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Central Extensions and Semi-infinite Wedge Representations of Krichever-Novikov Algebras for More than Two Points

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Abstract

For the generalized Krichever-Novikov algebras of meromorphic vector fields and their induced modules of weight λ a different basis is given. With respect to this basis the module structure is generalized graded. "Local" central extensions of these algebras and their representations on the space of semi-infinite wedge product of forms of weight λ are studied. In this generalization one obtains again $c = -2(6\lambda^2 - 6\lambda + 1)$ as value for the central charge.

1. Introduction

In [1,2] Krichever and Novikov studied the algebra of meromorphic vector fields which have poles only at two generic but fixed points on a Riemann surface of arbitrary genus g. This algebra (with or without) central extension is now usually called Krichever-Novikov algebra (short KN algebra). It is a generalization of the Virasoro algebra to higher genus and it is of considerable interest in conformal field theory (see references [6-14] in [3]).

In [3] I generalized the situation by allowing the vector fields and forms to have poles at more than two points. By Riemann-Roch type arguments [6,3] a certain set of generators for the space of meromorphic forms of weight λ were constructed. Explicit expressions for the generators in terms of theta functions and prime forms ($g \geq 1$) were given in the following paper [4]. Let N be the number of points where poles are allowed. In [3] a certain subset

(1) $\quad \{\, f_\alpha(\lambda) \mid (\alpha = (n,l),\ n \in \mathbf{Z},\, n < 0\,, l = 1, \ldots, N-1) \text{ or } (\alpha = (n,1),\ n \in \mathbb{N})\,\}$

of the generators were chosen to give a basis of the KN modules. The case of the algebra itself is included as $\lambda = -1$. We set $e_\alpha = f_\alpha(-1)$. Let $C^\gamma_{\alpha,\beta}(\lambda)$ be the structure constants of the of the module defined by

$$e_\alpha \cdot f_\beta(\lambda) = \sum_\gamma C^\gamma_{\alpha,\beta} f_\gamma(\lambda) \tag{2}$$

then in [3,(35)-(40)] a finite region for the occuring γ values was given. In the case $g = 0$ and the algebra itself, the structure constants were explicitly calculated. Independently R. Dick [7,8] gave similar sets of generators and basis elements. He also calculated the structure constants in the case $g = 0$. Guo, Na, Shen, Wang and Yu got similar results in the $g = 0$ case [9] and very recently for higher genus (and $\lambda \neq 0, 1$) [10].

The basis (1) chosen in [3] (and in all other references above) has some advantages. For example, a subset of the basis is a basis of the global holomorphic forms (in physicist's language: the zero modes). Additionally it is easy to see how the KN algebra of a subset of the N points lies in the whole algebra. Unfortunately for $N > 2$ the algebra (and the modules) are not generalized graded (see (27) below for the definition) with a grading in which the above basis elements are homogeneous elements. For example, for certain $\alpha = (-n, p)$ and $\beta = (-m, r)$ we will get that the number of elements on the right hand side of (2) will be proportional to $n + m$ as formula (40) or (48) in [3] shows. This makes such a generalized grading impossible. The missing of the generalized grading causes problems for the construction of the semi-infinite wedge representations as they were studied for $N = 2$ by Krichever and Novikov in [1,2]. In this paper I give a different subset of the generators as basis together with a natural $\mathbf{Z}$ – degree, such that there is generalized grading of the algebra and its modules induced by the degree of the generators. This we will do for an arbitrary splitting of the N points into two non-empty subsets (the "in" and "out" points). We study central extensions and show that the geometric cocycle ([3,(72)] or (49) below) obeys some "locality" condition. The action of the algebra elements on the forms of fixed weight λ induces an action of a central extension of the algebra on the semi-infinite wedge products of these forms. Here we use the generalized graded module structure. If we require that the extension given as basis and cocycle should be the same for all weights λ we can show in this general setting also that the central element operates with central charge

$$c(\lambda) = -2(6\lambda - 6\lambda + 1) \tag{3}$$

on the "vacuum vectors ".

Details of the proofs and the calculations can be found in [5]. To give an idea of the technique involved I will do the case $N = k + l$ with $k = l$ in some more detail.

For the notation I refer to [3]. Let me just repeat the fundamental definitions. Let X be a Riemann surface of genus g with N fixed points which are in general position ($N \geq 2$). The set of these points we will denote by A. Let A be splitted into two non-empty disjoint subsets I and O

$$I = \{P_1, P_2, \ldots, P_k\}, \qquad O = \{Q_1, Q_2, \ldots, Q_l\}, \quad N = k + l\ . \tag{4}$$

The points in I we will call "in"- points, the points in O "out"- points. We fix local coordinates

$$z_i\,, \quad i = 1, \ldots, k, \qquad\qquad w_i\,, \quad i = 1, \ldots, l \tag{5}$$

around the points $P_i \in I$ (resp. $Q_i \in O$). As we will see later on they will not enter the picture in an essential way.

