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Notes on Tolerances Relations of Lattices I

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by

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0. Introduction

A *tolerance relation* Θ on a lattice L is a reflexive and symmetric binary relation satisfying the substitution property.

In 1982, G. Czédli [1] proved that, for a lattice L and a tolerance relation Θ, the maximal Θ-connected subsets of L form a lattice. He considered lattices as algebras of type (2,2) and gave an algebraic proof. In Section 1, we investigate tolerances from the point of view of partial ordering in detail; in particular, we give an order-theoretical proof of Czédli's result. Our proof avoids Zorn's axiom needed by Czédli. Some results on Θ-block fixing sets and consequences thereof are added.

Tolerances can be viewed as quotients of congruences in a natural way. Using this fact, we extend the Second Isomorphism Theorem from congruences to tolerance relations in Section 2.

In connection with the extended Second Isomorphism Theorem, a question on the product of lattice varieties arises naturally. In Section 3 we answer it partially and illustrate the situation with examples.

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1. The lattice L/Θ

For concepts and notations not defined in this paper, see G. Grätzer [1].

Let $L = (L; \leq)$ be a lattice and Θ a tolerance on L. $x\Theta y \quad (x, y \in L)$ denotes, as usual, that $(x, y) \in \Theta$ holds; also, $H_1 \Theta H_2 \quad (H_1, H_2 \subseteq L)$ denotes that $x\Theta y$ holds for every $x \in H_1, y \in H_2$.

The following two lemmata are useful in many situations.

Lemma 1:

Let $x \leq x'$ and $y \leq y'$ be elements of L

with $x'\Theta x$, $y'\Theta y$, $x\Theta y'$, $y\Theta x'$. Then $(x' \vee y')\Theta(x \wedge y)$.

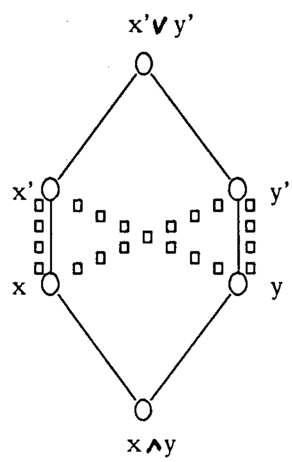

Diagram 1

Proof:

$x\Theta y', x\Theta x'$ imply that $x\Theta(x' \vee y')$.

Similarly, $y\Theta(x' \vee y')$ holds. Thus, $(x \wedge y)\Theta(x' \vee y')$.

q.e.d.

Lemma 2:

Let x, y, x', y' be elements of L

with $x\Theta x'$, $y\Theta y'$ and $x, y \leq x' \wedge y'$,

Then $x\Theta y$ and $x'\Theta y'$.

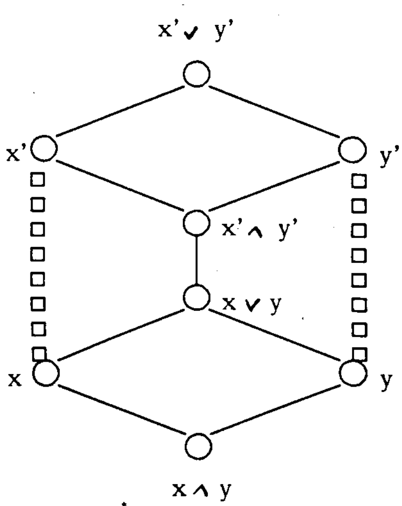

Diagram 2

Proof:

$x\Theta x', y\Theta y'$ imply that $(x \vee y)\Theta(x' \vee y')$.

From $x \vee y \leq x', y' \leq x' \vee y'$, we conclude that $x'\Theta y'$.

The second assertion follows analogously.

q.e.d.

We use the following notations and terminology:

A subset H of L is called Θ *-connected* if $x\Theta y$ holds for all $x, y \in H$. If H is an arbitrary subset of L, then we define $C_H := \{x \in L; x\Theta h \text{ for all } h \in H\}$.

C_H is either empty or it is a convex sublattice of L. C_H is not necessarily Θ -connected.

We further define $(H] := \{x \in L; x \leq h \text{ for some } h \in H\}$ and

$[H) := \{x \in L; x \geq h \text{ for some } h \in H\}$. Finally, $H_\Theta := C_H \cap (H]$ and $H^\Theta := C_H \cap [H)$.

<u>Lemma 3:</u>

Let H be a subset of the lattice L.

(1) H_Θ is a Θ - connected , convex, $\wedge$-closed subset of L. If H is upward directed, then H_Θ is either empty or it is a sublattice of L.

(2) H^Θ is a Θ-connected, convex, $\vee$-closed subset of L. If H is downward directed, then H^Θ is either empty or it is a sublattice of L.

<u>Proof:</u>

We only prove (1). H_Θ is clearly convex and $\wedge$-closed. To show that it is Θ-connected, let $x, y \in H_\Theta$. There are $x', y' \in H$ with $x \leq x'$, $y \leq y'$, $x\Theta x'$, $y\Theta x'$, $x\Theta y'$, $y\Theta y'$. By Lemma 1, $x\Theta y$. If H is upward directed, we can choose $x' = y'$ and obtain $x \vee y \leq x'$, hence $x \vee y \in (H]$. Since $x \vee y \in C_H$ is clear, we get $x \vee y \in H_\Theta$.

