Anomalies from Geometric Quantization of Fermionic Field Theories

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ABSTRACT

Geometric quantisation on (infinite dimensional) graded symplectic manifolds is elaborated for a restricted class of phase spaces. The formalism includes the treatment of femionic field theories. The chiral anomaly (U(1)-anomaly) as well as the non abelian (covariant) anomaly of D-dimensional non Abelian gauge theories is calculated in this framework. Thus previous work of the present authors is generalised.

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1. Introduction

Using symplectic geometry on the classical phase space, geometric quantization [1] provides a coordinate independent quantization scheme avoiding the ambiguity of operator ordering. In [2] it has been suggested to consider field theoretic anomalies in the context of this scheme. However it is not clear in the literature [3], to what extent geometric quantization is applicable to field theories. In [3] it was claimed (without proof) to yield the correct quantum field theory for linear systems and semiclassical approximations in general.

In previous work [4] the authors have contributed to this discussion: Considering the non conservation of the quantized chiral charge in time they have shown, how to calculate the chiral $U(1)$ anomaly of a non Abelian gauge theory in 4 dimensions within the geometric quantization scheme. As the chiral anomaly is a well established feature of gauge theories, one can regard its determination to be a significant test for the application of geometric quantization to field theories.

In [3] the space $\mathcal{V}$ of solutions of the Dirac equation in a gauge background has been taken as the classical phase space for the Dirac system. In [4] a slightly different approach has been chosen: As in [3] the solutions $\Psi \in \mathcal{V}$ have been represented by their initial values $\Psi(x,t)|_{t=0} =: \psi_0(x)$. However in accordance to the usual treatment of Fermionic field theories the $\psi_\tau(x)$ have been regarded as *anticommuting* coordinates on a *graded* symplectic manifold.

Although graded manifolds are extensively used in the physics literature [6,7], the subject of geometric quantization on such manifolds has been investigated systematically (to our knowledge) only in [5] and only up to the prequantum level. Hence it suggests itself to deal with the formalism of geometric quantization on phase spaces with the structure of the one used in [4]. This will be done in section 2 of the present paper. More strictly speaking we will consider the quantization of a graded symplectic manifold $(X, \mathcal{A}, \omega)$ (in the notion of [5]), where X is pointlike and $\mathcal{A}$ is the exterior algebra over the dual of a vectorspace. Results of [5] will be revisited as far as necessary to keep the paper self contained. However the consideration of a complex structure and the induced polarization as well as the construction of a quantum Hilbert space exceeds the material presented in [5].

In section 3 of the present paper the geometric quantization formalism developed in section 2 is applied to Dirac theory in even dimensions D. An appropriate polarization for a Dirac theory with gauge background is presented and the Fock space structure of the quantum Hilbert space is lined out.

A shortcoming of [4] was the restriction to the chiral anomaly in 4 spacetime dimensions. In section 4 the chiral U(1) anomaly in arbitrary even D dimensions as well as the (covariant) non abelian anomaly are calculated. Generalizing [4] the results are in full agreement with the standard ones [15]. The calculation shows that the half form contribution, corresponding to the transformation property of the measure in the Hilbert space, plays a crucial role in determining field theoretical anomalies. In an appendix we will point out technical details of the calculations done in section 4.

2. Graded Manifolds and Geometric Quantization

Geometric quântization on the one hand and the theory of graded manifolds on the other hand are well established in the physics as well as in the mathematics literature. Already in 1975 Kostant showed in a remarkable work [5] that the notion of graded symplectic manifolds induces a natural connection between these two fields. However with few exceptions (c.f. [16]) this connection has not been paid much attention to in the literature. Hence to the extent we need later we will start this section repeating the main ideas of [5] in short. For more details on graded manifolds (supermanifolds) in finite and also in infinite dimensions we refer to [6] and [7].

Let A be an algebra decomposed into $A = A_0 \oplus A_1$ such that $A_i \cdot A_j \subset A_{i+j}$, $i, j \in \mathbb{Z}_2$. We call $a_i \in A_i$ homogeneous element of A with degree $\mathrm{gr}(a_i) = i$. A is a graded (commutative) algebra over $\mathbb{Z}_2$, if the product of each two homogeneous elements $a, b \in A$ is graded commuting, i.e.

$$a \cdot b = (-1)^{\mathrm{gr}(a)\mathrm{gr}(b)} b \cdot a \qquad (2.1).$$

In this sense A_0 and A_1 are referred to as the even respectively the odd part of the algebra A.

Let X be a smooth manifold and $\{U_i\}$ the set of all open subsets of X. Let A be a graded algebra, equipped with an appropriate topology and consider smooth functions $f_i : U_i \to A$. The set of these functions also forms a graded algebra under pointwise operations, denoted by $\mathcal{A}(U_i)$. For a pair $U_j \subset U_i$ the (natural) restriction

$$\begin{aligned} \rho_{U_j}^{U_i} &: \mathcal{A}(U_i) \to \mathcal{A}(U_j) \\ \rho_{U_j}^{U_i}(f_i) &= f_i|_{U_j} \end{aligned} \qquad (2.2)$$

is an algebra homomorphism and the tupel $(X, \mathcal{A}(U_i), \rho_{U_j}^{U_i})$ is a special example of a sheaf. (For our purpose it is sufficient to consider a sheaf as an object of this type, for the exact definition we refer to [8].) If for an atlas $\{U_\alpha\}$ of X any function $f_\alpha \in \mathcal{A}(U_\alpha)$ can be written as

$$\tilde{f_\alpha}(x) = \tilde{f_\alpha}(x) + \sum_{n>0} \sum_{j_1 \dots j_n} (\tilde{g_\alpha}(x))_{j_1 \dots j_n} \theta_{j_1} \cdot \ldots \cdot \theta_{j_n} \qquad (2.3)$$

the sheaf $(X, \mathcal{A}(U_i), \rho_{U_j}^{U_i})$ together with this decomposition defines a graded manifold denoted by $(X, \mathcal{A})$. In (2.3) $x \in U_\alpha$ is a point, $\tilde{f_\alpha}, (\tilde{g_\alpha})_{j_1 \dots j_n}$ respectively are usual $C^\infty(U_\alpha)$ functions and $\theta_{j_1} \cdot \ldots \cdot \theta_{j_n}$ are the generators of A. Note that $A = \mathbb{R}$ also fits into the definition of a graded algebra (with trivial odd part), hence each usual manifold X can also be considered as a graded manifold (X, C^∞). If in contrast $A = \mathrm{Gr}$ is a Graßmann algebra the corresponding $(X, \mathcal{G}r)$ is also called a supermanifold.

For the application we have in mind we let V be a vector space and V^* its dual. In the case of infinite dimensional V let the dual be defined with respect to some pairing (e.g. in the sense of [9]). The exterior algebra $\bigoplus_n \bigwedge^n(V^*)$ over this dual space is a $\mathbb{Z}_2$ graded algebra, with respect to the $\wedge$ product. If we consider $W := \bigoplus_n \bigwedge^n(V^*)$ as a (trivial) sheaf over the pointlike manifold $X = \{p\}$, this defines a graded manifold

$$M_V := (\{p\}, W) \tag{2.4}$$

Here the splitting (2.3) holds trivially, considering the elements of $V = \bigwedge^1(V^*)$ as odd generators of W. As shown by Batchelor [7] the fact that V may be infinite dimensional does not spoil the construction. (In the same way as constructed above $\bigoplus_n \mathcal{S}^n(V^*)$, the symmetric tensor algebra of V^* defines a graded manifold $(\{p\}, \bigoplus_n \mathcal{S}^n(V^*))$ with trivial odd part. This example should be of interest in geometric quantization of Bosonic field theories, however it will be considered elsewhere [17].)

For graded manifolds the notion of a tangent space is not so natural as for a usual manifold. Nevertheless it is possible to do differential geometry and proceed with geometric quantization by considering the space of all superderivations instead of $T(X)$. This space of superderivations $\mathrm{Der}(\mathcal{A}) \subset \mathrm{End}(\mathcal{A})$ over the algebra of functions $\mathcal{A}$ is defined as the space of all linear maps $\delta : \mathcal{A} \to \mathcal{A}$ obeying a graded Leibnitz rule :

$$\mathrm{Der}(\mathcal{A}) = \left\{ \delta = \delta_0 + \delta_1 \in \mathrm{End}(\mathcal{A}) | \delta_k(f \cdot g) = \delta_k(f) \cdot g + (-1)^{\mathrm{gr}(f)\mathrm{gr}(\delta_k)} f \cdot \delta_k(g) \quad k \in \mathbb{Z}_2 \right\} \tag{2.5}$$

where $\delta = \delta_0 + \delta_1$ is understood with respect to the naturally induced ($\mathbb{Z}_2$-)grading of $\mathrm{End}(\mathcal{A})$. $\mathrm{Der}(\mathcal{A})$ does not define the tangent space of the graded manifold $(X, \mathcal{A})$, but generalizes the (algebraic) definition of $T(X)$ as the space of all derivations on $C^\infty(X)$. In coordinates (x_i, θ_j) on $(X, \mathcal{A})$ we have

$$\delta = \sum_i a_i \frac{\partial}{\partial x_i} + \sum_j b_j \frac{\partial}{\partial \theta_j} =: \sum_i a_i \partial_{x_i} + \sum_j b_j \partial_{\theta_j} \tag{2.6}$$

with coefficients $a_i, b_j \in \mathcal{A}$. For $\delta, \tilde{\delta} \in \mathrm{Der}(\mathcal{A})$ the commutator between (homogeneous) superderivations naturally generalizes the commutator between vector fields

$$[\delta, \tilde{\delta}]_\pm := \delta\, \tilde{\delta} + (-1)^{\mathrm{gr}(\delta)\mathrm{gr}(\tilde{\delta})+1}\, \tilde{\delta}\, \delta \tag{2.7}.$$

In the example (2.4) superderivations δ of M_V are completely determined by their action on a base of V^* via linearity and Leibnitz rule. Thus $\mathrm{Der}(W)$ may be identified with $W \otimes V$ where the elements of V act on W as superderivations of homogeneous grading 1.

