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On Gallai's Decomposition Theorem for Graphs

von

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A note on Gallai's decomposition theorem for graphs

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§1. Introduction

T. Gallai (3) proved a decomposition theorem for finite graphs in terms of their quasimaximal strongly autonomous vertex sets. It was reviewed in D. Kelly's review paper on "Comparability graphs" (5). Apart from reviewing existing results on comparability graphs D. Kelly generalized these results, as far as possible, to the case of infinite graphs. Gallai's theorem is not among them. The theorem is, however, true for a large class of infinite graphs - the so-called "Non Limit - graphs", shortly "NL-graphs". Although partial aspects of the result can be found in W.H. Cunningham & J. Edmonds (2) and L.N. Shevrin & N.D. Filippov (6) (both papers make no reference to Gallai's original paper) there is no theorem in the literature that displays all features of Gallai's useful theorem in the infinite case:as, e.g. the role of strongly autonomous subsets and their full interaction with Gallai's "edge-classes". Therefore we formulate it (Theorem 1) and give a short direct proof.It is based both on the"edge-vertex-lemma" (Lemma 1) that Gallai derived (I quote) as a"remarkable consequence" from his finitary decomposition theorem and on a useful lemma of D. Kelly (Lemma 2). In this way a new proof arises even for the finitary case. As indicated above, parts of theorem 1 can be found in L.N. Shevrin & N.D. Filippov (6) (see §3, Proposition 3 and §4, Lemma 16) in a different language. They use Zorn's Lemma, we don't. Theorem 2 appears as a natural and new supplement of Gallai's theorem: In order to formulate it, we define a "Gallai-decomposition" as a maximal decomposition into autonomous subgraphs such that the "external edge-classes" constitute a single edge-class. Theorem 2 states that exactly the NL-graphs have such a decomposition.

Two examples illustrate how the infinite version of Gallai's theorem (i.e. theorems 1 and 2) can be usefully applied. The note ends with a remark on graphs G with connected G and $\overline{G}$ that do not contain P_4. This remark analyses an example of D. Kelly in (5). (The referee pointed out to me that R. Möhring announced a related result in Methods of Operations Research 45 (1983), 287-291.)

As for the concepts "autonomous", "strongly autonomous", "in general position", "edge-class" etc... see D. Kelly (5).

If G is a graph, V(G) denotes the set of vertices of G and E(G) the set of edges of G. $\overline{G}$ denotes the complemented graph of G. "NL-graphs" are graphs that contain at least one quasimaximal strongly autonomous set Q of vertices (i.e. Q is maximal among all proper strongly autonomous subsets of V(G)). For $A \subseteq V(G)$, $G[A]$ denotes the full subgraph of G with vertex set A. If π is a partition of V(G) into autonomous subsets then G^{π} denotes the corresponding quotient graph. For $a,b \in V(G)$, "$a \sim b$" indicates the existence of the edge $\overline{ab}$. For $A,B \subseteq V(G)$, "$A \sim B$" means "$a \sim b$ for all $a \in A$ and $b \in B$" and "$A \not\sim B$" means "$a \not\sim b$ for all $a \in A$ and $b \in B$".

§2. The decomposition theorem and applications

Theorem 1 (Gallai's decomposition theorem):

Let G be an arbitrary non-trivial NL-graph:

(I) V(G) is the disjoint union of all its quasimaximal strongly autonomous subsets A_i, $i \in I$.

($\pi(G) := \{A_i; i \in I\}$ is called the "canonical decomposition of G". An edge class D is called "internal" if $D \subseteq E(G[A_i])$ for some i, otherwise "external".)

(II) $\pi(G)$ is obtained as follows:

(1) If G is not connected, then $\pi(G)$ consists of the com= ponents of G. There is no external edge class.

(2) If $\overline{G}$ is not connected, then $\pi(G)$ consists of the com= ponents of $\overline{G}$. For any fixed $i,j \in I$, $i \neq j$, we have $A_i \sim A_j$, and the set E_{ij} of all A_iA_j -edges constitutes an edge class. The classes E_{ij} are exactly all external edge classes.

(3) If G and $\overline{G}$ are connected, then $\pi(G)$ is the unique largest partition of V(G) into proper subsets which satisfies the following two properties (a), (b):

(a) $A_i \sim A_j$ or $A_i \not\sim A_j$ for all $i,j \in I$, $i \neq j$.

(b) The set $C := \bigcup(E_{ij}; i,j \in I, i \neq j)$ is a single edge class (in this case $V(C) = V(G)$).

