. The other arguments carry over by simple substitution. ∎

If D is countable then in $(Idl(\mathcal{D}), \subseteq)$ the concept of directed sets $S \subseteq Idl(\mathcal{D})$ reduces to the abilities of ω-chains in $Idl(\mathcal{D})$. We cite [AJ92] for the following theorem:

Theorem 1.10 (Directed sets and ω-chains)
If $(\mathcal{D}, \sqsubseteq)$ is a countable poset then every directed subset of $Idl(\mathcal{D})$ contains an ω-chain with the same supremum.

Corollary 1.11 (Isomorphism between chain and ideal completion)
If $(\mathcal{D}, \sqsubseteq)$ is a countable poset then $Ch(\mathcal{D}) = Idl(\mathcal{D})$ and therefore $(Chain(\mathcal{D}), \sqsubseteq_c) \cong (Idl(\mathcal{D}), \subseteq)$.

Proof: Let $I \in Idl(\mathcal{D})$. As $Idl(\mathcal{D})$ is d-algebraic there exists a directed set $S \subseteq \iota_d(\mathcal{D})$ with $\bigsqcup S = I$. Using theorem 1.10 we obtain an ω-chain $(C_i) \subseteq S$ with $\bigsqcup C_i = \bigsqcup S = I$ – all suprema computed in $Idl(\mathcal{D})$.

Now we look for the least upper bound of the obtained ω-chain (C_i) in $Ch(\mathcal{D})$. We know that $\forall i \in \mathbb{N} : C_i \in Ch(\mathcal{D})$ and that $(Ch(\mathcal{D}), \subseteq)$ is an ωc-cpo. Thus the ω-chain (C_i) has a supremum $C \in Ch(\mathcal{D})$. In both cpos, $Idl(\mathcal{D})$ and $Ch(\mathcal{D})$, the suprema are computed by $\bigcup C_i$. Therefore we have $C = I$. ■

1.3.1 Example: Isomorphism classes of plain trees – Part I

To give an example of a poset $(\mathcal{D}, \sqsubseteq)$ with $Ch(\mathcal{D}) \neq Idl(\mathcal{D})$ we introduce isomorphism classes of plain trees using the notions (for the most part word for word) of [BMC94].

A *plain tree* over a set of actions A and a set *Nodes* is a quadrupel $t = (N, E, l, v_0)$ consisting of a set $N \subseteq Nodes$ of nodes, a set $E \subseteq N \times N$ of edges, a labelling function $l : E \to A$ and a node $v_0 \in N$ such that (N, E) is a tree with root v_0 in the graphtheoretical sense, i.e. for each node $v \in N$ there exists a unique path from the root v_0 to v. The *depth* of a node $v \in N$ is the length of the (unique) path from the root to v. The *height* of a plain tree t it the length of a longest path in t. We denote the set of all plain trees over A and *Nodes* by $tree(A, Nodes)$.

Let $t = (N, E, l, v_0)$, $t' = (N', E', l', v_0')$ be plain trees over the same set of actions and over possibly different sets *Nodes* and *Nodes'*. An *embedding* $f : t \to t'$ is an injective function $f : N \to N'$ with $f(v_0) = v_0'$ such that the following condition is satisfied:

$$\text{If } (v, w) \in E \text{ then } (f(v), f(w)) \in E' \text{ and } l(v, w) = l(f(v), f(w)).$$

We call f an *isomorphism* from t to t' iff f is a bijecitive function $N \to N'$ such that f and f^{-1} are embeddings. $TREE(A)$ denotes the set of all ismorphism classes of plain trees over A and a countable set *Nodes*.

Let $t = (N, E, l, v_0)$ be a plain tree. A subset N' of N is called *leftclosed* iff N' is nonempty and for all $v \in N'$ the set of all predecessors of v is contained in N'. In this case

$$t\lceil N' := (N', E \cap (N' \times N'), l_{|E \cap (N' \times N')}, v_0)$$

is a plain tree.

Using this notion we define a partial order on $tree(A, Nodes)$. Let s and $t = (N, E, l, v_0)$ be plain trees then

$$s \sqsubseteq_{pt} t :\iff \exists N' \subseteq N : N' \text{leftclosed} \wedge s = t\lceil N'.$$

Now we can give a definition of a partial order on $TREE(A)$. Let $S, T \in TREE(A)$ be isomorphism classes of plain trees. Then we define:

$$S \sqsubseteq T :\Longleftrightarrow \exists s \in S, t \in T : \; s \sqsubseteq_{pt} t.$$

In general the relation $\sqsubseteq$ is not antisymmetric. But restricted to isomorphism classes of finitely branching trees it is a partial order.

We use the abbreviation $T_\perp$ for the isomorphism class of $(\{v_0\}, \emptyset, \emptyset, v_0)$. $\sum_{i=1}^{n} a_i.T_i$ describes the isomorphism class of a plain tree which has n branches at the root, each labelled with an action $a_i \in A$ and completed by an isomorphism class T_i. A detailed discussion of this notion can be found in [BMC94].

Using these definitions from [BMC94] we are able to present our example for a partial order $(\mathcal{D}, \sqsubseteq)$ with $Ch(\mathcal{D}) \neq Idl(\mathcal{D})$. Consider the poset $(\mathcal{D}, \sqsubseteq)$ where

$$\mathcal{D} := \{T \in TREE(\mathbb{R}) \,|\, T \text{ finitely branching and } height(T) \leq 1\}$$

and $\sqsubseteq$ is the above mentioned partial order on isomorphism classes of plain trees redustricted to $\mathcal{D}$.

We claim that $\mathcal{D}$ is a directed set. To prove this let $S = \sum_{i=1}^{n} a_i.T_\perp$, $T = \sum_{j=1}^{m} b_j.T_\perp \in \mathcal{D}$. We define $U := \sum_{i=1}^{n} a_i.T_\perp + \sum_{j=1}^{m} b_j.T_\perp$. Obviously we have $U \in \mathcal{D}$ and $S, T \sqsubseteq U$.

$\mathcal{D}$ is not countable because it has for example $E := \{r.T_\perp \,|\, r \in \mathbb{R}\}$ as subset. Consider the ideal completion of $\mathcal{D}$. As $Idl(\mathcal{D})$ is a d-cpo it contains $I := \bigsqcup \iota_d(\mathcal{D}) = \bigcup_{d \in \mathcal{D}} \iota_d(d)$. As we have especially $E \subseteq I$ the set I is not countable.

In the case of $Ch(\mathcal{D})$ we claim that all its elements are countable sets. To prove this we study first sets of the form $\downarrow S$ in $\mathcal{D}$, where $S \in \mathcal{D}$ is an ismormism class of plain trees. Let $S \in \mathcal{D}$. If $S = T_\perp$ we get $|\downarrow T_\perp| = 1$. If $S = \sum_{i=1}^{n} a_i.T_\perp$ then the set $\downarrow S = \{T_\perp\} \cup \{\sum_{i \in I_n} a_i.T_\perp \,|\, I_n \subseteq \{1, 2, \ldots, n\}\}$ is finite. Let $C = \bigcup_{i \in \mathbb{N}} \downarrow T_i \in Ch(\mathcal{D})$. As the sets $\downarrow T_i$ are finite and the index set is countable, the set C is countable.

Therefore we know that the above constructed set I is not in $Ch(\mathcal{D})$ and hence $Ch(\mathcal{D}) \neq Idl(\mathcal{D})$. ■

1.3.2 Example: Finite Strings over an alphabet A

With corollary 1.11 the question arises whether we have always $Ch(\mathcal{D}) \neq Idl(\mathcal{D})$ if $\mathcal{D}$ is not countable. The following example shows that this is not the case.

Let A be an alphabet, $\mathcal{D} := A^*$ the set of all finite words over A including the empty word ϵ. We use the prefix relation to define a partial order on $\mathcal{D}$. For all $u, v \in \mathcal{D}$ we define:

$$u \sqsubseteq_{prefix} v :\Longleftrightarrow \exists w \in A^* : \; uw = v.$$

Lemma 1.12 (Ideals in $(A^*, \sqsubseteq_{prefix})$)
Let A be an alphabet, $\mathcal{D} := A^*$ ordered by $\sqsubseteq_{prefix}$ and $I \subseteq \mathcal{D}$ an ideal. Then I is an ω-chain.

Proof: Let $I \subseteq D$ be an ideal, $x, y \in I$. Then there exists $z \in I$ with $x, y \sqsubseteq_{prefix} z$. Therefore there exist $s, t \in \mathcal{D}$ such that $xs = z = yt$. Let w.o.l.g. $length(s) \geq length(t)$. Then we get $x \sqsubseteq_{prefix} y$. Thus I is totally ordered by $\sqsubseteq_{prefix} \cap I \times I$.

An element of I is uniquely determined by its length: Let $x, y \in I$ with $length(x) = length(y)$. Then $x = y$ because we may assume w.l.o.g. $x \sqsubseteq_{prefix} y \Rightarrow x\epsilon = x = y$.

To prove that I is countable we introduce $s := \sup\{length(x) \mid x \in I\}$. If $s < \infty$ there exists an element $d \in I$ with $length(d) = s$. In this case we have $I = \{\epsilon, x_1, x_2, \ldots, x_s\}$, where x_i is the prefix of d with length i. Thus I is a finite set.

If $s = \infty$ we claim that $I = (a_i)$ for an ω-chain (a_i) where $\forall i \in \mathbb{N} : length(a_i) = i$. As $s = \infty$ there exists for all $n \in \mathbb{N}$ an element $a_m \in I$ with $length(a_m) \geq n$. With a_m all its prefixes are elements of I because an ideal is especially leftclosed. As an element of I is uniquely determined by its length we may conclude $I = (a_i)$. ■

Theorem 1.13 $(Ch(A^*) = Idl(A^*))$
Let A be an alphabet, $\mathcal{D} := A^*$ ordered by $\sqsubseteq_{prefix}$. Then $Ch(A^*) = Idl(A^*)$ and therefore $(Chain(A^*), \sqsubseteq_c) \simeq (Idl(A^*), \subseteq)$.

Proof: Using theorem 1.9 we have $Ch(A^*) \subseteq Idl(A^*)$. Lemma 1.12 gives us the inclusion in the other direction. ■

1.4 Metric concepts on pointed posets

So far we dealed exclusively with order-theoretical concepts. Now we introduce a metric on pointed posets with length. The definitions and theorems presented in this subsection are completely due to [MCB94] (for the most part word for word). They can just so be found – in a slightly different manner – in [BMC95]. The only new part is the straight forward definition of the length ρ^+ and the metric d_ρ^+ on $Ch(\mathcal{D})$.

