A Nonlinear Theorem of the Alternative
without Regularity Assumption

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Abstract: A nonlinear theorem of the alternative is proposed which needs no regularity assumption. Several equivalent formulations are derived under various additional hypotheses.

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1. Introduction and Problem Statement

Throughout this paper let us make the following *assumptions*:

> Y and Z are real topological vector spaces, with Y being locally convex;
> $P \subset Y$ and $Q \subset Z$ are nonvoid convex cones with P closed and $\operatorname{int} Q \neq \emptyset$[1];
> $S \subset Y \times Z$ is a nonvoid convex set, and
> $S_Y := \{ y \in Y \mid (y,z) \in S,\ z \in Z \}$.

We want to obtain a necessary and sufficient condition of Lagrangean type for the non-existence of a solution $(y,z) \in Y \times Z$ of the following system

(1) $\qquad (y,z) \in S, \quad y \in -P, \quad z \in \operatorname{int}(-Q).$

Such a characterization, which should not require any additional regularity hypotheses, is desirable, since many notions of optimality, efficiency, or infeasibility reduce to the inconsistency of a system like (1). For any real topological vector space E let us denote by E* the continuous dual. For $x \in E$ and $x^* \in E^*$ we write $\langle x^*, x\rangle$ instead of $x^*(x)$, and if $K \subset E$ is a convex cone, we denote the polar cone of K by

$$K^* := \{ x^* \in E^* \mid \langle x^*, x\rangle \geq 0 \quad \forall x \in K \} .$$

Then the classical Lagrangean condition concerning the inconsistency of (1) may be formulated as follows: There exists $(y^*,z^*) \in Y^* \times Z^*$ such that

(2) $\qquad y^* \in P^*,\ z^* \in Q^*,\ z^* \neq 0,\ 0 \leq \langle y^*,y\rangle + \langle z^*,z\rangle \quad \forall\, (y,z) \in S.$

It is obvious that the consistency of (2) is a sufficient condition for the inconsistency of (1). Indeed, (1) and (2) cannot have solutions at the same time (note that $z \in \operatorname{int}(-Q)$ and $z^* \in Q^*$, $z^* \neq 0$ imply $\langle z^*,z\rangle < 0$). However, the consistency of (2) is not a necessary condition for the inconsistency of (1), unless an additional regularity assumption is imposed. A classical example of such a regularity condition is the following:

(3) $\qquad \operatorname{int} P \neq \emptyset$, and $0_Y \in \operatorname{int}(S_Y + P)$.

Under (3) it can be shown that (2) has a solution if (1) is inconsistent. Hence, under the regularity assumption (3) we have a *theorem of the alternative*: Of the two systems (1) and (2) one, and only one, has a solution.

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1) Throughout the paper, "int" denotes the interior, "cl" the closure, and "conv" the convex hull.

Theorems of the alternative furnish a convenient tool to derive optimality conditions for many types of optimization problems. We refer the reader to [2], [3] for a comprehensive bibliography of theorems of the alternative in connection with optimality conditions. Unfortunately, assumption (3) is too strong for many purposes. So, there have been numerous attempts to weaken this assumption. In this note we want to propose a necessary and sufficient condition for the inconsistency of (1), which is close in form to (2) and which needs no regularity assumption at all. This condition is given in Theorem 1 below. Moreover, under the classical assumption (3) this condition is readily shown to be equivalent to the classical Lagrangean condition (2). This equivalence is established in Theorem 3. In the linear case, i.e., if $S := (A\times B)(X)$, where X is a locally convex topological vector space and $A : X \to Y$, $B : X \to Z$ are continuous linear mappings, our condition is equivalent to the following statement: There exists $z^* \in Z^*$ such that

$$z^* \in Q^*, \quad z^* \neq 0, \quad -B^*z^* \in \text{weak*-cl } A^*(P^*)$$

(where A^*, B^* are the adjoints of A and B). A similar equivalence holds in the affine case and is established in Theorem 4.

2. The general case

We turn now to the proposed theorem of the alternative. The relevant assumptions on Y, Z, P, Q and S have been collected at the beginning of Section 1.