<u>q.e.d.</u>

<u>Lemma 4:</u>

Let H be a Θ-connected subset of L.

(1) $(H^\Theta)^\Theta = H^\Theta$ and $(H_\Theta)_\Theta = H_\Theta$.

(2) $H \subseteq H_\Theta \subseteq (H_\Theta)^\Theta = ((H_\Theta)^\Theta)_\Theta = \ldots\ldots$, and $(H_\Theta)^\Theta$ is a Θ-connected, convex sublattice of L, if $H \neq \emptyset$.

Proof:

(1) is clear. As to (2): Lemma 3 yields that $H \subseteq H_\Theta \subseteq (H_\Theta)^\Theta \subseteq ((H_\Theta)^\Theta)_\Theta$. Let $x \in ((H_\Theta)^\Theta)_\Theta$. Then $x \leq y \geq u \leq v$ for some $y \in (H_\Theta)^\Theta, u \in H_\Theta$ and $v \in H$. We claim that $x \wedge u \in H_\Theta$. Indeed, clearly, $x \wedge u \leq v$, hence $x \wedge u \in (H]$. If $h \in H$, then $x\Theta h$. Together with $u\Theta h$ we get $(x \wedge u)\Theta h$; hence, $x \wedge u \in C_H$ and $x \wedge u \in H_\Theta$, as claimed. Now $x \in [H_\Theta) \cap C_{H_\Theta} = (H_\Theta)^\Theta$, and the first part of (2) has been proved.

H_Θ is Θ - connected and $\wedge$-closed, by Lemma 3. Hence, again by Lemma 3, $(H_\Theta)^\Theta$ is a Θ-connected, convex sublattice of L.

q.e.d.

The significance of $(H_\Theta)^\Theta$ comes from the next lemma.

Lemma 5:

Let X be a subset of L. The following two statements are equivalent:

(1) X is a maximal Θ-connected subset of L.

(2) $X = (H_\Theta)^\Theta$ for some non-empty Θ -connected $H \subseteq L$.

Proof:

(1) implies (2) follows by taking $H = X$ and by Lemma 4. In order to prove that (2) implies (1), we choose $u \in L$ with $u\Theta(H_\Theta)^\Theta$. For every $x \in (H_\Theta)^\Theta$, we get $u \wedge x \in ((H_\Theta)^\Theta)_\Theta$. From $u\Theta((H_\Theta)^\Theta)_\Theta$, we get $u \in (((H_\Theta)^\Theta)_\Theta)^\Theta = (H_\Theta)^\Theta$. Hence, $(H_\Theta)^\Theta$ is a maximal Θ-connected subset of L.

q.e.d.

In view of the last lemma, we call subsets of the form $(H_\Theta)^\Theta$ for Θ- connected subsets H of L $\Theta-$*blocks of* L.

The Θ-blocks are convex sublattices of L. They enjoy a useful property with respect to two natural preorderings on L. In order to prove it, we use the next trivial lemma.

Lemma 6:

For $A, B \subseteq L$, define $A \mathbin{\mathring{\vee}} B := \{a \vee b; a \in A, b \in B\}$ and $A \mathbin{\mathring{\wedge}} B := \{a \wedge b; a \in A, b \in B\}$. If A and B are Θ-connected, then so are $A \mathbin{\mathring{\vee}} B$ and $A \mathbin{\mathring{\wedge}} B$.

Definition 1:

For $A, B \subseteq L$ we define the following three binary relations:

(1) $A \circ\!\leq B :\Leftrightarrow$ For all $b \in B$ there is an $a \in A$ with $a \leq b$.

(2) $A \leq\!\circ B :\Leftrightarrow$ For all $a \in A$ there is a $b \in B$ with $a \leq b$.

(3) $A \leq B :\Leftrightarrow$ $A \circ\!\leq B$ and $A \leq\!\circ B$.

In general, the relations $\leq\!\circ$ and $\circ\!\leq$ are distinct. On convex subsets of L, the relation $\leq$ is a partial ordering. For Θ- blocks, the three relations coincide:

Lemma 7:

If A, B are Θ-blocks of L, then $A \circ\!\leq B$, $A \leq\!\circ B$ and $A \leq B$ are equivalent.

Proof:

Assume that $A \leq\!\circ B$, i.e., for every $a \in A$, there is a $b \in B$ with $a \leq b$. Hence, $a = a \wedge b \in A \mathbin{\mathring{\wedge}} B$. Thus, $A \subseteq A \mathbin{\mathring{\wedge}} B$. Since $A \mathbin{\mathring{\wedge}} B$ is Θ-connected by Lemma 6 and A is a maximal Θ-connected subset of L, we conclude that $A = A \mathbin{\mathring{\wedge}} B$. Hence, if $b \in B$ is given, then $b \mathbin{\mathring{\wedge}} a \in A$ for all $a \in A$. Thus, $A \circ\!\leq B$. The converse is analogous.

q.e.d.