First we give the definitions of different kinds of a length on a pointed poset. Such a length can be used to enrich a poset with a pseudo ultrametric or – under certain circumstances – with an ultrametric or even a complete ultrametric. Finally we show in which way a finite length on a pointed poset $\mathcal{D}$ induces an ultrametric on $Ch(\mathcal{D})$, $Idl(\mathcal{D})$ and $P_\downarrow(\mathcal{D})$. This is the starting point of chapter two where we relate the concepts of chain, ideal and metric competion.

Let $(\mathcal{D}, \sqsubseteq)$ be a pointed poset. We call a function $\rho : (\mathcal{D}, \sqsubseteq) \to (\mathbb{N}_0 \cup \{\infty\}, \leq)$ *length* on

$\mathcal{D}$, iff for all $x, y \in \mathcal{D}$ holds:

$$\rho(x) = 0 \iff x = \perp \quad \text{and} \quad x \sqsubseteq y \Rightarrow \rho(x) \leq \rho(y).$$

For $x \in \mathcal{D}$, $n \in \mathbb{N}$ we define:

$$\downarrow^n (x) := \{y \in \mathcal{D} \mid y \sqsubseteq x \wedge \rho(y) \leq n\}, \quad \downarrow^{fin} (x) := \bigcup_{n \in \mathbb{N}} \downarrow^n (x).$$

ρ is called *finite*, iff for all $x \in \mathcal{D}$ we have $\rho(x) < \infty$. An element $x \in \mathcal{D}$ is called *approximable*, iff x is the least upper bound of $\downarrow^{fin} (x)$. $M(\mathcal{D})$ denotes the set of approximable elements.

Let $(\mathcal{D}, \sqsubseteq)$ be a pointed poset. A *weight* is a lenght ρ on $(\mathcal{D}, \sqsubseteq)$ such that for all $x \in \mathcal{D}$ and $n \geq 0$ the set $\downarrow^n (x)$ has a greatest element which we denote by $x[n]$. $x[n]$ is called the n-cut of x with respect to ρ. We call a weight ρ on an ωc-cpo $(\mathcal{D}, \sqsubseteq)$ *ωc-continuous* iff for all $n \in \mathbb{N}$ the function $f_n : \mathcal{D} \to \mathcal{D}$, $x \mapsto x[n]$, is ωc-continuous. We call a weight ρ on a d-cpo $(\mathcal{D}, \sqsubseteq)$ *d-continuous* iff for all $n \in \mathbb{N}$ the function $f_n : \mathcal{D} \to \mathcal{D}$, $x \mapsto x[n]$, is d-continuous.

Let $(\mathcal{D}, \sqsubseteq)$ be a pointed poset with a length ρ. Then

$$d[\rho] : \begin{cases} \mathcal{D} \times \mathcal{D} & \to \quad \mathbb{R}_{\geq 0} \\ (x, y) & \mapsto \quad d[\rho](x, y) := \inf \{ \frac{1}{2^n} \mid \downarrow^n (x) = \downarrow^n (y) \} \end{cases}$$

is a pseudo ultrametric on D and an ultrametric on $M(\mathcal{D})$. If ρ is finite then $d[\rho]$ is an ultrametric on $\mathcal{D}$.

The following theorem shows that the length covers enough information of the partial order and thus enforces that under certain circumstances limit and least upper bound coincide.

Theorem 1.14 (Limit of monotone Cauchy-sequences)
Let $(\mathcal{D}, \sqsubseteq)$ be a d-cpo, ρ a d-continuous weight on $\mathcal{D}$. Then the induced ultrametric space $(M(\mathcal{D}), d[\rho])$ is complete. For each monotone Cauchy-sequence (x_n) in $M(\mathcal{D})$ we get

$$\lim_{n \to \infty} x_n = \bigsqcup x_n.$$

To give some examples for weights we cite [MCB94]:

> *The concept of a finite weight can be realized on various domains, e.g.*
>
> - *finite strings over some alphabet A (endowed with the prefixing ordering and the weight $\rho(x) = |x|$ where $|x|$ means the usual length of a string x),*

- *trees of finite heigth endowed with Winskel's partial order [Win84] and height as underlying weight and*
- *prime event structures of finite depth with Winskel's partial order [Win82] and the depth as underlying weight.*

Mazurkiewicz traces [Maz89] yield an example for a length which is not a weight.[5]

Given a length ρ on a pointed poset $(\mathcal{D}, \sqsubseteq)$ we are interested to carry over this concept to $Ch(\mathcal{D})$ and $Idl(\mathcal{D})$. Here we use the set $P_\downarrow(\mathcal{D})$ as common platform.

Let $(\mathcal{D}, \sqsubseteq)$ be a pointed poset with length ρ. Then

$$\rho_\downarrow : \begin{cases} (P_\downarrow(\mathcal{D}), \subseteq) & \rightarrow \quad (\mathbb{N}_0 \cup \{\infty\}, \leq) \\ X & \mapsto \quad \sup\{\, \rho(x) \mid x \in X \,\} \end{cases}$$

is a d-continuous weight on $P_\downarrow(\mathcal{D})$. The n-cut of an element $X \in P_\downarrow(\mathcal{D})$ with respect to $\rho_\downarrow$ may be computed by

$$X[n] = \{x \in X \mid \rho(s) \leq n\}.$$

If ρ is finite then all elements of $P_\downarrow(\mathcal{D})$ are approximable with respect to $\rho_\downarrow$ and therefore $(P_\downarrow(\mathcal{D}), d[\rho_\downarrow])$ is a complete ultrametric space. The metric $d[\rho_\downarrow]$ on $P_\downarrow(\mathcal{D})$ may be computed by the formula

$$d[\rho_\downarrow](X, Y) = \inf\{\, \frac{1}{2^n} \mid X[n] = Y[n] \,\}.$$

If $(X_n)_{n \in \mathbb{N}} \subseteq P_\downarrow(\mathcal{D})$ is a Cauchy-sequence with $d[\rho_\downarrow](X_n, X_m) \leq \frac{1}{2^n}$ for all $n \leq m \in \mathbb{N}$ then its limit is

$$\lim_{n \to \infty} X_n = \bigcup_{n \in \mathbb{N}} X_n[n]. \tag{1}$$

Let ρ be a finite length on a pointed poset $(\mathcal{D}, \sqsubseteq)$. Then we denote the restriction of $\rho_\downarrow$ to $Ch(\mathcal{D})$ by ρ^+ and the restriction of $d[\rho_\downarrow]$ to $Ch(\mathcal{D})$ by d^+_ρ. Analogous we denote the restriction of $\rho_\downarrow$ to $Idl(\mathcal{D})$ by ρ^* and the restriction of $d[\rho_\downarrow]$ to $Idl(\mathcal{D})$ by d^*_ρ. As $(P_\downarrow(\mathcal{D}), d[\rho_\downarrow])$ is a metric space $(Ch(\mathcal{D}), d^+_\rho)$ and $(Idl(\mathcal{D}), d^*_\rho)$ are just so.

We use the following notions: Let (M, d) be a metric space. Then the metric completion of (M, d) is denoted by $(\overline{M}, \overline{d})$. We assume that $M \subseteq \overline{M}$ and that d is the restriction of $\overline{d}$ on M. If (N, d') is a metric space and $f : M \rightarrow N$ a non-distance-increasing function then $\overline{f}$ denotes the unique non-distance-increasing function $\overline{M} \rightarrow \overline{N}$ with $\overline{f}(x) = f(x)$ for all $x \in M$.

2 $Ch(\mathcal{D})$ and $Idl(\mathcal{D})$ as metric spaces

The aim of this paper is to give a basis for studies concerning consistency between denotational semantics which employ different concepts to realize recursion: On the one hand there are semantics which use partial order techniques on the other hand there are semantics based on metric concepts. In this section we present completions of semantic domains which contain both: The supremum of an ω-chain and the limit of a Cauchy sequence.

We begin with a survey on the main concepts of chapter one, i.e. chain, ideal and metric completion of a pointed poset $(\mathcal{D}, \sqsubseteq)$ with length ρ. Before getting started we convince ourselves that the introduced completion concepts are really different. Then we look for conditions which ensure that chain completion $(Ch(\mathcal{D}), d_\rho^+)$ and ideal completion $(Idl(\mathcal{D}), d_\rho^*)$ are complete metric spaces. Finally we study the relation between the metric completion $(\overline{\mathcal{D}}, \overline{d[\rho]})$ and the order theoretical completions $Ch(\mathcal{D})$ and $Idl(\mathcal{D})$ under the condition that the latter are complete metric spaces.

2.1 Synopsis: Completions on a pointed poset with length

This section gives a synopsis on the so far introduced completions. We start with an overview of the relevant order theoretical concepts and conclude with the metric situation.

Let $(\mathcal{D}, \sqsubseteq)$ be a poset. From the order theoretical point of view we have the following situation:

$$\iota_d : \ (\mathcal{D}, \sqsubseteq) \ \rightarrow \ (Ch(\mathcal{D}), \subseteq) \ \subseteq \ (Idl(\mathcal{D}), \subseteq) \ \subseteq \ (P_\downarrow(\mathcal{D}), \subseteq).$$

The function $\iota_d : \mathcal{D} \rightarrow Ch(\mathcal{D})$ is monotone.

Let $(\mathcal{D}, \sqsubseteq)$ be a pointed poset with finite length ρ. Looking from the metric setting we find:

$$\begin{array}{ccccccccc}
\iota_d : & (\mathcal{D}, d[\rho]) & \rightarrow & (Ch(\mathcal{D}), d_\rho^+) & \subseteq & (Idl(\mathcal{D}), d_\rho^*) & \subseteq & (P_\downarrow(\mathcal{D}), d[\rho_\downarrow]) \\
 & \subseteq & & \subseteq & & \subseteq & & = \\
\overline{\iota_d} : & (\overline{\mathcal{D}}, \overline{d[\rho]}) & \rightarrow & (\overline{Ch(\mathcal{D})}, \overline{d_\rho^+}) & \subseteq & (\overline{Idl(\mathcal{D})}, \overline{d_\rho^*}) & \subseteq & (\overline{P_\downarrow(\mathcal{D})}, \overline{d[\rho_\downarrow]}).
\end{array}$$

The function $\iota_d : \mathcal{D} \rightarrow Ch(\mathcal{D})$ is an isometric embedding of the metric space $(\mathcal{D}, d[\rho])$ into the metric space $(Ch(\mathcal{D}), d_\rho^+)$ and hence its canonical extension $\overline{\iota_d} : \overline{\mathcal{D}} \rightarrow \overline{Ch(\mathcal{D})}$ is just so.