To generalize the definition of differential forms to graded manifolds (c.f. [5]) we consider (for all open $U_i \subset X$) the tensor algebra $\mathrm{T}(U_i)$ of $\mathrm{Der}(\mathcal{A}(U_i))$ with coefficients in $\mathcal{A}(U_i)$ and denote by $\mathrm{T}^m(U)$ the space of all m-tensors. Then the space of differential m-forms $\Omega^m(U, \mathcal{A}(U_i))$ is given as the set of all $\mathcal{A}(U_i)$-valued linear forms on $T^m(U_i)$ obeying a graded symmetry, specified below. Using the sheaf structure of $(X, \mathcal{A})$ we get globally $\Omega^m(X, \mathcal{A})$ as the space of all m-linear maps on $\mathrm{Der}(\mathcal{A})$ with values in $\mathcal{A}$, characterized by the additional graded symmetry condition on $\alpha \in \Omega^m(X, \mathcal{A})$

$$\alpha(\xi_1, \ldots, \xi_j, \xi_{j+1}, \ldots, \xi_m) = (-1)^{(\mathrm{gr}(\xi_j)+1)(\mathrm{gr}(\xi_{j+1})+1)} \alpha(\xi_1, \ldots, \xi_{j+1}, \xi_j, \ldots, \xi_m) \tag{2.8}$$

with $\xi_i \in \mathrm{Der}(\mathcal{A})$ homogeneous. (Note that our sign conventions coincide with [6] but not with [5].) For M_V all elements in V are of homogenous degree 1, so $\Omega^m(M_V)$ simplifies to the space of symmetric m-forms over V with values in W. Hence, denoting by $\mathcal{S}^m(V^*)$ the space of symmetric m-tensors over V^*

$$\Omega^m(M_V) = \mathcal{S}^m(V^*) \otimes W \tag{2.9}$$

This may become more apparent in a coordinate description : Assume a basis set $\{e_i\}$ is given on V, so each $v \in V$ may be written as

$$v = \sum_t \theta_i(v) e_i \tag{2.10}$$

The set of coordinates $\{\theta_i\}$ may be identified with the corresponding dual basis on V^*, i.e. $\theta_i(e_j) = \delta_{ij}$. Regarding θ_i as Graßmann numbers (anticommuting variables) elements of W become polynomials in θ_i. In these coordinates $\mathrm{Der}(W)$ is spanned by $\{\partial_{\theta i}\}$ with

$$\partial_{\theta i}(\theta_j) = \delta_{ij} \tag{2.11}$$

Note that $\{\partial_{\theta i}\}$ also determines a base of V that is anticommuting in contrast to $\{e_i\}$. For the construction of $\Omega^m(M_V)$ we denote the basis elements of $\mathcal{S}^1(V^*)$ by $d\theta_i$ with

$$d\theta_j(\partial_{\theta i}) = \partial_{\theta i} \rfloor d\theta_j = \delta_{ij} \tag{2.12}$$

This notation becomes consistent if we take $d\theta_i$ to be commuting, in contrast to the θ_i. Then the symbol d coincides with the exterior derivative on W, in coordinates

$$d = d\theta_i \otimes \partial_{\theta i} \tag{2.13}$$

that acts as a derivative of grading 1 and is nilpotent ($d^2 = 0$). On a graded manifold we also have the notion of an interior derivative $\mathbf{i}_\xi$ with degree $\mathrm{gr}(\mathbf{i}_\xi) = \mathrm{gr}(\xi) + 1$ defined as on a usual manifold by

$$\mathbf{i}_\xi \alpha(\xi_1, \ldots, \xi_{m-1}) = \xi \rfloor \alpha(\xi_1, \ldots, \xi_{m-1}) = \alpha(\xi, \xi_1, \ldots, \xi_{m-1}) \tag{2.14}$$

Now let $A_{\mathbb{C}} = A \otimes \mathbb{C}$ be the complexification of the algebra A and let $\{U_\alpha\}$ be an open covering of X. Then a (complex) line bundle sheaf L over the graded manifold $(X, \mathcal{A})$ is locally determined as

$$L(U_\alpha) = \mathcal{A}_{\mathbb{C}}(U_\alpha) \otimes \tau_\alpha \tag{2.15}$$

Here τ_α are even generators of $A_{\mathbb{C}}$ with invertible transition functions $c^{\alpha\beta} \in \mathcal{A}(U_\alpha \cap U_\beta)$ given by $\tau_\alpha = c^{\alpha\beta}\tau_\beta$. Using the sheaf structure of $(X, \mathcal{A})$ this can be globalized. The space of sections of a line bundle L over a graded manifold is defined as in the usual case and will be denoted by $\Gamma(L) \equiv L(X)$. For geometric quantization L has to carry a Hermitian structure, i.e. a bilinear, Hermitian operation

$$(.,.) L \times L \to \mathcal{A}_{\mathbb{C}} \tag{2.16}$$

mapping pairs of sections $\mathcal{S}(x), \tilde{\mathcal{S}}(x) \in \Gamma(L)$ smoothly to a section $(\mathcal{S}, \tilde{\mathcal{S}})(x)$ of the trivial line bundle $\mathcal{A}_{\mathbb{C}}$ over $(X, \mathcal{A})$. As for usual manifolds a connection ∇ on a line bundle sheaf L can be written as a map $\nabla_\xi : L \to L$, locally given by

$$\nabla_\xi \mathcal{S} = \xi \mathcal{S} + (\mathrm{i}_\xi \vartheta)\mathcal{S} \quad \text{for any} \quad \xi \in \mathrm{Der}(\mathcal{A}) \tag{2.17}$$

where $\vartheta \in \Omega^1(\mathcal{A})$ has degree $\mathrm{gr}(\vartheta) = 1$. The curvature of the connection then is $\mathrm{curv}\nabla = d\vartheta$ (with d given by (2.13)) and a Hermitian structure on a line bundle sheaf is said to be compatible with the connection, if

$$\xi(\mathcal{S}, \tilde{\mathcal{S}}) = (\nabla_\xi \mathcal{S}, \tilde{\mathcal{S}}) + (\mathcal{S}, \nabla_\xi \tilde{\mathcal{S}}) \tag{2.18}$$

For more details we refer to [5].

Symplectic mechanics on a graded manifold proceeds as for usual manifolds : $\omega \in \Omega^2(X, \mathcal{A})$ is called a graded symplectic form, if it is even with respect to the grading of $\mathcal{A}$, closed ($d\omega = 0$) and (weakly) nondegenerated on $\mathrm{Der}(\mathcal{A})$, (i.e. if $\omega(V, W) = 0$ for all $V \in \mathrm{Der}(\mathcal{A})$ then $W = 0$, c.f. [9]). Then due to the graded Darboux theorem there exist local coordinates with

$$\omega = \frac{1}{2} \sum_{ij} f^{ij} dx_i dx_j + \frac{1}{2} \sum_{ij} g^{ij} d\theta_i d\theta_j \tag{2.19}$$

where the matrices f^{ij} (antisymmetric) and g^{ij} (symmetric) are constant and of grading 0. Also there is a graded Poincare lemma that (locally) guarantees the existence of Θ with $\omega = d\Theta$. For M_V the Darboux theorem and the Poincare lemma hold globally and the f^{ij} in (2.19) vanish. To generalize the Poisson algebra from $C^\infty(X)$ to $\mathcal{A}$ one assigns via

$$\xi_F \rfloor \omega + dF = 0 \tag{2.20}$$

a Hamiltonian vectorfield $\xi_F \in \mathrm{Der}(\mathcal{A})$ to each observable $F \in \mathcal{A}$. Then the Poisson algebra over $\mathcal{A}$ is given by

$$\{F, G\} = (-1)^{\mathrm{gr}(F)} \xi_F G = (-1)^{\mathrm{gr}(F)+1} \xi_F \rfloor \xi_G \rfloor \omega \qquad (2.21),$$

with $\xi_F G = \xi_F \rfloor dG$. In Darboux coordinates this is

$$\{F, G\} = \sum_{ij} (f^{ij})^{-1} \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j} + (-1)^{\mathrm{gr}(F)} \sum_{ij} (g^{ij})^{-1} \frac{\partial F}{\partial \theta_i} \frac{\partial G}{\partial \theta_j} \tag{2.22}$$

This includes the usual Poisson bracket and also gives an anticommutator on the level of classical mechanics.