Theorem 1 (see G. Czédli[1])

If Θ is a tolerance on the lattice L, then L/Θ, the set of Θ-blocks, forms a lattice with respect to the ordering $\leq$.

In addition, we have $A \vee B = (A \curlyvee B)^{\Theta}$ and $A \wedge B = (A \curlywedge B)_{\Theta}$ for all $A, B \in L/\Theta$.

Proof:

We prove that $A \vee B$ exists and equals $(A \curlyvee B)^{\Theta}$; the second formula follows by duality. If $C \geq A, B$ for a Θ -block C, then trivially $C \geq_0 A \curlyvee B$, hence $C \geq_0 (A \curlyvee B)^{\Theta}$. Assuming that $(A \curlyvee B)^{\Theta}$ has been shown to be a Θ-block, we are finished, since then $C \geq (A \curlyvee B)^{\Theta}$ and, hence, $(A \curlyvee B)^{\Theta} = A \vee B$.

Since $(A \curlyvee B)^{\Theta}$ is Θ-connected, we only have to show that $(A \curlyvee B)^{\Theta}$ is a maximal Θ-connected set. Let $D \supseteq (A \curlyvee B)^{\Theta}$ be a Θ-connected subset of L. By Lemma 6, $D \curlywedge A$ and $D \curlywedge B$ are Θ-connected sets. Since $A \curlyvee B \subseteq D$, we obtain $A \subseteq D \curlywedge A$ and $B \subseteq D \curlywedge B$ and, hence, $A = D \curlywedge A$, $B = D \curlywedge B$. For $d \in D, a \in A, b \in B$, we get $d \wedge a \in A$, $d \wedge b \in B$ and $d \geq (d \wedge a) \vee (d \wedge b) \in A \curlyvee B$. Now $d \Theta D$ implies that $d \Theta (A \curlyvee B)$, hence $d \in (A \curlyvee B)^{\Theta}$. Thus, $D = (A \curlyvee B)^{\Theta}$, as claimed.

q.e.d.

The description of $A \vee B$ and $A \wedge B$ in Theorem 1 can be generalized.

Remark to Theorem 1:

If $A_1, A_2, \ldots\ldots\ldots, A_n$ are Θ-blocks, then $A_1 \vee A_2 \vee \ldots. \vee A_n = (A_1 \curlyvee A_2 \curlyvee \ldots \curlyvee A_n)^{\Theta}$ and $A_1 \wedge A_2 \wedge \ldots \wedge A_n = (A_1 \curlywedge A_2 \curlywedge \ldots. \curlywedge A_n)_{\Theta}$.

Proof:

(By induction on n) For $n = 1, 2$ we know the result. For $n \geq 3$ we obtain

$$A_1 \vee .... \vee A_{n-1} \vee A_n = ((A_1 \vee ... \vee A_{n-1}) \dot{\vee} A_n)^{\Theta} =$$

$$((A_1 \dot{\vee} ... \dot{\vee} A_{n-1})^{\Theta} \dot{\vee} A_n)^{\Theta} \text{ (by the induction hypothesis) } \subseteq ((A_1 \dot{\vee} ... A_{n-1} \dot{\vee} A_n)^{\Theta}.$$

The maximality of $A_1 \vee ... \vee A_n$ implies then that $A_1 \vee ... \vee A_n = (A_1 \dot{\vee} ... \dot{\vee} A_n)^{\Theta}$.

The second assertion follows by duality.

q.e.d.

We add a few observations.

If $H \subseteq L$ is a non-empty, Θ-connected subset of L, then both $(H_{\Theta})^{\Theta}$ and $(H^{\Theta})_{\Theta}$ are Θ-blocks containing H. As the next lemma shows, the first block is the smallest and the second block is the largest Θ-block containing H.

Lemma 8:

Let $H \subseteq L$ be a non-empty, Θ-connected set.

(1) If $D \supseteq H$ is a Θ-connected subset of L, then $H^{\Theta} \dot{\vee} D \subseteq H^{\Theta}$ and $H_{\Theta} \dot{\wedge} D \subseteq H_{\Theta}$.

(2) If D is a Θ-block with $D \supseteq H$, then $(H_{\Theta})^{\Theta} \leq D \leq (H^{\Theta})_{\Theta}$.

Proof:

(1): Let $x \in H^{\Theta} \dot{\vee} D$, i.e., $x = z \vee d$ for some $z \in H^{\Theta}$ and $d \in D$. Then we have $x \geq z \geq y$ for some $y \in H$ and $x \Theta H$ (since $z \Theta H$ and $d \Theta H$ hold true). Thus, $x \in H^{\Theta}$, and so $H^{\Theta} \dot{\vee} D \subseteq H^{\Theta}$. The proof of $H_{\Theta} \dot{\wedge} D \subseteq H_{\Theta}$ is analogous.

