A functional analytic description of normal spaces

by

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Throughout the paper X will denote a completely regular (Hausdorff) topological space and $C(X)$ the R-algebra of all real-valued continuous functions on X. When this algebra carries the continuous convergence structure [1], we write $C_c(X)$. We note that $C_c(X)$ is a complete [5] convergence R-algebra [1].

Our description of normality reads as follows: A completely regular topological space X is normal if and only if $C_c(X)/J$ (endowed with the obvious quotient structure, see section 1) is complete for every closed ideal $J \subset C_c(X)$.

1. Residue class algebras

For a closed non-empty subset $A \subset X$, let $I(A)$ denote the ideal in $C(X)$ consisting of all functions in $C(X)$ vanishing on A. Since the kernel of the restriction map

$$r: C(X) \longrightarrow C(A),$$

sending each $f \in C(X)$ into its restriction $f|A$, is $I(A)$, we have the following commutative diagram of R-algebra homomorphisms:

$$\text{(I)} \qquad \begin{array}{ccc} C(X) & \xrightarrow{\;r\;} & C(A) \\ \downarrow{\scriptstyle \pi} & \nearrow{\scriptstyle \bar{r}} & \\ C(X)/I(A) & & \end{array},$$

where π is the natural projection map and $\bar{r}$ the unique map factoring r. With $C_c(X)/I(A)$ we denote $C(X)/I(A)$ endowed with the natural quotient structure (in the category of convergence spaces) of $C_c(X)$ with respect to π. This means that a filter converges to zero in $C_c(X)/I(A)$ if and only if it is finer than the image (under π) of a filter converging to zero in $C_c(X)$. Endowing $C(X)$ and $C(A)$ with the continuous convergence structure and $C(X)/I(A)$ with this quotient structure, all the maps in diagram (I) are continuous.

Proposition 1. *The* R-*algebra monomorphism* $\bar{r}$ *is a homeomorphism from* $C_c(X)/I(A)$ *onto a subspace of* $C_c(A)$.

Proof. All we have to show is that a filter $\bar{\Theta}$ on $C(X)/I(A)$ for which $\bar{r}(\bar{\Theta})$ converges to zero in $C_c(A)$ also converges to zero in $C_c(X)/I(A)$. That is, we must construct a filter Θ on $C_c(X)$ converging to zero with the property that $\pi(\Theta)$ is coarser than $\bar{\Theta}$.

Let $\bar{\Theta}$ be a filter on $C(X)/I(A)$ with $\bar{r}(\bar{\Theta})$ convergent to zero in $C_c(A)$. Hence for each $p \in A$ and each positive real number ε, there is a neighborhood $U_{p,\varepsilon}$ of p in X and an $F'_{p,\varepsilon} \in \bar{r}(\bar{\Theta})$ contained in $r(C(X))$ with

$$|f'(q)| \leq \varepsilon$$

for all $f' \in F'_{p,\varepsilon}$ and all $q \in U_{p,\varepsilon} \cap A$. Without loss of generality, we can assume that each $U_{p,\varepsilon}$ is a cozero-set in X. To facilitate the construction of our filter, we choose inside of each $U_{p,\varepsilon}$ a zero-set neighborhood $\tilde{U}_{p,\varepsilon}$ in X of p. Furthermore to each y in $X \setminus A$ there exists, disjoint from A , a cozero-set neighborhood V_y of y in X inside of which we fix a zero-set neighborhood $\tilde{V}_y$ of y in X . We intend to show that all the sets of the form

$$(*) \quad F_{p,y,\varepsilon} = \{f \in C(X):\ f|A \in F'_{p,\varepsilon}\ ,\ f(\tilde{U}_{p,\varepsilon}) \subset [-2\varepsilon, 2\varepsilon]\ ,\ \text{and}\ f(\tilde{V}_y) = \{0\}\ \}$$

