Integrated supply and demand management in operations

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Sandra Transchel
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Dekan:	Prof. Dr. Hans H. Bauer
Erstberichterstatter:	Prof. Dr. Stefan Minner
Zweitberichterstatter:	Prof. Dr. Christian Hofmann
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Für meine Eltern

Antje und Rainer Transchel

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List of Symbols and Abbreviations

$\bar{\Psi}$	Random variable
$\bar{P}$	Reservation price
$\bar{P}^k$	Reservation price of product k
$\bar{Q}_l$	Minimum order quantity if the discount rate is r_l (breakpoint quantity)
δ	Auxiliary variable
$\kappa(N)$	Menu costs associated with N price changes
$\lambda(t)$	Costate variable
μ	Mean of Ψ
μ_i	Lagrangian multiplier
Π	Average profit
Π^M	Average profit of marketing
Π_j^k	Profit of product k in time interval j
Ψ	Random market potential
σ	Standard deviation of Ψ
$\Theta(K)$	Expected shortfall
ε	Precision criterion
ε_P	Price elasticity
$\varepsilon_s(P, C)$	Lost sales elasticity given price P and capacity C
$\vec{P}$	Price vector
$\vec{T}$	Time vector
A	Lower bound of Ψ
a	Market potential
a^k	Market potential of product k
AC^O	Average cost function of operations
B	Upper bound of Ψ
b	Price-sensitivity of demand
b^k	Price-sensitivity of demand of product k
C	Capacity
c	Variable procurement cost
c^k	Variable procurement cost of product k

List of Symbols and Abbreviations

c_0	Regular purchasing price
c_H	Variable production cost of product H
c_L	Variable production cost of product L
c_l	Reduced unit procurement price if the discount rate is r_l
c_o	Overage cost
c_p	Variable production cost
c_u	Underage cost
$D(P)$	Demand rate depending on price P
$D(t)$	Demand rate at time t
$D^k(t)$	Demand rate of product k at time t
F	Fixed ordering cost
$F(z)$	Distribution function
$f(z)$	Density function
F^k	Fixed ordering cost of product k
$F_{D(P_H)}$	Distribution function of the random demand $D(P_H)$
H	Hamiltonian
h	Inventory holding cost per unit and unit of time
h^k	Inventory holding cost per unit and unit of time of product k
h_l	Inventory holding cost per unit and unit of time if the discount rate is r_l
K	Number of products
L	Lagrangian function
L_i	Hamilton-Lagrange function of retailer i
N	Number of price changes
P	Selling price
$P(t)$	Price at time t
$P^k(t)$	Selling price of product k at time t
P_H	Selling price charged to customer class H
P_L	Selling price charged to customer class L
Q	Lot-size
Q^k	Lot-size of product k
r	Discount rate
$r(z)$	Hazard or failure rate
r_l	Discount rate at stage l
S	Storage capacity
s^k	Unit storage requirement of product k
T	Cycle length
t_i	Point in time where price P_i is changed to P_{i+1}
T_j	Time interval between the replenishment of product j and $j+1$

List of Symbols and Abbreviations

$y(t)$	Inventory level at time t
$y^k(t)$	Inventory level of product k at time t
y_j^k	Inventory level of product k when product j is ordered
$y_{0_j}^k$	Initial inventory level of product k in subinterval j
$y_{T_j}^k$	Final inventory level of product k in subinterval j
CAB	Civil Aviation Board
CS	Customer segmentation
CV	Coefficient of variation
EOQ	Economic order quantity
GFR	Generalized failure rate
IPE	Increasing price elasticity
KKT	Karush-Kuhn-Tucker
NCS	Non-customer segmentation
NPV	Net present value
OM	Operations Management
RM	Revenue Management
SCM	Supply Chain Management
w.r.t.	with respect to
WSP	Warehouse Scheduling Problem

1 Introduction

1.1 Motivation

The past decades have witnessed that *revenue management* (RM) has not only revolutionized industries like airlines, hotels, or car rental agencies, but also in retailing and manufacturing, the application of RM methods offers promising potential to improve supply chain profitability. For example, the integration of innovative pricing strategies, e.g., price changes over time or price differentiation between different customer classes with *operations management* (OM) decisions such as inventory control, manufacturing, or capacity management, are more and more employed to improve the overall performance and to increase profits.

A main reason for RM activities, such as dynamic pricing strategies, being more frequently used in business-to-business (B2B) and business-to-consumer (B2C) markets is the advent of the Internet and electronic commerce. This novel information technology has simplified transactions and reduced transaction cost. For instance, in terms of dynamic pricing companies are able to change their prices at very low (or zero) changeover (menu) costs (Elmaghraby and Keskinocak, 2003).

Dell Computers, for example, introduced a *Direct Business Model* which completely bypasses the dealer channel and directly sells their products via Internet to the end-customers. This strategy allows Dell to have a closer relationship with its customers and thus more valuable information. Using this information, Dell has the ability to segment its customers based on several criteria so that they are able to sell exactly the same product at different prices depending on whether the order is made by a private customer, a small, medium, or large company, or another business. Moreover, Dell achieves a better performance in matching their available production capacity to demand (McWilliams, 2001).

Another example is Ford Motor Company. They implemented an innovative pricing strategy which distinguishes cars in terms of their respective demand popularity. They followed the idea "Why waste cash on vehicles that sell well without it?" and increased prices for low-margin vehicles such as Escorts and Aspires, which led to decreasing sales, and cut prices for high-margin vehicles such as Escapes, which led to increasing sales. This strategy brought Ford a higher utilization of their production capacities and increasing profits. Moreover, they stepped up market research to find features that "the customers were willing to pay for but the industry was slow to deliver" for example, more comfortable supercabs on trucks. They set up their sales units as businesses and told them which vehicles and option packages give Ford the highest profit. Ford was able to better coordinate the demand side (revenue) with the supply side (manufacturing cost), see Welch (2003) and Coy (2000).

A popular example for markdown pricing is the fashion industry. At the end of the selling season, it can often be observed that fashion retailers decrease their selling prices in order to clear their inventories and release space for the next selling season. However, it is not unlikely that this kind of discounts results in selling prices which are below procurement or production costs so that markdowns are apparently rather a revision of an initial decision than a profitable strategy. Zara, the flagship brand of the Spanish retail group *Inditex*, maintains a high brand profile in the retail clothing sector. Zara's strategy differs from other fashion retailers in the way that they optimally match their selling prices with the offered quantity and the market requirement in order to cut down discounts. Only about 18 % of Zara's clothing do not work with their customers and must be discounted. That is a half of the industry average of 35 % (Dutta, 2002).

These recent developments of integrating RM with OM activities and the resulting successes have indicated the tremendous potential to improve the supply chain performance. However, in retailing and manufacturing the integration of supply and demand-oriented activities is still at an early stage. Fleischmann et al. (2004) review linkages between marketing and operational decisions and discuss several drivers that call for a coordination of supply and demand-oriented activities. Especially, the interaction of inventory/capacity issues, such as when and how much to order, together with pricing issues, such as when to charge which price, is not yet well understood. This thesis investigates this problem with the target to give detailed insights into this interrelation.

Holding inventory can have various motives, the *transaction*, *safety*, and *speculation* motive (Silver et al., 1998). The transaction motive results from the fact that driven by economies of scale or technical restrictions a company is forced to order/produce in batches instead of ordering/producing continuously. The resulting inventory is called *cycle stock*. *Safety stocks* are the result of the safety motive which is induced by uncertainty in required data, e.g., demand or lead time. A third motive is the speculation motive. In this case firms build up inventory in expectation of changing prices, i.e., early replenishments in expectation of increasing purchase prices or postponement of sales in expectation of increasing selling prices. However, this motive is not considered in this thesis.

Two elementary models considering economies of scale and uncertainty are undoubtedly the *economic order quantity* (EOQ) model and the *newsvendor model*, respectively. The EOQ model is concerned with answering the question of how much and equivalently how frequently inventory should be replenished taking economies of scale into account and given that the demand rate is accurately known. The objective is to identify the size of the order that minimizes the average costs consisting of the sum of fixed ordering costs (any fees associated with placing orders, such as delivery charges) and inventory holding costs (costs of storage as well as cost of capital) per period. It can be argued that the real world is seldom as well-behaved as deterministic models and real-world situations are more realistically described by stochastic models. However, deterministic models are often excellent approximations and represent a good starting-point to describe inventory phenomena.

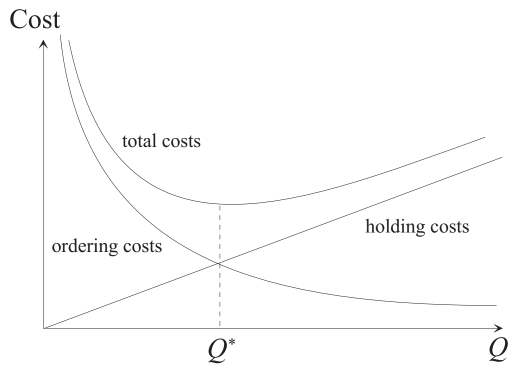

Figure 1.1: EOQ cost functions

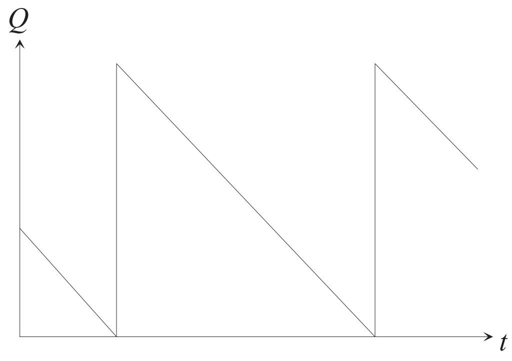

Figure 1.2: EOQ inventory cycle

Figure 1.1 illustrates the impact of order size Q on ordering and holding costs. While ordering costs per unit decrease exponentially with the size of orders, holding costs increase with an increasing order quantity. The optimal order size Q^* is equal to the value where the ordering cost function intersects the function of holding costs. The presence of fixed costs associated with ordering or production encourages firms to exploit economies of scale by ordering or producing larger lots resulting in inventories. The behavior of the inventory level applying an EOQ model is illustrated in Figure 1.2.

Two fundamental assumptions of the traditional EOQ problem are a fixed and exogenously given selling price and a constant and known demand rate. These assumptions yield that the revenue that is generated by each customer demanding a product is identical, but the costs caused by ordering and storing products are different. Assume that the unit selling price is P, procurement cost is c per unit, and inventory holding cost is h per unit and time unit. The order of size Q takes place at $t_0 = 0$. Thus, independent from the date of sales all units generate an identical revenue P. However, since the period of time where the units are kept in stock differs across the units, the profit margin is different across the units that are sold at different points in time.

A unit that is sold at time t_1 generates a profit margin of $(P-c-ht_1)$. However, a second unit that is sold at time t_2 with $t_1 < t_2$ generates a profit margin of

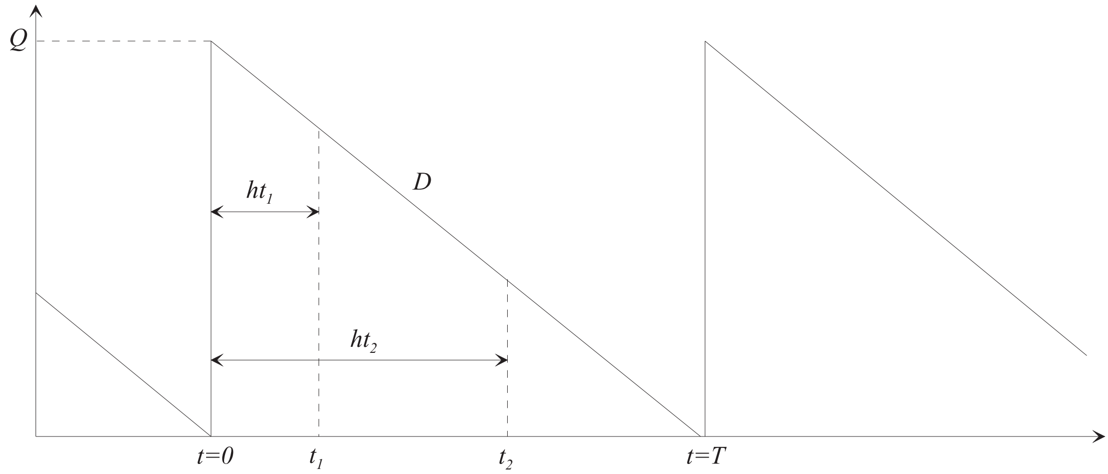

Figure 1.3: EOQ replenishment

$(P - c - ht_2)$ that is lower than the unit sold at t_1. From an economic point of view, firms would prefer customers buying early in an order cycle to those customers arriving late because the longer units are kept in stock the higher are the inventory holding costs. Thus, a firm has an incentive to stimulate customers to buy early in order to reduce stocks.

One method to control incoming demands is the application of *dynamic pricing* strategies. Dynamic pricing is as old as commerce itself. In most retail and industrial trades, firms use various forms of dynamic pricing, e.g., markdowns, sales promotions, coupons, and auctions in order to respond to dynamically changing market environments. The area by far the most mature in dynamic pricing is RM which is concerned with pricing a perishable resource to maximize revenue. Prices are adjusted dynamically as a function of inventory/capacity level and time left in the selling season. However, it is assumed that the initial inventory or capacity level is exogenous and cannot be replenished. Industries where RM is typically applied are airline, hotel, and car rental industries. However, the interrelation between dynamic pricing strategies and inventory management, for the case that replenishment is allowed, is not yet well understood.

The elementary model which is concerned with the question of how much capacity has to be acquired in the presence of demand uncertainty is the classical *newsvendor problem* (Silver et al., 1998). Capacity, in this context, is widely defined and can be, for instance, inventory, production, workforce, or transportation capacity. This problem is traced back to the problem faced by the owner of a newsstand who has to decide how many newspapers to stock in the morning before observing demand. If the newsvendor orders too much, excess inventory has to be scrapped or, if possible, can be sold at a salvage value. However, the unit salvage value is likely to be below the unit cost of acquisition so that in both cases the newsvendor faces a loss per unit. On the other hand, if he orders too little, the newsvendor faces lost sales for each unmet demand. Costs associated with excess demand may simply be the lost profit. However, additional costs may arise, for instance, for added costs acquiring the product from a second channel or goodwill costs. That is, both cases ordering too much and ordering too little are subject to costs which are called *overage costs* and *underage costs*, respectively. The objective of the newsvendor is to determine the capacity which optimally matches with demand, in particular, the capacity level that specifies the best trade-off between overage costs and underage costs.

Two fundamental assumptions of the classical newsvendor problem are a given demand distribution and an exogenous selling price. These assumptions give that the demand, which is uncertain, is exogenous and cannot be influenced. Therefore, the primary goal is to minimize the quantity gap between supply and demand. However, the mismatch of supply and demand is not just caused by quantity gaps. A "wrong" price, resulting in a too high or too low demand yields lower profit margins. An extension that incorporates pricing issues into the classic newsvendor problem is called the *price-setting newsvendor problem* where demand is given by a price-response function. The price-setting newsvendor problem is concerned with optimally matching supply with demand by simultaneous optimization of price and capacity.

Another cause of the supply-demand mismatch is when customers are characterized by a different willingness to pay, but the firm is unable to segment the customers. In this case, the firm is unable to charge the maximum amount that a customer is willing to pay so that the firm loses revenue. In traditional RM, there exist methods, such as product differentiation and price discrimination which are seeking to match a fixed supply with demand by discriminate prices between different customer classes. It is obvious that both the classical newsvendor problem and price discrimination within the RM context where capacity is assumed to be given are only suboptimal compared to a decision-making framework which simultaneously optimizes price strategy and capacity planning.

This thesis contributes to the emerging field of literature investigating the interaction between revenue management methods and operations management activities. In particular, it centers around the analysis of two general topics:

(A) interaction of dynamic pricing and inventory management,

(B) interaction of price discrimination between different customer classes and capacity planning in the presence of demand uncertainty.

1.2 Research questions

The central research question that is addressed in this thesis is how firms can benefit from coordinating demand and supply management activities and how these activities interact in a coordinated decision-making. The special focus in terms of demand management activities is on dynamic pricing strategies and customer segmentation while the special focus in terms of supply management is on inventory management and capacity planning under demand uncertainty, respectively.

The first major research question that is addressed is how firms can benefit from coordinating dynamic pricing strategies and inventory replenishment. In particular, what is the gain from a coordinated decision-making where price strategy and replenishment policy are optimized simultaneously compared to decentralized decision-making where marketing optimizes prices and operations optimizes replenishment policies. Furthermore, this thesis clarifies what the gain from applying dynamic pricing compared to a constant pricing strategy is. It is already known from literature that dynamic pricing strategies may increase the company performance. However, this thesis answers the question how dynamic pricing influences replenishment policies, particularly, order quantities and order frequencies. The specific research questions that are dealt with are:

1. What is the benefit of simultaneous decision-making of price strategy and replenishment policy compared to decentralized decision-making where marketing decides on the pricing strategy and operations optimizes the replenishment policy?
2. How do dynamically changing selling prices and the replenishment strategy interact?
3. What is the benefit of dynamic pricing compared to a constant pricing strategy?

These research questions are investigated under several business environments. Starting with a simple single-product monopoly setting, the analysis is extended to problems incorporating a supplier quantity discount, multiple products that share a warehouse with limited storage capacity, and a competitive environment.

The second major research question that is tackled in this thesis is how firms can benefit from coordinating customer segmentation either by price discrimination or product differentiation and capacity acquisition in the presence of demand uncertainty. In particular, it is clarified what the gain of coordinated decision-making of price discrimination and capacity acquisition is compared to (i) the case where the firm is not able to discriminate prices and (ii) the case of a decentralized decision-making framework where independent sales managers are responsible for price management and capacity acquisition.

The specific research questions that are addressed concerning the second topic are:

1. How does customer segmentation, in particular price discrimination influence price decision and capacity acquisition in an integrated planning approach?
2. What is the impact of capacity costs and demand uncertainty on price and capacity decisions?
3. What is the benefit of price discrimination compared to a single-pricing strategy?
4. What is the benefit of centralized decision-making compared to decentralized decisions where two individual product managers decide separately on price and capacity?

1.3 Structure and overview

The thesis is divided into 5 chapters. Chapter 2 provides a literature overview on the topics closely related to this thesis. Since there is huge body of literature on integrated RM and OM problems, Section 2.2.1 first presents comprehensive surveys that classify contributions of various research streams in this field. Sections 2.2.2 and 2.2.3 give a detailed overview of papers that are related to the problems analyzed in Chapter 3 and 4, respectively. Section 2.2.2 focuses on dynamic pricing problems in inventory and production, in particular, on problems where fixed ordering or setup costs play a significant role. Moreover, this section provides specific overviews on pricing and inventory control literature considering a supplier quantity discount, multiple products, and competition. Section 2.2.3 provides a survey of stochastic models that integrate pricing and capacity decisions.

Chapter 3 studies the benefit of dynamically changing prices to achieve operational efficiency in the EOQ model. Following an introduction of preliminaries and notation in Section 3.2, Section 3.3 is based on Transchel and Minner (2008) and analyzes an integrated pricing and EOQ replenishment problem of a monopolistic retailer who is allowed to change the selling price over time. First, a dynamic pricing model with continuous price changes is developed and analyzed. Since in practice there is, typically, a limit on the number of times where the price can be adjusted, the model is generalized to that the number of price changes over an order cycle are optimized. Besides, providing further evidence for the benefits of dynamic pricing, this part especially points out the impact of dynamic pricing on operational decisions such as order quantity and order frequency. It is established that the trade-off between fixed ordering and inventory holding costs yields that the optimal selling price increases over an order cycle and triggers higher demand rates when inventories are high. Furthermore, it is shown that the optimal cycle length and the order quantity are increasing in the number of price variations. Thus, the order frequency of the retailer is lower if more price adjustments are allowed over an order cycle. For the case of a linear price-response function, analytical solutions for the optimal prices, the optimal times where the price is adjusted, and the optimal cycle length are found. Moreover, it can be shown that the time intervals where a particular price is charged are equally. In the case of an exponential price-response, the length of the time intervals where a particular selling price is charged is increasing over the order cycle. In this case, if the price is low, the demand rate responds more sensitively to price variations than at higher prices. Therefore, the price jumps are lower at the beginning of the order cycle than at the end. Since for exponential price-response functions no closed-form solutions can be derived, an algorithm is developed to determine the optimal solution numerically. It is widely accepted that the *net present value* (NPV) approach is the right framework for valuing inventories. However, average cost or average profit approaches are simpler and thus more widely used. For the classical EOQ model that minimizes average costs, it can be verified that the average cost approach gives an approximately optimal solution for the NPV approach. This section shows that also in case of profit maximization for both constant and dynamic pricing, the outcomes of maximizing the average profit are approximately optimal compared to maximizing the discounted cash-flows, i.e., maximizing the NPV.

