Common Extensions of Order Bounded Vector Measures

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For algebras $\mathcal{M}$ and $\mathcal{N}$ of subsets of some set Ω and an order complete Riesz space $\mathbb{G}$, we give a condition on the algebras under which any two consistent order bounded vector measures $\mu : \mathcal{M} \longrightarrow \mathbb{G}$ and $\nu : \mathcal{N} \longrightarrow \mathbb{G}$ possess a common extension to an order bounded vector measure $\varphi : 2^{\Omega} \longrightarrow \mathbb{G}$.

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1. Introduction

Throughout this paper, let Ω be a set, let M and N be algebras of subsets of Ω, and let $\mathbb{G}$ be an order complete Riesz space. Two vector measures $\mu : M \longrightarrow \mathbb{G}$ and $\nu : N \longrightarrow \mathbb{G}$ are consistent if $\mu(A) = \nu(A)$ holds for all $A \in M \cap N$, and they have a common extension if there exists a vector measure $2^{\Omega} \longrightarrow \mathbb{G}$ extending both μ and ν. Obviously, consistency is a necessary condition for a common extension to exist, and this condition is also sufficient; see [3]. For consistent vector measures which are positive or order bounded, however, there need not exist a common extension which is positive or order bounded as well; see [2].

In the present paper we study order bounded vector measures: We introduce a new condition on the algebras M and N under which any two consistent order bounded vector measures on M and N possess an order bounded common extension. Our result extends and unifies the result proven by Lipecki [2] in the case $\mathbb{G} = \mathbb{R}$.

Let us now recall some definitions and facts which will be needed in the sequel:

For a Riesz space $\mathbb{H}$, a linear operator $T : \mathbb{H} \longrightarrow \mathbb{G}$ is order bounded if it maps the order bounded subsets of $\mathbb{H}$ into the order bounded subsets of $\mathbb{G}$. For further details on Riesz spaces and linear operators, see [1].

For an algebra $\mathcal{F}$ of subsets of Ω , a vector measure $\varphi : \mathcal{F} \longrightarrow \mathbb{G}$ is order bounded if it maps $\mathcal{F}$ into an order bounded subset of $\mathbb{G}$. Let

$$\mathbb{E}(\mathcal{F}) \ := \ \mathrm{lin}\, \{ \chi_A \mid A \in \mathcal{F} \}$$

and define $\chi : \mathcal{F} \longrightarrow \mathbb{E}(\mathcal{F})$ by letting

$$\chi(A) \ := \ \chi_A$$

for all $A \in \mathcal{F}$, where χ_A denotes the indicator function of A . Then $\mathbb{E}(\mathcal{F})$ is a Riesz space with order unit χ_Ω , and χ is a vector measure. Moreover, each vector measure $\varphi : \mathcal{F} \longrightarrow \mathbb{G}$ defines its representing linear operator $T : \mathbb{E}(\mathcal{F}) \longrightarrow \mathbb{G}$, given by

$$T\Big(\sum_{i=1}^{n} \alpha_i \chi_{A_i} \Big) \ := \ \sum_{i=1}^{n} \alpha_i \varphi(A_i) \quad ,$$

and each linear operator $T : \mathbb{E}(\mathcal{F}) \longrightarrow \mathbb{G}$ defines a vector measure $\varphi : \mathcal{F} \longrightarrow \mathbb{G}$, given by

$$\varphi \ := \ T \circ \chi \quad .$$

It is not hard to see that φ is order bounded if and only if T is order bounded, and in this case $|\varphi| := \varphi \vee (-\varphi)$ and $|T| := T \vee (-T)$ exist and satisfy

$$|\varphi| \ = \ |T| \circ \chi \quad ;$$

see [1; Theorem 1.18] and [5; Theorem 4.1.2 and its proof].

2. Order bounded operators

For a vector space $\mathbb{E}$, a mapping $P : \mathbb{E} \longrightarrow \mathbb{G}$ is sublinear if $P(x+y) \leq P(x) + P(y)$ and $P(\lambda x) = \lambda P(x)$ holds for all $x, y \in \mathbb{E}$ and $\lambda \in \mathbb{R}_+$.

