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Constitutive laws of bounded smoothly deformable media

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0. Introduction

Let us think of a material body moving and deforming in the Euclidean space $\mathbb{R}^n$. We make the geometric assumption that at any time the body is a n–dimensional, compact, connected, oriented and smooth manifold with boundary. The boundary shall be oriented too, but shall not necessarily be connected. The material should be a deformable medium. The deformable medium forming the boundary may differ from the one forming the inside of the body.

During the motion of the body the diffeomorphism type of the manifold with boundary is assumed to be fixed. Hence we can think of a standard body M, which from a geometrical point of view is a manifold diffeomorphic to the one moving and deforming in $\mathbb{R}^n$.

Thus a configuration is a smooth embedding from M into $\mathbb{R}^n$. The configuration space is hence the collection $E(M,\mathbb{R}^n)$ of all smooth embeddings of M into $\mathbb{R}^n$.
This set equipped with Whitney's C^∞–topology is a Fréchet manifold (cf.[Bi,Sn,Fi]). A smooth motion of the body in $\mathbb{R}^n$ therefore is described by a smooth curve in $E(M,\mathbb{R}^n)$. The calculus on Fréchet manifolds used in the sequel is the one presented in [Bi,Sn,Fi], which in our setting coincides with the one developed in [Fr,Kr].

The physical quality of the deforming medium certainly enters the work $F(J)(L)$ needed to deform (infinitesimally) the material at any configuration $J \in E(M,\mathbb{R}^n)$ in any direction L.
The directions are tangent vectors to $E(M,\mathbb{R}^n)$. Since the ladder space is open in the Fréchet space $C^\infty(M,\mathbb{R}^n)$ of all smooth $\mathbb{R}^n$–valued functions endowed with the C^∞–topology (cf.[Hi]), a tangent vector is thus nothing else but a function in $C^\infty(M,\mathbb{R}^n)$ and vice versa.

In the following we take F, which is an one–form on $E(M,\mathbb{R}^n)$, as a constitutive law. We do not discuss the question as to whether F characterizes the material fully or not. Throughout these notes we assume that F is smooth.

To allow only internal physical properties of the material to enter F, we have to specify the constitutive law somewhat more. Basic to this specification is the fact that these sorts of constitutive properties should not be affected by the particular location of the body in $\mathbb{R}^n$. Thus F has to be invariant under the operation of the translation group. Moreover if L is any constant map, we assume that $F(J)(L) = 0\,,\ \forall\, J \in E(M,\mathbb{R}^n)$ also.
The forms F, which have these two properties, can be regarded as one forms on $\{dJ \mid J \in E(M,\mathbb{R}^n)\}$, where dJ is the differential of any J. This set of differentials is equipped with the C^∞–topology as well and is denoted by $E(M,\mathbb{R}^n)/_{\mathbb{R}^n}$. The latter space is a Fréchet manifold, too. It admits a natural weak Riemannian metric of an L_2–type.

A smooth one–form on $E(M,\mathbb{R}^n)/_{\mathbb{R}^n}$ will be denoted by $F_{\mathbb{R}^n}$. Hence we deal with one–forms of the type $F = d^* F_{\mathbb{R}^n}$.

To handle this one–form F we assume that $F_{\mathbb{R}^n}$ can be represented via the metric mentioned by an integral, which we call the Dirichlet integral used in the field of partial differential equations on parts of $\mathbb{R}^n$.
The integral kernel of F is a differential of some smooth map $\mathfrak{H} \in C^\infty(E(M,\mathbb{R}^n)/_{\mathbb{R}^n}, C^\infty(M,\mathbb{R}^n))$, called a constitutive map.
Hence in our setting we characterize the medium as far as the internal physical properties enter $\mathfrak{H}$.
The constitutive function $\mathfrak{H}$ determines at any $dJ \in E(M,\mathbb{R}^n)/_{\mathbb{R}^n}$ two smooth force densities $\Phi(dJ)$ and $\varphi(dJ)$ linked to $\mathfrak{H}$ by :

$$\Delta(J)\mathfrak{H}(dJ) = \Phi(dJ)$$

and

$$d\mathfrak{H}(dJ)(N) = \varphi(dJ)$$

and the integrability condition necessary to solve this von Neumann problem.

Here $\Delta(J)$ is the Laplacian determined by the Riemannian metric $J^* < , >$, where $< , >$ is the fixed scalar product on $\mathbb{R}^n$. N is the positively oriented unite normal of ∂M in M. Vice versa any pair of force densities (Φ, φ) satisfying the integrability condition for the von Neumann problem determines some constitutive map $\mathfrak{H}$ of the above mentioned type.
Let us point out here that in these notes we neither discuss any dynamics nor do we study equilibrium conditions ! We only investigate the notion of a constitutive law in the above sense.

Since F is affected by the material forming the boundary, we treat in an analogous way the boundary material and exhibit in analogy to $\mathfrak{H}$ a characteristic constitutive map $\mathfrak{h}$. Thus $\Delta(j)\mathfrak{h}(dJ)$ with $j := J|\partial M$ and $J \in E(M,\mathbb{R}^n)$ describes the force density $\tilde{\varphi}(dJ)$ up to a constant force along ∂M. However $d\mathfrak{H}(dJ)(N)$ also determines force densities which can not be of the form $\tilde{\varphi}(dJ)$. Any specific properties of the boundary enter additively in $\mathfrak{h}$. An additive part of $\mathfrak{h}$ is the constitutive map for the boundary material thought to be detached from the body. Hence the rest of $\mathfrak{h}$ describes the influence of the body material to the boundary material implemented into the body.

Finally we show that both $\mathfrak{H}$ and $\mathfrak{h}$ are structured in the following sense :
In $\mathfrak{H}$ and in $\mathfrak{h}$ is, from a mathematical point of view, generically and naturally encoded the

work needed to deform volume, area and shape of the body and boundary respective. The shape is partly expressed in the unite normal vector field N(j) along the embedding of the boundary. Here $N(j) = \pm dJ\,N$; the plus–sign holds if J is orientation preserving, the minus–sign otherwise.

The procedure to decode the influence mentioned is to use an L_2–splitting of $d\mathfrak{H}(dJ)$.

1. The space of configurations, the phase space and geometric preliminaries

Let us think of a material body moving and deforming in the Euclidean space $\mathbb{R}^n$. We make the geometric assumption that at any time the body is a n–dimensional, compact, connected, oriented and smooth manifold with boundary. The boundary shall be oriented too, but shall not necessarily be connected. The material should be a deformable medium. The deformable medium forming the boundary may differ from the one forming the inside of the body.

During the motion of the body the diffeomorphism type of the manifold with boundary is assumed to be fixed.

Hence we can think of a standard material body M. By this we mean the following :
The underlying point set of the body is a smooth, compact, oriented and connected manifold with oriented boundary ∂M. Let us assume that the orientation on ∂M is the one induced by the orientation of M. The dimension of M is assumed to be n. The body constitutes of a deformable medium and we use M to denote both, the manifold with boundary and the material body.

From this situation we read off what we mean by a configuration :
A configuration is a smooth embedding

$$J : M \longrightarrow \mathbb{R}^n .$$

Hence the space of configurations is $E(M,\mathbb{R}^n)$, the collection of all smooth embeddings of M into $\mathbb{R}^n$.

Clearly each $J \in E(M,\mathbb{R}^n)$ induces a smooth embedding

$$J|\partial M : \partial M \longrightarrow \mathbb{R}^n .$$

This we call a configuration of the boundary of the body. Let us denote the collection of all smooth embeddings of ∂M into $\mathbb{R}^n$ by $E(\partial M,\mathbb{R}^n)$.

To see what the phase space is, let us first of all observe that the set $E(M,\mathbb{R}^n)$ is obviously a subset of $C^\infty(M,\mathbb{R}^n)$, the collection of all smooth $\mathbb{R}^n$–valued maps of M. Clearly $C^\infty(M,\mathbb{R}^n)$ is a $\mathbb{R}$–vector space.
We equip it with the C^∞–topology, also called the Whitney topology in [Hi]. Since M is compact, $C^\infty(M,\mathbb{R}^n)$ is a complete metrizable locally convex space, a so–called Fréchet space.

$E(M,\mathbb{R}^n)$, an open in $C^\infty(M,\mathbb{R}^n)$, inherits hence the C^∞–topology too. The phase space is therefore

$$TE(M,\mathbb{R}^n) = E(M,\mathbb{R}^n) \times C^\infty(M,\mathbb{R}^n) .$$

Proceeding for ∂M as for M we obtain $E(\partial M,\mathbb{R}^n)$ as an open subset of the Fréchet space $C^\infty(\partial M,\mathbb{R}^n)$ (cf.[Hi]). Hence also $E(\partial M,\mathbb{R}^n)$ is a Fréchet manifold with obviously trivial tangent bundle. The phase space for the boundary is then $E(\partial M,\mathbb{R}^n) \times C^\infty(\partial M,\mathbb{R}^n)$.

The next Lemma shows the relation between the two configuration spaces, i.e. the two spaces of embeddings :

Lemma 1.1 :

The restriction map

$$R : C^\infty(M,\mathbb{R}^n) \longrightarrow C^\infty(\partial M,\mathbb{R}^n) ,$$

assigning to each $J \in C^\infty(M,\mathbb{R}^n)$ the map $J|\partial M$, is surjective. The image $R(E(M,\mathbb{R}^n))$ is open in $E(\partial M,\mathbb{R}^n)$. Hence

$$TR : E(M,\mathbb{R}^n) \times C^\infty(M,\mathbb{R}^n) \longrightarrow R(E(M,\mathbb{R}^n)) \times C^\infty(\partial M,\mathbb{R}^n)$$

has the form $TR = R \times R$ and is surjective.

Proof :

Let $j \in C^\infty(\partial M,\mathbb{R}^n)$. By the collar theorem (cf.[Hi], p.113) ∂M admits a collar in M. This is to say that there is an open neighborhood $S \subset M$ of ∂M which is C^∞–diffeomorphic to $\partial M \times [0,\infty)$ via a map ρ, say. For simplicity we identify S with $\partial M \times [0,\infty)$ via ρ. Given any $l \in C^\infty(\partial M,\mathbb{R}^n)$, let $L \in C^\infty(S,\mathbb{R}^n)$ be defined by

$$L(p,s) = \psi(s)\cdot l(p) , \quad \forall\, s \in [0,\infty) \text{ and } p \in \partial M ,$$

where $\psi : [0,\infty) \longrightarrow \mathbb{R}$ is a smooth map being identical to one on [0,1] and which vanishes on $[2,\infty)$.

The map L extends l to all of S. The map L itself extends to all of M by putting it identically zero on the complement of S in M. Clearly $L \in C^\infty(M,\mathbb{R}^n)$ and is such that $R(L) = l$. Let us prove the second assertion next :

Let $J \in E(M,\mathbb{R}^n)$ be given and let us call $R(J)$ by j. For any $\lambda \in [0,\infty)$ we let $j(\lambda) := J | \partial M \times \{\lambda\}$. Clearly the family $j(\lambda)$ depends smoothly on λ. It obviously defines a smooth curve with $j(0) = j$. Let us choose an open convex neighborhood $O \subset E(\partial M,\mathbb{R}^n)$ of $j \in E(\partial M,\mathbb{R}^n)$ and let $\lambda_0 \in [0,\infty)$ be such that $j(\lambda_0) \in O$. We deviate now from the curve induced by J as follows : We extend the curve $j(\lambda)$ at λ_0 by a straight line along its tangent up to $j_2 \in O$, say. From here we pass on with a straight line segment to any given $j^1 \in O$. Clearly we can smooth out this curve at j_2 without affecting $j(\lambda)$ with $\lambda > 2\lambda_0$. Hence we have a smooth curve σ linking $j(\lambda_0)$ with j^1. By construction $\sigma(0) = j$. The smooth embedding

$$\tilde{J} : \partial M \times [0,\infty) \longrightarrow \mathbb{R}^n$$

defined by

$$\tilde{J}(p,\lambda) = \begin{cases} J(p,\lambda) & \lambda > 2\lambda_0 \quad \forall p \in M \\ \sigma(\lambda)(p) & \lambda \leq 2\lambda_0 \quad \forall p \in M \end{cases} \tag{1.1}$$

smoothly links with $J | (M \setminus \partial M \times [0,\infty))$. Thus we have a smooth $J^1 \in E(M,\mathbb{R}^n)$ such that $J^1 | \partial M = j^1$. The remaining assertions are obvious.

In the sequel of these notes we write O_∂ instead of $R(E(M,\mathbb{R}^n))$.