2.2 A first reflection

Before we present theorems on the relations of the above described completions of a pointed poset $(\mathcal{D}, \sqsubseteq)$ with a length ρ we should convince ourselves that

1. in general neither $(Idl(\mathcal{D}), d_\rho^*)$ nor $(Ch(\mathcal{D}), d_\rho^+)$ are complete metric spaces and that

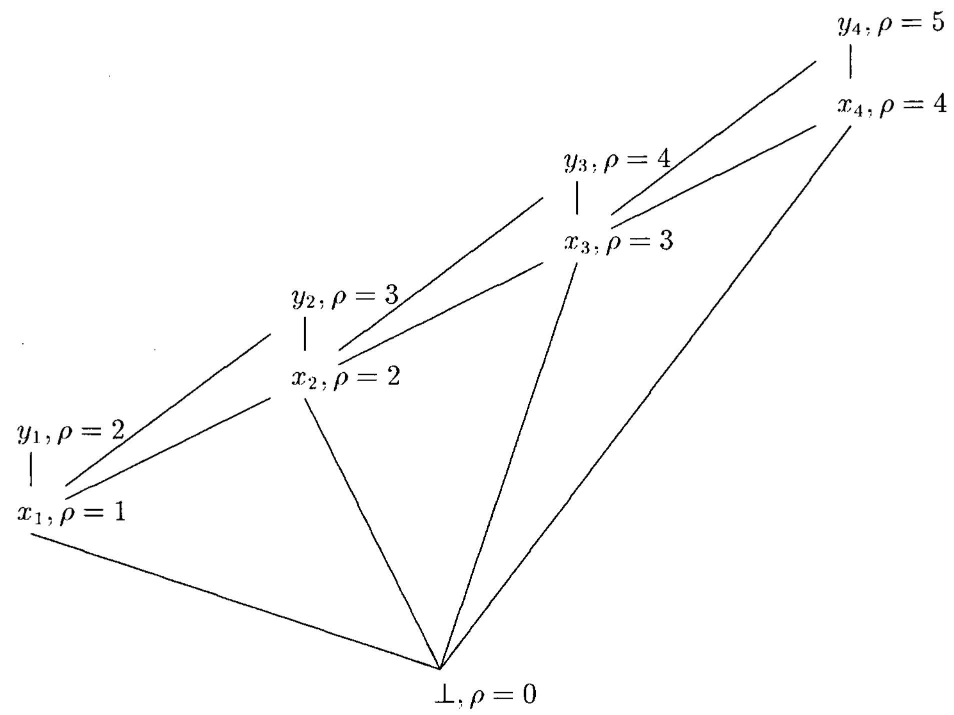

Figure 2: A domain with a Cauchy sequence which is not an ω-chain

2. the concepts of an ω-chain in $(\mathcal{D}, \sqsubseteq)$ and of a Cauchy sequence in $(\mathcal{D}, d[\rho])$ are really different.

For the first item we refer to [MCB94]. They give an example of a pointed poset $(\mathcal{D}, \sqsubseteq)$ with finite length ρ where $Idl(\mathcal{D}, d^*_\rho)$ is not a complete metric space. The same example holds for $(Ch(\mathcal{D}), d^+_\rho)$ because the chosen set $\mathcal{D}$ is countable and thus we have $Idl(\mathcal{D}) = Ch(\mathcal{D})$.

For the second item we give two examples of our own. The first example shows a Cauchy sequence in a metric space $(\mathcal{D}, d[\rho])$ which is not an ω-chain with respect to the partial order $\sqsubseteq$ on $\mathcal{D}$. It is moreover a demonstration of a poset $(\mathcal{D}, \sqsubseteq)$ where the chain completion is a complete metric space. The second example exhibits an ω-chain in another poset $(\mathcal{D}, \sqsubseteq)$ with length ρ which is not a Cauchy sequence with respect to induced metric $d[\rho]$.

2.2.1 Example: A Cauchy sequence which is not an ω-chain

To present an example of a Cauchy sequence which is not an ω-chain we introduce a pointed poset $(\mathcal{D}, \sqsubseteq)$ with length where

$$\mathcal{D} := \{\perp\} \cup \{x_n, y_n \,|\, n \in \mathbb{N}\}$$

and where $\sqsubseteq$ is the smallest partial order on $\mathcal{D}$ which satisfies:

$$\forall n \in \mathbb{N} : \perp \sqsubseteq x_n \;\wedge\; x_n \sqsubseteq x_{n+1} \;\wedge\; x_n \sqsubseteq y_n.$$

We define a finite length $\rho : \mathcal{D} \to \mathbb{N}_0$ by

$$\rho(\perp) := 0 \text{ and for all } n \in \mathbb{N}: \; \rho(x_n) := n, \; \rho(y_n) := n+1.$$

Figure 2 shows a small part of the poset $\mathcal{D}$. It is easy to prove that ρ is a finite weight:

- $\downarrow^k (\perp) = \{\perp\}$ for all $k \in \mathbb{N}$.
- For all $n \in \mathbb{N} : \downarrow^k (x_n) = \begin{cases} \downarrow x_n, & k > n \\ \downarrow x_k, & 1 \leq k \leq n \\ \{\perp\}, & k = 0. \end{cases}$
- For all $n \in \mathbb{N} : \downarrow^k (y_n) = \begin{cases} \downarrow y_n, & k > n \\ \downarrow x_k, & 1 \leq k \leq n \\ \{\perp\}, & k = 0. \end{cases}$

We claim that $(y_i)_{i\in\mathbb{N}}$ is a Cauchy sequence but not an ω-chain in $\mathcal{D}$. To prove this we use the above computed results concerning y_m, y_n with $m > n \in \mathbb{N}$. They imply that the sets $\downarrow^k (x_m)$ and $\downarrow^k (x_n)$ coincide exactly for $0 \leq k \leq n$. Thus for the distance holds $d[\rho](y_n, y_m) = \frac{1}{2^n}$, $n < m \in \mathbb{N}$. As neither $y_i \sqsubseteq y_{i+1}$ nor $y_{i+1} \sqsubseteq y_i$ the Cauchy sequence (y_i) is not an ω-chain.

Next we study whether the ideal completion of $\mathcal{D}$ contains a limit for the Cauchy sequence $(\iota_d(y_i))$. In order to compute $Idl(\mathcal{D})$ we may use the identity $Ch(\mathcal{D}) = Idl(\mathcal{D})$ of corollary 1.11 because $\mathcal{D}$ is countable. Theorem 1.8 tells us via the characterization of $Ch(\mathcal{D}) = Im(f)$ that the only "new" elements in $Ch(\mathcal{D})$ arise from non-stationary ω-chains in $(\mathcal{D}, \sqsubseteq)$. ω-chains which contain an y_i have to become stationary, as there are no elements above y_i. $[(x_i)] \in Chain(\mathcal{D})$ is the only class of non-stationary ω-chains in $\mathcal{D}$ built from $\perp$ and elements of $\{x_n \,|\, n \in \mathbb{N}\}$. Thus we have

$$Idl(\mathcal{D}) = Ch(\mathcal{D}) = \{\iota_d \,|\, d \in \mathcal{D}\} \cup \{f([(x_i)])\}.$$

$f([(x_i)]) = \bigcup \iota_d(x_i)$ is both: The least upper bound of $(\iota_d(x_i))_{i\in\mathbb{N}}$ thought as ω-chain and limit of the same sequence understood as Cauchy sequence. Furthermore it is the limit of the Cauchy sequence $(\iota_d(y_i))_{i\in\mathbb{N}}$. Thus $(Ch(\mathcal{D}), \subseteq)$ is an ωc-cpo and $(Ch(\mathcal{D}), d_\rho^+)$ is a complete metric space - this result coincides with theorem 2.5 which we will present in section 2.3.3.

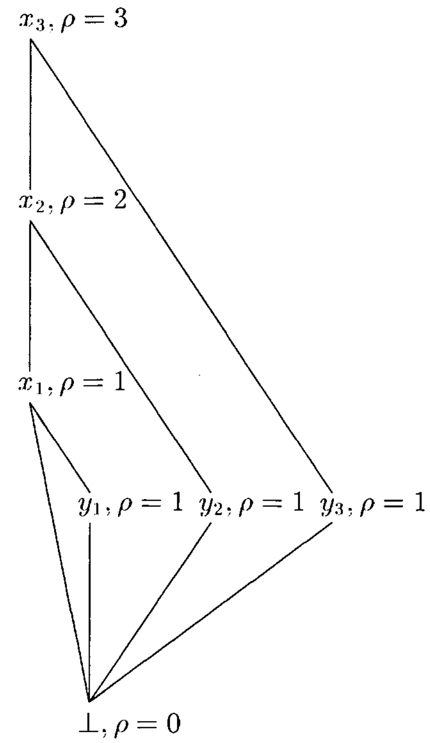

Figure 3: A domain with an ω-chain which is not a Cauchy sequence

2.2.2 Example: An ω-chain which is not a Cauchy sequence

To present an example of an ω-chain which is not a Cauchy sequence we introduce a pointed poset $(\mathcal{D}, \sqsubseteq)$ with length where

$$\mathcal{D} := \{\perp\} \cup \{x_n, y_n \mid n \in \mathbb{N}\}$$

and where $\sqsubseteq$ is the smallest partial order on $\mathcal{D}$ which satisfies:

$$\forall n \in \mathbb{N} : \perp \sqsubseteq x_n, y_n \ \wedge \ x_n \sqsubseteq x_{n+1} \ \wedge \ y_n \sqsubseteq x_n.$$

We define a finite length $\rho : \mathcal{D} \to \mathbb{N}_0$ by

$$\rho(\perp) := 0 \text{ and for all } n \in \mathbb{N} : \ \rho(x_n) := n, \ \rho(y_n) := 1.$$

Figure 3 shows a small part of the poset $\mathcal{D}$. Obviously (x_n) is an ω-chain in $(\mathcal{D}, \sqsubseteq)$, but we claim that it is not a Cauchy sequence with respect to $d[\rho]$. To prove this we compute the distance $d(x_m, x_n)$ for $m > n \in \mathbb{N}$. The sets $\downarrow^k (x_m)$ and $\downarrow^k (x_n)$ coincide for $k = 0$. But for $k > 0$ we have $(y_m) \in \downarrow^k (x_m)$ and $(y_m) \notin \downarrow^k (x_n)$. Therefore $d(x_m, x_n) = \inf\{ \frac{1}{2^k} \mid \downarrow^k (x_m) = \downarrow^k (x_n) \} = 1$ and (x_n) is not a Cauchy sequence.

