Daugavet's Equation and Orthomorphisms

Klaus D. Schmidt

96-1989

Daugavet's equation and orthomorphisms

Klaus D. Schmidt

Abstract. The main result of this paper asserts that each Dunford-Pettis operator on an AL-space having no discrete elements satisfies Daugavet's equation $\| I + T \| = 1 + \| T \|$; this extends a result of Holub on weakly compact operators. The proof is based on some properties of orthomorphisms in a Banach lattice, which also yield a short proof of another result of Holub on Daugavet's equation for bounded operators on an arbitrary AL- or AM-space.

Received by the editors

1980 Mathematics Subject Classification (1985 Revision).

Primary 47A30, 47B55; Secondary 47B38.

Keywords and phrases. Banach lattices, Dunford-Pettis operators, orthomorphisms, Daugavet's equation.

1. Introduction

A linear operator $T : \mathbb{E} \longrightarrow \mathbb{E}$ on a Banach space $\mathbb{E}$ satisfies Daugavet's equation if

$$\| I + T \| = 1 + \| T \|$$

holds, where $I : \mathbb{E} \longrightarrow \mathbb{E}$ denotes the identity operator. Daugavet's equation clearly fails for $T := -I$, but it holds under suitable conditions on $\mathbb{E}$ and T.

The first results on Daugavet's equation were obtained by Daugavet [6] and Lozanovskii [13] who proved that the identity $\| I + T \| = 1 + \| T \|$ holds for each compact operator on $C[0,1]$ or $L^1[0,1]$. These results were subsequently extended into various directions [4,5,7-12,16]; in particular, it follows from results of Foias and Singer [8] and Holub [9,10] that Daugavet's equation remains valid for each weakly compact operator on $C[0,1]$ or $L^1[0,1]$, and that each bounded operator on these spaces satisfies at least one of the identities $\| I + T \| = 1 + \| T \|$ and $\| I - T \| = 1 + \| T \|$.

In the present paper we shall study Daugavet's equation for linear operators on a Banach lattice. Using some properties of orthmorphisms, we shall prove that Daugavet's equation holds for each Dunford-Pettis operator on an AL-space having no discrete elements, and that each bounded operator on an arbitrary AL- or AM-space satisfies at least one of the identities $\| I + T \| = 1 + \| T \|$ and $\| I - T \| = 1 + \| T \|$. The first of these results extends a result of Holub [10] on weakly compact operators; the second is essentially due to Holub [9,10] and has recently been given a short proof by Abramovich [1], but the proof given here is equally short and avoids the use of representation theorems.

Throughout this paper, let $\mathbb{E}$ be a Banach lattice, let $L(\mathbb{E})$ denote the normed ordered vector space of all bounded operators $\mathbb{E} \longrightarrow \mathbb{E}$, and let $I : \mathbb{E} \longrightarrow \mathbb{E}$ denote the identity operator. A linear operator $Q : \mathbb{E} \longrightarrow \mathbb{E}$ is an orthomorphism if it is order bounded and if $Q(B) \subseteq B$ holds for each band B of $\mathbb{E}$. Let $\mathrm{Orth}(\mathbb{E})$ denote the Riesz space [3; Theorem 8.9] of all orthomorphisms $\mathbb{E} \longrightarrow \mathbb{E}$. If $\mathbb{E}$ is either an AL-space or an order complete AM-space with unit, then $L(\mathbb{E})$ is an order complete Banach lattice [3; Theorem 15.3 and the remark preceding it] and $\mathrm{Orth}(\mathbb{E})$ agrees with the (projection) band generated by I in $L(\mathbb{E})$ [3; Theorem 8.11]. This property of $\mathrm{Orth}(\mathbb{E})$ together with Lemma 2.3 below indicates a natural connection between Daugavet's equation and orthomorphisms on Banach lattices.

2. The results

We start with a simple but useful lemma on positive operators:

2.1. Lemma. Let $\mathbb{E}$ be an AL- or AM-space. Then Daugavet's equation holds for each positive $T \in L(\mathbb{E})$.

Proof. Suppose first that $\mathbb{E}$ is an AL-space and consider a positive operator $T : \mathbb{E} \longrightarrow \mathbb{E}$. Then

$$\| (I+T)z \| = \| z \| + \| Tz \|$$

holds for each $z \in \mathbb{E}_+$, and this yields

$$\| I + T \| = 1 + \| T \| .$$

In the case where $\mathbb{E}$ is an AM-space, the assertion now follows by duality. □

Our next result concerns bounded operators which are not necessarily positive:

2.2. Theorem. Let $\mathbb{E}$ be an AL- or AM-space. Then the identity

$$\max \{ \| I+T \| , \| I-T \| \} = 1 + \| T \|$$

holds for each $T \in L(\mathbb{E})$.