Section 3.4 is based on Transchel and Minner (2008a) and extends the model of Section 3.3 to the case where the supplier offers an all-units quantity discount. The benefits of coordinated decision-making and dynamic pricing are investigated compared to a decentralized decision framework. Three decision frameworks are distinguished: a decentralized decision-making strategy where price and replenishment decisions are made independently, a coordinated strategy with a constant selling price where a central decision maker decides simultaneously on a single price and the replenishment policy, and a coordinated strategy where the retailer is allowed to implement a finite number of price adjustments within each order cycle. While various findings state that decentralized decision-making leads to an underestimation of selling prices, in case that the supplier offers a quantity discount it is shown that two different effects influence the outcome of decision-making: the *overhead cost* and the *discount* effect. The overhead cost effect results from the fact that a centralized decision maker takes all relevant costs (fixed ordering cost and inventory holding cost) into account. This yields a larger selling price, a lower demand rate, and a lower lot-size compared to decentralized decision making where the price is set disregarding fixed ordering and inventory holding costs. The discount effect, on the other hand, results from the fact that a decentralized decision maker who decides on prices does not take a supplier quantity discount into account. Therefore, costs are overestimated which, in turn, yields a selling price that is too high. Depending on which effect dominates, selling price, demand rate, and order quantity increase or decrease.

In Section 3.5 that is based on Transchel and Minner (2006), the interaction between pricing and procurement decisions of multiple products that share a common warehouse with limited storage capacity is analyzed. Based on the traditional warehouse scheduling problem (WSP) that considers a coordinated planning of order sizes and delivery scheduling of multiple products with limited storage capacity, the interaction between dynamic pricing and replenishment decisions is analyzed. A common cycle approach is assumed that considers that all products have an order cycle with an identical length. A two-stage optimization model that integrates pricing and replenishment considerations is developed. At the second stage the order cycle is given and the objective is to determine the optimal price trajectories for each product. At the first stage, the replenishment policy for each product is optimized anticipating the optimal price strategy from the second stage and taking the limited storage capacity into account. A comparison between decentralized decision-making where marketing and operations optimize selling prices and replenishment policies independently and coordinated decision-making where constant and dynamic pricing are distinguished is made. The results indicate that by simultaneous decision-making a firm can reduce order quantities and hence required minimum storage capacity. Based on the results of Section 3.3.1 where in a single-product problem the selling price increases continuously over an order cycle, it is shown that in case of multiple products with limited storage capacity, the selling price does not increase continuously. At the point in time where one product is ordered, the price for this product decreases instantaneously. At the same time, the price for the other products instantaneously increases. This opposed pricing effect yields a better matching of demand with available inventory and an optimal utilization of the limited capacity.

In Section 3.6, based on Transchel and Minner (2008b), two retailers are competing with each other for the same potential buyers. The retailers are allowed to change their sales quantity dynamically over time, however, the retailers differ in their respective replenishment cost. Retailer 1 orders in batches (EOQ policy) whereas retailer 2 follows a just-in-time (JIT) strategy. The primary goal of this study is to analyze the optimal replenishment policy and the equilibrium output strategy. A differential game where both retailers repeatedly interact over the order cycle is developed and an open-loop Nash equilibrium is derived. It is shown that both retailers follow contrary output strategies. While retailer 1 decreases his output over an order cycle, retailer 2 increases his output. However, the diminishing rate of retailer 1 is larger than the enhancing rate of retailer 2 such that the total output decreases over the order cycle. Moreover, a numerical example indicates that if EOQ and JIT replenishment result in identical average profits in a monopoly, in a competitive environment where one firm follows an

EOQ and the other firm follows an JIT policy, EOQ replenishment leads to a higher average profit than JIT replenishment.

Chapter 4 investigates a joint pricing and capacity planning problem in the presence of demand uncertainty. Section 4.2 introduces the well-known *price-setting newsvendor problem* that studies a single-product, single-period capacity and price decision problem. Structural properties concerning the existence and uniqueness of the optimal solution are derived. Moreover, an algorithm is presented that determines the optimal price and capacity for a special class of additive demand functions.

Section 4.3 is based on Transchel et al. (2007) and extends the price-setting model of Section 4.2 to the case where market demand can be segmented into two customer classes differing in their willingness to pay. While Section 4.2 basically reviews existing results, Section 4.3 provides novel insight. The major contribution is to investigate the impact of customer segmentation on price strategy and capacity decision. Customer segmentation can be achieved by either price discrimination or product differentiation. If price discrimination is not possible because of arbitrage and cannibalization, the firm can use product differentiation, e.g., by different brands or different quality levels. In particular, the interaction of the prices that are charged from each customer class and the initial capacity investment is studied.

A stochastic model is developed which simultaneously optimizes selling prices and capacity acquisition. Structural properties are derived which show that under certain circumstances a unique optimal solution exists. Based on these results, an algorithm is developed, which determines the optimal solution efficiently. In order to examine the interaction of price decision and capacity planning, we analyze the decision problem for different capacity costs and different levels of demand uncertainty. It is shown that an integrated decision-making where price decisions are jointly made with the capacity decision provides a firm a higher flexibility to match supply with demand. This flexibility is characterized by the fact that the firm is able to adjust both capacity and selling prices simultaneously. Hence, prices can be adjusted differently between customer classes by taking their respective price-sensitivity and other demand characteristics into account.

Furthermore, we illustrate the benefit of customer segmentation and price discrimination and its impact on price and capacity decision compared to a single-pricing strategy. It is shown that the higher flexibility achieved by customer segmentation yields a risk reduction in terms of underage and overage costs, which, in turn, leads to increasing profit. Last but not least, we will show the benefit of centralized compared to decentralized decision-making. It is common in practice that different customer classes are served by different sales managers where each sales manager decides on the selling price and the required capacity reservation with the objective to maximize his individual profit.

Chapter 5 concludes this thesis. It summarizes the major findings and discusses implications and extensions for future research.

2 Fundamentals and literature review

The goal of this section is to integrate this thesis with the wide range of academic literature. It provides fundamentals and reviews related literature.

2.1 Fundamentals

Both in academia and practice, the following question often arises: What is *Supply Chain Management*? If you ask five different people, you will probably get five different answers about what the scope of Supply Chain Management (SCM) is.

Firms within a supply chain are linked through physical, information, and financial flows. Product flows include the movement of goods from the raw material supplier to the end-customer as well as any customer returns or service needs. Information flows involve sharing forecasts, transmitting orders and updating the status of delivery, and financial flows include credit terms, payment schedules, and consignment and ownership arrangements. Hence, the fundamental goal of SCM is the integration and coordination of all supply management and demand management activities, i.e., matching supply with demand (Cachon and Terwiesch, 2006). A definition of Chopra and Meindl (2007) points out that all intra-firm activities are included in SCM.

Supply Chain Management integrates supply and demand management within and across companies from raw material supplier to end customer.

Operations Management (OM) and *Revenue Management* (RM) are two organizational functions within a firm which are concerned with supply and demand management activities. While OM addresses supply and operations decisions and processes with the objective of lowering costs of production and delivery, RM includes market segmentation and price differentiation in order to manage the firm's interface with the market with the objective of increasing revenue. In this thesis, we propose that SCM integrates the two streams OM and RM, as illustrated in Figure 2.1, as two functions that operate supply and demand-oriented, respectively.

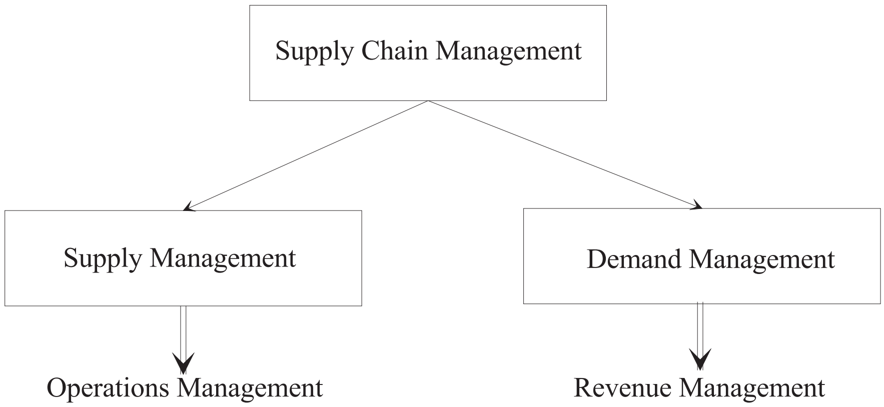

Figure 2.1: Supply chain management framework

Operations Management is the business function which is concerned with supply management activities (procurement and production) of goods and services and involves the responsibility of ensuring that the transformation process is efficient and effective. That is, OM includes the management of resources as well as the distribution of goods and services to customers. In particular, OM is concerned with questions of how supply processes can be optimally organized, structured, and managed to make them more efficient meeting demands (Reid and Sanders, 2007). Planning problems range from strategic to tactical and operational levels. Representative strategic issues include determining the size and location of manufacturing plants or service facility, deciding on the structure of service networks, and designing technologies. Tactical issues include plant layout and structure as well as equipment selection and replacement. Additionally, operational issues include, for instance, inventory replenishment, production scheduling and control, and materials handling.

Revenue Management includes mainly demand management activities in understanding, anticipating, and influencing consumer behavior. The most common RM definition is that RM "is a method which can help a firm to sell the right inventory unit to the right type of customer, at the right time, and for the right price" (Kimes, 1989). In particular, RM deals with modeling and optimization of pricing strategies and traditional issues of capacity allocation for a fixed capacity with the objective to maximize the overall revenue. RM began with the airline deregulation act in 1978. With this act, the US Civil Aviation Board (CAB) lost the control of airline prices which were strictly regulated and based on standardized prices and profitability targets. From this point in time onwards, established carriers have been free to set their prices individually without CAB approval. One of the greatest success stories is the American Airlines RM system "DINAMO". By carefully controlling the availability of various fare-products on their network via DINAMO, American Airlines estimates that they added 1.4 billion dollars to their bottom line for the period from 1989-92 (Talluri and van Ryzin, 2004). This was the beginning of intensive developments of RM techniques.

Two RM methods that are particularly focused on in this thesis are *dynamic pricing* and *customer segmentation*. *Dynamic pricing* can formally be defined as the ability to change selling prices over time (inter-temporal price changes) in response to changing supply and demand characteristics such as variations in inventory levels, production capacities, and demand.

Customer segmentation is another important characteristic inherent in effective RM meaning the ability to segment the market into customer classes and to charge a different price from each customer class (price discrimination). A common mechanism to segment the market is differentiating the customers according to their willingness to pay. However, customer segmentation can be difficult because arbitrary price discrimination is not realizable. It is necessary that one or more attributes used to segment the customers truly differentiate the products or services (Weatherford and Bodily, 1992). It has to be avoided that customers resell products after purchasing, otherwise arbitrage is possible.

Price discrimination can be classified into three types:

- In *first-degree price discrimination*, the company is able to separate the whole market into each individual customer and charges them the price he is willing to pay. This type is also called *perfect price discrimination*.
- In *second-degree price discrimination*, the company offers a price menu to the customers and depending on their preferences, they get an incentive for self-selection. For example, a firm introduces different prices for different quantities (quantity discounts).
- The *third-degree price discrimination* is the form of price discrimination most frequently found and involves charging different prices for the same product in different segments of the market.

In the real world, however, first-degree price discrimination is rather impossible to achieve unless the firm knows every consumer's preferences. This knowledge, however, would be associated with high transactions costs involved in finding out through market research what each customer is willing to pay. More common are the second and third-degree price discrimination. The latter is basically applied in traditional RM.

Figures 2.2 and 2.3 represent for a simple linear price-response function the maximum revenue obtained by selling at a single price and by selling at multiple prices. The striped area in both figures represents the revenue that is obtained. Obviously, the revenue obtained by applying price discrimination is significantly greater than the revenue obtained at a single price. Intuitively, the more different prices are charged, the greater the revenue that is generated and as the number of prices tends to infinity, first-degree price discrimination is achieved.

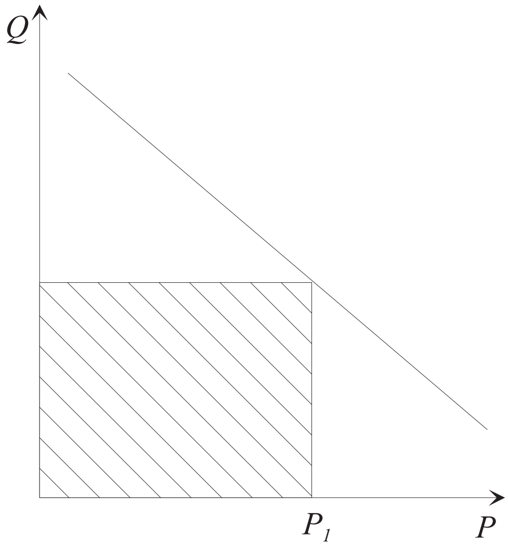

Figure 2.2: Revenue by selling a product at a single price, Talluri and van Ryzin (2004)

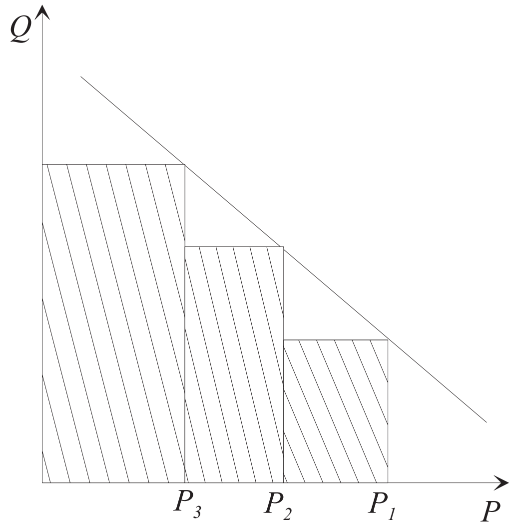

Figure 2.3: Revenue by selling a product at multiple prices, Talluri and van Ryzin (2004)

Several conditions are required to implement price discrimination. First, price discrimination requires market segmentation regarding customers' willingness to pay. It should be impossible to resell the product after purchasing, otherwise arbitrage is possible. Additionally, firms should have some degree of monopoly power to sustain a structure of market segmentation because in case of perfect competition, firms have no power to vary prices. Nevertheless, monopoly power need not be absolute when products are differentiated in some way or sold in dispersed markets. By this strategy called *product differentiation*, it is possible to maintain price discrimination even with limited market power (Talluri and van Ryzin, 2004).

Differentiated products are those which are in the same product group, yet are not identical. Product differentiation can be observed in practice either by *vertical product differentiation* or *horizontal product differentiation*. Products are vertically differentiated if all consumers agree on which product is better, if their prices are identical, e.g., consumers prefer the product with the higher quality level. On the other hand, products are said to be horizontally differentiated if consumers are different in their taste and product differentiation is based on appearance, e.g., color of the product. Durable goods manufacturer often design product lines by segmenting their market on quality attributes that exhibit a "more is better" property for all customers. By differentiating products, a firm is able to decrease the substitutability of their products and customize offers to the requirements of customers or market segments (Choi et al., 1997).

Typically, OM and RM are independent business functions. However, since de-

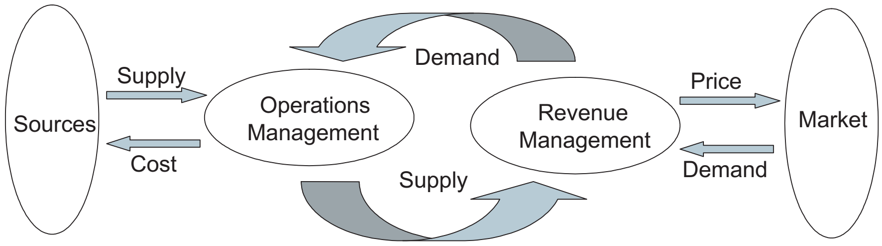

Figure 2.4: Interaction of OM and RM

mand management activities generate demands and OM is responsible to fulfill these demands in order to maximize the overall company profit, supply and demand activities have to be coordinated. Figure 2.4 illustrates this interrelation.

2.2 Literature review

The coordination of revenue management and inventory and manufacturing decisions has received extensive treatment in the literature for more than 50 years. One of the earliest contributions integrating price and inventory/manufacturing decisions is by Whitin (1955) who provides extensions of two fundamental inventory models. Whitin (1955) investigates, in both the deterministic EOQ and the stochastic "newsvendor" model, the benefits of simultaneous decision-making of price and order quantity. Since that time, many researchers have extended, modified, or more deeply analyzed these problems in order to achieve detailed insights into the interaction of pricing and operations strategies. This literature overview does not cover all streams of research in this field but focuses exclusively on closely-related contributions concerning the investigations in Chapter 3 and Chapter 4.

Section 2.2.1 presents comprehensive surveys that classify contributions of various research streams in this field. Section 2.2.2 focuses on contributions that analyze dynamic pricing problems in inventory and production, in particular, continuous time problems and problems where fixed ordering or setup costs play a significant role. Section 2.2.3 provides a survey of stochastic models that integrate pricing and capacity decisions.

2.2.1 Comprehensive overview

A thorough review of literature on integrated pricing and production models until the early nineties appears in Eliashberg and Steinberg (1993). Eliashberg and Steinberg (1993) focus on contributions that compare decentralized and coordinated decision-making of marketing and operations decisions. The papers that are reviewed in their survey explicitly outline the benefits of a coordinated system. A special issue of the *International Journal of Production Economics*, Vol. 37, No. 1, from 1994 deals with the coordination of sales and manufacturing decision. This special issue includes papers that describe specific company approaches of manufacturing-sales coordination, works that describe general requirements for an effective manufacturing sales coordination as well as works of specific coordination aspects.

Bitran and Caldentey (2003) provide a review of research results of dynamic pricing strategies and its relation to revenue management. Their survey is based on traditional RM problems in which a perishable and non-renewable set of resources is used to satisfy stochastic and price-sensitive demand.

Elmaghraby and Keskinocak (2003) provide a comprehensive overview of dynamic pricing strategies and its interaction with inventory management. They constitute a review of current practices in dynamic pricing. In order to better understand the state-of-the-art of dynamic pricing practices, they conduct interviews with directors of marketing and operations. The existing literature is classified according to three main market environments: replenishment vs. non-replenishment of inventory (R vs. N), dependent vs. independent demand over time (D vs. I), and myopic vs. strategic customers (M vs. S). Elmaghraby and Keskinocak (2003) identify that the majority of the literature can be assigned to two major groups: NR-I (this includes NR-I-M and NR-I-S) and R-I-M. Typical NR-I products are those with a short selling season and a long lead time, e.g., fashion apparel. R-I-M markets, on the other hand, mostly belong to the category of nondurable products such as consumer-packaged goods and fresh produce.

Compared to the two previous surveys, Chan et al. (2004) are the first who consider topics like demand learning or pricing and inventory control across multiple products. They classify papers basically according to the length of planning horizon underlying the reviewed models. Their survey is split up in (1) models to explain price realizations, (2) general pricing and production models assuming either variable production costs or fixed production set-up costs, (3) models dealing with markdown pricing and promotions, and (4) fixed pricing models. Moreover, Chan et al. (2004) distinguish papers according to their pricing mechanisms and production cost functions and provide an overview of multi-period models classified according to various assumptions, e.g., demand type (deterministic vs. stochastic and linear vs. exponential), dealing with excess demand (lost sales vs. backlogging), replenishment option, capacity limits, and number of products.

Yano and Gilbert (2004) basically focus their review on integrated pricing and production/procurement problems. They review research that involves both supply chain coordination and price competition. The authors identify models that deviate in their perception of operational costs. Three groups are identified: (1)

models with setup cost emphasis and time-invariant demand, (2) models with demand/production smoothing concerns, and (3) models dominated by demand uncertainty. To be more explicit, the cost trade-offs balance inventory holding costs vs. (1) fixed ordering or setup cost, (2) cost of varying the rate of production, and (3) cost of producing/ordering too much or too little when demand is uncertain. This classification shows that besides operational considerations the characteristics of demand play an important role.

Finally, Talluri and van Ryzin (2004) provide a detailed overview of the development of RM applications in theory and practice. Their book comprehensively covers theory and practice of the entire field. They distinguish RM between quantity-based RM that is concerned with questions like whether to accept or reject an offer to buy and price-based RM that is concerned with questions like how to price over time or across products.

2.2.2 Dynamic pricing in inventory and production

This section provides an overview on contributions that analyze the impact of dynamic pricing strategies in inventory and production with a special focus on continuous time models and models where fixed ordering or setup costs play a significant role. It is closely related to Chapter 3.

Pricing and replenishment in a monopoly environment

Section 3.3 considers a continuous time inventory replenishment and pricing problem. Models that consider fixed ordering costs and time-invariant demand are typically based on a discrete time framework. Thomas (1970) analyzes an integrated pricing and production planning decision model. He provides an analysis and a solution algorithm for a model characterized by a set of discrete time periods where the demand in each period is a downward-sloped function of the price. The model proposed by Thomas (1970) sets only a single price in any given period. Kunreuther and Schrage (1973) develop an algorithm for determining the optimal pricing and ordering decisions for a lot-sizing problem in a multi-period environment where demand differs from period to period. They provide bounds on the optimal solution under time-varying production cost assumptions.

A variety of models are using continuous time-optimal control. However, these models typically assume no fixed ordering costs, e.g., see Pekelman (1974), Feichtinger and Hartl (1985), Eliashberg and Steinberg (1987), Gaimon (1988), Jørgensen and Kort (2002), and Kogan and Spiegel (2006). Models that incorporate fixed ordering costs in a continuous time model are rather rare. Rajan et al. (1992) derive simultaneous pricing and ordering policies for a retailer under standard EOQ assumptions. They consider a problem of a monopolistic retailer who faces a known demand rate for a product exhibiting physical decay and value loss, and continuous pricing is applied throughout the season. Several parameters are varied to investigate how the optimal profit and the optimal cycle length change. They prove that the optimal cycle length is decreasing as the market potential increases for both linear and exponential demand. From their observations, they conclude that the optimal profit increases with the reservation price and decreases as procurement cost and inventory holding cost increase. Abad (1996) formulates a generalized model of dynamic pricing and lot-sizing for a reseller who sells perishable goods. He presents a simple solution procedure for solving the optimization problem. Abad (1997) provides a model that determines the optimal reseller response to a temporary price reduction by the supplier.