Let ρ be the unique meromorphic differential, holomorphic on $X \setminus A$ with poles of order -1 at A with

$$\operatorname{res}_P(\rho) = -\frac{1}{k}, \quad P \in I\,, \qquad\qquad \operatorname{res}_Q(\rho) = -\frac{1}{l}, \quad Q \in O \tag{6}$$

which has only imaginary periods [3]. We fix another point $B \in X \setminus A$. Now

$$u(P) := \mathbf{Re} \int_B^P \rho \tag{7}$$

is a well defined harmonic function on $X \setminus A$ with

$$\lim_{P \to P_i} u(P) = -\infty, \quad P_i \in I \qquad \text{and} \qquad \lim_{P \to Q_i} u(P) = \infty, \quad Q_i \in O. \tag{8}$$

The level line for $\tau \in \mathbb{R}$ is defined as

$$C_\tau := \{P \in X \setminus A \mid u(P) = \tau\} . \tag{9}$$

Varying τ defines a global fibration of the surface $X \setminus A$. For $\tau \ll 0$ ($\tau \gg 0$) the level line C_τ splits into disjoint components diffeomorphic to S^1 around the points $P \in I$ (resp. $P \in O$).

One might perhaps interpret this configuration as a closed string configuration at proper time τ with incoming free strings at the points $P \in I$ and outgoing free strings at the points $P \in O$ [2]. Such an interpretation gives a natural separation of the set A into I and O. Of course, it is possible without changing anything in the following to prescribe arbitrary real residues at the points $P \in A$ as long as there sum vanishes.

Let us denote by $\mathcal{KN}$ the Krichever-Novikov algebra $\mathrm{KN}(P_1, P_2, \ldots, P_k, Q_1, \ldots, Q_l)$. It is the Lie algebra of meromorphic vector fields on X which are holomorphic on $X \setminus A$. Let $\mathcal{F}^\lambda$ be the vector space of meromorphic forms of weight $\lambda \in \mathbf{Z}$ which are holomorphic on $X \setminus A$. Here we understand by a meromorphic form of weight λ a meromorphic section of the line bundles K^{λ}. K denotes the canonical line bundle. Its sections are the (1-) differentials. We use the same symbol K to denote its associated divisor class or just one associated divisor [6]. By taking the Lie derivative with respect to the vector fields in $\mathcal{KN}$ the vector space $\mathcal{F}^\lambda$ will become a Lie algebra module over $\mathcal{KN}$. It is called the Krichever-Novikov module of weight λ.

We will use the following abbreviation

$$M(\lambda) := (2\lambda - 1)(g - 1) - 1 \tag{10}$$

and calculate

$$M(-1) = -3g + 2 \qquad M(\lambda) + M(1 - \lambda) = -2 . \tag{11}$$

If s is a meromorphic section of a line bundle then we use $\mathrm{ord}_P(s)$ for the order of the zero of the section s at the point P. As usual, poles are zeros of negative order [6].

2. A special case: k=l.

We consider first the case of an even number of points in A separated into two subsets of equal size. Here the principle of the construction is not obscured by technicalities. In this section we study the generic situation $g = 0$ and all $\lambda \in \mathbf{Z}$ or $g \geq 2$ and $\lambda \neq 0, 1$. For all $n \in \mathbf{Z}$ and $p = 1, \ldots, k$ let $f_{n,p}(\lambda)$ be an element of $\mathcal{F}^\lambda$ obeying the conditions

$$\begin{aligned} \mathrm{ord}_{P_i}(f_{n,p}(\lambda)) &= n - \delta_{i,p}, \quad P_i \in I \\ \mathrm{ord}_{Q_i}(f_{n,p}(\lambda)) &= -n, \quad Q_i \in O \setminus \{Q_k\} \\ \mathrm{ord}_{Q_k}(f_{n,p}(\lambda)) &= -n + M(\lambda) + 1 \end{aligned} \tag{12}$$

which has the local form around P_p

$$f_{n,p}(\lambda)_{|} = z_p^{n-1}(1+O(z_p))\,(dz_p)^{\lambda} \ . \tag{13}$$

We calculate

$$\sum_{Q\in A} \operatorname{ord}_Q(f_{n,p}(\lambda)) = M(\lambda) \ .$$

Hence, an element obeying (12) is a generator showing up in the list of [**3**]. It exists and is unique up to multiplication with a constant. Condition (13) fixes this constant.

