Note on Pareto Optimality and Duality
for Certain Nonlinear Systems

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Abstract. We characterize the inconsistency of certain nonlinear systems under mild convexity requirements and without need for a regularity assumption. The result is used to establish a duality result for Pareto optimal points.

1. The present note is a continuation of [5]. There for Y a real topological vector space, $P \subset Y$ a nonvoid convex cone, and $S \subset Y \times I\!R$ a nonvoid set, the inconsistency of the system

$$(y,t) \in S\ ,\quad y \in -P\ ,\quad t < 0$$

had been characterized by the existence of $y^* \in P^*$ (P^* the polar cone of P) such that

$$0 \le t \quad \text{for all} \quad (y,t) \in S \quad \text{satisfying} \quad \langle y^*, y\rangle \le 0.$$

In order to make the necessity part of this characterization valid one needs first a convexity assumption, namely that the set $D := \{y \in Y | (y,t) \in S, t < 0\}$ is convex, and second one needs a so called regularity assumption, which may take various forms. The simplest regularity assumption, but also the least practical for many applications, requires the set D to be open in Y. Another regularity assumption, which in essence goes back to [3], requires that D is open in S_Y (S_Y the projection of S onto Y) and S_Y is convex with $0_Y \in \text{int } S_Y$. This assumption is more practical, but still has its drawbacks. Here, similarly to [1], we want to describe a simple approach which does not need any regularity assumption at all, yet gives a necessary and sufficient condition for the inconsistency of the above system. $I\!R$ is replaced by a more general vector space Z, permitting the consideration of Pareto optima. We conclude with a duality result in scalar and vectorial form respectively.

2. From now on we shall make the following *assumptions* :
Y, Z are real topological vector spaces, with Y being locally convex;
$P \subset Y$ and $Q \subset Z$ are nonvoid convex cones, with P *closed* , Q *open*, and $Q \neq Z$;
$P^+ \subset Y^*$ and $Q^+ \subset Z^*$ are the nonnegative polar cones of P and Q;
$S \subset Y \times Z$ is a given nonvoid set;
$V := \{z \in Z | (y,z) \in S, y \in -P\}$ is convex, and for all $z^* \in Q^+ \setminus \{0_{Z^*}\}$ the set
$D := \{y \in Y | (y,z) \in S, \langle z^*, z\rangle < 0\}$ is convex;
S^f denotes the collection of all *finite, nonempty* subsets of S.

Note that $y \in -P$ and $y^* \in P^+$ imply $\langle y^*, y\rangle \le 0$, whereas $z \in -Q$ and $z^* \in Q^+ \setminus \{0_{Z^*}\}$ imply $\langle z^*, z\rangle < 0$. For simplicity we write $\{0\}$ instead of $\{0_{Z^*}\}$.

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Theorem 1. The system

$$(1) \qquad (y,z)\in S, \quad y\in -P, \quad z\in -Q$$

has no solution if, and only if, there exists $z^* \in Q^+ \setminus \{0\}$ with the property that

$$(2) \qquad \begin{cases} \text{for all} \quad \Omega\in S^f \quad \text{there exists} \quad y^*\in P^+ \text{ such that} \\ \langle z^*,z\rangle \geq 0 \quad \text{for all} \quad (y,z)\in\Omega \quad \text{satisfying} \quad \langle y^*,y\rangle \leq 0. \end{cases}$$

Proof: a) Assume that (2) is satisfied with some $z^* \in Q^+ \setminus \{0\}$. Then (1) cannot have a solution $(\overline{y},\overline{z})$. Otherwise with $\Omega := \{(\overline{y},\overline{z})\}$ we would have for $y^* \in P^+$ as given by (2) that $\langle y^*,\overline{y}\rangle \leq 0$ and thereby $\langle z^*,\overline{z}\rangle \geq 0$, whereas from $\overline{z}\in -Q$ and $z^*\in Q^+\setminus\{0\}$ follows $\langle z^*,\overline{z}\rangle < 0$, a contradiction.
b) Assume that (1) has no solution. Then the convex set V introduced in the assumptions is disjoint from the open convex cone $-Q$. Hence by the weak separation theorem for convex sets there exists $z^*\in Q^+\setminus\{0\}$ such that $\langle z^*,z\rangle \geq 0$ for all $z\in V$ (if V is empty, then choose $z^*\in Q^+\setminus\{0\}$ arbitrarily, which is possible since $Q\neq Z$). Then the system

