ON RELATIONS BETWEEN X AND $C_c(X)$

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ABSTRACT

The algebra of all continuous real-valued functions on a space X endowed with the continuous convergence structure is denoted by $C_c(X)$. Relationships between a space X and its associated convergence algebra $C_c(X)$ are investigated. After appropriate definitions, the following two theorems are proved: (1). A c-embedded convergence space X is Lindelöf if and only if $C_c(X)$ is first countable (this has a generalization to upper λ-compact spaces). (2). A c-embedded convergence space X has weight λ if and only if $C_c(X)$ has weight λ. With the help of (2), it is shown that a completely regular topological space X is separable and metrizable if and only if $C_c(X)$ is second countable. A type of Stone-Weierstrass theorem proved by E. Binz is extended to deal with questions of density. This extension is utilized to provide another characterization of separable metrizable spaces, and to show that the algebraic tensor product of $C(X)$ and $C(Y)$ may be regarded as a dense subalgebra of $C_c(X \times Y)$.

An inductive limit (in the category of convergence spaces) of certain locally convex topological vector spaces is constructed. This inductive limit proves to be a useful approximation of $C_c(X)$. However, for a wide class of topological spaces, it is shown that $C_c(X)$ can not even be realized as an inductive limit of topological vector spaces.

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INTRODUCTION

We will consider relationships between a space X and the corresponding algebra $C(X)$, consisting of all continuous real-valued functions on X. It is well-known that the algebraic properties of $C(X)$ are not, in general, sufficient to determine the space X. Thus, in order to obtain information meaningful for a wide class of spaces, we must consider more than the strictly algebraic properties of $C(X)$. It turns out that the continuous convergence structure on $C(X)$ (see 0.2), which we denote by $C_c(X)$, is particularly well suited for our work.

Chapter 0 provides a brief summary of the concepts needed throughout the paper. We point out in 0.7 that the c-embedded convergence spaces form a natural class of spaces for investigating the interplay between X and $C_c(X)$. Furthermore, topological spaces whose topology is determined by $C(X)$, namely completely regular spaces (see 0.5), are c-embedded.

In chapter 1, after generalizing certain topological concepts, we prove that a c-embedded convergence space X is upper $\aleph$-compact if and only if $C_c(X)$ is $\aleph$-countable. With the help of theorem 2, a characterization of c-embedded convergence spaces having weight $\aleph$, we show that a completely regular topological space X is separable and metrizable if and only if $C_c(X)$ is second countable. Section 1.3 provides generalizations of some familiar topological results and examples to show that our extended definitions are not vacuous.

The problem of dense subsets in $C_c(X)$ leads us to theorem 1 in chapter 2, which is a generalization of a type of Stone-Weierstrass theorem proved in [5]. Using theorem 1, we give a characterization of separable metrizable spaces in terms of countable dense subsets of $C_c(X)$ (theorem 3). Furthermore, a general criterion for the separability of completely regular topological spaces is provided. Theorem 1 also allows us to investigate both the algebraic tensor product of function algebras (section 2.3) and the tensor product in a certain category of convergence algebras (section 2.4).

$C_c(X)$ is not, in general, a topological space. In chapter 3 we attempt to approximate $C_c(X)$ by an inductive limit of locally convex topological vector spaces (in the category of convergence spaces). Specifically, given a completely regular topological space X, we consider the inductive limit of the topological algebras $C_c(\beta X \setminus K)$ for all compact subsets K of $\beta X \setminus X$, and denote this limit by $C_I(X)$. The convergence algebra $C_I(X)$ provides a useful approximation of $C_c(X)$. We show, for example, that

$C_I(X)$ has the same closed ideals, the same continuous homomorphisms, and the same dual space as $C_c(X)$. Furthermore, $C_I(X)$, like $C_c(X)$, is always complete and is topological if and only if $C_c(X)$ is topological. On the other hand, $C_I(X)$ does not coincide with $C_c(X)$ in general, and moreover, for a large class of topological spaces, $C_c(X)$ can not be realized as an inductive limit of topological vector spaces (theorem 6). The last section in chapter 3 is devoted to investigating the locally convex inductive limit of the algebras $C_c(\beta X \setminus K)$.

0. BACKGROUND

0.1. Convergence spaces

Before introducing the concept of a convergence space, we will briefly clarify our notations in dealing with filters.

Let $F(X)$ denote the collection of all filters on a set X (in the sense of Bourbaki, [6], I, p. 57). Given filters ϕ and ψ on X, we write $\phi \leq \psi$ if ϕ is coarser than ψ (or ψ is finer than ϕ). If a non-empty collection $\mathcal{J}$ of subsets of X has the property that the intersection of any finite number of elements in $\mathcal{J}$ is not empty, then the coarsest filter containing $\mathcal{J}$ is called the filter generated by $\mathcal{J}$. If a collection $\mathcal{B}$ of subsets of X generates a filter ϕ and has the property that each $A \in \phi$ contains an element $B \in \mathcal{B}$, then $\mathcal{B}$ is said to be a base (or basis) for the filter ϕ. For a point $x \in X$, let $\dot{x}$ denote the trivial ultrafilter generated by $\{x\}$. Finally, for two filters ϕ and ψ in $F(X)$, $\phi \wedge \psi$ is the finest filter coarser than both ϕ and ψ (i.e., the filter generated by all the sets $A \cup A'$, for $A \in \phi$ and $A' \in \psi$).

A convergence structure (Limitierung, [1]) on a set X is a map Λ from X into the power set of $F(X)$ that satisfies the following conditions for each point $x \in X$:

(i) If $\phi \in \Lambda(x)$ and $\phi \leq \psi$ for $\psi \in F(X)$, then $\psi \in \Lambda(x)$.

(ii) If $\phi \in \Lambda(x)$ and $\psi \in \Lambda(x)$, then $\phi \wedge \psi \in \Lambda(x)$.

(iii) $\dot{x} \in \Lambda(x)$.

The pair (X,Λ) is called a convergence space (Limesraum, [1]). Every topological space X is, in a natural way, a convergence space. For each $x \in X$, $\Lambda(x)$ is simply the collection of all filters on X that converge to x in the topological space X . In analogy with topological spaces, we often denote a convergence space (X,Λ) by the symbol X alone. In this case, for a filter $\phi \in \Lambda(x)$, where $x \in X$, we say ϕ converges to x and write $\phi \longrightarrow x$. Thus, ϕ is a convergent filter in the convergence space X if $\phi \longrightarrow x$ for some $x \in X$.

A map f from a convergence space X into a convergence space Y is said to be continuous if for every convergent filter ϕ on X ,

$$f(\phi) \longrightarrow f(x)$$

in Y , where $\phi \longrightarrow x$ in X . By $f(\phi)$, we mean the filter generated on Y by

$$\{f(A): A \in \phi\}.$$

Obviously, for topological spaces the definition coincides with the usual concept of continuity.

The identity map from a convergence space X onto itself is continuous and further, given convergence spaces X, Y, and Z and continuous maps $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$, the map $g \circ f$ is continuous from X into Z. Therefore, we can speak of the category $\mathcal{L}$, whose objects are convergence spaces, and whose morphisms are continuous maps. We call an isomorphism in the category $\mathcal{L}$ a homeomorphism. Clearly, the category of topological spaces (morphisms, continuous maps) can be regarded as a (full) subcategory of $\mathcal{L}$.

We can extend the concept of a closure operator to the category $\mathcal{L}$. For a subset S of a convergence space X, the adherence of S, which we denote by $a(S)$, is the set of all points $x \in X$ with the property that there exists a convergent filter ϕ on X such that $\phi \longrightarrow x$ and ϕ has a trace on S. A filter ϕ on X is said to have a trace on a subset $S \subset X$ if every set $A \in \phi$ has a non-empty intersection with S. We say that a subset S of X is closed if $a(S) = S$. In general, the adherence operator is not idempotent, and thus the adherence of a subset S of X need not be closed.

A convergence space X is called separated if whenever a convergent filter ϕ on X converges to both x and y, then $x = y$. We say a separated convergence space is regular if for each convergent filter

ϕ on X, the filter generated by

$$\{a(A) : A \in \phi\}$$

is convergent in X. On the subcategory of topological spaces, these definitions agree with the usual concepts of separated (i.e., Hausdorff) and regular.

0.2. The continuous convergence structure

Given two convergence structures Λ and Λ' on the set X, the convergence space (X,Λ) is said to be finer than (X,Λ') (or (X,Λ') is coarser than (X,Λ)) if the identity map

$$\mathrm{id}: (X,\Lambda) \longrightarrow (X,\Lambda')$$

is continuous.

A subset S of a convergence space X is called a subspace of X (or carries the convergence structure inherited from X) if S is endowed with the coarsest of all convergence structures Λ for which the inclusion map

$$i: (S,\Lambda) \longrightarrow X$$

is continuous.

Given convergence spaces X and Y, we define the product convergence space $X \times Y$ to be the cartesian product of X and Y together with the coarsest of all convergence structures making the projection maps onto X and Y continuous. Obviously, we could extend this definition to the product of an arbitrary family of convergence spaces. For a convergence space Z, a map f from Z into $X \times Y$ is continuous if and only if $p_X \circ f$ and $p_Y \circ f$ are both continuous, where p_X and p_Y are the projections onto X and Y respectively.

If X and Y are non-empty convergence spaces, then the collection of all continuous maps from X into Y, which we denote by $C(X,Y)$, is not empty. Thus, for convenience, we restrict ourselves to non-empty convergence spaces. In particular, $\mathcal{C}$ will denote the category of convergence spaces excluding the empty set. Now, let w denote the natural evaluation map

$$w: C(X,Y) \times X \longrightarrow Y ,$$

defined by $w(f,x) = f(x)$ for every $f \in C(X,Y)$ and for every $x \in X$. Among all the convergence structures Λ on $C(X,Y)$ making the map w from $(C(X,Y),\Lambda) \times X$ into Y continuous, there exists a coarsest convergence structure Λ_c (see [1]). We call Λ_c the <u>continuous convergence structure</u> (Limitierung der stetigen Konvergenz, [1]), and we denote the convergence space $(C(X,Y),\Lambda_c)$ by $C_c(X,Y)$. The convergence space $C_c(X,Y)$ is separated if and only if Y is separated.

0.3. Convergence algebras and function algebras

The set $C(X,\mathbb{R})$ consisting of all continuous real-valued functions on a convergence space X, we denote simply by $C(X)$. Under the pointwise defined operations, $C(X)$ is an associative, commutative, unitary $\mathbb{R}$-algebra. The function $\underline{1}$ of constant value 1 is the unity element, and the function $\underline{0}$ of constant value 0 is the zero element. If a function $f \in C(X)$ has a multiplicative inverse in the algebra $C(X)$, we denote it with the suggestive notation $1/f$. Any algebra of the form $C(X)$ for a convergence space X is said to be a <u>function algebra</u>. We will be primarily concerned with the function algebra $C(X)$ together with the continuous convergence structure which we denote by $C_c(X)$.

A convergence space G, which is also a group, is said to be a <u>convergence group</u> if:

(1). The map

$$\cdot : G \times G \longrightarrow G ,$$

sending each $(g_1, g_2) \in G \times G$ to the group product $g_1 \cdot g_2$, is continuous.

(2). The map

$$^{-1} : G \longrightarrow G ,$$

sending each element in G to its inverse, is continuous.

It is evident that the convergence structure on a convergence group is determined by the filters convergent to the identity element. A convergence space V, which is also a vector space over $\mathbb{R}$, is a convergence vector space if V is a convergence group with respect to the underlying group structure, and scalar multiplication is continuous (i.e., the map from $\mathbb{R} \times V$ into V defined by scalar multiplication is continuous). Further, if the convergence vector space V is also an algebra over $\mathbb{R}$, then V is said to be a convergence algebra if the multiplication is continuous.

Since for topological spaces X and Y the product convergence space $X \times Y$ is simply the usual cartesian product of X and Y, the concepts of topological groups, vector spaces, and algebras are consistent with the above definitions. In particular, $C_k(X)$ and $C_s(X)$, the algebra $C(X)$ endowed with the compact-open topology and the topology of pointwise convergence respectively, are both topological (i.e., convergence) algebras for any convergence space X. For a definition of compactness in a convergence space, see [8], p. 277 .

It is not difficult to show (see [1]) that $C_c(X)$ is a convergence algebra for any convergence space X . Therefore, the continuous convergence structure on $C(X)$ is determined by the filters convergent to $\underline{0}$. Specifically, a filter Θ on $C_c(X)$ converges to $\underline{0}$ if and only if $w(\Theta \times \phi)$ converges to 0 in $\mathbb{R}$ for every convergent filter ϕ on X ($\Theta \times \phi$ denotes the filter generated on $C(X) \times X$ by the sets $A \times B$ for every $A \in \Theta$ and every $B \in \phi$). With this characterization it is easy to see that $C_c(X)$ is always finer than $C_k(X)$.

Remark. For a completely regular topological space X , the convergence algebra $C_c(X)$ is equal to $C_k(X)$ if and only if X is locally compact (see [6], II, p. 329).

We call a subset $A \subset C(X)$ a subalgebra of of $C(X)$ if A , with the inherited algebraic operations, is an algebra containing $\underline{1}$. It will often be helpful to consider the subalgebra $C^0(X)$, consisting of all bounded functions in $C(X)$. Here, we can define the sup-norm by

$$\|f\| = \sup_{x \in X} |f(x)|$$

for each $f \in C^0(X)$. We will denote by $C^0_n(X)$ the algebra $C^0(X)$ together with the sup-norm. Of course $C^0_n(X)$ is a Banach algebra.

Function algebras have the following useful algebraic structure. There is a natural partial ordering on $C(X)$ for a convergence space X defined by: $f \geq g$ if $f(x) \geq g(x)$ for every $x \in X$. With this ordering $C(X)$ is a partially ordered algebra (see [9], p. 11), and in addition, a lattice. In particular,

$$(f \vee g)(x) = f(x) \vee g(x)$$

for every $x \in X$, where "$\vee$" is the lattice operation in $\mathbb{R}$ (i.e., $a \vee b = \max\{a,b\}$ for a and b in $\mathbb{R}$). Similarly, $(f \wedge g)(x) = f(x) \wedge g(x)$ for every $x \in X$. The function $|f|$ is defined by

$$|f| = f \vee (-f) ,$$

and it follows immediately that for each $x \in X$

$$|f|(x) = |f(x)| .$$

Since $|f| \in C(X)$ and

$$f \vee g = \frac{1}{2} \{ (f+g) + |f-g| \} ,$$

the function $f \vee g$ and dually $f \wedge g$ are indeed continuous (i.e., elements of $C(X)$). If a subalgebra A of $C(X)$ is also a sublattice of $C(X)$, then A is said to be a lattice subalgebra.

0.4. Functorial properties

By a homomorphism between two associative, commutative, unitary $\mathbb{R}$-algebras, we will mean an algebra homomorphism taking unity to unity. Let $\mathcal{A}$ be the category of associative, commutative, unitary convergence algebras over $\mathbb{R}$. The morphisms in $\mathcal{A}$ are continuous homomorphisms. For convergence spaces X and Y , a continuous map $t: X \longrightarrow Y$ induces a homomorphism

$$t^*: C(Y) \longrightarrow C(X) .$$

defined by $t^*(f) = f \circ t$ for every $f \in C(Y)$. In fact,

$$t^*: C_c(Y) \longrightarrow C_c(X)$$

is continuous (see [2]). Therefore, we have a contravariant functor $\mathcal{C}_c$ from $\mathcal{L}$ into $\mathcal{A}$, where $\mathcal{C}_c$ takes each object X to $C_c(X)$ and each morphism t to t^* .

The set of all homomorphisms from an $\mathbb{R}$-algebra A onto $\mathbb{R}$ (i.e., taking unity to unity) we denote by Hom A . For $A \in \mathcal{A}$, let $\mathcal{H}om\,A$ be the subset of all continuous homomorphisms from A onto $\mathbb{R}$. To indicate the continuous convergence structure on $\mathcal{H}om\,A$ (inherited from $C_c(A)$) we write $\mathcal{H}om_c A$. Similarly, let the spaces $\mathcal{H}om_s A$ and $Hom_s A$ carry the topology of pointwise convergence on the sets in question. Given two convergence algebras A and B in $\mathcal{A}$, a homomorphism u from A into B induces a map

$$u^*: \mathrm{Hom}\,B \longrightarrow \mathrm{Hom}\,A$$

defined by $u^*(h) = h \circ u$ for each $h \in \mathrm{Hom}\,B$. In addition, if u is continuous (i.e., a morphism in $\mathcal{A}$), then $u^*|\mathcal{H}om\,B$, which we denote again by u^*, maps $\mathcal{H}om\,B$ into $\mathcal{H}om\,A$, and

$$u^*: \mathcal{H}om_c B \longrightarrow \mathcal{H}om_c A \quad \text{and} \quad u^*: \mathcal{H}om_s B \longrightarrow \mathcal{H}om_s A$$

are both continuous (see [2]). Clearly,

$$u^*: \mathrm{Hom}_s B \longrightarrow \mathrm{Hom}_s A$$

is continuous even if u is not continuous.

