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Isometric Euclidean Embeddings of a Compact Manifold form a Frêchet Manifold

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ISOMETRIC EUCLIDEAN EMBEDDINGS OF A COMPACT MANIFOLD FORM A FRÉCHET MANIFOLD

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Abstract: Let $E(M,\mathbb{R}^n)$ be the collection of all smooth embeddings of a compact smooth manifold M into $\mathbb{R}^n$. Given a fixed scalar product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$, the pull-back of $\langle\ ,\ \rangle$ by $j \in E(M,\mathbb{R}^n)$ is denoted by $m(j)$. We show that $m^{-1}(m(j))$ is a Fréchet manifold for any $j \in E(M,\mathbb{R}^n)$. This manifold is infinit dimensional if the codimension of M in $\mathbb{R}^n$ is large enough. The result links with Einstein's evolution equation and with elasticity theory.

Introduction: Throughout these notes M is a compact smooth manifold. The collection of all smooth embeddings of M into $\mathbb{R}^n$ is called $E(M,\mathbb{R}^n)$. This set is equipped with Whitney's C^∞-topology and since it is open in the Fréchet space $C^\infty(M,\mathbb{R}^n)$ (consisting of all smooth $\mathbb{R}^n$-valued maps of M and carrying Whitney's C^∞-topology) it is evidently a smooth Fréchet manifold.

Given a fixed scalar product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$ each $j \in E(M,\mathbb{R}^n)$ defines a Riemannian metric $m(j)$ on M, namely the pull-back of $\langle\ ,\ \rangle$ by j. It assigns to any two tangent vectors $v,w \in T_pM$ the real number $\langle djv,djw\rangle$ for any $p \in M$. Here dj denotes the $\mathbb{R}^n$-valued one form, called the differential of j, which locally represented is the Fréchet differential of j.

What we show is that $m^{-1}(m(i))$ the set of all $j \in E(M,\mathbb{R}^n)$ with $m(i) = m(j)$ for a fixed $i \in E(M,\mathbb{R}^n)$ is a Fréchet manifold when endowed with the C^∞-topology, a result which holds verbatim if $E(M,\mathbb{R}^n)$ is replaced by the collection of all smooth immersions from M into $\mathbb{R}^n$.

At this point let us refer to [Ja], who showed that $m^{-1}(m(i))$ admits isometric deformations. These beautiful investigations are based on an extension of Nash's implicit function theorem. Let us refer therefore to any $j \in m^{-1}(m(i))$ as an isometric deformation of i.

The method by which we establish the manifold structure of $m^{-1}(m(i))$ is the following one:
If j is near enough to i then its differential dj is represented by

$$dj = g \cdot di \cdot f.$$

Here $g \in C^\infty(M,SO(n))$ and f is a smooth strong bundle isomorphism of TM self-adjoint and positive definite (fibre-wise) with respect to m(i). If $v_p \in T_pM$ then the above equation means

$$dj\ v_p = g(p)(di\ f(v_p))$$

holding for any $p \in M$. Hence $j \in m^{-1}(m(i))$ iff $f = id_{TM}$. The parameter space of a chart at i in $m^{-1}(m(i))$ consists of the Fréchet space formed by all $h \in C^\infty(M,\mathbb{R}^n)$ for which the derivative $Dm(i)(h)$ of m at i in the direction of h vanishes. This means that

$$Dm(i)(h)(v_p,w_p) = \langle dh\ v_p, di\ w_p \rangle + \langle di\ v_p, dh\ w_p \rangle = 0$$

holds for all v_p, $w_p \in T_pM$ and all $p \in M$. Such an h is called an infinitesimal isometric deformation of i.

Let us point out that the calculus used in locally convex vector spaces is the one of [Gu] or [Mi]. In this context we refer via [Fr] also to the beautiful calculus of Frölicher and Kriegel.