$$(y,z)\in S, \quad y\in -P, \quad \langle z^*,z\rangle < 0$$

has no solution. For the convex set D introduced in the assumptions this means that $D\cap -P=\emptyset$. Let Ω be a finite, nonempty subset of S. Let $D_\Omega := \{y\in Y \,|\, (y,z)\in\Omega, \langle z^*,z\rangle<0\}$. If $D_\Omega=\emptyset$, then choose $y^*=0$. If $D_\Omega\neq\emptyset$, then from the convexity of D follows $\operatorname{conv} D_\Omega\subset D$. Hence $\operatorname{conv} D_\Omega\cap -P=\emptyset$. Since $\operatorname{conv} D_\Omega$ is convex, compact and P is convex, closed and Y is locally convex, the strong separation theorem gives $y^*\in P^+$ such that $\langle y^*,y\rangle>0$ for all $y\in\operatorname{conv} D_\Omega$, hence for all $y\in D_\Omega$. So from $\langle y^*,y\rangle\leq 0$ and $(y,z)\in\Omega$ follows $y\notin D_\Omega$, i.e., $\langle z^*,z\rangle\geq 0$. q.e.d.

We mention that in order to obtain in (2) the slightly stronger but more familiar "Lagrangian" statement $0\leq\langle y^*,y\rangle+\langle z^*,z\rangle$ for all $(y,z)\in\Omega$, not only stronger convexity requirements are needed (e.g. S convex), but also a regularity assumption – see [1].

Of particular interest is the case that

$$S := (f\times g)(X) + (P\times \operatorname{cl} Q),$$

where X is a convex set and $f: X\to Y, g: X\to Z$ are given mappings. With this specialization, since $P+P=P$ and $\operatorname{cl} Q+Q=Q$, the inconsistency of (1) means the inconsistency of the system

$$x\in X, \quad f(x)\in -P, \quad g(x)\in -Q.$$

Statement (2) takes the following form :

$$\begin{aligned} &\text{For all } W\in X^f \text{ there exists } y^*\in P^+ \text{ such that} \\ &\langle z^*,g(x)\rangle\geq 0 \quad \text{for all} \quad x\in W \quad \text{satisfying} \quad \langle y^*,f(x)\rangle\leq 0. \end{aligned}$$

The convexity of D resp. V in this case is satisfied if for the multivalued mappings

$\Psi(\cdot) := f(g^{-1}(\cdot)) + P$ resp. $\Psi(\cdot) := g(f^{-1}(\cdot)) + \mathrm{cl}\, Q$ one has that $\Psi(C)$ is convex for all convex subsets $C \subset Z$ resp. $C \subset Y$.

Motivated by [7] we give a vector-valued version of Theorem 1.

Theorem 2. The system (1) has no solution if, and only if,

$$(3) \qquad \begin{cases} \text{for all } \Omega \in S^f \ \text{ there exists } \ y^* \in P^+ \ \text{ such that} \\ z \notin -Q \ \text{ for all } \ (y,z) \in \Omega \ \text{ satisfying } \langle y^*, y \rangle \leq 0. \end{cases}$$

Proof: If (1) has no solution, then there exists $z^* \in Q^+ \setminus \{0\}$ such that (2) is satisfied, and this implies (3) since otherwise $z \in -Q$ would imply $\langle z^*, z \rangle < 0$. Conversely, let (3) hold. Then (1) has no solution $(\overline{y}, \overline{z})$. Otherwise we would set $\Omega := \{(\overline{y}, \overline{z})\}$ and obtain from(3) an $y^* \in P^+$ such that $\langle y^*, \overline{y} \rangle \leq 0$, hence $\overline{z} \notin -Q$. This contradicts $(\overline{y}, \overline{z})$ being a solution of (1). q.e.d.

3. Theorem 1 and Theorem 2 give rise to a duality theorem (compare [2] and [4]) in scalar and in vectorial form respectively. We first turn to the vectorial case, starting from Theorem 2. As before we let

$$V := \{b \in Z | (y,b) \in S, y \in -P\},$$

and we let

$$W := \{b \in Z | \text{ for all } \Omega \in S^f \text{ there exists } y^* \in P^+ \text{ such that}$$
$$z - b \notin -Q \text{ for all } (y,z) \in \Omega \text{ satisfying } \langle y^*, y \rangle \leq 0\}.$$

$\overline{b} \in Z$ is called *Pareto minimal* in V iff $\overline{b} \in V$ and $b - \overline{b} \notin -Q$ for all $b \in V$.
$\overline{b} \in Z$ is called *Pareto maximal* in W iff $\overline{b} \in W$ and $b - \overline{b} \notin Q$ for all $b \in W$.