2.1. Proposition.

Let $\mathbb{E}$ be a vector space, let $\mathbb{F}$ be a subspace of $\mathbb{E}$, and let $S : \mathbb{F} \longrightarrow \mathbb{G}$ be a linear operator.
If there exists a sublinear mapping $P : \mathbb{E} \longrightarrow \mathbb{G}$ satisfying $Sx \leq P(x)$ for all $x \in \mathbb{F}$, then there exists a linear operator $T : \mathbb{E} \longrightarrow \mathbb{G}$ satisfying $Tx = Sx$ for all $x \in \mathbb{F}$ and $Tx \leq P(x)$ for all $x \in \mathbb{E}$.

For a proof of the previous Hahn-Banach theorem, see [1; Theorem 2.1].

If $\mathbb{E}_1$ and $\mathbb{E}_2$ are subspaces of $\mathbb{E}$, then two linear operators $T_1 : \mathbb{E}_1 \longrightarrow \mathbb{G}$ and $T_2 : \mathbb{E}_2 \longrightarrow \mathbb{G}$ are consistent if $T_1 x = T_2 x$ holds for all $x \in \mathbb{E}_1 \cap \mathbb{E}_2$, and they have a common extension if there exists a linear operator $T : \mathbb{E} \longrightarrow \mathbb{G}$ satisfying $Tx = T_i x$ for all $i \in \{1,2\}$ and $x \in \mathbb{E}_i$.

2.2. Theorem.

Let $\mathbb{E}$ be an Archimedean Riesz space with order unit $e \in \mathbb{E}_+$, and let $\mathbb{E}_1$ and $\mathbb{E}_2$ be Riesz subspaces of $\mathbb{E}$ satisfying $e \in \mathbb{E}_1 \cap \mathbb{E}_2$.
If there exists some $\alpha \in \mathbb{R}_+$ such that for all $x \in \mathrm{lin}(\mathbb{E}_1 \cup \mathbb{E}_2)$ satisfying $|x| \leq e$ there exist $x_i \in \mathbb{E}_i$ satisfying $x = x_1 + x_2$ and $|x_1| \vee |x_2| \leq \alpha e$, then any two consistent order bounded operators $T_1 : \mathbb{E}_1 \longrightarrow \mathbb{G}$ and $T_2 : \mathbb{E}_2 \longrightarrow \mathbb{G}$ have an order bounded common extension $T : \mathbb{E} \longrightarrow \mathbb{G}$.

Proof. Define $\mathbb{F} := \mathrm{lin}(\mathbb{E}_1 \cup \mathbb{E}_2)$. Since T_1 and T_2 are consistent, the mapping $S : \mathbb{F} \longrightarrow \mathbb{G}$, given by

$$Sx := T_1 x_1 + T_2 x_2$$

for all $x \in \mathbb{F}$ and arbitrary $x_i \in \mathbb{E}_i$ satisfying $x = x_1 + x_2$, is well-defined and linear.

Furthermore, since $\mathbb{E}$ is Archimedean, the Minkowski functional $\rho : \mathbb{E} \longrightarrow \mathbb{R}_+$, given by

$$\rho(x) := \inf \{ \lambda \in \mathbb{R}_+ \mid |x| \leq \lambda e \} ,$$

satisfies $\rho(x) = 0$ if and only if $x = 0$, as well as

$$\left| \frac{1}{\rho(x)} x \right| = 1$$

for all $x \in \mathbb{E} \setminus \{0\}$. Define now

$$u := 2\alpha(|T_1|e \vee |T_2|e) .$$

Then the mapping $P : \mathbb{E} \longrightarrow \mathbb{G}$, given by

$$P(x) := \rho(x)\, u ,$$

is sublinear. To see that $Sx \leq P(x)$ holds for all $x \in \mathbb{F}$, consider first $x \in \mathbb{F}$ satisfying $|x| \leq e$. By assumption, there exist $x_i \in \mathbb{E}_i$ satisfying $x = x_1 + x_2$ and $|x_1| \vee |x_2| \leq \alpha e$, and this yields

$$\begin{aligned} Sx &= T_1 x_1 + T_2 x_2 \\ &\leq |T_1||x_1| + |T_2||x_2| \\ &\leq 2\alpha(|T_1|e \vee |T_2|e) \\ &= u . \end{aligned}$$

Therefore, we have

$$\begin{aligned} Sx &= \rho(x)\, S\left(\frac{1}{\rho(x)} x\right) \\ &\leq \rho(x)\, u \\ &= P(x) \end{aligned}$$

for all $x \in \mathbb{F} \setminus \{0\}$, and thus

$$Sx \leq P(x)$$

for all $x \in \mathbb{F}$.