Now, given a continuous map t from a convergence space X into a convergence space Y, it makes sense to speak of the continuous maps

$$t^{**}: \mathcal{H}om_c C_c(X) \longrightarrow \mathcal{H}om_c C_c(Y)$$

and

$$t^{**}: \mathcal{H}om_s C_c(X) \longrightarrow \mathcal{H}om_s C_c(Y) .$$

Similarly, for a continuous homomorphism

$$u: A \longrightarrow B ,$$

where A and B are elements in $\mathcal{A}$, we can speak of the continuous homomorphisms

$$u^{**}: C_c(\mathcal{H}om_c A) \longrightarrow C_c(\mathcal{H}om_c B)$$

and

$$u^{**}: C_c(\mathcal{H}om_s A) \longrightarrow C_c(\mathcal{H}om_s B) .$$

Finally, given a continuous function g in $C(\mathbb{R})$, one obtains a continuous map

$$g_*: C_c(X) \longrightarrow C_c(X) ,$$

for any convergence space X, defined by $g_*(f) = g \circ f$

for each $f \in C(X)$.

0.5. Associated topological structures

In 0.1 we introduced the concept of a closed subset of a convergence space X . Therefore, a subset U of X is called open if U is the complement of a closed set. The collection of all open subsets of X defines a toplogy on the set X , which we refer to as the associated topology on X .

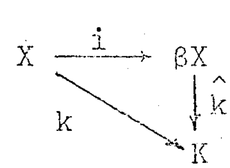
For our purposes, we wish to associate to each convergence space a completely regular topological space. Given an arbitrary convergence space X , let $X' = \mathcal{H}om_s C_c(X)$. We call X' the associated completely regular space of X . E. Binz has shown in [3] that the map

$$i_X : X \longrightarrow \mathcal{H}om\, C_c(X) ,$$

sending each $x \in X$ to the continuous homomorphism of point evaluation by x (i.e., $i_X(x)(f) = f(x)$ for each $f \in C(X)$), is surjective. Thus X' may be regarded as the space obtained by identifying the points in X which can not be distinquished by functions in $C(X)$, and giving this set the weak topology induced by $C(X)$ (considered as functions on the set X with the above identifications). Clearly for any convergence space X, the function algebra $C(X)$ is isomorphic to $C(X')$. Indeed, i_X is a continuous map onto X' and i_X^* is a continuous isomorphism from $C_c(X')$ onto $C_c(X)$.

0.6. Compactifications

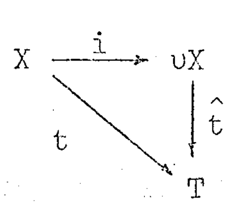
Completely regular topological spaces are characterized by the fact that they are precisely the subspaces of compact topological spaces. Specifically, for a completely regular topological space X, we will denote the Stone-Čech compactification of X by βX (see [9], p. 86). By a compactification of X, we mean a compact space which contains a homeomorphic copy of X as a dense subset. βX is the unique compactification of X, up to homeomorphism, satisfying the following universal property: Every continuous map k from X into any compact space K has a continuous extension $\hat{k}$ from βX into K. That is, if i is the natural embedding map from X into βX, the following diagram is commutative:



Furthermore, $C(\beta X)$ is isomorphic to the subalgebra of bounded functions, $C^0(X)$, via the canonical monomorphism i^* . We remark that the compactification βX can be realized as $\mathrm{Hom}_s C^0(X)$.

A completely regular topological space X is called realcompact if every homomorphism from $C(X)$ onto $\mathbb{R}$ can be represented by a point evaluation by an element in X (i.e., X is homeomorphic to $\mathrm{Hom}_s C(X)$). For example, every compact topological space is realcompact. It is not difficult to verify that two realcompact spaces X and Y are homeomorphic if and only if the algebras $C(X)$ and $C(Y)$ are isomorphic (see [9], p. 115).

By a realcompactification of a completely regular topological space X , we mean a realcompact space containing a homeomorphic copy of X as a dense subset. Let υX denote the Hewitt realcompactification of X (see [9], p. 118). In analogy to the Stone-Čech compactification, υX is the unique realcompactification of X , up to homeomorphism, satisfying the following universal property: Every continuous map t from X into a realcompact space T has a continuous extension $\hat{t}$ from υX into T . Thus, if i is the embedding map, the following diagram is commutative:



Moreover, X can be realized as $\mathrm{Hom}_s C(X)$, and thus it is homeomorphic to a subspace of βX . Now it is easy to verify that the map i^* is an isomorphism from $C(\upsilon X)$ onto $C(X)$.

0.7. c-embedded spaces

We have seen that for realcompact spaces, the function algebra $C(X)$ determines the space X . Similarly, we seek the largest class of convergence spaces such that the convergence algebra $C_c(X)$ determines the space X . We call a convergence space X c-embedded if X is homeomorphic to $\mathcal{H}om_c C_c(X)$. E. Binz has shown in [3] that $C_c(X)$ is bicontinuously isomorphic to $C_c(\mathcal{H}om_c C_c(X))$ via the map i_X^* for any convergence space X . Convergence algebras A and B are said to be bicontinuously isomorphic if there exists a homeomorphism of A onto B which is also an isomorphism. Indeed, the c-embedded convergence spaces are precisely the spaces we desire. Specifically, two c-embedded convergence spaces X and Y are homeomorphic if and only if $C_c(X)$ and $C_c(Y)$ are bicontinuously isomorphic (see [3], Satz 5). Further, every completely regular topological space is c-embedded. Thus,

$$X \approx \mathcal{H}om_c C_c(X) \approx \mathcal{H}om_s C_c(X) \approx \mathcal{H}om_s C_s(X)$$

for a completely regular topological space X , where "$\simeq$" means homeomorphic. In the case of a c-embedded convergence space X , clearly the associated completely regular space ($\mathcal{H}om_s C_c(X)$) can be regarded as a topological structure on the same underlying set.

1. AXIOMS OF COUNTABILITY

1.1. The aim of this section is to characterize Lindelöf and more generally upper $\aleph$-compact spaces.

We will first generalize a few topological concepts. By a <u>covering system</u> $\mathcal{S}$ of a convergence space X, we mean a collection of subsets of X with the property that for every convergent filter ϕ on X, there exists an $S \in \mathcal{S}$ such that $S \in \phi$. A <u>basic subcovering</u> of a covering system $\mathcal{S}$ is a subfamily $\mathcal{S}'$ of $\mathcal{S}$ with the property that for every convergent filter ϕ on X, there exists a finite number of elements in $\mathcal{S}'$, $\{S_i\}_{i=1}^{n}$, such that $\bigcup_{i=1}^{n} S_i \in \phi$.

<u>Definition</u> 1.

Let $\aleph$ be an arbitrary infinite cardinal number. A convergence space X is said to be <u>upper</u> $\aleph$-<u>compact</u> if every covering system of X has a basic subcovering of cardinal number less than or equal to $\aleph$. In particular, X is <u>Lindelöf</u> if it is upper $\aleph_0$-compact.

Definition 2.

A convergence space X is said to be *first countable* (respectively *$\aleph$-countable*) if for any point $x \in X$ and any filter ϕ convergent to x in X, there exists a coarser filter ϕ' such that $\phi' \to x$ and ϕ' has a countable basis (respectively a basis of cardinal number less than or equal to $\aleph$).

It is evident that our definitions correspond to the usual definitions in the case of topological spaces.

Given a convergence group G, we note that G is $\aleph$-countable if and only if the condition in definition 2 holds for filters convergent to the identity element in G.

We need the following two technical results.

Lemma 1.

Let X be a c-embedded convergence space and X' its associated completely regular space. If ϕ is a convergent filter in X, then the filter $\overline{\phi}$ generated by

$$\{\overline{M}^{X'} : M \in \phi\},$$

where $\overline{M}^{X'}$ is the closure of M in X', is also convergent in X.

Let $\phi \to x$ in X for some $x \in X$. We can consider ϕ convergent to x in $\mathcal{H}om_c C_c(X)$. This means that for every convergent filter Θ in $C_c(X)$, say $\Theta \to f$, and for every $\varepsilon > 0$, there exists a $T \in \Theta$ and an $M \in \phi$ such that

$$w(T \times M) \subset \{f(x) + [-\varepsilon , \varepsilon]\} ,$$

where w is the evaluation map as in 0.2 (i.e., $|g(y) - f(x)| \leq \varepsilon$ for every $g \in T$ and every $y \in M$). Since X' carries the weak topology induced by all the functions in $C(X)$,

$$w(T \times \overline{M}^{X'}) \subset \{f(x) + [-\varepsilon , \varepsilon]\} .$$

Hence $\overline{\phi}$ converges to x in X.

We say that $\mathcal{R}$ is a <u>refinement</u> of a covering system $\mathcal{S}$, if $\mathcal{R}$ is a covering system with the property that each $R \in \mathcal{R}$ is contained in some element of $\mathcal{S}$.

<u>Lemma</u> 2.

<u>Let</u> X <u>be a</u> c-<u>embedded convergence space</u>. <u>Every covering system of</u> X <u>has a refinement consisting of sets closed in the associated completely regular space</u>.

Let $\mathcal{S}$ be a covering system of X and let Φ denote the collection of all convergent filters in X. For $\phi \in \Phi$, lemma 1 implies $\overline{\phi} \in \Phi$. Therefore, there exists an $S \in \mathcal{S}$ such that $S \in \overline{\phi}$. Since $\overline{\phi}$ has a basis consisting of sets closed in X', we can choose a set $B_\phi \in \overline{\phi}$ such that B_ϕ is closed in X' and $B_\phi \subset S$. Of course $\overline{\phi}$ is coarser than ϕ and hence $\{B_\phi\}_{\phi \in \Phi}$ is indeed a refinement of $\mathcal{S}$.

Theorem 1.

A c-embedded convergence space X is upper $\aleph$-compact (respectively Lindelöf) if and only if $C_c(X)$ is $\aleph$-countable (respectively first countable).

Proof. Assume X is upper $\aleph$-compact. Again, denote by Φ the collection of all convergent filters in X. Let Θ be an arbitrary filter in $C_c(X)$ convergent to $\underline{0}$. This means that for every $1/n$, where $n \in \mathbb{N}$, and every $\phi \in \Phi$ there exists a $T_{1/n,\phi} \in \Theta$ and an $M_{1/n,\phi} \in \phi$ so that

$$w(T_{1/n,\phi} \times M_{1/n,\phi}) \subset \left[-\frac{1}{n}, \frac{1}{n}\right] .$$

For a fixed $n \in \mathbb{N}$, the collection

$$\{M_{1/n,\phi} : \phi \in \Phi\}$$

is a covering system of X and by assumption admits a basic subcovering

$$\mathcal{L}_n = \{M_\alpha : \alpha \in \mathcal{A}_n\}$$

of cardinal number less than or equal to $\aleph$. Let T_α be the element of Θ that corresponds to M_α as above. That is,

$$w(T_\alpha \times M_\alpha) \subset \left[-\frac{1}{n} , \frac{1}{n}\right] .$$

It follows that

$$\{T_\alpha : \alpha \in \bigcup_{n=1}^{\infty} \mathcal{A}_n\}$$

generates a filter Θ' coarser than Θ. Obviously Θ' has a basis of cardinal number $\leq \aleph$. It only remains to verify that $\Theta' \to \underline{0}$. Given $1/n$ for $n \in \mathbb{N}$ and $\phi \in \Phi$ there exists a finite subset of $\mathcal{A}_n$, $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, such that $\bigcup_{i=1}^{k} M_{\alpha_i} \in \phi$. Now $T = \bigcap_{i=1}^{k} T_{\alpha_i}$ is an element of Θ' with the property that

$$w(T \times \bigcup_{i=1}^{k} M_{\alpha_i}) \subset \left[-\frac{1}{n} , \frac{1}{n}\right] ,$$

and hence Θ' converges to $\underline{0}$ in $C_c(X)$.

Conversely, assume $C_c(X)$ is $\aleph$-countable. Let

$$\mathcal{S} = \{S_\alpha\}_{\alpha \in \mathcal{A}}$$

be an arbitrary covering system of X. Because of lemma 2, we can assume that the elements of $\mathcal{S}$ are closed in the associated completely regular space. We will prove that $\mathcal{S}$ has a basic subcovering of cardinal number less than or equal to $\aleph$. For each $S_\alpha \in \mathcal{S}$, set

$$T_\alpha = \{f \in C(X) : f(S_\alpha) = \{0\}\} .$$

Clearly the collection of all sets T_α for $\alpha \in \mathcal{A}$ generates a filter Θ that converges to $\underline{0}$ in $C_c(X)$. By assumption, there exists a filter Θ' coarser than Θ, convergent to $\underline{0}$ in $C_c(X)$, and having a base of cardinal number less than or equal to $\aleph$. Let

$$\{D_\beta : \beta \in \mathcal{B}\}$$

be a basis for Θ', where the cardinal number of the index set $\mathcal{B}$ is less than or equal to $\aleph$. Since $\Theta' \rightarrow \underline{0}$, for every $\phi \in \Phi$ there exists a $D_\beta \in \Theta'$ and an $L_\phi \in \phi$ such that

I) $$w(D_\beta \times L_\phi) \subset [-1 , 1] .$$

For a fixed $\beta \in \mathcal{B}$, let the union of all sets L_ϕ that correspond to D_β in the sense of (I) be denoted by R_β. It follows that

$$R = \{R_\beta : \beta \in \mathcal{B}\}$$

is a covering system for X. Since $\Theta' \leq \Theta$, for a given $\beta \in \mathcal{B}$, there exists a finite subset $\mathcal{A}_\beta$ of $\mathcal{A}$ such that

$$D_\beta \supset \bigcap_{\alpha \in \mathcal{A}_\beta} T_\alpha .$$

We claim that

II) $$R_\beta \subset \bigcup_{\alpha \in \mathcal{A}_\beta} S_\alpha .$$

Assume to the contrary, that there exists a point $x \in R_\beta \setminus \bigcup_{\alpha \in \mathcal{A}_\beta} S_\alpha$, where "$\setminus$" denotes the set theoretic difference. The fact that $\bigcup_{\alpha \in \mathcal{A}_\beta} S_\alpha$ is closed in the associated completely regular space X' implies that there exists a function $f \in C(X')$ such that

$$f(x) = 2 \quad \text{and} \quad f(\bigcup_{\alpha \in \mathcal{A}_\beta} S_\alpha) = \{0\} .$$

Because of the natural isomorphism from $C(X')$ onto $C(X)$ (see 0.5), we can assume $f \in C(X)$. Clearly $f \in \bigcap_{\alpha \in \mathcal{A}_\beta} T_\alpha$ but, in view of (I), the function $f \notin D_\beta$. This contradicts the fact that $D_\beta \supset \bigcap_{\alpha \in \mathcal{A}_\beta} T_\alpha$, and hence our claim is established. Now, it follows from the inclusion (II) that the collection

$$\mathcal{S}' = \{S_\alpha : \alpha \in \bigcup_{\beta \in \mathcal{B}} \mathcal{A}_\beta\}$$

is a basic subcovering of X. Furthermore, the cardinality of $\mathcal{S}'$ is less than or equal to $\aleph$, and thus X is upper $\aleph$-compact.

Corollary.

Let X be a c-embedded convergence space. If $C_c(X)$ is Lindelöf, then X is first countable.

If $C_c(X)$ is Lindelöf, then $C_c(C_c(X))$ is first countable. Since X is c-embedded, it is homeomorphic to a subspace of $C_c(C_c(X))$, and thus first countable.

In section 1.3 we will provide examples of Lindelöf convergence algebras $C_c(X)$.

1.2. Here, we obtain a characterization of separable metrizable topological spaces.

Let X be a convergence space. By a <u>basis</u> for X , we mean a collection $\mathcal{J}$ of subsets of X with the following property: For any convergent filter ϕ on X , say $\phi \rightarrow x$, there exists a coarser filter ϕ' such that $\phi' \rightarrow x$ and ϕ' has a basis consisting of sets in $\mathcal{J}$.

<u>Definition</u> 3.