Theorem 1. The following two statements are equivalent:

(i) System (1) has no solution.

(ii) There exists $(z^*,t^*) \in Z^* \times \mathbb{R}$ such that

$$(4)\quad \begin{cases} z^* \in Q^*,\ t^* \geq 0,\ (z^*,t^*) \neq 0; \\ \text{for all } \varepsilon > 0 \text{ and for all finite subsets } \mathcal{W} \subset S \\ \text{there exists } y^* \in P^* \text{ such that} \\ t^* - \varepsilon \leq \langle y^*,y\rangle + \langle z^*,z\rangle \quad \forall\, (y,z) \in \mathcal{W}. \end{cases}$$

Proof. a) Let $(z^*,t^*) \in Z^* \times \mathbb{R}$ satisfy (4). From (4) it follows in particular that $t^* \leq \langle z^*,z\rangle$ for all $(y,z) \in S$ with $y \in -P$. Let $(\bar{y},\bar{z})$ be a solution of (1). Then $t^* - \langle z^*,\bar{z}\rangle \leq 0$. But here both terms on the left hand side are nonnegative, and from $(z^*,t^*) \neq 0$ at least one is positive, a contradiction, i.e., (1) has no solution.

b) For the converse implication let (1) have no solution. We have to consider two different cases.

Case 1: $0 \notin S_Y + P$. In this case for any finite subset $\mathcal{Y} \subset S_Y$ the compact convex set conv $\mathcal{Y}$ is disjoint from the closed convex cone $-P$. Since Y is locally convex, the strong separation theorem [9, p.65] is applicable and yields $y^* \in P^*$ and $k > 0$ such that $\langle y^*,y\rangle \geq k \ \ \forall\, y \in \operatorname{conv} \mathcal{Y}$. Since (y^*,k) can be normalized such that $k = 1$, (4) is satisfied with $z^* = 0$, $t^* = 1$.

Case 2: $0 \in S_Y + P$. In this case the convex set $V := \{z \in Z \mid (y,z) \in S,\ y \in -P\}$ is non-empty, and since (1) has no solution, V is disjoint from the convex cone int $(-Q)$. The weak separation theorem [9, p.64] yields $z^* \in Q^* \setminus \{0\}$ such that $\langle z^*,z\rangle \geq 0 \ \ \forall\, z \in V$. It follows that the system

$$(y,z) \in S, \quad y \in -P, \quad \langle z^*,z\rangle < 0$$

has no solution. Fix $\mathcal{W} := \{(y_i,z_i) \mid i = 1,\ldots,n\}$, a finite subset of S, and $\varepsilon > 0$. Choose $(y_o,z_o) \in S$ with $y_o \in -P$ (which is possible due to the hypothesis of case 2). Set $t_i := \langle z^*,z_i\rangle \quad (i = 0,1\ldots,n)$, and $\mathcal{V} := \operatorname{conv}\{(y_i,t_i) \mid i = 0,1,\ldots,n\} \subset Y \times \mathbb{R}$. It follows that the system

$$(y,t) \in \mathcal{V}, \quad y \in -P, \quad t \leq -\varepsilon$$

has no solution either. Hence the compact convex set $\mathcal{V} + (0,\varepsilon)$ is disjoint from the closed convex cone $-P \times \mathbb{R}_-$, and the strong separation theorem yields $(y^*,\tau^*) \in P^* \times \mathbb{R}_+$ such that $\langle y^*,y\rangle + \tau^*(t+\varepsilon) > 0 \quad \forall (y,t) \in \mathcal{V}$. If $\tau^* = 0$, then in particular $\langle y^*,y_o\rangle > 0$, contradicting the fact that $y^* \in P^*$ and $y_o \in -P$. Consequently $\tau^* > 0$, and we normalize (y^*,τ^*) in such a way that $\tau^* = 1$. We obtain $\langle y^*,y\rangle + t \geq -\varepsilon \; \forall (y,t) \in \mathcal{V}$, hence in particular

$$\langle y^*,y\rangle + \langle z^*,z\rangle \geq -\varepsilon \quad \forall (y,z) \in \mathcal{W},$$

i.e., (4) is satisfied with $z^* \neq 0$, $t^* = 0$. q.e.d.