PROPOSITION 1. *We have the duality*

$$\frac{1}{2\pi \mathrm{i}} \oint_{C_\tau} f_{n,p}(\lambda) f_{1-m,r}(1-\lambda) = \delta_{m,n}\cdot\delta_{r,p}, \qquad n,m\in \mathbf{Z},\ r,p=1,\ldots,k \ . \tag{14}$$

This can be proven by a simple calculation of residues alternatively around the points $P_i \in I$ and $Q_i \in O$.

PROPOSITION 2. *The set*

$$\{\ f_{n,p}(\lambda) \mid \ n\in\mathbf{Z},\ p=1,\ldots,k\ \} \tag{15}$$

is a basis of $\mathcal{F}^{\lambda}$ (as vector space).

PROOF: By duality (14) it is clear that they are linearly independent. It remains to show that they generate $\mathcal{F}^{\lambda}$. We set for $n\in\mathbb{N}$

$$V(n) := H^0(X, \lambda\cdot K + D(n)) \tag{16}$$

with

$$D(n) := \sum_{i=1}^{k}(n+1)P_i + \sum_{i=1}^{k-1} nQ_i + (n-1-M(\lambda))\cdot Q_k \ .$$

The points $P\in A$ are chosen to be in general position. With the arguments used in [**3,5**] we calculate

$$\begin{aligned}\dim V(n) &= \dim H^0(X,\lambda\cdot K - M(\lambda)\cdot Q_k) + \deg\,(D + M(\lambda)\cdot Q_k)\\ &= k(2n+1)\end{aligned}$$

using $\dim H^0(X,\lambda\cdot K - M(\lambda)\cdot Q_l) = 1$ [**3**,prop.1]. The divisor $D(n)$ is the maximal polar divisor of section allowed in (16). Obviously, the elements

$$f_{m,p}(\lambda), \quad -n\le m\le n,\ p=1,\ldots,k$$

are in $V(n)$. Because this are the right number of elements, they are a basis of $V(n)$. Now every $v \in \mathcal{F}^\lambda$ lies in some $V(n)$. This proves the claim.

Of course, the n can be explicitly given by the order of v at the points $P \in A$. For $k = 1$ this basis is exactly (up to some index shift) the basis given by Krichever and Novikov [**1,2**] and coincide also with the basis in [**3**]. Note that for $k > 1$ there is in general no subset of this basis which also is a basis for the globally holomorphic forms (if there are any at all).

Combining proposition 1 and 2 we get

PROPOSITION 3. *Every form $v \in \mathcal{F}^\lambda$ can be given as a finite linear combination*

$$v = \sum_{n \in \mathbb{Z}} \sum_{p=1}^{k} A_{n,p} f_{n,p}(\lambda), \quad A_{n,p} \in \mathbb{C} \tag{17}$$

with

$$A_{n,p} = \frac{1}{2\pi \mathrm{i}} \oint_{C_\tau} v \cdot f_{1-n,p}(1-\lambda) \ . \tag{18}$$

Due to the special importance of certain weights we introduce

$$e_{n,p} := f_{n,p}(-1), \quad A_{n,p} := f_{n,p}(0), \quad \omega_{n,p} := f_{n,p}(1), \quad \Omega_{n,p} := f_{n,p}(2) \tag{19}$$

(assuming the result of the next section already). The Lie derivative $e \,.\, v$ of a form $v \in \mathcal{F}^\lambda$ with respect to a vector field $e \in \mathcal{KN}$ can be given in local terms as follows [**3**]. Let z be a local coordinate at some point P and

$$e_| = g(z)\frac{\partial}{\partial z}, \quad v_| = h(z)(dz)^\lambda$$

then

$$e \,.\, v = (g(z)\frac{\partial h}{\partial z} + \lambda h(z)\frac{\partial g}{\partial z})(dz)^\lambda \ . \tag{20}$$

In particular, $e \,.\, v$ lies again in $\mathcal{F}^\lambda$.

Now we come to the main advantage of this basis.

PROPOSITION 4.

$$e_{n,p} \cdot f_{m,r}(\lambda) = \sum_{h=n+m-2}^{n+m-2+L} \sum_{s=1}^{k} C_{(n,p),(m,r)}^{(h,s)}(\lambda) \cdot f_{h,s}(\lambda) \tag{21}$$

with

$$L = \begin{cases} 3 & , g = 0 \text{ and } k > 1 \\ 3g & , \text{otherwise} \end{cases} \tag{22}$$

and structure constants $C^{\cdot\cdot}_{\cdot\cdot\cdot} \in \mathbb{C}$ *with the boundary case*

$$C^{(n+m-2,s)}_{(n,p),(m,r)}(\lambda) = \delta_{p,r}\delta_{p,s}((m-1) + \lambda(n-1)) \ . \tag{23}$$

PROOF: One calculates using (20) the lowest term of the local form of $e \,.\, v$ at $P \in A$. Multiplies that with $f_{q,t}(1-\lambda)$. By considering which residues vanish we see for which q and t

$$\frac{1}{2\pi i} \oint_{C_\tau} (e_{n,p} \cdot f_{m,r}(\lambda)) \cdot f_{q,t}(1-\lambda)$$

will vanish. Duality (14) says which terms on the right hand side of (21) appear. The expression (23) is gained by calculating the exact residue.

