Second order optimality conditions for infinite programs

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Abstract. The paper deals with the minimization problem of a marginal function over a subset C of a space X. Unlike the existing papers, X is not assumed to be a finitedimensional space and C is a geometric constraint which may not coincide with X. Second order necessary and sufficient optimality conditions are written in terms of some approximations of the data of the problem.

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1. Introduction

Given subsets T, C and a function $f : T \times C \rightarrow I\!R$, consider the following infsup-Problem (P)

$$\text{minimize } \{\sup_{t \in T} f_t(x) : x \in C\}$$

where $f_t(x) = f(t, x)$.

If C is a subset of a finitedimensional space X then (P) is known in the literature as a semi-infinite program (SIP) whose theory is given in [2,5,6,9,12,14,19-21,23]. Second order necessary and sufficient optimality conditions for SIP are obtained in [9,23] under the assumption that $f(t, x)$ are twice continuously differentiable in both variables t, x and T is defined as a solution set of finitely many equalities and inequalities involving twice continuously differentiable functions. The case of SIP where $f(t, x)$ is not necessarily differentiable in t and T is an arbitrary compact set is treated in [2,5,6,12,19,21,23]. The reader who is interested in the general theory of higher order optimality conditions for mathematical programming is referred to [1,3,11,15,17,19].

The aim of the present paper is also to give second order necessary and sufficient optimality criteria for the problem (P) but, unlike [2,5,6,9, 12,14,17,23], X is of infinite dimension and C is a geometric constraint which may not coincide with X. (For the first order necessary conditions of such a problem, see [8].) Instead of assuming the differentiability properties of $f(t, \cdot)$ we shall introduce its approximations, suitable for deriving the desired results. We shall see that the first and second derivatives of $f(t, \cdot)$ can be used as approximations in our sense.

The organization of the paper is as follows. Section 2 is devoted to the discussion of three necessary optimality criteria for the infsup- Problem (P) where X is an arbitrary topological vector space. In the first criteria we prove that the " Lagrange multipliers " exist for any finitedimensional subspace of X. The second one is obtained by using the continuous dual of the Banach space of the continuous functions defined on a compact set. In the third criteria which is valid for a Banach reflexive space X necessary conditions are written in terms of ε-objects. Section 3 deals with sufficient optimality conditions which correspond to the necessity part stated in the first two criteria. Section 4 discusses the results in the case X being a finitedimensional space. In Section 5 we give some examples of approximations of the data used in Sections 2 and 3. In particular, we shall show that the approximating functions introduced in [12] to write optimality conditions for SIP can be served as approximations in our sense. For the sake of completeness we prove in the Appendix two assertions of Convex Analysis which are needed for deriving the results of Section 1.

We conclude the introduction by recalling that in the present paper by a cone H we mean a subset of a linear space such that $\lambda x \in H$ for all $\lambda \geq 0, x \in H$. A function $f : H \longrightarrow \mathbb{R}$ is said to be positively homogeneous of degree 1 (resp. degree 2) if $f(\lambda x) = \lambda f(x)$ (resp. $f(\lambda x) = \lambda^2 f(x)$) for all $\lambda \geq 0, x \in H$. The symbol $o(\cdot)$ will denote a function of $\varepsilon > 0$ such that $\lim_{\varepsilon \to 0} o(\varepsilon)\varepsilon^{-1} = 0$. For different functions having the just written property we shall use the symbols $o^1(\cdot), o^2(\cdot), o_t(\cdot)$...

2. Second order necessary conditions

In this section we shall consider the infsup-Problem (P) under the assumption that

T is an arbitrary compact set,

C is a subset of a topological vector space X

and

$f(t, x)$ is a function which is u.s.c. in t.

We say that $x_0 \in C$ is a solution (resp. strict solution) of (P) if there is a neighbourhood V of x_0 such that $f(x_0) \leq f(x)$ (resp. $f(x_0) < f(x)$) whenever $x \in C \cap V$ and $x \neq x_0$. Here $f(x) = \sup_{t \in T} f_t(x)$. Let x_0 be a solution of (P). Without loss of generality we may assume that $f(x_0) = 0$. In view of the upper semicontinuity of $f_t(x_0)$ as a function of t the set

$$T_0 := \{t \in T : f_t(x_0) = f(x_0) = 0\}$$

is compact.

Necessary conditions for the optimality of x_0 will be written in terms of some approximations of the data of (P). Examples of such approximations can be found in Section 4. They show that our assumptions are fulfilled for a large class of infinite programs.

Let $H \subset X$ be a convex cone and $D \subset H$ be a cone (not necessarily convex) such that there is a map $h : H \longrightarrow C$ with the property that

$$\forall d \in D,\ \forall x \in H,\ \forall \varepsilon > 0 \text{ sufficiently small: } h(\varepsilon d + \varepsilon^2 x) - (x_0 + \varepsilon d + \varepsilon^2 x) = o(\varepsilon^2)$$

where $o(\cdot)$ may depend on d and x. In other words, we require that

(2.1) $\forall d \in D, \forall x \in H, \exists o(\cdot)$ such that $x_\varepsilon \in C$ for ε sufficiently small where

$$x_\varepsilon = x_0 + \varepsilon d + \varepsilon^2 x + o(\varepsilon^2) \tag{2.2}$$

and

$$\lim_{\varepsilon \to 0} o(\varepsilon)\varepsilon^{-1} = 0. \tag{2.3}$$

Let $f_t^1 : \overline{H} \to \mathbb{R}$ ($\overline{H}$ being the closure of H) and $f_t^2 : D \to \mathbb{R}$ such that the following requirements are satisfied:

(1) $f_t^1(x)$ is l.s.c., convex, positively homogeneous of degree 1 in x and u.s.c. in t.

(2) $f_t^2(x)$ is positively homogeneous of degree 2 in x and u.s.c. in t.

(3) $\forall d \in D, \forall x \in H, \exists o_t(\cdot)$ such that

$$f_t(x_\varepsilon) - f_t(x_0) \leq f_t^1(\varepsilon d + \varepsilon^2 x) + f_t^2(\varepsilon d) + o_t(\varepsilon^2) \ , \forall t \in T \ , \tag{2.4}$$

where x_ε is given by (2.2) and

$\lim_{\varepsilon \to 0} o_t(\varepsilon)\varepsilon^{-1} = 0$ uniformly with respect to $t \in T$.