It now follows from Proposition 2.1 that there exists a linear operator $T : \mathbb{E} \longrightarrow \mathbb{G}$ satisfying $Tx = Sx$ for all $x \in \mathbb{F}$, and hence $Tx = T_i x$ for all $i \in \{1,2\}$ and $x \in \mathbb{E}_i$, as well as

$$Tx \leq P(x)$$

for all $x \in \mathbb{E}$. To see that T is order bounded, it is sufficient to show that T maps the order interval $[-e,e]$ into an order bounded set of $\mathbb{G}$, and this is true since

$$|Tx| \leq P(x) \leq P(e) = u$$

holds for all $x \in [-e,e]$, by the definition of P. □

Theorem 2.2 is related to a result of Ptak [4] concerning common extensions of linear functionals on closed subspaces of a Banach space.

3. Order bounded vector measures

A partition of Ω is a finite collection of mutually disjoint nonempty subsets of Ω whose union is equal to Ω .

Let $\mathcal{G}$ and $\mathcal{H}$ be partitions of Ω . For $k \in \mathbb{N}$, a finite sequence $\{ G_i \in \mathcal{G} \mid i = 1,\dots,k \}$ is an $(\mathcal{H},k)$-bridge, or simply an $\mathcal{H}$-bridge, from G_1 to G_k if

(i) $G_i \neq G_j$ holds for all $i,j \in \{1,\dots,k\}$ satisfying $1 \leq |i-j| \leq k-2$, and

(ii) for all $i \in \{1,\dots,k-1\}$ there exists some $H_i \in \mathcal{H}$ satisfying $G_i \cap H_i \neq \emptyset$ and $G_{i+1} \cap H_i \neq \emptyset$.

Two sets $G, G' \in \mathcal{G}$ are $(\mathcal{H},k)$-equivalent if there exists an $(\mathcal{H},k)$-bridge from G to G' , and they are $\mathcal{H}$-equivalent if they are $(\mathcal{H},k)$-equivalent for some $k \in \mathbb{N}$; in this case we shall write $G \sim_{\mathcal{H}} G'$.

3.1. Lemma.

$\sim_{\mathcal{H}}$ is an equivalence relation on $\mathcal{G}$.

Proof. It is immediate from the definitions that $\sim_{\mathcal{H}}$ is reflexive and symmetric. To see that $\sim_{\mathcal{H}}$ is also transitive, consider $G, G', G'' \in \mathcal{G}$ satisfying $G \sim_{\mathcal{H}} G'$ and $G' \sim_{\mathcal{H}} G''$. Obviously, $G \sim_{\mathcal{H}} G''$ holds whenever at least two of the sets G, G', G'' are identical. Let us now assume that these sets are all distinct, let $\{ G_i \in \mathcal{G} \mid i = 1,\dots,k \}$ be an $\mathcal{H}$-bridge from G to G' , and let $\{ G'_j \in \mathcal{G} \mid j = 1,\dots,k' \}$ be an $\mathcal{H}$-bridge from G' to G'' . Since $G_k = G' = G'_1$, there exists a smallest $i \in \{1,\dots,k\}$ satisfying $G_i = G'_j$ for some $j \in \{1,\dots,k'\}$, and j is unique since, by assumption, the sets $G'_1, \dots, G'_{k'}$ are all distinct. Define $k'' := i + k' - j$ and, for all $h \in \{1,\dots,k''\}$, define

$$G''_h := \begin{cases} G_h & , \text{ if } h \in \{1,\dots,i\} \\ G'_{h-i+j} & , \text{ if } h \in \{i+1,\dots,k''\} . \end{cases}$$

Then $\{ G''_h \mid h = 1,\dots,k'' \}$ is an H-bridge from G to G'', and we have $G \sim_H G''$. Therefore, $\sim_H$ is transitive. □

The algebras M and N are <u>weakly independent</u> if for any two partitions $G \subseteq M$ and $H \subseteq N$ there exist $G' \in G$ and $H' \in H$ satisfying $G' \cap H \neq \emptyset$ for all $H \in H$ and $G \cap H' \neq \emptyset$ for all $G \in G$, and they have a <u>controlling constant</u> if there exists some $k \in \mathbb{N}$ such that for any two partitions $G \subseteq M$ and $H \subseteq N$ either any two H-equivalent sets in G are (H,k')-equivalent for some $k' \in \{1,\dots,k\}$ or any two G-equivalent sets in H are (G,k')-equivalent for some $k' \in \{1,\dots,k\}$.