Without a proof we claim that like in example 2.2.1 ρ is a finite weight. The set $\mathcal{D}$ is countable, therefore $Idl(\mathcal{D})$ and $Ch(\mathcal{D})$ coincide. Using theorem 2.5 from section 2.3.3 again we establish that $(Ch(\mathcal{D}), d_\rho^+)$ is a complete metric space.

2.3 $(Ch(\mathcal{D}), d_\rho^+)$ and $(Idl(\mathcal{D}), d_\rho^*)$ as complete metric spaces

The aim of this section is to find properties of a pointed poset $(\mathcal{D}, \sqsubseteq)$ with a finite length ρ or a finite weight ρ which ensure that $(Ch(\mathcal{D}), d_\rho^+)$ respective $(Idl(\mathcal{D}), d_\rho^*)$ are complete metric spaces. The central idea to establish such a characteristic is based on the computation of limites in $(P_\downarrow(\mathcal{D}), d[\rho_\downarrow])$ which we presented in section 1.7 as equation (1): The limit X of a Cauchy sequence $(X_n) \subseteq P_\downarrow(\mathcal{D})$ with $d[\rho_\downarrow](X_m, X_n) \leq \frac{1}{2^n}$ for all $n \leq m \in \mathbb{N}$ may be computed as

$$\lim_{n\to\infty} X_n = \bigcup_{n\in\mathbb{N}} X_n[n] =: X.$$

Thus to establish that $(Ch(\mathcal{D}), d_\rho^+)$ is a complete metric space we have to prove that for any Cauchy sequence $(X_n) \subseteq Ch(\mathcal{D})$ its limit $X \in P_\downarrow(\mathcal{D})$ is of the form $\bigcup \iota_d(c_i)$ for an ω-chain $(c_i) \subseteq \mathcal{D}$ and therefore an element of $Ch(\mathcal{D})$.

For the ideal completion we formulate: To establish that $(Idl(\mathcal{D}), d_\rho^*)$ is a complete metric space we have to prove that for any Cauchy sequence $(X_n) \subseteq Idl(\mathcal{D})$ its limit $X \in P_\downarrow(\mathcal{D})$ is an ideal in $(\mathcal{D}, \sqsubseteq)$ and therefore an element of $Idl(\mathcal{D})$.

This section is divided in three parts: First we deal with pointed posets which are equipped with a finite length. Then we present Mazurkiewicz traces as an application of our theoretical results. The last part studies the situation when the poset exhibits a finite weight.

2.3.1 Starting with a length

Concerning the ideal completion of a pointed poset $(\mathcal{D}, \sqsubseteq)$ with length ρ we cite a result of [MCB94]:

Theorem 2.1 (The ideal completion as cms induced by a length)
Let $(\mathcal{D}, \sqsubseteq)$ be a pointed fb-cpo with a finite length ρ. Then $(Idl(\mathcal{D}), d_\rho^*)$ is a complete metric space and $\overline{\iota_d} : (\overline{\mathcal{D}, d[\rho]}) \to (Idl(\mathcal{D}), d_\rho^*)$ is an isometric embedding.

In order to establish an analog result for the chain completion we provide a simple lemma which is a slight modification of theorem 1.10: While we requested there that the poset $\mathcal{D}$ should be countable, we now assume that an ideal $I \subseteq \mathcal{D}$ is countable.

Lemma 2.2 (Countable ideals and ω-chains)
Let $(\mathcal{D}, \sqsubseteq)$ be a poset, $I \in Idl(\mathcal{D})$ be a countable set. Then there exists an ω-chain $(c_i) \subseteq \mathcal{D}$ such that $I = \bigcup \iota_d(c_i)$, i.e. $I \in Ch(\mathcal{D})$.

Proof: Let (x_i) be an enumeration of I. Define the ω-chain (c_i) by induction on i, starting with $c_1 := x_1$. Assume that we have defined the elements of this chain up to c_i. Consider the set $S := \{k \in \mathbb{N} \mid x_k \notin \bigcup_{j=1}^{i} \iota_d(c_i)\}$. If this set is empty let $c_{i+1} := c_i$ else let $l := \min S$. As I is directed, c_i and x_l are in I, there exists $z \in I$ with $c_i \sqsubseteq z$ and $x_l \sqsubseteq z$. Let $c_{i+1} := z$. Obviously we get $\bigcup \iota_d(c_i) \subseteq I$. To prove the inclusion the other way round let $x \in I$. Then there exists an index $n \in \mathbb{N}$ such that $x = x_n$. By construction we have $x_n \sqsubseteq c_n$, thus $x_n \in \bigcup \iota_d(c_i)$. ■

Theorem 2.3 (The chain completion as cms induced by a length)
Let $(\mathcal{D}, \sqsubseteq)$ be a pointed fb-cpo with a finite length ρ such that for all $C \in Ch(\mathcal{D})$, $n \in \mathbb{N}$ holds: the set $C[n] := \{c \in C \mid \rho(c) \leq n\}$ is countable. Then $(Ch(\mathcal{D}), d_\rho^+)$ is a complete metric space and $\overline{\iota_d} : (\overline{\mathcal{D}}, \overline{d[\rho]}) \to (Ch(\mathcal{D}), d_\rho^+)$ is an isometric embedding.

Proof: Using theorem 2.1 we know: Under the choosen assumptions on $(\mathcal{D}, \sqsubseteq)$ and ρ the ideal completion $(Idl(\mathcal{D}), d_\rho^*)$ is a complete metric space, i.e. the limit I of any Cauchy sequence sequence $(I_n) \subseteq Idl(\mathcal{D})$ is an ideal in $\mathcal{D}$.

Let $(C_i) \subseteq Ch(\mathcal{D})$ be a Cauchy sequence. As $Ch(\mathcal{D}) \subseteq Idl(\mathcal{D})$ we know that $C := \lim_{n\to\infty} C_n$ – computed in $Idl(\mathcal{D})$ – is an ideal in $\mathcal{D}$.

We claim that C is countable: By assumption for all $n \in \mathbb{N}$ the sets $C_n[n]$ are countable. In $Idl(\mathcal{D})$ the limit C is computed as the union of all these sets and therefore countable. Using lemma 2.2 this establishes $C \in Ch(C)$ and therefore $(Ch(\mathcal{D}), d_\rho^+)$ is a complete metric space. ■

2.3.2 Example: Mazurkiewicz traces – Part I

To present an application of theorem 2.3 we introduce the domain of Mazurkiewicz traces [Maz89]. A *concurrent alphabet* is a pair (A, Ind) consisting of a set of actions A and an independence relation $Ind \subseteq A \times A$ which is irreflexive and symmetric. Let for $x, y \in A^*$

$$x \equiv' y \;:\Longleftrightarrow\; \exists a, b \in A\, \exists u, v \in A^* :\; (a, b) \in Ind \wedge x = uabv \wedge y = ubav.$$

We define an equivalence relation $\equiv$ on A^* as the reflexive and transitive closure of the relation $\equiv'$ and denote the induced equvalence classes by $[x]$ for $x \in A^*$. The set of all Mazurkiewicz traces on a concurrent alphabet (A, Ind) is given by

$$MT(A, Ind) := \{[x] \mid x \in A^*\}.$$

Using the prefix ordering on A^* – see section 1.3.2 – we define a partial order $\sqsubseteq$ on $MT(A, Ind)$. Let $[x], [y] \in MT(A, Ind)$:

$$[x] \sqsubseteq [y] \;:\Longleftrightarrow \exists u \in [x],\, v \in [y]: \; u \sqsubseteq_{prefix} v.$$

The length ρ of an element $[x]$ of $MT(A, Ind)$ is given by $\rho([x]) := |x|$. Thus we gained a pointed poset with length. For further details see [Maz89].

If A contains more than one element and $Ind \neq \emptyset$ the function ρ is not a weight. For this result we cite an example of [MCB94]: Let $A := \{\alpha, \beta\}$, $Ind := \{(\alpha, \beta), (\beta, \alpha)\}$. Consider the set $\downarrow^1([\alpha\beta]) = \{\perp, [\alpha], [\beta]\}$. It contains no greatest element since $[\alpha]$ and $[\beta]$ are incomparable.

[Kwi91] has shown that $(MT(A, Ind), \sqsubseteq)$ is an fb-cpo. Thus we get with theorem 2.1: $(Idl(MT(A, Ind)), d^*_\rho)$ is a complete metric space. If $MT(A, Ind)$ is countable its chain completion coincides with its ideal completion. There remain two problems:

1. Do the Mazurkiewicz traces fulfill the requirements of theorem 2.3?

2. Is there a concurrent alphabet (A, Ind) with $Ch(MT(A, Ind)) \neq Idl(MT(A, Ind))$?

As we will show in the sequel the answer is "yes" for both questions.

Concerning the first question we prove a stronger property than the required one in theorem 2.3. We claim that for any concurrent alphabet (A, Ind) the elements $C \in Ch(MT(A, Ind))$ are countable sets.

Before we begin with the proof we define: Let $[x] \in MT(A, Ind)$ be a Mazurkiewicz trace. $action([x]) := \{a \in A \mid \exists 1 \leq j \leq |x| : \; x_j = a\}$ denotes the set of all actions to be found in $[x]$.

Let $C = \bigcup \iota_d(c_i) \in Ch(MT(A, Ind))$. For all $i \in \mathbb{N}$ the set $action(c_i)$ is finite. Thus their union $\bigcup action(c_i) =: B$ is countable. This implies that B^* is countable and therefore $MT(B, Ind \cap (B \times B))$ is countable.

Let $[x] \in C$. Then there exists $k \in \mathbb{N} : [x] \sqsubseteq c_k$, i.e. there exist representants $x' \in [x]$, $c'_k \in c_k$ such that $x' \sqsubseteq_{prefix} c'_k$. This means especially that $[x]$ consists only from actions in B and is therefore an element of $MT(B, Ind \cap (B \times B))$. Thus the set C is countable.

This result imlies that independent of the cardinality of the set A in the concurrent alphabet (A, Ind) the chain completion of Mazurkiewicz traces $(Ch(MT(A, Ind), d^+_\rho))$ is a complete metric space.

To answer the second question we give an example of a concurrent alphabet (A, Ind) with $Ch(MT(A, Ind)) \neq Idl(MT(A, Ind))$. In order to etablish this inequality we use the property that elements $C \in Ch(MT(A, Ind))$ are countable sets and construct an ideal $I \in Idl(MT(A, Ind))$ that is not countable.

Let (A, Ind) be a concurrent alphabet with both, A and Ind, not countable. We use the notion $\pi_i(Ind) := \{a_i \in A \mid (a_1, a_2) \in Ind\}$, $i \in \{1, 2\}$, to denote the projection of Ind on its i^{th} component.