It is possible to give a short proof of the next lemma based on the "forbidden pattern lemma" (Arditti & Jung (1) and Gilmore & Hoffman (3), corollary 1 to Lemma 3). We leave it as an exercise.

Lemma 1 (Edge-Vertex-Lemma of Gallai): If G is a graph and E_1, E_2 are edge-classes, then $V(E_1)=V(E_2)$ implies $E_1=E_2$.

We include a proof of the next lemma, since the original proof in D. Kelly (5) contains a minor mistake.

Lemma 2 (D. Kelly (5)): Let $A \subseteq V(G)$ be a non-trivial, independent and autonomous vertex set of the graph G. There exists a strongly autonomous vertex set B with (i) $A \subseteq B$, (ii) $(B \setminus A) \not\sim A$.

proof of lemma 2:

If A is strongly autonomous, we take $B = A$. Otherwise let $\mathbb{F} = \{C_i ; i \in I\}$ be the set of all autonomous subsets of $V(G)$ that are in general position with A and form $H := \bigcup (C_i;\ i \in I)$.
$B := H \cup A$ is autonomous and $A \subsetneq B$. We have $(B \setminus A) \not\sim A$, since $(C_i \setminus A) \not\sim A$ is easily seen to hold true for every $i \in I$. We will show that B is strongly autonomous and are done:
Assume that there is some autonomous X in general position with B. If $X \cap A = \emptyset$, then $X \cap H \neq \emptyset$, i.e. $X \cap C_i \neq \emptyset$ for some $i \in I$. Thus, $X \cup C_i = C_j \subseteq B$ for some $j \in I$, a contradiction.
We conclude $X \supsetneq A$ (otherwise $X \in F$, i.e. $X \subseteq B$ which is impossible). $(B \setminus A) \not\sim A$ implies $(X \setminus A) \not\sim A$ from which we deduce that $L_i := C_i \cup (X \setminus A)$ is autonomous. $L_i \in F$ implies $L_i \subseteq B$, hence $X \subseteq B$, a contradiction.
q.e.d.

proof of theorem 1:

(I) follows, as in the finite case, immediately from the fact that two strongly autonomous subsets $X, Y \subseteq V(G)$ are either disjoint or comparable.

(II): (1) is trivial. (2): Since G and $\overline{G}$ have the same autonomous subsets, $\pi(G)$ consists of the components of $\overline{G}$. Hence, $A_i \sim A_j$ for all $i,j \in I$, $i \neq j$. This fact shows that $ab \wedge bc$, $a \in A_i$, $b \in A_j$ $(i \neq j)$ implies $c \in A_i$, i.e. $[ab]_{\equiv} \subseteq E_{ij}$. If $c \in A_i$ is arbitrary with $c \neq a$, then a and c are in the same component of $\overline{G}\,[N_G(b)]$ (where $N_G(b)$ denotes the neighborhood of b in G), hence $ab \wedge bc$. Thus, $E_{ij} = [ab]_{\equiv}$. (3): We first show that $\pi(G)$ has properties (a) and (b). (a) is clear. As to (b): If $ab \in E_{rs}$, then $ab \wedge bc$ implies $c \notin A_s$, hence $[ab]_{\equiv} \subseteq C$. Since $V([ab]_{\equiv}) = \bigcup(A_i; i \in I_1)$ for some $I_1 \subseteq I$ with $\{r,s\} \subseteq I_1$, we switch over to G^{π} via the projection $p\colon G \dashrightarrow G^{\pi}$ and study $T := \{A_i; i \in I_1\}$. We are done if we can prove that $T \subsetneq \pi(G)$ is impossible. For then $V([ab]_{\equiv}) = V([cd]_{\equiv}) = V(G)$ for all $cd \in E_{uv}$ $(u,v \in I,\ u \neq v)$, and lemma 1 settles the matter. Thus, assume $T \subsetneq \pi(G)$. Since $|T| \geq 2$, T is not strongly autonomous (otherwise $p^{-1}(T)$ would be strongly autonomous in G).
Let X be an autonomous subset of $\pi(G)$ in general position with T and consider $X \cap T = \{A_i; i \in I_2\}$ with $I_2 \subseteq I_1$. $X \cap T$ and $T \setminus X$ are autonomous. Thus, neither $X \cap T$ nor $T \setminus X$ contain an edge from $[ab]_{\equiv}$; and $X \cap T \sim (T \setminus X)$, i.e. $X \cap T \not\sim (T \setminus X)$ in $\overline{G}^{\pi}$. We can apply case (2) of our theorem to the graph $G^{\pi}[T]$, since its complemented graph is not connected. The edge classes of $G^{\pi}[T]$ fall into two categories: internal ones and external ones. $[ab]_{\equiv}$ cannot be aninternal one, since $V([ab]_{\equiv}) = \bigcup T$. Thus, it is external and -(again since $V([ab]_{\equiv}) = \bigcup T$)- the only external one. Thus, $X \cap T$ and $T \setminus X$ are the two components of $\overline{G^{\pi}[T]}$, and $[ab]_{\equiv}$ consists exactly of all the $(X \cap T, T \setminus X)$-edges. I.p., every autonomous set of G^{π} intersecting both $X \cap T$ and $T \setminus X$ must contain all of T. Thus, there is no autonomous Y in general position with $X \cap T$, resp. $T \setminus X$, since otherwise $Y \cup (T \setminus X)$, resp. $Y \cup (X \cap T)$ would be autonomous.
We conclude that $X \cap T$ and $T \setminus X$ are strongly autonomous, i.e. they are singleta. Thus, T consists of 2 elements. Hence, T is independent in $\overline{G^{\pi}}$ and autonomous. By lemma 2, there exists a strongly autonomous set D with $T \subseteq D$ and $T \not\sim (D \setminus T)$ in $\overline{G^{\pi}}$. Since $\overline{G^{\pi}}$ is connected, we get $D \subsetneq \pi(G)$, i.e. $|T| = |D| = 1$, a contradiction.
Thus, $T = \pi(G)$, and we are done.