The first aim of geometric quantization is to associate to each observable $F \in \mathcal{A}$ an operator $\mathcal{O}_F$, acting on sections of a complex line bundle sheaf L over the graded symplectic manifold $(X, \mathcal{A}, \omega)$ such that a represention of the Poisson algebra is provided and the unit element $1 \in \mathcal{A}$ is represented as the unit operator :

$$\begin{aligned} [\mathcal{O}_F, \mathcal{O}_G]_\pm &= -i\hbar \mathcal{O}_{\{F,G\}} \\ \mathcal{O}_1 &= \mathbf{1} \end{aligned} \tag{2.23}$$

Such a representation is called prequantization. If we consider a usual symplectic manifold with $[\omega]$ integral Weil's theorem guarantees that there exists a Hermitian line bundle L over X with a connection ∇ such that ∇ is compatible with the Hermitan structure on L and induces the symplectic form by $\omega = \mathrm{curv}(\nabla)$. For a graded manifold $(X, \mathcal{A})$ such a Hermitian line bundle sheaf has been shown in [5] (c.f. section 6.3) to exist if X has trivial cohomology. This is the case for our application (2.4), moreover the line bundle over M_V can be chosen trivial, i.e. $L_V = W \otimes \mathbb{C}$. The prequantum operator on such a line bundle sheaf is then given by

$$\begin{aligned} \mathcal{O}_F : \Gamma(L) \to \Gamma(L) \\ \mathcal{O}_F = -i\hbar\nabla_{\xi_F} + F \end{aligned} \tag{2.24}$$

where the covariant derivative ∇_ξ may be written as

$$\nabla_\xi = \xi - \frac{i}{\hbar}\xi \rfloor \Theta \tag{2.25}.$$

However full quantization demands an irreducible representation of the Heisenberg subalgebra (c.f. [1]), not given by (2.24). On a usual manifold X this problem is solved by choosing a polarization (Lagrangian subspace) $P \subset T^{\mathbb{C}}(X)$ of the complexified tangent space. An appropriate polarization for geometric quantization is provided by a Kähler structure [10] on X. We use the notion of [3] and define an (almost) Kähler structure on X as a linear involution $J : T(X) \to T(X)$ with

$$\begin{aligned} J^2 = -1 \\ \omega(J(\xi), J(\eta)) = \omega(\xi, \eta) \end{aligned} \tag{2.26}.$$

If one can choose on X local coordinates $\{z_k, z_k^+\}$ solving over $T^{\mathbb{C}}(X)$ the eigenvalue problem

$$J\left[\frac{\partial}{\partial z_k}\right] = +i\left(\frac{\partial}{\partial z_k}\right) \quad J\left[\frac{\partial}{\partial z_k^+}\right] = -i\left(\frac{\partial}{\partial z_k^+}\right) \tag{2.27}.$$

This defines a Kähler polarization P spanned by the eigenvectors $\frac{\partial}{\partial z_k^+} =: \partial_{z_k^+}$. This description of a Kähler structure easily generalizes to graded manifolds given by an automorphism $J : \mathrm{Der}(\mathcal{A}) \to \mathrm{Der}(\mathcal{A})$ obeying (2.26). A Kähler polarization on a graded manifold is then determined by

$$P = \mathrm{Span}\left\langle \left\{ \partial_{z_k^+} \right\} \right\rangle \subset \mathrm{Der}(\mathcal{A}) \quad \text{with} \quad J\left[\partial_{z_k^+}\right] = -i\left(\partial_{z_k^+}\right) \tag{2.28}.$$

To fulfill the irreducibility condition we have to represent classical observables as operators on the space of polarized sections

$$\Gamma^P(L) = \{\, \mathcal{S} \in \Gamma(L) \mid \nabla_\xi \mathcal{S} = 0 \text{ for all } \xi \in P \} \tag{2.29}.$$

For a Kähler polarization this means that the wave functions $\mathcal{S} \in \Gamma^P(L)$ have to be holomorphic sections, i.e. covariantly constant under $\nabla_{z_k^+}$.

On a usual (2m-dimensional) manifold X the symplectic form ω induces a natural volume element $(\omega)^m$. Using this for integration over X the Hermitian structure (2.16) on L extends to an inner product on $\Gamma(L)$ by

$$\begin{aligned} <.,.>: \ \Gamma(L) \times \Gamma(L) \rightarrow \mathbb{C} \\ <\xi,\eta> = \int (\xi,\eta)(\omega)^m \end{aligned} \qquad (2.30).$$

Such is not the case on a graded manifold, where integration over forms is not defined directly. Due to Berezin [6] integration over anticommuting variables is identified with differentiation. A naive identification would yield a coordinate dependent integral. However considering the symplectic graded manifold M_V and a complex structure J defined on it Berezin's idea can be used to determine a coordinate independent integration : On M_V the symplectic form ω determines a map τ between superderivations and one forms by

$$\begin{aligned} \tau : \mathrm{Der}(M_V) \rightarrow \Omega^1(M_V) \\ \tau(\xi) := \xi \rfloor \omega \end{aligned} \qquad (2.31).$$

On the other hand the symplectic form and the complex structure yield an antisymmetric tensor field g on $\mathrm{Der}(M_V)$ by

$$g(\xi,\eta) = \omega(J(\xi),\eta) \qquad \xi,\eta \in \mathrm{Der}(M_V) \qquad (2.32).$$

We note that $\omega \in \Omega^2(M_V)$ and hence $g \in W \otimes \bigwedge^2(V^*)$, so we can define an antisymmetric form $\omega' \in W \otimes \bigwedge^2(V)$

$$\omega'(\alpha,\beta) = g(\tau^{-1}\alpha, \tau^{-1}\beta) \quad \alpha,\beta \in \Omega^1(M_V) \qquad (2.33).$$

For $2m$ dimensional V the m-fold tensor product $(\omega')^m \in W \otimes \bigwedge^{2m}(V)$ provides a natural volume element for the integration of functions over M_V, i.e. integration of sections $F \in \Gamma(L_V)$ in the following way : We have

$$F \cdot (\omega')^m \in W \otimes \bigwedge{}^{2m}(V) \qquad (2.34)$$

and the integration is carried out applying the $\bigwedge^{2m}(V)$ part as product of superderivations to W. This yields a coordinate independent map

$$\int \cdot \cdot (\omega')^m : \Gamma(L_V) \rightarrow \mathbb{C} \qquad (2.35)$$

that gives in coordinates the Berezin integral with $\sqrt{\det g_{ij}}$ used as integration measure. We note that this generalizes to infinite dimensions (c.f. chapter I.3 of [12]).

In contrast to $\Gamma(L)$ on the space of polarized sections $\Gamma^P(L)$ the natural volume element $(\omega)^m$ (respectively $(\omega')^m$) does in general not induce a pairing by integration. Therefore it is necessary to introduce the notion of half forms [1]. Essentially a half form on a usual manifold X is a function on the bundle of frames $\mathcal{F}^P(X)$ spanning the polarization P

$$\nu : \mathcal{F}^P(X) \to \mathbb{C} \tag{2.36}$$

which transforms under right group actions g on P according to

$$\nu \circ g = (\det_P g)^{-1/2} \nu \tag{2.37}$$

Roughly speaking ν reflects the transformation property of the measure in the Hilbert space build from the space of polarized sections $\Gamma^P(L)$. For infinite dimensional manifolds one furthermore has to choose a proper regularization to make the determinant well defined. The notion of half forms can also be applied to our graded manifold $M_V(\{p\}, W)$:

$$\nu : \mathcal{F}^P(W) \to W \tag{2.38}$$

where $\mathcal{F}^P$ is the frame bundle of the polarization $P \subset \mathrm{Der}(W)$ and ν transforms under group actions according to (2.37).

Quantum states are now taken as products of a (normalizes) polarized section $\mathcal{S} \in \Gamma^P(L)$ and a half form ν corresponding to the polarization P. The quantum operator $\hat{F}$ of a classical observable $F \in \mathcal{A}$ then becomes the sum of the prequantum action $\mathcal{O}_F$ on $\mathcal{S}$ and the Lie derivative of ν with respect to the Hamiltonian vector field ξ_F :

$$\hat{F}(\mathcal{S}\nu) = \mathcal{O}_F\mathcal{S} \cdot \nu + i\mathcal{S} \cdot \mathcal{L}_{\xi_F} \nu \tag{2.39}$$

Note that this gives the right quantum operator only if F respects the polarization in the sense that

$$[\xi_F, P] \subset P \qquad (2.40).$$

In the case of a Kähler polarization a holomorphic projection [18] is needed to obtain the correct quantum operator for observables not respecting the polarization. However this will not be crucial for our following considerations.