The least infinite cardinal number of a basis for X is called the <u>weight</u> of X . In particular, X is <u>second countable</u> if it has weight $\aleph_0$.

It is easy to verify that our definitions of basis, weight, and second countable coincide with the usual concepts in the case of topological spaces.

The following generalization of a topological result is evident.

Remark. a) Let X be a convergence space having weight $\aleph$. Then any subspace of X has weight less than or equal to $\aleph$ (and is $\aleph$-countable).

b) Any subspace of a second countable convergence space is second countable.

c) A second countable convergence space is first countable.

Theorem 2.

A c-embedded convergence space X has weight $\aleph$ (respectively is second countable) if and only if $C_c(X)$ has weight $\aleph$ (respectively is second countable).

Proof. Assume X has weight $\aleph$. Let

$$\mathcal{J} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$$

be a basis for X of cardinal number $\aleph$. Given $\alpha \in \mathcal{A}$, $r \in \mathbb{Q}$ (the rational numbers), and $n \in \mathbb{N}$, we define the following subset of $C(X)$:

$$M_{\alpha,r,n} = \{f \in C(X):\ f(U_\alpha) \subset \left[r - \frac{1}{n}\ ,\ r + \frac{1}{n}\right]\}\ .$$

Denote by $\mathcal{M}$ the collection of all finite intersections of sets of the form $M_{\alpha,r,n}$, for $\alpha \in \mathcal{A}$, $r \in \mathbb{Q}$, and $n \in \mathbb{N}$. Clearly the cardinality of $\mathcal{M}$ is still $\aleph$. We now show that $\mathcal{M}$ is indeed a basis for $C_c(X)$.

Let Θ be an arbitrary convergent filter in $C_c(X)$. Say $\Theta \to f$. Our assumption implies that for any convergent filter ϕ in X, say $\phi \to x$, there exists a convergent filter ϕ' which is coarser than ϕ, and has a base consisting of sets in $\mathcal{U}$. Thus, we can find a $U_\alpha \in \phi$, and a $T \in \Theta$ such that

$$w(T \times U_\alpha) \subset \{f(x) + \left[\frac{-1}{2n}, \frac{1}{2n}\right]\} .$$

Now choose an $r \in Q$ so that

$$|f(x) - r| \leq \frac{1}{2n} .$$

Because of our construction, there exists an $M_{\phi,n} \in \mathfrak{M}$ ($M_{\phi,n} = M_{\alpha,r,n}$) such that for every $g \in M_{\phi,n}$ and every $y \in U_\alpha$,

$$|g(y) - f(x)| \leq |g(y) - r| + |r - f(x)| \leq \frac{2}{n}$$

or

$$w(M_{\phi,n} \times U_\alpha) \subset \{f(x) + \left[\frac{-2}{n}, \frac{2}{n}\right]\} .$$

We observe that $M_{\phi,n} \supset T$, since

$$|g(y) - r| \leq |g(y) - f(x)| + |f(x) - r| \leq \frac{1}{n}$$

for every $g \in T$ and every $y \in U_\alpha$. Therefore, the collection of all $M_{\phi,n}$, for ϕ a convergent filter on X and $n \in \mathbb{N}$, generates a filter Θ' coarser than Θ with a basis consisting of sets in $\mathfrak{M}$. It is also clear that Θ' converges to f. Further, there can exist no basis $\mathfrak{M}'$ for $C_c(X)$ of cardinality strictly less than $\aleph$. If such an $\mathfrak{M}'$ existed, then, as we have just proved, $C_c(C_c(X))$ would have a basis of cardinality strictly less than $\aleph$. Because of the preceeding remark and the fact that X is homeomorphic to a subspace of $C_c(C_c(X))$, X would have weight unequal to $\aleph$.

Conversely, assume $C_c(X)$ has weight $\aleph$. Then, as above, X must have weight less than or equal to $\aleph$. The necessity of the theorem implies that X has weight exactly $\aleph$.

Since a completely regular topological space is separable and metrizable if and only if it is second countable (see [7], p. 187 & p. 195), we have the following result.

Theorem 3.

A completely regular topological space X is separable and metrizable if and only if $C_c(X)$ is second countable.

Corollary.

Let X be a completely regular topological space. $C_c(X)$ is a separable and metrizable topological space if and only if X is separable, metrizable, and locally compact.

In view of the remark in 0.3 and the discussion preceeding the last theorem, the proof is immediate.

1.3. We will extend two results that are known for topological spaces to the class of convergence spaces. These will prove useful in analysing the continuous convergence structure on $C(X)$.

Theorem 4.

Let X be a convergence space that has weight $\aleph$ (respectively is second countable). Then any subspace of X is upper $\aleph$-compact (respectively Lindelöf).

Because of the remark in section 1.2 , it suffices to show that X itself is upper $\aleph$-compact. Consider $\mathcal{T} = \{T_\alpha\}$ to be a basis for X of cardinal number $\aleph$. Let $\mathcal{S}$ be an arbitrary covering system for X . For each $T_\alpha \in \mathcal{T}$, choose S_α to be a fixed element in $\mathcal{S}$ such that $S_\alpha \supset T_\alpha$ if such an element S_α exists.

Denote by $\mathcal{S}'$ the collection of these S_α. Clearly $\mathcal{S}'$ is a collection of cardinal number less than or equal to $\aleph$. We will verify that $\mathcal{S}'$ is actually a basic subcovering of $\mathcal{S}$. Let ϕ be an arbitrary convergent filter in X, say $\phi \to x$. By assumption, there exists a filter ϕ' coarser than ϕ such that $\phi' \to x$ and ϕ' has a basis consisting of sets in $\mathcal{T}$. Since $\mathcal{S}$ is a covering system, there exists an S in $\mathcal{S}$ with $S \in \phi'$. Because S must contain some element $T_{\alpha_0} \in \mathcal{T}$, where T_{α_0} is also in ϕ', we can find an $S_{\alpha_0} \in \mathcal{S}'$ such that $S_{\alpha_0} \supset T_{\alpha_0}$. Thus S_{α_0} is an element of both ϕ' and ϕ.

Examples.

It is now easy to demonstrate that there exist convergence spaces that are upper $\aleph$-compact (respectively Lindelöf) and not topological, namely, $C_c(X)$ for X a completely regular topological space having weight $\aleph$ (respectively second countable) and not locally compact. Moreover, such a $C_c(X)$ has weight $\aleph$ (respectively is second countable) but is not topological.

For an example of a first countable convergence space that is neither second countable nor topological, consider $C_c(X)$ where X is a completely regular topological space which is Lindelöf and neither second countable nor locally compact.

In analogy with topological spaces, we say a subset S is <u>dense</u> in a convergence space Y if the adherence of S is Y. The space Y is said to be <u>separable</u> if it contains a countable dense subset.

Theorem 5.

Any subspace of a second countable convergence space is separable.

Let Y be a second countable convergence space with

$$\mathcal{J} = \{T_i\}_{i=1}^{\infty}$$

a countable basis. In light of the remark in section 1.2, it is sufficient to prove that Y is separable. For each $T_i \in \mathcal{J}$, pick a $y_i \in Y$ such that $y_i \in T_i$. We claim that $\{y_i\}_{i=1}^{\infty}$ is dense in Y. Given $y \in Y$, there exists a filter ϕ convergent to y in Y with the property that ϕ has a basis consisting of sets in $\mathcal{J}$. Hence ϕ has a trace on $\{y_i\}_{i=1}^{\infty}$, which completes the proof.

Remark. We have shown (theorems 3, 4, and 5) that if X is a separable, metrizable topological space, then $C_c(X)$ is second countable, first countable, Lindelöf, and separable.

In the next chapter we will study density and separability in a more general setting.

2. SEPARABILITY AND DENSITY

2.1. A certain type of Stone-Weierstrass theorem has been proved by E. Binz in [5] for closed subalgebras of $C_c(X)$. In order to investigate questions of density, we must develop a more general type of theorem, as it is not known when the adherence operator in $C_c(X)$ is idempotent.

Let X be a completely regular topological space. We say a subset M of $C(X)$ is <u>topology generating</u> if the weak topology induced on X by M coincides with the given topology. Recall that a set $M \subset C(X)$ is said to be dense in $C_c(X)$ if the adherence of M is $C(X)$ (see 1.3). Also, by definition (see 0.3), a subalgebra of $C(X)$ contains the unity element $\underline{1}$. We will show that if the bounded functions in a subalgebra A are topology generating, then A is dense in $C_c(X)$.

For a subalgebra $A \subset C(X)$, let

$$A^0 = A \cap C^0(X)$$

(i.e., the collection of all bounded functions in A). We remark that if A is a lattice subalgebra of $C(X)$, then A is topology generating if and only if

A^0 is topology generating. In what follows, " $\overline{}$ " will always denote the closure operator in $C_n^0(X)$ (the sup-norm closure).

Lemma 1.

Let A be a subalgebra of $C(X)$. The set $\overline{A^0}$ is a lattice subalgebra of $C(X)$ with the property that if $f \in \overline{A^0}$ and $\|f\| \geq \delta$ for some $\delta > 0$, then $1/f$ is in $\overline{A^0}$.

It is straight forward to verify that $\overline{A^0}$ is a lattice subalgebra (see, for example, [9], p. 241). To prove the inversion property, we first assume that $f \in \overline{A^0}$ and $f \geq \delta \underline{1}$ for $\delta > 0$. Thus, there exist m and n in $\mathbb{N}$ such that $(1/n)\underline{1} \leq f \leq m\underline{1}$. Since the Taylor expansion for the real-valued function $1/(1 - t)$ defined on $[0, r] \subset \mathbb{R}$ is uniformly convergent for $r < 1$,

$$m \frac{1}{f} = \frac{1}{1 - (1 - \frac{f}{m})}$$

can be uniformly approximated by polynomials in $(1 - \frac{f}{m})$. This implies $m/f \in \overline{A^0}$, and thus $1/f \in \overline{A^0}$. For an arbitrary $f \in \overline{A^0}$ bounded away from zero (i.e., $\|f\| \geq \delta$ for $\delta > 0$), $\frac{1}{f} = \frac{f}{f^2}$ and hence $1/f \in \overline{A^0}$.

For each point $x \in X$, we can define the point evaluation homomorphism $i_X(x) \in \mathrm{Hom}_s \overline{A^0}$ by

$$i_X(x)f = f(x)$$

for every $f \in \overline{A^0}$. Furthermore, it is evident that the map

$$i_X : X \longrightarrow \mathrm{Hom}_s \overline{A^0}$$

is continuous.

Lemma 2.

$i_X(X)$ is a dense subset of $\mathrm{Hom}_s \overline{A^0}$ for any subalgebra A of $C(X)$.

It suffices to show that a basic open neighborhood V in $\mathrm{Hom}_s \overline{A^0}$ intersects $i_X(X)$. We can assume

$$V = \bigcap_{i=1}^{n} \{k \in \mathrm{Hom}_s \overline{A^0} : |k(f_i) - h(f_i)| < \varepsilon\} ,$$

where $f_i \in \overline{A^0}$ for $i \in \{1, 2, \ldots, n\}$, $h \in \mathrm{Hom}_s \overline{A^0}$, and $\varepsilon > 0$. Now if

$$g = \sum_{i=1}^{n} (f_i - h(f_i)\underline{1})^2 ,$$

then $h(g) = 0$. Thus g can not be a unit in $\overline{A^0}$, and by lemma 1, there exists a point $p \in X$ such that $g(p) < \varepsilon^2$. This means

$$|f_i(p) - h(f_i)| < \varepsilon$$

for every $i \in \{1, 2, \ldots, n\}$, and hence $i_X(p) \in V$.

Lemma 3.

$Hom_s \overline{A^0}$ is a compact topological space for any subalgebra A of $C(X)$.

The proof consists of showing that $Hom_s \overline{A^0}$ is homeomorphic to a closed subspace of a product of closed intervals. For an arbitrary $f \in \overline{A^0}$, there exists an $n_f \in \mathbb{N}$ such that

$$f(X) \subset [-n_f , n_f] .$$

Since, by lemma 2, $i_X(X)$ is dense in $Hom_s \overline{A^0}$, it follows that $|h(f)| \leq n_f$ for every $h \in Hom_s \overline{A^0}$. Now, the map sending each $h \in Hom_s \overline{A^0}$ to $(h(f))_{f \in \overline{A^0}}$ is a homeomorphism of $Hom_s \overline{A^0}$ into

$$\Pi \{[-n_f , n_f] : f \in \overline{A^0}\} ,$$

where each n_f is chosen as above. It is easy to verify that if a point $(r_f)_{f \in \overline{A^0}}$ is an accumulation point of $\mathrm{Hom}_s \overline{A^0}$, embedded in the cartesian product, then the map sending each $f \in \overline{A^0}$ to r_f is a homomorphism on $\overline{A^0}$. Thus, the image of $\mathrm{Hom}_s \overline{A^0}$ is closed in the cartesian product which is compact by Tychonoff's theorem.

Let A be a subalgebra of $C(X)$ for a completely regular topological space X. Since $\mathrm{Hom}_s \overline{A^0}$ is compact, the universal property of the Stone-Čech compactification (see 0.6) implies that the map i_X can be extended to a continuous map from βX into $\mathrm{Hom}_s \overline{A^0}$. We denote this unique extension by π, and note that the following diagram is commutative:

$$\begin{array}{ccc} \beta X & \xrightarrow{\ \pi\ } & \mathrm{Hom}_s \overline{A^0} \\ {\scriptstyle i}\uparrow & \nearrow_{i_X} & \\ X & & \end{array} ,$$

where i is the natural inclusion map. In fact, π is surjective since $i_X(X)$ is dense in $\mathrm{Hom}_s \overline{A^0}$.

There is a Gelfand map

$$d \colon \overline{A^0} \longrightarrow C(\mathrm{Hom}_s \overline{A^0}) ,$$

defined for each $f \in \overline{A^0}$, by

$$d(f)h = h(f)$$

for every $h \in \mathrm{Hom}_s \overline{A^0}$. It is easy to see that d is a monomorphism into $C(\mathrm{Hom}_s \overline{A^0})$, and further, since $i_X(X)$ is dense in $\mathrm{Hom}_s \overline{A^0}$, d is an isometry from $\overline{A^0}$ regarded as a subspace of $C_n^0(X)$ into $C_n(\mathrm{Hom}_s \overline{A^0})$. Clearly $d(\overline{A^0})$ separates the points in $\mathrm{Hom}_s \overline{A^0}$, and thus the Stone-Weierstrass theorem implies that d is actually a surjection.

For $f \in C(X)$, where X is a completely regular topological space, let $\overline{f}$ denote the unique continuous extension of f to a map from βX into $\hat{\mathbb{R}}$, the one point compactification of $\mathbb{R}$ (see [7], p. 246). We say a subset $B \subset C(X)$ separates the points in βX from those in X if for each point $p \in \beta X$ and each point $x \in X$, there exists a function $f \in B$ such that

$$\overline{f}(p) \neq \overline{f}(x).$$

Proposition 1.

Let A be a subalgebra of $C(X)$ where X is a completely regular topological space. A^0 is topology generating if and only if A^0 separates the points in βX from those in X.

Assume A^0 is topology generating. Let $p \in \beta X$ and $y \in X$. In βX we can choose a closed neighborhood N_y of y such that $p \notin N_y$. Now, $N = N_y \cap X$ is a neighborhood of y in X. Since A^0 is topology generating, we can find a finite set $\{f_1, f_2, \ldots, f_n\}$ of functions in A^0 with the property that for each f_i, we have $f_i(y) = 0$, and

$$V = \bigcap_{i=1}^{n} \{x \in X: |f_i(x)| < 1\}$$

is a neighborhood of y contained in N. For

$$f = \sum_{i=1}^{n} f_i^2 ,$$

$$V' = \{x \in X: f(x) < 1\}$$

is a neighborhood of y such that $V' \subset V \subset N$. It only remains to show that $\bar{f}(p) \geq 1$. Let $\mathcal{U}$ be the collection of all neighborhoods of p in βX disjoint from N_y. Since X is dense in βX, the set $U \cap X$ is non-empty for each $U \in \mathcal{U}$, and of course

$$(U \cap X) \cap V' = \emptyset .$$

This implies

$$f(U \cap X) \subset [1\ ,\ \infty)\ .$$

Since the filter generated by $\mathcal{U}$ converges to p in βX and has a trace in X, we conclude that $\overline{f}(p) \geq 1$.