Remark 2.1.Observe that $f_t^2(0) = 0$. Hence, by setting $d = 0$ in (2.3) we obtain the following relation between f_t and f_t^1:

$$f_t(x_0 + \varepsilon x + o(\varepsilon)) - f_t(x_0) \leq f_t^1(\varepsilon x) + o_t(\varepsilon), \forall t \in T.$$

Definition 2.1. $d \in X$ is called a critical direction if

$$d \in D,$$

$$f_t(x_0) + \delta f_t^1(d) \leq 0, \forall t \in T, \tag{2.5}$$

where δ is a positive constant not depending on $t \in T$.

Remark 2.2. It is a simple matter to check that (2.5) implies

$$f_t(x_0) + \varepsilon f_t^1(d) \leq 0, \forall t \in T, \forall \varepsilon \in [0, \delta]. \tag{2.6}$$

In the sequel we shall need the following result.

Lemma 2.1 . Assume that x_0 is a solution of (P). Then for any critical direction d we have

$$q(x, d) := \sup_{t \in T_0} \{f_t^1(x) + f_t^2(d)\} \geq 0, \forall x \in H. \tag{2.7}$$

Proof. Let us set

$$q_t(x, d) = f_t^1(x) + f_t^2(d)$$

and assume the contrary that there is a point $\hat{x} \in H$ such that $q_t(\hat{x}, d) < 0$ for all $t \in T_0$. From the compactness of T_0 it follows the existence of a positive constant γ such that

$$q_t(\hat{x}, d) < -\gamma \quad , \forall t \in T_0.$$

The set

$$T_1 := \{t \in T : q_t(\hat{x}, d) < -2^{-1}\gamma\}$$

is open and contains T_0. Hence $T_2 := T \setminus T_1$ is compact and disjoint from T_0. Therefore for some $k_0 > 0$ we have

$$\sup_{t \in T_2} f_t(x_0) \leq -k_0.$$

Let

$$k_1 = \sup_{t \in T_2} f_t^1(d) \leq +\infty,$$
$$k_2 = \sup_{t \in T_2} q_t(\hat{x}, d) \leq +\infty.$$

Setting $x = \hat{x}, x_\varepsilon = x_0 + \varepsilon d + \varepsilon^2 \hat{x} + o(\varepsilon^2)$ and taking account of the sublinearity of $f_t^1(\cdot)$ and the homogeneity of $f_t^2(\cdot)$ we obtain from (2.4)

$$f_t(x_\varepsilon) \leq f_t(x_0) + \varepsilon f_t^1(d) + \varepsilon^2 q_t(\hat{x}, d) + o_t(\varepsilon^2) \tag{2.8}$$

For $t \in T_1$ it follows from the just written inequality that

$$f_t(x_\varepsilon) \leq \varepsilon^2 q_t(\hat{x}, d) + o_t(\varepsilon^2) < -2^{-1}\gamma\varepsilon^2 + o_t(\varepsilon^2)$$

since d is a critical direction. Hence there is $\delta_1 > 0$ such that

$$f_t(x_\varepsilon) < 0, \forall t \in T_1, \forall \varepsilon \in (0, \delta_1).$$

For $t \in T_2$ inequality (2.8) shows that

$$f_t(x_\varepsilon) \leq -k_0 + \varepsilon k_1 + \varepsilon^2 k_2 + o_t(\varepsilon^2)$$

which implies the existence of $\delta_2 > 0$ such that

$$f_t(x_\varepsilon) < 0, \forall t \in T_2, \forall \varepsilon \in (0, \delta_2).$$

We have thus obtained that $x_\varepsilon \in C$ and

$$\max_{t \in T} f_t(x_\varepsilon) < 0 = \max_{t \in T} f_t(x_0)$$

for all ε sufficiently small. This contradicts the local optimality of x_0.

Before going further, let us formulate the following lemma which can be established by using the same argument as in [16,Theorem 1] or [4,p. 99- 100]. For the sake of completeness we include its proof in the Appendix.

Lemma 2.2. Let $\tilde{H} \subset I\!R^n$ be a nonempty closed convex set, $\tilde{T}$ be an arbitrary compact set and $\varphi : \tilde{T} \times \tilde{H} \to I\!R$ be a function, convex and l.s.c. in $x \in \tilde{H}$ and u.s.c. in $t \in \tilde{T}$. If the system

$$x \in \tilde{H}, \varphi(t, x) < 0, \forall t \in \tilde{T}, \tag{2.9}$$

is inconsistent (that is, it has no solution) then there are a finite subset $\tau \subset \tilde{T}$ with $|\tau| \leq 1+n$, such that the system

$$x \in \tilde{H},\ \varphi(t,x) < 0,\ \forall t \in \tau,$$

is inconsistent. Here $|\tau|$ denotes the number of the elements of τ.

We recall also a result of Fan-Glicksberg-Hoffman which can be found in [18, p. 65]. For the sake of completeness we include in the Appendix a short proof based on the separation theorem in $\mathbb{R}^n$.

__Lemma 2.3__. Let $\tilde{H}$ be a convex set of a linear space X, $\tilde{T}$ be a finite set and $\varphi : \tilde{T} \times \tilde{H} \to \mathbb{R}$ be a function, convex in $x \in \tilde{H}$. If the system (2.9) is inconsistent then there are nonnegative numbers $\lambda_t (t \in \tau)$ satisfying

$$\sum_{t\in\tau} \lambda_t = 1, \tag{2.10}$$

$$\sum_{t\in\tau} \lambda_t \varphi(t,x) \geq 0, \forall x \in \tilde{H}. \tag{2.11}$$