In order to prove the fact that $\pi(G)$ is the (unique) largest partition with properties (a),(b), we choose some partition $\rho(G) = \{B_j; j \in J\}$ of $V(G)$ satisfying (a),(b) and denote by $\underline{R}$ the set of all B_iB_j -edges. No B_j is in general position with any A_i. Hence, every B_i is either the disjoint union of suitable A_j's or it is contained in some A_k which then, in turn, is the disjoint union of suitable B_l's. $R \cap C \neq \emptyset$ is clear, hence R=C. Assume $B_j = \bigcup(A_i;\ i \in I_1)$ for some $I_1 \subseteq I$ with $|I_1| \geq 2$. Then $C = R$ forces the independence of $\{A_i;\ i \in I_1\}$ in G^{π}. By lemma 2, we obtain a strongly autonomous $D \subseteq \pi(G)$ with $\{A_i; i \in I_1\} \subseteq D$. The connectedness of G^{π} forces $D \neq \pi(G)$ contradicting the fact that G^{π} has no non-trivial strongly autonomous vertex sets. Hence, $\rho(G) \leq \pi(G)$. q.e.d.

Let us call a decomposition $\rho(G) = \{A_i; i \in I\}$ of a graph G a "Gallai-decomposition" if it is maximal among all partitions satisfying the following two properties:

(a) $A_i \sim A_j$ or $A_i \not\sim A_j$ for all $i,j \in I$, $i \neq j$.

(b) The set of all A_iA_j-edges $(i,j \in I,\ i \neq j)$ constitutes a single edge class $\underline{C}$ (i.p. $|I| \geq 2$).

The next theorem constitutes a natural supplement of Gallai's decomposition theorem.

Theorem 2: Let G be a graph and $\rho(G)$ a Gallai-decomposition. Then G is an NL-graph and we have the following situation:

(1) $V(C)$ is a component of G and $V(C) = \bigcup(A_i; i \in I_1)$ for some $I_1 \subseteq I$. If $I_1 \subsetneq I$, then $\rho(G) = \{A_i; i \in I_1\} \cup \{D\}$ where D is the join of all components $\neq V(C)$.

(2) If G is connected, then $\rho(G) = \pi(G)$, and all proper autonomous subsets of $V(G)$ are contained in a quasimaximal strongly autonomous set.

proof:

(1): Assume that G is not connected and C_j, $j \in J$, are the com= ponents. Clearly, $V(C) = \bigcup(A_i; i \in I \wedge A_i \cap V(C) \neq \emptyset)$. Since $V(C)$ is connected, there is some $j_0 \in J$ with $V(C) \subseteq C_{j_0}$ and, hence, $V(C) = C_{j_0}$. Since $\sigma(G) := \{A_i;\ A_i \subseteq C_{j_0}\} \cup \{\bigcup(C_j; j \in J,\ j \neq j_0)\}$ is a partition satisfying (a),(b) and since $\rho(G) \leq \sigma(G)$, we con= clude $\rho(G) = \sigma(G)$.