<u>3.2. Lemma.</u>

If either

(a) M and N are weakly independent, or

(b) M or N is finite,

then M and N have a controlling constant.

The proof of Lemma 3.2 is immediate.

<u>3.3. Lemma.</u>

If M and N have a controlling constant, then there exists some $\alpha \in \mathbb{R}_+$ such that for all $g \in \mathbb{E}(M)$ and $h \in \mathbb{E}(N)$ satisfying $|g+h| \leq \chi_\Omega$ there exist $g' \in \mathbb{E}(M)$ and $h' \in \mathbb{E}(N)$ satisfying $g' + h' = g + h$ and $|g'| \vee |h'| \leq \alpha\chi_\Omega$.

Proof. Define

$$\alpha := 2k - 1 ,$$

where $k \in \mathbb{N}$ is a controlling constant of $\mathcal{M}$ and $\mathcal{N}$.
Consider $g \in \mathbb{E}(\mathcal{M})$ and $h \in \mathbb{E}(\mathcal{N})$ satisfying

$$|g+h| \leq \chi_\Omega ,$$

and choose partitions

$$\mathcal{G} = \{G_1,\ldots,G_m\} \subseteq \mathcal{M} \quad \text{and} \quad \mathcal{H} = \{H_1,\ldots,H_n\} \subseteq \mathcal{N}$$

satisfying

$$g = \sum_{i=1}^{m} \gamma_i \chi_{G_i} \quad \text{and} \quad h = \sum_{j=1}^{n} \eta_j \chi_{H_j}$$

for suitable $\gamma_1,\ldots,\gamma_m,\eta_1,\ldots,\eta_n \in \mathbb{R}$. Without loss of generality, we may assume that any two $\mathcal{H}$-equivalent sets in $\mathcal{G}$ are $(\mathcal{H},k')$-equivalent for some $k' \in \{1,\ldots,k\}$. Let $\mathcal{G}_1, \ldots, \mathcal{G}_l$ denote the equivalence classes of $\mathcal{G}$ with respect to $\sim_{\mathcal{H}}$.
Fix $p \in \{1,\ldots,l\}$. Choose $i_p \in \{1,\ldots,m\}$ satisfying

$$G_{i_p} \in \mathcal{G}_p ,$$

and define

$$M_p := \bigcup_{G \in \mathcal{G}_p} G$$

and

$$N_p := \bigcup_{\substack{H \in \mathcal{H} \\ H \cap M_p \neq \emptyset}} H .$$

For $H \in \mathcal{H}$ satisfying $H \cap M_p \neq \emptyset$, there exists some $G' \in \mathcal{G}_p$ satisfying $H \cap G' \neq \emptyset$, and for each $G \in \mathcal{G}$ satisfying $H \cap G \neq \emptyset$ we have $G \sim_{\mathcal{H}} G'$ and hence $G \in \mathcal{G}_p$. This yields

$$H \subseteq \bigcup_{\substack{G \in \mathcal{G} \\ H \cap G \neq \emptyset}} G \subseteq \bigcup_{G \in \mathcal{G}_p} G = M_p ,$$

hence

$$N_p \subseteq M_p ,$$

and thus

$$N_p = M_p \in \mathcal{M} \cap \mathcal{N} ,$$

since $\mathcal{H}$ is a partition.

Define now

$$g' := \sum_{i=1}^{m} \gamma_i \chi_{G_i} - \sum_{p=1}^{l} \gamma_{i_p} \chi_{M_p}$$

and

$$h' := \sum_{j=1}^{n} \eta_j \chi_{H_j} + \sum_{p=1}^{l} \gamma_{i_p} \chi_{N_p} \quad .$$