There are also expressions for the other boundary case[5]. Here in general the coefficients for all possible (p, r, s) combinations will not vanish.

Now we define

$$\deg f_{n,p}(\lambda) = n \tag{24}$$

and set

$$\mathcal{KN}(n) := \langle e_{n,1}, e_{n,2}, \ldots, e_{n,k} \rangle \quad \mathcal{F}^\lambda(n) := \langle f_{n,1}(\lambda), f_{n,2}(\lambda), \ldots, f_{n,k}(\lambda) \rangle \ . \tag{25}$$

Obviously, we have

$$\mathcal{KN} = \bigoplus_{n \in \mathbb{Z}} \mathcal{KN}(n), \quad \mathcal{F}^\lambda = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}^\lambda(n) \ . \tag{26}$$

By proposition 4 we get

$$\begin{aligned} [\mathcal{KN}(n), \mathcal{KN}(m)] &\subset \bigoplus_{h=(n+m)-K_0}^{(n+m)+K_1} \mathcal{KN}(h) \\ \mathcal{KN}(n) \,.\, \mathcal{F}^\lambda(m) &\subset \bigoplus_{h=(n+m)-L_0}^{(n+m)+L_1} \mathcal{F}^\lambda(h) \end{aligned} \tag{27}$$

with K_0, K_1, L_0, L_1 integers not depending on n and m. But (26) and (27) is exactly the condition that $\mathcal{F}^\lambda$ is a generalized graded module over $\mathcal{KN}$ with the grading (25).

For the above basis we can use "explicit" expressions given in [**4**] and calculate (at least in principle) the structure constants using theta functions, prime forms and their derivatives (for $g \geq 1$), the Weierstraß's $\sigma-$ function and its derivatives ($g = 1$) and rational functions ($g = 0$).

3. k=1. The exceptional λ cases.

As it was explained in [**3,4**] due to the fact that the divisors K and O (O is the divisor class corresponding to the trivial bundle) are special divisors we have to modify certain generators to ensure their existence and uniqueness (in an appropriate sense). Hence we have to modify our basis for $g \geq 2$ and $\lambda = 0$ or 1 and $g = 1$ and all λ. In the latter case only $\lambda = 0$ is of importance. In this case we have $K = \mathrm{O}$. Hence by constructing a basis $\{A_{n,p}\}$ of $\mathcal{F}^0$ we get a basis

$$\{ \, f_{n,p}(\lambda) := A_{n,p}(dz)^\lambda \, \}$$

for every $\mathcal{F}^\lambda$. We calculate

(28)
$$M(0) = -g, \qquad M(1) = g - 2 \ .$$

The replacement for $k = 1$ has already be done in [**3**]. Let me just quote from there the elements which have been modified (attention: there is an index shift). Again we always assume the normalization (13).

$\boxed{\lambda = 1, g \geq 2, k = 1}$ ω_n for $1 \leq n \leq g$ is given by a

(29)
$$\mathrm{ord}_{P_1}(\omega_n) = n - 1, \quad \mathrm{ord}_{Q_1}(\omega_n) = g - n$$

with a suitable fixing (13) of the multiplicative constant. As ω_0 we take the differential ρ defined in (6).

$\boxed{\lambda = 0, k = 1}$ We set $A_1 = 1$ and choose A_n, $-g < n \leq 0$ with

(30)
$$\mathrm{ord}_{P_1}(A_n) = n - 1, \quad \mathrm{ord}_{Q_1}(A_n) = -g - n \ .$$

In this region the A_n have one additional degree of freedom: we are allowed to add an arbitrary constant. This constant can be fixed by setting for $g \geq 2$

$$A_n = A'_n - \frac{1}{2\pi \mathrm{i}} \oint_{C_\tau} A'_n \cdot \omega_0$$

and for $g = 1$

$$A_0 = A'_0 - \frac{1}{2}\frac{1}{2\pi \mathrm{i}} \oint_{C_\tau} A'_0 \cdot A'_0 \, dz$$

starting from elements A'_n satisfying (30). Again we have duality as given in (14).

For the following we assume $k \geq 2$. We take as $\omega_{n,p}$ and $A_{n,p}$ exactly the elements defined in section 2 except in the following cases. In these cases the generators are given by the requirement (13) and the prescribed orders.