3. Geometric Quantization of Dirac Theory

To elaborate geometric quantization for a Dirac field we consider (according to [3]) the space of solutions of the (massless) Dirac equation

$$\gamma^{\mu}(i\partial_{\mu} + A_{\mu}(x,t))\Psi(x,t) = 0 \tag{3.1}$$

in D spacetime dimensions in a non Abelian background. The elements Ψ of this space are complex D-spinors and the field $A(x,t)$ is regarded as an external gauge connection $A(x,t) = A_a(x,t)T^a$ with T^a generating the gauge group. Our conventions are similar as in [4] and can be found in the appendix (A1). A solution of (3.1) is uniquely determined by its value $\psi_\tau(x)$ at a fixed time τ via

$$\Psi(x,t)|_{t=\tau} =: \psi_\tau(x) \tag{3.2}$$

what respects the linear structure of the solution space. An inner product between solutions of (3.1) is given by :

$$\Psi \odot \tilde{\Psi} := \int\limits_{\Sigma_\tau} \psi_\tau^+(x)\, \tilde{\psi_\tau}(x) d^{D-1}x \qquad (3.3),$$

where Σ_τ denotes the $t = \tau$ hypersurface. This fixes the space under consideration

$$\mathcal{V} := \{\Psi \text{ solution of (3.1)} \mid \Psi \odot \Psi < \infty\} \tag{3.4}$$

As explained above (2.4), this yields a graded manifold with the dual $\mathcal{V}^*$ determined by (3.3) :

$$M_{\mathcal{V}} = (\{p\}, \bigoplus_n \bigwedge{}^n(\mathcal{V}^*)) \tag{3.5}$$

The $(D-1)$ dimensional δ-functions span (formally) $\mathcal{V}^*$ assigning to each $\Psi \in \mathcal{V}$ its value $\psi_\tau(x)$ at some space point x. As explained in (2.10), (2.11) we use $\psi_\tau(x)$ as anticommuting coordinates on $M_{\mathcal{V}}$. (Note that our notation does not distinguish between $\psi_\tau(x)$ as elements of $\mathcal{V}^*$ and as functions on Σ_τ!) With the symplectic form

$$\omega = i \int\limits_{\Sigma_\tau} d\psi_\tau^+(x) d\psi_\tau(x) d^{D-1}x \tag{3.6}$$

on $M_{\mathcal{V}}$ the Poisson bracket (2.22) yields the well known equal time anticommutator

$$\{\psi_\tau(x), \psi_\tau^+(x')\}_+ = -\xi_{\psi_\tau(x)}\psi_\tau^+(x') = +i\delta(x-x') \qquad (3.7).$$

$\mathcal{V}$ is per construction a complex vector space, but the Kähler polarization with respect to the natural complex structure is not acceptable from the physical point of view : It would lead to an energy spectrum which is unbounded from below. For the free theory ($A = 0$) an appropriate polarization is given in [1]. There the operator

$$B_{\rm fr} = \gamma^0 \gamma^j i\partial_j \tag{3.8}$$

is used to split the space of solutions of the free Dirac equation $\mathcal{V}$ into a positive and a negative frequency part in order to define a complex structure by

$$J_{\rm fr}[\psi_\lambda] = i\text{sign}(\lambda)\psi_\lambda \quad \text{for eigenstates} \quad B_{\rm fr}\psi_\lambda(x) = \lambda\psi_\lambda(x) \qquad (3.9).$$

A natural generalization of (3.8) for a theory in a background field is

$$B_t = \gamma^0\gamma^j(i\partial_j + A_j(x,t)) \qquad (3.10).$$

We proceed in analogy to the free case and decompose at $t = \tau$ the function ψ_τ into a formal sum of eigenfunctions φ_n^τ of the Hermitian operator B_τ

$$\psi_\tau = \sum_n c_n^\tau \varphi_n^\tau \quad \text{with} \quad B_\tau \varphi_n^\tau(x) = \lambda_n^\tau \varphi_n^\tau(x) \qquad (3.11)\ .$$

In contrast to the free case λ_n^τ determines the time evolution of φ_n^τ only up to first order, nevertheless φ_n^τ provides a basis of $\mathcal{V}$. Considering (3.11) one should note that the decomposition is not discrete, so the sum over φ_n is only formal and has to be understood as an integration.

In the corresponding coordinate system $\{c_n^\tau\}$ we now can define the complex structure J^τ by

$$J^\tau\left[\frac{\partial}{\partial c_n^\tau}\right] = +i\ \text{sign}(\lambda_n^\tau)\frac{\partial}{\partial c_n^\tau} \qquad J^\tau\left[\frac{\partial}{\partial c_n^{\tau+}}\right] = -i\ \text{sign}(\lambda_n^\tau)\frac{\partial}{\partial (c_n^\tau)^+} \qquad (3.12).$$

This complex structure explicitly depends on τ. As τ can be chosen arbitrarily it defines a time dependent complex structure $J(t)$ by $J(t)|_{t=\tau} := J^\tau$. To describe this in a small neighborhood of τ, i.e. for $t = \tau + \delta t$, we use the (unitary) transformation matrix $\beta_{mn}^{(\tau,t)}$ between the eigenstates of B_τ and B_t

$$\varphi_n^t = \sum_m \beta_{mn}^{(\tau,t)} \varphi_m^\tau \qquad (3.13).$$

Then we have in $\{c_n^\tau\}$-coordinates the complex structure

$$J(\tau+\delta t)\left[\frac{\partial}{\partial c_n^\tau}\right] = i\sum_{lm}\beta_{nl}^{(\tau,t)}\ \text{sign}\ (\lambda_l^t)(\beta_{lm}^{(\tau,t)})^{-1}\frac{\partial}{\partial c_m^\tau} + o(\delta t^2) \qquad (3.14)$$

with a similar expression for $c_n^{\tau+}$. The complex structure is also a functional of the gauge background ($J(t) = J(t)[A]$) and transforms covariantly with respect to local (fixed time) gauge transformations $\alpha(x) = \alpha_a(x)T^a$

$$e^{i\alpha(x)}J(t)[A]e^{-i\alpha(x)} = J(t)[\rho_\alpha A] \qquad (3.15).$$

The Kähler polarization P^τ, determined by $J(t)|_{t=\tau}$ is then given as

$$P^\tau = \mathrm{span}\left\langle \left\{ \frac{\partial}{\partial c_n^\tau} \middle| \lambda_n^\tau < 0 \right\} \oplus \left\{ \frac{\partial}{\partial c_n^{\tau +}} \middle| \lambda_n^\tau > 0 \right\} \right\rangle \tag{3.16}$$

and naturally induces holomorphic (anticommuting) coordinates

$$z_n^\tau = \begin{cases} c_n^\tau & \lambda_n^\tau > 0 \\ c_n^{\tau +} & \lambda_n^\tau < 0 \end{cases} \qquad (z_n^\tau)^+ = \begin{cases} c_n^\tau & \lambda_n^\tau < 0 \\ c_n^{\tau +} & \lambda_n^\tau > 0 \end{cases} \tag{3.17}$$

In order to simplify our notation we will suppress the index τ in the sequel whenever this is possible.

We proceed in the coordinates $\{z_n, z_n^+\}$ on the graded manifold $M_{\mathcal{V}}$ where the symplectic form (3.6) is

$$\omega = i \sum_n dz_n^+ dz_n \tag{3.18}$$

and Θ can be chosen as

$$\Theta = \frac{i}{2}\left(\sum_n z_n^+ dz_n + \sum_n dz_n^+ z_n \right) \tag{3.19}.$$

According to (2.29) polarized sections $\mathcal{S} \in \Gamma^P(L_{\mathcal{V}})$ have to obey

$$\nabla_{\partial_{z_k^+}} \mathcal{S}(z, z^+) = 0 \tag{3.20}$$

Hence with (2.25) and Θ given in (3.19) (see also [1]) we obtain

$$\mathcal{S}(z, z^+) = \sigma(z) \exp(-1/2 \sum_n z_n z_n^+) \tag{3.21}$$

where the $\sigma(z)$ are holomorphic functions. On the (trivial) line bundle $L_{\mathcal{V}}$ over $M_{\mathcal{V}}$ the Hermitian structure defined by

$$(\mathcal{S}, \tilde{\mathcal{S}}) = \mathcal{S}^+ \cdot \tilde{\mathcal{S}} \quad \mathcal{S}, \tilde{\mathcal{S}} \in \Gamma(L_{\mathcal{V}}) \tag{3.22}$$

is compatible with the covariant derivative (2.25) on $L_{\mathcal{V}}$ (c.f.[1] respectively [5]). It extends (formally) to the inner product on the space Γ of sections of $L_{\mathcal{V}}$:

$$< \mathcal{S}(z, z^+), \tilde{\mathcal{S}}(z, z^+) >= \lim_{m \to \infty} \int (\omega')^m (\mathcal{S}, \tilde{\mathcal{S}})(z, z^+) \tag{3.23}$$

To make this formal definition meaningful we can approximate $\mathcal{V}$ as a sequence of finite dimensional vectorspaces V_n as proposed in [12] and [3]. However as a pairing between sections $\mathcal{S} \in \Gamma^P(L)$ (3.23) is well defined if it is understood in terms of the Fock space structure given below.