It can be easily verified that under certain conditions for the traditional EOQ problem the results of the *average cost* (AC) approach are approximately optimal to the *net present value* (NPV) approach which maximizes discounted cash-flows. However, several authors have shown that for more complex frameworks the results based on AC and NPV approach may differ. Beranek (1967) investigates how different payment arrangements with customers and suppliers affect the cash-flows and thus the value of the firm. Trippi and Lewin (1974) provide an intuitive derivation of the classical EOQ problem based on a present value approach. They show that the optimal order quantity from the present value point of view is always less than the optimal order quantity derived from the average cost approach. They also show that the present value function is even less sensitive to parameter errors than the average cost function. Hofmann (1998) compares a cash-flow-oriented and a cost-oriented formulation of an inventory problem that analyzes investments in setup and production processes. He investigates the effects of reduced setup costs between a cash-flow-oriented and a cost-oriented approach. The analysis shows that there are only minor differences between the cash-flow and the cost approach. Van der Laan and Teunter (2002) show that for two-source inventory problems there can be considerable gaps between AC and NPV approaches.

Pricing and replenishment with a supplier quantity discount

Section 3.4 investigates the impact of a supplier quantity discount on pricing and replenishment strategy. Benton and Park (1996) and Munson and Rosenblatt (1998) provide literature reviews on quantity discounts where they classify the literature w.r.t. several discount schemes, different perspectives (buyer, supplier, joint), and other criteria like planning horizon and number of products. The most common discount policies in the literature are the *all-units* and the *incremental-units* discount policy. When the supplier offers an all-units quantity discount (AQD), the reduced purchasing price applies to the entire order quantity once the order quantity reaches a critical breakpoint. An incremental-units discount, however, only applies to all units in excess of a particular breakpoint. Hadley and Whitin (1963) develop a procedure for determining the optimal economic order quantity for both all-units and incremental-units discount schemes. This approach is included in almost every textbook on inventory management and assumes that the demand rate is known and constant over an infinite planning horizon and that the decision maker follows the objective to minimize average costs. Gupta (1988) provides an improved procedure for determining the optimal lot-size by considering an upper bound for the relevant cost and Goyal and Gupta (1990) propose a further simplification which requires only a few EOQ calculations for determining the optimal lot-size.

In Abad (1988a) and Abad (1988b), simultaneous optimization of lot-size and selling price when the supplier offers an all-units or an incremental-units discount is analyzed. He develops a procedure for determining the optimal lot-size and the optimal selling price. Burewell et al. (1991) extend the model of Abad (1988a) and allow for planned inventory shortages. They derive a similar procedure as Abad (1988a) to determine the optimal lot-size and selling price for two classes of demand functions, iso-elastic and linear. More recently, discount pricing schedules have received growing attention to improve the coordination between vendors and buyers, e.g., see Weng (1995), Rubin and Benton (2003), and Wang (2005). Inventory problems with simultaneous optimization of the replenishment policy and a dynamic pricing strategy where only a limited number of price changes is allowed has been investigated less frequently. The main motivation for considering a limited number of price changes are organizational costs associated with each price change. Abad (1997) formulates a model where the reseller responds to a temporary price reduction of the supplier by an adjustment of the own selling price. Abad (1997) considers that the reseller is allowed to charge two selling prices in each order cycle. The presented model optimizes the first (discounted) selling price and fixes the second selling price to the optimal constant price. He presents a procedure where a selling price temporarily reduced yields a higher cycle profit than in the case of a static selling price.

Pricing and replenishment in a capacitated multi-product environment

Section 3.5 analyzes the interplay between pricing and procurement decisions of multiple products that share a common warehouse with limited storage capacity. Multi-product replenishment together with the issue of sharing a common resource has been widely addressed in the literature. Under the assumption that the selling price is exogenous and the demand rates of the products are constant over time, the problem of when and how much should be procured is known as the *warehouse scheduling problem* (WSP). Several approaches to this problem, which is an extension of the traditional EOQ problem, have been suggested. One approach suggested by Hadley and Whitin (1963) and Johnson and Montgomery (1974) is to model this problem as an aggregate of multiple single-product economic order quantity (EOQ) problems with the single constraint that at each time where a product is replenished, the sum of the required storage volume (EOQ multiplied with the required unit capacity requirement) does not exceed the limited storage capacity. Then, the problem is solved by a *Lagrange multiplier* approach. Even ordering all products at the same time provides a feasible solution of this approach.

Other approaches improve the usage of capacity. In the *common cycle* or *rotation cycle* approach, it is assumed that all products have the same order cycle length and are ordered once in the order cycle. In this approach, the capacity usage is improved by phasing the replenishments in an optimal manner (e.g., see Page and Paul (1976), Hall (1988), Rosenblatt and Rothblum (1990), Hariga and Jackson (1996)). Hartley and Thomas (1982) analytically compare the Lagrange multiplier approach and the common cycle approach to the optimal replenishment and staggering policy of the two-product problem. Anily (1991) shows that for identical products the common cycle approach is optimal, however, if the products are not identical, the worst case bound of this approach is infinity. Gallego et al. (1992) show that the staggering problem is NP-complete. Gallego et al. (1996) develop a heuristic for the problem of staggering the order arrivals to minimize the maximum resource utilization.

Cheng (1990) presents a multi-product EOQ model that integrates the price decision of a constant selling price into an order quantity problem. He determines a solution in which all products are ordered with the same frequency and the lot-sizes satisfy a storage capacity constraint. Later, Chen and Min (1994) revise the work of Cheng (1990). They analyze *Karush-Kuhn-Tucker* conditions for the problem and derive closed-form solutions in the case of linear price-response functions. The majority of contributions that relax the time-invariant assumption of demand base on models that assume a discrete time framework.

Gilbert (2000) considers a problem of jointly determining prices and production schedules for multiple products that are produced on the same resource with limited capacity. He uses a set of numerical examples to derive insights into the relationship between prices, holding costs, and the seasonal pattern of demand. One observation indicates that the optimal price for a product will tend to increase if it has larger holding cost and/or contributes more to the seasonality of aggregate demand. The other observation demonstrates that, among products that experience demand peaks during the firm's busy season, those that peak early in the busy season should be priced more aggressively than those that peak later.

Bertsimas and de Boer (2005) study a periodic multi-product pricing and inventory control problem with finite production capacity. At the beginning of each period, both production quantities and prices of all products have to be determined. The authors formulate the periodical multi-product pricing and inventory control problem as a stochastic dynamic program which allows for a variety of demand models. To overcome the problem of increasing dimensionality, they propose and test a heuristic solution method which combines linear and dynamic programming.

Zhu and Thonemann (2005) analyze a joint pricing and inventory control problem for a retailer who orders two products and where the stochastic demand of a particular product depends on the prices of both products (cross-price effect). They investigate the benefits of a joint optimization and dynamic pricing of all products in a periodic review environment.

Pricing and replenishment in a competitive environment

Section 3.6 focuses on the interaction of a dynamically changing sales quantity and the replenishment policy in a competitive environment. Cost and market structure have a fundamental impact on operations and pricing decisions and the overall company performance.

Gaimon (1989) analyzes a differential game of two competing retailers who choose price and capacity. The acquisition of new technology reduces the firm's unit operating costs. An open-loop and a closed-loop Nash equilibrium is derived concerning the optimal price and capacity strategies of both retailers. Gaimon (1989) shows that the dynamic Nash strategies obtained for the closed-loop model exhibit a more restrictive new technology acquisition and a greater reduction of existing capacity relative to the open-loop strategies. Furthermore, the prices that are charged in a closed-loop Nash equilibrium are higher compared to an open-loop Nash equilibrium strategy.

Eliashberg and Steinberg (1991) study a problem of determining production and marketing equilibrium strategies for two competing firms under dynamically changing demand. The firms differ with their production costs. One firm faces convex production costs and linear holding costs whereas the second firm faces linear production costs and will not hold inventory. Both firms are allowed to vary their production rates and their selling prices continuously over a finite planning horizon. The authors characterize and compare the equilibrium strategies of the two firms. It is shown that the firm facing the convex production costs will build up inventory at the beginning of the planning horizon, then continue by drawing down inventory until it reaches zero, and finally, the firm will follow a "zero-inventory" policy until the end of the planning horizon. Furthermore, they show that this production policy is robust with respect to the market structure (monopoly and oligopoly).

Min (1992) extends the profit-maximizing economic order quantity model to the case of a symmetric oligopoly. He derives economic implications regarding selling prices, demand elasticities, the number of competitors, marginal and average costs, and average holding costs.

Lederer and Li (1997) study a competition problem where firms compete for customers by setting prices, production rates for each type of customers, and a production schedule where the customers are sensitive regarding price and delay time. Customers are either homogeneous or they can be differentiated by their respective demand function and delay sensitivity. They show the existence of a competitive equilibrium which is well defined as to whether or not the firm can differentiate between customers.

Cachon and Harker (2002) analyze an economic order quantity game between two retailers with fixed-ordering costs and price-sensitive consumers. They investigate motivations for outsourcing if a supplier exists who is able to manage a firm's operations and charges a constant fee per unit of demand for that service. They show that there are contracts that give the supplier a positive profit and yield a higher profit to either retailer than if the retailers do not outsource.

Ha et al. (2003) analyze the role of delivery frequency in supplier competition. They examine how the nature of competition (price or delivery) and the decision rights (who is responsible for handling logistics, making price, and making the delivery decision) influence supply chain performance. They show that when suppliers compete through prices, higher delivery frequencies may result in more intensive price competition, which is beneficial to the customers.

Adida and Perakis (2007) investigate a manufacturing system where two firms compete through pricing and inventory control. Both firms only differ in their production capacity. They study a decentralized Nash equilibrium game as well as the centralized problem where an authority controls the entire system. Beside some intuitive results, e.g., that in both settings, the firm with lower production capacity charges higher prices, produces less, and generates less profit compared to the firm with higher production capacity, they show, at first sight, a rather counter-intuitive result that in the decentralized setting, the firm with the lower capacity may want to restrict its capacity even when additional capacity is available at zero costs.

Federgruen and Meissner (2008) develop a competitive pricing model for a problem considering a time-varying demand as well as a time-varying cost structure including both fixed and variable procurement costs. They establish the existence of a price-equilibrium and the associated optimal dynamic lot-sizing policy. Furthermore, they design efficient procedures to compute the equilibrium prices and dynamic lot-sizes.

2.2.3 Pricing and capacity planning under demand uncertainty

There is extensive research in the field of integrated pricing and capacity/inventory management under demand uncertainty. This section provides an overview on the literature closely related to Chapter 4. It mainly refers to two streams of operations management literature. The first stream investigates the single-product single-period pricing and capacity problem under demand uncertainty which in the literature is called *price-setting newsvendor* problem. The second stream which has received much less attention focuses on joint capacity and price setting problems with multiple products. This research stream investigates the benefits of capacity and price flexibility as two methods to hedge against demand uncertainty.

The price-setting newsvendor problem

The initial work of endogenous pricing and capacity/inventory models was by Whitin (1955). He considers a newsvendor-type problem where a firm simultaneously decides on price and order quantity. He qualitatively discusses how price changes affect both the marginal revenue and the marginal salvage value. Mills (1959) analyzes the price-setting newsvendor model when demand is composed of a deterministic price-response function and a price-independent random variable. He shows that the optimal price is lower than the price in the absence of demand uncertainty.

Karlin and Carr (1962) identify a different impact of uncertainty on the price decision for additive and multiplicative demand functions. While for additive demand functions the selling price decreases with increasing uncertainty, in case of a multiplicative demand function, the optimal selling price increases with increasing uncertainty. Lau and Lau (1988) develop a solution procedure to find the optimal solution for objective functions that are unimodal in quantity and price. In a numerical experiment, they confirm the result of Mills (1959) that increasing uncertainty leads to decreasing prices. Furthermore, they analyze the problem under the objective of maximizing the probability of attaining a given profit level. For this problem, they show analytically that given the shortage costs are zero, uncertainty in demand only influences the value of the objective, but does not influence the optimal price and order quantity.

Petruzzi and Dada (1999) investigate in detail inconsistencies between models with additive and multiplicative demand functions and review and extend the results of a single-period pricing/order quantity problem to a multi-period problem. They provide a unified framework to compare the selling price in a deterministic framework with the selling price under uncertainty and they investigate the impact of additive and multiplicative demand models on pricing and capacity decisions.

Van Mieghem and Dada (1999) provide insights into the economic and operational value of price and production postponement strategies. Price postponement considers that the firm is able to set prices when all demand information is known whereas production postponement considers that the production quantity can be postponed until accurate demand information is known. Both postponement strategies provide a certain degree of flexibility for a firm. Their results show that price postponement makes capacity and production decisions relatively insensitive to demand uncertainty. However, capacity investments under production postponement are always sensitive to demand uncertainty. Furthermore, this insensitivity result directly shows that if price postponement is possible, additional production postponement has a relatively small incremental value.

Khouja (2000) extends the price-setting newsvendor such that a firm can use multiple discount prices in order to sell excess inventory. The problem is solved by a two-stage solution approach. At the second stage, the optimal discounting scheme is determined for a given order quantity. At the first stage, the firm determines the optimal order quantity anticipating the optimal discount decision. Given that each price discount is subject to costs, larger costs lead to a smaller number of prices. Furthermore, his results show that discounting schemes are more significant for higher demand elasticity and smaller fixed discounting costs.

More recently, Arcelus et al. (2005) consider a price-setting newsvendor problem where a profit-maximizing retailer faces a manufacturer-trade incentive in form of a direct price discount and can set a rebate to the end-customer with the objective to jointly determine the optimal price and ordering policy.

Raz and Porteus (2006) develop a method for the determination of the optimal selling price and order quantity that applies for additive, multiplicative, and combined demand functions. This method regards an order quantity as a fractile of the demand probability distribution for a given price. They develop a solution procedure for general stochastic price-response functions where the demand distribution is approximated with a finite number of representative fractiles and linear interpolation.

Bell and Zhang (2006) analyze the decisions that a firm has to make regarding the implementation of simultaneous decision-making of price and capacity. They examine several criteria that a firm faces referring to this implementation. These include, for instance, the choice of the demand function (i.e., additive vs. multiplicative), how often prices should be changed, and the level of effort to devote demand forecasting and cost analysis. They analyze the trade-off between effort and benefit of, e.g., a detailed data research, use of complicated stochastic models compared to simpler deterministic models, and the usage of revenue maximization or contribution maximization approaches.

Zhan and Shen (2005) analyze structural properties of the price-setting newsvendor problem and obtain properties of the solution as well as geometric explanations. Based on these results, they propose two algorithms: an iterative and a simulation-based algorithm. While Zhan and Shen (2005) only analyze the problem structure for additive demand function, Yao et al. (2006) provide a detailed analysis on how the price-setting newsvendor problem can be solved without using specific demand functions.

Kocabiyikoglu and Popescu (2007) introduce a measure of elasticity of lost sales. This concept provides a framework to characterize structural results for the joint pricing and capacity decision problem. They identify bounds of the lost sales elasticity which ensure the concavity and submodularity of the profit function as well as the monotonicity of prices and capacity policies.

Cachon and Kök (2007) investigate the impact of the salvage value in the newsvendor problem. They demonstrate that several intuitive methods estimating the salvage value might lead to overestimated order quantities and thus to a profit loss. They note that in practice the fixed salvage value assumption is questionable when a clearance price is rationally chosen in response to the events observed during the selling season. The authors discuss how to estimate a salvage value that leads to the optimal or nearly optimal order quantity w.r.t. maximizing the total profit.

The interrelation between capacity planning and pricing is also investigated in other research areas. From an accounting point of view, this research question is analyzed with regard to the impact of different cost-allocation strategies (full-cost and marginal-cost pricing) on the capacity investment. Balakrishnan and Sivaramakrishnan (2002) provide a critical overview of research that investigates the role of cost allocation in decision-making.

Göx (2002) analyzes a capacity planning and pricing problem of a monopolist facing uncertain demand where the capacity constraint is modeled both by hard and by soft capacity constraints. He studies the capacity and pricing problem w.r.t. different information about future demand. In scenario I, both capacity and price decision are made under complete certainty about future demand. In scenario II, both decisions have to be made under uncertainty and in Scenario III, the capacity has to be set under uncertainty whereas the price can be determined when the demand uncertainty is resolved. He identifies that when pricing and capacity decisions base on the same information, capacity costs are relevant for pricing decisions. However, when a firm has more information on demand at the time where the price is decided then marginal cost pricing is relevant.

Multi-product problems

So far, mainly single-product problems have been analyzed. This paragraph focuses on multi-product problems that integrate pricing and capacity decisions in the presence of uncertain demand which have received much less attention in literature. The particular focus is on research that investigates the benefits of capacity and price flexibility as two methods to hedge against demand uncertainty.

Van Mieghem (1998) investigates the interaction between capacity investment and product prices in a two-product framework given the future demand is uncertain. He considers a firm that has the option either to invest in two product-dedicated resources or in a single flexible resource that is able to produce both products. This type of flexibility is called product mix flexibility where a company can switch production between different products without high costs. Although not jointly optimized, Van Mieghem (1998) highlights the important role of the selling prices which significantly affect the investment decision and the value of flexibility.

Birge et al. (1998) consider a joint capacity and pricing problem of a firm that produces two substitutable products that are produced with two resources. They analyze several cases where at least two of the four decision variables (selling prices and capacities for the two products) are fixed. Furthermore, they compare a decentralized decision-making framework where two brand managers maximize each product's profit independently to the case of centralized decision-making. They show how price and capacity decisions are affected by parameter changes.

Similarly, Bish and Wang (2004) and Chod and Rudi (2005) consider a price-setting firm that can invest either in two dedicated resources or in a single fully flexible, but more expensive resource. While the capacity decision has to be made under demand uncertainty, the selling prices can be set after the demand uncertainty is resolved. They characterize under which conditions a firm prefers to invest in the fully flexible resource rather than in two separate dedicated resources and quantify the value of resource flexibility with regard to demand uncertainty and demand correlation. Bish and Wang (2004) assume that the demand for one product is not influenced by the price of the other product. They show that it can be optimal for the firm to invest in a flexible resource even with perfectly positively correlated demands. The reason for this effect is a kind of financial pooling. Chod and Rudi (2005), however, do allow cross-price demand dependencies and prove that the optimal resource level is increasing in both demand variability and correlation.

Biller et al. (2006) compare the performance of price postponement compared to a fixed price policy in a flexible production environment. Concerning the interaction of capacity flexibility and price postponement they find that the optimal investment in flexible capacity decreases if the company postpones the pricing decision until all demand information is known. Moreover, price postponement can reduce the financial risk of capacity investment. Bish and Hong (2006) study a capacity investment and pricing problem of a firm that produces two products of varying complexity. They assume that the resource that produces the higher level product can also produce the lower level product, but not vice versa. Bish and Hong (2006) consider a firm that produces two products 1 and 2 and has the option to invest in a flexible resource and that can produce both products and a dedicated resource that can produce product 2 only. The capacity decisions have to be made under demand uncertainty whereas the prices are optimized when the demand uncertainty is resolved. They analyze the impact of centralized decision-making when the resources are owned by a single decision maker compared to a decentralized system when the capacity investment decision for each resource is made separately. Their findings are that capacity investments in flexible resources are larger under centralized decision-making than under decentralized decisions whereas capacity investments in dedicated resources are lower in the centralized than in the decentralized system.

Tomlin and Wang (2008) consider a co-production system in which two products are simultaneously produced in a single production run that is characterized by a random production output. After the production process, it is considered that the product with higher quality can be downgraded to the lower quality product. They assume that the firm has to decide about the prices, capacity levels, downconversion, as well as the capacity allocation. They establish that downconversion will never occur in the single-class case if both prices are set optimally. In contrast, they show that downconversion can be optimal in the two-class case even if prices are set optimally.

3 Dynamic pricing under EOQ replenishment

3.1 Introduction

In this chapter, continuous time inventory replenishment problems are considered where dynamic pricing is applied to coordinate demand and supply decisions when a firm faces the trade-off between fixed ordering and inventory holding costs. Inventory models incorporating fixed ordering cost and dynamic issues, e.g., when the demand is allowed to vary over time, are mostly based on a discrete time framework (Kunreuther and Schrage, 1973). However, a discretization of time implies a restriction of the points in time where a decision can be made. To avoid the problem of having an incorrectly discretized time horizon, one option is a fine discretization of time which, however, causes an exploding problem size. Another option are continuous time models where inventory can be seen as a continuous flow where there is no need to predetermine the points in time where decisions have to be made. The problems that are analyzed in this section base on EOQ-based models with the objective to maximize the average profit by optimizing both the order quantity and the price strategy.

The fundamental research question of this chapter is how firms can benefit from coordinating dynamic pricing and replenishment in order to balance the trade-off between fixed ordering and inventory holding cost. Moreover, it is clarified how coordinated decision-making affects pricing and replenishment decision compared to decentralized decision-making. Under decentralized decision-making, which is common in practice, marketing (or sales) and operations operate as independent business units. While marketing determines the optimal selling price without anticipating the accurate operational costs, operations optimizes the replenishment strategy for a fixed demand. Since marketing and operations face different incentive structures, i.e., marketing is rewarded for sales and satisfying customers whereas operations is rewarded for efficiency, decentralized decision-making will not lead to overall optimal company performance (Jerath et al., 2007).