$\boxed{\lambda = 1, n = 0, g \geq 2, k \geq 2}$

$$\begin{aligned} &\mathrm{ord}_{P_i}(w_{0,p}) = -\delta_{i,p}, \quad P_i \in I, \\ &\mathrm{ord}_{Q_1}(w_{0,p}) = -1, \\ &\mathrm{ord}_{Q_i}(w_{0,p}) = 0, \quad Q_i \in O \setminus \{Q_1, Q_l\} \\ &\mathrm{ord}_{Q_l}(w_{0,p}) = g\ . \end{aligned} \tag{31}$$

$\boxed{\lambda = 1, n = 1, g \geq 2, k = 2}$.

$$\begin{aligned} &\mathrm{ord}_{P_i}(w_{1,p}) = 1 - \delta_{i,p}, \quad i = 1, 2 \\ &\mathrm{ord}_{Q_1}(w_{1,p}) = 0, \\ &\mathrm{ord}_{Q_2}(w_{1,p}) = g - 1\ . \end{aligned} \tag{32}$$

(Remark: for $k \geq 3$ there is no change necessary because the requirement in [**3**,(23)] can be replaced by the requirement that there is not just one pole of order one and no other poles at all.)

$\boxed{\lambda = 0, n = 0, g \geq 1}$. We first choose generators $A'_{0,p}$ defined by

$$\begin{aligned} &\mathrm{ord}_{P_i}(A'_{0,p}) = -\delta_{i,p}, \quad P_i \in I, \\ &\mathrm{ord}_{Q_i}(A'_{0,p}) = 0, \quad Q_i \in O \setminus \{Q_l\}, \\ &\mathrm{ord}_{Q_l}(A'_{0,p}) = -g\ . \end{aligned} \tag{33}$$

By these they are not completely fixed. We calculate

$$\gamma_{p,r} := \begin{cases} \dfrac{1}{2\pi \mathrm{i}} \displaystyle\oint_{C_\tau} A'_{0,p}\omega_{0,r} & , g \geq 2 \\ \dfrac{1}{2}\dfrac{1}{2\pi \mathrm{i}} \displaystyle\oint_{C_\tau} A'_{0,p}A'_{0,r}\, dz & , g = 1 \end{cases} \tag{34}$$

and set

$$A_{0,p} := A'_{0,p} - \sum_{r=1}^{k} \gamma_{p,r} A_{1,r}\ . \tag{35}$$

With this modified $A_{n,p}$ and $\omega_{n,p}$ the proposition 1 (duality), proposition 2 (basis) and proposition 3 (expansion formula) are also valid. Again we have the structure equation (21) in proposition 4. The coefficients (23) keep the same. Only L has to be modified.

4. The general case N=k+l.

To avoid too many technicalities we will give here only the definition of the generic cases. For $g = 1$ or $g \geq 2$, $\lambda = 0$ or 1 the definition has to be modified for a finite set of degrees n of the generators.

We start with $k \geq l$. Let $a = (k-l)+1$. Then we define $f_{n,p}(\lambda)$, $n \in \mathbf{Z}, p = 1, \ldots, k$ as the unique generator given by

$$
\begin{aligned}
&\operatorname{ord}_{P_i}(f_{n,p}(\lambda)) = n - \delta_{i,p}, \quad P_i \in I, \\
&\operatorname{ord}_{Q_i}(f_{n,p}(\lambda)) = -n, \quad Q_i \in O \setminus \{Q_l\}, \\
&\operatorname{ord}_{Q_l}(F_{n,p}(\lambda)) = -a \cdot n + M(\lambda) + 1
\end{aligned}
\tag{36}
$$

and the local condition (13).

For $k < l$ we set $b = (l-k)+1$. Here again $f_{n,p}(\lambda)$, $n \in \mathbf{Z}, p = 1, \ldots, k$ is the unique generator given by the the condition (13) and the prescribed orders as follows

$$
\begin{aligned}
&\operatorname{ord}_{P_i}(f_{n,p}(\lambda)) = n - \delta_{i,p}, \quad P_i \in I, \\
&\operatorname{ord}_{Q_i}(f_{n,p}(\lambda)) = -n, \quad i = 1, \ldots, k-1 .
\end{aligned}
\tag{37}
$$

Before we can prescribe the order for the remaining points $Q_i \in O$ we introduce $\epsilon_n, \overline{\epsilon_n} \in \mathbf{Z}$ with

$$
\begin{aligned}
&\epsilon_n \equiv n \mod b, \quad \epsilon_n \in \{0, 1, \ldots, a-1\} \\
&\overline{\epsilon_n} \equiv n \mod b, \quad \overline{\epsilon_n} \in \{-a+1, \ldots, -1, 0\}
\end{aligned}
\tag{38}
$$