Geometric quantization of the Dirac equation means to determine the quantum operators (2.39) of any observable and apply it to polarized sections $\mathcal{S} \in \Gamma^P(L_{\mathcal{V}})$. For the coordinate functions z_k, z_k^+ as classical observables the half form contribution $\mathcal{L}_{X_F}\nu$ in

(2.39) vanishes for an appropriate normalization of the half form ν (c.f. (3.33)) and we obtain

$$
\begin{aligned}
\hat{z}_k^+ \mathcal{S}(z,z^+)\,\nu &= (\partial_{z_k}\sigma(z))\ \exp(-1/2\sum_n z_n z_n^+)\,\nu \\
\hat{z}_k \mathcal{S}(z,z^+)\,\nu &= (z_k\cdot\sigma(z))\ \exp(-1/2\sum_n z_n z_n^+)\,\nu
\end{aligned}
\tag{3.24}
$$

This coincides with the well known holomorphic representation of Fermionic field theory (c.f. [11],[12]). We define the vacuum state $|0>\in\Gamma^P(L_\mathcal{V})$ by

$$
|0>=\left(\prod_n z_n\right)\exp(-1/2\sum_m z_m z_m^+)\,\nu=\left(\prod_n z_n\right)\nu \tag{3.25}
$$

where $\prod z_n$ means the formal product over all coordinates z_n. This yields formally

$$
<0|0>=1 \tag{3.26}
$$

what may be regarded either as a definition or as the result of a limiting procedure defining (3.23) and (3.25) properly. Then (3.24) gives the interpretation of $\hat{z}_k^+$ and $\hat{z}_k$, respectively, as creation and annihilation operators

$$
\begin{aligned}
\hat{z}_k|0> &= 0 \\
\hat{z}_k^+|0> &=: |k>
\end{aligned}
\tag{3.27}
$$

for they fulfill the (usual) anticommutation relations

$$
\begin{aligned}
[\hat{z}_k,\hat{z}_l]_+ &= [\hat{z}_k^+,\hat{z}_l^+]_+ = 0 \\
[\hat{z}_k^+,\hat{z}_l]_+ &= \delta_{kl}
\end{aligned}
\tag{3.28}
$$

Together with (3.26) this yields the orthonormality relation

$$
<k|l>=\delta_{kl} \tag{3.29}
$$

The construction of the Fock space given above corresponds to the polarization P^t only at $t=\tau$. To extend this to a time $t=\tau+\delta t$ we use (3.14) to define holomorphic coordinates by

$$
z_n^t=\begin{cases} c_m^\tau\beta_{mn}^{(\tau,t)} & \lambda_n^t>0 \\ (\beta_{nm}^{(\tau,t)})^{-1}c_m^{\tau+} & \lambda_n^t<0\end{cases}\qquad (z_n^t)^+=\begin{cases} c_m^\tau\beta_{mn}^{(\tau,t)} & \lambda_n^t<0 \\ (\beta_{nm}^{(\tau,t)})^{-1}c_m^{\tau+} & \lambda_n^t>0\end{cases} \tag{3.30}
$$

what in some sense corresponds to the interaction picture of quantum mechanics. The dynamics of the system then is determined by the (time dependent) Hamiltonian

$$
H_t=\int_{\Sigma_t}\Psi^+(x,t)\gamma^0\gamma^j(i\partial_j+A_j(x,t))\Psi(x,t)d^3x=\sum_n|\lambda_n^t|(z_n^t)^+z_n^t \tag{3.31}
$$

where we choose the $A_0 = 0$ gauge for sake of simplicity. At $t = \tau$ the corresponding Hamiltonian vector field of H_t is given by

$$\xi_{H_\tau} = +i \sum_n |\lambda_n^\tau| \left((z_n^\tau)^+ \frac{\partial}{\partial (z_n^\tau)^+} - z_n^\tau \frac{\partial}{\partial z_n^\tau} \right) \tag{3.32}$$

preserving the Kähler polarization J^τ. To quantize H_τ we have to consider further the half form contributions. Choosing a reference half form ν_0 on P^τ (3.16) normed by $\nu_0(\partial_{z_1^+}, \ldots, \partial_{z_k^+}, \ldots) = 1$ quantum states are determined as

$$|\Sigma >= \mathcal{S}(z, z^+)\nu_0 = \exp(-1/2 \sum z_n^+ z_n)\sigma(z)\nu_0 \tag{3.33}$$

Then we obtain for the quantum operator $\hat{H}_\tau$

$$\begin{aligned} \hat{H}_\tau |\Sigma > &= -e^{-1/2 \sum z_n^+ z_n} \left(\sum_k |\lambda_k^\tau| z_k^\tau \frac{\partial}{\partial z_k^\tau} \sigma(z^\tau) \right) \nu_0 - \frac{1}{2} \left[\mathrm{Tr}_{P^\tau} \left(\mathcal{L}_{\xi_{H_\tau}} \right) \right] \cdot \nu_0 \\ &= +e^{-1/2 \sum z_n^+ z_n} \left(\sum_k |\lambda_k^\tau| \frac{\partial}{\partial z_k^\tau} z_k^\tau \sigma(z^\tau) \right) \nu_0 \; - \frac{1}{2} \sum_n |\lambda_n^\tau| \, |\Sigma > \end{aligned} \tag{3.34}$$

This confirms the interpretation (3.27) of $\hat{z}_k^+ \hat{=} \partial / \partial z_k$ as creation operator of a one particle state in Fock space with energy $\lambda_n > 0$. The vacuum contribution

$$< 0 | \hat{H}_\tau | 0 >_\tau = -(1/2) \sum_n |\lambda_n^\tau| \tag{3.35}$$

of the Hamiltonian may be compensated by a redefinition of the classical Hamiltonian due to $H_\tau \rightarrow H_\tau + 1/2 \sum_n |\lambda_n^\tau|$, what does not effect the dynamics of the system.

4. The Chiral Anomaly

a) U(1) Anomaly in 4 Dimensions

The chiral transformation on a Dirac field Ψ

$$\delta\Psi(x,t) = -\alpha\gamma^5\Psi(x,t) \tag{4.1}$$

is a symmetry of the equation of motion (3.1). Noether's theorem yields for the γ^5 current

$$(j^5)^\mu(x,t) = \Psi^+(x,t)\gamma^0\gamma^\mu(i\alpha\gamma^5)\Psi(x,t) \tag{4.2}$$

the conservation law

$$\partial_\mu(j^5)^\mu(x,t) = 0 \tag{4.3}$$

To obtain the anomaly of (4.3) we consider the nonconservation of the chiral charge, defined by

$$F^5 = \int\limits_{\Sigma_t}(j^5)^0(x,t)d^3x = \sum_{mn}\int\limits_{\Sigma_t}(\varphi_n^t)^+(x)i\alpha\gamma^5\varphi_m^t(x)d^3x\ c_n^+c_m \tag{4.4}$$

This is precisely the momentum map [13] of the chiral symmetry (4.1) with respect to the symplectic form ω_τ (3.6). To express (4.4) in the holomorphic coordinates (3.17) we note that γ^5 and B_τ commute, so they have a common eigenbase $\{\varphi_n^\tau\}$. With the notion

$$\Phi_{mn}^t := i\int\limits_{\Sigma_t}(\varphi_m^t)^+(x)\alpha\gamma^5\varphi_n^t(x)d^3x \tag{4.5}$$

we see that these matrix elements Φ_{mn}^t vanish if $\lambda_m^t \neq \lambda_n^t$ and obtain

$$F^5 = \sum_{mn}(z_n^t)^+z_m^t\left[\left(\Phi_{mn}^t\right)_{\lambda_n^t>0} - \left(\Phi_{nm}^t\right)_{\lambda_n^t<0}\right] \tag{4.6}$$

This yields the Hamiltonian vector field

$$\xi_{F^5} = +i\sum_{mn}\left[\left(\Phi_{mn}^t\right)_{\lambda_n^t>0} - \left(\Phi_{nm}^t\right)_{\lambda_n^t<0}\right]\left((z_n^t)^+\frac{\partial}{\partial(z_m^t)^+} - z_m^t\frac{\partial}{\partial z_n^t}\right) \tag{4.7}$$

As the chiral transformation (4.1) is a symmetry of the theory, the chiral charge (4.4) is conserved under the (classical) Hamiltonian dynamics (3.32) :

$$\left.\frac{d}{dt}\right|_{t=\tau}F^5 = \xi_{H_\tau}F^5 + \left.\frac{\partial}{\partial t}\right|_{t=\tau}F^5 = 0 \tag{4.8}$$

Note that the term $\partial/\partial t F^5$ occurs due to a possible explicit time dependence via the external field. Quantizing (4.8), i.e. considering the corresponding quantum relation

$$\left.\frac{d}{dt}\right|_{t=\tau}\hat{F}^5 = \left[\hat{H}_\tau, \hat{F}^5\right] + \left.\frac{\partial}{\partial t}\right|_{t=\tau}\hat{F}^5 \tag{4.9}$$

with (3.19) we obtain for the prequantum operators (2.24)

$$\mathcal{O}_{H_\tau} = -i\xi_{H_\tau} \qquad \mathcal{O}_{F^5} = -i\xi_{F^5} \qquad (4.10).$$

Furthermore ξ_{F^5} preserve the Kähler polarization provided by $J(t)$ (3.14). So we obtain at $t = \tau$ for a state $|\Sigma >$ given by (3.33)

$$\hat{F}^5|\Sigma >= -i(\xi_{F^5}\mathcal{S})\nu_0 - \frac{1}{2}\mathcal{S}\left[\mathrm{Tr}_{P^\tau}\left(\mathcal{L}_{\xi_{F^5}}\right)\right]\cdot\nu_0 \qquad (4.11).$$

Fixing $\tau = 0$ for the sequel and using (4.7) we have

$$< 0|\hat{F}^5|0 >_0 = < 0| - i\xi_{F^5}|0 >_0 = -\frac{i}{2}\sum_n \int\limits_{\Sigma_0} (\varphi_n^0)^+(x)\mathrm{sign}(\lambda_n^0)\alpha\gamma^5\varphi_n^0(x)d^3x \qquad (4.12).$$

Then the anomaly is determined by

$$\mathcal{A} = \left.\frac{d}{dt}\right|_{t=0} < 0|\hat{F}^5|0 >_t \qquad (4.13).$$