Proof. Let us first assume that $\mathbb{E}$ is an order complete AM-space with unit $e \in \mathbb{E}_+$.

For each $U \in \mathrm{Orth}(\mathbb{E})$, we have

$$|I+U| \vee |I-U| = I + |U|$$

and thus

(1) $$\max \{ \| I+U \| , \| I-U \| \} = \| |I+U| \vee |I-U| \| = \| I + |U| \| = 1 + \| U \| ,$$

by [3; Theorem 15.5] and Lemma 2.1.

Consider now $T \in L(\mathbb{E})$ and choose $S \in \mathrm{Orth}(\mathbb{E})$ and $R \in \mathrm{Orth}(\mathbb{E})^{\perp}$ satisfying

$$T = S + R .$$

Since $|R|e$ is dominated by a scalar multiple of e, there exists a positive $Q \in \mathrm{Orth}(\mathbb{E})$ satisfying

$$Qe = |R|e ,$$

by [3; Theorem 8.15]. Moreover, for each $P \in \mathrm{Orth}(\mathbb{E})$, we have

$$|P+Q| \vee |P-Q| = |P| + Q$$

and

$$|P| + |R| = |P+R| = |P-R| ,$$

hence

$$(|P+Q| \vee |P-Q|)e = |P+R|e = |P-R|e ,$$

and thus

$$\max \{ \|P+Q\| , \|P-Q\| \} = \| P+R \| = \| P-R \| . \tag{2}$$

Replacing P by S, $I+S$, and $I-S$ in (2), we obtain

$$\max \{ \|S+Q\| , \|S-Q\| \} = \| T \|$$

$$\max \{ \|I+S+Q\| , \|I+S-Q\| \} = \| I+T \|$$

$$\max \{ \|I-S+Q\| , \|I-S-Q\| \} = \| I-T \| ;$$

similarly, replacing U by $S+Q$ and $S-Q$ in (1), we obtain

$$\max \{ \|I+S+Q\| , \|I-S-Q\| \} = 1 + \| S+Q \|$$

$$\max \{ \|I+S-Q\| , \|I-S+Q\| \} = 1 + \| S-Q \| .$$

This yields

$$\begin{aligned} \max \{ \|I+T\| , \|I-T\| \} &= \max \{ \|I+S+Q\| , \|I+S-Q\| , \|I-S+Q\| , \|I-S-Q\| \} \\ &= 1 + \| T \| . \end{aligned}$$

In the case where $\mathbb{E}$ is an AL-space or an arbitrary AM-space, the assertion now follows by duality. □

The following result is another consequence of Lemma 2.1:

2.3. Lemma. Let $\mathbb{E}$ be an AL-space of an order complete AM-space with unit. Then Daugavet's equation holds for each $T \in \mathcal{L}(\mathbb{E})$ satisfying $I \wedge |T| = 0$.

Proof. By assumption, we have

$$|I+T| = I + |T|$$

and thus

$$\| I + T \| = \| I + |T| \| = 1 + \| T \| ,$$

by Lemma 2.1. □

We now turn to the main result of this paper. Recall that a linear operator $\mathbb{E} \longrightarrow \mathbb{E}$ is a Dunford-Pettis operator if it maps the weakly convergent sequences of $\mathbb{E}$ into the norm convergent sequences of $\mathbb{E}$, and that every Dunford-Pettis operator is bounded. Let $\mathcal{D}(\mathbb{E})$ denote the subspace of $\mathcal{L}(\mathbb{E})$ consisting of all Dunford-Pettis operators $\mathbb{E} \longrightarrow \mathbb{E}$. Also, recall that an element $u \in \mathbb{E}_+ \setminus \{0\}$ is discrete if the ideal generated by u in $\mathbb{E}$ agrees with the subspace generated by u in $\mathbb{E}$.

2.4. Theorem. Let $\mathbb{E}$ be an AL-space having no discrete elements. Then Daugavet's equation holds for each $T \in \mathcal{D}(\mathbb{E})$.

Proof. Consider $T \in \mathcal{D}(\mathbb{E})$ and define $S := I \wedge |T|$. Since $\mathrm{Orth}(\mathbb{E})$ as well as $\mathcal{D}(\mathbb{E})$ are bands of $\mathcal{L}(\mathbb{E})$ [3; Theorems 15.5 and 19.18], we have $S \in \mathrm{Orth}(\mathbb{E}) \cap \mathcal{D}(\mathbb{E})$.