As a consequence of Lemmas 2.1-2.3 we obtain

__Theorem 2.1__. Assume that x_0 is a solution of (P). Then for any critical direction d and any subspace $S \subset X$ with $\dim S < \infty$, there are a finite subset $\tau \subset T_0$ with $|\tau| \leq 1 + \dim S$, $\lambda \geq 0 (t \in \tau)$ satisfying (2.10) and

$$L^1(x) \geq 0, \forall x \in H \cap S, \tag{2.12}$$

$$L^2(d) \geq 0 \tag{2.13}$$

where

$$L^i(x) = \sum_{t\in\tau} \lambda_t f_t^i(x) \quad (i = 1,2). \tag{2.14}$$

Proof. Let d be a critical direction and S be a subspace of X such that $\dim X < \infty$. We derive from Lemma 2.1 that $q(x,d) \geq 0, \forall x \in H_1 := H \cap S$. Since S is of finite dimension and $q(x,d)$ is convex in x we have by [19,Corollary 7.3.3] that

$$(2.15) \qquad q(x,d) \geq 0, \forall x \in \overline{H}_1$$

where $\overline{H}_1$ denotes the closure of H_1 in the subspace S. Hence (2.15) means that the system (2.9) with $\tilde{H} = \overline{H}_1; \varphi(t,x) = q_t(x), \tilde{T} = T_0$ is inconsistent. Applying Lemmas 2.2 and 2.3 yields the existence of a finite subset $\tau \subset T_0$ with $|\tau| \leq 1 + \dim S$ and nonnegative numbers $\lambda_t (t \in \tau)$ such that (2.10) is satisfied and

$$(2.16) \qquad L^1(x) + L^2(d) \geq 0, \forall x \in \overline{H}_1.$$

Observe that $L_1(0) = 0$ (since $f_t^1(0) = 0, \forall t \in T$), we derive (2.13) from (2.16) by setting $x = 0$. The inequality (2.12) follows from (2.16) and the homogeneity of $f_t^1(\cdot)$.

To state the second optimality criteria we have to introduce some notations. Let $\tilde{T}$ be a nonempty compact set and $C(\tilde{T})$ be the Banach space of the continuous functions $F : \tilde{T} \to I\!R$ with the norm

$$\|F\| = \max_{t \in \tilde{T}} |F_t|.$$

We shall denote by $C^*(\tilde{T})$ the continuous dual of $C(\tilde{T})$:

$$C^*(\tilde{T}) = \{C(\tilde{T})\}^*.$$

For $\mu \in C^*(\tilde{T})$ we shall write $\mu \geq 0$ if $\langle \mu, F \rangle \geq 0$ for all $F \in C(\tilde{T})$ such that $F_t \geq 0, \forall t \in T_0$.

We shall need the following lemma.

Lemma 2.4. Let $\tilde{H}$ be an arbitrary set, $\tilde{T}$ be a nonempty compact set and $\varphi : \tilde{T} \times \tilde{H} \to I\!R$ be a function, continuous in $t \in \tilde{T}$. If the system (2.9) is inconsistent then there is $\mu \in C^*(\tilde{T})$ satisfying

(2.17) $$\mu \geq 0,\ \mu \neq 0,$$

$$\langle \mu, \varphi(t,x) \rangle \geq 0, \forall x \in \tilde{H}.$$

Proof. By assumption, for fixed $x \in \tilde{H}$, $\varphi(t,x)$ is continuous as a function of $t \in \tilde{T}$. Hence, $\varphi(t,x) =: F^1(x) \in C(T_0)$.

Let

$$Q_1 = \{F \in C(\tilde{T}) : F_t < 0, \forall t \in \tilde{T}\},$$

$$Q_2 = \{F \in C(\tilde{T}) : \exists x \in H \text{ such that } F \geq F^1(x)\}$$

where the inequality $F \geq G (F, G \in C(\tilde{T}))$ means that $F_t \geq G_t, \forall t \in \tilde{T}$. It can be verified that $Q_i (i = 1, 2)$ are nonempty convex sets and int $Q_1 \neq \emptyset$ where int Q_1 denotes the interior of Q_1 in the topology of $C(\tilde{T})$ induced by the norm defined above and $\emptyset$ stands for the empty set. On the other hand, the inconsistency of (2.9) shows that $Q_1 \cap Q_2 = \emptyset$. Hence, by the separation theorem we can find $\mu \in C^*(\tilde{T})$ satisfying the required conclusion of the lemma.

Theorem 2.2 . Assume in addition that $f_t^i(x) (i = 1, 2)$ are continuous in $t \in T$. If x_0 is a solution of IP then for any critical direction d there is $\mu \in C^*(T_0)$ satisfying (2.17) and

(2.18) $$L^1(x) \geq 0, \forall x \in H,$$

(2.19) $$L^2(d) \geq 0$$

where

(2.20) $$L^i(x) = \langle \mu, f_t^i(x) \rangle \ (i = 1, 2).$$

Proof. Take $\tilde{H} = H,\ \tilde{T} = T_0,\ \varphi(t,x) = f_t^1(d) + f_t^2(x)$ and observe by Lemma 2.1 that the system (2.9) is inconsistent. It remains to apply Lemma 2.4.

The following additional requirement will be used in Theorem 2.3:

(4) For any critical direction d the condition

$$q(x,d) \geq 0, \forall x \in H,$$

implies

$$q(x,d) \geq 0, \forall x \in \overline{H}.$$

Obviously, this requirement is satisfied if either of the following conditions holds:

(a) $q(x,d)$ is u.s.c. in x.

(b) H is closed.

(c) int $H \neq \emptyset$.

Theorem 2.3. Assume that X is a Banach reflexive space and the additional requirement (4) is satisfied. If x_0 is a solution of (P) then for any critical direction d and positive numbers ε, r there are a finite subset $\tau \subset T_0, \lambda_t \geq 0 (t \in \tau)$ satisfying (2.10) and

$$L^1(x) \geq -\varepsilon, \forall x \in B_r \cap \overline{H}, \tag{2.21}$$

$$L^2(d) \geq -\varepsilon \tag{2.22}$$

where $L^i(x)$ is defined by (2.14) and B_r stands for the closed ball with radius r around the origin.

Proof. We first claim that for any critical direction d and any $\varepsilon > 0$ there are a finite subset $\tau \subset T_0, \lambda_t \geq 0 (t \in \tau)$such that (2.10),(2.22) are satisfied and

$$L^1(x) \geq -\varepsilon |x|, \forall x \in \overline{H}. \tag{2.23}$$

Indeed, as T_0 is compact and $f_t^2(d)$ is u.s.c. in $t \in T_0$, there is a positive constant α such that

$$f_t^2(d) \leq \alpha, \forall t \in T_0. \tag{2.24}$$

Let $K = \overline{H} \cap B_\beta$ where $\beta = (\varepsilon + \alpha)\varepsilon^{-1} > 0$. The set K is weakly compact since X is a Banach reflexive space. Consider a family of sets $K_t (t \in T_0)$ defined as follows:

$$K_t = \{x \in K : f_t^1(x) + f_t^2(d) \le -\varepsilon\}.$$

Observe that $f_t^1(\cdot)$ being l.s.c. and convex is weakly l.s.c.. From this we can verify that K_t is a weakly closed set. On the other hand, by Lemma 1.1 and the requirement (4)