(2): $D \subseteq D \Wedge (H^{\Theta} \Vee D) \subseteq D \Wedge H^{\Theta} \subseteq D \Wedge (H^{\Theta})_{\Theta}$ implies that $D = D \Wedge (H^{\Theta})_{\Theta}$, i.e., $D \leq (H^{\Theta})_{\Theta}$. $D \subseteq D \Vee (H_{\Theta} \Wedge D) \subseteq D \Vee H_{\Theta} \subseteq D \Vee (H_{\Theta})^{\Theta}$ implies $D = D \Vee (H_{\Theta})^{\Theta}$, i.e., $D \geq (H_{\Theta})^{\Theta}$.

q.e.d.

Definition 2:

If L is a lattice and Θ is a tolerance on L, then we call a Θ-connected subset H of L a Θ-*block fixing set* if there exists exactly one Θ-block D with $H \subseteq D$.

Examples:

(1): If $A_1, \ldots, A_n$ are Θ-blocks, then $A_1 \Vee \ldots \Vee A_n$ and $A_1 \Wedge \ldots \Wedge A_n$ are Θ-block fixing sets.

Proof:

Let $D \supseteq A_1 \Vee \ldots \Vee A_n$ be a Θ-block, then $D \geq A_1 \vee \ldots \vee A_n = (A_1 \Vee \ldots \Vee A_n)^{\Theta}$ holds. If $d \in D$, then $d\Theta(A_1 \Vee \ldots \Vee A_n)$ and there are $a_i \in A_i$ with $d \geq a_1 \vee \ldots \vee a_n$; thus, $d \in (A_1 \Vee \ldots \Vee A_n)^{\Theta}$. We obtain $D \subseteq (A_1 \Vee \ldots \Vee A_n)^{\Theta}$ and therefore $D = A_1 \vee \ldots \vee A_n$. A dual argument shows that $A_1 \Wedge \ldots \Wedge A_n$ is a Θ-block fixing set.

q.e.d.

(2): If A, B, C are Θ-blocks, then $(A \Vee B) \Wedge C$ is not, in general, a Θ-block fixing set. The following example illustrates the claim:

The 8-element-lattice L of diagram 3 has a tolerance Θ which is given by the five Θ-blocks A, B, C, D, E.

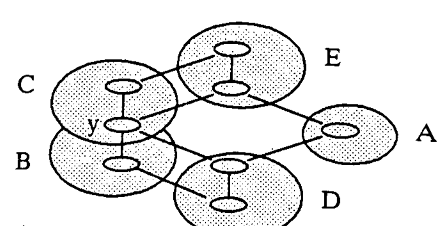

Diagram 3

Obviously, $(A \Vee B) \Wedge C = \{y\}$, but $\{y\}$ is not a Θ-block fixing set.

(3): If $H \subseteq L$ is a Θ-connected set, then H^{Θ} and H_{Θ} are Θ-block fixing sets.

Proof:

If $D \supseteq H^{\Theta}$ is a Θ-block, then $(H^{\Theta})_{\Theta} = ((H^{\Theta})_{\Theta})^{\Theta} \leq D \leq ((H^{\Theta})^{\Theta})_{\Theta} = (H^{\Theta})_{\Theta}$, by Lemma 8; thus, $D = (H^{\Theta})_{\Theta}$, and H^{Θ} is a Θ-block fixing set. Dually, H_{Θ} is a Θ-block fixing set as well.

q.e.d.

Theorem 2:

If $H \subseteq L$ is a Θ-connected set, then the following two statements are equivalent:

(1) H is a Θ-block fixing set.

(2) $(H_{\Theta})^{\Theta} = (H^{\Theta})_{\Theta}$.

Proof:

(1) clearly implies (2), and Lemma 8 shows that (2) implies (1).

q.e.d.

For $H \subseteq L$, let $[H]$ denote the sublattice of L generated by H.

Lemma 9:

Let Θ be a tolerance on the lattice L.

(1) $X, Y \subseteq L$ and $X \leq_{\circ} Y \subseteq [X]$ imply $X_{\Theta} \subseteq Y_{\Theta}$.

(2) If X, Y are Θ-connected subsets of L with $X_{\Theta} \subseteq Y_{\Theta}$, then $(X_{\Theta})^{\Theta} = (Y_{\Theta})^{\Theta}$.

Proof:

(1): Choose $a \in X_\Theta$. Then $a \Theta X$ holds and there is some $x \in X$ with $a \leq x$. We choose $y \in Y$ with $x \leq y$, hence $a \leq y$. $a\Theta X$ implies $a\Theta[X]$; thus, we have $a\Theta Y$. We conclude $a \in Y_\Theta$.

(2): X_Θ and Y_Θ are Θ-block fixing sets (see Example 3 preceeding Theorem 2). Because of $X_\Theta \subseteq (X_\Theta)^\Theta$ and $X_\Theta \subseteq (Y_\Theta)^\Theta$ we conclude $(X_\Theta)^\Theta = (Y_\Theta)^\Theta$.

q.e.d.

Lemma 9 is the crux of the next theorem.

Theorem 3:

Let L be a lattice and $X = \{x_1, \ldots\ldots, x_n\} \subseteq L, n \in \mathbb{N}$, a Θ-connected finite subset of L. Then $(X_\Theta)^\Theta = (\{x_1 \vee \ldots \vee x_n\}_\Theta)^\Theta$.