For $n \geq 0$ we set

$$
\begin{aligned}
&\operatorname{ord}_{Q_i}(f_{n,p}(\lambda)) = \frac{1}{b}(-n + \epsilon_n) \quad i = k, \ldots, k + |\overline{\epsilon_n}| - 1 \\
&\operatorname{ord}_{Q_i}(f_{n,p}(\lambda)) = \frac{1}{b}(-n + \overline{\epsilon_n}) \quad i = k + |\overline{\epsilon_n}|, \ldots, l-1 \\
&\operatorname{ord}_{Q_l}(f_{n,p}(\lambda)) = \frac{1}{b}(-n + \overline{\epsilon_n}) + M(\lambda) + 1 .
\end{aligned}
\tag{39}
$$

For $n < 0$ we interchange the role of ϵ_n and $\overline{\epsilon_n}$.

THEOREM. *For arbitrary $\lambda \in \mathbf{Z}$ and arbitrary splitting of the set of points A the following statements are valid.*
a) The set of the above generators

$$
f_{n,p}(\lambda), \quad n \in \mathbf{Z}, \ p = 1, \ldots, k
\tag{40}
$$

is a basis of the KN module $\mathcal{F}^{\lambda}$ (resp. for $\lambda = -1$ of the KN algebra $\mathcal{KN}$). The generators obey the duality relations

$$(41)\qquad \frac{1}{2\pi \mathrm{i}} \oint_{C_\tau} f_{n,p}(\lambda) f_{1-n,r}(1-\lambda) = \delta_{n,m} \cdot \delta_{p,r} \ .$$

Every element $v \in \mathcal{F}^{\lambda}$ can be written as a finite linear combination

$$(42)\qquad v = \sum_{n \in \mathbf{Z}} \sum_{p=1}^{k} A_{n,p} f_{n,p}(\lambda), \quad A_{n,p} \in \mathbf{C}$$

with

$$(43)\qquad A_{n,p} = \frac{1}{2\pi \mathrm{i}} \oint_{C_\tau} v \cdot f_{1-n,p}(1-\lambda) \ .$$

b) With respect of the grading induced by setting

$$n = \deg f_{n,p}(\lambda) = \deg e_{n,p}$$

the modules are generalized graded. Their structure is given by

$$(44)\qquad e_{n,p} \cdot f_{m,r}(\lambda) = \sum_{h=n+m-2}^{n+m+L_1} \sum_{s=1}^{k} C_{(n,p),(m,r)}^{(h,s)}(\lambda) \cdot f_{h,s}(\lambda)$$

with $C_{\cdot\cdot\cdot}^{\cdot\cdot} \in \mathbf{C}$. Here L_1 is a constant only depending on the genus g, the numbers k and l and the weight λ. The coefficients at the lower boundary are

$$(45)\qquad C_{(n,p),(m,r)}^{(n+m-2,s)}(\lambda) = \delta_{p,r}\delta_{p,s}((m-1) + \lambda(n-1)) \ .$$

The arguments for the proofs are essentially the same as the arguments used in section 2. and 3. For the non-exceptional λ values the constant L_1 does not depend on λ. Its value is given by the following table

(46)

$\boxed{k \geq l}$	-2	if $g = 0$	$k = 1$
	1	if $g = 0$	$k > 1$
	$1 + \frac{1}{a}(3g-3)$	if $g > 0$	
$\boxed{k < l}$	$-2 + 3b$	if $g = 0$	
	$-2 + (3g)b$	if $g \geq 1$	

On first sight one may ask why in the case $k < l$ we do not interchange the role of the "in" and "out" point, do the calculation and reverse the degree of the generators afterwards. But, this would also change the meaning of the grading. In contrast to the case $N = 2$ the grading is not symmetric with respect to interchanging I and O.

5. Central extensions.

Starting from the algebra $\mathcal{KN}$ we construct central extensions $\widehat{\mathcal{KN}}$ of it. Let $E_{n,p}$ be a fixed lift of $e_{n,p}$ to $\widehat{\mathcal{KN}}$. Then $\widehat{\mathcal{KN}}$ is generated by the elements $E_{n,p}$ $(n \in \mathbf{Z}, p = 1, \ldots, k)$ and a central element t. Its structure is given by

$$[E_{n,p}, t] = 0$$

(47)
$$[E_{n,p}, E_{m,r}] = \sum_{h=n+m-2}^{n+m+L_1} \sum_{s=1}^{k} C_{(n,p),(m,r)}^{(h,s)}(-1) \cdot E_{h,s} + \chi(e_{n,p}, e_{m,r}) \cdot t \ .$$

Here

(48)
$$\chi : \mathcal{KN} \times \mathcal{KN} \to \mathbb{C}$$

is a (2-)cocycle. A standard way to get such cocycles defining nontrivial central extensions is by using [2]