Remarks

1) Let $0 \in S_Y + P$ (i.e., let there exist $(y,z) \in S$ with $y \in -P$). Then it is obvious that (4) cannot be satisfied otherwise than with $z^* \neq 0$. Hence in this case we may replace the condition $(z^*,t^*) \neq 0$ by $z^* \neq 0$ in (4), and Theorem 1 continues to hold. Once $z^* \neq 0$ is ensured, we may of course set $t^* = 0$ as well in (4).

2) $0 \in S_Y + P$ (i.e., there does not exist $(y,z) \in S$ with $y \in -P$) if and only if there exists $t^* > 0$ such that (4) is satisfied with $z^* = 0$. The necessity of this condition has been shown in part b), case 1 of the proof of Theorem 1; the sufficiency is obvious.

3) Theorem 1 also gives a necessary and sufficient condition for the inconsistency of the system

$$x \in C, \qquad f(x) \cap (-P) \neq \emptyset, \qquad g(x) \cap \operatorname{int}(-Q) \neq \emptyset,$$

where C is a convex set and $f : C \rightrightarrows Y$, $g : C \rightrightarrows Z$ are multivalued mappings. For this purpose we have to assume that f is P-convex (which means $\{(x,y) \in C \times Y \mid y \in f(x) + P\}$ has to be convex) and that g is Q-convex (which means $\{(x,z) \in C \times Z \mid z \in g(x) + Q\}$ has to be convex). We set

$$S := (f \times g)(C) + (P \times Q).$$

Then S is convex. Since $P + P = P$ and $Q + \operatorname{int} Q = \operatorname{int} Q$, the inconsistency of the above system is then equivalent to the inconsistency of (1). It is easily seen that in this case in (4) we can replace the finite subsets $\mathcal{W} \subset S$ by the finite subsets $\mathcal{W} \subset (f \times g)(C)$, and Theorem 1 continues to hold. In particular, if f and g are single-valued, then it is enough to consider the sets $\mathcal{W} := (f \times g)(\mathcal{X})$, where $\mathcal{X}$ runs over all finite subsets of C.

4) If we assume that $Q \neq Z$, then we can replace $t^* \geq 0$ by $t^* = 0$ in condition (4) of Theorem 1. We only have to verify the necessity of this modified condition, for which we give an alternative *proof*:

Like in case 2 of the proof above we obtain $z^* \in Q^* \setminus \{0\}$ such that the system

$$(y,z) \in S, \quad y \in -P, \quad \langle z^*, z\rangle < 0$$

has no solution (if V is empty, then any $z^* \in Q^* \setminus \{0\}$ will do). Now let $\mathcal{W} := \{(y_i, z_i) \mid i = 1,\dots,n\} \subseteq S$ be finite. Then with $c := (c_1,\dots,c_n)$, where $c_i := \langle z^*, z_i\rangle$ $(i = 1,\dots,n)$, and with the continuous linear mapping $A : \mathbb{R}^n \to Y$ defined by $A u := \sum_{i=1}^{n} u_i y_i$, the system

$$(*) \qquad u \in \mathbb{R}^n, \quad u \geq 0, \quad A u \in -P, \quad \langle c, u\rangle < 0$$

has no solution either (because if u solves (*), then we may normalize $\Sigma_i u_i = 1$, so that $(y,z) := \Sigma_i u_i (y_i, z_i)$ solves the previous system). With A^* denoting the adjoint of A this implies that

$$(**) \qquad -c \in \mathrm{cl}(A^*P^* - \mathbb{R}^n_+).$$

Otherwise, the strong separation theorem would yield $u \in \mathbb{R}^n$ such that

$$0 < \langle -c, u\rangle,$$
$$0 \geq \langle A^*y^* - w, u\rangle = \langle y^*, A u\rangle - \langle w, u\rangle \qquad \forall\, y^* \in P^*, \quad \forall\, w \in \mathbb{R}^n_+,$$