$$\bigcap_{t \in T_0} K_t = \emptyset\ .$$

Hence, there is a finite subset $\tau \subset T_0$ such that

$$\bigcap_{t \in \tau} K_t = \emptyset$$

which means that the system

$$x \in K, f_t^1(x) + f_t^2(d) \le -\varepsilon, \forall t \in \tau,$$

is inconsistent. We invoke Lemma 2.3 to derive the existence of nonnegative numbers $\lambda_t (t \in \tau)$ satisfying (2.10) and

$$L^1(x) + L^2(d) \ge -\varepsilon, \forall x \in K. \tag{2.25}$$

Taking account of the fact that $0 \in K$ and $L^1(0) = 0$, we obtain (2.22) by setting $x = 0$ in (2.25). Observe now by (2.10) and (2.24) that $L^2(d) \le \alpha$. Consequently, we get from (2.25)

$$L^1(x) \ge -\varepsilon - \alpha, \forall x \in \overline{H} \cap B_\beta.$$

By the positive homogeneity of $L^1(\cdot)$ this yields

$$L^1(x) \ge -(\varepsilon + \alpha)\beta^{-1}|x| = -\varepsilon|x|, \forall x \in \overline{H}.$$

The above claim is thus established. To complete the proof of the theorem it remains to apply this claim with εr^{-1} in place of ε.

3. Second order sufficient conditions.

In this section we assume that X is a normed space, $C \subset X$ and T are arbitrary sets. Fix a point $x_0 \in C$ and assume as in the previous section that $\sup_{t\in T} f_t(x_0) = 0$. We shall introduce approximations quite different from those used to derive necessary conditions. Observe that the similar situation occurs in mathematical programming [3] and semi-infinite program [12].

Let there be given a cone $H \subset X$ and a map $h : C \to H$ such that

$$\lim_{\substack{x \to x_0 \\ x \in C}} |h(x) - (x - x_0)||x - x_0|^{-1} = 0. \tag{3.1}$$

In this case, Maurer and Zowe [19] say that C is approximated by H. This approximation property was used in [19] as a main tool to derive sufficiency theorems for mathematical programming. Details can be found in [19].

Let $f_t^i : H \to I\!R$ $(i = 1, 2)$ be such that the following requirements are satisfied:

(1) $f_t^1(x)$ is positively homogeneous of degree 1 in x.

(2) $f_t^2(x)$ is positively homogeneous of degree 2 in x.

(3) $\exists o_t^i(\cdot)$ such that $\forall t \in T_0, \forall x \in C$:

$$f_t(x) - f_t(x_0) \geq f_t^1(h(x)) + o_t^1(|h(x)|), \tag{3.2}$$

$$f_t(x) - f_t(x_0) \geq f_t^1(h(x)) + f_t^2(h(x)) + o_t^2(|h(x)|^2) \tag{3.3}$$

where $\lim_{\varepsilon \to 0} o_t^i(\varepsilon)\varepsilon^{-1} = 0 \quad (i = 1, 2)$.

<u>**Remark 3.1**</u>. If $f_t^2(\cdot)$ is l.s.c. at $h = 0$ then (3.2) is a consequence of (3.3). Indeed, from $f_t^2(0) = 0$ and the lower semicontinuity follows the existence of $\delta > 0$ such that $f_t^2(h) \geq -1$ for all $h \in H$ with $|h| < \delta$, hence by homogeneity $f_t^2(h) \geq -\delta^{-2}|h|^2$ for all $h \in H$. Therefore for all $h \in H$ we have

$$f_t^2(h) + o_t^2(|h|^2) \geq -\delta^{-2}|h|^2 + o_t^2(|h|^2) =: o_t^1(|h|).$$

This shows that (3.3) implies (3.2).

Definition 3.1. A sequence d_k is called weakly critical if

$$d_k \in H, |d_k| = 1, \forall k,$$

(3.4) $$\limsup_{k\to\infty} f_t^1(d_k) \le 0, \forall t \in T_0.$$

Theorem 3.1. Assume that, for every weakly critical sequence d_k, there are a finite subset $\tau \subset T_0$, $\lambda_t \ge 0$ $(t \in \tau)$ such that

(3.5) $$L^1(x) \ge 0, \ \forall x \in H,$$

(3.6) $$\limsup_{k\to\infty} L^2(d_k) > 0,$$

where $L^i(x)$ is defined by (2.14). Then x_0 is a strict solution of (P).

Proof. Assume the contrary that x_0 is not a strict solution of (P). Then there is a sequence $x_k \in C$ such that $q_k := x_k - x_0 \to 0$, $q_k \ne 0$ and $0 \ge f(x_k) - f(x_0)$, $\forall k$. The last inequality implies

(3.7) $$0 \ge f_t(x_k) - f_t(x_0), \ \forall k, \ \forall t \in T_0.$$

Let h be the map appeared in (3.1). Putting $h_k = h(x_k)$ it follows from (3.1) that

$$|h_k - q_k||q_k|^{-1} \to 0.$$

This implies, since $q_k \to 0$, that $|h_k - q_k| \to 0$, and therefore $h_k \to 0$. Hence $o_t^i(|h_k|)|h_k|^{-1} \to 0$ $(i = 1, 2)$. Combining (3.7) and (3.2) yields

(3.8) $$0 \ge f_t^1(h_k) + o_t^1(|h_k|), \ \forall k, \ \forall t \in T_0,$$

or, equivalently,

(3.9) $$0 \ge f_t^1(d_k) + o_t^1(|h_k|)|h_k|^{-1}, \ \forall k, \ \forall t \in T_0,$$

where

$$d_k := h_k |h_k|^{-1} \in H.$$

By letting $k \to \infty$ in (3.9) we get (3.4). This shows that d_k is a weakly critical sequence. By assumption there are a finite subset $\tau \subset T_0$, $\lambda_t \geq 0$ $(t \in \tau)$ satisfying (3.5) and (3.6). Putting $x = x_k$ in (3.3) and using (3.7) we have

$$0 \geq f_t^1(h_k) + f_t^2(h_k) + o_t^2(|h_k|^2), \ \forall k, \ \forall t \in \tau. \tag{3.10}$$

Multiplying (3.10) by λ_t and summing up we obtain

$$0 \geq L^1(h_k) + L^2(h_k) + o(|h_k|^2), \ \forall k, \tag{3.11}$$

where $o(\cdot)$ is some function satisfying (2.3). Dividing (3.11) by $|h_k|^2$ and taking account of (3.5) we find

$$0 \geq L^2(d_k) + o(|h_k|^2)|h_k|^{-2} \tag{3.12}$$

which implies

$$0 \geq \limsup_{k \to \infty} L^2(d_k),$$

contradicting (3.6). The theorem is thus proved.