First we claim that $\pi_1(Ind)$ is not countable. Assume that $\pi_1(Ind)$ is countable. As Ind is a symmetric relation we have $\pi_2(Ind) = \pi_1(Ind)$ and therefore the set $Ind' := \pi_1(Ind) \times \pi_2(Ind)$ is countable. As $Ind \subseteq Ind'$ the set Ind is countable – contradiction.

Now we define $I' := MT(\pi_1(Ind), Ind)$. As $\{[a] \mid a \in \pi_1(Ind)\} \subseteq I'$ the set I' is not countable. To show that it is directed let $[u], [v] \in I'$. Obviously we have $[u] \sqsubseteq [uv]$ and $[v] \sqsubseteq [vu]$. As $[u]$ and $[v]$ are built from actions in $\pi_1(Ind)$ we have $[uv] = [vu]$ and $[uv] \in I'$. Thus the set $I := \iota_d(I')$ is an ideal in $MT(A, Ind)$, therefore an element of $Idl(MT(A, Ind))$, and especially not countable.

2.3.3 Starting with a weight

If the pointed poset $(\mathcal{D}, \sqsubseteq)$ is equipped with a weight ρ the requirements to establish that the chain completion respective the ideal completion are complete metric spaces become less strong. Again we cite first [MCB94] for a theorem concerning the ideal completion before we present our result in the case of the chain completion.

Theorem 2.4 (The ideal completion as cms induced by a weight)
Let ρ be a finite weight on a pointed poset $(\mathcal{D}, \sqsubseteq)$. Then $(Idl(\mathcal{D}), d^*_\rho)$ is a complete metric space, $\overline{\iota_d} : (\overline{\mathcal{D}}, \overline{d[\rho]}) \to (Idl(\mathcal{D}), d^*_\rho)$ is an isometric embedding and ρ^* is a d-continuous weight.

Theorem 2.5 (The chain completion as cms induced by a weight)
Let ρ be a finite weight on a pointed poset $(\mathcal{D}, \sqsubseteq)$. Then $(Ch(\mathcal{D}), d^+_\rho)$ is a complete metric space, $\overline{\iota_d} : (\overline{\mathcal{D}}, \overline{d[\rho]}) \to (Ch(\mathcal{D}), d^+_\rho)$ is an isometric embedding and ρ^+ is an ωc-continuous weight.

Proof: First we claim that ρ^+ is a weight on $Ch(\mathcal{D})$, i.e. for all $C \in Ch(D)$, $n \in \mathbb{N}$, the set $C[n] := \{c \in C \mid \rho(c) \leq n\}$ is an element of $Ch(\mathcal{D})$:

Let $C \in Ch(D)$, $n \in \mathbb{N}$. As $Ch(D)$ is ωc-algebraic there exists an ω-chain $(c_i) \in \mathcal{D}$ such that $C = \bigcup \iota_d(c_i)$. By assumption ρ is a weight on $\mathcal{D}$, thus for all c_i their the n-cut $c_i[n]$ exists in $\mathcal{D}$. $(c_i[n])_{i \in \mathbb{N}}$ is an ω-chain in $\mathcal{D}$: Let $i \in \mathbb{N}$. As $\rho(c_i[n]) \leq n$ and $c_i \sqsubseteq c_{i+1}$ we get $c_i[n] \in \downarrow^n (c_{i+1})$. Therefore $c_i[n] \sqsubseteq c_{i+1}[n]$ because $c_{i+1}[n]$ is the greatest element of

$\downarrow^n (c_{i+1})$. Let

$$C' := \bigcup_{i \in \mathbb{N}} \iota_d(c_i[n]).$$

To prove that $C[n] = C'$ let $x \in C'$. Then there exists $i \in \mathbb{N}$ such that $x \sqsubseteq c_i[n]$. This implies $x \sqsubseteq c_i$ and $\rho(x) \leq n$. Thus we get $x \in C[n]$. To get the inclusion the other way round let $y \in C[n]$. Then we have $\rho(y) \leq n$ and there exists $j \in \mathbb{N}$ such that $y \sqsubseteq c_j$. This implies $y \sqsubseteq c_j[n]$. Thus we get $y \in C'$ and established therefore $C[n] \in Ch(\mathcal{D})$.

Let (C_n) be a Cauchy sequence in $Ch(D)$ with $d_\rho^+(C_m, C_n) \leq \frac{1}{2^n}$ for all $m > n \in \mathbb{N}$. As we mentioned above its limit in $(P_\downarrow(\mathcal{D}), d[\rho_\downarrow])$ is computed as $C = \lim_{n\to\infty} C_n = \bigcup C_n[n]$. We have just proved that the sets $C_n[n]$ are elements of $Ch(\mathcal{D})$. As $C_n[n] = C_m[n] \subseteq C_m[m]$ for all $m \geq n \in N$ they form an ω-chain in $Ch(\mathcal{D})$ and their least upper bound C is an element of $Ch(\mathcal{D})$.

Let for $n \in \mathbb{N}$ the function $f_n : Ch(\mathcal{D}) \to Ch(\mathcal{D})$, $C \mapsto C[n]$. We claim that all functions f_n are ωc-continuous. Let $C \in Ch(\mathcal{D})$. Then there exists an ω-chain $(c_i) \subseteq \mathcal{D}$ such that $C = \bigcup \iota_d(c_i)$. As we have proved above for the n-cut of C holds $C[n] = \bigcup \iota_d(c_i[n])$.

First we verify that the functions f_n are monotone. Let $n \in \mathbb{N}$ and $A = \bigcup \iota_d(a_i)$, $B = \bigcup \iota_d(b_j) \in Ch(\mathcal{D})$ with $A \subseteq B$. This implies that for all $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that $a_k \sqsubseteq b_l$. Therefore $a_k[n] \sqsubseteq b_l[n]$ and $f_n(A) = \bigcup \iota_d(a_i[n]) \subseteq \bigcup \iota_d(b_j[n]) = f_n(B)$. Thus f_n is monotone.

Now we establish $f_n(\bigsqcup C_i) = \bigsqcup f_n(C_i)$ for all ω-chains $(C_i) \subseteq Ch(\mathcal{D})$, $n \in \mathbb{N}$. Let $(C_i) \subseteq Ch(\mathcal{D})$ be an ω-chain, let $n \in \mathbb{N}$. Using corollary 1.5 and the isomorphy $(Chain(\mathcal{D}), \sqsubseteq_c) \simeq (Ch(\mathcal{D}), \subseteq)$ we may assumse w.l.o.g. that $C_i = \bigcup_{j\in\mathbb{N}} \iota_d(c_{i,j})$ and $C := \bigsqcup C_i = \bigcup_{i\in\mathbb{N}} \iota_d(c_{i,i})$ for ω-chains $(c_{i,j})_{j\in\mathbb{N}}$, $(c_{i,i})_{\in\mathbb{N}} \subseteq \mathcal{D}$. With this notions we can compute

$$\begin{aligned} f_n(\textstyle\bigsqcup C_i) = f_n(C) = C[n] = \bigcup_{i\in\mathbb{N}} \iota_d(c_{i,i}[n]) \quad \text{and} \\ \textstyle\bigsqcup_{i\in\mathbb{N}} f_n(C_i) = \bigsqcup_{i\in\mathbb{N}} C_i[n] = \bigcup_{i\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} \iota_d(c_{i,j}[n]). \end{aligned}$$

Studying the relations between the n-cuts $c_{i,j}[n]$ we get for all $i, j \in \mathbb{N}$: As $c_{i,j} \sqsubseteq c_{i,j+1}$ the n-cut $c_{i,j}[n]$ is an element of $\downarrow^n (c_{i,j+1})$ and therefore we get $c_{i,j}[n] \sqsubseteq c_{i,j+1}[n]$. The same argument establishes $c_{i,j}[n] \sqsubseteq c_{i+1,j}[n]$. Thus the ω-chains $(c_{i,j}[n])_{j\in\mathbb{N}}$ fulfill the requirements of corollary 1.5 and we may conclude $\bigcup_{i\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} \iota_d(c_{i,j}[n]) = \bigcup_{i\in\mathbb{N}} \iota_d(c_{i,i}[n])$. ■

2.4 Isometry between $(\overline{D}, \overline{d[\rho]})$ and $(Ch(\mathcal{D}), d_\rho^+)$ respective $(Idl(\mathcal{D}), d_\rho^*)$

In section 2.3 we have seen that for a domain $\mathcal{D}$ under certain circumstances the order theoretical completions $(Ch(\mathcal{D}), d_\rho^+)$ and $(Idl(\mathcal{D}), d_\rho^*)$ are complete metric spaces. Now we ask for a relation of these complete metric spaces to the metric completion $(\overline{\mathcal{D}}, \overline{d[\rho]})$.

In general the function $\overline{\iota_d}$ maps from the metric completion of $\mathcal{D}$ to the metric completion of $Ch(\mathcal{D})$ respective $Idl(\mathcal{D})$:

$$\overline{\iota_d}: \ \overline{\mathcal{D}} \ \rightarrow \ \overline{Ch(\mathcal{D})} \ \subseteq \ \overline{Idl(\mathcal{D})} \tag{2}$$

$\overline{\iota_d}$ is an isometric embedding. We try to characterize the situation when either $Ch(\mathcal{D})$ or $Idl(\mathcal{D})$ is a complete metric space and $\overline{\iota_d}$ is surjective.

Starting with a finite length ρ on a pointed fb-cpo $(\mathcal{D}, \sqsubseteq)$ theorem 2.1 claims that the ideal completion is a complete metric space. Concerning the chain completion as complete metric space we needed an additional condition in theorem 2.3. Thus we obtain the situation:

$$\overline{\iota_d}: \ \overline{\mathcal{D}} \ \rightarrow \ \overline{Ch(\mathcal{D})} \ \subseteq \ Idl(\mathcal{D}). \tag{3}$$

If $(Idl(\mathcal{D}), d^*_\rho)$ is isometric to $(\overline{\mathcal{D}}, \overline{d[\rho]}$ the ideal completion $Idl(\mathcal{D})$ coincides with the metric completion of $Ch(\mathcal{D})$.

Starting with a finite weight ρ on a pointed poset $(\mathcal{D}, \sqsubseteq)$ both ideal and chain completion are complete metric spaces. Thus we have:

$$\overline{\iota_d}: \ \overline{\mathcal{D}} \ \rightarrow \ Ch(\mathcal{D}) \ \subseteq \ Idl(\mathcal{D}).$$

If $(Idl(\mathcal{D}), d^*_\rho)$ is isometric to $(\overline{\mathcal{D}}, \overline{d[\rho]})$ ideal completion and chain completion coincide.