Consider now $z \in \mathbb{E}_+$. For each $y \in [0,Sz]$, there exists some $Q \in \mathrm{Orth}(\mathbb{E})$ satisfying

$$y = QSz$$

and $Qz \in [0,z]$, by [3; Theorem 8.15 and its proof]. Since S is an orthomorphism, we obtain

$$y = QSz = SQz \in S[0,z] ,$$

by [3; Theorems 8.24 and 8.21]. This yields

$$[0,Sz] = S[0,z] .$$

Since S is also a Dunford-Pettis operator, the set $S[0,z]$ is compact [3; Theorem 19.18]. Thus, the order interval $[0,Sz]$ is compact, hence Sz belongs to the band generated by the collection of all discrete elements of $\mathbb{E}$ [2; Theorem 21.12], and the assumption on $\mathbb{E}$ yields $Sz = 0$.
Therefore, we have $I \wedge |T| = S = 0$, and the assertion follows from Lemma 2.3. □

Under the assumption of Theroem 2.4, every weakly compact operator on $\mathbb{E}$ is a Dunford-Pettis operator, but the converse is not true; see [3; Theorems 19.6 and 19.23].

We finally note that Theorem 2.4 cannot be extended to arbitrary AL-spaces: In fact, if $\mathbb{E}$ is an AL-space having a discrete element $u \in \mathbb{E}_+ \setminus \{0\}$, then the band $B(\{u\})$ generated by u is a one-dimensional subspace of $\mathbb{E}$, by [14; Proposition 8.3]; consequently, the band projection $P : \mathbb{E} \longrightarrow B(\{u\})$ is compact, but Daugavet's equation fails for $T := -P$.

3. Remarks.

The following results can be proven in the same way as Lemmas 2.1 and 2.3:

3.1. Lemma. Let $\mathbb{E}$ be a Banach lattice satisfying $\| x \|^p + \| y \|^p = \| x+y \|^p$ for some $p \in [1,\infty)$ and all $x,y \in \mathbb{E}_+$. If $J \in L(\mathbb{E})$ is a positive isometry, the

$$\| J + T \| = 1 + \| T \|$$

holds for each positive $T \in L(\mathbb{E})$.

3.2. Lemma. Let $\mathbb{E}$ be an AL-space. If $J \in L(\mathbb{E})$ is a positive isometry, then

$$\| J + T \| = 1 + \| T \|$$

holds for each $T \in L(\mathbb{E})$ satisfying $J \wedge |T| = 0$.

Corresponding results hold in an AM-space or an order complete AM-space with unit, respectively, if in Lemmas 2.1 and 2.3 the identity operator is replaced by a surjective positive isometry.

Obviously, Daugavet's equation holds for $T := I$, and this implies that the condition of Lemma 2.3 is only sufficient, but not necessary, for Daugavet's equation to hold. Also, the fact that Daugavet's equation fails for $T := -I$ can be generalized as follows: If $T \in L(\mathbb{E})$ satisfies $0 < T \leq I$, then $\| I - T \|$ is strictly smaller than $1 + \| T \|$.

We conclude with a brief discussion of Daugavet's equation for almost integral and absolute kernel operators:

Let $\mathbb{E}$ be an AL-space or an order complete AM-space with unit. A linear operator $T : \mathbb{E} \longrightarrow \mathbb{E}$ is an <u>almost integral operator</u> if it is contained in the band of $L(\mathbb{E})$ which is generated by the linear operators $S : \mathbb{E} \longrightarrow \mathbb{E}$ satisfying $Sz = x'(z)y$ for some $x' \in \mathbb{E}$ and $y \in \mathbb{E}$ depending on S and all $z \in \mathbb{E}$. Synnatzschke [15] proved that $\mathbb{E}$ has no discrete elements if and only if each almost integral operator $T : \mathbb{E} \longrightarrow \mathbb{E}$ satisfies $I \wedge |T| = 0$, and in this case it follows from Lemma 2.3 that T satisfies Daugavet's equation. In the case where $\mathbb{E}$ is an AL-space, this result can also be deduced from [17; Theorem 123.5(ii)], [3; Theorem 19.18], and Theorem 2.4; in the case where $\mathbb{E}$ is an order complete AM-space with unit, it is due to Synnatzschke [16].