On the configuration space we have a natural action a by the translation groups $\mathbb{R}^n$ of $\mathbb{R}^n$ namely

$$a : E(M,\mathbb{R}^n) \times \mathbb{R}^n \longrightarrow E(M,\mathbb{R}^n)$$

assigning to each $J \in E(M,\mathbb{R}^n)$ and each $z \in \mathbb{R}^n$ the embedding $J + z$.
This action extends obviously to $C^\infty(M,\mathbb{R}^n)$. The translation group $\mathbb{R}^n$ acts accordingly on $E(\partial M,\mathbb{R}^n)$. This action restricts to O_∂ and obviously extend also to $C^\infty(\partial M,\mathbb{R}^n)$.
The orbit spaces of the respective actions are denoted by $C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n}$, $C^\infty(\partial M,\mathbb{R}^n)/_{\mathbb{R}^n}$, $E(M,\mathbb{R}^n)/_{\mathbb{R}^n}$, $E(\partial M,\mathbb{R}^n)/_{\mathbb{R}^n}$ and $O_\partial/_{\mathbb{R}^n}$.

The nature of these spaces are easily understood if we introduce for any $L \in C^\infty(M,\mathbb{R}^n)$ the differential dL which is locally given by the Fréchet derivative. Hence the tangent map TL of L is (L,dL). The respective notion of $l \in C^\infty(\partial M,\mathbb{R}^n)$ is introduced accordingly.

Hence the orbit spaces mentioned above are nothing else but spaces of differentials of the elements of those spaces on which $\mathbb{R}^n$ acts.

M and ∂M inherit via respective embedding into $\mathbb{R}^n$ some basic geometric structures described below.

Let us fix a scalar product and a normed determinant function $\underline{\Delta}$ (cf.[Gr]) on M.

Then each $j \in E(\partial M,\mathbb{R}^n)$ yields a unite normal vector field with

$$N(j) : \partial M \longrightarrow \mathbb{R}^n ,$$

with

$$<N(j),N(j)> = 1 ,$$

for which $j^* i_{N(j)}\underline{\Delta}$ determines the orientation class of ∂M. Here $j^* i_{N(j)}\underline{\Delta}$ denotes the pullback of the (n–1)–form $i_{N(j)}\underline{\Delta}$ to ∂M by j. Moreover $i_{N(j)}\underline{\Delta} := \Delta(N(j),\ldots)$.
Each $J \in E(M,\mathbb{R}^n)$ and each $j \in E(\partial M,\mathbb{R}^n)$ yield Riemannian metrics m(J) and m(j) on M and ∂M respectively. These metrics are defined by

$$m(J)(X,Y) = <dJX,dJY> , \quad \forall X,Y \in \Gamma TM \tag{1.2}$$

and

$$m(j)(X,Y) = <djX,djY> , \quad \forall X,Y \in \Gamma T\partial M . \tag{1.3}$$

Here ΓTQ denotes the collection of all smooth vector fields of any smooth manifold Q (with or without boundary).
Both m(J) and m(j) depend smoothly on its variables J and j.

For any $J \in E(M,\mathbb{R}^n)$ and any $j \in E(\partial M,\mathbb{R}^n)$ let us denote by $\mu(J)$ and $\mu(j)$ the Riemannian volume form determined by m(J) and the orientation of M respectively by m(j) and the orientation of ∂M. Clearly

$$i_{N(j)}\underline{\Delta} = \mu(j) \tag{1.4}$$

Let us point out that there is a normal vector field $N \in TM/\partial M$ such that

$$i_N\mu(J) = \mu(j) . \tag{1.5}$$

Hence we have

$$d\mathfrak{J}(N) = N(j)\,, \tag{1.6}$$

if $j := J\,|\,\partial M$ and if J is orientation preserving.
Clearly

$$\mu(J) = J^{*}\underline{\Delta}\,, \tag{1.7}$$

provided J preserves the orientation. These embeddings $J \in E(M,\mathbb{R}^n)$ for which (1.7) hold form an open set in $E(M,\mathbb{R}^n)$.

Associated with the metrics m(J) and m(j), we have the respective Levi–Civita connections $\nabla(J)$ on M and $\nabla(j)$ on ∂M. They are determined by

$$dJ\,\nabla(J)_X Y = d(dJY)(X)\,, \quad \forall\, X,Y \in \Gamma TM \tag{1.8}$$

and

$$dj\,\nabla(j)_X Y = d(djY)(X) - m(j)(W(j)X,Y)\cdot N(j)\,, \quad \forall\, X,Y \in \Gamma T\partial M. \tag{1.9}$$

By W(j) we mean the Weingarten map given by

$$dN(j)Z = djW(j)Z\,, \quad \forall\, Z \in \Gamma TM\,. \tag{1.10}$$

2. The constitutive law

We characterize the type of the material which constitutes the body M as far as it affects the work done if M is infinitesimally distorted (cf.[He], [E,S], [Bi], [Bi,Sc,So]). This idea is formalized by giving a smooth one–form on $E(M,\mathbb{R}^n)$, i.e. a smooth map

(2.1) $$F : E(M,\mathbb{R}^n) \times C^\infty(M,\mathbb{R}^n) \longrightarrow \mathbb{R} ,$$

which varies linearly in the second argument. We interpret $F(J)(L)$ as the work done if M at the configuration $J \in E(M,\mathbb{R}^n)$ is distorded by $L \in C^\infty(M,\mathbb{R}^n)$. We call the medium described by F a smoothly deformable medium.

It might be of physical significance that F depends on further parameters, so e.g. in case one likes to model a visco–elastic material (cf.[Bi,Sc,So]). However we restrict us for simplicity to forms of the type (2.1) since complications such as those just mentioned do not affect the basic apparatus.

It is intuitively clear that the work caused by internal physical processes, initiated by a distortion L at a particular configuration J, should not depend on the particular location of $J(M)$ within $\mathbb{R}^n$. That is to say this work is the same if J is replaced by $J + z$ for any $z \in \mathbb{R}^n$. Moreover a distortion by any $z \in \mathbb{R}^n$ should not cause any work due to these processes mentioned above. These ideas written more formally yield the following equations basic to our further development :

(2.2) $$F(J + z) = F(J) , \quad \forall J \in E(M,\mathbb{R}^n) , \forall z \in \mathbb{R}^n$$

and

(2.3) $$\text{(ii.)} \ F(J)(z) = 0 , \quad \forall J \in E(M,\mathbb{R}^n) , \forall z \in \mathbb{R}^n .$$

A one–form F on $E(M,\mathbb{R}^n)$ satisfying (2.2) and (2.3) is in the sequel called a *constitutive law.*

The lemma below is obvious :

Lemma 2.1 :

A smooth one–form

$$F : E(M,\mathbb{R}^n) \times C^\infty(M,\mathbb{R}^n) \longrightarrow \mathbb{R}$$

is a constitutive law iff it is of the form

(2.4) $$F = d^* F_{\mathbb{R}^n}$$

that is

(2.5) $$F(J)(L) = F_{\mathbb{R}^n}(dJ)(dL) , \qquad \forall J \in E(M,\mathbb{R}^n) \text{ and } \forall L \in C^\infty(M,\mathbb{R}^n) ,$$

where

(2.6) $$F_{\mathbb{R}^n} : E(M,\mathbb{R}^n)/_{\mathbb{R}^n} \times C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n} \longrightarrow \mathbb{R}$$

is a smooth one–form.

3. Integral representation of constitutive laws, the Dirichlet integral

The purpose of this section is to define what is meant by an integral representation of a one–form $F_{\mathbb{R}^n}$ on $E(M,\mathbb{R}^n)$.

In order to define this representation we first will introduce a quadric structure on $E(M,\mathbb{R}^n) \times A^1(M,\mathbb{R}^n)$, which is based on the dot product of any two $\mathbb{R}^n$–valued one–forms on M relative to an embedding of M into $\mathbb{R}^n$. We denote the smooth $\mathbb{R}^m$–valued one–form of any smooth manifold Q by $A^1(M,\mathbb{R}^m)$.
Let $\gamma \in A^1(M,\mathbb{R}^n)$ and $J \in E(M,\mathbb{R}^n)$ be given. The two tensor $<\gamma,dJ>$ determined by γ and J shall be given by $<\gamma X,dJY>$ for all $X,Y \in \Gamma TM$. This two tensor $<\gamma,dJ>$ yields a unique strong bundle map $A(\gamma,dJ)$ of TM defined by

(3.1) $$<\gamma X,dJY> = m(J)(A(\gamma,dJ)X,Y)\,, \quad \forall X,Y \in \Gamma TM.$$

From this equation we read off :

(3.2) $$\gamma X = dJA(\gamma,dJ)X\,, \quad \forall X \in \Gamma TM\,.$$

For any two one–forms $\gamma_1, \gamma_2 \in A^1(M,\mathbb{R}^n)$ and an embedding $J \in E(M,\mathbb{R}^n)$ we define the above mentioned dot product of γ_1 and γ_2 relative to J by

(3.3) $$\gamma_1 \cdot \gamma_2 := \operatorname{tr} A(\gamma_1,dJ)\cdot \tilde{A}(\gamma_2,dJ)\,.$$

Here $\tilde{A}(\gamma_2,dJ)$ is the adjoint of $A(\gamma_1,dJ)$ formed fibre–wise with respect to m(J).
Associated with this product is a weak scalar product $G_{\mathbb{R}^n}(dJ)$ on $A^1(M,\mathbb{R}^n)$ defined by

(3.4) $$G_{\mathbb{R}^n}(J)(\gamma_1,\gamma_2) := \int_M \gamma_1\cdot\gamma_2\,\mu(J)\,.$$

As mentioned before $\mu(J)$ denotes the Riemannian volume form determined by m(J) and the given orientation of M.

Weak means here, that $G_{\mathbb{R}^n}(J)$ does not determine the dual space of $C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n}$ to $A^1(M,\mathbb{R}^n)$.

We equip $A^1(M,\mathbb{R}^n)$ with the C^∞–topology (cf.[Bi,Sn,Fi]). The real number $G_{\mathbb{R}^n}(dJ)(\gamma_1,\gamma_2)$

depends smoothly on all its variables dJ, γ_1 and γ_2.

Since $C^{\infty}(M,\mathbb{R}^n)/_{\mathbb{R}^n} \subset A^1(M,\mathbb{R}^n)$ the quadric structure on $E(M,\mathbb{R}^n) \times A^1(M,\mathbb{R}^n)$ $G_{\mathbb{R}^n}$ yields a weak Riemannian structure on $E(M,\mathbb{R}^n)/_{\mathbb{R}^n}$ again denoted by $G_{\mathbb{R}^n}$.

We say that $F_{\mathbb{R}^n}$, a one–form on $E(M,\mathbb{R}^n)/_{\mathbb{R}^n}$, admits an integral repesentation if there exists a smooth map

$$\alpha : E(M,\mathbb{R}^n) \longrightarrow A^1(M,\mathbb{R}^n),$$

called the kernel of $F_{\mathbb{R}^n}$ such that for any choices of $dJ \in E(M,\mathbb{R}^n)/_{\mathbb{R}^n}$ and $dL \in C^{\infty}(M,\mathbb{R}^n)/_{\mathbb{R}^n}$

$$(3.5) \qquad F_{\mathbb{R}^n}(dJ)(dL) = \int_M \alpha(J) \cdot dL \; \mu(J) = G_{\mathbb{R}^n}(J)(\alpha(J), dJ)$$

holds true.

We speak of a constitutive law F with integral kernel α, if $F = d^* F_{\mathbb{R}^n}$ and $F_{\mathbb{R}^n}$ admits an integral representation with kernel α.

To discuss the uniqueness of α, if it exists of all, we first prove the following :

Theorem 3.1 :

Let $\gamma \in A^1(M,\mathbb{R}^n)$ and $J \in E(M,\mathbb{R}^n)$. There exists a uniquely determined differential $d\mathfrak{H} \in C^{\infty}(M,\mathbb{R}^n)$ called the exact part of γ and a uniquely determined $\beta \in A^1(M,\mathbb{R}^n)$ such that

$$(3.6) \qquad \gamma = d\mathfrak{H} + \beta,$$

where the exact part of β vanishes. Both $d\mathfrak{H}$ and β depend smoothly on J. If $\mathfrak{H}(p_0)$ for some $p_0 \in M$ is kept constant in J, then also $\mathfrak{H}$ varies smoothly in J.

Proof :

First let us construct $\mathfrak{H}$ and β. To this end we fix a basis $e_1,\ldots,e_n$ on $\mathbb{R}^n$, orthonormal with respect to $<\,,\,>$. Then

(3.7)
$$\gamma(X) = \sum_{r=1}^{n} \gamma^r(X)\, e_r\,, \qquad \forall\, X \in \Gamma TM\,,$$

with $\gamma^r \in A^1(M,\mathbb{R}^n)$ for all $r=1,\ldots,n$. Since for each r

(3.8)
$$\gamma^r(X) = m(J)(Y^r,X)\,, \qquad \forall\, X \in \Gamma TM$$

holds true for a well defined $Y^r \in \Gamma TM$, we find due to Hodge's decomposition (cf.[A,M,R]) a function $\tau^r \in C^\infty(M,\mathbb{R}^n)$ and a uniquely determined vector field $Y_0^r \in \Gamma TM$ such that the following three equations are satisfied

(3.9)
$$Y^r = \mathrm{grad}_J \tau^r + Y_0^r$$

and

(3.10)
$$\mathrm{div}_J Y_0^r = 0$$

together with the boundary condition

$$m(J)(Y_0^r,N) = 0 \ \text{ along } \partial M.$$

Here the indices J in grad_J and div_J mean that the respective operations are formed with respect to $m(J)$.