Conversely, assume A^0 separates the points in βX from those in X. We will show that for an arbitrary function $f \in C^0(X)$ and a point $y \notin Z(f)$, where $Z(f) = f^{-1}(0)$, we can find a closed set F in the topology generated by A^0 on X such that $F \supset Z(f)$ and $y \notin F$. Without loss of generality, we can assume $f(y) = 1$. Let π be the continuous surjection from βX onto $\mathrm{Hom}_s\overline{A^0}$ defined above. Since A^0 separates the points in βX from those in X,

$$\pi(y) \cap \pi(\overline{Z(f)}^{\beta X}) = \emptyset\ ,$$

where X is considered as a subspace of βX and $\overline{Z(f)}^{\beta X}$ is the closure of $Z(f)$ in βX. Clearly we can choose a function $g \in C(\mathrm{Hom}_s\overline{A^0})$ such that

$$g(\pi(\overline{Z(f)}^{X}) = \{-1\} \quad \text{and} \quad g(y) = 2\ .$$

Since $d(A^0)$ is dense in $C_n(\mathrm{Hom}_s\overline{A^0})$ there exists a $k \in A^0$ so that

$$d(k)(\pi(\overline{Z(f)}^{X})) \subset (-\infty , 0]$$

and

$$d(k)(y) > 1 .$$

It is now clear that the set

$$F = \{x \in X: k(x) \leq 0\}$$

has the desired property. That is, $y \notin F$ and $F \supset Z(f)$, which completes the proof.

Given a subset $S \subset C(X)$, let $a_c(S)$ be the adherence of S in $C_c(X)$.

Proposition 2.

Let X be a convergence space. For a subset $S \subset C^0(X)$,

$$a_c(S) = a_c(\overline{S}) ,$$

where $\overline{S}$ is the closure of S in $C^0_n(X)$.

Clearly $a_c(\overline{S}) \supset a_c(S)$. To prove the other inclusion, assume $f \in a_c(\overline{S})$. This means there exists a filter Θ in $C_c(X)$ such that $\Theta \to f$ and Θ has a basis in $\overline{S}$. Denote the collection of all convergent filters on X by Φ . Now for each $\varepsilon > 0$ and each $\phi \in \Phi$, say $\phi \rightarrow x$, there exists an $N_{\phi,\varepsilon} \in \phi$ and a $T_{\phi,\varepsilon} \in \Theta$ such that

$$w(T_{\phi,\varepsilon} \times N_{\phi,\varepsilon}) \subset \left[f(x) - \frac{\varepsilon}{2} , f(x) + \frac{\varepsilon}{2}\right] .$$

Set

$$D_{\phi,\varepsilon} = \{g \in S: g(N_{\phi,\varepsilon}) \subset [f(x) - \varepsilon , f(x) + \varepsilon]\},$$

and consider the collection

$$\mathcal{D} = \{D_{\phi,\varepsilon}: \phi \in \Phi \text{ and } \varepsilon > 0\} .$$

We will show that for a finite number of elements $D_{\phi_i,\varepsilon_i} \in \mathcal{D}$, $i \in \{1, 2, ..., n\}$,

$$\bigcap_{i=1}^{n} D_{\phi_i,\varepsilon_i} \neq \emptyset .$$

First, choose a function $t \in \bigcap_{i=1}^{n} T_{\phi_i,\varepsilon_i}$. Without loss of generality, we can assume $t \in \overline{S}$, and of course

$$t(N_{\phi_i,\varepsilon_i}) \subset \left[f(x_i) - \frac{\varepsilon_i}{2} , f(x_i) + \frac{\varepsilon_i}{2}\right] ,$$

where $\phi_i \to x_i$. Since $t \in \overline{S}$, there exists a $g \in S$ such that $\|g - t\| \leq \varepsilon_i/2$ for every $i \in \{1, 2, \ldots, n\}$. Now for each $i \in \{1, 2, \ldots, n\}$, we have

$$|g(p) - f(x_i)| \leq |g(p) - t(p)| + |t(p) - f(x_i)| \leq \varepsilon_i$$

for every $p \in N_{\phi_i, \varepsilon_i}$ and thus $g \in \bigcap_{i=1}^{n} D_{\phi_i, \varepsilon_i}$. It is easy to verify that the filter generated by $\mathcal{D}$ converges to f and has a basis in S . Hence $f \in a_c(S)$ as desired.

We now consider the case of a subalgebra $A \subset C(X)$, where X is a completely regular topological space. Here, a subset $S \subset \beta X$ is said to be π-closed if S is closed in βX and $\pi^{-1}(\pi(S)) = S$. The following lemma is due to E. Binz (see [5], lemma 4).

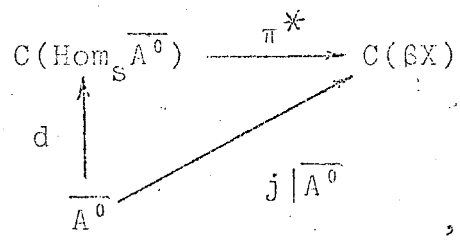
Lemma 4.

If S_1 and S_2 are two disjoint π-closed subsets of βX , then given any two functions g_1 and g_2 in $\overline{A^0}$ there exists a function $g \in \overline{A^0}$ such that

$$\overline{g}|S_1 = \overline{g}_1|S_1 \quad \text{and} \quad \overline{g}|S_2 = \overline{g}_2|S_2 \ .$$

The lemma can be proved by applying the Tietze extension theorem to $C(\mathrm{Hom}_s \overline{A^0})$, and recalling that

d is an isomorphism in the following commutative diagram:



where j is the canonical isomorphism from $C^0(X)$ onto $C(\beta X)$.

Theorem 1.

Let A be a subalgebra of $C(X)$, for a completely regular topological space X . If A^0 , the algebra of all bounded functions in A , is topology generating, then A is dense in $C_c(X)$.

In view of proposition 2, it is sufficient to show that $a_c(\overline{A^0}) = C(X)$. We utilize a technique that appears in the proof of theorem 5 in [5]. Let f be an arbitrary element in $C(X)$. We will construct a filter Θ on $C(X)$ that converges to f in $C_c(X)$ and has a basis in $\overline{A^0}$. For a point $p \in X$, let $g_p \in \overline{A^0}$ such that $g_p(p) = f(p)$. Define

$$V_{p,\varepsilon} = \{y \in \beta X: \ \overline{f}(y) \subset (\overline{g}_p(y) - \varepsilon \ , \ \overline{g}_p(y) + \varepsilon)\}$$

for $\varepsilon > 0$. Now $V_{p,\varepsilon}$ is an open neighborhood of p in βX , and thus $X \setminus V_{p,\varepsilon}$ is a compact subset of βX . Since, by proposition 1, A^0 separates the points in βX from those in X , the set $\pi(\beta X \setminus V_{p,\varepsilon})$ is disjoint from $\pi(p)$. In $\mathrm{Hom}_s \overline{A^0}$, we choose a closed neighborhood N of $\pi(p)$ disjoint from $\pi(\beta X \setminus V_{p,\varepsilon})$. It follows that $\pi^{-1}(N)$ is a π-closed neighborhood of p contained in $V_{p,\varepsilon}$. Let $W_{p,\varepsilon} = \pi^{-1}(N)$, and set

$$T_{p,\varepsilon} = \{g \in \overline{A^0} : |\overline{g}(y) - \overline{f}(y)| < \varepsilon \text{ for every } y \in W_{p,\varepsilon}\} .$$

Consider the collection $\mathcal{J}$ of all sets $T_{p,\varepsilon}$ for all $p \in X$ and $\varepsilon > 0$. Clearly each element $T_{p,\varepsilon} \in \mathcal{J}$ is not empty, since it contains at least the function g_p . We will show that for a finite number of elements $T_{p_i,\varepsilon_i} \in \mathcal{J}$, $i \in \{1, 2, \ldots, n\}$,

$$\bigcap_{i=1}^{n} T_{p_i,\varepsilon_i} \neq \emptyset .$$

For convenience we can assume $\varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_n$. Since we know T_{p_1,ε_1} is non-empty, we assume $\bigcap_{i=1}^{m-1} T_{p_i,\varepsilon_i} \neq \emptyset$ for $m \in \{2, 3, \ldots, n\}$, and prove that $\bigcap_{i=1}^{m} T_{p_i,\varepsilon_i} \neq \emptyset$. Let $L = \bigcup_{i=1}^{m-1} W_{p_i,\varepsilon_i}$. We might as well assume $W_{p_m,\varepsilon_m} \setminus L \neq \emptyset$, for otherwise our proof would be complete. Since the union of a finite number of π-closed sets is π-closed, L is a π-closed set. Thus, $\pi^{-1}(\pi(y))$ is a π-closed set disjoint from L for every $y \in W_{p_m,\varepsilon_m} \setminus L$. Let Ω be the collection of all sets $\pi^{-1}(\pi(y))$ for $y \in W_{p_m,\varepsilon_m} \setminus L$. For the following calculation we will denote the elements in Ω by Greek letters.

First, we choose

$$g_1 \in \bigcap_{i=1}^{m-1} T_{p_i,\varepsilon_i} \quad \text{and} \quad g_2 \in T_{p_m,\varepsilon_m} .$$

Now for each σ and ξ in Ω , lemma 4 allows us to pick a function $g_{\sigma,\xi} \in \overline{A^0}$ which extends both $\overline{g}_1|L$ and $\overline{g}_2|\sigma \cup \xi$. Let

$$M = \bigcup_{i=1}^{m} W_{p_i,\varepsilon_i} .$$

(i.e., $M = L \cup W_{p_m,\varepsilon_m}$). Choose an integer k such that

$$k > \varepsilon_m + \|g_1\| + \|g_2\| ,$$

and set

$$f' = ((f \wedge k\underline{1}) \vee -k\underline{1}) .$$

Clearly $\overline{f}'|M = \overline{f}|M$, and thus the set

$$U_\sigma^\xi = \{y \in \beta X : \overline{g}_{\sigma,\xi}(y) < \overline{f}'(y) + \varepsilon_m\}$$

is an open neighborhood of $\sigma \cup \xi \cup L$. For a fixed ξ , the collection $\{U_\sigma^\xi\}_{\sigma \in \Omega}$ is an open covering of the compact set M . Hence, there exists a finite subset Σ_1 of Ω such that $\{U_\sigma^\xi\}_{\sigma \in \Sigma_1}$ covers M . The function

$$g_\xi = \bigwedge_{\sigma \in \Sigma_1} g_{\sigma,\xi}$$

is an element of $\overline{A^0}$ and has the property that

$$\overline{g}_\xi|L = \overline{g}_1|L \quad , \qquad \overline{g}_\xi|\xi = \overline{g}_2|\xi \quad ,$$

and

$$\overline{g}_\xi(y) < \overline{f}'(y) + \varepsilon_m .$$

for every $y \in M$. Now for each $\xi \in \Omega$, let

$$U_\xi = \{y \in \beta X : \overline{g}_\xi(y) > \overline{f}'(y) - \varepsilon_m\} .$$

U_ξ is an open neighborhood of $\xi \cup L$, and thus $\{U_\xi\}_{\xi \in \Omega}$.

is an open covering of M. Again, there exists a finite subcovering, $\{U_\xi\}_{\xi \in \Sigma_2}$ for Σ_2 a finite subset Ω. The function

$$g = \bigvee_{\xi \in \Sigma_2} g_\xi$$

is an element of $\overline{A^0}$ and enjoys the property that

$$\overline{g}|L = \overline{g}_1|L \quad \text{and} \quad |\overline{g}(y) - \overline{f}'(y)| < \varepsilon_m$$

for every $y \in M$. Hence $g \in \bigcap_{i=1}^{m} T_{p_i, \varepsilon_i}$ as desired. It is straight forward to verify that $\mathcal{J}$ generates a filter that converges to f in $C_c(X)$ and has a basis in $\overline{A^0}$.

If X is a convergence space, the canonical map from X onto its associated completely regular space X', induces a continuous isomorphism from $C_c(X')$ onto $C_c(X)$ (see 0.5). Thus, in view of proposition 1, we have the following:

Corollary.

Let A be a subalgebra of $C(X)$ for a convergence space X. If A^0 separates the points in $\beta X'$ from those in X', then A is dense in $C_c(X)$.

Proposition 3.

If A is a subalgebra of $C(X)$ for a convergence space X, then $a_c(A)$ is a lattice subalgebra of $C(X)$.

It is evident that the adherence of A in $C_c(X)$ is a subalgebra. To prove $a_c(A)$ is a lattice, it suffices to show $|f|$ is an element of $a_c(A)$ whenever f is in A, since

$$f \vee g = \frac{1}{2}(f + g + |f - g|) .$$

Let $f \in a_c(A)$, and let Θ be a filter convergent to f in $C_c(X)$ with a base in A. We denote the collection of all convergent filters on X by Φ. Now for each $\phi \in \Phi$, say $\phi \to x$, and each $\varepsilon > 0$ there exists an $N_{\phi,\varepsilon} \in \phi$ and a $T_{\phi,\varepsilon} \in \Theta$ such that

$$w(T_{\phi,\varepsilon} \times N_{\phi,\varepsilon}) \subset (f(x) - \frac{\varepsilon}{2} , f(x) + \frac{\varepsilon}{2}) .$$

Define

$$D_{\phi,\varepsilon} = \{g \in A: g(N_{\phi,\varepsilon}) \subset (|f|(x) - \varepsilon , |f|(x) + \varepsilon) .$$

We will show that $D_{\phi,\varepsilon}$ is not empty. Indeed, we will demonstrate that for finitely many $\phi_i \in \Phi$ and $\varepsilon_i > 0$, where $i \in \{1, 2, \ldots, n\}$, the set

$$\bigcap_{i=1}^{n} D_{\phi_i, \varepsilon_i}$$

is not void. Let t be a fixed element in $\bigcap_{i=1}^{n} T_{\phi_i, \varepsilon_i} \cap A$. Obviously

$$t(N_{\phi_i, \varepsilon_i}) \subset \left(f(x_i) - \frac{\varepsilon_i}{2}, f(x_i) + \frac{\varepsilon_i}{2}\right) ,$$

where $\phi_i \to x_i$ for each $i \in \{1, 2, \ldots, n\}$. In particular, there exists an integer k such that

$$t\left(\bigcup_{i=1}^{n} N_{\phi_i, \varepsilon_i}\right) \subset [-k , k] .$$

Now the binomial expansion for $(1 - s)^{1/2}$ (the function from $\mathbb{R}$ into $\mathbb{R}$) converges uniformly for $|s| \leq 1$. Thus there exists a polynomial P with the property that

$$|(1 - s)^{1/2} - P(s)| < \frac{\varepsilon'}{2k} ,$$

where

$$\varepsilon' = \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\} .$$

This means that

$$\left| \left|\frac{t}{k}\right|(x) - P(\underline{1}-(\frac{t}{k})^2)(x) \right|$$

$$= \left| \{1 - (\underline{1}-(\frac{t}{k})^2)(x)\}^{\frac{1}{2}} - P(\underline{1}-(\frac{t}{k})^2)(x) \right| < \frac{\varepsilon'}{2k}$$

for every $x \in \bigcup_{i=1}^{n} N_{\phi_i, \varepsilon_i}$. Furthermore, for each $i \in \{1, 2, \ldots, n\}$, we have

$$\left| |f|(x_i) - kP(\underline{1}-(\frac{t}{k})^2)(x) \right|$$

$$\leq \left| |f|(x_i) - |t|(x) \right| + \left| |t|(x) - kP(\underline{1}-(\frac{t}{k})^2)(x) \right| < \varepsilon_i$$

for every $x \in N_{\phi_i, \varepsilon_i}$. Hence $kP(\underline{1} - (\frac{t}{k})^2)$ is an element of $\bigcap_{i=1}^{n} D_{\phi_i, \varepsilon_i}$. Now the collection of sets $D_{\phi, \varepsilon}$, for $\phi \in \Phi$ and $\varepsilon > 0$, generates a filter convergent to $|f|$ in $C_c(X)$ with a basis in A , and thus $|f| \in a_c(A)$ which completes the proof.

Because a lattice subalgebra $A \subset C(X)$ is topology generating if and only if the subalgebra A^0 consisting of all bounded functions in A is topology generating, proposition 3 and theorem 1 yield:

Theorem 2.

If A is a topology generating subalgebra of C(X), for a completely regular topological space X, then $a_c(A)$ is dense in $C_c(X)$.

2.2. In chapter 1 we provided a characterization of separable metrizable topological spaces (theorem 3). Here, using the results of the last section, we will prove the following:

Theorem 3.

For a completely regular topological space X, the following statements are equivalent:

(1). X *is separable and metrizable.*

(2). $C_c(X)$ *is second countable.*

(3). C(X) *contains a countable, topology generating subset.*

(4). $C_c(X)$ *contains a countable, dense, topology generating subset.*

(5). C(X) *contains a countable subset which separates the points in* βX *from those in* X.