The specific research questions that are addressed in this chapter are:

1. What is the benefit of simultaneous decision-making of price strategy and replenishment policy compared to decentralized decision-making where mar-

keting decides on the pricing strategy and operations optimizes the replenishment policy?

2. How do dynamically changing selling prices and the replenishment strategy interact?
3. What is the benefit of dynamic pricing compared to constant pricing?

Starting with a simple single-product monopoly setting, the analysis is extended to problems incorporating a supplier quantity discount, multiple products that share a warehouse with limited storage capacity, and a competitive environment. Furthermore, we differentiate models where continuous price adjustments are allowed to the case where only a limited number of price changes is allowed. A continuous price adjustment is often either not feasible or too costly. For instance, catalog retailers incur significant costs each time they change prices because new catalogs have to be printed. Another example are general retailers who have to communicate price changes by means of advertising which is associated with significant costs (Netessine, 2006).

3.2 Preliminaries

3.2.1 Assumptions and notation

Throughout this chapter, an infinite and continuous planning horizon is considered. All information about future demand and cost parameters are given and constant. It is assumed that customer demand follows a function of the selling price P and arrives continuously at a rate of $D(P)$ with $D(P) \geq 0$ and $D'(P) < 0$. There exists a critical price $\bar{P} \in (0, \infty)$ such that $D(P) > 0$ for $P \in [0, \bar{P})$ and $D(P) = 0$ for $P \in [\bar{P}, \infty)$. $D(P)$ is a on the interval $[0, \bar{P}]$ differentiable and strictly decreasing in P. Furthermore, it is assumed that $D(P)$ satisfies $2 - \frac{D(P)D''(P)}{D'(P)^2} \geq 0$. This condition holds for the majority of price-response function, e.g., linear, iso-elastic, and exponential. We denote the optimal decision variables by a superscript "$*$". Furthermore, let ε_P define the price elasticity of $D(P)$ with

$$\varepsilon_P = -\frac{D'(P)}{D(P)} P \tag{3.1}$$

which is the percentage change in demand in response to a percentage change in price. Based on the price elasticity, we define a class of price-response functions as follows:

Definition 1. *A price-response function $D(P)$ has an increasing price elasticity (IPE), if $\frac{\partial \varepsilon_P}{\partial P} \geq 0$.*

The intuition behind the IPE property is that with a price increase by a certain percentage demand decreases by a larger percentage.

At every point in time, the demand rate depends solely on the current price, i.e., the customers are myopic and effects of forward-buying or postponement in the case of dynamically changing prices are not incorporated. This myopic customer behavior can be appropriate in several settings:

- the products that are sold are necessity products and customers cannot wait for a price drop or
- price changes are small enough such that strategic waiting for lower prices does not provide much value.

This category captures most nondurable products such as grocery items, produce, and pharmaceutical products (Elmaghraby and Keskinocak, 2003).

Following the assumptions of the EOQ model, the retailer has to place replenishment orders in batches of size Q every T periods during the infinite planning horizon. With the release of any single order, there are associated ordering costs F and variable procurement cost c per unit. Ordering costs are the sum of all fixed costs that are incurred when an order is placed and are independent of the order size. For products that are ordered externally, these costs would include, for instance, costs to enter the requisition, to process the receipts, incoming inspection, invoice processing, and vendor payment. Depending on who has to bear the costs for transportation, also inbound freight may be included. In manufacturing these costs are called setup costs. These costs would include opportunity costs of time to initiate the work order, production scheduling time, machine setup time, as well as inspection and cleaning time. Variable procurement costs include any unit costs associated with purchasing a single unit, e.g., unit material costs and unit transportation costs.

Products delivered but not yet sold are kept in inventory subject to a holding cost h per unit and unit of time. Essential components of holding costs are cost of capital, insurance, taxes, and storage costs. That is, total cost of holding inventory is the sum of carrying costs and cost of capital (Timme and Williams-Timme, 2003). It is common in research and industry to apply holding costs as a percentage of the inventory value. While the calculation of the EOQ itself is fairly simple, determining the correct input data is far more complex. Timme and Williams-Timme (2003) present a methodology that supports managers to establish their accurate total inventory holding costs. Another method to overcome this holding cost valuation problem is to optimize the *net present value* (NPV) as the sum of discounted cash-flows. In academia the NPV approach is widely accepted as the right framework for studying inventory control and production planning problems. However, in a later analysis it is shown that applying certain transformations of the holding cost parameter, the average profit approach of the EOQ-type problem gives approximately near optimal results with respect to maximizing the NPV (Trippi and Lewin, 1974).

A fundamental assumption is that the supplier has no capacity constraints and the overall order quantity is delivered in one shipment with lead time zero. Since all demand and cost data are known, general validity is not lost by assuming zero lead time. Furthermore, it is assumed that backorders are not permitted. This assumption is reasonable if it is expensive when customers cannot be satisfied immediately but have to wait until the next order arrives. The "no-backorder" constraint ensures that the inventory level $y(t)$ at any time t has to be nonnegative.

In the traditional EOQ model where the demand rate is exogenous, only the order quantity and thus the order frequency is controlled in order to match supply with demand. However, if the demand is endogenous, also the demand rate can be controlled by the selling price.

3.2.2 Constant pricing and replenishment in a monopoly

This section introduces the fundamental pricing and replenishment problem in an EOQ-type framework which is presented first by Whitin (1955). The following analysis (see Eliashberg and Steinberg (1993)) illustrates the necessity of an integration of price and inventory management by a simple EOQ model with a constant selling price.

It is common in practice that price and replenishment decisions are made by independent business units. Marketing is responsible for price management whereas operations is responsible for replenishment decisions. Marketing maximizes the average profit Π^M only taking variable procurement costs into account:

$$\Pi^M(P) = (P - c)D(P). \tag{3.2}$$

Maximizing (3.2) yields that the optimal price has to satisfy the following first-order condition

$$P' + \frac{D(P')}{D'(P')} = c. \tag{3.3}$$

Given the optimal price P' and the resulting demand rate D', operations determines the optimal replenishment policy with the objective to minimize the average costs AC^O regarding the trade-off between fixed ordering costs and inventory holding costs:

$$AC^O(Q) = \frac{h}{2}Q + \frac{FD'}{Q}. \tag{3.4}$$

The outcomes are the economic order quantity, the economic order interval, and minimized costs per period as follows

$$Q' = \sqrt{\frac{2FD'}{h}}, \qquad T' = \sqrt{\frac{2F}{hD'}}, \qquad AC^{O'} = \sqrt{2FhD'}. \tag{3.5}$$

A decision maker who simultaneously optimizes the order quantity Q or equivalently the order interval T and the selling price P faces the following profit function

$$\Pi(T, P) = (P - c)D(P) - \frac{h}{2}D(P)T - \frac{F}{T}. \tag{3.6}$$

The optimal order interval and the optimal price resulting from the first-order conditions $\frac{\partial \Pi(T,P)}{\partial T} = 0$ and $\frac{\partial \Pi(T,P)}{\partial P} = 0$ can be determined by solving the following system of nonlinear equations

$$T^* = \sqrt{\frac{2F}{hD(P^*)}} \qquad \text{and} \qquad P^* + \frac{D(P^*)}{D'(P^*)} = c + \frac{h}{2}T^*. \tag{3.7}$$

A comparison of (3.3) and (3.7) gives that decentralized decision-making yields an underestimation of P, which, in turn, leads to an overestimated demand rate. Consequently, decentralized decision-making yields a larger order quantity compared to simultaneous decision-making. The reason for this suboptimality of decentralized decision-making is that marketing disregard fixed ordering and inventory holding cost. To overcome the suboptimality of decentralized decision-making it is not essential to simultaneously decide. If marketing correctly anticipates operations optimal costs, i.e., marketing maximizes

$$\max \Pi^M(P) = (P - c)D(P) - \sqrt{2FhD(P)}, \tag{3.8}$$

this solution is optimal for the entire firm. This result is easy to verify by optimizing (3.8) (Whitin (1955) and Eliashberg and Steinberg (1993)).

3.3 Dynamic pricing and replenishment in a monopoly

This section presents an EOQ model that considers coordinated dynamic pricing and lot-sizing decisions. A retailer procures a single product from an external supplier and sells it on a single market without competition. The retailer's objective is to maximize the average profit by choosing an optimal lot-size and pricing strategy where the retailer may vary the selling price over time. The contribution is to investigate the replenishment policy and pricing strategy in an EOQ framework when the demand rate is endogenous, i.e., it can be controlled by selling price. Within this context, two pricing frameworks are considered: a continuous price adjustment and a limited number of price changes within an order cycle. The continuous pricing framework is analyzed by a model based on *optimal control theory*. The outcome of this model reflects the results of Rajan et al. (1992) although they incorporate perishable products. The contribution of this section is to generalize the assumption of continuous price adjustments to an optimized number of price changes which is reasonable if each price change is associated with costs.

3.3.1 Model with continuous price adjustment

This section considers that the retailer is allowed to vary the selling price continuously, i.e., at any time t, the retailer is allowed to charge a different price $P(t)$. The objective is to maximize the average profit Π by determining the optimal price trajectory $P^*(t)$ and the optimal replenishment policy which is characterized by the optimal cycle length T^* and the optimal order quantity Q^*. In real-world problems, continuous price adjustments are less realistic, however, given the trend that in various businesses price changes are almost costless, this model provides an upper bound for the average profit.

The problem can be considered as a two-stage hierarchical optimization problem. At the second stage, we determine the optimal price trajectory P^* given a fixed cycle length T and without consideration of F by using *Pontryagin's Maximum Principle* (Kamien and Schwartz, 1991). By substituting the optimal price $P^*(t)$ for all t and for a given T into the overall objective function and integrating with respect to t, we get the first-stage average profit as a function of T, $\Pi(T)$. Then, the optimal cycle length T^* is determined by maximizing $\Pi(T \mid P^*)$ (Rajan et al., 1992).

Optimal control problem - second stage

The second-stage optimization problem is given by the following optimal control problem, which maximizes the cycle profit Π_T for a given cycle length T:

$$\Pi_T^* = \max_{P(t)} \int_0^T [(P(t)-c)D(P(t)) - hy(t)]\, dt, \tag{3.9}$$

$$\dot{y}(t) = -D(P(t)), \tag{3.10}$$

$$y(0) = Q,\ y(T) = 0, \tag{3.11}$$

$$0 \leq P(t) \leq \bar{P}. \tag{3.12}$$

$y(t)$ represents the inventory level at time t and $\dot{y}(t)$ denotes the *state transition*, i.e., $\dot{y}(t) = \frac{\partial y(t)}{\partial t}$, which is equal to the negative demand rate at time t. Hence, the change of inventory at time t equals the demand rate at time t. Every replenishment cycle has the *initial* condition $y(0) = Q$ and *terminal* condition $y(T) = 0$, i.e., the inventory level at the beginning of a replenishment cycle is equal to the lot-size and at the end of a replenishment cycle the inventory level is equal to zero (eq. (3.11)). When the inventory level drops to zero, the next order arrives immediately. (3.12) ensures the non-negativity of the selling price and the demand rate. From (3.10) and (3.11) it follows that $y(0) = \int\limits_0^T D(P(t))dt$ and $y(t) = \int\limits_t^T D(P(s))ds = Q - \int\limits_0^t D(P(s))ds$.

Let $f(P,y,t) = (P(t)-c)D(P,t) - hy(t)$ denote the profit function at a particular time t with the assumption that $f(P,y,t)$ is continuously differentiable and concave. The *Hamiltonian* is

$$H := H(P,y,\lambda(t),t) = f(P,y,t) - \lambda(t)D(P,t) \tag{3.13}$$

with the costate variable $\lambda(t)$ which represents the shadow price of the state variable $y(t)$. The *Lagrange function* is

$$L := L(P,y,\lambda,\mu_1,\mu_2,t) = f(P,y,t) - \lambda(t)D(P,t) + \mu_1(t)(\bar{P} - P(t)) + \mu_2(t)P(t) \tag{3.14}$$

with the *Lagrangian multipliers* $\mu_1(t)$ and $\mu_2(t)$ associated with (3.12). For notational simplicity, we omit the argument t. The maximum principle states that a necessary condition for P^* to be an optimal control is

$$\frac{\partial L}{\partial P} = D(P) + (P-c-\lambda)\frac{\partial D(P)}{\partial P} - \mu_1 + \mu_2 \stackrel{!}{=} 0 \tag{3.15}$$

where μ_1 and μ_2 must satisfy the complementary slackness conditions

$$\mu_1 \geq 0 \quad \text{and} \quad \mu_1(\bar{P} - P) = 0 \tag{3.16}$$
$$\mu_2 \geq 0 \quad \text{and} \quad \mu_2 P = 0. \tag{3.17}$$

The following necessary condition defines the costate variable λ as a function of t

$$\dot{\lambda} = -\frac{\partial L}{\partial y} = h \Rightarrow \lambda(t) = ht + \lambda_0. \tag{3.18}$$

$\lambda(t)$ is the value of the costate variable at time t measuring the value of an additional unit of inventory along the optimal path (Feichtinger and Hartl, 1985) and is a linearly increasing function of t with a slope equal to the inventory holding cost h. That is, the value of an additional unit of inventory at time t is equal to the cost of holding this unit over a time period $[0, t]$. Therefore, it follows that $\lambda(T) = hT$ (Kamien and Schwartz, 1991) and using (3.18) it follows that $\lambda_0 = 0$ and

$$\lambda(t) = ht. \tag{3.19}$$

From (3.15), (3.16), (3.17), and (3.19) (*Karush-Kuhn-Tucker-conditions*) (KKT) it follows that the cases $\mu_1^* > 0$ and $\mu_2^* = 0$, $\mu_1^* = 0$ and $\mu_2^* > 0$, as well as $\mu_1^* > 0$ and $\mu_2^* > 0$ lead to infeasible solutions. In detail, if $\mu_1 > 0$, then from (3.16) it follows that $P^* = \bar{P}$, which, in turn, gives that $D(P^*) = 0$. Moreover, (3.15) gives that $\mu_1 = (\bar{P} - c - \lambda)\frac{\partial D(\bar{P})}{\partial P} < 0$ which, in turns, gives that the *Karush-Kuhn-Tucker-conditions* are not satisfied. The interpretation that $\mu_2^* = 0$ follows analogously. Therefore, an inner solution that satisfies $c^k < P^* < \bar{P}$ with $\mu_1^* = \mu_2^* = 0$ has to be optimal and the optimal price is expressed by an implicit function

$$P + \frac{D(P)}{D'(P)} = c + ht. \tag{3.20}$$

(3.20) establishes that the optimal selling price at time t satisfies that the marginal revenue is equal to the marginal costs at time t. Thus, for the first unit that is sold, the marginal costs are equal to c while the last unit sold at the end of an order cycle, after being kept in inventory T units of time, causes marginal costs of $c + hT$. Within the order cycle, the valuation of holding costs through the adjoint variable $\lambda(t)$ increases linearly following (3.19). By substitution of $P^*(t)$, $D(P^*(t))$, and $y^*(T, t) = \int_t^T D(P^*(s))ds$ into (3.9), we get the profit as a function of an order cycle of length T:

$$\Pi_T^* = \int_0^T \left[(P^*(t) - c)D(P^*(t)) - h \int_t^T D(P^*(s))ds \right] dt. \tag{3.21}$$

Optimization problem - first stage

The average profit of the first stage is determined by the nonlinear optimization problem

$$\Pi(T) = \max_{T} \frac{1}{T}\left(\Pi_T^* - F\right) \tag{3.22}$$

$$s.t. \quad T \geq 0 \tag{3.23}$$

where Π_T^* is the optimal cycle profit given a cycle length T.

Proposition 1. *There exists a unique optimal T^* which satisfies the condition that the marginal profit at T^* is equal to the average profit, i.e.,*

$$\Pi(T) \stackrel{!}{=} (P^*(T) - c - hT)D(P^*(T)).$$

Proof. The first-order condition of (3.22) gives

$$\frac{\partial \Pi(T)}{\partial T} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \Pi(T) = \frac{\partial \Pi_T^*}{\partial T} = (P^*(T) - c - hT)D(P^*(T)). \tag{3.24}$$

That is, any intersection point of $\Pi(T)$ and $MP(T)$ must be equal to a local extreme point, i.e., a local maximum or a local minimum. In order to demonstrate that there exists a unique T, we show that the marginal profit is a decreasing function of T.

Let $MP(T) := (P^*(T) - c - hT)D(P^*(T))$ denote the marginal profit as a function of T. The first derivative gives

$$\frac{\partial MP(T)}{\partial T} = \left(\frac{\partial P^*(T)}{\partial T} - h\right) D(P^*(T)) + (P^*(T) - c - hT)D'(P^*(T))\frac{\partial P^*(T)}{\partial T}. \tag{3.25}$$

Since the optimal second-stage decision is anticipated, from (3.20) it can be derived that

$$(P^*(T) - c - hT)D'(P^*(T)) = -D(P^*(T)). \tag{3.26}$$

Substituting (3.26) into (3.25) we get

$$\frac{\partial MP(T)}{\partial T} = -hD(P^*(T)) \leq 0. \tag{3.27}$$

The second-order derivative $\frac{\partial^2 MP(T)}{\partial T^2} = -hD'(P^*(T))\frac{\partial P^*(T)}{\partial T} \geq 0$ gives that $MP(T)$ is a decreasing and convex function in T such that only a single intersection point between $MP(T)$ and $\Pi(T)$ can exist. $\square$

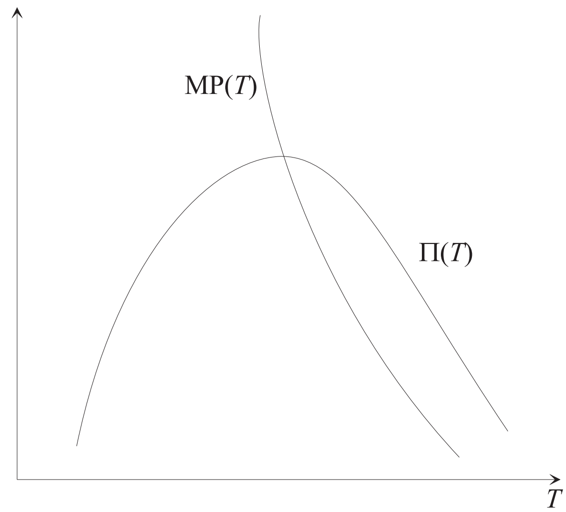

Figure 3.1: Average profit and marginal profit as functions of T

3.3.2 Model with a discrete number of price changes

This section considers that the number of times where the retailer is allowed to change the selling price is limited. The main motivation for considering a limited number of price changes are organizational costs associated with each price change. As we will show, the majority of benefits can be captured by a few different price levels. Therefore, a discrete number of changes balances benefits and costs of price changes. The retailer has to decide how to set the price for the product in each time interval where the price is fixed and when to switch from one price to another. That is, the retailer determines the optimal number of different prices N per order cycle of length T. The administrative costs associated with price changes are denoted by $\kappa(N)$ and are a non-decreasing function of N. For a given N, the retailer has to establish the time intervals $[t_{i-1}, t_i)$ and the associated prices P_i. The time t_N is equal to the cycle length T and $t_0 = 0$.

This problem can be solved by a two-stage hierarchical optimization approach. At the second stage, the retailer optimizes the prices for a given number N, the points in time where the prices are adjusted, and the optimal cycle length. At the first stage, the number of price settings N will be optimized anticipating the optimal timing and sizing of prices for a given N. The objective is to maximize the average profit per unit of time and can be formulated as follows:

Stage 1:

$$\Pi^* = \max_N \left[\left\{ \Pi^{(N)^*} - \kappa(N) \right\} \right], \tag{3.28}$$

Stage 2:

$$\Pi^{(N)^*} = \max_{P_1,\cdots,P_N,t_1,\cdots,t_N} \frac{1}{t_N}\Bigg[\sum_{i=1}^{N}(P_i - c)D(P_i)\,(t_i - t_{i-1}) \tag{3.29}$$

$$-\frac{h}{2}\sum_{i=1}^{N} D(P_i)\,(t_i - t_{i-1})^2 - h\sum_{i=1}^{N-1}\left((t_i - t_{i-1})\sum_{j=i+1}^{N} D(P_j)\,(t_j - t_{j-1})\right) - F\Bigg]$$

$$s.t. \qquad 0 \leq P_i \leq \bar{P} \qquad \forall \qquad i = 1,\cdots,N, \tag{3.30}$$

$$t_{i-1} - t_i \leq 0 \qquad \forall \qquad i = 1,\cdots,N. \tag{3.31}$$

Equation (3.28) represents the first-stage problem that optimizes N anticipating the optimal price and timing decision at the second stage. (3.29) - (3.31) represent the optimization problem at the second stage where we determine the optimal prices $P^* = (P_1^*,\cdots,P_N^*)$ and the associated timing of price changes $t^* = (t_1^*,\cdots,t_N^*)$ simultaneously for a given N. The average profit in (3.29) is given by the revenue minus purchasing cost, inventory holding cost, and setup cost over the order cycle, divided by the cycle length. The first term is the unit revenue minus direct purchasing cost for each interval multiplied by the respective demand rate and the length of the interval. In the second and third term we subtract the inventory holding cost. These consist of the average of initial and final inventory in each interval (triangles) and the inventory that has to be carried over the entire interval $(t_i - t_{i-1})$ to cover the demand of all subsequent intervals $j = i+1,\cdots,N$ with an amount of $\sum_{j=1}^{N} D(P_j)(t_j - t_{j-1})$ (rectangles). Constraint (3.30) ensures that price and demand in each interval are nonnegative and (3.31) guarantees that all intervals are mutually exclusive and exhaustive. The path of inventory deployment for the case $N = 3$ is illustrated in Figure 3.2. The optimal order quantity is then:

$$Q^* = \sum_{i=1}^{N} D(P_i^*)\left(t_i^* - t_{i-1}^*\right). \tag{3.32}$$

For a concave profit function $\Pi^{(N)}$ and a linear cost function $\kappa(N)$, a simple incremental search for the optimal value of N can be conducted. In general, for the stage 1 problem only a few integer values are relevant such that this problem can be solved by enumeration and we can restrict the analysis to the inner problem for a given number of price changes N.