To compute $< 0|\hat{F}^5|0 >_t$ at $t \neq 0$ we have to use the coordinates provided by (3.30) because of the time dependent polarization P^t. By the classical conservation law it is clear that the prequantum operators commute ($[\mathcal{O}_{H_\tau}, \mathcal{O}_{F^5}] = 0$) so we obtain

$$\begin{aligned}\mathcal{A} &= \left.\frac{\partial}{\partial t}\right|_{t=0}\sum_n\left(\frac{i}{2}\int\limits_{\Sigma_0}(\varphi_n^0)^+(x)(\beta_{nm}^{(t,0)})^{-1}\mathrm{sign}(\lambda_m^t)\alpha\gamma^5\beta_{mn}^{(t,0)}\varphi_n^0(x) + \sigma(\delta t)^2\right)\\ &\simeq \left.\frac{\partial}{\partial t}\right|_{t=0}\sum_n\frac{i}{2}\int\limits_{\Sigma_0}(\varphi_n^0)^+(x)B_t(B_t)^{-1/2}\alpha\gamma^5\varphi_n^0(x)\end{aligned} \qquad (4.14).$$

Here $\simeq$ refers to replacing the eigenvalue expression $\mathrm{sign}(\lambda^t) = \lambda^t\cdot(\lambda^t)^{-1/2}$ by the corresponding formal series in the operator B_t. This an identity in (4.14). However the infinite potentially divergent series demands a regularization. Thus we start the summation over the φ_n^0 from small energy eigenvalues λ_n^0, i.e. we choose a regulator

$$\mathcal{R}_0 = \exp\left(-\frac{(\lambda_n^0)^2}{M^2}\right) \simeq \exp\left(-\frac{B_0^2}{M^2}\right) \tag{4.15}$$

and take the limit $M \to \infty$ after the summation:

$$\mathcal{A} = \frac{\partial}{\partial t}\bigg|_{t=0} \lim_{M\to\infty} \sum_n \frac{i}{2} \int_{\Sigma_0} d^3x \, (\varphi_n^0)^+(x) i B_t (B_t^2)^{-1/2} \alpha\gamma^5 \mathcal{R}_0 \varphi_n^0(x) \qquad (4.16).$$

This expression is well defined and we can proceed in analogy to [14], changing the basis set to plane waves. We let Tr refer to both the trace over gauge group $\mathrm{tr}_{\mathcal{G}}$ and the γ indices tr_γ and define

$$\mathcal{K}^4(x,t) := \lim_{M\to\infty} \mathrm{Tr} \frac{i}{2} \int \frac{d^3k}{(2\pi)^3} e^{+ikx} B_t \left(B_t^2\right)^{-1/2} \alpha\gamma^5 \exp\left(-\frac{B_0^2}{M^2}\right) e^{-ikx} \qquad (4.17)$$

to obtain

$$\mathcal{A} = \frac{\partial}{\partial t}\bigg|_{t=0} \int_{\Sigma_0} d^3x \mathcal{K}^4(x,t) \qquad (4.18).$$

To calculate $\mathcal{K}^4(x,t)$ we define the operator

$$B_t(k,x) := \gamma^0\gamma^j \left(k_j + \frac{i\partial_j}{M} + \frac{A_j(x,t)}{M}\right) \qquad (4.19),$$

substitute $\vec{k} \to \vec{k}M$, eliminate the plane wave from the k integral and obtain

$$\mathcal{K}^4(x,t) = \lim_{M\to\infty} \mathrm{Tr} M^3 \frac{i}{2} \int \frac{d^3k}{(2\pi)^3} B_t(k,x) \left(B_t^2(k,x)\right)^{-1/2} \alpha\gamma^5 \exp(-B_0^2(k,x)) \qquad (4.20).$$

$B_t^2(k,x)$ contains the gauge curvature $F_{jk}(x,t) = i\partial_j A_k - i\partial_k A_j + [A_j, A_k]$:

$$B_t^2(k,x) = \left(k^2 + \frac{2}{M}(\vec{k}\vec{A}(x,t) + i\vec{k}\vec{\partial}) + \frac{1}{M^2}(2i\vec{A}(x,t)\vec{\partial} + i\vec{\partial}\vec{A} + \vec{A}^2 + \vec{\partial}^2) - \frac{\gamma^j\gamma^k}{2M^2} F_{jk}(x,t)\right) \qquad (4.21).$$

Expanding $\mathcal{K}^4(x,t)$ in $\frac{1}{M}$ and using results of the appendix (A6), we see that only terms proportional to $\epsilon^{ijk} A_i F_{jk}$ will contribute in the limit $M \to \infty$. Then with (A10) the Taylor expansion yields

$$\begin{aligned}
\mathcal{K}^4(x,t) = -2i\epsilon^{ijk}\mathrm{tr}_{\mathcal{G}} \Bigg(&\frac{1}{3} \int \frac{d^3k}{(2\pi)^3} |k| e^{-k^2} \alpha F_{jk}(x,0) A_i(x,0) \\
&- \frac{1}{3} \int \frac{d^3k}{(2\pi)^3} \frac{1}{|k|} e^{-k^2} A_i(x,t) \alpha F_{jk}(x,0) \\
&+ \frac{1}{6} \int \frac{d^3k}{(2\pi)^3} \frac{1}{|k|} e^{-k^2} F_{jk}(x,t) \alpha A_i(x,0) \\
&- \frac{1}{4} \int \frac{d^3k}{(2\pi)^3} \frac{1}{|k|^3} e^{-k^2} A_i(x,t) F_{jk}(x,t) \alpha \\
&+ \frac{1}{4} \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{|k|^5} e^{-k^2} A_i(x,t) F_{jk}(x,t) \alpha \Bigg)
\end{aligned} \qquad (4.22).$$

The last two terms are logarithmic divergent but can be properly regulated using (A14). With a cyclic σ permutation and the integration (A11) for the convergent terms we have

$$\begin{aligned}\mathcal{K}^4(x,t) = -\alpha\frac{i\epsilon^{ijk}}{12\pi^2}\mathrm{tr}_{\sigma}\big(&2A_i(x,0)F_{jk}(x,0) - 2A_i(x,t)F_{jk}(x,0) \\ &+A_i(x,0)F_{jk}(x,t) - A_i(x,t)F_{jk}(x,t)\big)\end{aligned} \qquad (4.23),$$

what determines the integrated anomaly to be

$$\begin{aligned}\mathcal{A} &= \frac{i\epsilon^{0ijk}}{4\pi^2}\mathrm{tr}_{\sigma}\int\limits_{\Sigma_0} d^3x\,\alpha\dot{A}_i(x,0)F_{jk}(x,0) \\ &= \frac{i\epsilon^{\mu\nu\rho\sigma}}{16\pi^2}\mathrm{tr}_{\sigma}\int\limits_{\Sigma_0} d^3x\,\alpha F_{\mu\nu}F_{\rho\sigma}\end{aligned} \qquad (4.24)$$

From the computations it is clear that $< 0|F^5|0 >_{t=0} = 0$. Furthermore we can repeat the above calculations for the other components of the γ^5 current :

$$\int\limits_{\Sigma_t}(j^5)^k(x,t)d^3x = \sum_{mn}\int\limits_{\Sigma_t}(\varphi_n^t)^+(x)\gamma^0\gamma^k(i\alpha\gamma^5)\varphi_m^t(x)d^3x\ c_m^+c_n \qquad (4.25).$$

Using (A2) one can show

$$\left\langle 0\left|\int_{\Sigma_t}(\hat{j^5})^k(x,t)d^3x\right|0\right\rangle_{t=0} = 0 \qquad (4.26)$$

All the calculations hold even if the transformation parameter α in (4.4) is taken to be local, i.e. $\alpha = \alpha(x)$. Choosing $\alpha(x) = \delta(x-y)$ this allows to quantize also the local relation (4.3) and derive the nonintegrated form of the anomaly, what coincides with the celebrate result (c.f.[15])

$$\partial_\mu < 0|(j^5)^\mu(x,t)|0 >_{t=0} = \frac{i}{16\pi^2}\mathrm{tr}_{\sigma}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \qquad (4.27).$$

b) The Chiral U(1) Anomaly in D Dimensions

To determine the chiral anomaly in D dimensions (D even) we replace γ^5 by $(-(i)^{D/2}\gamma^{D+1})$ and all above considerations naturally generalize from the 4 dimensional case. However to compute

$$\mathcal{K}^D(x,t) = -\lim_{M\to\infty} \mathrm{Tr} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{+ikx} B_t \left(B_t^2\right)^{-1/2} (i)^{D/2} \alpha \gamma^{D+1} \exp\left(-\frac{B_0^2}{M^2}\right) e^{-ikx} \tag{4.28}$$

explicitly is a more tedious work. Again we make use of the appendix (A6) to argue that in the limit $M \to \infty$ only terms proportional to

$$\epsilon^{ij_1\dots j_{D-1}} A_i F_{j_1 j_2} \dots F_{j_{D-2}j_{D-1}} \tag{4.29}$$

will contribute. With the Taylor coefficients of $(1-x)^{-1/2}$ given by

$$b_n = \frac{(2n)!}{(n!)^2 4^n} \tag{4.30}$$

the expansion of (4.28) yields

$$\begin{aligned}\mathcal{K}^D(x,t) = -(2i)^{D/2} \epsilon tr_{\sigma} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{e^{-k^2}}{|k|} \Bigg(& \sum_{n=0}^{N} A(x,t) b_n \left(\frac{F(x,t)}{2k^2}\right)^n \alpha \frac{1}{(N-n)!} \left(\frac{F(x,0)}{2}\right)^{N-n} \\ & - \sum_{n=0}^{N} \frac{2b_{n+1}(n+1)}{D-1} A(x,t) \left(\frac{F(x,t)}{2k^2}\right)^n \alpha \frac{1}{(N-n)!} \left(\frac{F(x,0)}{2}\right)^{N-n} \\ & - \sum_{n=0}^{N} b_n \left(\frac{F(x,t)}{2k^2}\right)^n \alpha \frac{2(N-n+1)}{(N-n+1)!} \frac{k^2}{D-1} \left(\frac{F(x,0)}{2}\right)^{N-n} A(x,0) \Bigg)\end{aligned} \tag{4.31}$$