(6). $C_c(X)$ *contains a countable, dense subset which separates the points in* βX *from those in* X.

The equivalence of (1) and (2) is simply a restatement of theorem 3 in 1.2.

Clearly (6) implies (5). We first prove that (5) implies (4). Assume D is a countable subset of $C(X)$ which separates the points in βX from those in X. Without loss of generality, we can assume $D \subset C^0(X)$. For otherwise,

$$\{((-n\underline{1}) \vee f) \wedge (n\underline{1})\}_{n \in \mathbb{N}}$$

could replace each unbounded $f \in D$, and this new set of bounded functions would have the required properties. Now proposition 1 implies that the subalgebra A generated by D is topology generating. Furthermore, by theorem 1, A is dense in $C_c(X)$. We consider the set $\hat{D}$ consisting of all functions of the form

$$P(f_1, f_2, \ldots, f_n),$$

where $f_i \in D \cup \underline{1}$ and P runs through all polynomials (without constant term) in $n \geq 1$ indeterminates with rational coefficients. Clearly the set $\hat{D}$ is still countable. We will show that $\hat{D}$ satisfies the conditions of statement (4). To this end, we prove first that $\hat{D}$ is dense in A with respect to the sup-norm topology (i.e., the subspace topology on A inherited from $C_n^0(X)$).

Let

$$\sum_{i=1}^{n} a_i \prod_{k=1}^{m_i} f_{i_k}$$

for $a_i \in \mathbb{R}$ and $f_{i_k} \in D \cup \underline{1}$ be an arbitrary element in A. Since all the functions in question are bounded, given $\varepsilon > 0$ there exist rational numbers $r_1, r_2, \ldots, r_n$ so that

$$\left\| \sum_{i=1}^{n} a_i \prod_{k=1}^{m_i} f_{i_k} - \sum_{i=1}^{n} r_i \prod_{k=1}^{m_i} f_{i_k} \right\|$$

$$\leq \left(\sum_{i=1}^{n} |a_i - r_i| \right) \left\| \sum_{i=1}^{n} \prod_{k=1}^{m_i} f_{i_k} \right\| < \varepsilon .$$

Therefore, it follows from proposition 2 that $\hat{D}$ is dense in $C_c(X)$. It only remains to verify that $\hat{D}$ is topology generating. Since A is a topology generating subalgebra, any neighborhood of a point $x \in X$ contains $f^{-1}(-1, 1)$ for some $f \in A$. In fact, we can assume $f(x) = 0$. Let $g \in \hat{D}$ such that $\|g - f\| < 1/2$. Thus $g^{-1}(-1/2, 1/2)$ is a neighborhood of x contained in $f^{-1}(-1, 1)$ as desired.

Of course (4) implies (3) trivially. To prove (3) implies (1), assume B is a countable, topology generating subset of $C(X)$. Since B is topology generating, the map sending each point $x \in X$ into $(f(x))_{f \in B}$ is a homeomorphism of X into $\mathbb{R}^{\mathbb{N}}$, the cartesian product of a countable collection of real lines. Now statement (1) follows from the fact that $\mathbb{R}^{\mathbb{N}}$ is separable and metrizable.

It only remains to prove (1) implies (6). Let d denote a metric on X that induces the given topology, and let $\{x_n\}_{n \in \mathbb{N}}$ be a countable dense subset of X. We define, for each $n \in \mathbb{N}$, the function $\tilde{x}_n \in C^0(X)$ by

$$\tilde{x}_n(y) = \min\{d(x_n, y) , 1\}$$

for all $y \in X$. Let A be the subalgebra of $C(X)$ generated by $\{\tilde{x}_n\}_{n \in \mathbb{N}}$. Clearly A is topology generating, and thus, by proposition 1, the algebra A separates the points in βX from those in X. We consider the set E consisting of all functions of the form

$$P(\tilde{x}_{n_1}, \ldots, \tilde{x}_{n_k}) ,$$

where $\tilde{x}_{n_i} \in \{\tilde{x}_n\}_{n \in \mathbb{N}} \cup \underline{1}$ and P ranges through all polynomials (without constant term) in $k \geq 1$ indeterminates with rational coefficients. Arguing as above, E is dense in A with the sup-norm topology. Now an easy calculation shows that E separates the points in βX from those in X. Theorem 1 implies that A is dense in $C_c(X)$, and by appealing to proposition 2, we conclude that E itself is dense in $C_c(X)$. Since the set E is countable, the proof is complete.

We conclude this section with a characterization of separability.

Proposition 4.

A completely regular (respectively realcompact) topological space X is separable if and only if there exists a continuous monomorphism (respectively a monomorphism) from $C_c(X)$ into $C_c(Y)$, where Y is a countable discrete topological space.

Let X be a completely regular, separable topological space, and let Y be a countable dense subset of X . Give Y the discrete topology, and denote the inclusion map from Y into X by i . Since i is continuous, the induced map

$$i^*: C_c(X) \longrightarrow C_c(Y) ,$$

sending each function $f \in C(X)$ to the function $f \circ i$, is a continuous homomorphism. Furthermore, since $i(Y)$ is dense in X , the homomorphism i^* is injective.

Conversely, assume first that X is a completely regular topological space, and u is a continuous monomorphism from $C_c(X)$ into $C_c(Y)$, where Y is a countable discrete space. Now, the map

$$u^*: \mathcal{H}om_c C_c(Y) \longrightarrow \mathcal{H}om_c C_c(X) ,$$

sending each homomorphism $h \in \mathcal{H}om\, C(Y)$ to the homomorphism $h \circ u$, is continuous. Since both X and Y are c-embedded convergence spaces, u^* can be regarded as a continuous map from Y into X . It is easy to verify that the induced map u^{**} must be equal to u . To prove that X is separable, assume that the countable set $u^*(Y)$ is not dense in X . Thus, there exists an open set U in X disjoint form the closure of $u^*(Y)$. Since X is a completely regular space, we can find a function $f \in C(X)$ such that $f \neq \underline{0}$ while $f(U^c) = \{0\}$ ($U^c = X \setminus U$). This means that $u(f) = \underline{0}$ which contradicts the fact that u is injective. Therefore, $u^*(Y)$ is indeed dense in X . Finally, assume X is realcompact and u is a monomorphism of C(X) into C(Y) , where Y is a countable discrete space. Now

$$u^*: \mathrm{Hom}_s C(Y) \longrightarrow \mathrm{Hom}_s C(X)$$

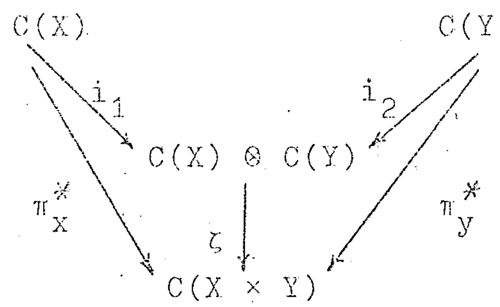
is continuous. Y is Lindelöf, and thus theorem 8.2, p. 115 in [9] implies that Y is realcompact. Since X is realcompact by assumption, u^* can be regarded as a continuous map from Y into X. Again, $u^{**} = u$, and arguing as above, $u^*(Y)$ must be dense in X.

2.3. In this section we will investigate the algebraic tensor product of $C(X)$ and $C(Y)$ for completely regular topological spaces X and Y. For a definition of the tensor product of two algebras, see, for example, [12], p. 420.

In the usual manner, we write basis elements of $C(X) \otimes C(Y)$ in the form $f \otimes g$ for $f \in C(X)$ and $g \in C(Y)$. The canonical monomorphism i_1 from $C(X)$ into $C(X) \otimes C(Y)$ is defined by

$$i_1(f) = f \otimes \underline{1}$$

for each $f \in C(X)$. Similarly, i_2 sending g to $\underline{1} \otimes g$ is a monomorphism of $C(Y)$ into the tensor product. Let π_x and π_y be the projections of $X \times Y$ onto X and Y respectively. Since the projections are continuous and surjective, π_x^* (respectively π_y^*) is a continuous monomorphism from $C_c(X)$ (respectively $C_c(Y)$) into $C_c(X \times Y)$. Now, by the universal property of the tensor product (see [12], p. 420), there exists a unique homomorphism ζ, making the following diagram commutative:



It is therefore clear that for a basis element $f \otimes g \in C(X) \otimes C(Y)$,

$$\zeta(f \otimes g) = \pi_x^*(f) \cdot \pi_y^*(g) \quad .$$

Thus the image of an arbitrary element in $C(X) \otimes C(Y)$ can be calculated by linearity.

Lemma 5.

Given $C(X)$ and $C(Y)$, the map ζ is a monomorphism from $C(X) \otimes C(Y)$ into $C(X \times Y)$.

Suppose that $\sum_{i=1}^{n} (f_i \otimes g_i)$ is send to $\underline{0}$ under ζ. Without loss of generality, we can assume $f_1, f_2, \ldots, f_n$ are linearly independent. By definition,

$$\sum_{i=1}^{n} (f_i(x) \cdot g_i(y)) = 0$$

for every $(x , y) \in X \times Y$. Assume that there exists a

$g_i \in \{g_1, \ldots, g_n\}$ and $y \in Y$ such that $g_i(y) \neq 0$. This implies that

$$\sum_{i=1}^{n} g_i(y) f_i = \underline{0} ,$$

which contradicts the fact that $f_1, \ldots, f_n$ are linearly independent. Hence ζ is indeed injective.

Since ζ is a monomorphism into $C(X \times Y)$, we can regard $C(X) \otimes C(Y)$ as a subalgebra of $C(X \times Y)$.

<u>Theorem</u> 4.

<u>If</u> X <u>and</u> Y <u>are completely regular topological spaces</u>, <u>then</u> $C(X) \otimes C(Y)$ <u>is a dense subalgebra of</u> $C_c(X \times Y)$.

In view of theorem 1, it is sufficient to prove that the collection of bounded functions in $C(X) \otimes C(Y)$ is topology generating. The topology on $X \times Y$ is simply the coarsest topology such that the projections are continuous. Since $C^0(X)$ and $C^0(Y)$ generate the topologies of X and Y respectively, the collection of all functions $f \circ \pi_X$ and $g \circ \pi_Y$, for $f \in C^0(X)$ and $g \in C^0(Y)$, generate the topology of $X \times Y$. Furthermore, $f \circ \pi_X = \pi_X^*(f)$, which means $f \circ \pi_X = f \otimes \underline{1}$ (regarded as an element in $C(X \times Y)$). Similarly, $g \circ \pi_Y = 1 \otimes g$.

Since π_x^* and π_y^* take bounded funtions to bounded functions, the subalgebra

$$(C(X) \otimes C(Y)) \cap C^0(X \times Y)$$

is topology generating.

Let $[C(X) \otimes C(Y)]_c$ denote the subalgebra $C(X) \otimes C(Y)$ together with the convergence structure inherited from $C_c(X \times Y)$.

Proposition 5.

For X *and* Y *completely regular topological spaces,* $\mathcal{H}om_c[C(X) \otimes C(Y)]_c$ *is homeomorphic to* $X \times Y$.

We will first show that as sets $\mathcal{H}om\ [C(X) \otimes C(Y)]_c$ can be identified with $X \times Y$. Consider the map

$$i_{X\times Y}: X \times Y \longrightarrow \mathcal{H}om_c[C(X) \otimes C(Y)]_c$$

sending each $(x , y) \in X \times Y$ to the homomorphism of point evaluation by (x , y) . In view of theorem 4, the subalgebra $C(X) \otimes C(Y)$ separates the points in $X \times Y$, and thus $i_{X\times Y}$ is injective. For the following proof of the surjectivity of $i_{X\times Y}$ we are indebted to E. Binz and K. Kutzler. Assume there exists an

$h \in \mathcal{H}om\ [C(X) \otimes C(Y)]_c$ such that h is not an element of $i_{X\times Y}(X \times Y)$. For convenience, we denote the subalgebra $C(X) \otimes C(Y)$ by A. As noted in the proof of theorem 4, the subalgebra A^0 consisting of all bounded functions in A is dense in $C_c(X \times Y)$, and thus A^0 is dense in $[C(X) \otimes C(Y)]_c$. This means that $h|A^0$ can not be realized as a point evaluation. For if $h|A^0$ were a point evaluation, the density of A^0 in $[C(X) \otimes C(Y)]_c$ would imply that h itself is a point evaluation. Now let $\overline{A^0}$ denote the sup-norm closure of A^0 in $C_n^0(X \times Y)$. The homomorphism $h|A^0$ can be extended to a continuous homomorphism $h' : \overline{A^0} \longrightarrow \mathbb{R}$ with respect to the sup-norm topology. Furthermore, $\overline{A^0}$ is a lattice subalgebra of $C(X \times Y)$ (see lemma 1), and it is easy to verify that h' is a lattice homomorphism (i.e., $h'(f \wedge g) = h'(f) \wedge h'(g)$ and $h'(f) \vee h'(g) = h'(f \vee g)$ for every f and g in $\overline{A^0}$). Since h' is not a point evaluation homomorphism, for each point $z \in X \times Y$ we can choose a function $f_z \in \overline{A^0}$ such that

$$f_z(z) = 0 \quad \text{and} \quad h'(f_z) = 1 .$$

Because h' is a lattice homomorphism, we can assume each $f_z \geq 0$. Now for each $z \in X \times Y$ and each $\varepsilon > 0$ there exists a neighborhood $U_{z,\varepsilon}$ of z so that

$$f_z(U_{z,\varepsilon}) \subset [0\ ,\ \tfrac{\varepsilon}{2})\ .$$

Define

$$D_{z,\varepsilon} = \{f \in A^0 : f(U_{z,\varepsilon}) \subset (-\varepsilon\ ,\ \varepsilon) \text{ and } h(f) > \tfrac{1}{2}\}\ .$$

Let $\mathscr{D}$ denote the collection of all sets $D_{z,\varepsilon}$ for $z \in X \times Y$ and $\varepsilon > 0$. Given a finite number of elements D_{z_1,ε_1} , D_{z_2,ε_2} , ..., D_{z_n,ε_n} in $\mathscr{D}$, we claim that

$$\bigcap_{i=1}^{n} D_{z_i,\varepsilon_i} \neq \emptyset\ .$$

Now the function

$$g = \bigwedge_{i=1}^{n} f_{z_i,\varepsilon_i}$$

is in $\overline{A^0}$ with the property that $h'(g) = 1$ and

$$g(U_{z_i,\varepsilon_i}) \subset [0\ ,\ \tfrac{\varepsilon_i}{2})$$

for each $i \in \{1, 2, ..., n\}$. If

$$\tilde{\varepsilon} = \min\{\varepsilon_1, \varepsilon_2, ..., \varepsilon_n\}\ ,$$

we can find a function g' in the subalgebra A^0 with

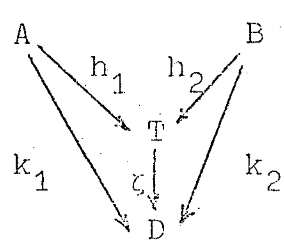
$||g' - g|| < \varepsilon/4$ and $h'(g') > 1/2$. It is evident that $g' \in \bigcap_{i=1}^{n} D_{z_i,\varepsilon_i}$. Thus the collection $\mathcal{D}$ generates a filter Θ that converges to $\underline{0}$ in $[C(X) \otimes C(Y)]_c$. On the other hand, $h(\Theta)$ doesn't converge to 0 since for every set $T \in \Theta$ there exists a function $f \in T$ such that $h(f) > 1/2$. This contradicts the fact that h is continuous, and thus $i_{X\times Y}$ is surjective. Now, to show that the spaces in question are homeomorphic, consider the following commutative diagram:

$$\text{I)} \quad \begin{array}{ccc} \mathcal{H}om_c C_c(X \times Y) & \xrightarrow{\;i^*\;} & \mathcal{H}om_c [C(X) \otimes C(Y)]_c \\ & \searrow^{\text{id}} & \downarrow \text{id} \\ & & \mathcal{H}om_s [C(X) \otimes C(Y)]_c \end{array} ,$$

where i^* is the map induced by the inclusion i from $[C(X) \otimes C(Y)]_c$ into $C_c(X \times Y)$ and id denotes the identity map. It follows from the proof of theorem 4 that $C(X) \otimes C(Y)$ is topology generating, and thus $\mathcal{H}om_s [C(X) \otimes C(Y)]_c$ is homeomorphic to $X \times Y$. Since all the maps in (I) are continuous and $X \times Y$ is c-embedded, we conclude that $\mathcal{H}om_c [C(X) \otimes C(Y)]_c$ is homeomorphic to $X \times Y$.