Now dh of each $h \in C^\infty(M,\mathbb{R}^n)$ can be written as

$$dh = s \cdot di$$

with $s \in C^\infty(M,\mathrm{End}(\mathbb{R}^n))$. Hence $Dm(i)(h) = 0$ iff

$$\langle s \cdot di\ v_p, di\ w_p \rangle + \langle di\ v_p, s \cdot di\ w_p \rangle = 0$$

for all v_p, $w_p \in T_pM$ and all $p \in M$.
The construction of a chart at i in $m^{-1}(m(i))$ is based on the observation that if dh satisfies $Dm(i)(h) = 0$ then

$$(\exp \circ s) \cdot di$$

is the differential of an embedding $j \in m^{-1}(m(i))$ if $s \in C^\infty(M,\mathrm{End}(\mathbb{R}^n))$ is near enough to zero. exp is the usual exponential given on $\mathrm{End}(\mathbb{R}^n)$. Hence our construction of charts in $m^{-1}(m(i))$ is based on an integration scheme of infinitesimal Euclidean isometric deformations.

If the codimension of M is high enough then $m^{-1}(m(i))$ is not finite dimensional.

The fact that $m^{-1}(m(i))$ is a Fréchet manifold plays a crucial role if isometric deformations of embeddings have to be considered. Hence it links in particular to Einstein's evolution equation formulated on Euclidean immersions and to elasticity theory where the space of configurations consists of all embeddings of a body M into $\mathbb{R}^n$. In the first case the above result allows the rigid evolution given by an equation for infinitesimal isometric deformations to be realized by space-like sections of the four manifold. We refer to [Bi,1] and [Bi,2] as well as to the excellent book [Ma,Hu] for studies in those areas.

The fact that $m^{-1}(m(i))$ forms a Fréchet manifold for each $i \in E(M,\mathbb{R}^n)$ reveals a structure additional to the principal bundle structure of $E(M,\mathbb{R}^n)$ described in [Bi,Fi].

1) The space of embeddings $E(M,\mathbb{R}^n)$

$E(M,\mathbb{R}^n)$ denotes the collection of all smooth embeddings of a compact m-dimensional manifold M into $\mathbb{R}^n$. As shown e.g. in [Hi] this set is open in $C^\infty(M,\mathbb{R}^n)$ which is endowed with the C^∞-topology. Since $C^\infty(M,\mathbb{R}^n)$ is, due to the compactness of M, a complete metrizable locally convex topological vector space, a so called Fréchet space, $E(M,\mathbb{R}^n)$ is obviously a smooth Fréchet manifold.

Any $h \in C^\infty(M,\mathbb{R}^n)$ defines a smooth symmetric two tensor m(h) given by

$$m(h)(X,Y) = \langle dh\, X, dh\, Y \rangle$$

for all pairs $X,Y \in \Gamma TM$. By ΓTM we denote the collection of all smooth vector fields on M, a Fréchet space when endowed with the C^∞-topology. dh is the differential of h, which locally is given by the Fréchet derivative of h. Clearly the tangent map Th decomposes into $Th = (h,dh)$.

Evidently m(j) is a smooth Riemannian metric if $j \in E(M,\mathbb{R}^n)$.

If $S^2(M)$ stands for the collection of all smooth symmetric two tensors of M equipped with the C^∞-topology

$$m: C^\infty(M,\mathbb{R}^n) \longrightarrow S^2(M)$$

is a smooth map in the sense of [Gu] or [Mi] as a routine calculation shows (cf. [Sch]).

By $\mathfrak{M}(M)$ we denote the collection of all smooth Riemannian metrics endowed with the C^∞-topology. This set $\mathfrak{M}(M)$ is an open cone in $S^2(M)$ and hence a Fréchet manifold. Clearly $m(E(M,\mathbb{R}^n)) \subset \mathfrak{M}(M)$ and

$$m:E(M,\mathbb{R}^n) \longrightarrow \mathfrak{M}(M)$$

is a smooth map. It is generally not surjective as the theorem of Nash (cf. [Ja]) tells us.