the latter inequality implying in particular that $u \geq 0$ and, since P is closed and Y is locally convex, $A u \in -P^{**} = -P$. Hence, altogether, u would solve (*), a contradiction. Now, if $\mathcal{U}$ denotes the set of all unit vectors in $\mathbb{R}^n$, it follows from (**) that for all $\varepsilon > 0$ there exists $y^* \in P^*$ and $w \in \mathbb{R}^n_+$ such that for all $u \in \mathcal{U}$, $|\langle -c - A^*y^* + w, u\rangle| \leq \varepsilon$. Hence, since $w \geq 0$, $-\varepsilon \leq \langle y^*, A u\rangle + \langle c, u\rangle$ for all $u \in \mathcal{U}$, i.e., $-\varepsilon \leq \langle y^*, y_i\rangle + \langle z^*, z_i\rangle$ $(i = 1,\dots,n)$. q.e.d.

In particular, if $Y = \mathbb{R}^n$ and $P = P^* = \mathbb{R}^n_+$, then $A^*P^* - \mathbb{R}^n_+$ is a finitely generated cone, and therefore closed. In this case the closure in (**) may be omitted, and condition (4) remains even true with $\varepsilon = 0$.

3. Equivalent formulations

In this section we establish, under additional regularity assumptions, two equivalent - and more familiar - versions of statement (ii) in Theorem 1. Our overall assumption, stated in the introduction, remains in force.

Theorem 2. Assume that there exists a finite subset $\mathcal{Y} \subset S_Y$ such that

(5) $\quad 0 \in \operatorname{int}\,(\operatorname{conv} \mathcal{Y} + P)$.

Then the following statements are equivalent:

(i) There exists $(z^*,t^*) \in Z^* \times \mathbb{R}$ satisfying (4).

(ii) There exists $(y^*,z^*) \in Y^* \times Z^*$ satisfying (2).

Proof. Obviously, if (y^*,z^*) satisfies (2), then z^* and $t^* := 0$ satisfy (4). Conversely, let (z^*,t^*) satisfy (4). From (5) it follows that $0 \in S_Y + P$, hence $z^* \neq 0$ (see remark 1 following Theorem 1). Choose a finite subset $\mathcal{W}^o \subset S$ such that $\{y \in Y \mid (y,z) \in \mathcal{W}^o,\ z \in Z\} = \mathcal{Y}$. We consider the family of sets

$$Y^*(\mathcal{W},\varepsilon) := \{y^* \in P^* \mid -\varepsilon \leq \langle y^*,y\rangle + \langle z^*,z\rangle \quad \forall\, (y,z) \in \mathcal{W}\},$$

where $\mathcal{W}^o \subset \mathcal{W} \subset S$, $\mathcal{W}$ finite, and $0 < \varepsilon \leq 1$. From (4) all sets $Y^*(\mathcal{W},\varepsilon)$ are nonempty, and this implies then also that any finite collection of these sets has nonempty intersection. The sets $Y^*(\mathcal{W},\varepsilon)$ are clearly weak*-closed, and they are contained in the set

$$K := \{y^* \in Y^* \mid -1-\alpha \leq \langle y^*,y\rangle \quad \forall\, y \in \operatorname{conv} \mathcal{Y} + P\},$$

where $\alpha := \max\,\{\langle z^*,z\rangle \mid (y,z) \in \mathcal{W}^o\}$. By assumption (5) $\operatorname{conv}\mathcal{Y} + P$ is a neighborhood of the origin, and K - as a polar of this neighborhood - is then weak*-compact from Alaoglu's Theorem [6, p.70]. It follows from these facts that the entire family of the sets $Y^*(\mathcal{W},\varepsilon)$ has nonempty intersection, i.e., there exists $y^* \in Y^*$ such that

$$y^* \in \bigcap\,\{Y^*(\mathcal{W},\varepsilon) \mid \mathcal{W}^o \subset \mathcal{W} \subset S,\ \mathcal{W} \text{ finite},\ 0 < \varepsilon \leq 1\}.$$

This y^* together with z^* satisfies (2). q.e.d.