2.4. Let $\mathcal{T}_c$ be the subcategory of $\mathcal{A}$ consisting of all convergence algebras of the form $C_c(X)$, for a completely regular topological space X. Here, with the help of theorem 4, we will determine the tensor product in the category $\mathcal{T}_c$.

Let A and B be objects in an arbitrary category $\mathfrak{A}$. An object T in $\mathfrak{A}$ together with morphisms $h_1: A \to T$ and $h_2: B \to T$ is said to be a <u>coproduct</u> of A and B if the following universal property is satisfied: Given an object $D \in \mathfrak{A}$ and morphisms $k_1: A \to D$ and $k_2: B \to D$, there exists a unique morphism $\zeta: T \to D$ making the following diagram commutative:



We call such an object T a <u>tensor product</u> of A and B.

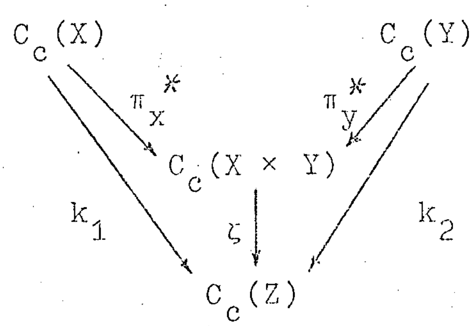
By standard categorical arguments, it is clear that any two tensor products are isomorphic. Thus if tensor products exist, we can speak of <u>the</u> tensor product.

<u>Theorem 5.</u>

<u>Let</u> $C_c(X)$ <u>and</u> $C_c(Y)$ <u>be objects in</u> $\mathcal{T}_c$. <u>The tensor product (in</u> $\mathcal{T}_c$ <u>) of</u> $C_c(X)$ <u>and</u> $C_c(Y)$ <u>is</u> $C_c(X \times Y)$.

We will show that $C_c(X \times Y)$ together with the morphisms π_x^* and π_y^*, as defined in the last section, is a coproduct of $C_c(X)$ and $C_c(Y)$. As mentioned in 0.4, induced maps such as π_x^* and π_y^* are morphisms (i.e., continuous) in the category $\mathcal{F}_c$. Given an object $C_c(Z) \in \mathcal{F}_c$ and morphisms $k_1 : C_c(X) \to C_c(Z)$ and $k_2 : C_c(Y) \to C_c(Z)$, we construct a unique morphism ζ so that the following diagram is commutative:

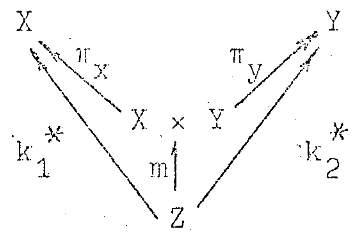
I)



The induced map $k_1^* : \mathcal{H}om_s C_c(Z) \longrightarrow \mathcal{H}om_s C_c(X)$ can be regarded as a continuous map from Z into X, since the spaces in question are completely regular (see 0.7). Similarly, k_2^* can be regarded as a continuous map from Z into Y. We now define a map m from Z into $X \times Y$ by

$$m(z) = (k_1^*(z) , k_2^*(z))$$

for every $z \in Z$. Clearly m is continuous, and further, the following diagram is commutative:



We claim that $\zeta = m^*$ is the desired map (i.e., $\zeta(f) = f \circ m$ for each $f \in C(X \times Y)$). We note that the induced map k_1^{**} from $C_c(X)$ into $C_c(Z)$ is just k_1 , and similarly, $k_2^{**} = k_2$. It is now easy to verify that ζ makes the diagram (I) commutative. It only remains to prove that ζ is unique. Clearly on elements of the form $\pi_x^*(f)$ and $\pi_y^*(g)$, for $f \in C(X)$ and $g \in C(Y)$, the map ζ is unique. It follows that ζ is completely determined on the subalgebra $C(X) \otimes C(Y)$ as $\pi_x^*(f) = f \otimes \underline{1}$ and $\pi_y^*(g) = \underline{1} \otimes g$. Since $C(X) \otimes C(Y)$ is dense in $C_c(X \times Y)$ by theorem 4, the proof is complete.

Remark. Let $\mathcal{T}_k$ and $\mathcal{T}_s$ be the subcategories of $\mathcal{A}$ consisting of all topological algebras of the form $C_k(X)$ and $C_s(X)$ respectively, for a completely regular topological space X . It is now easy to show that the tensor product of objects $C_k(X)$ and $C_k(Y)$ in $\mathcal{T}_k$ (respectively $C_s(X)$ and $C_s(Y)$ in $\mathcal{T}_s$) is $C_k(X \times Y)$ (respectively $C_s(X \times Y)$).

3. INDUCTIVE LIMITS

3.1. We introduce the concept of an inductive limit in the category of convergence spaces.

Consider a non-empty family $\{Y_a\}_{a \in \mathcal{A}}$ of convergence spaces. Assume the index set $\mathcal{A}$ is directed, and denote the preorder relation by " $\leq$ ". We require that for every $(a, a') \in \mathcal{A} \times \mathcal{A}$, the family $\{Y_a\}_{a \in \mathcal{A}}$ satisfies the following two conditions:

(i). If $a \leq a'$ then $Y_a \subset Y_{a'}$ (as sets).

(ii). If $a \leq a'$ then the natural inclusion map from Y_a into $Y_{a'}$ is continuous.

Let $Y = \bigcup_{a \in \mathcal{A}} Y_a$, and let i_a be the natural inclusion map,

$$i_a : Y_a \longrightarrow Y .$$

The set Y together with the finest of all convergence structures making the inclusion maps i_a for every $a \in \mathcal{A}$ continuous is called the <u>inductive limit</u> (induktiver Limes [11]) of the family $\{Y_a\}_{a \in \mathcal{A}}$. We denote this space by $\operatorname{ind}_{a \in \mathcal{A}} Y_a$.

Even if all the members of a family $\{Y_a\}_{a \in \mathcal{A}}$ are topological spaces, the inductive limit will not in general be a topological space, as we are working in the category of convergence spaces. In fact, we have the following characterization of convergent filters in $\operatorname{ind}_{a \in \mathcal{A}} Y_a$ (see [11]):

Proposition 1.

A filter ϕ on Y converges to y in $\operatorname{ind}_{a \in \mathcal{A}} Y_a$ if and only if there exists a filter ϕ_a on Y_a, for some $a \in \mathcal{A}$, such that $\phi_a \rightarrow y$ in Y_a and ϕ_a is a basis for the filter ϕ in Y (i.e., $i_a(\phi_a) = \phi$).

It is now easy to see that the inductive limit of a family $\{Y_a\}_{a \in \mathcal{A}}$ is separated if and only if every Y_a is separated.

By appealing to proposition 1, one can verify the following universal property for an inductive limit (see [11]).

Proposition 2.

A map t from the $\operatorname{ind}_{a \in \mathcal{A}} Y_a$ into a convergence space X is continuous if and only if the composition map,

$$t \circ i_a : Y_a \longrightarrow X ,$$

is continuous for each $a \in \mathcal{A}$.

We will now consider the special case of a family $\{L_a\}_{a \in \mathcal{A}}$ of convergence vector spaces. Here, we demand that the family $\{L_a\}_{a \in \mathcal{A}}$ satisfies conditions (i) and (ii), and in addition, the inclusion maps of condition (ii) must be linear. This means $L = \bigcup_{a \in \mathcal{A}} L_a$ is, in a natural way, a vector space and each L_a is a linear subspace of L (i.e., the maps i_a are linear). Whenever we speak of the inductive limit M of a family $\{L_a\}_{a \in \mathcal{A}}$, where each L_a is a convergence vector space, we require that the above conditions are satisfied, which quarantees that M itself is a convergence vector space. In this case, we write

$$M = \operatorname{Ind}_{a \in \mathcal{A}} L_a \ ,$$

or simply $M = \operatorname{Ind} L_a$. If each L_a is a locally convex topological vector space, then M is called a <u>Marinescu-space</u> (Marinescu-Raum, [10]) or M has a <u>Marinescu-convergence structure</u>.

3.2. In this section we will define a Marinescu-convergence structure on $C(X)$.

Let X be a completely regular topological space. We regard X as embedded in its Stone-Čech compactification denoted by βX . Given any compact subset K of βX

such that $K \subset \beta X \setminus X$, the natural inclusion map

$$\mu_K : X \longrightarrow \beta X \setminus K$$

is obviously continuous, and thus induces a homomorphism

$$\mu_K^* : C(\beta X \setminus K) \longrightarrow C(X) ,$$

defined by $\mu_K^*(f) = f \circ \mu_K$ for every $f \in C(\beta X \setminus K)$. In fact, μ_K^* is a monomorphism as X is dense in $\beta X \setminus K$. For convenience, we will identify the topological algebra $C_c(\beta X \setminus K)$ with its restriction to a subalgebra of $C(X)$ via the map μ_K^* (i.e., retaining the same topology). Since $\beta X \setminus K$ is a locally compact topological space, the remark in 0.3 implies that

$$C_c(\beta X \setminus K) = C_k(\beta X \setminus K)$$

(the compact-open topology).

Consider the family $\{C_c(\beta X \setminus K)\}_{K \in \kappa}$, where κ is the collection of all compact subsets of $\beta X \setminus X$. Since the union of two elements in κ is again in κ , the collection κ is a directed set under the preorder of inclusion. Given $K_1 \subset K_2$ for K_1 and K_2 in κ , we have

$$(\beta X\setminus K_2) \subset (\beta X\setminus K_1) .$$

This natural inclusion, call it j, induces a continuous homomorphism

$$j^*: C_c(\beta X\setminus K_1) \longrightarrow C_c(\beta X\setminus K_2) ,$$

and j^* is also injective as $\beta X\setminus K_2$ is dense in $\beta X\setminus K_1$. Indeed, j^* is simply the inclusion map from $C_c(\beta X\setminus K_1)$ into $C_c(\beta X\setminus K_2)$ (as subalgebras of $C(X)$) . Thus, we can speak of the inductive limit of the family $\{C_c(\beta X\setminus K)\}_{K\in\kappa}$. Since each member of this family is a locally convex topological vector space, $\operatorname{Ind}_{K\in\kappa} C_c(\beta X\setminus K)$ is a Marinescu-space.

We claim that the $\operatorname{Ind} C_c(\beta X\setminus K)$ is actually a Marinescu-convergence structure on $C(X)$. That is,

$$\bigcup_{K\in\kappa} C(\beta X\setminus K) = C(X) .$$

One inclusion is clear, and thus it is sufficient to show that every function in $C(X)$ has an extension to $C(\beta X\setminus K)$ for some $K\in\kappa$. Given $f\in C(X)$, consider $\overline{f}$, the continuous extension of f to a map from βX into $\tilde{R}$ as in 2.1. Since f is real-valued, $\overline{f}^{-1}(\infty)\subset \beta X\setminus X$. Furthermore, the continuity of $\overline{f}$

implies that $\overline{f}^{-1}(\infty)$ is a closed and hence a compact subset of βX. Thus f has an extension to $C(\beta X \setminus \overline{f}^{-1}(\infty))$, and $\overline{f}^{-1}(\infty)$ is a member of κ.

To simplify the notation, we set

$$C_I(X) = \operatorname*{Ind}_{K \in \kappa} C_c(\beta X \setminus K) \quad .$$

Since all the inclusion maps j^* are homomorphisms, it is easy to verify that $C_I(X)$ is also a convergence algebra.

The maps

$$\mu_K^* : C_c(\beta X \setminus K) \longrightarrow C_c(X)$$

are continuous for every $K \in \kappa$, and hence proposition 2 implies that the identity,

$$\mathrm{id}: C_I(X) \longrightarrow C_c(X) \; ,$$

is always continuous.

3.3. The concept of completeness in topological vector spaces can be extended to convergence vector spaces (see [5]). A filter Θ in a convergence vector space V is said to be Cauchy if $\Theta - \Theta$ converges to the zero element, where " - " is the operation on filters induced by the subtraction in V. Thus, we call V complete if every Cauchy filter converges to an element in V.

Theorem 1.

$C_I(X)$ is complete for any completely regular topological space X.

Assume Θ is a Cauchy filter in $C_I(X)$. Since $\Theta - \Theta$ converges to $\underline{0}$ in $C_I(X)$, there exists a filter Ψ convergent to $\underline{0}$ in $C_c(\beta X \setminus K)$ for some $K \in \kappa$, with the property that Ψ is a basis for $\Theta - \Theta$ in $C_I(X)$. Thus, there exist sets M and N in Θ such that $(M - N) \in \Psi$. Consider a fixed element f in M. Given any function $g \in N$, we know $(f - g) \in C(\beta X \setminus K)$ which implies

$$\bar{g}^{-1}(\infty) \subset \bar{f}^{-1}(\infty) \cup K \quad .$$

Therefore, the set N is contained in $C(\beta X \setminus K')$, where

$$K' = \bar{f}^{-1}(\infty) \cup K \quad .$$

This means the filter Θ has a basis in $C(\beta X \setminus K')$. Specifically, the filter Θ' in $C(\beta X \setminus K')$, consisting of all sets $B \cap C(\beta X \setminus K')$ for $B \in \Theta$, is a basis for Θ

in $C(X)$. Furthermore, $\Theta' - \Theta'$ converges to $\underline{0}$ in $C_c(\beta X \setminus K')$, as it is the image of Ψ under the continuous inclusion map

$$j^*: C_c(\beta X \setminus K) \longrightarrow C_c(\beta X \setminus K')$$

(i.e., $j^*(\Psi) = \Theta' - \Theta'$). Now it is well-known that $C_c(\beta X \setminus K')$ is complete (e.g., see [5]), and thus $\Theta' \to k$ for some function $k \in C(\beta X \setminus K')$. It follows that the filter Θ converges to k in $C_I(X)$, and hence $C_I(X)$ is complete.

3.4. Here, we will investigate the structure of closed ideals in $C_I(X)$.

For a non-empty subset M of a convergence space X , we define the ideal $I(M)$ in $C(X)$ by

$$I(M) = \{f \in C(X): f(M) = \{0\}\} .$$

Similarly, we define the ideal $I^0(M)$ in $C^0(X)$, the bounded functions in $C(X)$, by

$$I^0(M) = \{f \in C^0(X): f(M) = \{0\}\} .$$

An ideal J is said to be <u>full</u> if

$$J = I(M)$$

for some subset M of X .

It is easy to verify the following:

Lemma 1.

Let X be a completely regular topological space. If J is a full ideal in $C(X)$, then J is closed in $C_I(X)$.

Given a completely regular topological space X , we will denote the convergence structure on $C^0(X)$ inherited as a subspace of $C_I(X)$ by $C_I^0(X)$. It is straight forward to verify that $C_I^0(X)$ is bicontinuously isomorphic to the inductive limit of the family $\{C_c^0(\beta X\setminus K)\}_{K\in\kappa}$, where $C_c^0(\beta X\setminus K)$ carries the subspace topology inherited from $C_c(\beta X\setminus K)$.

For an ideal J in $C(X)$ or in $C^0(X)$, we define

$$N_X(J) = \{x\in X\colon f(x) = 0 \text{ for every } f\in J\} \quad ,$$

and refer to this set as the null-set of J . In terms of zero-sets,

$$N_X(J) = \bigcap_{f\in J} Z(f) \quad .$$

By the zero-set of a function $f \in C(X)$, which we denote by $Z(f)$, we mean $\{x \in X: f(x) = 0\}$.

Proposition 3.

If J is an ideal in $C(X)$, then

$$N_X(J) = N_X(J \cap C^0(X)) .$$

Let P denote the ideal $J \cap C^0(X)$ in $C^0(X)$. Since $P \subset J$, it is clear that $N_X(J) \subset N_X(P)$. On the other hand, assume $x \notin N_X(J)$. Therefore $x \notin Z(f)$ for some $f \in J$. Further, there exists a unit u (an invertible function) in $C(X)$ such that

$$((-\underline{1} \vee f) \wedge \underline{1}) = uf$$

(see [9], p. 21). Now x is not in $N_X(P)$ since

$$Z(uf) = Z(f)$$

and $uf \in P$.

Before showing that a closed ideal in $C_I(X)$ is full, we need the following result.

Lemma 2.

If J is a closed ideal in $C_I(X)$, then $N_X(J)$ is not empty, for any completely regular topological space X.