(In words: All finitely generated Θ-blocks are principal Θ-blocks.)

Proof:

By Lemma 9, $X \leq_\circ \{x_1 \vee \ldots\ldots \vee x_n\} \subseteq [X]$ implies $X_\Theta \subseteq \{x_1 \vee \ldots\ldots \vee x_n\}_\Theta$; thus, $(X_\Theta)^\Theta = (\{x_1 \vee \ldots\ldots \vee x_n\}_\Theta)^\Theta$.

q.e.d.

2. Quotients of Tolerances and the Second Isomorphism Theorem

Let L be a lattice, $ConL$ the lattice of congruences on L and $TolL$ the lattice of tolerances on L.

If Θ and Φ are tolerances on L, then we define the binary relation Θ/Φ on L/Φ as follows: $A\Theta/\Phi B$ holds if and only if there are $a \in A, b \in B$ with $a\Theta b$.

If Θ and Φ are congruences and $\Theta \geq \Phi$, then Θ/Φ is a congruence on L/Φ, and the well-known Second Isomorphism Theorem states that $L/\Theta \cong (L/\Phi)/(\Theta/\Phi)$ holds. In general, Θ/Φ is only a tolerance on L/Φ. We will show that every tolerance on an arbitrary lattice L' is of the form Θ/Φ for congruences Θ and Φ on a suitable lattice L.

Thus, let L' be a lattice and let Θ' be a tolerance relation on L'.

We define the lattice L as sublattice of the direct product $L' \times (L'/\Theta')$ on the carrier set $L := \{(a, A); A \in L'/\Theta' \text{ and } a \in A\}$. If $\pi_1 : L \longrightarrow L'$ and $\pi_2 : L \longrightarrow L'/\Theta'$ are the restrictions of the two canonical projections from $L' \times (L'/\Theta')$ onto L' and L'/Θ', resp., then π_1 and π_2 are lattice epimorphisms. We define $\Theta := kernel(\pi_2)$ and $\Phi := kernel(\pi_1)$. The homomorphism theorem yields $L/\Phi \cong L'$, and the corresponding isomorphism identifies $a \in L'$ with $\pi_1^{-1}(a) \in L/\Phi$.

Under this identification we get that $L' = L/\Phi$ and $\Theta' = \Theta/\Phi$.

<u>Definition 3:</u>

Let L' be a lattice and Θ' a tolerance on L'. The lattice L and the congruences Θ, Φ just constructed are called *the lattice, resp. the congruences associated with* (L', Θ').

We summarize:

Theorem 4:

Let L' be a lattice, Θ' a tolerance on L'. Let L be the lattice and Θ, Φ the congruences associated with (L', Θ'). The canonical identification makes the following two statements true:

(i) $L/\Phi = L'$, (ii) $\Theta/\Phi = \Theta'$.

In case of a congruence Θ', we get $\Phi = \omega, L = L'$ and $\Theta = \Theta'$ in Theorem 4. In case of a tolerance Θ', we have no natural correspondence between L/Θ and $(L/\Phi)/(\Theta/\Phi)$, but a suitable modification yields a generalized version of the Second Isomorphism Theorem. In order to derive the result, we find another way of interpreting a tolerance Θ' on a lattice L'. This interpretation associates Θ' on L' with $\Phi \circ \Theta \circ \Phi$ on L.

Lemma 10:

Let L be a lattice and Θ, Φ tolerances on L. Then the following are true:

$\Phi \circ \Theta \circ \Phi \in TolL$, $\Theta \leq \Phi \circ \Theta \circ \Phi$ and $\Phi \circ \Theta \circ \Phi/\Phi = \Theta/\Phi \in TolL/\Phi$.

An Example:

Quite different from the situation for congruences, not every tolerance Θ on a lattice L is of the form $\Phi \circ \Xi \circ \Phi$ for suitable tolerances Φ, Ξ. E.g., the tolerance Θ on the lattice L of Diagram 3 is not of that kind.

Theorem 5 (The Second Isomorphism Theorem):

Let L be a lattice, $\Phi \in ConL$ and $\Theta \in TolL$.

Then $L/\Phi \circ \Theta \circ \Phi \;\cong\; (L/\Phi)/(\Theta/\Phi) \;=\; (L/\Phi)/(\Phi \circ \Theta \circ \Phi/\Phi)$.

Proof:

Define $L' := L/\Phi$, $\Theta' := \Theta/\Phi$ and let $\pi : L \longrightarrow L'$ be the natural projection. If we extend π to the respective powersets in the canonical way, then we obtain the mapping $\bar{\pi} : Pot(L) \longrightarrow Pot(L')$ and, by restriction, $\bar{\pi} : L/\Phi \circ \Theta \circ \Phi \longrightarrow Pot(L')$.

(i) Claim: $\mathbf{A} \in L/\Phi \circ \Theta \circ \Phi$ implies that $\bar{\pi}(\mathbf{A}) \in L'/\Theta'$.