Remark 3.2. If $o_t^1(\cdot)$ in (3.2) is independent of $t \in T_0$ then the following condition is sufficient for the assumption of Theorem 3.1 to be satisfied: there are positive numbers δ, γ, a finite subset $\tau \subset T_0$, $\lambda_t \geq 0$ $(t \in \tau)$ such that (3.5) holds and $L^2(d) > \gamma$ for all $d \in H_\delta$ where

$$H_\delta = H \cap \{x : |x| = 1, \ f_t^1(x) \leq \delta, \ \forall t \in T_0\}.$$

Indeed, if d_k is a weakly critical sequence then, for all k sufficiently large, $d_k \in H_\delta$ hence $L^2(d_k) > \gamma$ which implies (3.6).

Theorem 3.2. Assume that $f_t^i(x)$ $(i = 1, 2)$ are continuous in t and $o_t^2(\cdot)$ in (3.3) is independent of $t \in T_0$. Assume that for every weakly critical sequence d_k there is a functional $\mu \in C^*(T_0)$, $\mu \geq 0$, such that (3.5) and (3.6) are satisfied where $L^i(x) = \langle \mu, f_t^i(x) \rangle$ $(i = 1, 2)$. Then x_0 is a strict solution of (P).

The proof is similar to that of Theorem 3.1 and will be omitted.

Remark 3.3. If $o_t^1(\cdot)$ in (3.2) is also independent of $t \in T_0$ then the conclusion similar to that of Remark 3.2 can be stated for Theorem 3.2.

Remark 3.4. The results of Sections 2 and 3 can be used to obtain optimality conditions for the problem (P_0):

$$\text{minimize } \{f_0(x) : x \in C,\ f_t(x) \leq 0,\ \forall t \in T\}$$

where $f_0 : C \to I\!R$ is a given function. To see this, it is enough to consider the following problem (P_1) having the same structure as (P):

$$\text{minimize } \{ \sup_{t \in T} (f_0(x) - f_0(x_0); f_t(x)) : x \in C\}$$

and observe [12] that

x_0 is a solution of (P_0) if it is a solution of (P_1);

x_0 is a strict solution of (P_0) if it is a feasible point for (P_0) and is a strict solution of (P_1).

4. Finitedimensional case

In this section we assume that $\dim X < \infty$. Applying Theorem 1.1 to the subspace $S = X$ we get

Theorem 4.1. If x_0 is a solution of (P), then for any critical direction d there are a finite subset $\tau \subset T_0$ with $|\tau| \leq 1 + \dim$, $\lambda_t \geq 0$ $(t \in \tau)$ satisfying (2.10) and

(4.1) $$L^1(x) \geq 0,\ \forall x \in \overline{H}.$$

(4.2) $$L^2(d) \geq 0.$$

where $L^i(\cdot)$ is given by (2.14).

Turning now to sufficient conditions, we first introduce

Definition 4.1. A vector $d \in X$ is called a weakly critical direction if

$$d \in \overline{H},\ |d| = 1,$$

$$f_t^1(d) \le 0,\ \forall t \in T_0.$$

Here we assume that the functions $f_t^i(\cdot)$ $(i = 1, 2)$ are defined on $\overline{H}$ and satisfy the requirements (1)-(3) of Section 3. Obviously, if $d \in H$ is a weakly critical direction then the stationary sequence $d_k \equiv d$ is a weakly critical sequence (see Def. 3.1).

Theorem 4.2. Assume that $f_t^i(x)$ $(i = 1, 2)$ are l.s.c. in x. If for every weakly critical direction d there are a finite subset $\tau \subset T_0$, $\lambda_t \ge 0$ $(t \in \tau)$ such that

$$L^1(x) \ge 0,\ \forall x \in H, \tag{4.3}$$

$$L^2(d) > 0, \tag{4.4}$$

then x_0 is a strict solution of (P).

Proof. Take an arbitrary weakly critical sequence d_k. By the compactness of the unit sphere in X and the lower semicontinuity of $f_t^1(\cdot)$ we may assume that d_k converges to some weakly critical direction d satisfying (4.3) and (4.4) for suitable $\tau \subset T_0$, $\lambda_t \ge 0$ $(t \in \tau)$. Since (4.4) implies (3.6), it remains to apply Theorem 3.1.

5. Examples

This section gives some examples of approximations of the data of (P) used in Sections 2 and 3. Clearly, combining these examples with the previous results yields various optimality conditions for the problem under consideration.

Example 5.1. Assume that C is a convex set of a topological vector space X. Let

$$D = H = \text{cone}\,[C - x_0] := \{\lambda(c - x_0) : \lambda \ge 0,\ c \in C\}.$$

We shall show that $o(\cdot) \equiv 0$ satisfies (2.1) for all $d \in D, x \in H$. Indeed, take $\alpha > 0$ and $\beta > 0$ such that $d_1 := x_0 + \alpha^{-1}d \in C$ and $x_1 = x_0 + \beta^{-1}x \in C$. Since $x_0 + \varepsilon d + \varepsilon^2 x = (1 - \alpha\varepsilon - \beta\varepsilon^2)x_0 + \alpha\varepsilon d_1 + \beta\varepsilon^2 x_1 \in C$ for ε sufficiently small, the desired conclusion follows.

<u>Example 5.2</u>. Let $C = \{x : g(x) = 0\}$ where $g : X \to Y$ is a map between Banach spaces X and Y. Assume that g is of the class C^2 (that is, g is twice continuously differentiable in the sense of Frechet) and the first derivative $g'(x_0)$ is surjective. Take

$$D = \{d : g'(x_0)d = 0, g''(x_0)[d, d] = 0\},$$

$$H = \{x : g'(x_0)x = 0\}$$

where $g''(x_0)$ denotes the second derivative of g at x_0. That the condition (2.1) is satisfied follows from a result of Ljusternik [13,p. 30].