Here we suppressed the indices, set $N := D/2 - 1$ and used (A7). The $n = N$ terms of the first and second sum in (4.31) are infrared divergent and have to be integrated with (A14). The rest is a usual Gauß integral (A11) and yields

$$\begin{aligned}\mathcal{K}^D(x,t) = -\epsilon \left(\frac{i}{\pi}\right)^{D/2} tr_{\sigma} \frac{(D/2-1)!}{(D-1)!} \Bigg(& 2b_N A(x,t)(F(x,t))^N \alpha + 2b_N (F(x,t))^N \alpha A(x,0) \\ & + \sum_{n=0}^{N-1} \left(\frac{D-4-2n}{2}\right)! \frac{b_n(D-1)}{(N-n)!} A(x,t)(F(x,t))^n \alpha (F(x,0))^{N-n} \\ & - \sum_{j=0}^{N-1} \left(\frac{D-4-2n}{2}\right)! \frac{2(n+1)b_{n+1}}{(N-n)!} A(x,t)(F(x,t))^n \alpha (F(x,0))^{N-n} \\ & - \sum_{n=0}^{N-1} \left(\frac{D-2-2n}{2}\right)! \frac{2b_n}{(N-n)!} (F(x,t))^n \alpha (F(x,0))^{N-n} A(x,0) \Bigg)\end{aligned} \tag{4.32}$$

After a cyclic σ_J permutation this determines the anomaly in D dimensions by

$$\frac{i}{2}\frac{\partial}{\partial t}\bigg|_{t=0} \mathcal{K}^D(x,t) = -i\kappa_D \epsilon^{ij_1\dots j_{D-1}} \mathrm{tr}_{\sigma_J}\left(\alpha \dot{A}_i(x,0) F_{j_1 j_2}(x,0) \dots F_{j_{D-2} j_{D-1}}(x,0)\right) \quad (4.33)$$

where the coefficient κ_D computes from (4.32) to

$$\kappa_D = \left(\frac{i}{\pi}\right)^{D/2} \frac{(D/2-1)!}{(D-1)!} \left(b_N + \sum_{n=0}^{N-1} b_n\right) \quad (4.34).$$

This induces a recursion formula for κ_D that will be solved by

$$\kappa_D = 2D\left(\frac{i}{4\pi}\right)^{D/2} \frac{1}{(D/2)!} \quad (4.35).$$

Hence the integrated anomaly in D dimensions is determined by

$$\mathcal{A} = -2\frac{(i)^{D/2+1}\epsilon^{\mu_0\dots\mu_{D-1}}}{(D/2)!\,(4\pi)^{D/2}} \int_{\Sigma_0} d^{D-1}x\, \alpha F_{\mu_0\mu_1} \dots F_{\mu_{D-2}\mu_{D-1}} \quad (4.36).$$

c) Non Abelian Anomaly

Also the non Abelian anomaly [15] can be discussed in this framework. Introducing the pair of orthogonal projection operators

$$\Pi_L = \frac{1}{2}\left(1 - i\gamma^5\right) \qquad \Pi_R = \frac{1}{2}\left(1 + i\gamma^5\right) \tag{4.37}$$

the space of solutions of the Dirac equation is split into the direct sum $\mathcal{V} = \mathcal{V}_L \oplus \mathcal{V}_R$ of left and right handed spinors. Now we consider $\mathcal{V}_L$ to be the space of left handed solutions only. As Π_L commutes with B_t (because γ^5 does so) we can choose the base $\{\varphi_n\}$ in (3.12) to be given by eigenstates of Π_L. Thus a base of $\mathcal{V}_L$ is provided by the eigenstates to the eigenvalue 1, denoted by $\{\varphi_n^L\}$. The gauge transformation

$$\delta\Psi(x,t) = -i\alpha_a T^a \Psi(x,t) \tag{4.38}$$

yields for the Noether current

$$j^\mu(x,t) = -\Psi^+(x,t)\gamma^0\gamma^\mu(\alpha_a T^a)\Psi(x,t) \tag{4.39}$$

on classical level a covariant conservation law

$$\mathcal{D}_\mu j^\mu(x,t) = 0 \tag{4.40}.$$

On the quantum level we may obtain the (integrated) non abelian anomaly from

$$\mathcal{A}^{\mathcal{G}} = \int \left(\partial_0 < 0|\hat{j}^0|0 >_t + < 0|[\hat{H},\hat{j}^0]|0 >_t + [A^i(x,t), < 0|\hat{j}^i|0 >_t\right) d^3x \tag{4.41}.$$

The last two terms on the r.h.s can be shown to vanish at $t = 0$. Thus for a theory with left handed Fermions only we have

$$\mathcal{A}_L^{\mathcal{G}} = \left.\frac{\partial}{\partial t}\right|_{t=0} < 0|\hat{F}_L^{\mathcal{G}}|0 >_t \tag{4.42},$$

where similar to (4.4)

$$F_L^{\mathcal{G}} = \sum_{\varphi_m,\varphi_n \in \mathcal{V}_L} \left(\int_{\Sigma_0} (\varphi_n^0)^+(x)\alpha_a T^a \gamma^5 \varphi_m^0(x) d^3x \right) c_m^+ c_n \tag{4.43}.$$

For Π_L eliminates $\mathcal{V}_R$ we can rewrite the summation over $\mathcal{V}_L$ as a sum over all of $\mathcal{V}$ and obtain

$$F_L^{\mathcal{G}} = \sum_{m,n} \left(\int_{\Sigma_0} (\varphi_n^0)^+(x)\alpha_a T^a \, \Pi_L \, \varphi_m^0(x) d^3x \right) c_m^+ c_n \tag{4.44}.$$

The technical calculations of the anomaly now proceeds in the same way as above. However the matrix elements Φ_{mn} (4.5) have to be replaced by

$$\Phi_{mn} \rightarrow \int\limits_{\Sigma_0} \varphi_m^+(x)\alpha_a T^a \left(\frac{1-i\gamma^5}{2}\right) \varphi_n(x) d^3x \tag{4.45}$$

Thus we have to compute instead of (4.17)

$$\mathcal{K}^{\mathcal{G}}(x,t) = \lim_{M\to\infty} \mathrm{Tr} \int \frac{d^3k}{(2\pi)^3} e^{+ikx} B_t \left(B_t^2\right)^{-1/2} \alpha_a T^a \left(\frac{1-i\gamma^5}{2}\right) \exp(-B_0^2/M^2) e^{-ikx} \tag{4.46}$$

With B_t^2 from (4.21) and the properties (A2) on the gamma trace we see that only the part containing γ_5 will contribute. After a cyclic $\mathcal{G}$ permutation we obtain

$$\mathcal{A}_L^{\mathcal{G}} = \frac{i\epsilon^{\mu\nu\rho\sigma}}{32\pi^2} \mathrm{tr}_{\mathcal{G}} \int\limits_{\Sigma_0} d^3x \, \alpha_a T^a F_{\mu\nu} F_{\rho\sigma} \tag{4.47}$$

On the other hand the non Abelian chiral transformation

$$\delta\Psi(x,t) = -\alpha_a T^a \gamma^5 \Psi(x,t) \tag{4.48}$$

can be discussed similarly. For the matrix elements Φ_{mn} we have instead of (4.45)

$$\Phi_{mn} \rightarrow \int\limits_{\Sigma_0} \varphi_m^+(x) i\gamma^5 \alpha_a T^a \left(\frac{1-i\gamma^5}{2}\right) \varphi_n(x) d^3x \tag{4.49}$$

and we obtain for the left handed Fermions

$$\mathcal{A}_L^5 = \frac{i\epsilon^{\mu\nu\rho\sigma}}{32\pi^2} \mathrm{tr}_{\mathcal{G}} \int\limits_{\Sigma_0} d^3x \, \alpha_a T^a F_{\mu\nu} F_{\rho\sigma} \tag{4.50}$$

The same considerations made for a theory with right handed Fermions only yield

$$\begin{aligned} \mathcal{A}_R^{\mathcal{G}} &= -\frac{i\epsilon^{\mu\nu\rho\sigma}}{32\pi^2} \mathrm{tr}_{\mathcal{G}} \int\limits_{\Sigma_0} d^3x \, \alpha_a T^a F_{\mu\nu} F_{\rho\sigma} \\ \mathcal{A}_R^5 &= \frac{i\epsilon^{\mu\nu\rho\sigma}}{32\pi^2} \mathrm{tr}_{\mathcal{G}} \int\limits_{\Sigma_0} d^3x \, \alpha_a T^a F_{\mu\nu} F_{\rho\sigma} \end{aligned} \tag{4.51}$$

In a theory with different gauge connections A_L and A_R for the left handed and right handed Fermions we thus obtain the (covariant) gauge anomaly and the chiral anomaly, respectively as

$$
\begin{aligned}
\mathcal{A}^{\mathcal{V}} &= \frac{i\epsilon^{\mu\nu\rho\sigma}}{32\pi^2} \int\limits_{\Sigma_0} d^3x \; \alpha_a T^a \left(F_{\mu\nu}[A^L] F_{\rho\sigma}[A^L] - F_{\mu\nu}[A^R] F_{\rho\sigma}[A^R] \right) \\
\mathcal{A}^5 &= \frac{i\epsilon^{\mu\nu\rho\sigma}}{32\pi^2} \int\limits_{\Sigma_0} d^3x \; \alpha_a T^a \left(F_{\mu\nu}[A^L] F_{\rho\sigma}[A^L] + F_{\mu\nu}[A^R] F_{\rho\sigma}[A^R] \right)
\end{aligned}
\tag{4.52}
$$

For the calculation of the consistent anomaly one notes, that the current (4.39) is defined by the gauge transformation (4.38) only up to a constant (in the phase space). Thus the anomaly is determined in our framework only up to the covariant derivative of a (local) polynomial in the gauge field. As shown in [15] the difference between covariant and consistent anomaly is an expression of this type.