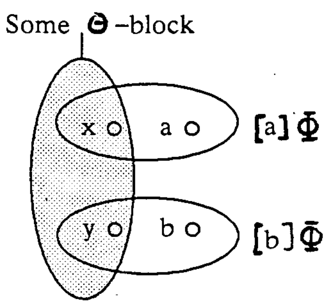

Diagram 4

Assume that $a\Phi \circ \Theta \circ \Phi b$, i.e., $a\Phi x\Theta y\Phi b$ for suitable $x \in [a]\Phi, y \in [b]\Phi$ $(a, b \in L)$.
By the definition of Θ', we get $([a]\Phi)\Theta'([b]\Phi)$.
Thus, $\bar{\pi}(\mathbf{A})$ is Θ'-connected.

To show the maximality of $\bar{\pi}(\mathbf{A})$ with respect to Θ'-connectedness, let $[x]\Phi \in L'$ be an element with $([x]\Phi)\Theta'([a]\Phi)$ for all $a \in \mathbf{A}$. Thus, for every $a \in \mathbf{A}$, there are elements $x' \in [x]\Phi$, $a' \in [a]\Phi$ with $x'\Theta a'$. We conclude that $x\Phi x'\Theta a'\Phi a$, i.e., $x\Phi \circ \Theta \circ \Phi a$, and, hence, $x\Phi \circ \Theta \circ \Phi\mathbf{A}$. We deduce that $x \in \mathbf{A}$ and, hence, $[x]\Phi \in \bar{\pi}(\mathbf{A})$. Thus, $\bar{\pi}(\mathbf{A})$ is a Θ'-block.

(ii) Claim: $A \in L'/\Theta'$ implies that $\bar{\pi}^{-1}(A) \in L/\Phi \circ \Theta \circ \Phi$.

Let $\mathbf{A} := \bar{\pi}^{-1}(A)$. Clearly, $\bar{\pi}(\mathbf{A}) = A$. Choose $a, b \in \mathbf{A}$ arbitrarily.
Then $[a]\Phi, [b]\Phi \in A$ implies that $([a]\Phi)\Theta'([b]\Phi)$; thus, $a\Phi x\Theta y\Phi b$ holds for suitable $x \in [a]\Phi, y \in [b]\Phi$, i.e., we have $a\Phi \circ \Theta \circ \Phi b$. Thus, $\mathbf{A}$ is $\Phi \circ \Theta \circ \Phi$-connected.
We embed $\mathbf{A}$ in some $\Phi \circ \Theta \circ \Phi$-block $\mathbf{A}^\star$ and obtain $\bar{\pi}(\mathbf{A}) = A \subseteq \bar{\pi}(\mathbf{A}^\star)$. Since A and, by (i), $\bar{\pi}(\mathbf{A}^\star)$ are Θ'-blocks, we obtain $A = \bar{\pi}(\mathbf{A}^\star)$ and, hence, $\mathbf{A}^\star \subseteq \bar{\pi}^{-1}(A) = \mathbf{A}$.
Thus, $\mathbf{A} = \mathbf{A}^\star$.

(i), (ii) and the surjectivity of π immediately imply (iii) and (iv) below.

(iii) $\bar{\pi}^{-1}(\bar{\pi}(\mathbf{A})) = \mathbf{A}$ holds for all $\Phi \circ \Theta \circ \Phi$-blocks $\mathbf{A}$ of L.

(iv) $\bar{\pi}(\bar{\pi}^{-1}(A)) = A$ holds for all Θ'-blocks A of L'.

(v) Statements (i) to (iv) prove that the restriction $\bar{\pi} : L/\Phi \circ \Theta \circ \Phi \longrightarrow L'/\Theta'$ is a bijection. Finally, we show that $\bar{\pi}$ is even a lattice-homomorphism :

$\bar{\pi}^{-1}(A) = \{a \in L;\ [a]\Phi \in A\}$ holds for every $A \in L'/\Theta'$. If, therefore, $A, B \in L'/\Theta'$ are arbitrarily chosen, then we get :

$$\begin{aligned} \bar{\pi}^{-1}(A \vee B) &= \{x \in L; [x]\Phi \in A \vee B\} \\ &\supseteq \{a \vee b;\ a, b \in L \text{ and } [a]\Phi \in A, [b]\Phi \in B\} \\ &= \bar{\pi}^{-1}(A) \mathbin{\dot\vee} \bar{\pi}^{-1}(B). \end{aligned}$$

The last set is a $\Phi \circ \Theta \circ \Phi$-block fixing set. Thus, there is exactly one $\Phi \circ \Theta \circ \Phi$- block of L containing $\bar{\pi}^{-1}(A) \mathbin{\dot\vee} \bar{\pi}^{-1}(B)$, namely $\bar{\pi}^{-1}(A) \vee \bar{\pi}^{-1}(B)$. Thus, we proved that

$\bar{\pi}^{-1}(A \vee B) \quad = \quad \bar{\pi}^{-1}(A) \vee \bar{\pi}^{-1}(B)$ holds.

Similarly, $\bar{\pi}^{-1}(A \wedge B) \quad = \quad \bar{\pi}^{-1}(A) \wedge \bar{\pi}^{-1}(B)$ holds.

q.e.d.