We now turn to examples of approximations of the function $f_t(x)$. Throughout the forthcoming, unless otherwise specified, we shall assume that X is a normed space and $f_t(\cdot)$ is of the class C^1 (that is, $f_t(\cdot)$ is continuously differentiable in the sense of Frechet). In Examples 5.3 -5.6 we shall set

$$D = H = X$$

(5.1) $$f_t^1(x) = f_t'(x_0)x$$

where $f_t'(x_0)$ denotes the first derivative of $f_t(\cdot)$ at $x = x_0$. Since in these examples the requirements (1) and (2) of Section 2 are obviously satisfied, we shall verify only the requirement (3). We emphasize that this requirement will be fulfilled for all $d \in X$, $x \in X$ and $o(\cdot)$ being an arbitrary function with property (2.3). So our approximations $f_t^i(x)$ are independent of the approximations D and H of the constraint set C. Hence they can be used to formulate optimality conditions for various approximations of C.

<u>Example 5.3</u>. Assume that $f_t(\cdot)$ is of the class C^2 and the derivatives (in x) $f_t'(x)$, $f_t''(x)$ are jointly continuous in (t, x). Taking

(5.2) $$f_t^2(x) = 2^{-1}f_t''(x_0)x ,$$

we have by the second order Taylor expansion theorem

$$f_t(x_\varepsilon) - f_t(x_0) = f_t'(x_0)\ [\varepsilon d + \varepsilon^2 x] + 2^{-1} f_t''(x_0)\ [\varepsilon d, \varepsilon d] + 2^{-1}\varphi_t(\varepsilon)$$

where

$$x_\varepsilon = x_0 + \varepsilon d + \varepsilon^2 x + o(\varepsilon^2)\ ,$$

$$\varphi_t(\varepsilon) = 2f_t'(x_0)o(\varepsilon^2) + \{f_t''(x_0 + \theta(x_\varepsilon - x_0))\ [x_\varepsilon - x_0, x_\varepsilon - x_0] - \varepsilon^2 f_t''(x_0)\ [d, d]\ \}$$

and $\theta \in [0,1]$ is a number depending on t and ε. Evidently, as $\varepsilon \to 0$, $\varphi_t(\varepsilon)$ converges to zero uniformly with respect to $t \in T$. Hence (2.4) is satisfied.

Example 5.4. Assume that $f_t'(x)$ is jointly continuous in (t, x) and, for any $d \in X$, the convergence of the limit

$$g_t(x_0; d) := \limsup_{\varepsilon \to 0} \varepsilon^{-2}[f_t(x_0 + \varepsilon d) - f_t(x_0) - \varepsilon f_t'(x_0)d] \tag{5.3}$$

is uniform with respect to t. We have

$$\varepsilon^{-2}[f_t(x_\varepsilon) - f_t(x_0) - f_t'(x_0)\ (\varepsilon d + \varepsilon^2 x) - g_t(x_0; \varepsilon d)] = \tag{5.4}$$

$$\varepsilon^{-2}[f_t(x_0 + \varepsilon d) - f_t(x_0) - \varepsilon f_t'(x_0)\ d] - g_t(x_0; d) + \varphi_t(\varepsilon)$$

where

$$\varphi_t(\varepsilon) = \varepsilon^{-2}[f_t(x_\varepsilon) - f_t(x_0 + \varepsilon d)] - f_t'(x_0)x.$$

By the mean value theorem

$$\varphi_t(\varepsilon) = f_t'(x_0 + \varepsilon d + \theta[\varepsilon^2 x + o(\varepsilon^2)])x - f_t'(x_0)x$$

where $\theta \in [0,1]$ is some suitable number depending on t and ε. Note that $\varphi_t(\varepsilon) \to 0$ uniformly with respect to t. Therefore, by taking

$$f_t^2(x) = g_t(x_0; x) \tag{5.5}$$

we derive from (5.4) and the uniform convergence in (5.3) that (2.4) is satisfied.

__Examples 5.5__. Let V be a (convex) neighbourhood of $x_0 \in X = \mathbb{R}^n$. Following [10] we denote by $C^{11}(V)$ the class of functions F which are differentiable in V and whose gradient $F'(x)$ is locally Lipschitz in V. Let $\delta^2 F(x_0)$ be the generalized Hessian matrix of F at x_0, defined as in [10]:

$$\delta^2 F(x_0) = \text{ co } \{M : \exists x_i \to x_0 \ , \ F''(x_i) \text{ exists and } F''(x_i) \to M\}.$$

Here the space of $n \times n$ matrices M is topologized by taking some matricial norm $\|\cdot\|$ on it and "co" stands for the convex hull. Observe [10] that $\delta^2 F(\cdot)$ has the following properties:

1. $\delta^2 F(x_0)$ is locally bounded in the sense that

$$\|M\| \leq \alpha \ , \ \forall M \in \delta^2 F(x_0) \ , \ \forall x \in V,$$

where α is a Lipschitz constant for $F'(\cdot)$ on V.

2. $$\max_{M \in \delta^2 F(x_0)} \langle Me, d \rangle = \limsup_{\substack{x \to x_0 \\ \varepsilon \to 0}} \ \varepsilon^{-1}[F'(x + \varepsilon e)d - F'(x)d] \ , \ \forall e, d \in X \ ,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product.

3. The second order Taylor expansion theorem is true:

$$\forall x_i \in V \quad (i = 1, 2), \quad \exists \xi \in [x_1, x_2], \quad \exists M \in \delta^2 F(\xi)$$

such that

$$F(x_2) - F(x_1) = F'(x_1)(x_2 - x_1) + 2^{-1}\langle M(x_2 - x_1), x_2 - x_1 \rangle.$$

Assume now that $f_t(\cdot)$ is of the class $C^{11}(V)$ and

(!) $f'_t(x)$ is jointly continuous in (t, x) and is locally Lipschitz in x uniformly with respect to t (that is, there is a positive constant α not depending on $t \in T$ such that $|f'_t(x_1) - f'_t(x_2)| \leq \alpha |x_1 - x_2|$, $\forall t \in T$, $\forall x_1, x_2 \in V$).