(49)
$$\chi(e, h) = \frac{c}{24\pi \mathrm{i}} \oint_{C_\tau} \left(\frac{1}{2}(f'''g - fg''') - R \cdot (f'g - fg') \right) \ .$$

Here e and h are vector fields with local representations

$$e_| = f(z)\frac{\partial}{\partial z}, \qquad h_| = g(z)\frac{\partial}{\partial z},$$

c is a constant and R is a meromorphic projective connection, holomorphic on $X \setminus A$ (see [3,section 10] for details). If we restrict ourselves to projective connections R with maximal pole order 2 at each of the points $P \in A$ we can calculate [5]

(50)
$$\chi(e_{n,p}, e_{m,r}) = 0 \quad \text{for} \quad (n+m) \geq 5 \quad \text{or} \quad (n+m) \leq T \leq 3 \ .$$

Here T is a number calculated in [5]. It depends only on the genus g of X and the values k and l. For example, if $k = l$ we get

$$T = \begin{cases} -3 & , g = 0, k > 1 \\ -6g + 3 & , \text{otherwise.} \end{cases}$$

We call a cocycle local if it obeys (50). The corresponding extension is called a local extension. Hence an extension defined by (49) is a local extension. By some modification of the arguments in [1] one can show that the converse is also true[1]. For later use we calculate for a local central extension χ with $c = 1$

$$\chi(e_{2-i,p}, e_{2+i,r}) = \left(-\frac{1}{12}(i^3 - i - 4\alpha_p i)\right)\delta_{p,r} \tag{51}$$

where the projective connection R has the local representation:

$$R_| = \alpha_p z_p^{-2} + z_p^{-1}(O(1)) \tag{52}$$

around the point P_p. Using (45) we see

$$[E_{2-i,p}, E_{2+i,p}] = 2iE_{2,p} + \sum_{n\geq 3}\sum_{r} C_{\cdot\cdot}^{\cdot\cdot} E_{n,r} + \chi(e_{2-i,p}, e_{2+i,p})\cdot t\ . \tag{53}$$

Hence we can replace $E_{2,p}$ by $E'_{2,p} = E_{2,p} + 1/6\alpha_p t$ and get as cocycle

$$\chi(e_{2-i,p}, e_{2+i,r}) = -\left(\frac{1}{12}(i^3 - i)\right)\delta_{p,r} \tag{54}$$

which defines the same extension class. There are similar expressions for the other boundary case in (50).

6. Semi-infinite wedge representations.

In this section we show that the action of $\mathcal{KN}$ on $\mathcal{F}^\lambda$ will lift to an action of a local central extension on the space of semi-infinite wedge products. This central extension can be simultaneously defined for every weight λ.

The vector space of semi-infinite wedge products **[1-2,3,11]** is defined to be the vector space generated by the formal elements (dropping the λ in the notation)

$$w = f_{i_1,p_1} \wedge f_{i_2,p_2} \wedge \cdots f_{m,1} \wedge f_{m,2} \cdots \wedge f_{m+1,1} \wedge \cdots\ . \tag{55}$$

The multi-indices are in strictly increasing lexicographical order and it is required that starting from some index (m,p) all indices $(m',p') > (m,p)$ will occur.

We want to transfer the operation of $e_{n,p}$ on $\mathcal{F}^\lambda$ to $\mathcal{H}^\lambda$. We try the following naive definition (Leibniz rule)

$$\begin{aligned} e_{n,p}\cdot w := {} & (e_{n,p}\cdot f_{i_1,p_1}) \wedge f_{i_2,p_2} \wedge \cdots \\ & + f_{i_1,p_1} \wedge (e_{n,p}\cdot f_{i_2,p_2} \wedge \cdots \\ & + f_{i_1,p_1} \wedge f_{i_2,p_2} \wedge \cdots\ . \end{aligned} \tag{56}$$

[1] Due to lack of time, not every detail of the argument has yet been written up. Nevertheless, one should perhaps look for a different proof which stays completely inside the analytic category

Here $\wedge$ indicates how we should transform the result into the standard basis by using the properties of the wedge product (linearity, alternating). For example, if the same element appears at two different positions then the semi-infinite form will be zero. By the structure equation (44) we see that for $e_{n,p}$ with $n \geq 3$ or $n \leq (-1 - L_1)$ there will be only a finite number of elements on the right hand side. We define

$$
\begin{aligned}
\mathcal{KN}^+ &= \langle e_{n,p} \mid n \geq 3,\ p = 1, \ldots, k \rangle \\
\mathcal{KN}^- &= \langle e_{n,p} \mid n \leq -1 - L_1,\ p = 1, \ldots, k \rangle \ .
\end{aligned}
\tag{57}
$$