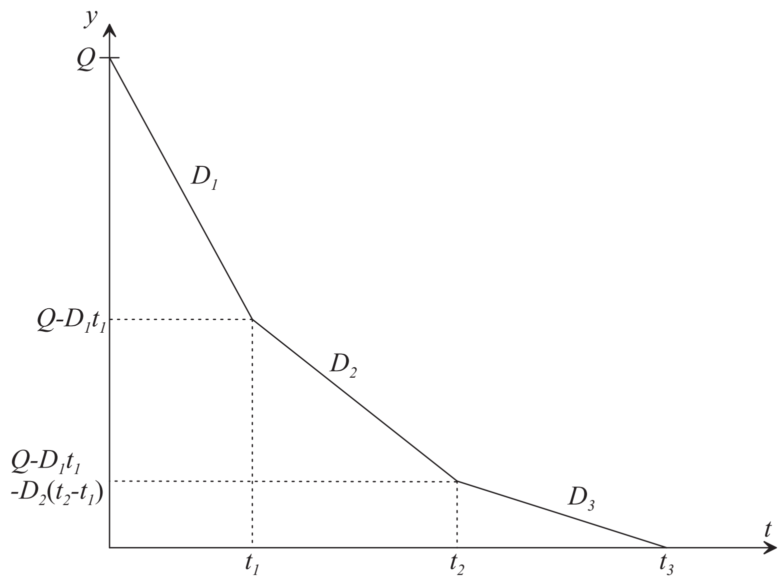

Figure 3.2: Illustration of the inventory level

After rearranging terms in (3.29), the *Lagrangian* can be expressed as

$$
\begin{aligned}
L^{(N)} = & \frac{1}{t_N}\left[\sum_{i=1}^{N}\left(P_i - c - \frac{h}{2}(t_i + t_{i-1})\right) D(P_i)\,(t_i - t_{i-1}) - F\right] \\
& + \sum_{i=1}^{N} \lambda_i(\bar{P} - P_i) + \sum_{i=1}^{N} \mu_i(t_i - t_{i-1}).
\end{aligned}
\tag{3.33}
$$

where λ_i and μ_i for $i = 1, \cdots, N$ represent the *Lagrangian multipliers* of constraints (3.30) and (3.31), respectively.

Proposition 2. *The optimal prices P_i^* are strictly increasing over time, i.e., $P_i^* < P_{i+1}^* < \bar{P}$ for all $i = 1, \cdots, N-1$.*

The proof is given in Appendix A.1. The general intuition behind this result is that it is beneficial to enhance demand when inventories are high in order to reduce holding costs. Note that due to the replenishments which are repeated over an infinite horizon and the assumption of deterministic demand this intuition is different from slow-moving, single order, stochastic demand clearance pricing models where prices will decrease over time to salvage excessive inventories. An optimal pricing policy is illustrated in Figure 3.3.

Corollary 1. *An optimal solution $[P_1^*, \cdots, P_N^*, t_1^*, \cdots, t_N^*]$ that maximizes (3.33) is an interior solution, i.e., $\mu_i^* = \lambda_i^* = 0$ for all $i = 1, \cdots, N$.*

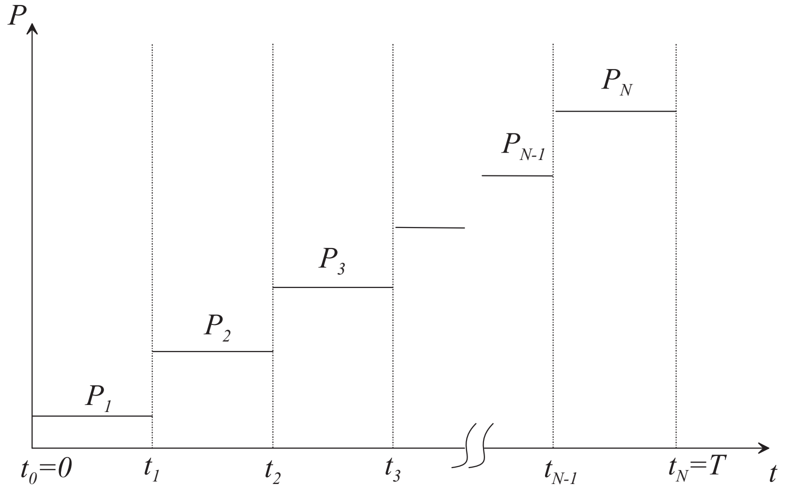

Figure 3.3: Illustration of the pricing policy

The proof of Corollary 1 follows directly from Proposition 2. Thus, the second-stage problem is reduced to

$$\Pi^{(N)} = \frac{1}{t_N}\left[\sum_{i=1}^{N}\left(P_i - c - \frac{h}{2}(t_i + t_{i-1})\right) D(P_i)\,(t_i - t_{i-1}) - F\right]. \tag{3.34}$$

In order to maximize the average profit, we differentiate (3.34) with respect to P_i and t_i for $i = 1, \cdots, N$. The necessary first-order conditions for the optimal prices $\frac{\partial \Pi^{(N)}}{\partial P_i} \overset{!}{=} 0$ and the optimal times for price changes $\frac{\partial \Pi^{(N)}}{\partial t_i} \overset{!}{=} 0$ for $i = 1, \cdots, N$ are characterized by:

$$P_i^* + \frac{D(P_i^*)}{D'(P_i^*)} = c + \frac{h}{2}\left(t_i^* + t_{i-1}^*\right), \quad \text{for } i = 1, \cdots, N, \tag{3.35}$$

$$t_i^* = \frac{(P_i^* - c)D(P_i^*) - (P_{i+1}^* - c)D(P_{i+1}^*)}{h(D(P_i^*) - D(P_{i+1}^*))} \quad \text{for } i = 1, \cdots, N-1, \tag{3.36}$$

and

$$\Pi^{(N)} = \left(P_N^* - c - \frac{h}{2}((t_N^*)^2 - (t_{N-1}^*)^2)\right) D(P_N^*). \tag{3.37}$$

(3.35) reflects the well-known optimality condition of marginal revenue equals marginal cost. However, in joint replenishment and pricing optimization the marginal cost in interval i additionally includes the holding cost for an average item which is in stock for a duration of $\frac{t_i^* + t_{i-1}^*}{2}$. For interpretation purposes we rearrange (3.36) to $ht_i^*(D(P_i^*) - D(P_{i+1}^*)) = (P_i^* - c)D(P_i^*) - (P_{i+1}^* - c)D(P_{i+1}^*)$.

This gives that it is optimal to increase the time t_i for the price adjustment from P_i to P_{i+1} until the holding cost for the additional demand in interval i compared to interval $i+1$ $(ht_i(D(P_i)-D(P_{i+1})))$ is smaller than the difference of the profit margin per unit of time between intervals i and $i+1$, $(P_i-c)D(P_i)-(P_{i+1}-c)D(P_{i+1})$. Condition (3.37) gives that the optimal cycle length t_N^* is the time where the average profit is equal to the marginal profit. The intuition behind condition (3.37) is the usage of scale economies due to the fixed ordering costs. If the last increment of profit is larger than the average profit of all previous units, an additional unit of inventory leads to larger economies of scales and thus, to an increasing average profit. That is, as long as the marginal profit is larger than the average profit, it is optimal to increase the order cycle. However, if the marginal profit is lower than the average profit, then an additional unit will decrease the average profit.

By an iterative approach, we can show that P_i^* for $i=1,\cdots N$ and t_i^* for $i=1,\cdots N-1$ can be expressed as single variable functions of t_N. From (3.36) it can be observed that for all $i=1,\cdots,N-1$ the optimal time t_i^* is a function of the optimal price P_i^* of the current interval, P_{i+1}^* the optimal price of the next interval, and the corresponding optimal demand rates. Given the results of (3.35) and (3.36), it is apparent that t_i^* can be represented as a function of its predecessor t_{i-1}^* and its successor t_{i+1}^*. With the initial condition $t_0=0$, it follows that t_1^* is a function of its successor t_2^*. t_2^* is a function of t_1^* and t_3^*. By substituting $t_1^*(t_2^*)$ into $t_2^*(t_1^*,t_3^*)$ and some algebraic manipulations, the time t_2^* is represented as a single-variable function of t_3^*. An iterative transformation and substitution gives that from $[t_1^*(t_2^*),\ t_2^*(t_1^*,t_3^*),\ \cdots,t_N^*(t_{N-1}^*)]$ it follows that each t_i^* can be represented as a function only depending on its successor, i.e., $[t_1^*(t_2^*),\ t_2^*(t_3^*),\ \cdots,t_{N-1}^*(t_N^*)]$. Now, by backward insertion, each optimal time where the selling price is changed can be represented as a function $t_i^*(t_N^*)$ of the cycle length. Therefore, the optimal prices and the optimal demand rates can be reduced to functions that only depend on the cycle length t_N, e.g., $(P_i^*(t_N),D_i^*(t_N))$. In Section 3.3.3 , we show that for a linear price-response function this iterative procedure yields closed-form expressions $P_i^*(t_N)$ for all $i=1,\cdots,N$. For general price-response functions, this interrelation has to be solved numerically. Therefore, constraints (3.30) can be relaxed to $P_N(t_N)\leq\bar{P}$ only. By solving the equation $P_N^*(t_N)=\bar{P}$, we obtain an upper bound for the optimal cycle length denoted by t_N^{Max} i.e., $t_N^*\leq t_N^{Max}$.

The first-order condition for t_N disregarding the constraint $P_N\leq\bar{P}$ provides:

$$t_N'=\sqrt{\frac{2F}{hD(P_N^*)}-\frac{1}{D(P_N^*)}\sum_{i=1}^{N-1}(t_i^*)^2(D(P_i^*)-D(P_{i+1}^*))}. \tag{3.38}$$

The derivation of (3.38) is given in Appendix B.1. However, using the upper bound on t_N, the optimal cycle length is determined by

$$t_N^* = \begin{cases} t'_N & \text{if } t'_N \leq t_N^{Max} \\ t_N^{Max} & \text{otherwise.} \end{cases} \tag{3.39}$$

An intuition of (3.38) can be obtained by a transformation into

$$F = \frac{h}{2} \sum_{i=1}^{N} D(P_i)(t_i^2 - t_{i-1}^2).$$

This optimality condition is known from the EOQ model, namely that for an entire inventory cycle the total inventory holding costs are equal to the fixed cost for a single replenishment.

Compared to static pricing (see Section 3.2.2), at the beginning of a cycle a lower selling price results in a higher demand which yields a reduction of inventory holding costs. This, in turn, leads to a lower order frequency. The managerial intuition behind this effect is that the retailer places orders in lots such that the later an item is sold the higher are inventory holding costs for this item. For this reason, the retailer has an incentive to reduce inventories at the beginning of the order cycle.

3.3.3 Special price-response functions

Linear price-response

Assume that at any point in time t the market potential is denoted by a and an amount of bP customers decide that the price is too high and do not buy. In particular, the *reservation price* $\bar{P} = \frac{a}{b}$ denotes the price where the demand rate drops to zero.

$$D(P) = \begin{cases} a - bP & : \quad 0 \leq P \leq \frac{a}{b} \\ 0 & : \quad P > \frac{a}{b} \end{cases} . \tag{3.40}$$

A linear price-response function is often used in economics, marketing, and operations management literature. In the following, we analyze constant pricing, discrete time price changes, and continuous price adjustment for the linear price-response function.

Constant pricing

Analyzing the special case of constant pricing, we get that substituting (3.40) into the optimality condition (3.7) leads to

$$T^* = \sqrt{\frac{2F}{h(a - bP^*)}} \qquad \text{and} \qquad P^* = c + \frac{(a - bP^*)}{b} + \sqrt{\frac{Fh}{2(a - bP^*)}}. \quad (3.41)$$

In order to determine the optimal price, the second equation has to be transformed into the following cubic equation:

$$\begin{aligned} &- 8b^3P^3 + (16ab^2 + 8cb^3)P^2 - (10a^2b + 12ab^2c + 2b^3c^2)P \\ &\qquad + 2a^3 + 4a^2bc + 2ab^2c^2 - Fhb^2 = 0. \quad (3.42) \end{aligned}$$

Using the *Trigonometrical Solution Method* (see Bronshtein et al. (2004)), it can be shown that (3.42) has at most three different real roots. However, given that $0 \leq P \leq \bar{P}$ there exists a unique P^* (see Whitin (1955), Eliashberg and Steinberg (1991)).

Discrete price changes

In the case where price changes are limited to a given number N, the following three propositions provide results on the optimal cycle length and the optimal points in time where the price will be adjusted.

Proposition 3. *In an order cycle where N price variations are allowed and the price-response function is linear, the time intervals $[t_{i-1}, t_i)$ are equidistant: $\delta^* := t_i^* - t_{i-1}^* = t_{i+1}^* - t_i^*$ for all $i = 1, \cdots, N-1$.*

Proof. From (3.35) and (3.36) we find

$$P_i^* = \frac{1}{2}\left(\frac{a}{b} + c + \frac{h}{2}\left(t_i^* + t_{i-1}^*\right)\right) \qquad \text{and} \qquad t_i^* = \frac{1}{h}\left(P_i^* + P_{i+1}^* - \left(\frac{a}{b} + c\right)\right). \quad (3.43)$$

Inserting P_i^* and P_{i+1}^* into the equation for t_i^* leads to the condition: $t_i^* = \frac{t_{i-1}^* + t_{i+1}^*}{2} \iff t_i^* - t_{i-1}^* = t_{i+1}^* - t_i^* = \delta_t^*$. □

According to Proposition 3,

$$t_i^* = i\frac{t_N^*}{N} \qquad \text{and} \qquad P_i^* = \frac{1}{2}\left(\frac{a}{b} + c + \frac{h}{2}\frac{(2i-1)}{N}t_N^*\right). \quad (3.44)$$

Equation (3.44) indicates that the optimal price increases over the order cycle so that at every time t_i the retailer increases the price by a constant of $\frac{h}{2}\frac{t_N^*}{N}$. The condition $P_i \leq \frac{a}{b}$ gives

$$\frac{1}{2}\left(\frac{a}{b}+c+\frac{h}{2}\frac{(2N-1)}{N}t_N\right) \leq \frac{a}{b} \Longleftrightarrow \quad t_N \leq \frac{2(a-bc)}{hb}\frac{N}{(2N-1)} = t_N^{Max}. \quad (3.45)$$

Using (3.44) in (3.39), the optimal cycle length results from

$$\frac{4N^2-1}{N^2}t_N^3 - \frac{6(a-bc)}{hb}t_N^2 + \frac{24F}{h^2b} = 0. \quad (3.46)$$

Proposition 4. *If the response function is linear and N price changes are allowed in an order cycle, there exists a unique optimal cycle length $t_N^* \geq 0$ if*

$$F \leq \frac{4}{3}\frac{(a-bc)^3}{b^2h}\frac{N^4}{(4N^2-1)^2} =: F_{max} \quad (3.47)$$

and the optimal cycle length t_N^* is

$$t_N^* = \begin{cases} -2\frac{(a-bc)}{hb}\frac{N^2}{(4N^2-1)}\left(2\cos(\frac{\pi}{3}+\frac{\phi}{3})-1\right) & : F \leq \frac{2}{3}\frac{(a-bc)^3}{hb^2}\frac{N^4}{(4N^2-1)^2} \\ \phi = arccos\left(1-\frac{3F}{2}\frac{hb^2}{(a-bc)^3}\frac{(4N^2-1)^2}{N^4}\right) & \\ & \\ 2\frac{(a-bc)}{hb}\frac{N^2}{(4N^2-1)}\left(2\cos(\frac{\pi}{3}+\frac{\phi}{3})+1\right) & : F > \frac{2}{3}\frac{(a-bc)^3}{hb^2}\frac{N^4}{(4N^2-1)^2} \\ \phi = arccos\left(\frac{3F}{2}\frac{hb^2}{(a-bc)^3}\frac{(4N^2-1)^2}{N^4}-1\right). & \end{cases} \quad (3.48)$$

Furthermore, t_N^* is increasing in N.

The proof is given in Appendix A.2. The first part of Proposition 4 gives an upper bound for the ordering cost $F \leq F_{max}$, that is, if the ordering cost exceeds this upper bound, there is no positive cycle length and therefore no positive profit. The second part states the existence of a unique optimal cycle length. Finally, it is shown that the more price changes are allowed over an order cycle, the longer is the cycle length and the lower is the order frequency. As N increases, the necessity to compromise on inventory holding costs further reduces (see also the following inter-leaved property) and therefore the overall lot-sizing trade-off between fixed replenishment costs and inventory holding costs shifts in favor of larger replenishments and therefore larger cycles.

Proposition 5. *(Inter-leaved property) If the price-response function is linear in P, then for each $N > 1$ follows:*

$$P_i^*(N+1, t_{N+1}^*) \leq P_i^*(N, t_N^*) \leq P_{i+1}^*(N+1, t_{N+1}^*) \quad \forall i = 1, \cdots, N-1.$$

The proof of Proposition 5 is given in Appendix A.3. This property characterizes the pricing strategy when the number of prices increases. It indicates that if the retailer changes the selling price N times within an order cycle, the i-th selling price is between the i-th and the $(i+1)$-st price of a strategy where $N+1$ price changes are allowed. This form of inter-temporal price discrimination leads to a better trade-off between fixed and holding cost. Chen et al. (2004) show in a numerical example that this inter-leaved property also holds in a single-product-inventory problem with price-sensitive demand following a Brownian motion.

Continuous price adjustment

If the retailer is allowed to change the selling price continuously, then the optimal solution can be obtained by two alternative ways:

1. by applying the optimal control approach presented in Section 3.3.1 or
2. by analyzing the discrete model for $N \to \infty$.

Using (3.20) and (3.18), the optimal price at time t is

$$P^*(t) = \begin{cases} \frac{1}{2}\left(\frac{a}{b} + c + ht\right) & : \quad ht < \frac{a}{b} - c \\ \frac{a}{b} & : \quad else \end{cases} . \tag{3.49}$$

By integrating condition (3.10) in conjunction with the boundary conditions (3.11), the optimal lot-size Q^* and the inventory level at time t are

$$Q^* = \frac{b}{2}\left[\frac{a}{b} - c - \frac{h}{2}T\right]T \tag{3.50}$$

and

$$y(t)^* = \frac{b}{2}\left[\frac{a}{b} - c - \frac{h}{2}(T+t)\right](T-t). \tag{3.51}$$

Using (3.49) in the expression for the optimal profit (3.9), the average profit per time unit becomes

$$\Pi_T^* = \frac{1}{12}bh^2T^2 - \frac{h}{4}(a-bc)T + \frac{(a-bc)^2}{4b} - \frac{F}{T}. \tag{3.52}$$

Since P^* is increasing in t, the constraint $P^*(T) \leq \bar{P}$ gives an upper bound for the optimal cycle length, $T_{max} = \frac{a-bc}{bh}$. T^* is determined from $\frac{\partial \Pi}{\partial T} \stackrel{!}{=} 0$:

$$\frac{1}{6}bh^2T - \frac{h}{4}(a-bc) + \frac{F}{T^2} \stackrel{!}{=} 0. \tag{3.53}$$

Proposition 6. *If the response function is linear in P and the price adjustments occur continuously, there exists a unique $T^* \leq T_{max}$ with $\Pi(T^*) \geq 0$ only if $F \leq \frac{(a-bc)^3}{12b^2h} =: F^{\infty}_{Max}$.*
Furthermore, if $F \leq F^{\infty}_{Max}$, the optimal cycle length T^* is given by:

$$T^* = \begin{cases} -\frac{1}{2}\frac{(a-bc)}{hb}\left(2\cos(\frac{\pi}{3}+\frac{\phi}{3})-1\right) & : F \leq \frac{1}{24}\frac{(a-bc)^3}{b^2h} \\ \quad \phi = arccos\left(1-\frac{24Fhb^2}{(a-bc)^3}\right) & \\ & \\ \frac{1}{2}\frac{(a-bc)}{hb}\left(2\cos(\frac{\pi}{3}+\frac{\phi}{3})+1\right) & : F > \frac{1}{24}\frac{(a-bc)^3}{b^2h} \\ \quad \phi = arccos\left(\frac{24Fhb^2}{(a-bc)^3}-1\right). & \end{cases} \tag{3.54}$$

The proof is similar to Proposition 4 and can alternatively be obtained from analyzing the limiting case $N \to \infty$.