(!!) The function $s_t(x, d, d)$ is u.s.c. in x at x_0 uniformly with respect to t where

$$s_t(x, e, d) := \limsup_{\substack{u \to x \\ \varepsilon \to 0}} \ \varepsilon^{-1}[f'_t(u + \varepsilon e)d - f'_t(u)d]. \tag{5.7}$$

Observe that by (!) $s_t(x,e,d)$ is continuous as a function of t and the above mentioned property of the generalized Hessian matrix yields

(5.8) $$s_t(x,e,d) = \max_{M\in\delta^2 f_t(x)} \langle Me, d\rangle \ .$$

Also, (!!) is satisfied if either $s_t(x,d,d)$ is u.s.c. in (t,x) at every point (t,x_0), $t\in T$; or $\delta^2 f_t(\cdot)$ is u.s.c. at x_0 uniformly with respect to t.

Let us check (2.4) for

(5.9) $$f_t^2(x) := 2^{-1} \max_{M\in\delta^2 f_t(x_0)} \langle Mx, x\rangle \ \ (= 2^{-1}s_t(x_0,x,x)).$$

Condition (!) implies the existence of $\alpha > 0$ such that

$$|f_t'(x_0)| \le \alpha, \ \forall t\in T,$$

$$\|M\| \le \alpha, \ \forall M\in\delta^2 f_t(x), \ \forall x\in V, \ \forall t\in T.$$

This fact together with the second order Taylor expansion theorem gives

(5.10) $$f_t(x_\varepsilon) - f_t(x_0) = f_t'(x_0)[\varepsilon d + \varepsilon^2 x] + 2^{-1}\langle M\varepsilon d, \varepsilon d\rangle + f_t'(x_0)o(\varepsilon^2)$$

where M is a suitable element of $\delta^2 f_t(\xi_\varepsilon)$, $\xi_\varepsilon\in[x_0,x_\varepsilon]$.
It follows from (5.10)

(5.11)
$$f_t(x_\varepsilon) - f_t(x_0) \le f_t^1(\varepsilon d + \varepsilon^2 x) + f_t^2(\varepsilon d) + 2^{-1}\varepsilon^2[s_t(\xi_\varepsilon,d,d) - s_t(x_0,d,d)] + f_t'(x_0)o(\varepsilon^2).$$

On the other hand, by the continuity of $f_t'(x_0)$ as a function of $t\in T$

(5.12) $$\lim_{\varepsilon\to 0} f_t'(x_0)o(\varepsilon^2) = 0 \text{ uniformely with respect to } t$$

and by the condition (!!)

(5.13) $$\lim_{\varepsilon\to 0} \max \{0, s_t(\xi_\varepsilon, d, d) - s_t(x_0, d, d)\} = 0 \text{ uniformly with respect to } t$$

Combining (5.11), (5.12) and (5.13) proves (2.4).

Example 5.6. Assume that $f_t(\cdot)$ is of the class $C^{11}(V)$ and the condition (!) is satisfied. For $\eta > 0$, $e \in X$, $d \in X$ let us set

(5.14) $$\tilde{s}_t(\eta, e, d) = \sup_{\substack{|x - x_0| < \eta \\ |x + e - x_0| < \eta \\ 0 < \varepsilon < \eta}} \varepsilon^{-1}[f_t'(x + \lambda e)d - f_t'(x)d]$$

It is shown in [12] that for all $e \in X$, $d \in X$ we have

1. $\tilde{s}_t(\eta_1, e, d) \leq \tilde{s}_t(\eta_2, e, d)$ if $\eta_1 \leq \eta_2$.
2. $\tilde{s}_t(\eta, e, d) - \tilde{s}_t(\eta, \overline{e}, \overline{d}) \leq \alpha(|e - \overline{e}||d| + |d - \overline{d}||\overline{e}|)$, $\forall \overline{e}, \overline{d} \in X$.
3. $2^{-1}\tilde{s}_t(|d|, d, d) \geq f_t(x_0 + d) - f_t(x_0) - f_t'(x_0)d$.

We claim that (2.4) holds if

(5.15) $$f_t^2(x) = 2^{-1}\tilde{s}_t(\eta, x, x)$$

when η is an arbitrary positive constant. Indeed, setting $\overline{x}_\varepsilon = \varepsilon d + \varepsilon^2 x + o(\varepsilon^2)$ we see that $|\overline{x}_\varepsilon| \leq \eta$ for ε sufficiently small. For simplicity of notation we shall write $\tilde{s}_t(\eta, d)$ instead of $\tilde{s}_t(\eta, d, d)$. Using the above properties of $\tilde{s}_t(\eta, e, d)$ we obtain

$$\begin{aligned} f_t(x_\varepsilon) - f_t(x_0) &\leq f_t'(x_0)[\varepsilon d + \varepsilon^2 x] + f_t'(x_0)o(\varepsilon^2) + 2^{-1}\tilde{s}_t(|\overline{x}_\varepsilon|, x_\varepsilon), \\ \tilde{s}_t(|\overline{x}_\varepsilon|, \overline{x}_\varepsilon) &\leq \tilde{s}_t(\eta, \overline{x}_\varepsilon) = \tilde{s}_t(\eta, \varepsilon d) + [\tilde{s}_t(\eta, \overline{x}_\varepsilon) - \tilde{s}_t(\eta, \varepsilon d)] \\ &\leq \tilde{s}_t(\eta, \varepsilon d) + \alpha|\varepsilon^2 x + o(\varepsilon^2)|(|\overline{x}_\varepsilon| + |\varepsilon d|). \end{aligned}$$

From the just written inequalities it is clear that (2.4) is satisfied if we set

$$o_t(\varepsilon^2) = f_t'(x_0)o(\varepsilon^2) + \alpha|\varepsilon^2 x + o(\varepsilon^2)|(|\overline{x}_\varepsilon| + |\varepsilon d|).$$

Consider now examples of approximations of $f_t(\cdot)$ used in Section 3. We shall assume in Examples 5.7 - 5.9 that C is an arbitrary set of a normed space X. We take $H =$

cone $[C - x_0]$, $h(x) = x - x_0$, $f_t^1(x) = f_t'(x_0)x$ and we shall see that the requirements (1) - (3) of Section 3 are fulfilled by a suitable choice of $f_t^2(\cdot)$. All we have to check in these examples will be the inequality (3.3).

__Example 5.7__. Assume that $f_t(\cdot)$ is of the class C^2 and $f_t^2(\cdot)$ is given by (5.2). Then (3.3) follows directly from the twice continuity of $f_t(\cdot)$.