3. Products of lattice varieties

If $\mathbf{V}$ and $\mathbf{W}$ are two varieties of lattices, then the product $\mathbf{V} \circ \mathbf{W}$ consists of all lattices L for which there is some congruence Θ satisfying the following two properties:

(i) All Θ-blocks of L are in $\mathbf{V}$,

(ii) $L/\Theta \in \mathbf{W}$.

We combine these two conditions by saying that Θ *establishes that L is in* $\mathbf{V} \circ \mathbf{W}$.

G.Grätzer and D.Kelly [1] give an overview of these variety products.

$\mathbf{V} \circ \mathbf{W}$ is not, in general, a variety. However, one knows that the variety generated by $\mathbf{V} \circ \mathbf{W}$ is $\boldsymbol{H}(\mathbf{V} \circ \mathbf{W})$, the class of all homomorphic images of lattices in $\mathbf{V} \circ \mathbf{W}$.

R. N. McKenzie conjectured that the variety $\boldsymbol{H}(\mathbf{V} \circ \mathbf{W})$ can be characterized as follows: "A lattice L' is contained in $\boldsymbol{H}(\mathbf{V} \circ \mathbf{W})$ if and only if there is a tolerance Θ' on L' such that all Θ'-blocks of L' are in $\mathbf{V}$ and $L'/\Theta' \in \mathbf{W}$."

The next theorem answers one direction of the conjecture in the affirmative.

<u>Theorem 6:</u>

Let $\mathbf{V}$ and $\mathbf{W}$ be varieties of lattices.

Let L' be a lattice with a tolerance Θ' that satisfies the following two properties:

(i) All Θ'-blocks are in $\mathbf{V}$, (ii) $L'/\Theta' \in \mathbf{W}$.

Then $L' \in \boldsymbol{H}(\mathbf{V} \circ \mathbf{W})$.

Proof:

Let L be the lattice and let Θ, Φ be the congruences associated with (L', Θ'). Of course, $\Theta \cap \Phi = \omega$ and $L/\Theta \cong L'/\Theta'$. Thus, $L/\Theta \in \mathbf{W}$. If $\pi_1 : L \longrightarrow L'$ and $\pi_2 : L \longrightarrow L'/\Theta'$ are the projections yielding Φ and Θ, then the Θ -blocks of L are of the form $\pi_2^{-1}(A_0)$ for fixed $A_0 \in L'/\Theta'$. $\pi_2^{-1}(A_0) = \{(a, A_0);\ \ a \in A_0\} \cong A_0$ shows that $\pi_2^{-1}(A_0) \in \mathbf{V}$ holds. Thus, Θ establishes that $L \in \mathbf{V} \circ \mathbf{W}$. The projection π_1 yields $L' \in \boldsymbol{H}(L) \subseteq \boldsymbol{H}(\mathbf{V} \circ \mathbf{W})$.

q.e.d.

In order to tackle the opposite direction of the above conjecture, we begin with some fixed lattice $L' \in \boldsymbol{H}(\mathbf{V} \circ \mathbf{W})$. $L' \in \boldsymbol{H}(\mathbf{V} \circ \mathbf{W})$ means $L' \cong L/\Phi$ for some $L \in \mathbf{V} \circ \mathbf{W}$ and a suitable congruence Φ on L. $L \in \mathbf{V} \circ \mathbf{W}$ is established by some congruence Θ on L. Then $\Theta' := \Theta/\Phi$ is a tolerance on L', and McKenzie's conjecture seems to be based on the hope that (i) all Θ'-blocks of L' are in $\mathbf{V}$ and (ii) $L'/\Theta' \in \mathbf{W}$ is always true.
The next and last theorem states that the first assertion is valid. An example will show that the second one is, in general, not true. This suggests that the answer to McKenzie's conjecture is in the negative.

Theorem 7:

Let $\mathbf{V}, \mathbf{W}$ be lattice varieties and assume that $L' \in \boldsymbol{H}(\mathbf{V} \circ \mathbf{W})$.
Then $L' \cong L/\Phi$ for some $L \in \mathbf{V} \circ \mathbf{W}$ and some congruence Φ on L. Let Θ be a congruence on L establishing $L \in \mathbf{V} \circ \mathbf{W}$. If $\Theta' := \Theta/\Phi$, then all Θ'-blocks of L' are in $\mathbf{V}$.

Proof:

We will show that for every finite Θ'-connected set $\{[a_0]\Phi, ..., [a_n]\Phi\}$ of Φ-blocks there is a Θ-connected subset $\{u_0, ..., u_n\}$ of L with $u_i \in [a_i]\Phi$. This suffices, since then every Θ'-block satisfies every identity which is satisfied by every Θ-block.

Since tolerance blocks are sublattices, we may assume that $[a_0]\Phi < [a_i]\Phi$ holds for all $i = 1, 2, ..., n$.

By the definition of Θ', we find suitable $b_i \in [a_i]\Phi$ and $a_0^i \in [a_0]\Phi$ with $a_0^i < b_i$ and $a_0^i \Theta b_i$ $(i = 1, 2, ..., n)$. Let $a := a_0^1 \vee a_0^2 \vee .... \vee a_0^n \in [a_0]\Phi$.