In view of proposition 3, it is sufficient to prove that $N_X(P) \neq \emptyset$, where $P = J \cap C^0(X)$. By $N_{\beta X \setminus K}(P)$ for any $K \in \kappa$, we mean the null-set of P regarded as an ideal in $C^0(\beta X \setminus K)$. Of course the subalgebra $C(\beta X \setminus K)$ contains $C^0(X)$. It is easy to verify the following:

$$N_X(P) = N_{\beta X}(P) \cap X$$

and

$$N_{\beta X \setminus K}(P) = N_{\beta X}(P) \cap \beta X \setminus K \quad .$$

In particular, assume that $N_X(P)$ is empty. Then $N_{\beta X}(P)$, which we denote by K_o, is a subset of $\beta X \setminus X$, and further, K_o is compact in βX. This means K_o is an element of κ, and $N_{\beta X \setminus K_o}(P)$ is empty. If we let J' be the ideal $J \cap C(\beta X \setminus K_o)$ in $C(\beta X \setminus K_o)$, proposition 3 implies that

$$N_{\beta X \setminus K_o}(J') = N_{\beta X \setminus K_o}(P) = \emptyset \quad .$$

But J' is a closed ideal in $C_c(\beta X \setminus K_o)$, which contradicts the fact that the null-set of a closed ideal in the topological algebra $C_c(\beta X \setminus K_o)$ is never empty (see, for example, [4]). Thus $N_X(P)$ can not be empty, which completes the proof.

Lemma 3.

Let X be a completely regular topological space. If J is a closed ideal in $C_I(X)$, then J is full.

For $P = J \cap C^0(X)$, let $N = N_X(P)$ in the following proof. We know, by the previous lemma, that N is a non-empty subset of X. We will demonstrate that J is the full ideal I(N). First, we define $\overline{N}$ to be the closure of N in βX, and show that

$$\overline{N} = N_{\beta X}(P) \quad .$$

Assume equality does not hold; then there exists a $t \in \beta X$ such that $t \in N_{\beta X}(P) \setminus \overline{N}$. Further, we can choose a closed neighborhood V of t in βX so that $V \cap \overline{N} = \emptyset$. Denote $V \cap N_{\beta X}(P)$ by K'. Clearly K' is a compact subset of βX, and

$$K' \subset N_{\beta X}(P) \setminus \overline{N} \subset \beta X \setminus X \quad .$$

Now, we can find a function $g \in C(\beta X)$ with the property that

$$g(t) = 1 \quad \text{and} \quad g(\beta X \setminus V) = \{0\} .$$

For example, let U be an open neighborhood of t contained in V. Then by complete regularity, there exists a function $g \in C(\beta X)$ such that $g(t) = 1$ and $g(U^c) = \{0\}$. Since J is a closed ideal in $C_I(X)$, for each $K \in \kappa$ the ideal $J \cap C(\beta X \setminus K)$ is closed in $C_c(\beta X \setminus K)$. It is well-known that an ideal in the topological algebra $C_c(\beta X \setminus K)$ is closed if and only if it is full (see, for example, [4]). Since $C(\beta X \setminus K) \supset C^0(X)$, we conclude that

$$P = I^0(N_{\beta X \setminus K}(P))$$

for each $K \in \kappa$. Now

$$N_{\beta X \setminus K'}(P) = N_{\beta X}(P) \cap \beta X \setminus K' ,$$

and therefore the function g is an element of $I^0(N_{\beta X \setminus K'}(P))$ but $g \notin I\,(N_{\beta X}(P))$ which is impossible. Hence $\bar{N} = N_{\beta X}(P)$ which implies $I^0(N) = P$. To complete the proof, we show that J is equal to $I(N)$. Obviously, $J \subset I(N)$. On the other hand, given $f \in I(N)$, we have $((-\underline{1} \vee f) \wedge \underline{1}) = uf$ for a unit $u \in C(X)$ (see [9], p. 21).

Since $uf \in P$ and J contains P, the function $f = \frac{1}{u} uf \in J$. Thus $J = I(N)$.

We have now proved

Theorem 2.

For a completely regular topological space X, an ideal J in $C_I(X)$ is closed if and only if it is full.

Given a convergence space X, the topological algebra $C_s(X)$ is bicontinuously isomorphic to $C_s(X')$, where X' is the associated completely regular space of X. Since a full ideal in $C_s(X)$ is closed, we state

Corollary a.

For a convergence space X, the same ideals are closed under any convergence structure on $C(X)$ finer than $C_s(X)$ and coarser than $C_I(X')$ (regarded as a convergence structure on $C(X)$).

Let X be a completely regular topological space. Point evaluation by a point in X is a continuous homomorphism on $C_c(\beta X \setminus K)$ for every $K \in \kappa$. Thus, it follows from the universal property of the inductive limit (proposition 2) that X can be regarded as a subset of $\mathcal{H}om\ C_I(X)$. In fact, we will show the following:

Corollary b.

For a completely regular topological space X , the map

$$i_X : X \longrightarrow \mathcal{H}om\ C_I(X) \ ,$$

sending each $x \in X$ to the homomorphism of point evaluation by x , is a bijection.

Proof. For any element h in $\mathcal{H}om\ C_I(X)$, the ideal $h^{-1}(0)$, the kernel of h , is closed in $C_I(X)$. Since $h^{-1}(0)$ is also a maximal ideal, theorem 2 implies that $h^{-1}(0) = I(x)$ for some point $x \in X$. It follows that

$$h(f) = f(x)$$

for every $f \in C(X)$ as desired.

Theorem 3.

Every completely regular topological space X is homeomorphic to $\mathcal{H}om_c C_I(X)$.

We need only prove that $\mathcal{H}om_c C_I(X)$ is homeomorphic to $\mathcal{H}om_c C_c(X)$ as X is c-embedded. In view of the previous corollary, we know that X is also homeomorphic to $\mathcal{H}om_s C_I(X)$. Now the topology of point-wise convergence is always coarser than the continuous convergence structure, and thus the identity maps in the following commutative diagram are continuous:

$$\begin{array}{ccc} \mathcal{H}om_c C_c(X) & \xrightarrow{\ id^* \ } & \mathcal{H}om_c C_I(X) \\ & \searrow^{id} & \downarrow id \\ & & \mathcal{H}om_s C_I(X) \end{array} ,$$

where id^* is the map induced by the identity from $C_I(X)$ onto $C_c(X)$. Since id^* is also continuous, we conclude that X is homeomorphic to $\mathcal{H}om_c C_I(X)$.

3.5. In analogy with the functor $\mathcal{C}_c$, we introduce the functor $\mathcal{C}_I$. This allows us to characterize the continuous homomorphisms between algebras $C_I(X)$ and $C_I(Y)$ (theorem 4).

First, we prove the following.

Proposition 4.

Let X and Y be completely regular topological spaces.

(i). If s is a continuous function from $\mathbb{R}$ into $\mathbb{R}$, then

$$s_*: C_I(X) \longrightarrow C_I(X) ,$$

defined by $s_*(f) = s \circ f$ for every $f \in C(X)$, is continuous.

(ii). If t is a continuous map from X into Y, then the homomorphism

$$t^*: C_I(Y) \longrightarrow C_I(X) ,$$

defined by $t^*(f) = f \circ t$ for every $f \in C(Y)$, is continuous.

To prove part (i), let $K \in \kappa$, and consider the map

$$s_{K*}: C_c(\beta X \setminus K) \longrightarrow C_c(\beta X \setminus K) ,$$

where $s_{K*}(f) = s \circ f$ for every $f \in C(\beta X \setminus K)$. Now it is easy to verify that the following diagram is commutative:

$$\begin{array}{ccccc} C_c(\beta X\setminus K) & \xrightarrow{\ i\ } & C_I(X) & \xrightarrow{\ s_*\ } & C_I(X) \\ \Big\downarrow{\scriptstyle s_{K*}} & & & \nearrow{\scriptstyle i} & \\ C_c(\beta X\setminus K) & & & & \end{array} ,$$

where i is the natural inclusion map. s_{K*} is in fact continuous (see 0.4), and thus $s_* i$ is continuous. It follows from proposition 2 that s_* is continuous.

For part (ii), let i_y denote the natural inclusion map from Y into βY. Of course $i_y \circ t$ is a continuous map from X into βY, and by the universal property of the Stone-Cech compactification, it has a continuous extension t' from βX into βY. If K is a compact subset of $\beta Y\setminus Y$, then $t'^{-1}(K)$ is a closed and hence a compact subset of βX contained in $\beta X\setminus X$. Now set

$$t'_K = t' \,|\, (\beta X\setminus t'^{-1}(K)) .$$

It follows that

$$t'_K \colon \; (\beta X\setminus t'^{-1}(K)) \longrightarrow (\beta Y\setminus K)$$

is continuous, and thus the homomorphism

$$t'^{*}_K \colon C_c(\beta Y\setminus K) \longrightarrow C_c(\beta X\setminus t'^{-1}(K)) ,$$

sending f to $f \circ t'_K$ for each $f \in C(\beta Y \setminus K)$, is continuous. It is easy to verify that the following diagram is commutative:

$$\begin{array}{ccccc}
C_c(\beta Y \setminus K) & \xrightarrow{\ i\ } & C_I(Y) & \xrightarrow{\ t^*\ } & C_I(X) \\
\Big\downarrow {\scriptstyle t'^*_K} & & & \nearrow {\scriptstyle \tilde{i}} & \\
C_c(\beta X \setminus t'^{-1}(K)) & & & &
\end{array} ,$$

where i and $\tilde{i}$ are the inclusion maps. Since $\tilde{i}$ is obviously continuous, we conclude from proposition 2 that t^* itself is continuous.

Recall that $\mathcal{L}$ is the category of convergence spaces and $\mathcal{A}$ is the category of convergence algebras. Given spaces X and Y in $\mathcal{L}$, and a continuous map $t: X \to Y$, we identify t with the continuous map $t^{**}: X' \to Y'$, where X' and Y' are the associated completely regular spaces. Now it follows easily from proposition 4 that $\mathcal{C}_I$, which sends each object X in $\mathcal{L}$ to $C_I(X')$ in $\mathcal{A}$ and each morphism t in $\mathcal{L}$ to the induced morphism t^* in $\mathcal{A}$, is a contravariant functor from $\mathcal{L}$ into $\mathcal{A}$.

Given completely regular topological spaces X and Y, we now know that every continuous map $t: X \to Y$ induces a continuous map $t^*: C_I(Y) \to C_I(X)$ which is a homomorphism. On the other hand, assume u is a

continuous homomorphism from $C_I(Y)$ into $C_I(X)$.
The map

$$u^*: \mathcal{H}om_s C_I(X) \longrightarrow \mathcal{H}om_s C_I(Y) ,$$

defined by $u^*(h) = h \circ u$ for every $h \in \mathcal{H}om\, C_I(X)$, is continuous. Since X and Y are completely regular, it follows from corollary b of theorem 2 that u^* can be identified with a continuous map from X into Y (namely, the map $i_Y^{-1} \circ u^* \circ i_X$). Now it is evident that $u^{**} = u$, and hence we have proved the following:

Theorem 4.

A homomorphism

$$u: C_I(X) \longrightarrow C_I(Y) ,$$

where X and Y are completely regular topological spaces, is continuous if and only if $u = t^*$ for some continuous map.

$$t: Y \longrightarrow X .$$

3.6. It is natural to ask if or when the convergence structure of $C_I(X)$ coincides with that of $C_c(X)$. We will show, in fact, that for a wide class of spaces, $C_c(X)$ can not even be realized as an inductive limit of topological vector spaces. On the other hand, $C_I(X)$, like $C_c(X)$, is a topological space (namely, $C_k(X)$) if and only if X is locally compact.

A convergence vector space V is said to be a pseudo-topological union (pseudotopologische Vereinigung, [10]) if it is the inductive limit of topological vector spaces.

The following result is due to H.H. Keller (see [11]).

Proposition 5.

A pseudo-topological union, $V = \operatorname{Ind}_{a \in \mathcal{A}} V_a$, *is a topological vector space if and only if there exists an* $a' \in \mathcal{A}$ *such that* $V_a = V_{a'}$ *(as topological vector spaces) for every* $a \geq a'$.

For completeness, we include the following proof. Assume V is a topological vector space, and let $\mathcal{O}$ denote the neighborhood filter of zero in V. By definition, $\mathcal{O}$ has a basis in $V_{a'}$ for some $a' \in \mathcal{A}$. Because each neighborhood of zero is absorbent, it follows that $V_{a'} = V$ as vector spaces. Further, V_a for $a \geq a'$ can not be strictly coarser than $V_{a'}$, for then

$\mathfrak{O}$ would not be the neighborhood filter of zero in V. The sufficiency is clear.

Theorem 5.

For a completely regular topological space X, the following three statements are equivalent:

(i). $C_I(X)$ is a topological space.

(ii). $C_I(X)$ carries the compact-open topology (and is therefore bicontinuously isomorphic to $C_c(X)$).

(iii). X is locally compact.

Proof. It is easy to verify that $\beta X \backslash X$ is a compact subset of βX if and only if X is locally compact (see [9], p. 90). Now it follows from proposition 5 that $C_I(X)$ is a topological space if and only if X is locally compact. It is evident that if X is locally compact, then $C_I(X)$ is equal to $C_k(X)$. In this case, $C_c(X)$ also carries the compact-open topology since $C_c(X)$ is always coarser than $C_I(X)$ and finer than $C_k(X)$.

Theorem 6.

If X is a completely regular topological space with the property that there exists a point $p \in X$ such that the neighborhood filter of p has a countable base and p has no compact neighborhood, then $C_c(X)$ can not be a pseudo-topological union.

<u>Proof.</u> We claim that a pseudo-topological union $V = \text{Ind}_{a \in \mathcal{A}} V_a$ has the property that if a filter $\phi \to \underline{0}$ (the zero element in V), then there exists a coarser filter ϕ' with the property that ϕ' converges to $\underline{0}$ and

$$\lambda\phi' = \phi'$$

for every $\lambda \in \mathbb{R}\setminus\{0\}$. By $\lambda\phi'$ we simply mean $\{\lambda A : A \in \phi'\}$. Indeed, if $\phi \to \underline{0}$ in V, then $\phi \geq i(\mathcal{O}_a)$, where $\mathcal{O}_a$ is the neighborhood filter of $\underline{0}$ in V_a for some $a \in \mathcal{A}$, and i is the inclusion map from V_a into V. Since V_a is a topological vector space, $\lambda\mathcal{O}_a = \mathcal{O}_a$, and hence $\lambda i(\mathcal{O}_a) = i(\mathcal{O}_a)$ for each $\lambda \neq 0$.