This decomposition is obtained by solving the following von Neumann problem

(3.11)
$$\Delta(J)\tau^r = -\mathrm{div}_J Y^r$$

with the boundary condition

(3.12)
$$d\tau^r(N) = m(J)(Y^r,N)\,.$$

This problem has, according to [Hö], a solution τ^r unique up to a constant.

The desired function $\mathfrak{H}$ and the form β are defined by

$$\mathfrak{H} := \sum_r \tau^r \cdot e_r \tag{3.13}$$

and

$$\beta(X) := \sum_r m(J)(Y_0{}^r, X), \qquad \forall X \in \Gamma TM, \tag{3.14}$$

respectively. It is a matter of routine to show that $d\mathfrak{H}$ and β do not depend on the basis chosen. With these notions we immediately deduce

$$\gamma = d\mathfrak{H} + \beta. \tag{3.15}$$

To see that the exact part of β vanishes let us assume that $\psi^r \in C^\infty(M,\mathbb{R}^n)$ is such that for each r=1,...,n

$$\mathrm{grad}_J \psi^r + Y^r_{00} = Y^r_0, \tag{3.16}$$

for some divergence free vector field Y^r_{00} perpendicular to the normal field N. Then

$$\Delta(J)\, \psi^r = -\mathrm{div}_J \mathrm{grad}_J \psi^r = 0 \quad \text{and} \quad \psi^r = \text{const.}. \tag{3.17}$$

Thus the exact part of β vanishes.

To discuss smoothness properties of $\mathfrak{H}$ in J let us show next that both $\mathrm{grad}_J \tau^r$ and Y^r_0 depend smoothly on $J \in E(M,\mathbb{R}^n)$. To approach our goal, we consider a smoothly parameterized family $J(t) \in E(M,\mathbb{R}^n)$ with t varying in $\mathbb{R}$. We assume that J(0) coincides with a fixed $I \in E(M,\mathbb{R}^n)$. Thus

$$dJ(t) = dI\, A(dJ(t),dI), \quad \forall t \in \mathbb{R} \tag{3.18}$$

and hence

$$\nabla(J(t))_Y X = \nabla(I)_Y X + A(dJ(t),dI)^{-1} \nabla(I)_Y (A(dJ(t),dI))X \tag{3.19}$$

holds for any choice of $X,Y \in \Gamma TM$. Since $\nabla(J(t))$ is torsion free for any $t \in \mathbb{R}$ the following equation is valued for all $X,Y \in \Gamma TM$

$$\nabla(I)_Y (A(dJ(t),dI))X = \nabla(I)_X (A(dJ(t),dI))Y. \tag{3.20}$$

With these formulas we deduce immediately

$$\text{(3.21)} \qquad \operatorname{grad}_{J(t)}\tau = A(dJ(t),dI)^{-1}\cdot\tilde{A}(dJ(t),dI)^{-1}\operatorname{grad}_J\tau ,$$

for any $\tau \in C^\infty(M,\mathbb{R}^n)$ and

$$\text{(3.22)} \qquad \operatorname{div}_{J(t)}X = \operatorname{div}_I X + \operatorname{tr} A(dJ(t),dI)^{-1}\nabla(I)_X(A(dJ(t),dI)) ,$$

both holding for all t and all $X \in \Gamma TM$.

Let $Y \in \Gamma TM$. First we assume that the following three equations associated with the Hodge–decomposition

$$\text{(3.23)} \qquad Y = \operatorname{grad}_{J(t)}\tau(J(t)) + Y^0(J(t)) ,$$

where

$$\text{(3.24)} \qquad \operatorname{div}_{J(t)}Y^0(J(t)) = 0$$

and

$$\text{(3.25)} \qquad d\tau(J(t))(N) = m(J(t))(Y,N) ,$$

all depend smoothly on t. Then (3.18), (3.21) and (3.22) yield the next three equations

$$\text{(3.26)} \qquad d\dot{J}(0) = dI\; A(d\dot{J}(0),dI) ,$$

$$\text{(3.27)} \qquad \frac{d}{dt}\operatorname{grad}\Big|_{t=0}\tau = -2\,A(d\dot{J}(0),dI)_{sym}\operatorname{grad}_I\tau ,$$

$$\forall \text{ fixed } \tau \in C^\infty(M,\mathbb{R}) ,$$

where $A(d\dot{J}(0)dI)_{sym}$ denotes the selfadjoint part of $A(d\dot{J}(0),dI)$ formed with respect to $m(I)$ via the polar decomposition (cf.[Bi.Sn,Fi]) and finally

$$\text{(3.28)} \qquad \frac{d}{dt}\operatorname{div}_{J(t)}\Big|_{t=0} X = \operatorname{tr}\nabla(I)A(d\dot{J}(t),dI)X .$$

Using the last three formulas, the derivatives of (3.23), (3.24) and (3.25) with respect to t read therefore as

$$(3.29)\qquad 0 = -2\, A(d\dot{J}(0),dI)_{sym}\mathrm{grad}_I \tau(I) + \mathrm{grad}_I \dot{\tau}(I) + \dot{Y}^0(I)\,,$$

$$(3.30)\qquad \mathrm{div}_I \dot{Y}(I) = -\mathrm{tr}\; \nabla(I)A(d\dot{J}(0),dI)Y^0(I)$$

and

$$(3.31)\qquad d\dot{\tau}(I)(N) = \dot{m}(I)(Y,N)\,.$$

Applying div_I to (3.29) yields the equations

$$(3.32)\qquad \Delta(I)\dot{\tau}(I) = -2\, \mathrm{div}_I(A(d\dot{J}(0),dI)_{sym}\mathrm{grad}_I \tau(I)) + \mathrm{tr}\; \nabla(I)A(d\dot{J}(0),dI)Y^0(I)\,,$$

with its boundary condition

$$(3.33)\qquad d\dot{\tau}(I)N = \dot{m}(I)(Y,N)\,.$$

Turning back to the problem of showing the smoothness in t of (3.23), (3.24) and (3.25) the equations (3.32) and (3.33) pose a von Neumann problem with $\dot{\tau}(I)$ as the unknown, provided we drop the smoothness assumption in connection with (3.23) and (3.24). The right hand sides of both (3.32) and (3.33) are smooth. As we already know such problems have a solution unique up to a constant. Without loss of generality we may assume that for some $p_0 \in M$

$$(3.34)\qquad \tau(J(t))(p_0) = 0\,, \quad \forall t \in \mathbb{R}\,,$$

which in turn suggests that $\dot{\tau}(I)(p_0) = 0$.
Equation (3.32) produces a candidate for $\dot{\tau}(I)$ and if we insert $\dot{\tau}(I)$ into (3.29) we obtain a candidate for $\dot{Y}^0(I)$. Now it is a matter of routine to verify that these candidates in fact do satisfy

$$(3.35)\qquad \frac{1}{t}\lim_{t\to 0} (\tau(J(t))-\tau(I)) = \dot{\tau}(I)$$

and

$$(3.36)\qquad \frac{1}{t}\lim_{t\to 0} (Y^0(J(t))-Y^0(I)) = \dot{Y}^0(I)\,,$$

respectively. Since $I \in E(M,\mathbb{R}^n)$ was chosen arbitrarily we obtain $\dot{\tau}(t)$ and $\dot{Y}^0(J(t))$ for all $t \in \mathbb{R}$.

To show the existence of all higher derivatives we have to set up an induction procedure based on (3.26), (3.27), (3.28), (3.29), (3.30), (3.31) and (3.32), which to execute is left to the reader. Both, therefore $\tau(J(t))$ and $Y(J(t))$ depend smoothly on $t \in \mathbb{R}$. Since the parameterization in t was arbitrarily, we conclude by the criterion in the calculus presented in [Fr,Kr], that both $\tau(J)$ and $Y(J)$ depend smoothly on $J \in E(M,\mathbb{R}^n)$. This ends the proof.

Some of the calculation made in the proof above allow us to look at $G_{\mathbb{R}^n}(J)$ from another angle. Given $\gamma \in A^1(M,\mathbb{R}^n)$ and $J \in E(M,\mathbb{R}^n)$ we have according to (3.7) and (3.8) in above proof

(3.37) $$\gamma(X) = dJA(\gamma,dJ)X = \sum_{r=1}^{n} m(J)(Y^r,X)\, e_r\,, \quad \forall X \in \Gamma TM\,.$$

Let us denote $(dJ)^{-1}e_r$ by E_r, for all $r=1,\ldots,n$. Then we read off from the equation (3.37) that

(3.38) $$Y^r = \tilde{A}(\gamma,dJ)E_r\,, \quad \forall r=1,\ldots,n\,,$$

holds true. This remark yields the following observation :

Proposition 3.2 :

Given $\gamma_1, \gamma_2 \in A^1(M,\mathbb{R}^n)$, $J \in E(M,\mathbb{R}^n)$ and a fixed basis $e_1,\ldots,e_n$ on $\mathbb{R}^n$ orthonormal with respect to $<\,,\,>$, then there exist two sets

$$Y_1^{\,1},\ldots,Y_1^{\,n} \quad \text{and} \quad Y_2^{\,1},\ldots,Y_2^{\,n}$$

of vector fields in ΓTM, such that

(3.39) $$\gamma_1 \cdot \gamma_2 = \sum_{r=1}^{n} m(J)(Y_1^{\,r},Y_2^{\,r})$$

and hence

$$(3.40) \qquad G_{\mathbb{R}^n}(J)(\gamma_1,\gamma_2) := \int_M \gamma_1\cdot\gamma_2\,\mu(J) = \sum_{r=1}^{n} \int_M m(J)(Y_1^{\,r},Y_2^{\,r})\mu(J)\,.$$

If in addition $\gamma_1 = d\mathfrak{H}$ for some $\mathfrak{H} \in C^\infty(M,\mathbb{R}^n)$ then $G_{\mathbb{R}^n}(J)(d\mathfrak{H},\gamma_2) = 0$ provided that the exact part of γ_2 vanishes.

Proof :

Let $Y_i^{\,r} \in \Gamma TM$, r=1,...,n and i=1,2 be as in (3.8). Then

$$(3.41) \qquad \begin{aligned} \gamma_1\cdot\gamma_2 &= \operatorname{tr} A(\gamma_1,dJ)\cdot\tilde{A}(\gamma_2,dJ) \\ &= \sum_{r=1}^{n} m(J)(A(\gamma_1,dJ)\cdot\tilde{A}(\gamma_2,dJ)E_r,E_r) \\ &= \sum_{r=1}^{n} m(J)(Y_1^{\,r},Y_2^{\,r}) \end{aligned}$$

establishing (3.39). To show the last part of the proposition we use Gauss' theorem as follows :

$$(3.42) \qquad \begin{aligned} G_{\mathbb{R}^n}(dJ)(\gamma_1,\gamma_2) &= \sum_{r=1}^{n} \int_M m(J)(\operatorname{grad}_J\tau^r,Y_0^r) \\ &= \sum_{r=1}^{n} \int_M d\tau^r(Y_0^r)\mu(J) \\ &= \sum_{r=1}^{n} \int_M (\operatorname{div}_J(\tau^r\, Y_0^r) - \tau\cdot\operatorname{div}_J Y_0^r)\mu(J) \\ &= \sum_{r=1}^{n} \int_M m(J)(\tau^r\, Y_0^r,N)\mu(J) \\ &= 0\ . \end{aligned}$$

Here we have $\mathfrak{H} = \sum_{r=1}^{n} \tau^r e_r$ and $\operatorname{div}_J Y_0^r = 0$ as well as $m(J)(Y_0^r,N) = 0$.

In case γ_1 and γ_2 in the above proposition are exact, then the respective vector fields in (3.39) are gradients. Hence the right hand side of the integral in (3.40) is the classical Dirichlet integral (cf.[J]) for $\mathbb{R}^n$–valued functions.

The integral in the middle part of (3.40) hence generalizes and reformulates the Dirichlet integral. We call it therefore the Dirichlet integral of any two smooth $\mathbb{R}^n$–valued forms γ_1, γ_2 relative to $J \in E(M,\mathbb{R}^n)$.