Due to $b_i \Theta a_0^i$ and $a_0^i \Theta a_0^i$, we get $(b_i \vee a)\Theta a$ and $b_i \vee a \in [a_i]\Phi$ for $i = 1, 2, ...n$.

With $u_0 := a$ and $u_i := b_i \vee a$, $i = 1, 2, ..., n$, our claim has been proved.

q.e.d.

An Example:

We modify an example of G. Czédli[1] to show that, under the hypotheses of Theorem 7, we cannot, in general, conclude that $L'/\Theta' \in \mathbf{W}$.

To do so, we describe a distributive lattice L and two congruences Θ, Φ on L (with $\Theta \cap \Phi = \omega$) such that $L/\Phi \circ \Theta \circ \Phi$ $(\cong (L/\Phi)/(\Theta/\Phi)$, by Theorem 5) is not distributive.

Let L_5 be the 5-element lattice on $\{1, 2, 3, 4, 5\}$ with $1 < 2 < 3 < 4 < 5$.

Then $\underline{L} := (L_5 \times L_5) \setminus \{(4, 1), (5, 1)\} \in \mathbf{D}$ (variety of distributive lattices).

On L_5 we define the congruences $\Phi_1, \Phi_2, \Theta_1, \Theta_2$ via the corresponding congruence blocks listed below:

$$\Phi_1 ----\{\{1,2\},\{3,4\},\{5\}\}$$

$$\Phi_2 ----\{\{1\},\{2,3\},\{4\},\{5\}\}$$

$$\Theta_1 ----\{\{1\},\{2\},\{3\},\{4,5\}\}$$

$$\Theta_2 ----\{\{1,2\},\{3\},\{4,5\}\}$$

Then $\Phi_1 \times \Phi_2$, $\Theta_1 \times \Theta_2 \in Con(L_5 \times L_5)$, and we define $\Phi := \Phi_1 \times \Phi_2|_L$, $\Theta := \Theta_1 \times \Theta_2|_L$. Diagram 5 shows the Θ-blocks (indicated by bold borderlines) and the Φ-blocks (indicated by normal border lines) on L.

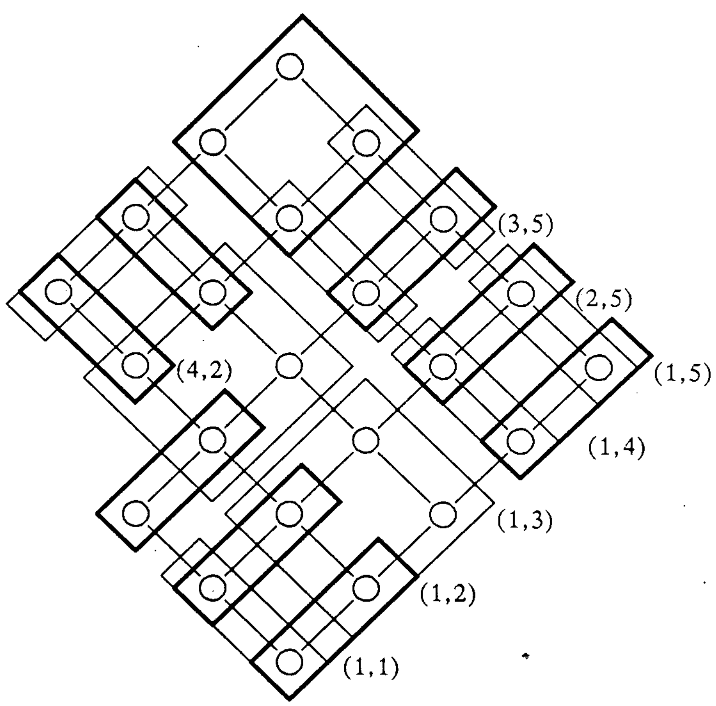

Diagram 5

Diagram 6 shows the $\Phi \circ \Theta \circ \Phi$-blocks on L. We recognize that $L/\Phi \circ \Theta \circ \Phi \cong N_5 \notin \mathbf{D}$.

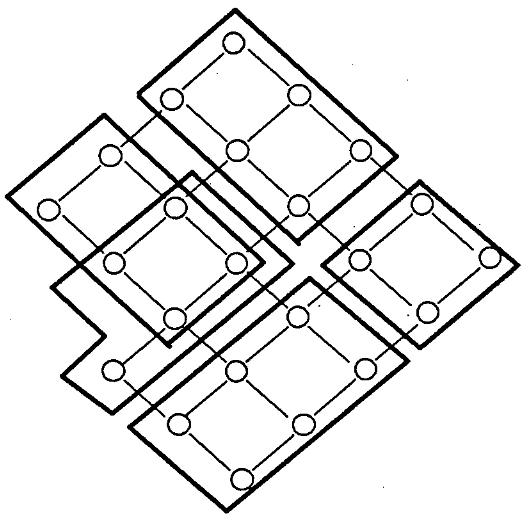

Diagram 6

<u>Note:</u> The conjecture referred to in this paper has in the meanwhile been answered in the negative by E. Fried and G. Grätzer[1].

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