Exponential price-response

A second class of demand functions assumes a nonlinear response to price variations. In the following, we assume an exponential relationship

$$D(P) = \begin{cases} ae^{-bP} & : \quad 0 \geq P \\ 0 & : \quad P < 0 \end{cases}. \tag{3.55}$$

The limiting value $\lim\limits_{P\to\infty} D(P) = 0$, i.e., the reservation price is infinite. For nonlinear price-response functions, in general, we do not obtain closed-form solutions for the optimal prices and the optimal cycle length.

Constant pricing

By inserting (3.55) into (3.35) and (3.39), the optimal constant price and the optimal cycle length are obtained from

$$T^* = \sqrt{\frac{2F}{hae^{-bP^*}}} \quad \text{and} \quad P^* = c + \frac{1}{b} + \frac{h}{2}\sqrt{\frac{2F}{hae^{-bP^*}}}. \tag{3.56}$$

Since a closed-form solution cannot be obtained, the equations have to be solved numerically.

Discrete price changes

In the case of a discrete number of N price adjustments, we can derive an upper bound for the ordering cost similar to the linear case.

Proposition 7. *If the demand decreases exponentially in the price according to (3.55), the profit is positive only if* $F \leq \frac{8a}{hb^2} e^{-(3+bc)}$.

If the ordering cost exceeds the upper bound, it is not beneficial to sell this product at all. The proof and an algorithm for determining the optimal price is given in Appendix A.4. From (3.35) it follows

$$P_i^* = c + \frac{1}{b} + \frac{h}{2}(t_i^* + t_{i-1}^*) \Longrightarrow D_i^* = ae^{-(1+bc+\frac{hb}{2}(t_i^*+t_{i-1}^*))} \tag{3.57}$$

for $i = 1, \cdots, N$. Therefore, the optimal time instants t_i^* in (3.36) for $i = 1, \cdots, N-1$ are given by

$$t_i^* = \frac{\left(\frac{1}{b} + \frac{h}{2}(t_i^* + t_{i-1}^*)\right) e^{-\frac{hb}{2}t_{i-1}^*} - \left(\frac{1}{b} + \frac{h}{2}(t_i^* + t_{i+1}^*)\right) e^{-\frac{hb}{2}t_{i+1}^*}}{h(e^{-\frac{hb}{2}t_{i-1}^*} - e^{-\frac{hb}{2}t_{i+1}^*})} \tag{3.58}$$

$$\Longrightarrow t_i^* = \frac{2}{hb} + \frac{t_{i-1}^* e^{-\frac{hb}{2}t_{i-1}^*} - t_{i+1}^* e^{-\frac{hb}{2}t_{i+1}^*}}{e^{-\frac{hb}{2}t_{i-1}^*} - e^{-\frac{hb}{2}t_{i+1}^*}}. \tag{3.59}$$

The optimal cycle length follows from (3.39) by

$$t_N^* = \sqrt{\frac{1}{ae^{-(1+bc+\frac{hb}{2}(t_N^*+t_{N-1}^*))}} \left(\frac{2F}{h} - \sum_{i=1}^{N-1} (t_i^*)^2 ae^{-(1+bc+\frac{hb}{2}t_i^*)} \left(e^{-\frac{hb}{2}t_{i-1}^*} - e^{-\frac{hb}{2}t_{i+1}^*}\right)\right)}. \tag{3.60}$$

A difference between the linear and nonlinear price-response function is the optimal timing of price adjustments. Proposition 8 characterizes the behavior of the optimal price changing strategy.

Proposition 8. *In an order cycle where* N *price variations are allowed and the price-response follows* $D(P) = ae^{-bP}$*, the length of time intervals* $[t_{i-1}, t_i)$ *where the price* P_i *is charged is non-decreasing in* i*, more specifically*

$$\frac{t_{i-1}}{t_i} \leq \frac{i-1}{i} \qquad \forall i = 1, \cdots, N. \tag{3.61}$$

The proof is given in Appendix A.5. Proposition 8 gives that $t_1 \leq (t_2 - t_1) \leq \cdots \leq (t_i - t_{i-1}) \leq \cdots \leq (t_N - t_{N-1})$. For a linear price-response function we showed that the time interval between two consecutive price changes is equidistant for all points in time, which is equivalent to $\frac{t_1^*}{t_2^*} = \frac{1}{2}, \frac{t_2^*}{t_3^*} = \frac{2}{3}, \cdots, \frac{t_{i-1}^*}{t_i^*} = \frac{i-1}{i}, \cdots, \frac{t_{N-1}^*}{t_N^*} = \frac{N-1}{N}$. In the exponential price-response case $\frac{t_{i-1}^*}{t_i^*} \leq \frac{i-1}{i}$ for all $i = 1, \cdots, N$ implies that the time intervals increase over the order cycle. As an example, let $N = 2$, with $\frac{t_1^*}{t_2^*} \leq \frac{1}{2}$ it follows that $t_1^* \leq \frac{t_2^*}{2} \longrightarrow (t_2^* - t_1^*) \geq \frac{t_2^*}{2}$. This behavior results from the properties of the exponential price-response function. When the price is low, the demand rate responds more sensitive to price variations. An increase in the selling price by one unit at the beginning of the order cycle where the price is low has a larger impact on the demand rate than at the end of the order cycle. This larger impact results in a lower interval length at the beginning of the order cycle.

For the case of an exponential price-response function, the optimal cycle length t_N^* cannot be obtained in closed form. The following algorithm describes an iterative procedure to determine the optimal solution numerically.

Algorithm

The algorithm for $N \geq 2$ works as follows. Equation (3.59) gives that t_i^* depends on its predecessor and successor t_{i-1}^* and t_{i+1}^*, respectively. Therefore, t_1^* only depends on t_2^* because $t_0^* = 0$. From Proposition 8 it follows that $t_{i-1}^* \leq t_i^* \frac{i-1}{i}$

$$\Longrightarrow t_1^* \leq \frac{1}{2} t_2^* \leq \frac{1}{2}\left(\frac{2}{3} t_3^*\right) = \frac{1}{3} t_3^* \leq \frac{1}{3}\left(\frac{3}{4} t_4^*\right) \leq \cdots \leq \prod_{i=2}^{N-1} \frac{i}{i+1} t_N^* \Longrightarrow t_1^* \leq \frac{1}{N} t_N^*.$$

Based on this result, the algorithm starts with an upper for t_1 with $t_1 := \frac{T}{N}$ where $T := t_{N=1}^*$, the optimal cycle length of the constant pricing case. Given t_1, the algorithm successively calculates t_i for $i = 2, \cdots, N$ by using (3.59). After that, the algorithm checks the optimality of this solution by using (3.60). The algorithm stops if for a particular t_1 and the following calculated $t_2, \cdots, t_N$ the equation (3.60) is satisfied (sufficiently exact) .

Furthermore, δ denotes the step size, ε denotes the precision criterion, and Λ and Ψ are auxiliary variables which are set sufficiently large.

REPEAT

  $t_1 := t_1 - \delta$

  FOR i from 2 to N

    $t_i := t_{i-1}$

    REPEAT

      $t_i = t_i + \delta$

      $\Lambda := t_{i-1} - \frac{2}{hb} - \frac{t_{i-2} e^{-\frac{hb}{2} t_{i-2}} - t_i e^{-\frac{hb}{2} t_i}}{e^{-\frac{hb}{2} t_{i-2}} - e^{-\frac{hb}{2} t_i}}$

    UNTIL $|\Lambda| \leq \varepsilon$.

  END FOR

FOR i from 1 to N

Determine P_i and D_i from (3.57) and (3.58).

END FOR

Determine by using (3.38) $\Psi := t_N - \sqrt{\frac{2F}{hD_N} - \frac{1}{D_N}\sum_{i=1}^{N-1}(t_i)^2(D_i - D_{i+1})}$

UNTIL $|\Psi| \leq \varepsilon$.

3.3.4 Numerical example

Consider a linear price-response function $D(P) = a - bP$ with $a = 500$ and $b = 20.5$. The setup cost is $F = 900$, purchasing cost $c = 15$ per unit, and inventory holding cost $h = 1.5$ per unit and time unit. Furthermore, we set the menu cost $\kappa(N) = 0$. This assumption provides that the optimal profit resulting from ($N = \infty$) gives an upper bound with respect to $N < \infty$.

N	Π^*	Q^*	t_N^*	P_{aver}^*
1	-14.45	274.05	4.38	21.34
2	1.05	288.65	4.98	21.25
5	6.39	294.81	5.34	21.22
10	7.23	295.88	5.42	21.21
∞	7.51	296.26	5.45	21.21

Table 3.1: Optimal profit, order quantity, cycle length, and average charged price when the number of price settings is increasing

The results in Table 3.1 and the behavior of the optimal average profit illustrated in Figure 3.4 indicate the potential for improvement even with only a few price adjustments. In this example, the results show that if the retailer optimizes the profit on the basis of constant pricing, the product is not profitable and the loss per time unit is 14.45. With a single price adjustment, the product generates a positive profit of 1.05 per time unit. Furthermore, the additional benefit is decreasing with the number of price changes. The order cycle and the order quantity increase with increasing N. Figures 3.6 and 3.7 illustrate the optimal price strategy and the corresponding cycle length for $N = 1$, 2, 3, and an infinite number of price changes for a linear and an exponential prices response function. In case of the exponential price-response function, we assumed $b = 0.13$. The results indicate that for both linear and exponential price-response function the optimal order interval is increasing in N and that the inter-leaved property holds in this numerical example for an exponential price-response function.

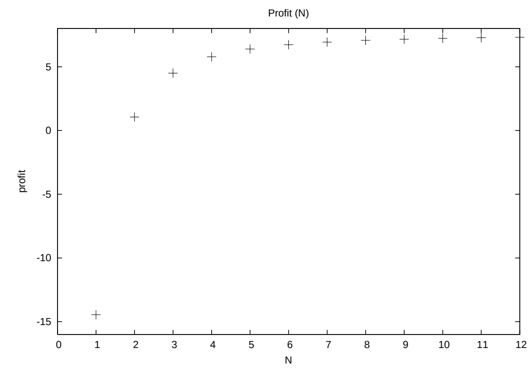

Figure 3.4: Optimal average profit as a function of N

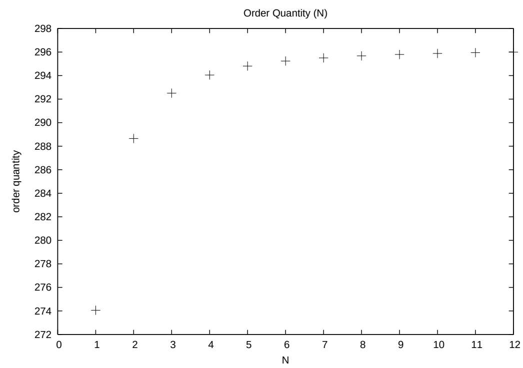

Figure 3.5: Optimal EOQ as a function of N

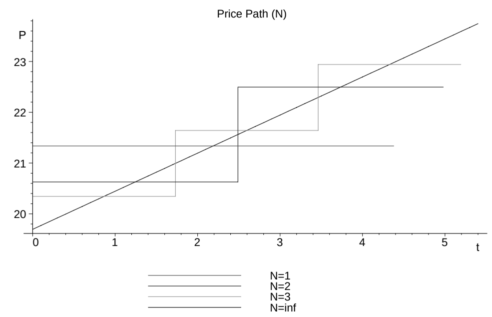

Figure 3.6: Optimal pricing strategies regarding N when the response function is linear in P

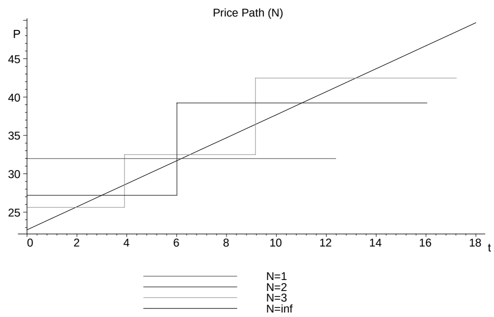

Figure 3.7: Optimal pricing strategies regarding N when the response function is exponential in P

Now we assume a menu cost of $\kappa = 1$ for every price adjustment. The results in Table 3.2 illustrate the effects of parameter variations on the optimal number of price changes, the optimal profit, the optimal order quantity, and the optimal cycle length. On the basis of the given data, we analyze the effect of a variation of setup cost F, market potential a, price-sensitivity b, inventory holding cost h, and procurement cost c.

An intuitive result is that the optimal profit decreases (increases) if the cost parameters F, h, and c increase (decrease). We observe that the optimal number of price changes increases with an increase of any of the cost parameters F, h, and c. Whenever costs increase, there are larger benefits from operational efficiency. With respect to order quantity and cycle length, we observe the same effects as in the traditional EOQ model, i.e., Q and T increase with an increase in F. However, an increase in h does not lead to the same effect as in the traditional EOQ model. As Table 3.2 shows, an increase in h decreases the order quantity, however, the effect on the cycle length is not unidirectional. For small

F	N^*	Π^*	Q^*	t_N^*		a	N^*	Π^*	Q^*	t_N^*
200	2	221.58	151.2	1.84		499	4	0.10	292.6	5.37
800	3	23.00	280.0	4.60		500	4	2.78	294.0	5.29
900	4	2.78	294.0	5.29		510	4	32.27	307.1	4.73
910	4	0.90	295.1	5.37		530	3	102.96	328.8	4.14
920	4	-0.93	296.3	5.45		750	2	1634.62	498.5	2.46

b	N^*	Π^*	Q^*	t_N^*		h	N^*	Π^*	Q^*	t_N^*
10.0	2	2386.62	448.1	2.72		0.60	2	149.11	494.8	6.48
18.0	3	215.53	342.0	3.83		1.00	3	63.76	363.3	5.41
19.5	3	71.90	314.8	4.39		1.30	3	26.73	319.2	5.18
20.2	4	21.16	301.2	4.91		1.50	4	2.78	294.0	5.29
20.5	4	2.78	294.0	5.29		1.51	4	1.09	292.1	5.31
20.6	4	-2.82	291.4	5.48		1.53	4	-0.57	290.2	5.33

c	N^*	Π^*	Q^*	t_N^*
10	2	566.32	489.2	3.82
14	3	81.75	331.4	4.44
15	4	2.78	294.0	5.29
15.1	4	-3.73	289.7	5.51

Table 3.2: Effect of parameter F, a, b, h, and c on N^*, Π^*,Q^*, and t_N^*

holding costs, the cycle length decreases as in the EOQ model. At the same time, the increase of the holding cost results in increasing prices and therefore decreasing demand rates. For significantly large holding costs, the latter effect compensates the former and results in an increasing order cycle. A decreasing demand rate, in turn, leads to a longer cycle length. The same observation can be made concerning the impact of increasing variable procurement costs c on t_N^*. With respect to variable procurement cost c, market potential a, and sensitivity b, we observe that the optimal number of price changes decreases with larger demand rates, that is, the faster a good is moving, the fewer price changes are required. Order quantity and cycle length show the same dependency on demand rate driven by the underlying parameters, that is, Q increases in a and decreases in c and b whereas the optimal length of the order cycle changes in the opposite direction.

3.3.5 Discounted cash-flow analysis

The objective of the majority of EOQ-based problems is to maximize average profit or minimize average costs. However, several authors complain that optimizing the average performance does not explicitly take into account the time value of money and that the *net present value* (NPV) approach, a standard method for using the time value of money to appraise long-term decision problems, is the right framework for studying inventory control and production planning problems. The NPV defines the total discounted cash-flows over a finite or infinite planning horizon. In the following, we will show that the average profit approach (AP) used in the previous section is a reasonable approximation of the NPV approach.

Constant pricing

Let r denote the discount rate. The NPV for a single order cycle of length T is determined by the initial procurement costs, i.e., variable and fixed ordering costs plus the discounted continuous payments over the order cycle. In order to keep the analysis simple and transparent, we omit out-of-pocket holding costs. Then,

$$NPV(T) = -cD(P)T - F + \int_0^T PD(P)e^{-rt}dt.$$

Over an infinite planning horizon, the NPV is the discounted sum of all single-cycle NPVs, i.e.,

$$\begin{aligned} NPV &= \sum_{n=0}^{\infty} NPV(T)e^{-nrT} = NPV(T)\frac{1}{1-e^{-rT}} \\ &= \frac{PD(P)}{r} - \frac{F + cD(P)T}{1-e^{-rT}} \end{aligned} \tag{3.62}$$

The first-order condition of (3.62) w.r.t. T gives

$$\frac{\partial NPV}{\partial T} = 0 \Leftrightarrow cD(P)(e^{rT} - 1) = r(F + cD(P)T). \tag{3.63}$$

In the following, we use a Maclaurin series, which is a special Taylor series expansion of a function about 0, in order to transform the first-order conditions (Bronshtein et al., 2004). The Maclaurin series of a function $w(x)$ is defined as

$$w(x) = w(0) + w'(0)x + \frac{w''(x)}{2!}x^2 + \ldots + \frac{w^{(n)}(x)}{n!}x^n + \ldots.$$

Using a second-order *Maclaurin series* approximation $e^{rT} \approx 1 + rT + \frac{r^2T^2}{2}$ to (3.63), we get

$$T^* = \sqrt{\frac{2F}{rcD(P)}}, \tag{3.64}$$

which is equal to (3.7) for $h = rc$.

The first-order condition of (3.62) w.r.t. P gives

$$\frac{\partial NPV}{\partial P} = 0 \quad \Leftrightarrow \quad \frac{1}{r}\left(D(P) + PD'(P)\right) = \frac{cD'(P)T}{1 - e^{-rT}}, \tag{3.65}$$

$$\Leftrightarrow \quad P + \frac{D(P)}{D'(P)} = \frac{rcT}{1 - e^{-rT}}. \tag{3.66}$$

The first-order Maclaurin series approximation w.r.t. r applied to $\frac{r}{(1-e^{-rT})}$ (see Grubbström and Thorstenson (1986), Corbey et al. (1999), and van der Laan and Teunter (2002)) give

$$\frac{r}{(1 - e^{-rT})} \approx \frac{r}{2} + \frac{1}{T}.$$

Hence, (3.66) can be transformed into

$$P + \frac{D(P)}{D'(P)} = c + \frac{rc}{2}T, \tag{3.67}$$

which is equal to (3.7) for $h = rc$.

Thus, when the holding cost rate is chosen as $h = rc$, the optimal solution, i.e., the optimal cycle length and optimal price of the average profit approach are approximately optimal for the NPV approach.

Dynamic pricing

For the case of dynamic pricing, we exemplarily present the analysis for $N = 2$. The analysis for an arbitrary N follows analogously. For a single order cycle

$$NPV^{(2)}(P_1, P_2, t_1, t_2) = -cQ - F + \int_0^{t_1} P_1 D(P_1) e^{-rt} dt + \int_{t_1}^{t_2} P_2 D(P_2) e^{-rt} dt$$

with $Q = D(P_1)t_1 + D(P_2)(t_2 - t_1)$. Thus, the NPV is determined as

$$NPV = \sum_{n=0}^{\infty} NPV^{(2)}(P_1, P_2, t_1, t_2) e^{-nrt_2} = \frac{1}{(1 - e^{-rt_2})} NPV^{(2)}(P_1, P_2, t_1, t_2). \tag{3.68}$$

The optimal solution has to satisfy the following first-order conditions

$$\frac{\partial NPV}{\partial t_1} = 0 \Leftrightarrow \frac{1}{(1-e^{-rt_2})}\Bigg[-c(D(P_1)-D(P_2)) \\ + (P_1D(P_1)-P_2D(P_2))e^{-rt_1}\Bigg] = 0, \quad (3.69)$$

thus,

$$t_1 = \frac{1}{r}\ln\left(\frac{P_1D(P_1)-P_2D(P_2)}{c(D(P_1)-D(P_2))}\right).$$

Using the Maclaurin series approximation for $\ln(1+x) \approx x$, it follows that

$$t_1 = \frac{(P_1-c)D(P_1)-(P_2-c)D(P_2)}{rc(D(P_1)-D(P_2))}$$

which is equal to (3.36) of the AP approach if $h = rc$. The first-order condition of (3.68) w.r.t. t_2 gives

$$\frac{\partial NPV}{\partial t_2} = 0 \quad \Leftrightarrow \quad \frac{-re^{-rt_2}}{(1-e^{-rt_2})^2}\Pi^{(2)} + \frac{(-cD(P_2)+P_2D(P_2)e^{-rt_2})}{(1-e^{-rt_2})}, \quad (3.70)$$

$$\Leftrightarrow \quad \frac{(e^{rt_2}-1)}{r}(P_2e^{-rt_2}-c)D(P_2) = NPV^{(2)} \quad (3.71)$$

Some algebraic transformations give

$$(P_1D(P_1)-P_2D(P_2))\left(\frac{1-e^{-rt_1}}{r}\right) + cD(P_2)\left(\frac{e^{-rt_1}-1}{r}\right) \\ = F + cD(P_1)t_1 + cD(P_2)(t_2-t_1).$$

Using the Maclaurin series approximation for e^{-rt_1}, and using the condition $(c(D(P_1)-D(P_2)) = (P_1D(P_1)-P_2D(P_2))e^{-rt_1})$ from (3.69), by some algebraic transformation it follows

$$t_2^2 = \frac{2F}{rcD(P_2)} - \frac{2(D(P_1)-D(P_2))}{rD(P_2)}\frac{rt_1^2}{2}.$$

Thus, the optimal cycle length is

$$t_2 = \sqrt{\frac{2F}{rcD(P_2)} - \frac{(D(P_1)-D(P_2))}{D(P_2)}t_1^2} \quad (3.72)$$

which is equal to (3.38) for $N = 2$ if $h = rc$. The first-order conditions of (3.68) w.r.t. P_1 and P_2 give

$$\frac{\partial NPV}{\partial P_1} = 0 \Leftrightarrow P_1 + \frac{D(P_1)}{D'(P_1)} = ct_1 \frac{r}{(1 - e^{-rt_1})}, \tag{3.73}$$

$$\frac{\partial NPV}{\partial P_2} = 0 \Leftrightarrow P_2 + \frac{D(P_2)}{D'(P_2)} = c(t_2 - t_1) \frac{r}{(e^{-rt_1} - e^{-rt_2})}. \tag{3.74}$$

Using the first-order Maclaurin series approximations w.r.t. r (together with *l'Hospital's* rule) $\frac{r}{(1-e^{-rt_1})} \approx \frac{r}{2} + \frac{1}{t_1}$ and $\frac{r}{(e^{-rt_1}-e^{-rt_2})} \approx \frac{1}{(t_2-t_1)}(1 + \frac{r}{2}(t_2 + t_1))$ give that the approximative representations of (3.73) and (3.74) are

$$P_1 + \frac{D(P_1)}{D'(P_1)} = c + \frac{rc}{2} t_1$$

and

$$P_2 + \frac{D(P_2)}{D'(P_2)} = c + \frac{rc}{2}(t_1 + t_2).$$

which are equal to (3.35) for $i = 1, 2$ if $h = rc$.