Proposition 3.2 also shows that the integral kernel of a constitutive law is not unique at all. We may add to any kernel a map which assumes as its values, one–forms of which the integrable part vanishes. However the following theorem guaranties us a uniqueness of a very specific type of kernel :

Theorem 3.3 :

Let F be a constitutive law with integral kernel. There exists a unique smooth map

$$\alpha : E(M,\mathbb{R}^n)/_{\mathbb{R}^n} \longrightarrow C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n} \subset A^1(M,\mathbb{R}^n)\,, \tag{3.43}$$

such that for any $J \in E(M,\mathbb{R}^n)$ and any $L \in C^\infty(M,\mathbb{R}^n)$

$$F(J)(L) = \int_M \alpha(dJ)\cdot dL\ \mu(J) \tag{3.44}$$

holds true. In fact there is a unique smooth map

$$\mathfrak{H} : E(M,\mathbb{R}^n)/_{\mathbb{R}^n} \longrightarrow C^\infty(M,\mathbb{R}^n)\,, \tag{3.45}$$

satisfying the following two equations

$$\alpha(dJ) = d\mathfrak{H}(dJ)\,, \quad \forall\, dJ \in E(M,\mathbb{R}^n)/_{\mathbb{R}^n} \tag{3.46}$$

and

$$\int_{\partial M} \langle \mathfrak{H}(dJ), z\rangle\, \mu(J) = 0\,, \quad \forall\, z \in \mathbb{R}^n. \tag{3.47}$$

Proof :

The existence of such a kernel is guaranteed by proposition 3.2. The uniqueness follows easily :
Let α_1 and α_2 be two kernels with values in $C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n}$. Then we would have

$$(3.48) \qquad \int_M (\alpha_1 - \alpha_2)(J) \cdot dL\, \mu(J) = 0 ,$$

for all $J \in E(M,\mathbb{R}^n)$ and all $dL \in C^\infty(M,\mathbb{R}^n)$. Since for all $dL = \alpha_1 - \alpha_2$, G is positive definite, we conclude $\alpha_1 = \alpha_2$. To show that $\mathfrak{H}$ exists and can be chosen to satisfy (3.47) we introduce $C^\infty_J(M,\mathbb{R}^n)$, the collection of all $L \in C^\infty(M,\mathbb{R}^n)$ satisfying

$$(3.49) \qquad \int_M \langle L,z\rangle \mu(J) = 0 , \quad \forall z \in \mathbb{R}^n ,$$

for a given $J \in E(M,\mathbb{R}^n)$. With this space at hand we have the splitting

$$(3.50) \qquad C^\infty(M,\mathbb{R}^n) = \mathbb{R}^n \oplus C^\infty_J(M,\mathbb{R}^n) .$$

Equipping $C^\infty_J(M,\mathbb{R}^n)$ with the C^∞–topology, yields a Fréchet space also denoted by $C^\infty_J(M,\mathbb{R}^n)$. Since for any two $I,J \in E(M,\mathbb{R}^n)$.

$$(3.51) \qquad m(J)(X,Y) = m(I)(B(dJ,dI)^2X,Y) , \quad \forall X,Y \in \Gamma TM,$$

holds for a uniquely determined smooth strong bundle isomorphism $B(dJ,dI)$ of TM, we conclude that

$$(3.52) \qquad C^\infty_J(M,\mathbb{R}^n) = \det(B(dJ,dI)) \cdot C^\infty_I(M,\mathbb{R}^n) .$$

Clearly

$$(3.53) \qquad d : C^\infty_J(M,\mathbb{R}^n) \to C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n}$$

is an isomorphism for each J. Let us denote it by d_J. The desired map $\mathfrak{H}$ is given by

$$\mathfrak{H}(dJ) := d_J^{-1} \alpha(dJ) .$$

Moreover we have the projection

$$P(J) : C^\infty(M,\mathbb{R}^n) \longrightarrow C^\infty_J(M,\mathbb{R}^n) \tag{3.54}$$

determined by (3.50). Equation (3.52) shows that $P(J)$ is smooth in $J \in E(M,\mathbb{R}^n)$. Hence if α is a kernel of the form (3.43), then $\mathfrak{H}(dJ)$ is smooth in $J \in E(M,\mathbb{R}^n)$, satisfies (3.46) as well as (3.47) and by construction

$$P(J) \circ \mathfrak{H}(J) = \alpha(J) , \quad \forall\, J \in E(M,\mathbb{R}^n), \tag{3.55}$$

holds true.

4. Force densities associated with constitutive laws admitting kernels

The purpose of this section is to associate with any constitutive law admitting integral kernels at any configuration some well defined force densities, one acting upon the whole body, and one acting upon the boundary only.

Throughout this section F is a constitutive law admitting a kernel α. By the previous theorem we may assume that α maps into $C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n}$.

To construct the force densities mentioned we use F in the form

$$(4.1) \qquad F(J)(L) = \int_M \operatorname{tr} A(\alpha(dJ),dJ)\cdot\tilde{A}(dL,dJ)\,\mu(J)\,,$$

holding for any of the variables of F. Writing any $L \in C^\infty(M,\mathbb{R}^n)$ relative to a given $J \in E(M,\mathbb{R}^n)$ in the form

$$(4.2) \qquad L = dJ\,X(L,J)\,,$$

with a unique $X(L,J) \in \Gamma TM$ we have

$$(4.3) \qquad dL\,X = dJ\,\nabla(J)_X X(L,J)\,, \qquad \forall X \in \Gamma TM\,.$$

and hence derive immediately

$$(4.4) \qquad A(dL,dJ) = \nabla(J)\,X(L,J)\,, \qquad \forall L \in C^\infty(M,\mathbb{R}^n)\,.$$

Thus if $e_1,\dots,e_n$ is a orthonormal basis of $\mathbb{R}^n$ and if we define $E_r \in \Gamma TM$ again by $dJ\,E_r = e_r$ for $r=1,\dots,n$ then

$$(4.5) \qquad F(J)(L) = \sum_{r=1}^{n} \int_M m(J)(\tilde{A}(\alpha(dJ),dJ)\cdot\nabla(J)_{E_r} X(L,J),E_r)\,\mu(J)\,.$$

Let us introduce the notion $\operatorname{div}_J T$, the divergence of a strong bundle endomorphism T of TM by

$$(4.6) \qquad \operatorname{div}_J T := \sum_{r=1}^{n} \nabla(J)_{E_r}(T)(E_r)\,.$$

This notion does not depend of the basis chosen. Equation (4.6) together with (4.5) imply

$$F(J)(L) = \int_M \mathrm{div}_J(\tilde{A}(\alpha(dJ),dJ)X(L,J))\,\mu(J) - \int_M m(J)(\mathrm{div}_J A(\alpha(dJ),dJ),X(L,J))\,\mu(J)\,. \tag{4.7}$$

To bring these formulas in a more familiar form we introduce the notions of $\Delta(J)K$ and $\Delta(J)\gamma$, the Laplacian, for any $K \in C^\infty(M,\mathbb{R}^n)$ and any $\gamma \in A^1(M,\mathbb{R}^n)$. In doing so we follow [Mat]. We set

$$d^* K = 0\,. \tag{4.8}$$

If $\gamma \in A^1(M,\mathbb{R}^m)$ for some natural number m, we set

$$d^* \gamma = -\sum_{r=1}^{n} \nabla(J)_{E_r}(\gamma)(E_r)\,. \tag{4.9}$$

Clearly

$$d^* \gamma = -\mathrm{div}_J Y\,, \tag{4.9a}$$

if

$$\gamma(X) = m(J)(Y,X)\,, \quad \forall X \in \Gamma TM.$$

$\Delta(J)$ is then defined by

$$\Delta(J) := dd^* + d^* d\,, \tag{4.10}$$

Consequently we have

$$\Delta(J)K = d^* dK = \sum_{r=1}^{n} \nabla(J)_{E_r}(dK)(E_r)\,. \tag{4.11}$$

Since the two expressions $\nabla(J)_Y(dK)X$ and $\nabla(J)_Y(T)X$ respectively formed for any $K \in C^\infty(M,\mathbb{R}^n)$ and any strong bundle map T of TM and any choices of $X,Y \in \Gamma TM$ are by definition $d(dK(X))Y - dK\,\nabla(J)_X Y$ and $\nabla(J)_Y(TX) - T\,\nabla(J)_Y X$, we find

$$(4.12)\qquad \Delta(J)K = -\Big(\sum_{r=1}^{n} d(dJ\,A(dK,dJ\,E_r))(E_r) - dJ\,A(dK,dJ)(\nabla(J)_{E_r}E_r)\Big)$$
$$= -\sum_{r=1}^{n} dJ\,\nabla(J)_{E_r}(A(dK,dJ))E_r$$
$$= -dJ\,\mathrm{div}_J A(dK,dJ)\,.$$

Hence equation (4.7) turns into

$$(4.13)\qquad F(J)(L) = \int_M \mathrm{div}_J \tilde{A}(\alpha(dJ),dJ)X(L,J)\,\mu(J) + \int_M <\Delta(J)\mathfrak{H}(dJ),L>\,\mu(J)\,,$$

with $\alpha(dJ) = d\mathfrak{H}(dJ)$ for some $\mathfrak{H} \in (C^\infty(E(M,\mathbb{R}^n)/_{\mathbb{R}^n},C^\infty(M,\mathbb{R}^n))$.

Using Gauss'theorem we derive with the help of theorem 3.3 the following

Proposition 4.1 :

Let F be a constitutive law admitting a kernel. Then for each $J \in E(M,\mathbb{R}^n)$ there exists a smooth map

$$\mathfrak{H} : E(M,\mathbb{R}^n) \to C^\infty(M,\mathbb{R}^n)$$

uniquely determined up to a smooth map from $E(M,\mathbb{R}^n)$ into $\mathbb{R}^n$ for which

$$(4.14)\qquad F(J)(L) = \int_M <\Delta(J)\mathfrak{H}(dJ),L>\mu(J) + \int_{\partial M} <d\mathfrak{H}(dJ)(N),L>i_N\,\mu(J)$$

and hence a Green's equation

$$(4.14a)\qquad \int_M <\Delta(J)\mathfrak{H}(dJ),L>\mu(J) - \int_M <\mathfrak{H}(dJ)\Delta(J),L>\mu(J)$$
$$= \int_{\partial M} <dL(N),\mathfrak{H}(dJ)>i_N\,\mu(J) - \int_{\partial M} <d\mathfrak{H}(dJ)(N),L>i_N\,\mu(J)$$

both holding for all variables of F. Here $i_N\,\mu(J)$ is the volume element on ∂M defined by

$\mu(J)$ and N, the positively oriented unite normal vector field of $\partial M \subset M$.

We call the map $\mathfrak{H}$ in proposition 4.1 a constitutive map because it fully determines the constitutive law.

The above proposition motivates us to set for any $J \in E(M,\mathbb{R}^n)$

(4.15) $$\Phi(J) := \Delta(J)\, \mathfrak{H}(dJ)$$

and

(4.16) $$\varphi(dJ) := d\mathfrak{H}(dJ)(N)\,,$$

with $\mathfrak{H}(dJ)$ as in (4.14).

We call the maps Φ and φ the force densities associated with F. These force densities determine F by

(4.17) $$F(J)(L) = \int_M <\Phi(dJ),L>\mu(J) + \int_{\partial M} <\varphi(dJ),L>i_N\mu(J)\,,$$

for all $J \in E(M,\mathbb{R}^n)$ and all $L \in C^\infty(M,\mathbb{R}^n)$.

Since $\mathfrak{H}$ is smooth both Φ and φ are smooth $C^\infty(M,\mathbb{R}^n)$–valued respectively $C^\infty(\partial M,\mathbb{R}^n)$–valued functions on $E(M,\mathbb{R}^n)/_{\mathbb{R}^n}$.

Given vice versa two smooth maps

(4.18) $$\Phi : E(M,\mathbb{R}^n)/_{\mathbb{R}^n} \longrightarrow C^\infty(M,\mathbb{R}^n),$$

(4.19) $$\varphi : E(M,\mathbb{R}^n)/_{\mathbb{R}^n} \longrightarrow C^\infty(\partial M,\mathbb{R}^n),$$

for which the integrability condition

(4.20) $$0 = \int_M <\Phi(dJ),z>\mu(J) + \int_{\partial M} <\varphi(dJ),z>i_N\mu(J)\,,\ \ \forall Z \in \mathbb{R}^n$$

holds, there exists for each $J \in E(M,\mathbb{R}^n)/_{\mathbb{R}^n}$ a smooth map

$$\overline{\mathfrak{H}}(dJ) : M \longrightarrow \mathbb{R}^n , \tag{4.21}$$

such that the von Neumann problem

$$\Delta(J)\overline{\mathfrak{H}}(dJ) = \Phi(dJ) \tag{4.22}$$

with the boundary condition

$$d\overline{\mathfrak{H}}(dJ)(N) = \varphi(dJ) \tag{4.23}$$

is solvable uniquely up to a constant. With these force densities we define a one–form F on $E(M,\mathbb{R}^n)$ by

$$F(J)(L) = \int_M <\Delta(J)\overline{\mathfrak{H}}(dJ),L>\mu(J) + \int_{\partial M} <d\overline{\mathfrak{H}}(dJ)(N),L> i_N\mu(J) , \tag{4.24}$$

for all $J \in E(M,\mathbb{R}^n)$ and for all $L \in C^\infty(M,\mathbb{R}^n)$. F is a constitutive law due to (4.20).