Then we have $g' \in \mathbb{E}(M)$ and $h' \in \mathbb{E}(N)$, as well as $g'+h' = g+h$ and

$$|g'+h'| \leq \chi_\Omega \quad .$$

Furthermore, for each $G \in \mathcal{G}$, there exists some $p \in \{1,\ldots,l\}$ with $G \in \mathcal{G}_p$ and an $(\mathcal{H},k')$-bridge $\{ G_i' \in \mathcal{G} \mid i = 1,\ldots,k' \}$ with $k' \leq k$ from G_{i_p} to G , and for each $i \in \{1,\ldots,k'-1\}$ there exists some $H_i' \in \mathcal{H}$ satisfying $G_i' \cap H_i' \neq \emptyset$ and $G_{i+1}' \cap H_i' \neq \emptyset$, hence

$$|g'(\omega)| \leq |h'(\omega')| + 1$$

for all $\omega \in G_i' \cup G_{i+1}'$ and all $\omega' \in H_i'$, and thus

$$|g'(\omega)| \leq |g'(\omega'')| + 2$$

for all $\omega \in G_i'$ and all $\omega'' \in G_{i+1}'$,

and this together with $G = G_k'$, $G_1' = G_{i_p}$, and $g'\chi_{G_{i_p}} = 0$ gives

$$|g'\chi_G| \leq 2(k'-1)\chi_\Omega \quad .$$

Since $G \in \mathcal{G}$ was arbitrary and since $\mathcal{G}$ is a partition, this yields

$$|g'| \leq 2(k'-1)\chi_\Omega \quad ,$$

and from $|h'| \leq |g'| + \chi_\Omega$ and $k' \leq k$ we obtain

$$|g'| \vee |h'| \leq \alpha \quad ,$$

which completes the proof. □

We can now state and prove the main result of this paper:

3.4. Theorem.

If M and N have a controlling constant, then any two consistent order bounded vector measures $\mu : M \longrightarrow \mathbb{G}$ and $\nu : N \longrightarrow \mathbb{G}$ have an order bounded common extension $\varphi : 2^{\Omega} \longrightarrow \mathbb{G}$.

Proof. Define $\mathbb{E}_1 := \mathbb{E}(M)$, $\mathbb{E}_2 := \mathbb{E}(N)$, and $\mathbb{E} := \mathbb{E}(2^{\Omega})$, and let $T_1 : \mathbb{E}_1 \longrightarrow \mathbb{G}$ and $T_2 : \mathbb{E}_2 \longrightarrow \mathbb{G}$ denote the representing linear operators of μ and ν , respectively. By Lemma 3.3 and and Theorem 2.2, T_1 and T_2 have an order bounded common extension $T : \mathbb{E} \longrightarrow \mathbb{G}$, and it now follows that the vector measure $\varphi : 2^{\Omega} \longrightarrow \mathbb{G}$, given by $\varphi := T \circ \chi$, is an order bounded common extension of μ and ν . □

By Lemma 3.2, Theorem 3.4 extends and unifies the results proven by Lipecki [2] in the case $\mathbb{G} = \mathbb{R}$.

4. Remarks

Let us now assume that $\mathbb{G}$ is a Banach lattice. For an algebra $\mathcal{F}$ of subsets of Ω , a vector measure $\varphi : \mathcal{F} \longrightarrow \mathbb{G}$ is <u>bounded</u> if it maps $\mathcal{F}$ into a norm bounded subset of $\mathbb{G}$, and it has <u>bounded variation</u> if $\sup \Sigma \|\varphi(A_i)\|$ is finite, where the supremum is taken over all partitions $(A_1, A_2, \ldots, A_n)$ in $\mathcal{F}$. If $\mathbb{G}$ is an order complete AM-space with unit, then a vector measure $\mathbb{E} \longrightarrow \mathbb{G}$ is bounded if and only if it is order bounded, and if $\mathbb{G}$ is an AL-space, then a vector measure $\mathbb{E} \longrightarrow \mathbb{G}$ has bounded variation if and only if it is order bounded; see [6]. Therefore, the following results are immediate from Theorem 3.4:

4.1. Corollary.

Let $\mathbb{G}$ be an order complete AM-space with unit.
If M and N have a controlling constant, then any two consistent bounded vector measures $\mu : M \longrightarrow \mathbb{G}$ and $\nu : N \longrightarrow \mathbb{G}$ have a bounded common extension $\varphi : 2^\Omega \longrightarrow \mathbb{G}$.

4.2. Corollary.

Let $\mathbb{G}$ be an AL-space.
If M and N have a controlling constant, then any two consistent vector measures $\mu : M \longrightarrow \mathbb{G}$ and $\nu : N \longrightarrow \mathbb{G}$ of bounded variation have a common extension $\varphi : 2^\Omega \longrightarrow \mathbb{G}$ of bounded variation.

It would be interesting to know whether Corollary 4.2 can be extended to a larger class of Banach lattices.

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