Remark. If $\operatorname{int} P \neq \emptyset$ or if Y is finite-dimensional, then

(6) $\quad 0 \in \operatorname{int}(S_Y + P)$

is a sufficient condition for the existence of $\mathcal{Y}$ satisfying (5).

Proof. a) Assume that $\operatorname{int} P \neq \emptyset$, and let (6) hold. Then $(S_Y + P) \cap \operatorname{int}(-P) \neq \emptyset$, and therefore $0 \in (S_Y + P) + \operatorname{int} P = S_Y + \operatorname{int} P$. Choose $y^o \in S_Y$ such that $0 \in y^o + \operatorname{int} P$. Then (5) is satisfied with $\mathcal{Y} := \{y^o\}$. b) Assume that Y is finite-dimensional, and let (6) hold. Then there exists a finite subset $\mathcal{V} \subset S_Y + P$ such that $0 \in \operatorname{int} \operatorname{conv} \mathcal{V}$, and obviously we can find $\mathcal{Y} \subset S_Y$ finite such that $\operatorname{conv} \mathcal{V} \subset \operatorname{conv} \mathcal{Y} + P$. $\mathcal{Y}$ fulfills (5). q.e.d.

Besides regularity assumptions of interior point type - such as (5) - regularity assumptions of closedness type are equally important. With regard to the latter we show the equivalence of statement (ii) in Theorem 1 with a condition which has been established in [8] under the additional hypothesis that S is closed and Z locally convex. For this purpose we adjoin to the convex set $S \subset Y \times Z$ the convex cone $K_S \subset Y \times Z \times \mathbb{R}$ given by

$$K_S := \{ t \cdot (y,z,1) \mid (y,z) \in S,\ t \geq 0 \} .$$

Let K_S^* be the polar cone of K_S. It is easily seen that

$$(y^*,z^*,t^*) \in K_S^* \iff \langle y^*,y\rangle + \langle z^*,z\rangle + t^* \geq 0 \quad \forall\, (y,z) \in S.$$

If S is closed, then $(y,z) \in S \iff (y,z,1) \in \operatorname{cl} K_S$. With the overall assumption still presupposed we have the following result.

<u>Theorem 3</u>. Let S be closed and let Z be locally convex. Then the following statements are equivalent:

(i) There exists $(z^*,t^*) \in Z^* \times \mathbb{R}$ satisfying (4).

(ii) There exists $(z^*,t^*) \in Z^* \times \mathbb{R}$ satisfying

$$(7) \qquad \begin{cases} z^* \in Q^*,\ t^* \geq 0,\ (z^*,t^*) \neq 0, \\ (0,-z^*,t^*) \in \text{weak*-cl}\ \Delta, \end{cases}$$

where $\Delta := -K_S^* + (P^* \times \{0_{Z^*}\} \times \{0_{\mathbb{R}}\}) \subset Y^* \times Z^* \times \mathbb{R}$.

Proof. a) Let (z^*,t^*) satisfy (7). Let $\mathcal{W}$ be an arbitrary finite subset of S, and let $\varepsilon > 0$. Then $(0,-z^*,t^*) \in \text{weak*-cl}\ \Delta$ implies that there exist $(\eta^*,\zeta^*,\tau^*) \in -K_S^*$ and $y^* \in P^*$ such that

$$\left| \langle \eta^* + y^*,y\rangle + \langle \zeta^* + z^*,z\rangle + \tau^* - t^* \right| \leq \varepsilon \quad \forall\, (y,z) \in \mathcal{W}.$$

Since $\langle \eta^*,y\rangle + \langle \zeta^*,z\rangle + \tau^* \leq 0 \ \forall\, (y,z) \in S$, it follows that

$$- \langle y^*,y\rangle - \langle z^*,z\rangle + t^* \leq \varepsilon \quad \forall\, (y,z) \in \mathcal{W},$$

i.e., (z^*,t^*) satisfies (4).