We now apply proposition 4.1 to obtain a smooth map

$$\mathfrak{H} : E(M,\mathbb{R}^n)/_{\mathbb{R}^n} \longrightarrow C^\infty(M,\mathbb{R}^n) ,$$

producing

$$\Phi(dJ) = \Delta(J)\ \mathfrak{H}(dJ) \tag{4.25}$$

and

$$\varphi(dJ) = d\mathfrak{H}(dJ)(N) , \tag{4.26}$$

for all $J \in E(M,\mathbb{R}^n)$. Thus we have the following

Theorem 4.2 :

Every constitutive law with integral kernel admits a smooth constitutive map

(4.27) $$\mathfrak{H} : E(M,\mathbb{R}^n)/_{\mathbb{R}^n} \longrightarrow C^\infty(M,\mathbb{R}^n)\ ,$$

uniquely determined up to a map in $C^\infty(E(M,\mathbb{R}^n)/_{\mathbb{R}^n},\mathbb{R}^n)$, such that the kernel of F is given by

(4.28) $$d\mathfrak{H} : E(M,\mathbb{R}^n)/_{\mathbb{R}^n} \longrightarrow C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n}$$

and which moreover satisfies

(4.29) $$F(J)(L) = \int_M <\Delta(J)\mathfrak{H}(dJ),L>\mu(J) + \int_{\partial M} <d\mathfrak{H}(dJ)(N),L> i_N\mu(J)$$

on all of $TE(M,\mathbb{R}^n)$. The map $\mathfrak{H}$ determines two smooth maps

$$\Phi : E(M,\mathbb{R}^n)/_{\mathbb{R}^n} \longrightarrow C^\infty(M,\mathbb{R}^n)\ ,$$

and

$$\varphi : E(M,\mathbb{R}^n)/_{\mathbb{R}^n} \longrightarrow C^\infty(\partial M,\mathbb{R}^n)\ ,$$

called the force densities associated with F which are given for all $J \in E(M,\mathbb{R}^n)$ by

(4.30) $$\Phi(dJ) = \Delta(J)\mathfrak{H}(dJ)$$

and

(4.31) $$\varphi(dJ) = d\mathfrak{H}(dJ)(N)$$

satisfying

(4.32) $$\int_M <\Phi(dJ),z>\mu(J) + \int_{\partial M} <\varphi(dJ),z> i_N\mu(J) = 0\ , \quad \forall z \in \mathbb{R}^n.$$

Vice versa, given two smooth maps of the form (4.18) and (4.19) satisfying (4.32), then there is a constitutive map $\mathfrak{H}$ of the form (4.27) for which (4.29) holds.

In conclusion of this section let us remark, that near a reference embedding $I \in E(M,\mathbb{R}^n)$ the force density $\Phi(dJ)$ can be linearized as

$$\begin{aligned}(4.33) \qquad \Phi(dI) + D\Phi(dI)(J-I) = \Delta(I)\mathfrak{H}(dI) + D\Delta(I)(J-I)\mathfrak{H}(dI) \\ + \Delta(I)D\mathfrak{H}(dI)(J-I)\,.\end{aligned}$$

Using (3.19) and (4.10) we find the somewhat lengthy formula

$$\begin{aligned}(4.34) \qquad D\Delta(I)\mathfrak{H}(dI)(L) = {} & \Delta(I)D\mathfrak{H}(dI)(L) + dI\, A(dL,dI)\Delta(I)\mathfrak{H}(dI) \\ & - \sum_r [\nabla(I)_{E_r}(A(dL,dI)),\nabla(I)X(\mathfrak{H}(dI),I)](E_r) \\ & + \sum_r \nabla(I)_{A_{sym}(dL,dI)E_r}(\nabla(I)X(\mathfrak{H}(dI),I)(E_r) \\ & + \sum_r \nabla(I)_{E_r}(\nabla(I)X(\mathfrak{H}(dI),I))(A_{sym}(dL,dI)E_r) \\ & + dI\, \mathrm{div}_I A(dL,dI)\nabla(I)X(\mathfrak{H}(dI),I)\,,\end{aligned}$$

with $E_1,\ldots,E_r$ a m(I)–orthonormal moving frame on M and with $A_{sym}(dL,dI)$ the m(I)–selfadjoint part of $A(dL,dI)$.

If in particular $\mathfrak{H}(dJ) := I + \nu L$, for all $L \in C^\infty(M,\mathbb{R}^n)$ and $\nu \in \mathbb{R}$ fixed, then $D\mathfrak{H}(dI)(L) = \nu L$ and $\nabla(I)X(I,I) = id$ yielding

$$(4.35) \qquad D\Delta(I)\mathfrak{H}(dI)(L) = \nu\cdot\Delta(I)L - \Delta(I)L\,.$$

Hence if $L \equiv J-I$, the linearized force density is

$$(4.36) \qquad \Phi(dI + dL) = (\nu-1)\cdot\Delta(I)L\,.$$

The linearized boundary condition is

$$(4.37) \qquad \varphi(dI + dL) \equiv \nu\, dL(N)\,.$$

5. Constitutive laws for the boundary

The task in this section is to study constitutive laws for the boundary, that is for a deformable medium forming a skin of which the underlying point set is the manifold ∂M. This skin is thought to be detached from the body. In doing so, we first formulate in analogy to sections two and three what is meant by a constitutive law with integral kernel for the boundary material.

Let us recall that the open set $O_\partial \subset E(\partial M, \mathbb{R}^n)$ is the collection of all $J|\partial M$ with $J \in E(M, \mathbb{R}^n)$. The constitutive laws mentioned above will be given on any open set $O \subset E(M, \mathbb{R}^n)$ and will later be specified on O_∂.

At the very first we introduce the notion corresponding to the Dirichlet integral :
Given any $l \in C^\infty(\partial M, \mathbb{R}^n)$ and any $j \in E(\partial M, \mathbb{R}^n)$ then for all $X, Y \in \Gamma T\partial M$

(5.1) $$\langle dl\, X, dj\, Y\rangle = m(J)(A(dl,dj)X, Y)$$

holds for some smooth strong bundle endomorphism $A(dl,dj)$ of $T\partial M$. Moreover there is a uniquely defined smooth map

(5.2) $$c(dl,dj) : \partial M \longrightarrow \mathbb{R}^n ,$$

satisfying the following two conditions

(5.3) $$c(dl,dj)dj(T_p\partial M) \subset \mathbb{R}\cdot N(j)(p) , \qquad \forall p \in \partial M$$

and

(5.4) $$c(dl,dj)N(j)(p) \subset djT_p\partial M , \qquad \forall p \in \partial M,$$

and such that the equation

(5.5) $$dl\, X = c(dj,dl)dj\, X + dj\, A(dl,dj)$$

holds true for any $X \in \Gamma TM$. We refer to [Bi,Sn,Fi] or [Bi,Sc,So] for more details. Based on (5.4) we introduce $U(dl,dj)$ by

(5.6) $$c(dl,dj)N(j) = dj\, U(dl,dj) .$$

This vector field $U(dl,dj) \in \Gamma T\partial M$ is obviously uniquely determined.

Splitting A(dl,dj) into its skew– respectively selfadjoint parts C(dl,dj) and B(dl,dj) formed pointwise with respect to m(j) we end up with

(5.7)
$$dl = c(dl,dj)\cdot dj + dj(C(dl,dj) + B(dl,dj))\,.$$

This decomposition generalizes in the obvious way to any $\gamma \in A^1(\partial M,\mathbb{R}^n)$ and reads as

(5.7a)
$$\gamma = c(\gamma,dj)\cdot dj + dj(C(\gamma,dj) + B(\gamma,dj))\,.$$

The metric $G^{\partial}_{\mathbb{R}^n}(dj)$ at $dj \in E(\partial M,\mathbb{R}^n)/_{\mathbb{R}^n}$ applied to any two $dl,dk \in C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n}$ is defined by integrating the function

(5.8)
$$\begin{aligned} dl\cdot dk := -\tfrac{1}{2}\,\mathrm{tr}\; &c(dl,dj)\cdot c(dk,dj) \\ &- \mathrm{tr}\; C(dl,dj)\cdot C(dk,dj) \\ &+ \mathrm{tr}\; B(dl,dj)\cdot B(dk,dj) \end{aligned}$$

with respect to $\mu(j)$, that is, it is defined by

(5.9)
$$\begin{aligned} G^{\partial}_{\mathbb{R}^n}(dj(dl,dk) &:= \int_M dl\cdot dk\;\mu(j) \\ &= -\frac{1}{2}\int_M \mathrm{tr}\; c(dl,dj)\cdot c(dk,dj)\mu(j) \\ &\quad - \int_M \mathrm{tr}\; C(dl,dj)\cdot C(dk,dj)\mu(j) \\ &\quad + \int_M \mathrm{tr}\; B(dl,dj)\cdot B(dk,dj)\mu(j)\,. \end{aligned}$$

Let $O \subset E(\partial M,\mathbb{R}^n)$ be any open set.

We now define a constitutive law F^{∂} on O in analogy to section two, that is we require

(5.10)
$$F_{\partial} = d^{*}F^{\partial}_{\mathbb{R}^n}\,,$$

for some one–form $F^{\partial}_{\mathbb{R}^n}$ on $O/_{\mathbb{R}^n}$. Accordingly F_{∂} is called a constitutive law with kernel α, if for some $\alpha \in C^\infty(O,A^1(\partial M,\mathbb{R}^n))$ the following equation holds true

(5.11)
$$F_{\partial}(dj)(dl) = \int_M \alpha(dj)\cdot dl\;\mu(j)\,,\quad \forall\, l \in C^\infty(\partial M,\mathbb{R}^n)\,,\forall\, dj \in O/_{\mathbb{R}^n}\,.$$

We introduce for any $j \in E(\partial M,\mathbb{R}^n)$ the Laplacian $\Delta(j)$ accordingly to (4.9) but require that

E_s in this case is a moving frame on ∂M.

With this notion at hand the constitutive laws on O are characterized in details in the next theorem :

***Theorem** 5.1 :*

Let F_∂ be a constitutive law on any open set $O \subset E(M,\mathbb{R}^n)$. The following are then equivalent :

(i.) F_∂ admits a kernel $\alpha \in C^\infty(O/_{\mathbb{R}^n}, A^1(\partial M,\mathbb{R}^n))$.

(ii.) There is a smooth map $\mathfrak{h} \in C^\infty(O/_{\mathbb{R}^n}, C^\infty(\partial M,\mathbb{R}^n))$ uniquely determined up to maps in $C^\infty(O/_{\mathbb{R}^n},\mathbb{R}^n)$, such that

$$(5.12) \qquad F_\partial(j)(l) = \int_{\partial M} d\mathfrak{h}(dj)\cdot dl\ \mu(j)\ , \qquad \forall j \in O, \ \forall l \in C^\infty(\partial M,\mathbb{R}^n)\ .$$

(iii.) There is a unique smooth map $\varphi \in C^\infty(O, C^\infty(\partial M,\mathbb{R}^n))$, such that

$$(5.13) \qquad F_\partial(j)(l) = \int_{\partial M} <\varphi(dj),l>\mu(j)\ , \quad \forall j \in O, \forall l \in C^\infty(\partial M,\mathbb{R}^n),$$

and which satisfies

$$(5.14) \qquad \int_{\partial M} <\varphi(dj),z>\mu(j) = 0\ , \qquad \forall j \in O, \forall z \in \mathbb{R}^n\ .$$

(iv.) There is a smooth map $\mathfrak{h} \in C^\infty(O/_{\mathbb{R}^n}, C^\infty(\partial M,\mathbb{R}^n))$ uniquely determined up to maps in $C^\infty(\partial M,\mathbb{R}^n)$, such that

$$(5.15) \qquad F_\partial(j)(l) = \int_{\partial M} <\Delta(j)\mathfrak{h}(dj),l>\mu(j)\ , \quad \forall j \in O, \forall l \in C^\infty(\partial M,\mathbb{R}^n).$$

Proof :

The equivalence of (i.) with (ii.) is the analogy of theorem 3.3. The proof of this sort of reduction theorem and can also be found in [Bi] or [Bi,Sc,So]. Let us pass next to the equivalence of (ii.) with (iv.) :