Our proof will consist of finding a filter Θ convergent to $\underline{0}$ in $C_c(X)$ that does not satisfy the above condition. We first construct inductively the following system of decreasing neighborhoods of p. Assume that p has no compact neighborhood and

$$\{Q_m\}_{m \in \mathbb{N}}$$

is a countable collection of open sets that form a base for the neighborhood filter at p. Set $N_1 = X$, and let

$$\{O_{1\alpha}\}$$

be an open covering of N_i with no finite subcovering. We define

$$U_1 = O_{1\alpha_p} \cap Q_1 ,$$

where $p \in O_{1\alpha_p} \in \{O_{1\alpha}\}$ and $Q_1 \in \{Q_m\}_{m \in \mathbb{N}}$. Asume $\{N_i , U_i\}$ have been constructed for $i \leq j - 1$. Choose N_j to be a closed neighborhood of p contained in U_{j-1}, and let

$$\{O_{j\alpha}\}$$

be a covering of N_j by open sets in X that admits no finite subcovering. We pick U_j to be an open neighborhood of p contained in

$$O_{j\alpha_p} \cap Q_j \cap N_j$$

where $p \in O_{j\alpha_p} \in \{O_{j\alpha}\}$ and $Q_j \in \{Q_m\}_{m \in \mathbb{N}}$. With this system of respectively closed and open neighborhoods of p,

$$N_1 \supset U_1 \supset N_2 \supset U_2 \supset \ldots ,$$

we construct our filter Θ. Let

$$T_n = \{f \in C(X): \ f(N_n) \subset \left[-\frac{1}{n}, \frac{1}{n}\right]\} ,$$

and let

$$T_x = \{f \in C(X): f(W_x) = \{0\} \}$$

for each $x \in X \setminus \{p\}$, where we choose W_x as follows: Since $x \neq p$, there exists an $r \in \mathbb{N}$ such that $x \in N_r \setminus N_{r+1}$. Let W_x be a closed neighborhood of x so that

$$W_x \subset (\bigcap_{i=1}^{r} O_{i\alpha_x} \cap N_{r+1}^c)$$

($N_{r+1}^c = X \setminus N_{r+1}$), where $x \in O_{i\alpha_x} \in \{O_{i\alpha}\}$ for each $i \in \{1, 2, \ldots, r\}$. It is easy to verify that the collection

$$\mathcal{J} = \{T_n\}_{n \in \mathbb{N}} \cup \{T_x\}_{x \in X \setminus \{p\}}$$

generates a filter Θ that converges to $\underline{0}$ in $C_c(X)$. Now, we show that there exists no coarser filter Θ' convergent to $\underline{0}$ with the property that $\lambda\Theta' = \Theta'$ for each $\lambda \neq 0$. Assume to the contrary, that such a filter Θ' exists. Since $\Theta' \to \underline{0}$, there exists a neighborhood V of p and an element $F \in \Theta'$ such that

$$w(F' \times V) \subset [-1\ ,\ 1]\ .$$

The fact that V is a neighborhood of p implies that $V \supset N_k$ for some $k \in \mathbb{N}$, and thus

$$w(\frac{1}{2k}\ F' \times N_k) \subset \left[\frac{-1}{2k}\ ,\ \frac{1}{2k}\right]\ .$$

By assumption, $\frac{1}{2k}\ F' \in \Theta'$ and $\Theta' \leq \Theta$, which means there exists an element $F \in \Theta$ such that $F \subset \frac{1}{2k}\ F'$. Without loss of generality, we can assume F is the intersection of a finite number of sets in $\mathcal{J}$, and therefore we can write

$$\frac{1}{2k}\ F' \supset F = \bigcap_{n \in \mathcal{N}} T_n \cap \bigcap_{x \in \mathcal{X}} T_x$$

for $\mathcal{N}$ a finite subset of $\mathbb{N}$ and $\mathcal{X}$ a finite subset of $X \backslash \{p\}$. Now, we claim that

$$N_k \not\subset \bigcup_{x \in \mathcal{X}} W_x \cup N_{k+1}\ .$$

Our construction guarantees that for a fixed W_x , either $W_x \subset N_k^c$ or $W_x \subset O_{k\alpha}$, where $O_{k\alpha}$ is an element of the open covering $\{O_{k\alpha}\}$. Furthermore, N_{k+1} is contained in $O_{k\alpha_p}$, where $O_{k\alpha_p} \in \{O_{k\alpha}\}$. Since the open covering $\{O_{k\alpha}\}$ of N_k has no finite subcovering, the claim is true. In fact, for a point

$$q \in N_k \setminus (N_{k+1} \cup \bigcup_{x \in \mathcal{X}} W_x) \quad ,$$

because X is completely regular, we can pick a function $f \in C(X)$ such that

$$\|f\| \leq \frac{1}{k} \quad , \quad f(q) = \frac{1}{k} \quad , \quad \text{and} \quad f(N_{k+1} \cup \bigcup_{x \in \mathcal{X}} W_x) = \{0\} \; .$$

It follows that f is an element of F. But $f \notin \frac{1}{2k} F'$, as $f(q) = \frac{1}{k}$, and this contradiction establishes the theorem.

Remark. The proof of theorem 6 reveals the following property of the continuous convergence structure: Given a filter Θ convergent to $\underline{0}$ in $C_c(X)$, there does not, in general, exist a coarser filter Θ' convergent to $\underline{0}$ such that $\lambda\Theta' = \Theta'$ for every $\lambda \in \mathbb{R}\setminus\{0\}$.

The following is an immediate corollary of theorem 6.

Corollary.

For a first countable, completely regular topological space X, the convergence algebra $C_c(X)$ is a pseudo-topological union if and only if X is locally compact.

3.7. In this section, we will examine the locally convex inductive limit of the family $\{C_c(\beta X\setminus K)\}_{K\in\mathcal{K}}$.

Let $\{L_a\}_{a\in\mathcal{A}}$ be a family of locally convex topological vector spaces satisfying the conditions in section 3.1. By the <u>locally convex inductive limit</u> of $\{L_a\}_{a\in\mathcal{A}}$, we mean the finest locally convex vector space topology making all the inclusion maps continuous (see [14], p. 78). We denote this by

$$\varinjlim L_a .$$

Given a convergence vector space V , its <u>associated locally convex topology</u> is the finest locally convex vector space topology on the linear space V which is coarser than the given convergence structure. Such a topology indeed exists, for it is the topology determined by all the continuous seminorms on V .

In view of proposition 2, it is easy to verify that the associated locally convex topology of $C_I(X)$ is just $\varinjlim C_c(\beta X\setminus K)$.

<u>Theorem</u> 7.

<u>For a completely regular topological space X , the locally convex inductive limit of the family $\{C_c(\beta X\setminus K)\}_{K\in\mathcal{K}}$ is the compact-open topology on $C(X)$.</u>

Proof. Let U be an arbitrary neighborhood of $\underline{0}$ in $\underset{\rightarrow}{\operatorname{Lim}}\, C_c(\beta X \setminus K)$. Without loss of generality, we can assume U is closed and convex. Since all the inclusion maps into $C_k(X)$ are continuous, it suffices to show that U is a neighborhood of $\underline{0}$ in $C_k(X)$. Clearly $U \cap C^0(X)$ is a neighborhood of $\underline{0}$ in $C^0_n(X)$, as $C_c(\beta X)$ is bicontinuously isomorphic to $C^0_n(X)$. Thus, there exists a $\delta > 0$ with the property that $f \in U$ whenever $\|f\| < \delta$ and $f \in C(X)$.

Here, we interrupt our proof to introduce the concept of a support set as developed in [13]. A support set for U is a compact subset $G \subset \beta X$ such that if $f \in C(X)$ and $\overline{f}$ vanishes on G, then $f \in U$ (here again, $\overline{f}$ is the unique extension of f to a continuous map from βX into $\tilde{R}$). Trivially, βX itself is a support set for U. Given any support set G for U, we claim that if $f \in C(X)$ and $\|\overline{f}\|_G \leq \delta/2$, where $\|\overline{f}\|_G = \sup_{x \in G} |\overline{f}(x)|$, then $f \in U$. Indeed, let

$$g = (f \vee \tfrac{\delta}{2}\, \underline{1}) + (f \wedge \tfrac{-\delta}{2}\, \underline{1}) .$$

Since by assumption $\|\overline{f}\|_G \leq \delta/2$, the function $2\overline{g}$ vanishes on G, and thus $2g \in U$. Further, $\|2(f - g)\| \leq \delta$, which implies $2(f - g) \in U$. Hence

$$f = \tfrac{1}{2}[2(f - g) + 2g] \in U ,$$

as U is convex. We will show that G is a support set for U if and only if G has the property that if $f \in C(X)$ and $\bar{f}$ vanishes on some neighborhood of G in βX, then f is in U. The necessity is obvious. For the sufficiency, assume $f \in C(X)$ and $\bar{f}(G) = \{0\}$. Again, define $g = (f \vee \frac{\delta}{2} \underline{1}) + (f \wedge \frac{-\delta}{2} \underline{1})$. Since $\bar{f}$ vanishes on G, the set

$$N = \bar{f}^{-1}(\frac{-\delta}{2}, \frac{\delta}{2})$$

is an open neighborhood of G such that $\bar{g}$ vanishes on N. By assumption, $2g \in U$, and as above, $2(f - g) \in U$. By convexity, f is an element of U. Hence G is indeed a support set. The collection of all support sets for U is, in fact, closed under finite intersections. It suffices to show that the intersection of two support sets, G_1 and G_2, is again a support set. Let W be an open neighborhood of $G = G_1 \cap G_2$, and f a function in $C(X)$ whose extension $\bar{f}$ vanishes on W. Since G_1 and $G_2 \setminus W$ are disjoint closed sets in βX, we can choose open neighborhoods W_1 and W_2 of G_1 and $G_2 \setminus W$ respectively with the property that there exists a function $k \in C^0(X)$ so that

$$\bar{k}(W_1) = \{1\} \quad \text{and} \quad \bar{k}(W_2) = \{0\}.$$

It follows that

$$2\overline{fk}(W \cup W_2) = \{0\} \quad \text{and} \quad 2\overline{f(\underline{1} - k)}(W_1) = \{0\} .$$

Since G_1 and G_2 are support sets, $2fk$ and $2f(\underline{1} - k)$ are both elements of U , and thus

$$f = \frac{1}{2}\{2fk + 2f(\underline{1} - k)\} \in U ,$$

which means $G = G_1 \cap G_2$ is a support set for U .
We are now prepared to show that there exists a unique smallest support set for U , which we denote by G_U .
We can write

$$G_U = \bigcap_{G \in \Gamma} G ,$$

where Γ is the collection of all support sets for U .
To verify that G_U is actually a support set, let f be an element of $C(X)$ such that $\overline{f}$ vanishes on some open neighborhood W of G_U . Since

$$W^c \cap \bigcap_{G \in \Gamma} G = \emptyset$$

and βX is compact,

$$W^c \cap \bigcap_{G \in \Gamma'} G = \emptyset ,$$

where Γ' is a finite subset of Γ . Thus W contains the support set $\bigcap_{G \in \Gamma'} G$ which implies $f \in U$.

Returning to our proof, we need only show that this smallest support set G_U is contained in X . For then,

$$\{f \in C(X) : \|f\|_{G_U} \leq \frac{\delta}{2}\}$$

would be a neighborhood of $\underline{0}$ in $C_k(X)$ contained in U . To this end, let p be an arbitrary point in $\beta X \setminus X$. Now

$$U \cap C(\beta X \setminus p)$$

is a neighborhood of $\underline{0}$ in $C_c(\beta X \setminus p)$. Therefore, there exists a compact subset $G' \subset \beta X \setminus p$ with the property that if $f \in C(\beta X \setminus p)$ and $\bar{f}$ vanishes on G', then $f \in U$. Consider any function $g \in C(X)$ such that $\bar{g}(G') = \{0\}$. The Fréchet filter Θ determined by the sequence

$$((n\underline{1} \wedge g) \vee -n\underline{1})_{n \in \mathbb{N}}$$

converges to g in $C_c(\beta X \setminus \bar{g}^{-1}(\infty))$, and hence Θ converges to g in $\varinjlim C_c(\beta X \setminus K)$. Since Θ has a trace on U and U is closed in $\varinjlim C_c(\beta X \setminus K)$, we conclude that $g \in U$. Thus, we have a support set G' for U disjoint from p . Because G_U is the intersection of all support sets for U , the point p is not in G_U which completes the proof.

Since $C_c(X)$ is coarser than $C_I(X)$ and finer than $C_k(X)$, we have an alternative proof for the following known result (to appear in the thesis of H.P. Butzmann, Universität Mannheim) without using integral representations.

Corollary a.

If X is a completely regular topological space, then $C_k(X)$ is the associated locally convex topology of $C_c(X)$.

For a convergence vector space V , let $L(V)$ denote the dual space of V (i.e., the vector space of all continuous linear functionals on V). It has been shown (to appear in the thesis of H.P. Butzmann) that $L(C_k(X)) = L(C_c(X))$ for any c-embedded convergence space X . In the case of a completely regular topological space X , we can extend this result to the finer convegence structure $C_I(X)$. Specifically, as an immediate corollary of theorem 7, we have

Corollary b.

If X is a completely regular topological space, then

$$L(C_I(X)) = L(C_c(X)) = L(C_k(X)) .$$

Remark. Theorem 7 tells us that $C_k(X)$, for any completely regular topological space X , can be realized as the locally convex inductive limit of a family, each of whose members is a function algebra on a locally compact topological space (with the compact-open topology).

We will consider the locally convex inductive limit of a subfamily of $\{C_c(\beta X\backslash K)\}_{K\in\kappa}$. Define

$$\mathcal{Z} = \{Z(f): f\in C(\beta X) \quad \text{and} \quad Z(f) \subset \beta X\backslash X\}$$

($Z(f) = f^{-1}(0)$). It is clear that $\mathcal{Z}$ is a subset of κ , and further, the family $\{C_c(\beta X\backslash Z)\}_{Z\in\mathcal{Z}}$ satisfies the conditions in section 3.1. Recall that υX denotes the Hewitt realcompactification.

Theorem 8.

For a completely regular topological space X , the locally convex inductive limit of the family $\{C_c(\beta X\backslash Z)\}_{Z\in\mathcal{Z}}$ is bicontinuously isomorphic to $C_k(\upsilon X)$.

We will first show that

$$\bigcup_{Z \in \mathcal{Z}} C(\beta X \setminus Z) = C(X) .$$

Clearly, it suffices to demonstrate that every $f \in C(X)$ is an element of $C(\beta X \setminus Z)$ for some $Z \in \mathcal{Z}$. Assume f is in $C(X)$, and set

$$g = |f| \vee 1 .$$

Now $\overline{g}^{-1}(\infty) = \overline{f}^{-1}(\infty)$, and furthermore, g has a bounded inverse (i.e., $1/g \in C^0(X)$). It follows that $Z(\overline{\frac{1}{g}}) = \overline{f}^{-1}(\infty)$, and hence $f \in C(\beta X \setminus Z(\overline{\frac{1}{g}}))$, where $Z(\overline{\frac{1}{g}}) \in \mathcal{Z}$. Now it is easy to verify that $Z \cap \upsilon X = \emptyset$ for every $Z \in \mathcal{Z}$ (see [9], p. 118). Thus, given $Z \in \mathcal{Z}$, the inclusion map from υX into $\beta X \setminus Z$ induces a continuous monomorphism from $C_c(\beta X \setminus Z)$ into $C_c(\upsilon X)$. Because of the canonical isomorphism between $C(X)$ and $C(\upsilon X)$, we can regard $\varinjlim C_c(\beta X \setminus Z)$ as a convergence structure on $C(\upsilon X)$. Since $C_k(\upsilon X)$ is coarser than $C_c(\upsilon X)$, the universal property of the locally convex inductive limit (see [14], p. 79) implies that the identity,

$$id: \varinjlim C_c(\beta X \setminus Z) \longrightarrow C_k(\upsilon X) ,$$

is continuous. Conversely, assume U is a neighborhood of $\underline{0}$ in $\underset{\rightarrow}{\text{Lim}}\ C_c(\beta X\setminus Z)$. With no loss of generality, we can assume U is closed and convex. As in the proof of theorem 7, the intersection of all support sets for U, which we denote by G_U, is again a support set for U. Further, there exists a $\delta > 0$ such that $f \in U$ whenever $f \in C(X)$ with $\|f\|_{G_U} \leq \delta$. It only remains to prove that G_U is contained in υX. For an arbitrary $t \in \beta X\setminus \upsilon X$, there exists a function $k \in C(\beta X)$ such that $k(t) = 0$ and $Z(k) \cap \upsilon X = \emptyset$ (see [9], p. 104). Since

$$U \cap C(\beta X\setminus Z(k))$$

is a neighborhood of $\underline{0}$ in $C_c(\beta X\setminus Z(k))$, there exists a compact subset $G' \subset \beta X\setminus Z(k)$ with the property that if $f \in C(\beta X\setminus Z(k))$ and $\overline{f}(G') = \{0\}$, then $f \in U$. Now, as in the proof of theorem 7, one can show that if g is any function in $C(X)$ and $\overline{g}$ vanishes on G' then $g \in U$. Therefore G' is a support set for U disjoint from t, which completes the proof.

BIBLIOGRAPHY

1. Binz, E. and H.H. Keller: Funktionenräume in der Kategorie der Limesräume, Ann. Acad. Scie. Fenn. A, I, 383, 1-21 (1966)
2. Binz, E.: Bemerkungen zu limitierten Funktionenalgebren, Math. Ann. 175, 169-184 (1968)
3. —— : Zu den Beziehungen zwischen c-einbettbaren Limesräumen und ihren limitierten Funktionenalgebren, Math. Ann. 181, 45-52 (1969)
4. —— : On Closed Ideals in Convergence Function Algebras, Math. Ann. 182, 145-153 (1969)
5. —— : Notes on a Characterization of Function Algebras, Math. Ann. 186, 314-326 (1970)
6. Bourbaki, N.: General Topology Parts I & II, Addison & Wesley (1966)
7. Dugundji, J.: Topology, Allyn & Bacon, Boston (1966)
8. Fischer, H.R.: Limesräume, Math. Ann. 137, 269-303 (1959)
9. Gillman, L. and M. Jerison: Rings of Continuous Functions, Van Nostrand, Princeton (1960)
10. Jarchow, H.: Marinescu-Räume, Comm. Math. Helv. 44, 138-163 (1969)
11. Keller, H.H.: Räume stetiger multilinearer Abbildungen als Limesräume, Math. Ann. 159, 259-270 (1965)

12. Lang, S.: Algebra, Addison-Wesley (1969)

13. Nachbin, L.: Topological vector spaces of continuous functions, Proc. N.A.S. 40, 471-474 (1954)

14. Robertson, A.P. and W.J. Robertson: Topological Vector Spaces, Cambridge Univ. Press (1964)