Concerning isometry between ideal completion and metric completion [MCB94] gives the following condition:

Lemma 2.6 (A condition on isometry in the case of ideal completion)
Let ρ be a finite length on a pointed poset $(\mathcal{D}, \sqsubseteq)$ such that:

- *$(Idl(\mathcal{D}), d^*_\rho)$ is a complete metric space.*
- *For all $I \in Idl(\mathcal{D})$, $n \in \mathbb{N}$: The set $I[n] := \{x \in I \mid \rho(x) \leq n\}$ is finite.*

Then $\overline{\iota_d} : \overline{\mathcal{D}} \rightarrow Idl(\mathcal{D})$ is an isometry.

The assumptions of this lemma are rather restrictive. They reduce the ideal completion to the chain completion:

Lemma 2.7 (Consequence of the isometry condition)
Under the assumptions of lemma 2.6 holds: Chain completion and ideal completion of $(\mathcal{D}, \sqsubseteq)$ coincide, i.e. $Ch(\mathcal{D}) = Idl(\mathcal{D})$.

Proof: Let $I \in Idl(\mathcal{D})$. Then I is especially an ideal in $\mathcal{D}$. As ρ is a finite length on $\mathcal{D}$ we have $I = \bigcup I[n]$. By assumption the sets $I[n]$ are finite. Thus I is countable. This implies with lemma 2.2 that $I \in Ch(\mathcal{D})$. ∎

It should be mentioned that lemma 2.7 is not a "natural" consequence of isometry. Equations (2) and (3) show that in general isometry between $\overline{\mathcal{D}}$ and $\overline{Idl(\mathcal{D})}$ concerns only the metric completion of $Ch(\mathcal{D})$ – not $Ch(\mathcal{D})$ itself.

Using lemma 2.6, lemma 2.7 and theorem 2.1 respective theorem 2.4 we summerize – following and completing two theorems of [MCB94] – for the ideal completion:

Theorem 2.8 (Isometry starting with a length in the case of ideal completion) *Let $(\mathcal{D}, \sqsubseteq)$ be a pointed fb-cpo with a finite length ρ such that for all $I \in Idl(\mathcal{D})$, $n \in \mathbb{N}$ holds: The set $I[n] := \{x \in I \mid \rho(x) \leq n\}$ is finite. Then $Idl(\mathcal{D}) = Ch(\mathcal{D})$, especially $Ch(\mathcal{D}) = \overline{Ch(\mathcal{D})}$, and $\overline{\iota_d} : (\overline{\mathcal{D}}, \overline{d[\rho]}) \to (Idl(\mathcal{D}), d^+_\rho)$ is an isometry.*

Theorem 2.9 (Isometry starting with a weight in the case of ideal completion) *Let ρ be a finite weight on a pointed poset $(\mathcal{D}, \sqsubseteq)$ such that for all $I \in Idl(\mathcal{D})$, $n \in \mathbb{N}$ holds: The set $I[n] := \{x \in I \mid \rho(x) \leq n\}$ is finite. Then $Idl(\mathcal{D}) = Ch(\mathcal{D})$ and $\overline{\iota_d} : (\overline{\mathcal{D}}, \overline{d[\rho]}) \to (Idl(\mathcal{D}), d^+_\rho)$ is an isometry.*

Theorem 2.9 confirms our result concerning the domain $\mathcal{D}$ of finite strings over some alphabet A from section 1.3.1 in a new way. We proved there that $Idl(\mathcal{D}) = Ch(\mathcal{D})$ even if A is not countable. As the n-cut of an ideal in $\mathcal{D}$ is a finite set $Idl(\mathcal{D})$ suffices the requirements of theorem 2.9 and we get: $Idl(\mathcal{D}) = Ch(\mathcal{D})$.

In the case of the chain completion we give the following condition on isometry:

Lemma 2.10 (A condition on isometry in the case of chain completion) *Let ρ be a finite length on a pointed poset $(\mathcal{D}, \sqsubseteq)$ such that:*

- *$(Ch(\mathcal{D}), d^*_\rho)$ is a complete metric space.*
- *For all $C \in Ch(\mathcal{D})$, $n \in \mathbb{N}$: The set $C[n] := \{x \in C \mid \rho(x) \leq n\}$ is finite.*

Then $\overline{\iota_d} : \overline{\mathcal{D}} \to Ch(\mathcal{D})$ is an isometry.

Proof: (Outline[6]) Let $C \in Ch(\mathcal{D})$. Then there exists an ω-chain $(c_i) \subseteq D$ such that $C = \bigcup \iota_d(c_i)$. We construct a subsequence (c'_i) of (c_i). For $x \in C$ we define $ind(x) := \min\{k \in \mathbb{N} \mid x \sqsubseteq c_k\}$. For $n \in \mathbb{N}$ let $num(n) := \max\{ind(x) \mid x \in C[n]\}$. As by assumption the sets $C[n]$ are finite for all $n \in \mathbb{N}$ this maximum exists. With this notions we define:

[6] This proof is a variation of the proof of lemma 2.6 which can be found in [MCB94].

$\forall i \in \mathbb{N} : c'_i := c_{num(i)}$. For this chain holds $\iota_d(c'_i) = C[i]$. Furthermore (c'_i) is a Cauchy sequence in $\mathcal{D}$ and $\overline{\iota_d}(\lim_{i\to\infty} c'_i) = C$. ■

Using lemma 2.10 and theorem 2.3 respective theorem 2.5 we summerize for the chain completion:

Theorem 2.11 (Isometry starting with a length in the case of chain completion) *Let $(\mathcal{D}, \sqsubseteq)$ be a pointed fb-cpo with a finite length ρ such that for all $C \in Ch(\mathcal{D})$, $n \in \mathbb{N}$ holds: The set $C[n] := \{c \in C \mid \rho(c) \leq n\}$ is finite. Then $\overline{\iota_d} : (\overline{\mathcal{D}}, \overline{d[\rho]}) \to (Ch(\mathcal{D}), d^+_\rho)$ is an isometry.*

Theorem 2.12 (Isometry starting with a weight in the case of chain completion) *Let ρ be a finite weight on a pointed poset $(\mathcal{D}, \sqsubseteq)$ such that for all $C \in Ch(\mathcal{D})$, $n \in \mathbb{N}$ holds: The set $C[n] := \{c \in C \mid \rho(c) \leq n\}$ is finite. Then $\overline{\iota_d} : (\overline{\mathcal{D}}, \overline{d[\rho]}) \to (Ch(\mathcal{D}), d^+_\rho)$ is an isometry.*

We conclude this section with two examples. The first continues section 1.3.1 on isomorphism classes of plain trees. It demonstrates that metric, chain and ideal completion may be different concepts even when we start with a finite weight on a pointed poset. The second example illustrates theorem 2.11. We establish for a special kind of concurrent alphabets an isometry for the domain $\mathcal{D}$ of Mazurkiewicz traces between $\overline{\mathcal{D}}$ and $Ch(\mathcal{D})$.

2.4.1 Example: Isomorphism classes of plain trees – Part II

Consider the poset $\mathcal{D} := \{T \in TREE(\mathbb{R}) \mid T \text{ finitely branching and } height(T) \leq 1\}$ from section 1.3.1 equipped with the there introduced partial order $\sqsubseteq$ and the height as length ρ.

We already know that $Ch(\mathcal{D}) \neq Idl(\mathcal{D})$. As ρ is a finite weight we conclude with theorem 2.5 that $Ch(\mathcal{D}) = \overline{Ch(\mathcal{D})}$ and with theorem 2.4 that $Idl(\mathcal{D}) = \overline{Idl(\mathcal{D})}$. $d[\rho]$ is a discrete metric on $\mathcal{D}$ therefore we gain $\mathcal{D} = \overline{\mathcal{D}}$.

The function $\iota_d : \mathcal{D} \to Ch(\mathcal{D})$ is not surjective – for example there is no isomorphism class S of finite branching plain trees with $\iota_d(S) = \bigcup_{n\in\mathbb{N}} \iota_d(\sum_{i=1}^{n} i.T_\perp)$. Thus we have the situatation that $\mathcal{D}$, $Ch(\mathcal{D})$ and $Idl(\mathcal{D})$ are all complete metric spaces which are not isometric. Therefore we may conclude that in general neither for $I \in Ch(\mathcal{D})$ nor for $I \in Idl(\mathcal{D})$ holds that $I[n]$ is a finite set for all $n \in \mathbb{N}$.

2.4.2 Example: Mazurkiewicz traces – Part II

Let (A, Ind) be a concurrent alphabet where Ind is a finite set. We claim that Mazurkiewicz traces on such a concurrent alphabet fulfill the requirements of theorem 2.11. We already know from part I that Mazurkiewicz traces form a fb-cpo. Thus it remains to prove that for all $C \in Ch(MT(A, Ind))$ and $n \in \mathbb{N}$ the set $C[n]$ is finite – independent of the cardinality of the alphabet A.

Let $C \in Ch(MT(A, Ind))$, $n \in \mathbb{N}$. Then there exist an ω-chain $(c_i) \subseteq MT(A, Ind)$ such that $C = \bigcup \iota_d(c_i)$. Let for $0 \leq j \leq n$

$$R_j := \{x \in C \mid \rho(x) = j\}.$$

With this notion holds $C[n] = \bigcup_{j=0}^{n} R_j$. Thus $C[n]$ is a finite set iff for all j the sets R_j are finite. We prove this by induction on j.

For the basis of the induction let $j = 0$. The only element in $MT(A, Ind)$ with length 0 is the equivalence class of the empty word ϵ which is simultaneous the bottom element of the partial order. Thus we have $R_0 = \{[\epsilon]\}$ and the basis holds.

For the induction step "$j \to j + 1$" we differentiate two situtations: If the set R_{j+1} is empty we are done. If R_{j+1} is not empty we find an element $u = [u_1 u_2 \ldots u_j u_{j+1}] \in R_{j+1}$. As $C[n]$ is leftclosed there exists a "corresponding" element $u' \in R_j$ with $u' = [u_1 u_2 \ldots u_j]$. By the induction hypothesis the set R_j is finite. We claim that there are only finitely many choices on u_{j+1} and that therefore the set R_{j+1} is finite.