Given $j \in O$. For simplicity we write k instead of $\mathfrak{h}(dj)$. We use the identity

$$(5.16) \qquad dlY = m(j)(\mathrm{grad}_j \theta(l,j) - W(j)X(l,j),Y)\cdot N(j) + dj(\nabla(j)_Y X(l,j) + \theta(l,j)\cdot W(j))Y\,, \quad \forall\, Y \in \Gamma TM\,,$$

holding for any $l \in C^\infty(\partial M,\mathbb{R}^n)$. Using any moving frame $E_1,\ldots,E_{n-1}$ on ∂M orthonormal with respect to $m(j)$ we verify the next set of equations

$$\begin{aligned}(5.17) \qquad dk\cdot dl = &-\tfrac{1}{2}\,\mathrm{tr}\; c(dk,dj)\cdot c(dl,dj)\\ &-\tfrac{1}{4}\,\mathrm{tr}(\nabla(j)X(dk,dj) - \tilde{\nabla}(j)X(dk,dj))\\ &\quad\cdot(\nabla(j)X(dl,dj) - \tilde{\nabla}(j)X(dl,dj))\\ &+\tfrac{1}{4}\,\mathrm{tr}(\nabla(j)X(dk,dj) + \tilde{\nabla}(j)X(dk,dj) + \theta(k,j)W(j))\\ &\quad\cdot(\nabla(j)X(dl,dj) + \tilde{\nabla}(j)X(dl,dj) + \theta(l,j)W(j))\\ = &-\tfrac{1}{2}\,\mathrm{tr}\; c(dk,dj)\cdot c(dl,dj)\\ &+\mathrm{tr}(\tilde{\nabla}(j)X(dk,dj)) + \theta(k,j)W(j))\\ &\quad\cdot(\nabla(j)X(dl,dj) + \theta(l,j)W(j))\\ = &-m(j)(\mathrm{grad}_j \theta(l,j) - W(j)X(l,j),U(k,j))\\ &+\sum_{i=1}^{n-1} m(j)(\tilde{\nabla}(j)X(dk,dj)) + \theta(k,j)W(j))\;\nabla(j)_{E_r}X(l,j),E_r)\\ &+\sum_{i=1}^{n-1} m(j)((\theta(l,j))W(j)(\nabla(j)X(dk,dj) + \theta(k,j)\cdot W(j))E_r,E_r)\,.\end{aligned}$$

The expression for $dk\cdot dl$ is therefore

(5.18) $$dk \cdot dl = m(j)(W(j)U(k,j),X(l,j) - d\theta(l,j)U(k,j)) + \mathrm{div}_j((\nabla(j)X(dk,dj) + \theta(k,j)W(j))\, X(l,j)) - m(j)(\mathrm{div}_j(\nabla(j)X(dk,dj) + \theta(k,j)W(j)),X(l,j)) + \theta(l,j)\, \mathrm{tr}\, W(j)(\nabla(j)X(dk,dj) + \theta(k,j)W(j))\, .$$

On the other hand let us repeat that $\Delta(j)$ be defined by

(5.19) $$\Delta(j) = d^*d + dd^*$$

with

(5.20) $$d^*l = 0\,, \qquad \forall\, l \in C^\infty(\partial M,\mathbb{R}^n)$$

and

(5.21) $$d^*\gamma = -\sum_{r=1}^{n-1} \nabla(j)_{E_r}(\gamma)(E_r)\,, \quad \forall\, \gamma \in A^1(M,\mathbb{R}^n)\,,$$

any moving frame $E_1,\dots,E_{n-1}$ of the above type. Thus the following is also easily verified :

(5.22) $$\begin{aligned}\Delta(j)k &= d^*dk \\ &= d^*(m(j)(\mathrm{grad}_j\theta(k,j) - W(j)X(k,j),\dots)\cdot N(j)) \\ &\quad + d^*(dj\, \nabla(j)X(k,j) + \theta(k,j)\cdot W(j)) \\ &= -d^*(m(j)(U(dk,dj),..)\cdot N(j)) + d^*(dj\, \nabla(J)X(k,j) + \theta(k,j)\cdot W(j)) \\ &= dj\, W(j)U(k,j) + (\mathrm{div}_jU(dk,dj))\cdot N(j) \\ &\quad - dj(\mathrm{div}_j\nabla(j)X(k,j) + \theta(k,j)\cdot W(j)) \\ &\quad + \mathrm{tr}\, W(j)(\nabla(j)X(k,j) + \theta(k,j)\cdot W(j))\cdot N(j)\, .\end{aligned}$$

(5.18) and (5.22) show the equivalence of (ii.) with (iv.).
Let us prove (ii.) $\Longrightarrow$ (iii.) :

Integrating both sides of (5.18) and posing the equation

$$\int dk\cdot dl\, \mu(j) = \int <\varphi(dj),l>\mu(j)$$

yields via Gauss' theorem the smooth $\varphi(dj)$ given for each $dj \in E(\partial M,\mathbb{R}^n)/_{\mathbb{R}^n}$ by

$$(5.23)\qquad \varphi(dj) = -dj\, \mathrm{div}_j(\nabla(j)X(dk,dj) + \theta(k,j)\cdot W(j)) - W(j)U(k,j) + (\mathrm{tr}\, W(j)(\nabla(j)X(dk,dj) + \theta(k,j)W(j)) + \mathrm{div}_j U(dk,dj))\cdot N(j)\,.$$

Hence we have

$$(5.24)\qquad \Delta(j)k = \varphi(dj)\,, \quad \forall\, j \in E(\partial M,\mathbb{R}^n)$$

and φ depends smoothly on dj. This implication can be reversed due to

$$\int_M <\Delta(j)k,z>\nu(j) = 0\,, \quad \forall\, z \in \mathbb{R}^n,$$

(cf.[Hö]). Finally we concentrate on the equivalence of (iii.) and (iv.). (iii.) yields a map $\mathfrak{h}$ by solving for each $j \in O$ the equation

$$(5.25)\qquad \Delta(j)\mathfrak{h}(dj) = \varphi(dj)\,,$$

with (5.14) as integrability condition (cf.[Hö]). Let us show that $\mathfrak{h}(dj)$ depends smoothly on j.

Without loss of generality we can assume that $\mathfrak{h}(dj) \in C^\infty_j(\partial M,\mathbb{R}^n)$, the subspace of $C^\infty(\partial M,\mathbb{R}^n)$ for which

$$(5.26)\qquad \int_{\partial M} <l,z>\mu(j) = 0\,, \quad \forall\, l \in V_j \text{ and } \forall\, Z \in \mathbb{R}^n,$$

holds. This map $\mathfrak{h}$, also satisfying (5.25), is uniquely determined.
Since $\Delta(j)$ is selfadjoint with respect to $\int < , >\mu(j)$, we also find

$$(5.27)\qquad F_{\mathcal{J}}(j)(l) = \int_{\partial M} <\mathfrak{h}(dj),\Delta(j)l>\mu(j)\,.$$

Let $j(t) \in O$ vary smoothly and let $j(t_0) = j$. Since

$$(5.28)\qquad F_\partial(j(t_0+t))(l) - F_\partial(j)(l)$$
$$= \int_{\partial M} <\mathfrak{h}(dj(t_0+t)),\Delta(j(t_0+t)l>\mu(j(t_0+t)$$
$$- \int_{\partial M} <\mathfrak{h}(dj),\Delta(j)l>\mu(j)$$
$$= \int_{\partial M} <\mathfrak{h}(dj(t_0+t))-\mathfrak{h}(dj),\Delta(j(t_0+t))l>\mu(j(t_0+t))$$
$$+ \int_{\partial M} <\mathfrak{h}(dj),\Delta(j(t_0+t))l>\mu(j(t_0+t))$$
$$- \int_{\partial M} <\mathfrak{h}(dj),\Delta(j)l>\mu(j)$$

and since F_∂, $\Delta(j)$ and $\mu(j)$ all vary smoothly in j, we conclude that for all $l \in C^\infty(\partial M,\mathbb{R}^n)$ the following limit

$$\lim_{t\to 0} \int_{\partial M} <\tfrac{1}{t}(\mathfrak{h}(dj(t_0+t))-\mathfrak{h}(dj)),\Delta(j(t_0))l>\mu(j(t_0))$$

exists.

An induction procedure shows that $\mathfrak{h}(dj(t))$ varies smoothly in t.

Thus by the differentiation theory of [Fr,Kr] not only $D\mathfrak{h}(j)$ exists, but we moreover are ensured that $\mathfrak{h}$ is even smooth. The reverse implication is obvious.

6. The interplay between constitutive laws of boundary and body

The deformable media forming the inside of the body and the boundary respectively may differ and each hence has to be described on one hand by different constitutive laws. On the other hand these materials together form one body and should be describable by only one constitutive law holding for the whole body.
The qualitative properties of the boundary material attached to the body may be influenced by the deformable material forming the body as a whole.

The purpose of this section is to study the influence of the constitutive properties of the deformable medium forming the body to the constitutive properties of the deformable medium forming the boundary of the body. In other words we will decode the constitutive properties of the boundary material attached to the body from the constitutive law describing the material of the body on the whole.

Let the constitutive law of the deformable medium forming the whole body by F again. Moreover F^{∂} denotes the constitutive law of the deformable medium forming the boundary only and which is thought to be detached from the rest of the body. Thus F_{∂} is a one–form on O_{∂}. Both, F and F_{∂}, are supposed to admit integral representations.
The constitutive law F is according to theorem 4.2 determined by a smooth map

$$\mathfrak{H} : E(M,\mathbb{R}^n)/_{\mathbb{R}^n} \longrightarrow C^{\infty}(M,\mathbb{R}^n) ,$$

the constitutive map of the deformable medium. We will first exhibit its influence to the constitutive entities of the material forming the boundary of the body :
This map yields according to theorem 4.2 force densities

(6.1) $$\Phi : E(M,\mathbb{R}^n) \longrightarrow C^{\infty}(M,\mathbb{R}^n)$$

and

(6.2) $$\varphi : E(M,\mathbb{R}^n) \longrightarrow C^{\infty}(\partial M,\mathbb{R}^n)$$

The ladder, the force density acting on ∂M, is defined by

(6.3) $$\varphi(dJ) = d\mathfrak{H}(dJ)(N) , \quad \forall\, dJ \in E(M,\mathbb{R}^n)/_{\mathbb{R}^n} .$$

Let us split this force density φ into

(6.4) $$\varphi(dJ) = \varphi_{\mathbb{R}^n}(dJ) + \psi(dJ) , \quad \forall\, dJ \in E(M,\mathbb{R}^n)/_{\mathbb{R}^n} ,$$

where $\varphi_{\mathbb{R}^n}(dJ)$ is characterized for each $dJ \in E(M,\mathbb{R}^n)/_{\mathbb{R}^n}$ by the equation

$$\int_{\partial M} < \varphi_{\mathbb{R}^n}(dJ), z > i_N \mu(J) = 0 , \quad \forall z \in \mathbb{R}^n \tag{6.5}$$

and where

$$\psi : E(M,\mathbb{R}^n) \longrightarrow \mathbb{R}^n ,$$

is a smooth map, which makes (6.4) to hold.
Let us remark that even if

$$dJ_1 | \partial M = dJ_2 | \partial M$$

for some $J_1, J_2 \in E(M,\mathbb{R}^n)$ we may not necessarily have

$$\varphi_{\mathbb{R}^n}(dJ_1) = \varphi_{\mathbb{R}^n}(dJ_2) .$$

The condition (6.5) allows us to choose some map

$$\mathfrak{h}_{\mathbb{R}^n} : E(M,\mathbb{R}^n)/_{\mathbb{R}^n} \longrightarrow C^\infty(\partial M,\mathbb{R}^n) , \tag{6.6}$$

such that for all $dJ \in E(M,\mathbb{R}^n)/_{\mathbb{R}^n}$ the equation

$$\varphi_{\mathbb{R}^n}(dJ) = \Delta(J | \partial M)\, \mathfrak{h}_{\mathbb{R}^n}(dJ) \tag{6.7}$$

holds true.We may choose $\mathfrak{h}_{\mathbb{R}^n}$ such that

$$\mathfrak{h}_{\mathbb{R}^n}(dJ) \in C^\infty_j(\partial M,\mathbb{R}^n) \text{ with } j \equiv J | \partial M , \tag{6.8}$$

for all $J \in E(M,\mathbb{R}^n)$. This map depends smoothly on its variable J, as shown in the proof of theorem 5.1.
Thus the constitutive law F is determined by a map

$$\mathfrak{H} : E(M,\mathbb{R}^n)/_{\mathbb{R}^n} \longrightarrow C^\infty(M,\mathbb{R}^n) ,$$

yielding a force density Φ for which

$$\Phi(J) = \Delta(J)\mathfrak{H}(dJ) , \quad \forall J \in E(M,\mathbb{R}^n)$$

holds and its boundary condition has the form

$$(6.9)\qquad d\mathfrak{H}(dJ)(N) = \Delta(J\,|\,\partial M)\,\mathfrak{h}_{\mathbb{R}^n}(dJ) + \psi(dJ)\,,$$

for some smooth maps

$$(6.10)\qquad \mathfrak{h}_{\mathbb{R}^n} : E(M,\mathbb{R}^n)/_{\mathbb{R}^n} \longrightarrow C^\infty(\partial M,\mathbb{R}^n)\,,$$

and

$$\psi : E(M,\mathbb{R}^n) \longrightarrow \mathbb{R}^n\,.$$

These boundary condition (6.9) obviously describes how the constitutive properties of the material forming the boundary of the body are encoded in $\mathfrak{H}$.