Thus, the average profit approach is approximately optimal and gives the same order quantity and the same selling price as in the NPV approach when the holding cost rate is chosen as $h = rc$.

3.3.6 Summary and implications

This section analyzed a problem of jointly determining the profit-maximizing pricing and lot-sizing policy with inter-temporal price discrimination in an EOQ framework. Besides providing further evidence for the benefits of dynamic pricing (which admittedly has already been pointed out in more complex environments), we especially show its impact on operational (order quantity and order cycle) decisions. In the general dynamic pricing model, we could not derive closed-form solutions for the optimal decisions. To gain more insights into structural results, we investigated the problem for a linear and an exponential price-response function. For the linear model, we found analytical solutions for the optimal prices, the optimal times where the price is adjusted, and the optimal cycle length, and we have proven that the time intervals where a particular price is charged are equidistant. In case of an exponential price-response, the length of the time intervals where a particular selling price is charged is increasing over the order cycle. In this case, if the price is low, the demand rate responds more sensitive to price variations than at higher prices. Therefore, the price jumps are lower at the beginning of the order cycle than at the end. The overall benefit from a discrete number of price changes in an economic order quantity context is to increase the demand rate when inventories are high such that operational efficiency is increased.

For both linear and exponential price-response and menu cost are equal to zero, the optimal profit is increasing and the marginal revenue is decreasing with the number of price changes. If there is no menu cost for price adjustment, the profit in case of continuous price adjustments is an upper bound. If the menu costs are strictly positive, then there exists an optimal N^* that maximizes the average profit per time unit. Further, it was shown that the optimal cycle length and the order quantity are increasing in the number of price variations. Thus, the order frequency of the retailer is lower if more price adjustments are allowed over an order cycle. In the numerical example, we showed the impact of changing parameters on the optimal number of price changes, order quantity, cycle length, and average profit. We observed that the optimal number of price changes increases with an increase of any of the cost parameters F, h, and c. Furthermore, increasing fixed costs have the same impact on the optimal lot-size and cycle length as in the traditional EOQ model, i.e., Q^* and T^* increase in F. The impact of increasing holding costs is not unique. Two complementary effects occur. On the one hand, increasing holding costs lead to a higher order frequency, just as in the traditional EOQ model. On the other hand, an increasing h leads to increasing prices and decreasing demand rates. These effects, in turn, lead to a longer cycle length. Depending on which effect dominates, the cycle length and the lot-size increase or decrease.

This model is based on several simplifying assumptions, e.g., deterministic environment, a single product, or a monopoly that allowed us to obtain detailed structural insights into the optimal pricing, timing, and inventory policies. The major implication is that using dynamic pricing strategies does not only influence demand but also affects the replenishment policy in such a way that order quantity and order cycle length increase with an increasing number of price changes. In the following, this problem is extended in several ways. In the next section a supplier quantity discount is incorporated. It is investigated how coordinated decision-making where the retailer optimizes price and replenishment policy simultaneously increases the performance compared to a decentralized decision making framework where price and replenishment decisions are made independently.

3.4 Dynamic pricing and replenishment with quantity discounts

This section, extends the model of Section 3.3 to the case where the supplier offers an all-units quantity discount (AQD). It is a common practice that suppliers offer a discount for large order quantities. For this kind of price discrimination (see Section 2.1), a supplier designs a menu of price-quantity pairs and customers select their optimal purchasing volume. Dolan (1987) and Wilcox et al. (1987) provide several reasons for a firm to offer quantity discounts from both a marketing and an operations management point of view. If the supplier faces high setup costs which lead to a large lot-size and high holding costs, quantity discounts may reduce the inventory level immediately after stocking due to larger customer orders. Furthermore, suppliers offer quantity discounts for a better utilization of idle capacity in order to achieve economies of scale in manufacturing. From a marketing perspective, quantity discounts are used to stimulate sales, e.g., Neslin et al. (1995). From a financial point of view, the time value of money is taken into consideration. Because of the offered quantity discount, buyers decide to buy earlier and a larger quantity. Therefore, revenues are available earlier for possible reinvestment (Beranek, 1967).

We compare a decentralized decision framework where first marketing determines the optimal pricing strategy and then operations optimizes the replenishment policy to coordinated decision-making where the retailer decides on pricing strategy and replenishment policy simultaneously. Hereby, we distinguish between two pricing strategies. In case of constant pricing, the retailer determines the optimal selling price that is constant over an infinite planning horizon. In case of dynamic pricing, the retailer varies the selling price over time. We analyze the benefits from coordinated dynamic pricing and replenishment compared to coordinated constant pricing and replenishment and to decentralized decision-making. We assume that the number of price changes over an order cycle is given. However, in order to determine the optimal number of price changes, the results of Section 3.3.2 can be used. There it is shown how dynamic pricing can enhance operational efficiency by increasing the demand rate when inventories are high. The benefits of exploiting supply quantity discounts significantly depend on whether variable purchasing price reductions can offset additional holding costs from ordering minimum required quantities. Therefore it appears promising that the use of dynamic pricing even stronger influences operational efficiencies than in case without a quantity discount. Furthermore, this section extends Eliashberg and Steinberg (1993) who compare sequential and simultaneous optimization of lot-size and (constant) selling price without quantity discounts. They show that the optimal selling price in the case of simultaneous optimization is larger than in the case of sequential optimization which, in turn, results in a lower order frequency. This property does not hold necessarily if the supplier offers a quantity discount.

We develop optimization models for three different decision frameworks, the *decentralized framework* where marketing and operations optimize independently, the *coordinated-constant framework* where the retailer optimizes a constant price and the order quantity simultaneously, and the *coordinated-dynamic framework* where the retailer employs a finite number of price changes over an order cycle. We provide analytical properties of the objective functions and present algorithms for determining the optimal replenishment policy and price strategy for the *coordinated-constant* and the *coordinated-dynamic framework*, respectively.

3.4.1 Model formulation

We follow the assumptions of the EOQ model described in Section 3.2. Additionally, it is assumed that the supplier offers a regular purchasing price c_0 per unit and an AQD schedule with $l = 0, \ldots, L$ different purchasing prices where the discount is r_l percent on c_0 per unit if the order quantity is larger than or equal to a breakpoint quantity $\bar{Q}_l$. The AQD policy with multiple breakpoints is characterized by a vector

$$\{(r_0, \bar{Q}_0), (r_1, \bar{Q}_1), \cdots, (r_L, \bar{Q}_L) \mid r_0 < r_1 < \cdots < r_L, \bar{Q}_0 < \bar{Q}_1 < \cdots < \bar{Q}_L\}$$

with $r_0 = \bar{Q}_0 = 0$. Let $c_l := (1 - r_l)c_0$ be the reduced procurement price for a unit if the order quantity $Q \in [\bar{Q}_l, \bar{Q}_{l+1})$ with $c_0 > c_1 > \cdots > c_L$ and $\bar{Q}_{L+1} = \infty$. Inventory holding costs depend, among others, on the cost of capital that, in turn, depends on the purchase price c_l and are denoted by h_l per unit and unit of time. We assume that h_l is an increasing function of c_l.

3.4.2 Decentralized decision-making

Assume that the selling price and the purchasing strategy are determined by separated decision-making units. First, marketing optimizes the selling price and generates customer demand. Then, given this demand rate, operations determines the optimal replenishment policy taking into account the supplier's quantity discount. Marketing does not take into account fixed ordering costs and does not anticipate purchasing price discounts as a result of order quantities because the discount actually applied is unknown until the operations decision is taken. Marketing's objective function is as follows:

$$\tilde{\Pi}(P) = (P - c_0)D(P). \tag{3.75}$$

The optimal selling price $\tilde{P}^*$ is obtained from the first-order condition $P + \frac{D(P)}{D'(P)} = c_0$. Given the demand rate $\tilde{D}^* = D(\tilde{P}^*)$, operations minimizes average costs of replenishment and inventory taking into account the supplier quantity discount.

Hadley and Whitin (1963) developed a two-stage algorithm in order to determine the optimal replenishment policy. A first stage iteratively calculates the constrained economic order quantity $\tilde{Q}_l^*$ and the resulting costs starting from the highest discount c_L until the first index l_0 is found where the solution satisfies $\tilde{Q}_{l_0}^* \geq \bar{Q}_{l_0}$ and $\tilde{Q}_l^* < \bar{Q}_l$ for all $l > l_0$. Thus,

$$\tilde{Q}_{l_0}^* = \sqrt{\frac{2FD(\tilde{P}^*)}{h_{l_0}}} \geq \bar{Q}_{l_0} \quad \text{and} \quad \tilde{C}_{l_0}^* = \sqrt{2Fh_{l_0}D(\tilde{P}^*)}. \tag{3.76}$$

At a second stage, the cost of this inner solution $\tilde{C}_{l_0}^*$ is compared with the costs of all breakpoint quantities larger than $\tilde{Q}_{l_0}^*$, i.e., $\tilde{C}_l(\bar{Q}_l)$ for $l = l_0+1, ..., L$ with

$$\tilde{C}_l(\bar{Q}_l) = c_l D(\tilde{P}^*) + \frac{h_l}{2}\bar{Q}_l + F\frac{D(\tilde{P}^*)}{\bar{Q}_l}.$$

Thus, the optimal profit in the case of decentralized decision-making is as follows:

$$\tilde{\Pi}^* = (\tilde{P}^* - c_0)D(\tilde{P}^*) - \min\{\tilde{C}_{l_0}^*, \tilde{C}_l(\bar{Q}_l) \mid l = l_0+1, ..., L\}.$$

3.4.3 Coordinated decision-making - constant pricing

In case of coordinated decision-making, we simultaneously optimize lot-size Q and selling price P. The optimization problem for a particular purchasing price c_l is given by

$$\Pi_l^*(P,Q) = \max_{P,Q} \left((P - c_l)D(P) - \frac{h_l}{2}Q - F\frac{D(P)}{Q} \right) \tag{3.77}$$

$$s.t. \qquad Q \geq \bar{Q}_l. \tag{3.78}$$

A relaxation of (3.78) and differentiating (3.77) with respect to Q and P yields the necessary first-order conditions for an inner solution (e.g., see Whitin (1955) and Eliashberg and Steinberg (1993)). Solving (3.79) and then determining $T^* = \sqrt{\frac{2F}{h_l D(P_l^*)}}$ leads to the same solution as solving (3.7) for $c = c_l$ and then determining $Q^* = \sqrt{\frac{2FD(P_l^*)}{h_l}}$.

$$Q_l^* = \sqrt{\frac{2FD(P_l^*)}{h_l}} \quad \text{and} \quad P_l^* + \frac{D(P_l^*)}{D'(P_l^*)} = c_l + \frac{F}{Q_l^*} \tag{3.79}$$

where the optimal selling price P_l^*, conditional that the purchasing price is c_l, is represented as an implicit function of the optimal order quantity. For a particular purchasing price c_l, the optimal order quantity is only feasible if $\bar{Q}_l \leq Q_l^*$. Given this constraint, a transformation of Q_l^* from (3.79) yields that an inner solution P_l^* is only feasible if P_l^* is lower than a break price $\bar{P}_l$ for a given unit purchasing cost c_l:

$$P_l^* \leq D^{-1}\left(\frac{h_l \bar{Q}_l^2}{2F}\right) =: \bar{P}_l \tag{3.80}$$

where $D^{-1}(.)$ denotes the inverse of the price-response function. The inverse function indeed exists due to the fact that $D(P)$ is a strictly monotone function. Using this result, the AQD policy can be characterized by

$$\{(r_0, \bar{P}_0), (r_1, \bar{P}_1), \cdots, (r_L, \bar{P}_L) \mid r_0 < r_1 < \cdots < r_L, \bar{P}_0 > \bar{P}_1 > \cdots > \bar{P}_L\}$$

where $\bar{P}_0 = \bar{P}$. Substituting the optimal order quantity into (3.77), the two-variable problem is reduced to a single-variable problem that only depends on P:

$$\Pi_l(P) = (P - c_l)D(P) - \sqrt{2Fh_l D(P)}, \tag{3.81}$$
$$s.t. \quad P \leq \bar{P}_l. \tag{3.82}$$

Properties of an optimal pricing and lot-sizing policy

The following properties characterize the profit function (3.81).

Proposition 9. *For $P \geq 0$, (3.81) is either a concave-convex function or a strictly concave function of P with $\lim_{P \to 0} \Pi_l(P) < 0$ and $\lim_{P \to \bar{P}} \Pi_l(P) = 0$.*

The proof of Proposition 9 is given in Appendix A.6. Abad (1988a) reduces (3.77) to a single-variable problem that only depends on the order quantity Q. He shows that if the first-order condition with respect to P yields a closed-form solution $P^*(Q)$, as it is the case for linear and iso-elastic price-response functions, the reduced profit function is a convex-concave function of Q and develops a procedure to determine the optimal price and lot-size. Based on this result, Abad (1988a) gives an algorithm to determine the overall optimal lot-size. Note that we do not need the requirements of having a closed-form solution for $P^*(Q)$.

Proposition 10. *For an arbitrary fixed selling price P, (3.81) is a strictly decreasing function of c_l. Therefore, the profit functions Π_l and $\Pi_{l'}$ for different unit purchasing costs c_l and $c_{l'}$ do not intersect and $\Pi_l(P) > \Pi_{l'}(P)$ for all P and $c_l < c_{l'}$.*

The proof follows from the first partial derivative of (3.81) with respect to c_l and the assumption that h_l increases in c_l. With this, it is easy to verify that $\frac{\partial \Pi_l}{\partial c_l} < 0$. Implicitly, it follows that the profit function $\Pi_l(P) > \Pi_{l'}(P)$ for $c_l < c_{l'}$.

Theorem 1. *Let l_0 be the largest index of a discount where the local optimum $Q^*_{l_0}$ is feasible, i.e., $\bar{Q}_{l_0} \leq Q^*_{l_0}$ and $\bar{Q}_l > Q^*_l$ for all $l = l_0 + 1, \cdots, L$. Then for all $l < l_0$, $\Pi_l(P) < \Pi_{l_0}(P^*_{l_0})$ for all P.*

The proof follows directly from Propositions 9 and 10 with the implication that all discounts that are lower than the discount r_{l_0} can be omitted from the determination of the optimal solution. If $\Pi_l(P^*_l) < 0$ for all c_l, then $Q^* = 0$. To find the optimal value of P, we have to find the profit maximizing price in each interval $(\bar{P}_{l+1}, \bar{P}_l]$ and compare these profits to determine the global optimum.

For the following illustration we assume that the supplier offers a single price break $\bar{Q}_1$. Then, there are 3 cases for the optimal price P^*.

1. The free local optimum P^*_1 for the reduced purchasing price c_1 is a feasible solution, i.e., $P^*_1 \leq \bar{P}_1$.
2. The free local optimum P^*_1 is infeasible and the breakpoint profit (the retailer orders the breakpoint quantity $\bar{Q}_1$ at the reduced purchasing price c_1) is larger than the optimal profit given the regular purchasing price c_0, $\Pi_1(\bar{P}_1) > \Pi_0(P^*_0)$.
3. The free local optimum P^*_1 is infeasible and the profit where the retailer orders the breakpoint quantity is lower than the optimal profit for the regular purchasing price, $\Pi_1(\bar{P}_1) < \Pi_0(P^*_0)$.

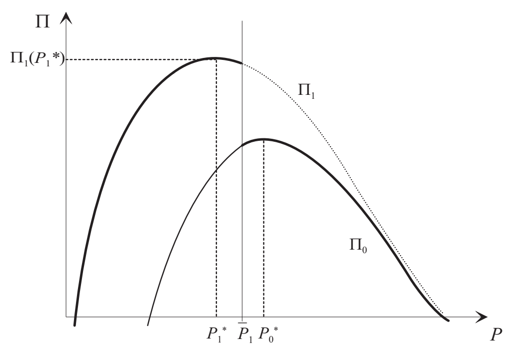

Figure 3.8: Average profit curve: Case 1

In Case 1, as shown in Figure 3.8, the breakpoint price $\bar{P}_1$ is large enough such that the optimal selling price P^*_1 is a feasible solution. Figures 3.9 and 3.10 illustrate the Cases 2 and 3 where P^*_1 is an infeasible solution. In Case 2, the breakpoint profit is larger than the optimal profit for the regular purchasing price c_0 and in Case 3 the opposite holds. If $\Pi_1(\bar{P}_1) = \Pi_0(P^*_0)$, the retailer is indifferent between ordering $\bar{Q}_1$ or Q^*_0. The bold lines represent the feasible profit curves. Propositions 9 and 10 and Theorem 1 allow the following algorithm for calculating the optimal selling price and lot-size.

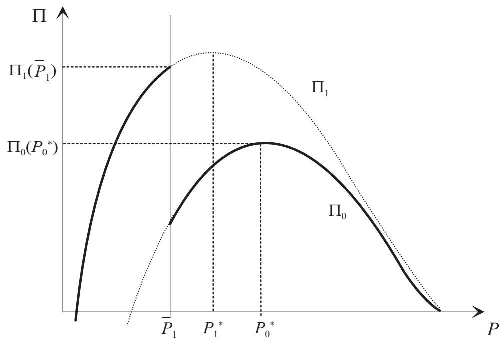

Figure 3.9: Average profit curve: Case 2

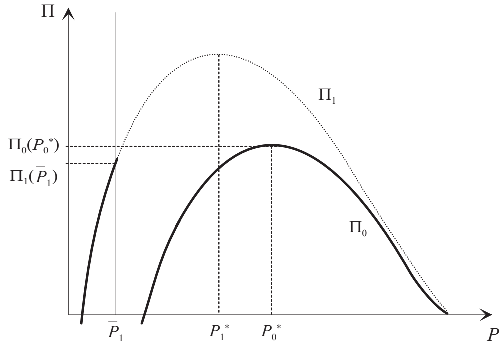

Figure 3.10: Average profit curve: Case 3

Algorithm

Set $l := L$,

REPEAT

Calculate the optimal selling price P_l^* by solving the first-order condition $\frac{\partial \Pi_l}{\partial P} \stackrel{!}{=} 0$,

$$\frac{D(P)}{D'(P)} + P - c_l - \sqrt{\frac{Fh_l}{2D(P)}} = 0,$$

calculate $\Pi_l^* := \Pi_l(\bar{P}_l)$,
$l := l - 1$,

UNTIL $P_{l+1}^* < \bar{P}_{l+1}$.

(The loop stops at the index l_0, the largest index for which $P_{l_0}^* \leq \bar{P}_{l_0}$, i.e., the local optimum $P_{l_0}^*$ is an inner solution.)
Calculate $\Pi_{l_0}^* := \Pi_{l_0}(P_{l_0}^*)$.

The optimal profit is determined from

$$\Pi^* = \begin{cases} \max\{\Pi_l^* \mid l \in \{l_0, \ldots, L\}\} & : \text{ if at least one } \Pi_l^* \geq 0 \\ 0 & : \text{ otherwise} \end{cases}.$$

Let $l^* := \operatorname{argmax}\{\Pi_l^*\}$. Then the optimal price and the optimal order quantity are given by

$$P^* = \begin{cases} \operatorname{argmax}\{\Pi_{l^*}^*(P)\}\} & : \Pi^* \geq 0 \\ \bar{P} & : \text{ otherwise} \end{cases},$$

and

$$Q^* = \begin{cases} \sqrt{\frac{2FD(P^*)}{h_{l^*}}}, & \text{if } P^* = P^*_{l_0} \\ \bar{Q}_l, & \text{if } P^* = \bar{P}_l \end{cases} .$$

END

The important property for this algorithm to work is that the profit curves do not intersect which gives that the profit function for a discount rate r_l is located above of profit functions for discount rates $r_{l'}$ with $l' < l$.