Appendix

1) For the calculations involving γ matrices in D (even) dimensions we use the the conventions

$$\begin{aligned} \{\gamma^\mu, \gamma^\nu\}_+ &= 2\eta^{\mu\nu} = 2\mathrm{diag}(+, - \ldots, -) \\ (\gamma^\mu)^+ &= \gamma^0\gamma^\mu\gamma^0 \\ \gamma^{D+1} &= \gamma^0\gamma^1 \ldots \gamma^{D-1} \ \Rightarrow \ (\gamma^{D+1})^+ = (-)^{D/2-1}\gamma^{D+1} \end{aligned} \qquad (A1).$$

From this one derives the trace formulas

$$\begin{aligned} \mathrm{tr}_\gamma\left(\gamma^{D+1}\gamma^0\gamma^{j_1}\ldots\gamma^{j_k}\right) &= \begin{cases} 0 & k < D-1 \\ 2^{D/2}\epsilon^{j_1,\ldots j_{D-1}} & k = D-1 \end{cases} \\ \mathrm{tr}_\gamma\left(\gamma^{D+1}\gamma^{j_1}\ldots\gamma^{j_k}\right) &= 0 \qquad \text{for } 0 \notin \{j_1, \ldots j_k\} \\ \mathrm{tr}_\gamma\left(\gamma^0\gamma^{j_1}\ldots\gamma^{j_k}\right) &= 0 \qquad \text{for } 0 \notin \{j_1, \ldots j_k\} \end{aligned} \qquad (A2)$$

2) In computing the integral $\mathcal{K}^D(x,t)$ one has not to take care on ultraviolet divergences because of the Gaußian regulator. However infrared divergences may appear from

$$B_t\left(B_t^2\right)^{-1/2} = \frac{\gamma^0\gamma^j}{M}\frac{k_j + D_j}{|k|} \cdot \left(1 + \frac{2k_jD_j}{k^2M} + \frac{\gamma^k\gamma^l F_{kl}(x,t) + D_jD_j}{k^2M^2}\right)^{-1/2} \qquad (A3)$$

with $D_j = A_j(x,t) + i\partial_j$. Each term of order $(\frac{1}{M})^n$ in the expansion of (A3) will contributes with factors $(\frac{1}{k})^{n-1}$ and $(\frac{1}{k})^n$. Hence for the resulting $D-1$ dimensional integral

$$M^{D-1}\int k^{D-2}dk \ \text{Polynom}\left(\frac{1}{kM}, k\right) \qquad (A4)$$

no IR divergences appear in order M^j for $j > 0$. In order M^0 there are logarithms divergent contribution and for negative powers of M rational divergences appear, what will be discussed below.

3) From (A2) it can be seen that in the $\frac{1}{M}$ expansion of $\mathcal{K}^D(x,t)$ only terms will contribute under the γ-trace containing at least $\frac{D-2}{2}$ factors

$$\frac{1}{M^2}\gamma^j\gamma^k F_{jk} \qquad (A5)$$

On the other hand no more then $\frac{D-1}{2}$ such factors can contribute in the limes $M \to \infty$. So performing the γ-trace and suppressing the indices all terms in the expansion will have the form

$$\lim_{M\to\infty} \mathrm{tr}_{\sigma j} M^{D-1} 2^{D/2}\epsilon \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{e^{-k^2}}{|k|}\mathcal{P}(k^2)\left(-\frac{F}{2M^2}\right)^{D/2-1}\frac{A}{M} \qquad (A6).$$

where $\mathcal{P}(k^2)$ is a Laurent polynom in k^2, determined by the M-expansion of $\mathcal{K}^D(x,t)$. Note that one has to take care on the order of terms in (A6), what will be considered below. For computing the polynom $\mathcal{P}$ explicitly we have to replace terms of the form $k_j(\vec{k}\vec{A})$ under the surface integral :

$$\int d^{D-1}k\ k_i(\vec{k}\vec{A}) = \frac{1}{D-1}\int d^{D-1}k\ k^2 A_i \tag{A7}$$

4) Determining $\mathcal{P}(k^2)$ in (A6) one gets from the expansion of $\mathcal{K}^D$ terms of the form

$$\epsilon^{j_1,\dots,j_n} F_{j_1 j_2}(t)\dots F_{\cdot j_{l-1}}(t)\alpha F_{j_{l+1}\cdot}(0)\dots F_{\cdot j_{k-1}}(0)[i\partial_{j_k} + A_{j_k}(0)]F_{j_{k+1}\cdot}(0)\dots F_{j_{n-1}j_n}(0) \tag{A8}$$

Then we can eliminate the spatial derivative ∂_j from the expression by using

$$\epsilon^{ijk}\left[i\partial_i F_{jk}(t)\right] = \epsilon^{ijk}\left[F_{ij}(t)A_k(t) - A_i(t)F_{jk}(t)\right] \tag{A9}$$

and shift at $t=0$ the field $A(0)$ to the right. So we get for (A8)

$$\epsilon^{j_1,\dots,j_n} F_{j_1 j_2}(t)\dots F_{j_{l-2}j_{l-1}}(t)\alpha F_{j_{l+1}j_{l+2}}(0)\dots F_{j_{n-2}j_{n-1}}(0)A_{j_n}(0) \tag{A10a}$$

After integration by parts ∂_{j_l} acts to the left, so we can use the same argument to show

$$\begin{aligned}&\epsilon^{j_1,\dots,j_n} F_{j_1 j_2}(t)\dots F_{\cdot j_{l-1}}(t)[i\partial_{j_l} + A_{j_l}(t)]F_{j_{l+1}\cdot}(t)\dots F_{\cdot j_k}(t)\alpha F_{j_{k+1}\cdot}(0)\dots F_{\cdot j_n}(0) = \\ &\epsilon^{j_1,\dots,j_n} A_{j_1}(t)F_{j_1 j_2}(t)\dots F_{j_{l-2}j_{l-1}}(t)\alpha F_{j_{l+1}j_{l+2}}(0)\dots F_{j_{n-2}j_{n-1}}(0)A_{j_n}(0))\end{aligned} \tag{A10b}$$

5) The Gaußian integrals in $D-1$ dimensions yield

$$\int \frac{d^{D-1}k}{(2\pi)^{D-1}}|k|^n e^{-k^2} = \left(\frac{1}{\pi}\right)^{D/2}\frac{(D/2-1)!}{2(D-2)!}\left(\frac{D-3+n}{2}\right)! \tag{A11}$$

for $D-2+n$ positive and odd.
Furthermore one has to consider the IR divergent integrals

$$\mathcal{I} = \int \frac{e^{-k^2}k^{D-2}dk}{|k|}\left[b_N\left(\frac{1}{k^2}\right)^N - b_{N+1}(N+1)\frac{2(\vec{k}\vec{k})}{D-1}\left(\frac{1}{k^2}\right)^{N+1}\right] \tag{A12}$$

with $N = D/2-1$ (c.f. (4.22) and (4.31)). To regulate the logarithmic divergence we substitute $|k| \to \sqrt{k^2+\epsilon-\epsilon}$ and expand the denominator around $k_\epsilon^2 := k^2+\epsilon$. This yields

$$
\begin{aligned}
\mathcal{I} = & \sum_j b_N \int e^{-k^2} k^{D-2} c_j^{2N+1} \epsilon^j \left(\frac{1}{k^2+\epsilon}\right)^{N+j+1/2} dk \\
& - \frac{2b_{N+1}(N+1)}{D-1} \sum_j \int e^{-k^2} k^D c_j^{2N+3} \epsilon^j \left(\frac{1}{k^2+\epsilon}\right)^{N+j+3/2} dk
\end{aligned} \tag{A13}
$$

with c_j^{2M+1} the Taylor coefficients of $(\sqrt{1-\epsilon})^{2M+1}$. More rigorously we would have to substitute $k^2 \rightarrow k^2 + M^2\epsilon - M^2\epsilon$ in (4.21) before the expansion of $B_t^{-1/2}$ in M. By this IR contributions are avoided not only for M^0 but in any order. However this also yields (A13). Expressing c_j^{2N+3} by c_j^{2N+1} and b_{N+1} by b_N we receive after an integration by parts of the second term of (A13) :

$$
\begin{aligned}
\mathcal{I} &= 2b_N \int e^{-k^2} k^D \sum_j \frac{c_j^{2N+1}}{D-1} \epsilon^j \left(\frac{1}{k^2+\epsilon}\right)^{N+j+1/2} dk \\
&= 2\frac{b_N}{D-1} \int e^{-k^2} k^D \frac{1}{k^{D-1}} dk \\
&= \frac{b_N}{D-1}
\end{aligned} \tag{A14}
$$

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