Let $v := [u_1 u_2 \ldots u_j u'_{j+1}]$ with $u'_{j+1} \in A$ be an element of R_{j+1}. Using the fact that u and v both are elements of C we get that there exists $m \in \mathbb{N}$ such that $u \sqsubseteq c_m$ and $v \sqsubseteq c_m$. Thus there exist $w, w' \in A^*$ with $[u_1 u_2 \ldots u_j u_{j+1} w] = c_m$ and $[u_1 u_2 \ldots u_j u'_{j+1} w'] = c_m$. We conclude $[u_1 u_2 \ldots u_j u_{j+1} w] = [u_1 u_2 \ldots u_j u'_{j+1} w']$ and[7] $[u_{j+1} w] = [u'_{j+1} w']$. If $u_{j+1} \notin \pi_1(Ind)$ then $u_{j+1} = u'_{j+1}$ and thus $u = v$. If $u_{j+1} \in \pi_1(Ind)$ then $u'_{j+1} \in \pi_1(Ind)$ and there are only finitely many choices for u'_{j+1}. Thus the set R_{j+1} is finite.

We conclude this section with a survey on results concerning metric, chain and ideal completion of Mazurkiewicz traces $MT(A, Ind)$ on a concurrent alphabet (A, Ind).

[Kwi91] showed:

- $(MT(A, Ind), \sqsubseteq)$ is an fb-cpo.
- If A is finite then $(Idl(MT(A, Ind)), d^*_\rho)$ is isometric to $(\overline{MT(A, Ind)}, \overline{d[\rho]})$.
- If A is countable then $(Idl(MT(A, Ind)), d^*_\rho)$ is a complete metric space.
- If A is infinite then an example shows that $(Idl(MT(A, Ind)), d^*_\rho)$ must not be isometric to $(\overline{MT(A, Ind)}, \overline{d[\rho]})$.

[MCB94] proved:

[7] – with proposition 2.2.5 of [Maz89] –

$$\begin{tikzcd} \mathcal{D} \arrow[r, "f"] \arrow[d, "\iota^{\mathcal{D}}"] & \mathcal{E} \arrow[d, "\iota^{\mathcal{E}}"] \\ \mathcal{I}(\mathcal{D}) \arrow[r, "f^{\mathcal{I}}"] & \mathcal{I}(\mathcal{E}) \end{tikzcd}$$

Figure 4: Extension of a monotone function f to a continuous function $f^{\mathcal{I}}$

- $(Idl(MT(A, Ind)), d_\rho^*)$ is a complete metric space – independent of the cardinality of the alphabet A.

Our results are:

- There is an example with $Ch(MT(A, Ind)) \neq Idl(MT(A, Ind))$ – see section 2.3.2.
- $(Ch(MT(A, Ind)), d_\rho^+)$ is a complete metric space – see section 2.3.2.
- If Ind is finite then $(Ch(MT(A, Ind)), d_\rho^+)$ is isometric to $(\overline{MT(A, Ind)}, \overline{d[\rho]})$ – independent of the cardinality of the alphabet A.

3 Denotational semantics on the different completions

This chapter is devoted to the application of our studies in denotational semantics. We begin with a discussion of different extensions of functions, whether they "coincide" and have "the same" fixed points. Finally we present two consistency results for a CCS-like language $\mathcal{L}$: We model the finite part of $\mathcal{L}$ in $\mathcal{D}$. Under some conditions we can show that the sematics of "full" $\mathcal{L}$ defined by structural induction on $\overline{\mathcal{D}}$ or on $Ch(\mathcal{D})$ respective $Idl(\mathcal{D})$ are consistent.

3.1 Canonical extensions of functions

Up to now we studied the relation between metric and order theoretical completions on the level of elements of a set, i.e. whether there is some sort of embedding or – the same question from another point – whether the desired sets fulfill a special completeness property.

Now we turn our discussion to functions and their extensions. Let $f : \mathcal{D} \to \mathcal{D}$ be a function which is monotone with respect to $\sqsubseteq$ and non-distance-increasing/contracting with respect to $d[\rho]$. Then there are canonical extenstions of f on $\overline{\mathcal{D}}$, $Ch(\mathcal{D})$ and $Idl(\mathcal{D})$. The questions is: How are these different extensions related?

First we compile some results on standard techniques to extend a monotone function $f : \mathcal{D} \to \mathcal{E}$ to a continuous function $f^{\mathcal{I}} : \mathcal{I}(\mathcal{D}) \to \mathcal{I}(\mathcal{E})$ where $(\mathcal{D}, \sqsubseteq_{\mathcal{D}})$ and $(\mathcal{E}, \sqsubseteq_{E})$ are posets and $\mathcal{I}$ is some completion operator. Figure 4 shows the general situtation. The first result of the following theorem concerns the choice $\mathcal{I} = Chain$ and is due to [Kni]. With theorem 1.9 we "translate" it for the case $\mathcal{I} = Ch$. The situation $\mathcal{I} = Idl$ is studied for example in [AJ92].

Theorem 3.1 (Continuous extension of a monotone function)
Let $(\mathcal{D}, \sqsubseteq_{\mathcal{D}})$, $(\mathcal{E}, \sqsubseteq_{\mathcal{E}})$ be posets, let $f : \mathcal{D} \to \mathcal{E}$ be a monotone function.

1. *Let $\iota_c^{\mathcal{D}} : \mathcal{D} \to Chain(\mathcal{D})$, $\iota_c^{\mathcal{E}} : \mathcal{E} \to Chain(\mathcal{E})$ be the canonical embeddings of $\mathcal{D}$ in $Chain(\mathcal{D})$ respective $\mathcal{E}$ in $Chain(\mathcal{E})$. Then*
$$f^{Chain} : \begin{cases} Chain(\mathcal{D}) & \to & Chain(\mathcal{E}) \\ [(c_i)_{i\in\mathbb{N}}] & \mapsto & [(f(c_i))_{i\in\mathbb{N}}] \end{cases}$$
is an ωc-continuous function with $f^{Chain} \circ \iota_c^{\mathcal{D}} = \iota_c^{\mathcal{E}} \circ f$.

2. *Let $\iota_d^{\mathcal{D}} : \mathcal{D} \to Ch(\mathcal{D})$, $\iota_d^{\mathcal{E}} : \mathcal{E} \to Ch(\mathcal{E})$ be the canonical embeddings of $\mathcal{D}$ in $Ch(\mathcal{D})$ respective $\mathcal{E}$ in $Ch(\mathcal{E})$. Then*
$$f^{Ch} : \begin{cases} Ch(\mathcal{D}) & \to & Ch(\mathcal{E}) \\ \bigcup_{i\in\mathbb{N}} \iota_d^{\mathcal{D}}(c_i) & \mapsto & \bigcup_{i\in\mathbb{N}} \iota_d^{\mathcal{E}}(f(c_i)) \end{cases}$$
is an ωc-continuous function with $f^{Ch} \circ \iota_d^{\mathcal{D}} = \iota_d^{\mathcal{E}} \circ f$.

3. *Let $\iota_d^{\mathcal{D}} : \mathcal{D} \to Idl(\mathcal{D})$, $\iota_d^{\mathcal{E}} : \mathcal{E} \to Idl(\mathcal{E})$ be the canonical embeddings of $\mathcal{D}$ in $Idl(\mathcal{D})$ respective $\mathcal{E}$ in $Idl(\mathcal{E})$. Then*
$$f^{Idl} : \begin{cases} Idl(\mathcal{D}) & \to & Idl(\mathcal{E}) \\ I & \mapsto & \iota_d^{\mathcal{E}}(f(I)) \end{cases}$$
is a d-continuous function with $f^{Idl} \circ \iota_d^{\mathcal{D}} = \iota_d^{\mathcal{E}} \circ f$.

4. *The extensions f^{Ch} and f^{Idl} coincide on $Ch(\mathcal{D})$, i.e. $f^{Ch} = f^{Idl}_{|Ch(\mathcal{D})}$.*

Proof: Above we gave references for the first three items. Thus it remains to prove that $f^{Ch} = f^{Idl}_{|Ch(\mathcal{D})}$. Let $C = \bigcup \iota_d^{\mathcal{D}}(c_i) \in Ch(\mathcal{D})$. We have to show that $\bigcup \iota_d^{\mathcal{E}}(f(c_i)) = \iota_d^{\mathcal{E}}(f(C))$.

Let $x \in \bigcup \iota_d^{\mathcal{E}}(f(c_i))$. Then there exists $i \in \mathbb{N}$ such that $x \sqsubseteq_{\mathcal{E}} f(c_i)$. As $c_i \in C$ we get $x \in \iota_d^{\mathcal{E}}(f(C))$ and therefore $\bigcup \iota_d^{\mathcal{E}}(f(c_i)) \subseteq \iota_d^{\mathcal{E}}(f(C))$.

$$\begin{tikzcd} \overline{\mathcal{D}} \arrow[r, "\overline{f}"] \arrow[d, "\overline{\iota_d}"] & \overline{\mathcal{D}} \arrow[d, "\overline{\iota_d}"] \\ \mathcal{I}(\mathcal{D}) \arrow[r, "f^{\mathcal{I}}"] & \mathcal{I}(\mathcal{D}) \end{tikzcd}$$

Figure 5: The commuting diagram in the case of a finite weight, $\mathcal{I} \in \{Idl, Ch\}$

Let $x \in \iota_d^{\mathcal{E}}(f(C))$. Then there exist $y \in f(C)$ with $x \sqsubseteq_{\mathcal{E}} y$. Furthermore there exists $i \in \mathbb{N}$ such that $y \sqsubseteq_{\mathcal{E}} f(c_i)$. Transitivity gives us $x \sqsubseteq_{\mathcal{E}} f(c_i)$ and we may conclude $x \in \bigcup \iota_d^{\mathcal{E}}(f(c_i))$. Therefore we get $\iota_d^{\mathcal{E}}(f(C)) \subseteq \bigcup \iota_d^{\mathcal{E}}(f(c_i))$. ∎

Let $(\mathcal{D}, \sqsubseteq)$ be a poset with finite length ρ. Let $f : \mathcal{D} \to \mathcal{D}$ be a function which is monotone with respect to $\sqsubseteq$ and non-distance-increasing with respect to $d[\rho]$. We study the relation between its canonical extension

$$\overline{f} : \begin{cases} \overline{\mathcal{D}} & \to \overline{\mathcal{D}} \\ x = \lim_{n\to\infty} x_n & \mapsto \overline{f}(x) := \lim_{n\to\infty} f(x_n) \end{cases}$$

concerning the metric completion $\overline{\mathcal{D}}$ and the continuous extensions f^{Ch} respective f^{Idl} concerning the order theoretical completions $Ch(\mathcal{D})$ respective $Idl(\mathcal{D})$.

In the case that ρ is a finite weight we get the expected result of a commuting diagram, see figure 5. If ρ is just a finite lenght this relation does not hold in general: [MCB94] gives a counterexample for both, $Ch(\mathcal{D})$ and $Idl(\mathcal{D})$.