(2): If G is connected, we have $V(C) = V(G)$ as the only compo= nent. Let D be an arbitrary autonomous set that intersects both A_{i_0} and A_{i_1} properly for some $i_0 \neq i_1$. Then $A_{i_0} \cup A_{i_1}$ is autonomous and, hence, $A_{i_0} \not\sim A_{i_1}$ (otherwise $V(G) = V(C) \subseteq D$). Hence, $\sigma(G) := \{A_{i_0} \cup A_{i_1}\} \cup \{A_i; i \in I \setminus \{i_0, i_1\}\}$ is a partition of $V(G)$ satisfying (a),(b) and $\sigma(G) > \rho(G)$, a contradiction. Thus, every autonomous set $D \subseteq V(G)$ intersects at most one A_{i_0} properly. Therefore we have either $D = \bigcup(A_i; i \in I_1)$ for some $I_1 \subseteq I$ or $D = (D \cap A_{i_0}) \cup (\bigcup(A_i; i \in I_2)$ for some $i_0 \notin I_2 \subseteq I$ with $\emptyset \neq D \cap A_{i_0} \subsetneq A_{i_0}$. In neither case can D contain an A_iA_j-edge, if D is a proper autonomous subset. For such D we have either in $\{D\} \cup \{A_i; i \in I \setminus I_1\}$ or in $\{D \cup A_{i_0}\} \cup \{A_i;\ i_0 \neq i \in I \setminus I_2\}$ a partition of $V(G)$ satisfying (a),(b). Hence, $D = A_{i_0}$, resp. $D \subsetneq A_{i_0}$ depending on the case. Therefore the sets A_i of $\rho(G)$ are (exactly all) quasimaximal strongly autonomous subsets of $V(G)$, and each proper autonomous set D is contained in one of them. q.e.d.

We contended in § 1 that the extension of Gallai's theorem to NL-graphs (theorem 1) and the complementing theorem 2 con= stitute the natural basis of other results also in the infinite case. To make out my case, let us prove two interesting results of D. Kelly ([2]) in the light of our theorems:

1) Lemma 3.2 in [2] reads (rephrased) as follows: "If G is a graph and $A \subsetneq V(G)$ an autonomous, connected subset, then each of the components of $\overline{G}[A]$ is a strongly autonomous subset of $V(G)$."

proof: Let $\{B_i; i \in I\}$ be the components of A in $\overline{G}[A]$. W.l.o.g. $|I| \geq 2$ (otherwise let X and A be in general position and autonomous in G; then $A \cap X$ and $A \setminus (X \cap A)$ are autonomous in G, a contradiction). By theorem 1, the B_i are the quasimaximal strongly autonomous sets of $G[A]$, we have $B_i \sim B_j$ for all $i \neq j$ and, for each $i \neq j$, the set F_{ij} of all B_iB_j-edges constitutes a single edge class of $G[A]$ and, hence, of G. Fix $i_0 \in I$ and let C be an autonomous vertex set of G properly intersecting B_{i_0}. Then $C \cap A = C \cap B_{i_0}$ and $\{B_{i_0} \cup C\} \dot{\cup} \{B_i;\ i \in I \setminus \{i_0\}\}$ are the components of $\overline{G}[A \cup C]$. Thus, the $(B_{i_0} \cup C, B_i)$-edges and the $B_{i_0}B_i$-edges coincide, since both sets constitute the same edge class by theorem 1. We conclude $C \subseteq B_{i_0}$. q.e.d.

2) Theorem 3.4 in [2] reads as follows: " A non-trivial connected graph G with a connected complement $\overline{G}$ has no proper autonomous vertex sets if it has no proper strongly autonomous vertex sets."

proof: G is an NL-graph with trivial canonical decomposition. Theorem 2.2 proves the result. q.e.d.

§3. A remark on P_4.

Let P_n denote the graph with n vertices $a_1, a_2, \ldots, a_n$ and edges a_1a_2, $a_2a_3, \ldots, a_{n-1}a_n$. D. Kelly [2] refers to various proofs for the fact that finite prime graphs (i.e. graphs without proper autonomous vertex sets) contain P_4 as a full subgraph (we write $P_4 \precsim G$).