__Example 5.8__. Assume that $f_t(\cdot)$ is of the class $C^{11}(V)$ and

$$f_t^2(x) = 2^{-1} \min_{M \in \delta^2 f_t(x_0)} \langle Mx, x\rangle \; (= -2^{-1} s_t(x_0, -x, x)). \tag{5.16}$$

We invoke the second order Taylor expansion theorem formulated in Example 5.5 to derive

$$f_t(x_0 + x) - f_t(x_0) = f_t'(x_0)x + 2^{-1}\langle Mx, x\rangle$$

for some $M \in \delta^2 f_t(\xi)$, $\xi \in [x_0, x]$.
Taking account of (5.8) and (5.16) we can write

$$\begin{aligned} f_t(x_0 + x) - f_t(x_0) &\geq f_t^1(x) + f_t^2(x) + 2^{-1}[s_t(x_0, -x, x) - s_t(\xi, -x, x)] \\ &\geq f_t^1(x) + f_t^2(x) + \varphi_t(x) \end{aligned}$$

where

$$\varphi_t(x) = 2^{-1} \min \{0, s_t(x_0, -x, x) - s_t(\xi, -x, x)\}.$$

Using (5.8) and the upper semicontinuity of $\delta^2 f_t(\cdot)$ (see [10]) we verify without difficulty that

$$\lim_{x \to 0} \varphi_t(x)|x|^{-2} = 0.$$

This shows that (3.3) holds.

__Example 5.9__. Assume that $f_t(\cdot)$ is of the class $C^{11}(V)$ and, for fixed positive number η, (5.17) $$f_t^2(x) = -2^{-1}\tilde{s}_t(\eta, -x, x)$$

The desired inequality (3.3) is an immediate consequence of the following inequality (see [12, Proposition 5])

$$f_t(x) - f_t(x_0) \geq f_t'(x_0)x - 2^{-1}s_t(|x|, -x, x)$$

and the observation that

$$s_t(|x|, -x, x) \leq s_t(\eta, -x, x)$$

if $|x| \leq \eta$.

Remark 5.1. The definition of the functions (5.7) and (5.14) is given in [12] to study semi-infinite programms with equality constraints.

Let us compare shortly the results of Section 4 with those of Ioffe [12] under the assumption that equality constraints are absent. Since in that case $C = X$ we can take $D = H = X$. Setting $f_t^1(x) = f_t'(x_0)x$ and defining $f_t^2(\cdot)$ by (5.9) or (5.10) we see that Theorem 3 (hence Theorem 4) in [12] is a consequence of our Theorems 4.1 and 4.2. (It should be noted [12] that optimality criteria proved in [2] for SIP is weaker than [12, Theorem 4]). Applying Theorem 4.1 to the case where $f_t^2(\cdot)$ is defined by (5.15) yields necessary conditions different from those of [12, Theorem 1]. The sufficiency Theorem 2 in [12] is stronger than Theorem 4.2 applied to (5.17). The reader is referred to [6,9,13] for similar sufficient optimality conditions for SIP.

APPENDIX

Proof of Lemma 2.2. For simplicity we assume that $\tilde{T}$ has at least $n+1$ elements.

Let $x_0 \in \tilde{H}$ and $\tilde{H}_\rho := \{x \in \tilde{H} : |x - x_0| \leq \rho\}$ $(\rho > 0)$. If (2.9) has no solution then for all $\varepsilon > 0, \rho > 0$ the system

$$x \in \tilde{H}_\rho,\ \varphi(t, x) \leq -\varepsilon,\ \forall t \in \tilde{T}$$

has no solution. Since the sets

$$\{x \in \tilde{H}_\rho : \varphi(t,x) \le -\varepsilon\}$$

are convex and compact for all $t \in \tilde{T}$ and have empty intersection it follows from [22, Corollary 21.3.2] that there is a subfamily of $n+1$ sets having empty intersection, i.e., there is $(t_1, t_2, ..., t_{n+1}) \in \tilde{T}^{n+1}$ such that the system

$$x \in \tilde{H}_\rho,\ \varphi(t_i, x) \le -\varepsilon\ (i = 1, 2, ..., n+1)$$

has no solution. Hence the sets

$$F(\varepsilon,\rho) := \{(t_1, t_2, ..., t_{n+1}) \in \tilde{T}^{n+1} : \max_{i=1,2,...,n+1} \varphi(t_i, x) \ge -\varepsilon,\ \forall x \in \tilde{H}_\rho\}$$

are nonempty for all $\varepsilon > 0,\ \rho > 0$. This implies at the same time that any finite collection of the sets $F(\varepsilon, \rho)$ has nonempty intersection since

$$\bigcap_i F(\varepsilon_i, \rho_i) \supset F(\min \varepsilon_i, \max \rho_i).$$

The sets $F(\varepsilon, \rho)$ are being closed subsets of the compact set $\tilde{T}^{n+1}$, the collection of all $F(\varepsilon, \rho)$ with $\varepsilon > 0, \rho > 0$ has nonempty intersection. But from

$$(\bar{t}_1, \bar{t}_2, ..., \bar{t}_n) \in \bigcap_{\substack{\varepsilon > 0 \\ \rho > 0}} F(\varepsilon, \rho)$$

follows $\max_{i=1,2,...,n+1} \varphi(\bar{t}_i, x) \ge,\ \forall x \in \tilde{H}$.
Hence, setting $\tau := \{\bar{t}_1, \bar{t}_2, ..., \bar{t}_{n+1}\}$ we obtain the desired conclusion of Lemma 2.2.

Proof of Lemma 2.3. Let $1, 2, ..., n$ be the elements of $\tilde{T}$ ($n = |\tilde{T}|$). From the hypothesis it follows that $0 \in \mathbb{R}^n$ is not an interior point of the convex set

$$Q := \{\zeta \in \mathbb{R}^n : \zeta_i \ge \varphi(i, x),\ i = 1, 2, ..., n;\ x \in \tilde{H}\}.$$

Hence there is $\mu = (\mu_1, \mu_2, ..., \mu_n) \in \mathbb{R}^n,\ \mu \ne 0$, such that $\langle \mu, \zeta \rangle \ge 0,\ \forall \zeta \in Q$. From $Q + \mathbb{R}^n_+ \subset Q$ follows $\mu_i \ge 0,\ \forall i$ ($\mathbb{R}^n_+$ being the nonnegative orthant of $\mathbb{R}^n$). We normalize μ such that $\sum \mu_i = 1$ and choose as $\zeta \in Q$ the point $\zeta = (\varphi(1, x), \varphi(2, x), ..., \varphi(n, x))$ with $x \in \tilde{H}$ to obtain the conclusion of the lemma.

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