b) Let (z^*,t^*) satisfy (4). Assume that (z^*,t^*) does not satisfy (7), i.e., $(0,-z^*,t^*) \notin \text{weak*-cl}\ \Delta$. Then the strong separation theorem provides $(y,z,t) \in Y \times Z \times \mathbb{R}$ such that $(y,z,t) \in -\Delta^*$ and $\langle -z^*, z\rangle + t^*t > 0$. $(y,z,t) \in -\Delta^*$ implies that $(y,z,t) \in K_S^{**} = \text{cl}\ K_S$ and $y \in -P^{**} = -P$. By definition of K_S we have $t \geq 0$. Let us first consider the case $t > 0$. Normalizing (y,z,t) such that $t = 1$ we obtain $(y,z,1) \in \text{cl}\ K_S$, and thereby, since S is closed, $(y,z) \in S$. Moreover $y \in -P$ and $t^* > \langle z^*, z\rangle$. But (4) implies that $t^* \leq \langle z^*, z\rangle$ for all $(y,z) \in S$ with $y \in -P$. Hence we have obtained a contradiction. In case $t = 0$ we have $0 > \langle z^*, z\rangle$, $(y,z,0) \in \text{cl}\ K_S$, $y \in -P$. Assume that there exists $(y^o, z^o, 1) \in \text{cl}\ K_S$ with $y^o \in -P$. Then for all $r \geq 0$ sufficiently large it follows that $(y^o + ry, z^o + rz) \in S$, $y^o + ry \in -P$, $t^* > \langle z^*, z^o + rz\rangle$; this contradicts (4) again. If there is no $(y^o, z^o, 1) \in \text{cl}\ K_S$ with $y^o \in -P$, then $z^* = 0$, $t^* = 1$ meets the requirements of (7). Otherwise, if $(0,0,1) \notin \text{weak*-cl}\ \Delta$, then as before the strong separation theorem provides $(y,z,t) \in Y \times Z \times \mathbb{R}$ such that $(y,z,t) \in \text{cl}\ K_S$, $y \in -P$ and $t > 0$. This gives a contradiction to the non-existence of $(y^o, z^o, 1)$ as above. So altogether we have obtained that (7) must be satisfied. q.e.d.

Remark. If Δ is weak*-closed, then statement (ii) in Theorem 4 is easily seen to be equivalent to the following: There exists $(y^*, z^*, t^*) \in Y^* \times Z^* \times \mathbb{R}$ such that

$$y^* \in P^*,\ z^* \in Q^*,\ t^* \geq 0,\ (z^*, t^*) \neq 0,$$
$$t^* \leq \langle y^*, y\rangle + \langle z^*, z\rangle \quad \forall\ (y,z) \in S.$$

By a result of Dieudonné ([1], see also [4, p.80]) the cone $\Delta = -K_S^* + (P^* \times \{0_{Z^*}\} \times \{0_{\mathbb{R}}\})$ is closed, if P^* is locally weak*-compact and $K_S^* \cap (P^* \times \{0_{Z^*}\} \times \{0_{\mathbb{R}}\})$ is a linear subspace. The latter condition amounts to the requirement that $\{y^* \in P^* \mid 0 \leq \langle y^*, y\rangle\ \ \forall y \in S_Y\}$ is a linear subspace of Y^*.

4. The affine case

In this section we specialize Theorem 1 to the affine case, i.e., we assume that

$$S := (A \times B)(X) - (a,b),$$

where

X is a real locally convex topological vector space,
$A : X \to Y$ and $B : X \to Z$ are continuous linear mappings,
$(a,b) \in Y \times Z$ is fixed.