The derivative of m at $j \in E(M,\mathbb{R}^n)$ in the direction of $h \in C^\infty(M,\mathbb{R}^n)$ is

$$Dm(j)(h)(X,Y) = \langle dh\,X, dj\,Y\rangle + \langle dj\,X, dh\,Y\rangle$$

for any two vector fields $X, Y \in \Gamma TM$.

2) The relative description of differentials of embeddings.

Any two $i,j \in E(M,\mathbb{R}^n)$ can be smoothly linked within $E(M,\mathbb{R}^n)$ only if j is in the connected component O_i of $i \in E(M,\mathbb{R}^n)$. Clearly any $E(M,\mathbb{R}^n)$-valued smooth curve σ of which the domain contains $[0,1]$ and which satisfies $\sigma(0) = i$ and $\sigma(1) = j$ is a homotopy. It is such a homotopy which allows us to describe dj of any $j \in O_i$ relatively to di. This description is done as follows:

Associated with $j \in E(M,\mathbb{R}^n)$ we have its Gauss map

$$\hat{j}:M \longrightarrow G(m,n)$$

mapping any $p \in M$ into $dj\, T_pM$ an element of the Grassmannian $G(m,n)$ consisting of all planes of dimension m in $\mathbb{R}^n$. This map $\hat{j}$ is smooth. On $G(m,n)$ we have two canonical bundles ξ and η namely the m-plane and the (n-m)-plane bundle respectively (cf. [G,H,V]). These two bundles add up to $G(m,n)\times\mathbb{R}^n$. Hence the Whitney sum of their pull-backs $\hat{j}^*\xi$ and $\hat{j}^*\eta$ with respect to the Gauss map is

$$\hat{j}^*\xi \oplus \hat{j}^*\eta \cong M \times \mathbb{R}^n.$$

Clearly $\hat{j}^*\xi \cong Tj\ TM$ and $\hat{j}^*\eta$ is isomorphic to $\nu(j)$, the normal bundle of $Tj\ TM$ in $\mathbb{R}^n\times\mathbb{R}^n$.
If thus $i, j \in O_i$ and therefore smoothly homotopic we have a smooth homotopy of their respective Gauss maps $\hat{i}$ and $\hat{j}$ which in turn yields a smooth strong bundle isomorphism (cf. [G,H,V])

$$\Psi: Ti\ TM \oplus \nu(i) \longrightarrow Tj\ TM \oplus \nu(j)$$

preserving the Whitney sums. Since moreover range and domain of Ψ are trivial bundles Ψ yields a smooth map

$$\varphi: M \longrightarrow GL(n)$$

such that

$$\varphi(p)(di\ v_p) = dj\ v_p$$

for all $v_p \in T_pM$ and all $p \in M$. We abbreviate the last equation by

$$dj = \varphi\cdot di.$$

This description of dj with respect to di however is not unique at all. To see this one chooses any strong smooth bundle isomorphism θ of $\nu(i)$ and extends it to all of $M\times\mathbb{R}^n$ by the requirement that $\theta|\hat{i}^*\xi = id$. Then we have:

$$\varphi\cdot di = (\varphi\cdot\theta)\cdot di$$

Decomposing φ polarly into $\varphi = g\cdot\bar{f}$ where $g(p) \in SO(n)$ and $\bar{f}(p)$ is self-adjoint with respect to $\langle\ ,\ \rangle$ for any choice of $p \in M$ implies

$$m(j)(v_p,w_p) = \langle g\cdot\bar{f}\cdot di\ v_p, g\cdot\bar{f}\cdot di\ w_p\rangle = \langle\bar{f}\cdot di\ v_p, \bar{f}\cdot di\ w_p\rangle$$

for all v_p, $w_p \in T_pM$. Representing $m(j)$ with respect to $m(i)$ by a smooth strong bundle isomorphism A of TM as

$$m(j)(v_p,w_p) = m(i)(A\ v_p,w_p)$$

provides us with $f := \sqrt{A}$, formed fibrewise. The smooth strong bundle isomorphism f of TM is evidently self-adjoint and positive definite with respect to $m(i)$. By what we have so far the following is easy to show:

Theorem 1: The differential dj of any $j \in O_i$ in the connected component O_i of a fixed $i \in E(M,\mathbb{R}^n)$ allows a description of the form

$$dj = g\cdot di\cdot f$$

with $g \in C^\infty(M,SO(n))$ and f being a smooth strong bundle isomorphism of TM fibrewise self-adjoint and positive definite with respect to $m(i)$. In this representation f is unique and determines the metric $m(j)$ via the equation

$$m(j)(v_p,w_p) = m(i)(f^2\ v_p,w_p)$$

holding for all v_p, $w_p \in T_pM$ and all $p \in M$. The factor g is not unique. However if $\bar{g}$: $M \longrightarrow End(\mathbb{R}^n)$ is given by

$$\bar{g}(p)_{|di\ T_pM} = g(p)_{|di\ T_pM} \qquad \text{and } \bar{g}_{|\nu(i)} = 0$$

then

$$dj = \bar{g}\cdot di\cdot f$$

is a unique representation of dj by di. If j and i vary smoothly, both $\bar{g}$ and f vary smoothly as well.

<u>Remark:</u> In general the exterior differential $\delta(di \cdot f)$ of the $\mathbb{R}^n$-valued one form $di \cdot f$ does not vanish. Thus g can be regarded as an "integrating factor" of $di \cdot f$.

We pause briefly to illustrate the above mechanism by looking at the covariant derivative of $m(j)$. If

$$P(j)\colon \mathbb{R}^n \times \mathbb{R}^n \longrightarrow TM$$

denotes the orthogonal projection along $\nu(j)$ followed by $(Tj)^{-1}$ then the covariant derivative of Levi-Cività of $m(j)$ is given by

$$\nabla(m(j))_X Y = P(j)(d(dj\ Y)(X)).$$

Hence for all $X, Y \in \Gamma TM$ the vector field $\nabla(m(j))_X Y$ is represented via $\nabla(m(i))$ by

$$\nabla(m(j))_X Y = f^{-1} \cdot \nabla(m(i))_X f \cdot Y + f^{-1} \cdot P(i) \cdot g^{-1} \cdot dg(X) \cdot di\ f \cdot Y.$$

Thus if we set

$$\nabla(g)_X Y = \nabla(m(i))_X Y + P(i) \cdot g^{-1} \cdot dg(X) \cdot di\ Y$$

then

$$\nabla(m(j))_X Y = f^{-1} \cdot \nabla(g)_X f \cdot Y$$

and thus its curvature $R(m(j))$ expressed by the curvature $R(g)$ of $\nabla(g)$ is

$$R(m(j))(X,Y)Z = f^{-1} \cdot R(g)(X,Y) f \cdot Z.$$

As it is easy to see g is in particular responsible for the vanishing of the torsion of the Levi-Cività connection $\nabla(m(j))$. For further remarks along these lines we refer to [Bi,Pe].

Next we investigate the infinitesimal situation and remind us of $T_iE(M,\mathbb{R}^n) = C^\infty(M,\mathbb{R}^n)$ for all $i \in E(M,\mathbb{R}^n)$. Given any $h \in C^\infty(M,\mathbb{R}^n)$ we choose a smooth curve σ defined on a neighbourhood of $0 \in \mathbb{R}$ with values in $E(M,\mathbb{R}^n)$ such that

$$\sigma(0) = i \text{ and } \dot\sigma(0) = h.$$

By the theorem above we have thus

$$d\sigma(t) = \bar{g}(t)\cdot di\cdot f(t)$$

and consequently

$$\frac{d}{dt}\, d\sigma(t)\Big|_{t=0} = d\left(\frac{d}{dt}\,\sigma(t)\Big|_{t=0}\right)$$

$$= dh$$

$$= \frac{d}{dt}\,\bar{g}(t)\Big|_{t=0}\cdot di + di\cdot\frac{d}{dt}\,f(t)\Big|_{t=0}.$$