These are subalgebras and the action of them on $\mathcal{H}^\lambda$ given by (56) is well defined. For the elements $e_{n,p}$ with $-L \leq n \leq 2$ the above action does not make sense. By standard techniques illustrated in [11] by Kac and Raina in the Virasoro case (which is included in the above setting) we can embed $\mathcal{KN}$, using its operation on the basis $f_{m,r}$ of $\mathcal{F}^\lambda$, into the Lie algebra $\overline{a_\infty}$. The latter is the Lie algebra of infinite matrices having only a finite number of nonzero diagonals. This works because $\mathcal{F}^\lambda$ is a generalized graded module over $\mathcal{KN}$. $\overline{a_\infty}$ has a unique central extension a_∞ and the action of $\overline{a_\infty}$ on $\mathcal{F}^\lambda$ (matrix multiplication) extends to an action of a_∞ on $\mathcal{H}^\lambda$. By the embedding of $\mathcal{KN}$ this defines an action of a central extension $\widehat{\mathcal{KN}}$ on $\mathcal{H}^\lambda$. By calculating the cocycle one sees that the extension is a local extension, hence can be given by a cocycle (49).[1] . Especially we have (54). This we can even show directly without using the argument of the last section. We can calculate explicitly the cocycle $\chi(e_{2-i,p}, e_{2+i,r})$ for each λ. After a suitable adjustment of the action (involving only the central element t and $E_{n,p}$ with $-L_1 \leq n \leq 2$) the cocycle has exactly the form (54) independent of λ.

By the extension the action of the subalgebras (57) keep the same. In the critical range $-L_1 \leq n \leq 2$ we have

$$
E_{n,p} \cdot w = e_{n,p} \vdots\ w + r(n,p)w \ . \tag{58}
$$

Here $r(n,p) \in \mathbb{C}$ and $e_{n,p} \vdots\ w$ is obtained by "throwing away" multiples of w appearing on the right hand side of (56). Let us consider

$$
\Phi_T = f_{T,1} \wedge f_{T,2} \cdots f_{T+1,1} \cdots \tag{59}
$$

where all indices $\geq (T,1)$ occur. We call Φ_T the "vacuum vector" of level T. $\mathcal{KN}^+$ operates trivially on it. Using (45) we get

$$
e_{2,p} \cdot f_{m,r} = \delta_{p,r}(m - 1 + \lambda) f_{m,p} + \sum_{n>m} \sum_{s} C_{..} f_{n,s} \ .
$$

[1] There is a different, but in principal equivalent, possibility [5] to show this by using formal power series calculation following an idea of R. Weissauer[12]

Hence $e_{2,p} \cdot \Phi_T = 0$ and $E_{2,p}$ operates as multiplication on Φ_T.
We define $h_\lambda(T,p)$ and c_λ by

$$E_{2,p} \,.\, \Phi_T = h_\lambda(T,p)\Phi_T, \qquad t \,.\, \Phi_T = c_\lambda \Phi_T \,. \tag{60}$$

c_λ is called the central charge. Using the cocycle (54) we see

$$-E_{3,p} \,.\, (E_{1,p} \,.\, \Phi_T) = [E_{1,p}, E_{3,p}] \,.\, \Phi_T = 2E_{2,p} \,.\, \Phi_T + 0t \,.\, \Phi_T = 2h_\lambda(T,p)\Phi_T$$
$$-E_{4,p} \,.\, (E_{0,p} \,.\, \Phi_T) = 4E_{2,p} \,.\, \Phi_T - \frac{1}{2}t \,.\, \Phi_T = (4h_\lambda(T,p) - \frac{1}{2}c_\lambda)\Phi_T$$

and calculate

$$h_\lambda(T,P) = -\frac{1}{2}(T-1)(T-2+2\lambda) \qquad c_\lambda = -2(6\lambda^2 - 6\lambda + 1) \tag{61 .}$$

Generated by the vacuum vector we got a representation of $\widehat{\mathcal{KN}}$ in such a way that $\mathcal{KN}^+$ operates trivially on the vacuum vector, $E_{2,p}$ operates as multiplication with a certain value on the vacuum vector and the central element operates as multiplication with a fixed central charge. Hence we can consider them as generalization of the Verma modules (better as quotients of the Verma modules). If one wants a representation for which $E_{-L_1,p}$ operates as multiplication on some vacuum and KN^- annihilates the vacuum one has to use left semi-infinite forms (i. e. decreasing indices).

In the same spirit one can now study the corresponding generalized Heisenberg algebras and use this to study operators on the string world sheet (see [**1**,**2**] for the case $N = 2$). A lot of work done in references [**6-14**] in the literature list of [3] can now be generalized to arbitrary N. For example, the $b-c$ systems as they were studied in [**13**]. Some of this will be done in [**5**].

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