for $p \in A$, $y \in X \setminus A$, and ε a real number greater than 0 , generate the desired filter. We first demonstrate that

$$(**) \qquad r\Big(\bigcap_{i=1}^{n} F_{p_i, y_i, \varepsilon_i}\Big) \supset \bigcap_{i=1}^{n} F'_{p_i, \varepsilon_i} \ ,$$

where p_i , y_i , and ε_i are as above. To this end, let $f' \in \bigcap_{i=1}^{n} F'_{p_i, \varepsilon_i}$ and j a fixed integer between 1 and n . We now choose an element $f \in C(X)$ for which $r(f) = f'$ and associate to this function the sets

$$P_j = \{q \in \tilde{U}_{p_j, \varepsilon_j} :\ |f(q)| \geq 2\varepsilon_j\}$$

and

$$Q_j = \{q \in X:\ |f(q)| \leq \varepsilon_j\} \cup (X \setminus U_{p_j, \varepsilon_j}) \ .$$

It is clear that $Q_j \supset A$, and furthermore, P_j and Q_j are disjoint zero-sets in X . Hence there is a function $h_j \in C(X)$ separating P_j and Q_j , that is,

$$h_j(q) = 0 \quad \text{for all} \quad q \in P_j$$

and

$$h_j(q) = 1 \quad \text{for all} \quad q \in Q_j \ .$$

Without loss of generality, we may assume that $h_j(X) \subset [-1,1]$.

Similarly, we pick a function $k_j \in C(X)$ with the property that

$$k_j(q) = 0 \quad \text{for all} \quad q \in \tilde{V}_{y_j} ,$$

$$k_j(q) = 1 \quad \text{for all} \quad q \in X \setminus V_{y_j} ,$$

and $k_j(X) \subset [-1,1]$. The function $g = f \cdot h_1 \cdot h_2 \cdot \dots \cdot h_n \cdot k_1 \cdot \dots \cdot k_n$ is an element of $\bigcap_{i=1}^{n} F_{p_i, y_i, \varepsilon_i}$ and extends f' . Now the filter Θ generated on $C(X)$ by all the sets of the form (*) obviously converges to zero in $C_c(X)$. Because (**) is satisfied, $\pi(\Theta)$ is coarser than $\overline{\Theta}$, and thus the proof is complete.

Next, we will investigate the universal representation [2] of $C_c(X)/I(A)$, i.e., the R-algebra $C_c(Hom_c C_c(X)/I(A))$ and the R-algebra homomorphism

$$d: C_c(X)/I(A) \longrightarrow C_c(Hom_c C_c(X)/I(A)) ,$$

where $Hom_c C_c(X)/I(A)$ denotes the space of all continuous R-algebra homomorphisms from $C_c(X)/I(A)$ onto R together with the continuous convergence structure. The map d sends each element $\overline{f} \in C_c(X)/I(A)$ to the function defined by $d(\overline{f})(h) = h(\overline{f})$ for each $h \in Hom_c C_c(X)/I(A)$.

We intend to establish a relationship between $Hom_c C_c(X)/I(A)$ and A . The homomorphism π induces a continuous map

$$\pi^*: Hom_c C_c(X)/I(A) \longrightarrow Hom_c C_c(X) ,$$

sending each $h \in Hom_c C_c(X)/I(A)$ to $h \circ \pi$. By $Hom_c C_c(X)$ we mean the collection of all continuous R-algebra homomorphisms from $C_c(X)$ onto R together with the continuous convergence structure. As pointed out in [3] the map