On the other hand we have F_∂, which is determined, according to theorem 6.1, by a smooth map

$$(6.11)\qquad h_\partial : O_\partial/\mathbb{R}^n \longrightarrow C^\infty(\partial M,\mathbb{R}^n)\,.$$

The force density defined on $O_\partial/_{\mathbb{R}^n}$ associated with $\mathfrak{h}_\partial$ will be called in the sequel by φ_∂. We choose an extension

$$(6.12)\qquad \mathfrak{H}_\partial : E(M,\mathbb{R}^n)/_{\mathbb{R}^n} \longrightarrow C^\infty(M,\mathbb{R}^n)$$

of $\mathfrak{h}_\partial$ by posing the following Višik problem, which according to [Hö] has a solution unique up to constants :

$$(6.13)\qquad \Phi_\partial = 0 = \Delta(J)\,\mathfrak{H}_\partial(dJ)$$

and

$$(6.14)\qquad d\mathfrak{H}_\partial(dJ)(N) = \Delta(J\,|\,\partial M)\,\mathfrak{h}_\partial(d(J\,|\,\partial M)) = \varphi_\partial(dJ)$$

together with

$$(6.15)\qquad \mathfrak{H}_\partial(dJ) = \mathfrak{h}_\partial(d(J\,|\,\partial M))\,,$$

all holding for any $J \in E(M,\mathbb{R}^n)$. Again $\mathfrak{H}_\partial$ depends smoothly on its variable. This is due to the fact that the constitutive law determined by $\mathfrak{H}_\partial$ only depends on its integral over the boundary ∂M and therefore is a reformulation of F_∂. Proposition 4.1 yields the smoothness of $\mathfrak{H}_\partial$. F^0 shall denote the constitutive law on $E(M,\mathbb{R}^n)$ determined by $\mathfrak{H}_\partial$.

$\mathfrak{h}_{\mathbb{R}^n} - R^* \mathfrak{h}_\partial$ and ψ show how the material forming the boundary of the body is affected by the fact that the boundary material is implemented into the body.

Without loss of generality we may think that $R^* \mathfrak{h}_\partial$ is an additive part of $\mathfrak{h}_{\mathbb{R}^n}$. This motivates us to write in the sequel $\mathfrak{h}$ only instead of $\mathfrak{h}_{\mathbb{R}^n}$.

What we have done in this section might be formulated in :

Theorem 6.1 :

Any smoothly deformable medium is characterized by a constitutive map

(6.16) $$\mathfrak{H} : E(M,\mathbb{R}^n) \longrightarrow C^\infty(M,\mathbb{R}^n) ,$$

determining itself two smooth maps

$$\mathfrak{h} : E(M,\mathbb{R}^n) \longrightarrow C^\infty(\partial M,\mathbb{R}^n)$$

and

$$\psi : E(M,\mathbb{R}^n) \longrightarrow \mathbb{R}^n ,$$

which are linked to $\mathfrak{H}$ by the boundary condition

(6.17) $$d\mathfrak{H}(dJ)(N) = \Delta(J\,|\,\partial M)\, \mathfrak{h}(dJ) + \psi(dJ) .$$

$\mathfrak{h}$ is unique up to $\mathbb{R}^n$–valued smooth maps of $E(M,\mathbb{R}^n)$ and ψ is unique. Moreover $\mathfrak{H}$ satisfies the integrability conditions

(6.18) $$0 = \int_M <\Delta(J)\mathfrak{H}(dJ),z>\, \mu(J) + \int_{\partial M} <\partial\mathfrak{H}(dJ)(N),z>\, i_N\mu(J) , \qquad \forall\, J \in E(M,\mathbb{R}^n) \text{ and } \forall\, z \in \mathbb{R}^n.$$

(6.18) equivalently formulated reads as

$$(6.19)\qquad 0 = \int_M <\Delta(J)\mathfrak{H}(dJ),z>\mu(J) + \int_{\partial M} <\psi(dJ),z> i_N\mu(J)\ ,$$

a boundary condition holding for $\mathfrak{H}$ and ψ together. The constitutive law on $E(M,\mathbb{R}^n)$ describing the constitutive properties of the materials forming body together with its boundary is thus given via the formula

$$(6.20)\qquad F(J)(L) = \int_M <\Delta(J)\mathfrak{H}(dJ),L>\mu(J) + \int_{\partial M} <d\mathfrak{H}(dJ)N,L> i_N\mu(J)\ ,$$

or equivalently by

$$(6.21)\qquad F(J)(L) = \int_M <\Delta(J)\mathfrak{H}(J),L>\mu(J) + \int_{\partial M} <\Delta(J|\partial M)\mathfrak{h}(J|\partial M) + \psi(dJ)),L> i_N\mu(J)\ ,$$
$$\forall\, J \in E(M,\mathbb{R}^n),\ \forall\, L \in C^\infty(M,\mathbb{R}^n).$$

The work of any distortion $l \in C^\infty(\partial M,\mathbb{R}^n)$ of the deformable material forming the boundary attached to the body is given by

$$(6.22)\qquad F_{\partial M}(dJ)(l) = \int_{\partial M} <\Delta(J|\partial M)\mathfrak{h}(dJ),l> i_N\mu(J)\ ,$$

for any $J \in E(M,\mathbb{R}^n)$.

Any constitutive properties describing the constitutive properties of the deformable medium of the boundary detached from the body, which is given by a smooth map $\mathfrak{h}_\partial \in C^\infty(O_\partial/\mathbb{R}^n, C^\infty(\partial M,\mathbb{R}^n))$ is additively incorporated into $\mathfrak{h}$ via the map $R^*\mathfrak{h}_\partial \in C^\infty(E(M,\mathbb{R}^n)/_{\mathbb{R}^n}, C^\infty(\partial M,\mathbb{R}^n))$. Hence $\mathfrak{h} - R^*\mathfrak{h}_\partial$ and ψ describe how the constitutive properties of the material forming the boundary of the body is affected by the fact that this material is incorporated into the material forming the whole body.

Simple *examples* :

Given $L \in C^\infty(M,\mathbb{R}^n)$ and $J \in E(M,\mathbb{R}^n)$ we may form according to (3.54) the map $P(J) \circ L$. If $L = J$ then let us write $\hat{J}$ instead of $P(J) \circ J$.

(i.) Let $\mathfrak{H}(dJ) = \hat{J}$ for all $J \in E(M,\mathbb{R}^n)$, then

$$d\hat{J} = dJ$$

and

(6.23)
$$\begin{aligned}
&\int_M dJ \cdot dL\, \mu(J) \\
&= \int_M <\Delta(J)J,L>\mu(J) + \int_{\partial M} <dJ(N),l>i_N\mu(J) \\
&= \int_M \mathrm{tr}\ A(dL,dJ)\mu(J) \\
&= \int_M \mathrm{tr}\ \nabla(J)X(L,J)\mu(J) \quad . \\
&= \int_M \mathrm{div}_J X(L,J)\mu(J) \\
&= \int_{\partial M} <N(j),l>\mu(j) \\
&= D(\int_M \mu(J))(L)\ .
\end{aligned}$$

Here $l := L|\partial M$ and $j := J|\partial M$. The above calculation shows

(6.24)
$$\Phi(dJ) = \Delta(J)J = 0$$

and

(6.25)
$$\varphi(dJ) = N(j)\ , \qquad \forall\, J \in E(M,\mathbb{R}^n)\ ;$$

$l = z$ with $z \in \mathbb{R}^n$ evidently implies

(6.26)
$$\int_M <N(j),z>\mu(j) = 0\ , \quad \forall\, z \in \mathbb{R}^n\ .$$

This shows that in this example

(6.27)
$$\varphi = \varphi_{\mathbb{R}^n}\ .$$

The map $\mathfrak{h}_{\mathbb{R}^n}$ in this case is thus given by

$$(6.28) \qquad N(j) = \Delta(j)\mathfrak{h}_{\mathbb{R}^n}(dJ)\,, \qquad \forall\, J \in E(M,\mathbb{R}^n) \text{ and } j := J|\partial M\,.$$

Since $\mathfrak{h}_{\mathbb{R}^n}$ depends here on dj only we have the situation that $F = F^0$.

Let us turn our attention to $\mathfrak{h}_\partial$ on O_∂, given by $\mathfrak{h}_\partial(dj) = \hat{j}$, $\forall\, j \in O_\partial$. Here $\hat{j} \in C^\infty_j(\partial M,\mathbb{R}^n)$ is the projection of j along $\mathbb{R}^n$. One easily verifies the following set of equations

$$(6.29) \qquad \int_{\partial M} <\Delta(j)\hat{j},l>\mu(j) = \int_{\partial M} dj\cdot dl\,\mu(j)$$
$$= \int_{\partial M} (\mathrm{tr}(\nabla(j)X(l,j) + \theta(l,j)\cdot W(j))$$
$$= \int_{\partial M} (\mathrm{div}_j X(l,j) + \theta(l,j)\cdot H(j))\mu(j)$$
$$= \int_{\partial M} \theta(l,j)\cdot H(j)\mu(j)$$
$$= \int_{\partial M} <\theta(l,j)\cdot H(j)\cdot N(j),l>\mu(j)$$
$$= D(\int_{\partial M} \mu(j))(l)\,.$$

Hence $\mathfrak{H}_\partial$ is given by

$$(6.30) \qquad 0 = \Delta(J)\mathfrak{H}_\partial(dJ)\,, \qquad \forall J \in E(M,\mathbb{R}^n)\,,$$

together with

$$(6.31) \qquad d\mathfrak{H}_\partial(dJ)(N) = \Delta(j)\hat{j} = H(j)\cdot N(j)\,, \qquad \forall\, J \in E(M,\mathbb{R}^n) \text{ and } J := J|\partial M\,.$$

Again $F = F^0$ here.

(ii.) Next let us consider quite another influence of the boundary by looking at $\mathfrak{h}_\partial : O_{\partial/\mathbb{R}^n} \to C^\infty(\partial M,\mathbb{R}^n)$ given by $\mathfrak{h}_\partial(dj) = N(j)$, $\forall\, j \in O_\partial$. Then the formula

$$(6.32) \qquad \Delta(j)N(j) = d^* dN(j)$$
$$= d^* dj\, W(j)$$
$$= -dj\, \mathrm{grad}_j H(j) + (\mathrm{tr}\, W(j)^2)\cdot N(j)$$

holds for any $j \in O_\partial$. In this case $\mathfrak{H}_\partial$ is given by the system

(6.33) $$0 = \Delta(J)\mathfrak{H}_{\partial}(dJ)\,, \qquad \forall J \in E(M,\mathbb{R}^n)\,,$$

(6.34) $$d\mathfrak{H}_{\partial}(N) = \Delta(j)\cdot N(j)\,, \qquad \forall\, J \in E(M,\mathbb{R}^n)$$

and $J := J\,|\,\partial M$.

Let us point out that $\Delta(j)N(j) \neq 0$ even if $j(\partial M) \subset \mathbb{R}^n$ is minimal, that is to say even if $H(j) = \text{const}$.
In the special case of $\dim \partial M = 2$ a topological constant, the Euler characteristic $\aleph(\partial M)$, enters the constitutive law F determined by N(j) for each $j \in O_\partial$. It is hidden in

(6.35) $$F(j)(N(j)) = \int_{\partial M} <\Delta(j)N(j),N(j)>\mu(j) = \int_{\partial M} \operatorname{tr} W(j)^2\mu(j)\,,$$

as seen as follows :
By the Cayley Hamilton theorem (cf.[Gr]) and the Gauss Bonnet theorem (cf.[G,H,V]), $F^{\partial}(j)(N(j))$ equation (6.35) can be expressed as

(6.36) $$F^{\partial}(j)(N(j)) = -4\pi\,\aleph(\partial M) + \int_{\partial M} H(j)^2\mu(j)\,.$$

Here we also have $F = F^0$.

7. A general decomposition of constitutive laws

In this section we will exhibit a decomposition of the constitutive map $\mathfrak{H}$. This decomposition is based on two specific one–forms on $E(M,\mathbb{R}^n)$ and $E(\partial M,\mathbb{R}^n)$ respectively, namely the derivatives of the volume function

(7.1) $$\mathfrak{V} : E(M,\mathbb{R}^n) \longrightarrow \mathbb{R} ,$$

assigning to any $J \in E(M,\mathbb{R}^n)$ the volume

(7.2) $$\mathfrak{V}(J) = \int_M \mu(J)$$

and of the area function

(7.3) $$\mathfrak{A} : E(\partial M,\mathbb{R}^n) \longrightarrow \mathbb{R} ,$$

sending any $j \in E(\partial M,\mathbb{R}^n)$ into

(7.4) $$\mathfrak{A}(j) = \int_{\partial M} \mu(j) .$$

As we know from the previous examples this derivates are

(7.5) $$\begin{aligned} D\mathfrak{V}(J)(L) &= \int_{\partial M} <N(J\,|\,\partial M),L> i_N\, \mu(J) \\ &= \int_M dJ \cdot dL\, \mu(J) \\ &= \int_{\partial M} <dJ(N),L> i_N\, \mu(J) \end{aligned}$$

and

(7.6) $$\begin{aligned} D\mathfrak{A}(j)(l) &= \int_{\partial M} H(j)<N(j),l>\mu(j) \\ &= \int_{\partial M} dj \cdot dl\, \mu(j) , \end{aligned}$$

holding for all $J \in E(M,\mathbb{R}^n)$, all $L \in C^\infty(M,\mathbb{R}^n)$, all $j \in E(\partial M,\mathbb{R}^n)$ and all $l \in C^\infty(\partial M,\mathbb{R}^n)$.
We will show in this section that $D\mathfrak{V}$ and $R^* D\mathfrak{A}$ multiplied with appropriate $\mathbb{R}$–valued maps are all part of any constitutive law F defined on $E(M,\mathbb{R}^n)$.