If the canonical extension $\overline{f}$ of f is contracting it has by Banach's fixed point theorem an unique fixed point $\mathit{fix}(\overline{f}) \in \overline{\mathcal{D}}$. On the other hand both f^{Ch} and f^{Idl} have a least fixed point $\mathit{lfp}(f^{Ch}) \in Ch(\mathcal{D})$ respective $\mathit{lfp}(f^{Idl}) \in Idl(\mathcal{D})$ by Tarski's fixed point theorem. Concerning these different fixed points we can establish under certain circumstances that

$$\overline{\iota_d}(\mathit{fix}(\overline{f})) = \mathit{lfp}(f^{\mathcal{I}}),\ \mathcal{I} \in \{Ch, Idl\} \tag{4}$$

for both, ρ a lenght and ρ a weight.

First we present the results in the case that ρ is a finite weight. For the ideal completion we cite [MCB94]:

Lemma 3.2 (Relation between $\overline{f}$ and f^{Idl} if ρ is a finite weight)
Let ρ be a finite weight on a pointed poset $(\mathcal{D}, \sqsubseteq)$. Then by theorem 2.4 $(Idl(D), d_\rho^*)$ is a complete metric space. If $f : \mathcal{D} \to \mathcal{D}$ is monotone and non-distance-increasing then

$\overline{\iota_d} \circ \overline{f} = f^{Idl} \circ \overline{\iota_d}$, see figure 5. Furthermore f^{Idl} is non-distance-increasing. If f is contracting then also f^{Idl} is contracting.

Part 4 of theorem 3.1 allows us to use lemma 3.2 to formulate an analog result for the chain completion. As $f^{Ch} = f^{Idl}_{|Ch(\mathcal{D})}$ we may conclude:

Lemma 3.3 (Relation between $\overline{f}$ and f^{Ch} if ρ is a finite weight)
Let ρ be a finite weight on a pointed poset $(\mathcal{D}, \sqsubseteq)$. Then by theorem 2.5 $(Ch(\mathcal{D}), d^+_\rho)$ is a complete metric space. If $f : \mathcal{D} \to \mathcal{D}$ is monotone and non-distance-increasing then $\overline{\iota_d} \circ \overline{f} = f^{Ch} \circ \overline{\iota_d}$, see figure 5. Furthermore f^{Ch} is non-distance-increasing. If f is contracting then also f^{Ch} is contracting.

If ρ is just a finite lenght [MCB94] shows by an example that in general f^{Idl} is not non-distance-increasing for a monotone and non-distance-increasing function f. As the chosen domain $\mathcal{D}$ is countable in this example it is also a counterexamble for f^{Ch}. Nevertheless [MCB94] gives a positive result concerning contracting functions:

Lemma 3.4 (Relation between *fix*$(\overline{f})$ and *lfp*(f^{Idl}) if ρ is a finite lenght)
Let ρ be a finite lenght on a pointed poset $(\mathcal{D}, \sqsubseteq)$ such that $(Idl(D), d^*_\rho)$ is a complete metric space. Let $f : \mathcal{D} \to \mathcal{D}$ be a monotone and contracting function. Then $\overline{f}$ is contracting with contracting constant $\frac{1}{2}$ and $\overline{\iota_d}(fix(\overline{f})) = lfp(f^{Idl})$.

For the chain completion we formulate without an explicit proof an analog lemma. The proof for the above lemma 3.4 in [MCB94] can be used word by word for our claim. As in the case of a finite length we do not know whether the ideal completion or the chain completion are in general complete metric spaces the relation between lemma 3.4 and 3.5 is different from those between lemma 3.2 and 3.4.

Lemma 3.5 (Relation between *fix*$(\overline{f})$ and *lfp*(f^{Ch}) if ρ is a lenght)
Let ρ be a finite lenght on a pointed poset $(\mathcal{D}, \sqsubseteq)$ such that $(Ch(D), d^+_\rho)$ is a complete metric space. Let $f : \mathcal{D} \to \mathcal{D}$ be a monotone and contracting function. Then $\overline{f}$ is contracting with contracting constant $\frac{1}{2}$ and $\overline{\iota_d}(fix(\overline{f})) = lfp(f^{Ch})$.

Lemma 3.4 and 3.5 concern not only the case of a finite length. As a weight is a lenght they provide especially a proof of equation (4).

3.2 The consistency of denotational semantics

Most of this section is completely due to [MCB94] - sometimes word for word. We follow their definitions of the language $\mathcal{L}$ and its denotational semantics on the metric completion and on the ideal completion. New is the straight forward definition of a denotational sematic on the chain completion and – of course – the theorem of consistency concerning the chain completion.

Let $(\mathcal{D}, \sqsubseteq)$ be pointed poset with a finite length ρ. We assume that $\mathcal{D}$ is a semantic domain for nonrecursive programs. We consider a language where recursion is modelled by declarations, i.e. a program is a pair $< s, \sigma >$. A statement s is built from operator symbols (like prefixing or sequential composition, nondeterministic choice, parallelism, etc.) or process variables. A declaration σ is a function which assigns to each process variable x a statement $\sigma(x)$. We denote the set of all statements s by $\mathcal{L}$.

For each operator symbol ω in $\mathcal{L}$ let $\omega_{\mathcal{D}}$ be a semantic operator on $\mathcal{D}$ which is monotone with respect to $\sqsubseteq$ and non-distance-increasing/contracting with respect to $d[\rho]$. Let $f : \mathcal{L} \to \mathcal{D}$ be any function. For a fixed declaration σ we may define a mapping

$$F : (\mathcal{L} \to \mathcal{D}) \to (\mathcal{L} \to \mathcal{D})$$

by structural induction on $s \in \mathcal{L}$:

- Let $F(f)(a) := a_{\mathcal{D}}$ for each constant symbol $a \in Lan$.
- Let $F(f)(x) := f(\sigma(x))$ for each process variable x.
- Let $F(f)(\omega(s_1, s_2, \ldots, s_n)) := \omega_{\mathcal{D}}(F(f)(s_1), F(f)(s_2), \ldots, F(f)(s_n))$ for each n-ary operator symbol ω in $\mathcal{L}$..

Similarly we ge mappings

- $F_{cms} : (\mathcal{L} \to \overline{\mathcal{D}}) \to (\mathcal{L} \to \overline{\mathcal{D}})$,
- $F_{Ch} : (\mathcal{L} \to Ch(\mathcal{D})) \to (\mathcal{L} \to Ch(\mathcal{D}))$ and
- $F_{Idl} : (\mathcal{L} \to Idl(\mathcal{D})) \to (\mathcal{L} \to Idl(\mathcal{D}))$

where we use the canonical extensions $\overline{\omega}$, ω^{Ch} respective ω^{Idl} as semantic operators.

Since F_{Ch} and F_{Idl} are ωc-continuous we have denotational cpo semantics on $Ch(\mathcal{D})$ respective $Idl(\mathcal{D})$:

$$Me_{Ch} : \begin{cases} \mathcal{L} & \to & Ch(\mathcal{D}) \\ s & \mapsto & lfp(F_{Ch})(s) \end{cases} \qquad Me_{Idl} : \begin{cases} \mathcal{L} & \to & Idl(\mathcal{D}) \\ s & \mapsto & lfp(F_{Idl})(s) \end{cases}$$

Under certain conditions (e.g. the guardedness of the statements $\sigma(x)$ in the sense of [Mil89]) the function F_{cms} is contracting and hence has a unique fixed point. In this case we get a metric denotational semantics on $\overline{\mathcal{D}}$:

$$Me_{cms} : \begin{cases} \mathcal{L} & \to & \overline{\mathcal{D}} \\ s & \mapsto & \mathit{fix}(F_{cms})(s) \end{cases}$$

[MCB94] gives the following consistency result for Me_{cms} and Me_{Idl} :

Theorem 3.6 (Consistency of Me_{cms} and Me_{Idl})
Let ρ be a finite length on a pointed poset $(D, \sqsubseteq)$ such that $(Idl(\mathcal{D}), d^*_\rho)$ is a complete metric space. Then

$$\overline{\iota_d} \circ Me_{cms} = Me_{Idl}.$$

We add a theorem for Me_{cms} and Me_{Ch} :

Theorem 3.7 (Consistency of Me_{cms} and Me_{Ch})
Let ρ be a finite length on a pointed poset $(D, \sqsubseteq)$ such that $(Ch(\mathcal{D}), d^+_\rho)$ is a complete metric space. Then

$$\overline{\iota_d} \circ Me_{cms} = Me_{Ch}.$$

Proof: For a proof we refer to the proof of above cited theorem 3.6 in [MCB94]. This proof does not use any specific property of the ideal completion and holds so as well in the case of the chain completion. ■

Conclusion

In this paper we successfully added the technique of chain completion to the theory of [MCB94] concerning the relation between the metric and the ideal completion of a pointed poset with finite length and its application in denotational semantics. We showed

1. that the usual chain completion $Chain(\mathcal{D})$ can be reformulated as an isomorphic completion $Ch(\mathcal{D})$ which is a subset of the ideal completion $Idl(\mathcal{D})$.

2. that the metric space $(Ch(\mathcal{D}), d^+_\rho)$ is complete under certain circumstances.

3. that under certain circumstances there is an isometry between the complete metric spaces $(Ch(\mathcal{D}), d^+_\rho)$ and $(\overline{\mathcal{D}}, \overline{d})$.

4. that under certain circumstances the denotational semantics on the metric completion and on the chain completions are consistent (see figure 6).

$$\begin{tikzcd}
\mathcal{L}' \arrow[rr, "Me"] & & (\mathcal{D}, \sqsubseteq, d) \arrow[dl] \arrow[dd, "\iota_d"] \\
& (\overline{\mathcal{D}}, \overline{d}) \arrow[dr, "\overline{\iota_d}"] & \\
\mathcal{L}' \cup \{\mathrm{rec}\} \arrow[ur, "Me_{cms}"] \arrow[rr, "Me_{Ch}"] \arrow[drr, "Me_{Idl}"'] & & (Ch(\mathcal{D}), \subseteq) \arrow[d, phantom, "\subseteq"] \\
& & (Idl(\mathcal{D}), \subseteq)
\end{tikzcd}$$

Figure 6: Consistency of denational semantics

The following aspects seem to be worth mentioning:

Probably the most important result of our work is that it is possible to "translate" the definitions and theorems from the ideal completion to the chain completion. This shows that the ideas of [MCB94] are of fundamental nature and not specific to the ideal completion.

It is not our intention to claim for semantics based on chain completion. The examples of domains where chain and ideal completion differ are probably not the standard situation in denotational semantics. But another approach to the ideal completion of a countable domain seems to be useful. Knowing that ideal and chain completion coincide gives a second description of the elements in the completed domain.

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