In the finite case the property $P_4 \precsim G$ holds even true for any graph G such that G and $\overline{G}$ are connected, and this is the basis of the finitary proofs. D. Kelly generalized the result to infinite prime graphs and showed via an example that the connectedness of G and $\overline{G}$ do not suffice any more. Thus, an alltogether new proof was asked for. Kelly's example (see [2]) is essentially the ordered set of diagram 3. Its comparability graph G is an L-graph such that G and $\overline{G}$ are connected and $P_4 \not\leq G$. This comparability graph G is the graph $G_{\aleph_0}$ on the vertex set $\{a_n; n \in \mathbb{N}\} \cup \{b_n; n \in \mathbb{N}\}$, defined as follows (see diagram 4):

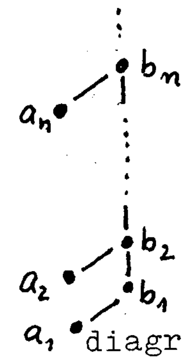

diagram 1

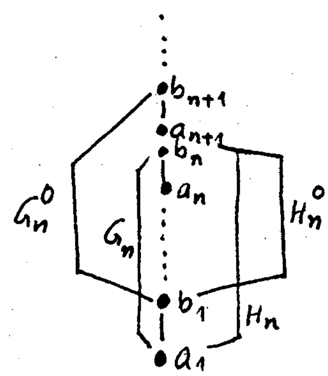

diagram 2

(1) $a_n \not\sim a_m, b_m$ for all $m < n$;

(2) $b_n \sim a_m, b_m$ for all $m < n$;

(3) $b_n \sim a_n$ for all $n \in \mathbb{N}$.

We denote by G_n, H_n, G_n^0, H_n^0 the following graphs:

$G_n := G_{\aleph_0}[\{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_n\}]$;

$H_n := G_{\aleph_0}[\{a_1, \ldots, a_{n+1}\} \cup \{b_1, \ldots, b_n\}]$;

$G_n^0 := G_{\aleph_0}[\{a_2, \ldots, a_{n+1}\} \cup \{b_1, \ldots, b_{n+1}\}]$;

$H_n^0 := G_{\aleph_0}[\{a_2, \ldots, a_{n+1}\} \cup \{b_1, \ldots, b_n\}]$.

Clearly, $\overline{G_n} = H_n^0$ and $\overline{H_n} = G_n^0$.

<u>Remark:</u> Let G be a non-trivial graph such that G and $\overline{G}$ are con= nected: $P_4 \not\leq G$ implies $G_{\aleph_0} \leq G$.

Thus, Kelly's example is the minimal model for his case.

Apart from this observation the result subsumes all finitary proofs.

proof of the remark:

Since $\overline{P}_4 = P_4$ we have $P_4 \not\leq \overline{G}$. Of course, $P_3 \leq G$ since G and $\overline{G}$ are connected. Let $G[a,b,c] = P_3$ with $a \sim c \sim b$ be given. Then $a \not\sim c$, $b \not\sim c$ in $\overline{G}$, and we choose a shortest path (in $\overline{G}$) from c to b. Since $P_4 \not\leq \overline{G}$, it is of the form $c \sim e \sim b$ with $a \sim e$ in $\overline{G}$. $G[a,b,c,e] \leq G$ shows $G\,[a,c,e] = H_1 \leq G$.

Assume H_n (see diagram 4) $\leq G$. Since G is connected, we have a path $a_{n+1} \sim b_{n+1} \sim b_n$ in G. Let $g \in \{b_j;\ j \leq n\} \cup \{a_j; j \leq n\}$. If $b_{n+1} \not\sim g$, then $G[a_{n+1}, b_{n+1}, b_n, g] = P_4 \leq G$, a contradiction. Thus, $b_{n+1} \sim g$ for all of the above g. Hence, $G[\{a_j, b_j;\ j \leq n+1\}] = G_{n+1} \leq G$. Then $\overline{G}_{n+1} = H^0_{n+1} \leq \overline{G}$. Since $\overline{G}$ is connected, we have a path of length 3, say $b_{n+1} \sim a_{n+2} \sim a_{n+1}$, from b_{n+1} to a_{n+1} in $\overline{G}$. As above, $\overline{G}[\{a_i;\ i \leq n+2\} \cup \{b_j;\ j \leq n+1\}] = G^0_{n+1} \leq \overline{G}$. Hence, $\overline{\overline{G}^0_{n+1}} = H_{n+1} \leq G$. By our construction, $H_n \leq H_{n+1} \leq G$. Thus, $G_{\aleph_0} = \bigcup(H_n; n \in \mathbb{N}) \leq G$. q.e.d.

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