$$i_X : X \longrightarrow Hom_c C_c(X) ,$$

defined by the relation $i_X(p)(f) = f(p)$ for all $f \in C(X)$ and all $p \in X$, is a homeomorphism. Hence the map $i_X^{-1} \circ \pi^*$ maps $Hom_c C_c(X)/I(A)$ continuously into X. In fact, the range of this map is in A since $(i_X^{-1} \circ \pi^*)(h)$ for any $h \in Hom_c C_c(X)/I(A)$ is sent to zero by all the functions in I(A) and A is a closed subset of a completely regular space. Next, we show that $i_X^{-1} \circ \pi^*$ is actually a bijection onto A. Because π is surjective, the map $i_X^{-1} \circ \pi^*$ is clearly injective. For the surjectivity, choose a point $p \in A$. The homomorphism $i_X(p): C_c(X) \longrightarrow R$ annihilates all the functions in I(A), and therefore can be factored to a continuous homomorphism h on $C_c(X)/I(A)$. It is clear that $(i_X^{-1} \circ \pi^*)(h) = p$.

<u>Proposition</u> 2. *The map*

$$i_X^{-1} \circ \pi^* : Hom_c C_c(X)/I(A) \longrightarrow A$$

is a homeomorphism.

<u>Proof</u>. Since $i_X^{-1} \circ \pi^*$ is a continuous bijection, it remains to verify that $(i_X^{-1} \circ \pi^*)^{-1}$ is also continuous. We have the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{i_A} & Hom_c C_c(A) \\ & {\scriptstyle (i_X^{-1}\circ\pi^*)^{-1}} \searrow \quad \swarrow {\scriptstyle \bar{r}^*} & \\ & Hom_c C_c(X)/I(A) & \end{array} ,$$

where $\bar{r}^*$ sends each $h \in Hom_c C_c(A)$ to $h \circ \bar{r}$. Since both i_A and $\bar{r}^*$ are continuous, the proposition is established.

2. <u>Closed C-embedded subsets</u>

A closed non-empty subset A of a space X is said to be C-*embedded* if every continuous real-valued function defined on A has a continuous extension to X , that is to say

$$r: C(X) \longrightarrow C(A)$$

is surjective. For example, every compact subset of X is C-embedded.

<u>Theorem</u> 1. *A closed non-empty subset* A *of a completely regular topological space* X *is* C-*embedded if and only if* $C_c(X)/I(A)$ *is complete.*

<u>Proof</u>. If A is a C-embedded subset of X , then the map $\bar{r}$ is a homeomorphism (see proposition 1) and hence $C_c(X)/I(A)$ is complete. Conversely, assume that $C_c(X)/I(A)$ is complete. Proposition 1 implies that $\bar{r}(C_c(X)/I(A))$ is a closed subalgebra of $C_c(A)$. By a type of Stone-Weierstrass theorem proved in [5], which states that a closed subalgebra

of $C_c(Y)$ that contains the constant functions and determines the topology (see [6], p. 39) of the completely regular topological space Y is all of $C(Y)$, we conclude that the map $\bar{r}$ is surjective. Thus A is C-embedded.

Proposition 3. *A closed non-empty subset* A *of a completely regular topological space* X *is compact if and only if* $C_c(X)/I(A)$ *is normable.*

Proof. For A compact, $C_c(A)$ is a normed algebra under the supremum norm. It follows from proposition 1 that $C_c(X)/I(A)$ is normable. On the other hand, if $C_c(X)/I(A)$ is normable, then $Hom_c C_c(X)/I(A)$ is a compact topological space (see [7]) and hence A is compact by proposition 2.

Corollary. *Let* A *be a closed non-empty subset of a completely regular topological space* X. *If* $C_c(X)/I(A)$ *is normable, then it is complete.*

3. Normal spaces

A completely regular topological space is normal if and only if every non-empty closed subset is C-embedded (see [6], p. 48). In view of theorem 1, we know that the space X is normal if and only if $C_c(X)/I(A)$ is complete for every non-empty closed subset $A \subset X$. Since every closed ideal in $C_c(X)$ is of the form $I(A)$ for a non-empty closed subset A of X (see [4]), we state

Theorem 2. *A completely regular topological space* X *is normal if and only if* $C_c(X)/J$ *is complete for every closed ideal* $J \subset C_c(X)$.

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