Consider now a σ-finite measure space $(\Omega, \mathcal{F}, \mu)$ and $p \in [1, \infty]$. A linear operator $T : L^p(\Omega, \mathcal{F}, \mu) \longrightarrow L^p(\Omega, \mathcal{F}, \mu)$ is an <u>absolute kernel operator</u> if there exists an $\mathcal{F} \times \mathcal{F}$-measurable function $t : \Omega \times \Omega \longrightarrow \mathbb{R}$ satisfying

$$\int_\Omega |t(.,\omega)| f(\omega) \, d\mu(\omega) \in L^p(\Omega, \mathcal{F}, \mu)$$

and

$$(Tf)(.) = \int_\Omega t(.,\omega) f(\omega) \, d\mu(\omega)$$

for all $f \in L^p(\Omega, \mathcal{F}, \mu)$. If $(\Omega, \mathcal{F}, \mu)$ has no atoms, then $L^p(\Omega, \mathcal{F}, \mu)$ has no discrete elements and a linear operator $L^p(\Omega, \mathcal{F}, \mu) \longrightarrow L^p(\Omega, \mathcal{F}, \mu)$ is an absolute kernel operator if and only if it is an almost integral operator [17; Theorem 94.7]. Thus, for $p \in \{1, \infty\}$, Daugavet's equation holds for each absolute kernel operator $L^p(\Omega, \mathcal{F}, \mu) \longrightarrow L^p(\Omega, \mathcal{F}, \mu)$ whenever $(\Omega, \mathcal{F}, \mu)$ has no atoms.

Acknowledgement.

Part of this work was done during my visit of Indiana University / Purdue University at Indianapolis in autumn 1988. I would like to express my gratitude to Y.A. Abramovich, C.D. Aliprantis, and O. Burkinshaw for their hospitality and stimulating discussions on the subject, and to the Deutsche Forschungsgemeinschaft for financial support.

References

[1] Y.A. Abramovich,
A new proof of a theorem of J. Holub,
preprint (1988).

[2] C.D. Aliprantis and O. Burkinshaw,
Locally Solid Riesz Spaces,
Academic Press, New York and London 1978.

[3] C.D. Aliprantis and O. Burkinshaw,
Positive Operators,
Academic Press, New York and London 1985.

[4] V.F. Babenko and S.A. Pichugov,
A property of compact operators in the space of integrable functions,
Ukrainian Math. J. 33 (1981), 374-376.

[5] P. Chauveheid,
On a property of compact operators in Banach spaces,
Bull. Soc. Roy. Sci. Liège 51 (1982), 371-378.

[6] I.K. Daugavet,
A property of compact operators in the space C ,
Uspekhi Mat. Nauk 18 (1963), 157-158. (Russian).

[7] U.I. Diallo and P.P. Zabreiko,
On the Daugavet-Krasnoselskii theorem,
Vestsi Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk (1987), 26—31. (Russian).

[8] C. Foias and I. Singer,
Points of diffusion of linear operators and almost diffuse operators in spaces of continuous functions,
Math. Z. 87 (1965), 434-450.

[9] J.R. Holub,
A property of weakly compact operators on C[0,1] ,
Proc. Amer. Math. Soc. 97 (1986), 396-398.

[10] J.R. Holub,
Daugavet's equation and operators on $L^1(\mu)$,
Proc. Amer. Math. Soc. 100 (1987), 295-300.

[11] H. Kamowitz,
A property of compact operators,
Proc. Amer. Math. Soc. 91 (1984), 231-236.

[12] M.A. Krasnoselskii,
A class of linear operators in the space of abstract continuous functions,
Math. Notes 2 (1967), 856-858.

[13] G.Y. Lozanovskii,
On almost integral operators in KB-spaces,
Vestnik Leningrad. Univ. (1966), no. 7, 35-44.
(Russian).

[14] H.H. Schaefer,
Banach Lattices and Positive Operators,
Springer, Berlin and New York 1974.

[15] J. Synnatzschke,
On almost integral operators in K-spaces,
Vestnik Leningrad Univ. Math. 4 (1977), 243-252.

[16] J. Synnatzschke,
On the adjoint of a regular operator and some of its applications to the question of complete continuity and weak complete continuity of regular operators,
Vestnik Leningrad Univ. Math. 5 (1978), 71-81.

[17] A.C. Zaanen,
Riesz Spaces II,
North-Holland, Amsterdam and New York 1983.

Klaus D. Schmidt
Seminar für Statistik
Universität Mannheim
A 5
6800 Mannheim
West Germany