We therefore can represent dh by

$$dh = a\cdot di + di\cdot b$$

with

$$a = \frac{d}{dt}\,\bar{g}(t)\Big|_{t=0} \quad \text{and } b = \frac{d}{dt}\,f(t)\Big|_{t=0} \;.$$

Let us point out that for all v_p, $w_p \in T_pM$ and $p \in M$

$$Dm(i)(h)(v_p,w_p) = \langle dh\, v_p, di\, w_p\rangle + \langle di\, v_p, dh\, w_p\rangle$$

$$= m(i)\left(\frac{d}{dt}\,f(t)\Big|_{t=0}\cdot v_p, w_p\right) + m(i)\left(v_p, \frac{d}{dt}\,f(t)\Big|_{t=0}\cdot w_p\right)$$

$$= m(i)(b\, v_p,w_p) + m(i)(v_p,b\, w_p)$$

implying

$$\langle a\cdot di, di\rangle + \langle di, a\cdot di\rangle = 0.$$

Clearly $a_{|\nu(i)} = 0$ and $b\colon TM \longrightarrow TM$ is self-adjoint with respect to $m(i)$. Let us investigate the above description of dh somewhat further.

There is a unique $C: TM \longrightarrow TM$, a smooth strong bundle endomorphism of TM (fibrewise) skew-adjoint with respect to $m(i)$ such that for all $v_p, w_p \in T_pM$

$$\langle a(p)(di\, v_p), di\, w_p\rangle + \langle di\, v_p, a(p)(di\, w_p)\rangle =$$

$$= m(i)(C(p)v_p, w_p) + m(i)(v_p, C(p)w_p).$$

Therefore we find a smooth $\mathrm{End}(\mathbb{R}^n)$-valued map

$$c: M \longrightarrow \mathrm{End}(\mathbb{R}^n)$$

such that

$$a(p)(di\, v_p) = c(p)(di\, v_p) + di\, C(p)v_p.$$

Without loss of generality we can assume that $c(M) \subset so(n)$ (where $so(n)$ is the Lie algebra of $SO(n)$).

We hence have furthermore

$$c(p)(di\, T_pM) \subset \nu(i)_p \text{ and } c(p)(\nu(i)_p) \subset di\, T_pM$$

for all $p \in M$. Here $\nu(i)_p$ denotes the normal of $di\, T_pM$. Thus c is uniquely determined by a and can be viewed as a vector field along $\hat{i}$. The following is thus easily verified:

Proposition 2: The differential dh of any $h \in C^\infty(M,\mathbb{R}^n)$ is uniquely represented by

$$dh = c \cdot di + di \cdot C + di \cdot b$$

where $i \in E(M,\mathbb{R}^n)$ is fixed. C and b are smooth strong bundle endomorphisms of TM which are skew- respectively self-adjoint with respect to m(i). For each $p \in M$ the map

$$c: M \longrightarrow so(n)$$

maps $di\, T_pM$ into $\nu(i)_p$ and vice versa. Hence c can be viewed as a vector field along the Gauss map $\hat{i}$. The maps c, C and b depend smoothly on h.

Corollary 3: The differential dh of any $h \in C^\infty(M,\mathbb{R}^n)$ can be represented via $s \in C^\infty(M,so(n))$ as

$$dh = s\cdot di \text{ where } s = a + \bar{C} + \bar{b}.$$

The maps $\bar{C}$ and $\bar{b}$ are given on di TM by

$$\bar{C}\cdot di = di\cdot C \text{ and } \bar{b}\cdot di = di\cdot b$$

respectively and are both supposed to vanish on $\nu(i)$.