If $b_1 \in V$ and $b_2 \in W$, then $b_1 - b_2 \notin -Q$. Indeed: For $b_1 \in V$ let $(y_1, b_1) \in S$ with $y_1 \in -P$. Then for all $y^* \in P^+$ we have $\langle y^*, y_1 \rangle \leq 0$. In particular for the $y^* \in P^+$ resulting from $b_2 \in W$ with $\Omega := \{(y_1, b_1)\}$ we have $\langle y^*, y_1 \rangle \leq 0$, implying $b_1 - b_2 \notin -Q$. From this it follows immediately :

If $\overline{b} \in V \cap W$, then $\overline{b}$ is Pareto minimal in V and Pareto maximal in W.

Theorem 3. $\overline{b} \in Z$ is Pareto minimal in V if, and only if, $\overline{b} \in V \cap W$.

Proof: a) Assume that $\overline{b}$ is Pareto minimal in V. Then $\overline{b} \in V$, and the system

$$(y,b) \in S, \quad y \in -P, \quad b - \overline{b} \in -Q$$

has no solution. By Theorem 2, where we have to replace S by $S - (0, \overline{b})$, we obtain that $\overline{b} \in W$.

b) That $\overline{b} \in V \cap W$ implies $\overline{b}$ being Pareto minimal in V has already been observed. q.e.d.

The scalar version is fully analogous. Again we let

$$V := \{b \in Z | (y,b) \in S, y \in -P\},$$

and for $z^* \in Q^+ \setminus \{0\}$ we let

$$W(z^*) := \{b \in Z | \text{ for all } \Omega \in S^f \text{ there exists } y^* \in P^+ \text{ such that } \langle z^*, z - b\rangle \geq 0 \text{ for all } (y, z) \in \Omega \text{ satisfying } \langle y^*, y\rangle \leq 0\}.$$

By a similar reasoning as above we obtain: If $b_1 \in V$ and $b_2 \in W(z^*)$, then $\langle z^*, b_1 - b_2\rangle \geq 0$. From this it follows immediately :

$$\text{If } \bar{b} \in V \cap W(z^*), \text{ then } \min_{b\in V}\langle z^*, b\rangle = \langle z^*, \bar{b}\rangle = \max_{b\in W(z^*)}\langle z^*, b\rangle .$$

Theorem 4. $\bar{b} \in Z$ is Pareto minimal in V if, and only if, there exists $z^* \in Q^+ \setminus \{0\}$ such that $\bar{b} \in V \cap W(z^*)$.

Proof: The proof is analogous to that of Theorem 3. Note that from $\langle z^*, b - \bar{b}\rangle \geq 0$ and $z^* \in Q^+ \setminus \{0\}$ follows $b - \bar{b} \notin -Q$. q.e.d.

For fixed $z^* \in Q^+ \setminus \{0\}$ let us consider the quantities

$$\alpha := \min_{z\in V}\langle z^*, z\rangle \quad , \quad \beta^* := \max_{z\in W(z^*)}\langle z^*, z\rangle.$$

There holds

$$\begin{aligned}
\beta^* &= \inf_{\Omega\in S^f} \sup_{y^*\in P^+} \left(\inf\{\langle z^*, z\rangle | (y, z) \in \Omega, \langle y^*, y\rangle \leq 0\}\right) \\
&\geq \sup_{y^*\in P^+} \inf_{\Omega\in S^f} \left(\inf\{\langle z^*, z\rangle | (y, z) \in \Omega, \langle y^*, y\rangle \leq 0\}\right) \\
&= \sup_{y^*\in P^+} \left(\inf\{\langle z^*, z\rangle | (y, z) \in S, \langle y^*, y\rangle \leq 0\}\right) \\
&=: \beta.
\end{aligned}$$

Hence in the situation of Theorem 4 one has $\alpha = \beta^* \geq \beta$. Under a suitable regularity assumption (see [5], [6]) one has even $\alpha = \beta$. But without such a regularity assumption one may have a duality gap $\alpha > \beta$, and the value β^* is designed so as to close eventually this gap.

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