The assumptions concerning Y, Z, P, Q remain as before. With these specifications system (1) becomes then

$$(8) \qquad x \in X, \quad Ax - a \in -P, \quad Bx - b \in \operatorname{int}(-Q),$$

and condition (4) becomes

$$(9) \qquad \begin{cases} z^* \in Q^*, \ t^* \geq 0, \ (z^*,t^*) \neq 0; \\ \text{for all } \varepsilon > 0 \text{ and all finite subsets } \mathfrak{X} \subset X \\ \text{there exists } y^* \in P^* \text{ such that} \\ t^* - \varepsilon \leq \langle y^*, Ax-a\rangle + \langle z^*, Bx-b\rangle \quad \forall x \in \mathfrak{X}. \end{cases}$$

From Theorem 1 we know that (8) has no solution if, and only if, there exists $(z^*,t^*) \in Z^* \times \mathbb{R}$ satisfying (9). Let $A^* : Y^* \to X^*$ and $B^* : Z^* \to X^*$ denote the adjoint mappings of A and B. Then we obtain the following equivalent characterization.

Theorem 4. The following statements are equivalent:
(i) There exists $(z^*,t^*) \in Z^* \times \mathbb{R}$ satisfying (9).
(ii) There exists $(z^*,t^*) \in Z^* \times \mathbb{R}$ satisfying

$$(10) \qquad \begin{cases} z^* \in Q^*, \ t^* \geq 0, \ (z^*,t^*) \neq 0; \\ (-B^*z^*, \ \langle b,z^*\rangle + t^*) \in \text{weak*-cl}\ \Gamma, \end{cases}$$

where $\Gamma := \{ (A^*y^*, \langle -a,y^*\rangle) \mid y^* \in P^* \} \subset X^* \times \mathbb{R}$.

Proof. Obviously, (10) implies (9); this is easily seen by spelling out the condition for being an element of the weak*-closure as in the proof of the preceding theorem. For the converse implication let $(z^*,t^*) \in Z^* \times \mathbb{R}$ satisfy (9).
Case 1: There exists $x^o \in X$, $Ax^o - a \in -P$. In this case we shall construct a suitable $\tau^* \in \mathbb{R}$ such that (z^*,τ^*) satisfies (10). From (9) follows that the system

$$x \in X, \quad Ax - a \in -P, \quad \langle z^*, Bx-b\rangle < \tau$$

has no solution for $\tau \leq t^*$. But it does have a solution for $\tau > \langle z^*, Bx^o - b\rangle$. Let τ^* be the infimum of all τ such that this system has a solution. Then $t^* \leq \tau^* < \infty$, hence $\tau^* \geq 0$ and $(z^*,\tau^*) \neq 0$. Moreover, by the definition of τ^* the system

$$(*) \qquad x \in X, \quad Ax - a \in -P, \quad \langle z^*, Bx-b\rangle < \tau^*$$

has no solution. Assume now that (z^*,τ^*) does not satisfy (10). Then $(-B^*z^*, \langle b,z^*\rangle + \tau^*) \notin$ weak*-cl Γ. By the strong separation theorem we obtain $(x,t) \in X \times \mathbb{R}$ such that

$$0 \geq \langle A^*y^*, x\rangle + \langle -a, y^*\rangle \cdot t = \langle y^*, Ax-at\rangle \quad \forall y^* \in P^*,$$
$$0 < \langle -B^*z^*, x\rangle + (\langle b, z^*\rangle + \tau^*) \cdot t = -\langle z^*, Bx-bt\rangle + \tau^* t.$$

Since $P^{**} = P$ this implies

$$Ax - at \in -P, \quad \langle z^*, Bx-bt\rangle < \tau^* t.$$

Clearly it is enough to consider the cases $t=1$, $t=0$, and $t=-1$. If $t=1$, then x solves (*), a contradiction. If $t=0$, then for all $r>0$ large enough it follows that $x^o + rx$ solves (*), again a contradiction. If $t=-1$, then there exists $\varepsilon > 0$ such that $Ax + a \in -P$, $\langle z^*, Bx+b\rangle < -\tau^* - \varepsilon$. Then by the definition of τ^* there exists $x^1 \in X$ such that $Ax^1 - a \in -P$, $\langle z^*, Bx^1 - b\rangle < \tau^* + \varepsilon$. It follows for all $r > 0$ large enough that $x^1 + r(x+x^1)$ solves (*), once more a contradiction. So (z^*,τ^*) satisfies (10).