3) An integration scheme.

As we saw in the previous section the differential of any $j \in O_i$ can be described by

$$dj = \varphi\cdot di \text{ with } \varphi \in C^\infty(M,GL(n)).$$

Similarly the differential dh of any $h \in C^\infty(M,\mathbb{R}^n)$ is given relatively to di by

$$dh = s\cdot di \text{ with } s \in C^\infty(M,End(\mathbb{R}^n)).$$

Let us call φ a di-factor of dj and s an infinitesimal di-factor of dh. In both cases we have not insisted on uniqueness in the description.

The question arises as to whether the di-factors can be computed via the infinitesimal ones. The answer will be prepared by the following:

<u>Lemma 4:</u> Given $h \in C^\infty(M,\mathbb{R}^n)$ of which the differential is given by

$$dh = s \cdot di$$

then for each natural number n

$$\delta(s^n \cdot di) = 0.$$

<u>Proof:</u> We form

$$F(s',j) := s' \cdot dj \text{ with } \partial F(s',j) = 0$$

where s' varies in $C^\infty(M,so(n))$ and j in O_i respectively. By proposition 2 we may assume that F depends smoothly on its variables. The total derivative od F at (s,i) in the direction of $s_1 \in C^\infty(M,so(n))$ and $k \in C^\infty(M,\mathbb{R}^n)$ is

$$DF(s,i)(s_1,k) = s_1 \cdot di + s \cdot dk = s_1 \cdot di + s \cdot s_2 \cdot di$$

where we set $dk = s_2 \cdot di$. We demand that $\partial(s_1 \cdot di) = 0$. Hence $\partial((s \cdot s_2)di) = 0$. A simple induction on the power r completes the proof.□

Now one immediately deduces the following:

<u>Corollary 5:</u> Given any $h \in C^\infty(M,\mathbb{R}^n)$ of which the differential dh is represented as

$$dh = s \cdot di$$

with $s \in C^\infty(M, \mathrm{End}(\mathbb{R}^n))$ the $\mathbb{R}^n$-valued one form

$$(\exp\circ s)\cdot di$$

is the differential of some smooth $\mathbb{R}^n$-valued immersion on M and is the differential of some $j \in O_1$, provided that s is near enough to zero.

Proof: By proposition 4 we know that $\delta((\exp\circ s)\cdot di) = 0$ showing that $(\exp\circ s)\cdot di$ can be regarded as the differential of an $\mathbb{R}^n$-valued smooth immersion $j(s)$ defined on the universal covering $\tilde{M}$ of M. Let

$$\pi\colon \tilde{M} \longrightarrow M$$

be the canonical projection. Then if t varies in $\mathbb{R}$

$$(j(t\cdot s) - i\circ\pi)\colon \tilde{M} \longrightarrow \mathbb{R}^n$$

is a smoothly parametrized family of maps with

$$dj(0) = di\circ T\pi.$$

Hence $dj(t\cdot s)$ (regarded as an $\mathbb{R}^n$-valued one form on M) and di belong to the same cohomology class in

$$H^1(M,\mathbb{R}^n) = H^1(M,\mathbb{R}) \otimes \mathbb{R}^n$$

showing that $j(s)$ factors to M and hence is an immersion of M. If s is near enough to zero then $j(s) \in O_1$.□

4) $m^{-1}(m(i))$ as a manifold.

The purpose of the next few developments is to show that given $i \in E(M,\mathbb{R}^n)$ the set $m^{-1}(m(i))$ is a Fréchet manifold when regarded as a topological subspace of $E(M,\mathbb{R}^n)$.
First we consider $S \subset C^\infty(M,\mathbb{R}^n)$ and identify $S/\mathbb{R}^n$ with the

set $\{dh/\ h \in S\}$, which carries hence the quotient topology determined by S.