Let us first concentrate on $D\mathfrak{V}$ and see how it is encoded in any constitutive map $\mathfrak{H}$.
To this end let F be determined by some $\mathfrak{H} \in C^\infty(E(M,\mathbb{R}^n)/_{\mathbb{R}^n}, C^\infty(M,\mathbb{R}^n))$. As we know from

the previous section it determines two maps $\mathfrak{h} \in C^\infty(E(\partial M,\mathbb{R}^n)/_{\mathbb{R}^n}, C^\infty(\partial M,\mathbb{R}^n))$ and $\psi \in C^\infty(E(M,\mathbb{R}^n)/_{\mathbb{R}^n},\mathbb{R})$, such that both are linked to $\mathfrak{H}$ by the equation

$$d\mathfrak{H}(dJ)(N) = \Delta(J|\partial M)\mathfrak{h}(dJ) + \psi(dJ)\,, \tag{7.7}$$

which holds for any $J \in E(M,\mathbb{R}^n)$.

Let us consider the real Hilbert-space H_J consisting of all $L,K : M \longrightarrow \mathbb{R}^n$, for which

$$<<L,K>> := \int_M <L,K>\mu(J) \tag{7.8}$$

exists.

Recalling that $C_J^\infty(M,\mathbb{R}^n)$ is the collection of all $L \in C^\infty(M,\mathbb{R}^n)$, such that

$$\int_M <L,z>\mu(J) = 0\,, \quad \forall\, z \in \mathbb{R}\,,$$

we restate the splitting

$$C^\infty(M,\mathbb{R}^n) = \mathbb{R}^n \oplus C_J^\infty(M,\mathbb{R}^n)\,. \tag{7.9}$$

This is a splitting as Fréchet spaces since the functional, assigning to any $J \in C^\infty(M,\mathbb{R}^n)$ the real $\int_M <\ ,z>\mu(J)$, is continuous on $C^\infty(M,\mathbb{R}^n)$ for any $z \in \mathbb{R}$. Moreover (7.9) is orthogonal with respect to $<<\ ,\ >>$, defined in (7.8).

The projection $\hat{J}$ along $\mathbb{R}^n$ of $J \in E(M,\mathbb{R}^n)$ in $C_J^\infty(M,\mathbb{R}^n)$ satisfies as we already know from the previous section

$$d\hat{J} = dJ \tag{7.10}$$

and

$$\Delta(J)\hat{J} = \Delta(J)J = 0\,. \tag{7.11}$$

Since both $\mathfrak{H}(dJ)$ and $\hat{J}$ belong to H_J we may take the component in H_J of $\mathfrak{H}(dJ)$ along $\hat{J}$. This component has the form $\pi^1(dJ)\cdot\hat{J}$ for some real number $\pi^1(dJ)$. Thus there is some $\mathfrak{H}^1 \in C^\infty(E(M,\mathbb{R}^n)/_{\mathbb{R}^n}, C^\infty(M,\mathbb{R}^n))$ such that

$$\mathfrak{H}(dJ) = \pi^1(dJ)\cdot\hat{J} + \mathfrak{H}^1(dJ) \tag{7.12}$$

is an orthogonal decomposition in H_J. We leave it to the reader to show that $\pi^1(dJ)$ and

$\mathfrak{H}^1(dJ)$ vary smoothly with dJ.

Clearly we have due to (7.7) and (7.11)

$$\Delta(J)\mathfrak{H}(dJ) = \Delta(J)\mathfrak{H}^1(dJ) \tag{7.13}$$

and

$$d\mathfrak{H}(dJ)(N) = \pi^1(dJ)N(J|\partial M) + d\mathfrak{H}^1(dJ)(N)\,, \tag{7.14}$$

both holding for all $J \in E(M,\mathbb{R}^n)$.

Let us denote by F^1 the constitutive law on $E(M,\mathbb{R}^n)$ determined by the map $\mathfrak{H}^1$. Equation (7.13) yields then

$$F = \pi^1 \cdot D\mathfrak{V} + F^1\,. \tag{7.15}$$

The map

$$\pi^1 \cdot N : E(M,\mathbb{R}^n)/_{\mathbb{R}^n} \longrightarrow C^\infty(\partial M,\mathbb{R}^n) \tag{7.16}$$

assigning to any dJ the map $\pi(dJ)\cdot N(j)$ with $j := J/\partial M$, yields a part of $\mathfrak{h}$. This part, called $\mathfrak{h}_N$, is produced by regarding $\pi(dJ)\cdot N(j)$ as a force density along ∂M, which according to (6.28) has to satisfy the equation

$$\Delta(j)\mathfrak{h}_N(dJ) = \pi^1(dJ)N(j)\,, \quad j := J|\partial M\,, \tag{7.17}$$

for all $J \in E(M,\mathbb{R}^n)$. The map $\mathfrak{h}_N(dJ) \in C^\infty_j(\partial M,\mathbb{R}^n)$ varies smoothly with $J \in E(M,\mathbb{R}^n)$. Hence we have the splitting

$$\mathfrak{h}(dJ) = \pi^1(dJ)\cdot\mathfrak{h}_N(dJ)) + \mathfrak{h}^1(dJ)\,, \quad \forall\, dJ \in E(M,\mathbb{R}^n)/_{\mathbb{R}^n}\,, \tag{7.18}$$

for some smooth $\mathfrak{h}^1(dJ) \in C^\infty_j(\partial M,\mathbb{R}^n)$. This example shows what we had in mind as we were claiming that $D\mathfrak{V}$ is part of F.

To get the full decomposition we broaden our scope a little and introduce first of all the Hilbert space A_i consisting of all maps $\gamma_1, \gamma_2 : TM \longrightarrow \mathbb{R}$ linear on the fibres of TM for which the right hand side of

(7.19) $$G_{\mathbb{R}^n}(dj)(\gamma_1,\gamma_2) := \int_{\partial M} \gamma_1 \cdot \gamma_2 \, \mu(j)$$

exists. Clearly $d\mathfrak{h}_N(dJ)$, dj and $dN(j)$ all belong to A_j and are generically linearly independent. The set O_3 of all $j \in E(\partial M, \mathbb{R}^n)$ for which these three differentials are linearly independent form a dense open set. In case $j(\partial M)$ is a $(n-1)$–sphere in $\mathbb{R}^n$ however, $N(j)$ is a real multiple, r say, of j and $\mathfrak{h}_N(J)$ is hence $\frac{r}{(n-1)} \cdot j$. This can be confirmed by looking at (6.28) and (6.31). In the case of linear independence the three above mentioned differentials are in general (with respect to $G_{\mathbb{R}^n}(dj)$) not orthogonal to each other, however.

We therefore orthogonalize them by using the method of Schmidt : This means we form

(7.20) $$d\mathfrak{h}_N(dJ),$$

$$dj + b_1 \, d\mathfrak{h}_N(dJ),$$

and

$$dN(dj) + b_2(dj + b_1 \, d\mathfrak{h}_N(dJ)) + b_3 \, d\mathfrak{h}_N(dJ)$$

with the following coefficients

(7.21) $$b_1 = -\frac{D\mathfrak{V}(J)(J)}{\|d\mathfrak{h}_N(dJ)\|^2},$$

$$b_2 = -\frac{D\mathfrak{A}(j)(N(j)) \cdot \|d\mathfrak{h}_N(dJ)\|^2 - D\mathfrak{V}(J)(J) \cdot \mathfrak{A}(j)}{D\mathfrak{A}(j)(j)\|d\mathfrak{h}_N(dJ)\|^2 - 2\,D\mathfrak{V}(J)(J) + \|d\mathfrak{h}_N(dJ)\|^2},$$

$$b_3 = -\frac{\mathfrak{A}(j)}{\|d\mathfrak{h}_N(dJ)\|^2},$$

Let us point out, that in case $j(\partial M)$ is an $(n-1)$–sphere then (7.20) reduces to $d\mathfrak{h}_N(dJ) = \frac{r}{n-1} \cdot dj$ since the other differentials vanish.

Hence generically $d\mathfrak{h}^1(dJ)$ in (7.18) splits in A_j orthogonally into

(7.22)
$$\begin{aligned} d\mathfrak{h}(dJ) &= \pi^1(dJ)\, d\mathfrak{h}_N(dJ) + \pi^2(dJ)(d\hat{j} + b_1 d\mathfrak{h}_N(dJ)) \\ &\quad + \pi^3(dJ)(dN(dj)) + b_2(d\hat{j} + b_1 d\mathfrak{h}_N(dJ)) + b_3 d\mathfrak{h}_N(dJ)) \\ &\quad + d\mathfrak{h}^2(dJ) \\ &= (\pi^1(dJ) + \pi^2(dJ)\cdot b_1 + \pi^3(dJ)\cdot b_1 \cdot b_2 + \pi^3(dJ)\cdot b_3)\, d\mathfrak{h}_N(dJ) \\ &\quad + (\pi^2(dJ) + \pi^3(dJ)\cdot b_2)\cdot d\hat{j} \\ &\quad + \pi^3(dJ)\cdot dN(j) + d\mathfrak{h}^2(dJ)\,, \end{aligned}$$

for some $\mathfrak{h}^2(dJ) \in C^\infty(\partial M,\mathbb{R}^n)$.
For our further investigation let us call the coefficients of $d\mathfrak{h}_N(dJ)$, dj and $dN(dj))$ in (7.22) by $a_1(dJ)$, $a_2(dJ)$ and $a_3(dJ)$ respectively.

Next we extend all maps $\mathfrak{h}_N(dJ)$, $\hat{j}$ and $N(j)$ to all of M in the following way :
Given $f \in C^\infty_j(\partial M,\mathbb{R}^n)$ we solve the following Višik problem

(7.23)
$$\begin{aligned} \Delta(J)f_M &= 0 \\ df_M(N) - \Delta(j)f &= 0\,, \end{aligned}$$

with $f_M \subset C^\infty(M,\mathbb{R}^n)$ and where $J \in E(M,\mathbb{R}^n)$ and $j := J\,|\,\partial M$. Clearly

$$(\mathfrak{h}_N(dJ))_M = \hat{J}\,.$$

All the splittings and extensions done to construct $\hat{j}_M$ and $N(j)_M$ depend smoothly on $j \in E(\partial M,\mathbb{R}^n)$. The above mentioned decomposition of $\mathfrak{H}$ is then described in the following theorem :

Theorem 7.1 :

Let F be a constitutive law on $E(M,\mathbb{R}^n)$ determined by $\mathfrak{H} \in C^\infty(E(M,\mathbb{R}^n)/_{\mathbb{R}^n}, C^\infty(M,\mathbb{R}^n))$. Then $\mathfrak{H}$ determines uniquely three smooth maps

$$a_1, a_2, a_3 : O_3 \subset E(M,\mathbb{R}^n)/_{\mathbb{R}^n} \longrightarrow \mathbb{R}$$

and uniquely two smooth maps

$$\mathfrak{h}, \mathfrak{h}^2 : O_3 \subset E(M,\mathbb{R}^n)/_{\mathbb{R}^n} \longrightarrow C^\infty(\partial M,\mathbb{R}^n) ,$$

such that the following splitting holds for any $dJ \in O_3 \subset E(M,\mathbb{R}^n)$

(7.24) $$\mathfrak{h}(dJ) = a_1(dJ)\cdot\mathfrak{h}_N(dJ) + a_2(dJ)\hat{j} + a_3(dJ)\cdot N(j) + \mathfrak{h}^2(dJ)$$

with $j := j/\partial M$. The differential $d\mathfrak{h}^2(dJ)$ is orthogonal with respect to $G(dj)_{\partial M}$ to the space spanned by $d\mathfrak{h}_N(dJ)$, dj and $dN(j)$.

The map $\mathfrak{H}(dJ)$ decomposes into

(7.25) $$\mathfrak{H}(dJ) = a_1(dJ)\cdot\hat{J} + a_2(dJ)\cdot\hat{j}_M + a_3(dJ)\cdot N_M(dJ) + \mathfrak{H}^2(dJ) ,$$

where $j := J\,|\,\partial M$.

The constitutive law F splits accordingly into

(7.26) $$F(J)(L) = a_1(dJ)\cdot D\mathfrak{B}(J)(L) + a_2(dJ)\cdot D\mathfrak{A}(j)(l) + F_N(dJ)(dL) + F^2(J)(L) ,$$

with $J := J\,|\,\partial M$ and $l := L\,|\,\partial M$ and with

(7.27) $$F_N(J)(L) = a_3(dJ)\cdot {}_M\!\int dN_M(dJ)\cdot dL\;\mu(J) .$$

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