Case 2: $x \in X$, $Ax - a \in -P$ has no solution. In this case (10) is satisfied for $z^* = 0$, $t^* = 1$. Otherwise we have $(0,1) \notin$ weak*-cl Γ, and similar to case 1 the strong separation theorem gives $(x,t) \in X \times \mathbb{R}$ such that $Ax - at \in -P$ and $t > 0$. This contradicts the hypothesis of case 2. q.e.d.

If $Y = \mathbb{R}^n$ and $P = P^* = \mathbb{R}^n_+$, then Γ is weak*-closed, and (10) simplifies in the same way as indicated in the remark following Theorem 3.

5. Characterization of weakly efficient points

The results obtained so far can be used to characterize *weakly efficient points*. Besides our overall assumption concerning Y, Z, P, Q we assume that

C is a convex set,
$f : C \to Y$ and $g : C \to Z$ are mappings which are P-convex and Q-convex respectively,
$D := \{x \in C \mid f(x) \in -P\}$.

We consider the problem

$$\text{(11)} \qquad \text{eff } \{g(x), Q \mid x \in D\},$$

where x^o is, by definition, a solution of (11), iff $x^o \in D$ and there does not exist $x \in D$ with $g(x)-g(x^o) \in \text{int } (-Q)$. So $x^o \in D$ is a solution of (11) if and only if the system

$$x \in C, \quad f(x) \in -P, \quad g(x)-g(x^o) \in \text{int } (-Q)$$

is inconsistent. Therefore a straightforward application of Theorem 1 (together with the remarks 1 and 3 following it) and Theorem 2 gives the following result.

Corollary. Let $x^o \in D$. Then x^o is a solution of (11) if, and only if, there exists $z^* \in Q^*$, $z^* \neq 0$, such that for all $\varepsilon > 0$ and for all finite subsets $\mathfrak{X} \subset C$ there exists $y^* \in P^*$ satisfying

$$\langle z^*, g(x^o)\rangle - \varepsilon \leq \langle y^*, f(x)\rangle + \langle z^*, g(x)\rangle \quad \forall x \in \mathfrak{X}.$$

If there exists a finite subset $\mathfrak{X}^o \subset C$ such that $0 \in \text{int } (\text{conv } f(\mathfrak{X}^o) + P)$, then $x^o \in D$ is a solution of (11) if, and only if, there exist $y^* \in P^*$, $z^* \in Q^*$, $z^* \neq 0$ such that

$$\langle z^*, g(x^o)\rangle \leq \langle y^*, f(x)\rangle + \langle z^*, g(x)\rangle \quad \forall x \in C.$$

In a previous paper [5] the authors have obtained similar results for problem (11) with $D := \{x \in C \mid f_t(x) \leq 0 \;\; \forall t \in T\}$, where $f_t(\cdot) : C \to \mathbb{R}$ is a convex function for all $t \in T$, and T is an arbitrary set.

References

[1] J. Dieudonné: Sur la séparation des ensembles convexes. Math. Annalen 163 (1966), 1-3.

[2] F. Giannessi: Theorems of the alternative for multifunctions with applications to optimization: General results. Research Report n. 127, Department of Mathematics, University of Pisa, 1985.

[3] F. Giannessi: Theorems of the alternative for multifunctions with applications to optimization: Necessary conditions. Research Report n. 131, Department of Mathematics, University of Pisa, 1986.

[4] J. Gwinner: Closed images of convex multivalued mappings in linear topological spaces with applications. J. Math. Analysis Applications 60 (1977), 75-86.

[5] G. Heinecke, W. Oettli: Characterization of weakly efficient points. Zeitschrift für Operations Research (ZOR), to appear.

[6] R. Holmes: Geometric Functional Analysis and its Applications. Springer, New York, 1975.

[7] R. Lehmann, W. Oettli: The theorem of the alternative, the key-theorem, and the vector-maximum problem. Math. Programming 8 (1975), 332-344.

[8] W. Oettli: A duality theorem for the nonlinear vector-maximum problem. Colloquia Mathematica Societatis János Bolyai 12 (1974), 697-703.

[9] H. Schaefer: Topological Vector Spaces. Springer, New York, 1971.