Due to proposition 2 and corollary 5 there is a map

$$\Omega_{exp}: C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n} \longrightarrow C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n}$$

given by

$$\Omega_{exp}(dh) = (\exp\circ s)\cdot di$$

where dh is uniquely represented as $s\cdot di$ with

$$s = c + \bar{c} + \bar{b}.$$

(Both $\bar{c}$ and $\bar{b}$ are as in corollary 3). The smoothness of this map follows by the so called Ω-lemma (cf. [Gu] or [Mi]).

Due to the continuity of Ω_{exp} we find an open neighbourhood $U \subset C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n}$ of zero for which

$$\Omega_{exp}(U) \subset O_i/_{\mathbb{R}^n}.$$

For $dh \in U$ we immediately deduce

$$\Omega_{exp}(dh) \subset m^{-1}(m(i))/_{\mathbb{R}^n} \text{ iff } Dm(i)(h) = 0$$

Thus we have the first part of

Proposition 6: There is an open neighbourhood

$$W \subset \{dh/\ Dm(i)(h) = 0\}$$

such that

$$\Omega_{exp}(W) \subset m^{-1}(m(i))/_{\mathbb{R}^n}$$

is an open neighbourhood of $di \in m^{-1}(m(i))/_{\mathbb{R}^n}$. If W is small enough, then

$$\Omega_{\exp|W}: W \longrightarrow m^{-1}(m(i))/\mathbb{R}^n$$

is a homeomorphism onto $\Omega_{\exp}(W)$.

To verify the last part of proposition 6 we have to choose W that small that the infinitesimal di-factor s of each dh ϵ W maps M into the domain of injectivity of

$$\exp: so(n) \longrightarrow SO(n)$$

which is possible, due to the continuity of $\Omega_{\exp}$.

Moreover the following is now evident:

Corollary 7: Given i ϵ $E(M,\mathbb{R}^n)$ then $m^{-1}(m(i))/\mathbb{R}^n$ is a smooth Fréchet manifold. An atlas is formed by taking for each dj ϵ $m^{-1}(m(i))/\mathbb{R}^n$ a suitable small open neighbourhood W in

$$T_{dj}(m^{-1}(m(i))/\mathbb{R}^n) := \{dh/\ Dm(j)(h) = 0\}$$

as a chart at dj and $\Omega_{\exp|W}$ as a chart map .

Finally our main theorem is immediate:

Theorem 8: Given i ϵ $E(M,\mathbb{R}^n)$ the set $m^{-1}(m(i))$ equipped with Whitney's C^∞-topology is a smooth Fréchet manifold.

Remarks:

i) The manifold $m^{-1}(m(i))$ is infinit dimensional if n is large enough. To see this let $n = n' + r$ for some natural number n'. If i ϵ $E(M,\mathbb{R}^{n'})$ then i ϵ $E(M,\mathbb{R}^{n'+r})$ and any h ϵ $C^\infty(M,\mathbb{R}^r)$ satisfies

$$Dm(i)(h) = 0.$$

Hence $m^{-1}(m(i))$ is infinite dimensional.

ii) On the other hand if the codimension of M is small enough and $i(M) \subset \mathbb{R}^n$ satisfies some additional geometric conditions then $m^{-1}(m(i))$ is diffeomorphic to the Euclidean group of $\mathbb{R}^n$ (cf. [B,B,G]).

iii) If $E(M,\mathbb{R}^n)$ is replaced by $I(M,\mathbb{R}^n)$, the collection of all smooth immersions of M into $\mathbb{R}^n$, and if m means the map

$$m: I(M,\mathbb{R}^n) \longrightarrow \mathfrak{M}(M)$$

given by restricting $m: C^\infty(M,\mathbb{R}^n) \longrightarrow S^2(M)$ to $I(M,\mathbb{R}^n)$, then $m^{-1}(m(i))$ is a Fréchet manifold as well. The proof follows exactly the same lines except for some (minor) simplifications.

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