Impact of coordinated decision-making on pricing and replenishment

This section demonstrates differences in the decisions between decentralized (sequential) and coordinated (simultaneous) decision-making on pricing and replenishment strategy. From Section 3.2.2 it is known that for a particular discount rate r_l, a comparison of the first-order condition of decentralized and coordinated decision-making gives

$$\frac{D(\tilde{P})}{D'(\tilde{P})} + \tilde{P} = c_0 \quad \text{and} \quad \frac{D(P)}{D'(P)} + P = c_l + \frac{F}{Q}. \tag{3.83}$$

Under decentralized decision-making, the optimal selling price $\tilde{P}^*$ is only based on the regular purchasing costs c_0. However, in the coordinated case, the price P^* takes into account both overhead costs and an eventually beneficial quantity discount.

Proposition 11. *If $D(P)$ has an increasing price elasticity and the retailer does not use a quantity discount ($l = 0$), then the optimal selling price in case of decentralized decision-making is lower than under simultaneous optimization, i.e., $\tilde{P}^* < P^*$ and $D(\tilde{P}^*) > D(P^*)$. If $l > 0$, the coordinated case yields a higher price in comparison to the decentralized system if the optimal coordinated selling price satisfies $P^*_l > D^{-1}\left(\frac{Fh_l}{2(c_0-c_l)^2}\right)$.*

The proof is given in Appendix A.7. Coordinated decision-making is influenced by two contrary effects, the *overhead cost effect* and the *discount effect*. The overhead cost effect implies that a coordinated decision-making takes overhead costs into account. These costs are disregarded in case of decentralized decision-making so that a coordinated decision-making yields that the selling price increases compared to decentralized decision-making. The discount effect, on the other hand, implies that marketing does not take into account that operations might use a quantity discount. This, in turn, leads to an overestimation of costs and thus to a higher price in case of decentralized decision-making compared to coordinated decision-making. Whether the optimal selling price is larger or smaller compared to the decentralized case depends on which effect dominates.

3.4.4 Coordinated decision-making - dynamic pricing

As in Section 3.3.2, we assume that the retailer can change the selling price N times, at times t_i, $i = 0, \ldots, N-1$ within each order cycle and each price change is subject to administrative costs. Like in Section 3.3.2, this problem can be formulated by a two-stage optimization problem. However, the optimal procurement price has additionally to be chosen implicitly by choosing the optimal order quantity. Using (3.34), the two-stage optimization problem can be formulated as follows

$$\Pi^* = \max_N \left[\max_{l, \mathbf{P}_N, \mathbf{t_N}} \left\{ \Pi_l^{(N)} - \kappa(N) \right\} \right], \tag{3.84}$$

with

$$\Pi_l^{(N)} = \frac{1}{t_N} \left[\sum_{i=1}^{N} \left(P_i - c_l - \frac{h_l}{2}(t_i + t_{i-1}) \right) D_i \left(t_i - t_{i-1} \right) - F \right]. \tag{3.85}$$

$$s.t. \qquad P_i \leq \bar{P} \qquad \forall \qquad i = 1, \ldots, N \tag{3.86}$$

$$\sum_{i=1}^{N} D_i (t_i - t_{i-1}) \geq \bar{Q}_l. \tag{3.87}$$

Equation (3.85) represents the average retailer profit given c_l, a price vector $\mathbf{P}_N = (P_1, \ldots, P_N)$ and a timing vector $\mathbf{t}_N = (t_1, \ldots, t_N)$. Constraint (3.86) ensures that the selling prices do not exceed the reservation price and by constraint (3.87) the lot-size must be larger than or equal to the breakpoint $\bar{Q}_l$ to ensure that the used unit price is attained.

Properties of an optimal pricing and lot-sizing policy

In order to maximize the average profit, we differentiate (3.34) with respect to P_i and t_i, $i = 1, \cdots, N$. The first-order conditions with respect to P_i give

$$\frac{\partial \Pi_l^{(N)}}{\partial P_i} = \frac{1}{t_N} \left[D_i \left(t_i - t_{i-1} \right) + \left(P_i - c_l - \frac{h_l}{2}(t_i + t_{i-1}) \right) D_i' \left(t_i - t_{i-1} \right) \right] \stackrel{!}{=} 0,$$

$$\Leftrightarrow P_i + \frac{D_i}{D_i'} \stackrel{!}{=} c_l + \frac{h_l}{2} \left(t_i + t_{i-1} \right). \tag{3.88}$$

Equation (3.88) corresponds to (3.35) when there is no quantity discount. The optimal price P_i^* in a particular time interval $[t_{i-1}, t_i)$ for $i = 1, \cdots, N$ must satisfy (3.88) and depends on the initial and final point of this interval. For price-response functions with a price-elasticity that is non-decreasing in price, the optimal selling price P_i^* is non-decreasing in i. From Section 3.3.2 it follows that for a given c_l and a given N, the optimal pricing policy and the optimal inventory development over an order cycle is as illustrated in Figures 3.11 and 3.12. The first-order condition of $\Pi_l^{(N)}$ with respect to t_i for $i = 1, ..., N-1$ gives

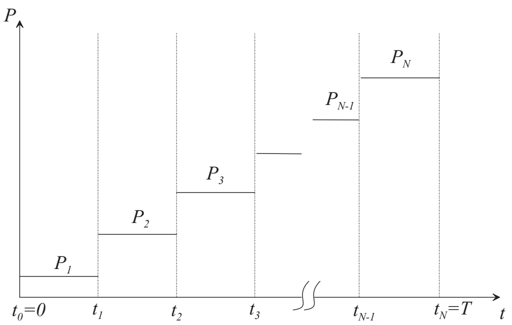

Figure 3.11: Illustration of the pricing policy

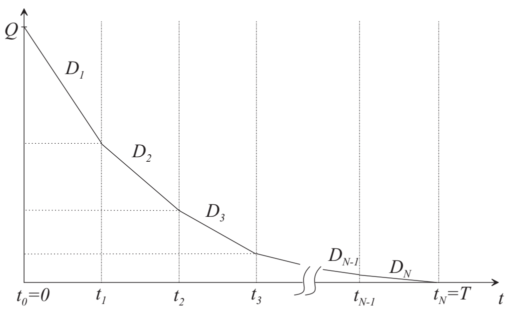

Figure 3.12: Illustration of inventory level

$$\frac{\partial \Pi_l^{(N)}}{\partial t_i} = \frac{1}{t_N} \left[(P_i - c_l)D_i - (P_{i+1} - c_l)D_{i+1} - h_l t_i (D_i - D_{i+1}) \right] \stackrel{!}{=} 0.$$

Therefore, the optimal times t_i^* must satisfy

$$t_i^* = \frac{(P_i^* - c_l)D_i - (P_{i+1}^* - c_l)D_{i+1}^*}{h_l(D_i^* - D_{i+1}^*)} \qquad i = 1, \cdots, N-1. \tag{3.89}$$

(3.89) corresponds to (3.36) when there is no quantity discount. From (3.89) we find that the optimal time t_i^* is a function of the optimal price P_i^* of the current interval, the optimal price P_{i+1}^* of the next interval, and the corresponding optimal demand rates. As shown in Section 3.3.2, the optimal prices and the optimal demand rates can be reduced to functions that only depend on the cycle length t_N, e.g., $(P_i^*(t_N), D_i^*(t_N))$, and the maximization problem reduces to the following single-variable problem:

$$\Pi_l^{(N)}(t_N) := \Pi_l^{(N)}(P_1^*(t_N), \cdots, P_N^*(t_N), t_1^*(t_N), \cdots, t_{N-1}^*(t_N), t_N) =$$

$$\frac{1}{t_N}\left[\sum_{i=1}^{N}\left(P_i^*(t_N) - c_l - \frac{h_l}{2}(t_i^*(t_N) + t_{i-1}^*(t_N))\right) D_i^*(t_N)(t_i^*(t_N) - t_{i-1}^*(t_N)) - F\right]. \tag{3.90}$$

Given linear price-response functions, this iterative procedure yields closed-form expressions for $P_i^*(t_N)$ for all $i = 1, \ldots, N$. In the general case, the dependencies have to be solved numerically. As for the static pricing problem, the profit function $\Pi_l^{(N)}(t_N)$ has to possess some particular characteristics that enable us to find the optimal solution efficiently.

Conjecture 1. *For an arbitrary feasible t_N $(0 \le P_i^*(t_N) \le \bar{P}, \forall i = 1, \cdots, N)$, (3.90) is a decreasing function of c_l. Therefore, if the profit function is concave, quasi-concave, or concave-convex, then $\Pi_l^{(N)}(t_N)$ and $\Pi_{l'}^{(N)}(t_N)$ do not intersect for $c_l \neq c_{l'}$, i.e., $\Pi_l^{(N)}(t_N) > \Pi_{l-1}^{(N)}(t_N)$ for all $l = 1, \ldots, N-1$.*

For general price-response functions, the proof of Conjecture 1 is analytically intractable. However, in the next section we show for linear price-response functions that Conjecture 1 holds. The shape of (3.90) depends on the price-response function. If $D(P)$ is such that (3.90) is concave, quasi-concave, or concave-convex, similar to the constant pricing case, for a fixed c_l, the following algorithm can be applied to determine the optimal solution. Otherwise, a complete enumeration over all inner solutions and breakpoints has to be used in order to determine the optimal solution.

This algorithm resembles the algorithm for the classical EOQ average cost minimization problem with an all-units quantity discount. It takes into account the interdependencies of the determination of the optimal order quantity, the optimal prices in the N time intervals, and the optimal times where the price is adjusted. In Step 1, we solve the constrained optimization problems backwards, starting with the highest discount rate r_L until an unconstrained solution is feasible for the first time. In Step 2, we compare the various breakpoint profits for L to l_0+1 to the first free optimal solution and identify the optimal discount rate r_l^* where the profit reaches the maximum. In Step 3, we determine the resulting optimal cycle length and the optimal order quantity.

Algorithm

Set $l := L$,

REPEAT

Solve the constrained optimization problem as in Section 3.3.2:

$$\max_{t_N} \quad \{\Pi_l^{(N)}(t_N)\} \tag{3.91}$$

$$s.t. \quad P_i(t_N) \leq \bar{P} \quad \forall i = 1, \cdots, N. \tag{3.92}$$

$l := l - 1$

UNTIL $\sum_{i=1}^{N} D_i^*(t_N^*)(t_i^*(t_N) - t_{i-1}^*(t_N)) \geq \bar{Q}_{l+1}$.

Let l_0 be the largest index for which the free optimum

$$Q_{l_0}^* = \sum_{i=1}^{N} D_i^*(t_N^*)(t_i^*(t_N^*) - t_{i-1}^*(t_N^*))$$

is a feasible solution, i.e., $Q_{l_0}^* \geq \bar{Q}_{l_0}$. Calculate the optimal profit

$$\Pi_{l_0}^{(N)^*} := \Pi_{l_0}^{(N)}[P_1^*(t_N^*), \cdots, P_N^*(t_N^*), t_1^*(t_N^*), \cdots, t_{N-1}^*(t_N^*), t_N^*]$$

and the breakpoint profits

$$\Pi_l^{(N)^*} := \Pi_l^{(N)}[P_1^*(t_{N_l}), \cdots, P_N^*(t_{N_l}), t_1^*(t_{N_l}), \cdots, t_{N-1}^*(t_{N_l}), t_{N_l}]$$

for all l from $l_0 + 1$ to L where t_{N_l} denotes the optimal cycle length given that the retailer orders the breakpoint quantity $\bar{Q}_l$.

The optimal profit is determined from

$$\Pi^{(N)^*} = \begin{cases} \max\{\Pi_l^{(N)^*} \mid l \in \{l_0, \ldots, L\}\} & : \text{ if at least one } \Pi_l^{(N)^*} \geq 0 \\ 0 & : \text{ otherwise} \end{cases}.$$

Let $l^* = \text{argmax}\{\Pi_l^{(N)^*}\}$. For the case $\Pi^{(N)^*} > 0$, the optimal cycle length $\hat{t}_N^*$ is either t_N^* if $l^* = l_0$ or t_{N_l} if $l^* > l_0$ and the optimal order quantity is determined by

$$Q^* = \sum_{i=1}^{N} D_i^*(\hat{t}_N^*)(t_i^*(\hat{t}_N^*) - t_{i-1}^*(\hat{t}_N^*)).$$

Otherwise, it is not optimal to order, i.e., $Q^* = 0$.

END

3.4.5 Linear price-response function

In this section, we detail the previous results for the special case of a linear price-response function. A detailed analysis of the decentralized framework and the coordinated-constant pricing framework for linear price-response functions is also given in Eliashberg and Steinberg (1993) and Abad (1988a). Let

$$D = D(P) = \begin{cases} a - bP & : \quad 0 \leq P \leq \frac{a}{b} \\ 0 & : \quad P > \frac{a}{b} \end{cases} \tag{3.93}$$

where a represents the market potential and the reservation price is $\bar{P} = \frac{a}{b}$. Using the results of Section 3.3, the optimal times t_i^* can be represented as a function of t_N

$$t_i^*(t_N) = i\frac{t_N}{N}. \tag{3.94}$$

Moreover, the optimal prices and demand rates are characterized by

$$\begin{aligned} P_i^*(t_N) \quad &= \tfrac{1}{2}\left(\tfrac{a}{b} + c_l + \tfrac{h_l}{2}(2i-1)\tfrac{t_N}{N}\right), & (3.95) \\ \Longrightarrow D_i^*(t_N) \quad &= \tfrac{b}{2}\left(\tfrac{a}{b} - c_l - \tfrac{h_l}{2}(2i-1)\tfrac{t_N}{N}\right), & (3.96) \end{aligned}$$

and by inserting (3.94), (3.95), and (3.96) into (3.90), the profit function for a given purchase price c_l can be transformed into the single-variable function

$$\Pi_l^{(N)}(t_N) = \frac{b}{4}\left[\left(\frac{a}{b} - c_l\right)\left(\frac{a}{b} - c_l - h_l t_N\right) + \frac{h_l^2 t_N^2}{12}\frac{(4N^2-1)}{N^2}\right] - \frac{F}{t_N}. \tag{3.97}$$

For a detailed derivation of (3.97) see Appendix B.2.

Proposition 12. *For an arbitrary t_N, (3.97) is a decreasing function of c_l. Therefore, the profit functions $\Pi_l^{(N)}(t_N)$ and $\Pi_{l'}^{(N)}(t_N)$ do not intersect for $c_l \neq c_{l'}$, i.e., $\Pi_l^{(N)}(t_N) > \Pi_{l-1}^{(N)}(t_N)$ for all $l = 1, \ldots, N-1$.*

Proposition 13. *If $D = D(P)$ is linear, the profit function $\Pi_l^{(N)}(t_N)$ is a concave-convex function of t_N.*

The proofs of Proposition 12 and 13 are given in Appendix A.8 and A.9. With the results of Proposition 12 and 13, all requirements to use the following Algorithm are satisfied.

Algorithm:

Step (1):
Solve the constrained optimization problem (3.91)-(3.92) from $l = L$ until the largest index l_0 is identified where $\bar{Q}_l \leq Q_{l_0}^* < \bar{Q}_{l_0+1}$ by using the *Lagrangian*

Multiplier method. We introduce the Lagrangian multipliers λ_1 and λ_2^i for $i = 1, \cdots, N$ related to the minimum order quantity constraint and the reservation price constraints for all charged selling prices over an order cycle. The Lagrangian function $\mathcal{L}_l$ for a particular purchasing cost c_l is defined by:

$$\mathcal{L}_l(P_1, \cdots, P_N, t_N, \lambda_1, \lambda_2^i) = \frac{1}{t_N}\left[\sum_{i=1}^{N}\left(P_i - c_l - \frac{h_l}{2}(2i-1)\frac{t_N}{N}\right)(a - bP_i)\frac{t_N}{N} - F\right]$$

$$+ \lambda_1\left[\frac{t_N}{N}\sum_{i=1}^{N}(a - bP_i) - \bar{Q}_l\right] + \sum_{i=1}^{N}\lambda_2^i\left[\frac{a}{b} - P_i\right],$$

with the *Karush-Kuhn-Tucker* conditions

$$\frac{\partial \mathcal{L}_l}{\partial P_i} \leq 0 \qquad P_i^*\frac{\partial \mathcal{L}_l}{\partial P_i} = 0 \quad \forall i = 1, \ldots, N$$

$$\frac{\partial \mathcal{L}_l}{\partial t_N} \leq 0 \qquad t_N^*\frac{\partial \mathcal{L}_l}{\partial t_N} = 0$$

$$\frac{\partial \mathcal{L}_l}{\partial \lambda_1} \geq 0 \qquad \lambda_1^*\frac{\partial \mathcal{L}_l}{\partial \lambda_1} = 0$$

$$\frac{\partial \mathcal{L}_l}{\partial \lambda_2^i} \geq 0 \qquad \lambda_2^{i*}\frac{\partial \mathcal{L}_l}{\partial \lambda_2^i} = 0 \quad \forall i = 1, \ldots, N$$

$$\lambda_1 \geq 0,\ \lambda_2^i \geq 0 \quad \forall i = 1, \ldots, N.$$

We seek the price vector $P_1^*, P_2^*, \ldots, P_N^*$ and the cycle length t_N^* that satisfy all these conditions. The partial derivation with respect to P_i for all $i = 1, \ldots, N$, and the partial derivation with respect to t_N give

$$\frac{\partial \mathcal{L}_l}{\partial P_i} = \frac{1}{N}\left[a - bP_i + (P_i - c_l - \frac{h_l}{2}(2i-1)\frac{t_N}{N})(-b) - \lambda_1 b t_N - \lambda_2^i N\right] \stackrel{!}{=} 0,$$

$$\Leftrightarrow P_i(t_N,\, \lambda_1,\, \lambda_2^i) = \frac{1}{2}\left(\frac{a}{b} + c_l + \frac{h_l}{2}(2i-1)\frac{t_N}{N} - \lambda_1 t_N - \lambda_2^i N\right), \tag{3.98}$$

$$\frac{\partial \mathcal{L}_l}{\partial t_N} = \frac{F}{t_N^2} - \frac{h_l}{2}\left(a - \frac{b}{N^2}\sum_{i=1}^{N}(P_i(2i-1))\right) + \lambda_1\left(a - \frac{b}{N}\sum_{i=1}^{N}P_i\right) \stackrel{!}{=} 0,$$

$$\Leftrightarrow t_N(P_1,\ldots,P_N,\,\lambda_1) = \sqrt{\frac{2F}{h_l\left(a-\frac{b}{N^2}\sum\limits_{i=1}^{N}(P_i(2i-1))\right)-2\lambda_1\left(a-\frac{b}{N}\sum\limits_{i=1}^{N}P_i\right)}}. \tag{3.99}$$

Subject to the *complementary slackness condition*, we obtain the first-order condition

$$\frac{\partial \mathcal{L}_l}{\partial \lambda_1} = \frac{t_N}{N}\sum_{i=1}^{N}(a-bP_i) - \bar{Q}_l \overset{!}{\geq} 0. \tag{3.100}$$

The two constraints (3.100) and $\lambda_1 \geq 0$ are complementary inequalities, i.e., $\frac{\partial \mathcal{L}_l}{\partial \lambda_1} > 0$ if and only if $\lambda_1 = 0$. Thus, if $\lambda_1 > 0$, it must be satisfied that $\frac{\partial \mathcal{L}_l}{\partial \lambda_1} = 0$. The value of λ_1 can be interpreted as *shadow price* that values a decrease of the breakpoint quantity $\bar{Q}_l$ by one unit. Equation (3.98) indicates that if $h_l > 0$, the optimal selling price is strictly increasing over the order cycle. Therefore, the reservation price constraints must not be binding for $i = 1,\ldots,N-1$, that is, $\lambda_2^i = 0$ for $i = 1,\ldots,\,N-1$. The only reservation price that may constrain the problem is for $P_N(t_N) \leq \bar{P}_N$:

$$\frac{\partial \mathcal{L}_l}{\partial \lambda_2^N} = \frac{a}{b} - P_N \overset{!}{\geq} 0. \tag{3.101}$$

A similar shadow price interpretation holds for λ_2^N. Here, (3.101) and $\lambda_2^N \geq 0$ are complementary inequalities, i.e., $\lambda_2^N > 0$ if and only if $P_N = \frac{a}{b}$. Substituting P_N from (3.98) into (3.101) gives an upper bound for the optimal cycle length

$$t_N \leq \frac{\frac{a}{b} - c_l + \lambda_2^N N}{h_l(\frac{2N-1}{N} - \lambda_1)} = t_{N_l}. \tag{3.102}$$

For determining the optimal solution $(P_1^*,\ldots,P_N^*,t_N^*,\lambda_1^*,\lambda_2^{N^*})$, we solve the system of equations and inequalities (3.98), (3.99), (3.100), and (3.102) under the condition $\lambda_1^* \geq 0$ and $\lambda_2^{N^*} \geq 0$.

Step (2) and Step (3) follow straight just as described in Section 3.4.4.

3.4.6 Numerical example

The following numerical example shows the benefit of coordinated planning of price and order quantity compared to a decentralized planning approach and the impact of the ratio of fixed and holding cost, price-sensitivity of customers, and the supplier's quantity discount. The demand follows the linear structure $D(P) = a - bP$. The parameters are set to $a := 1000$, $c_0 := 5$, and $h_l := 0.02c_l$. The parameters F, $\bar{Q}_1$, r_1, and b are varied in order to show the impact on lot-sizing and pricing. In the case of a dynamic pricing strategy, we assume that the retailer changes the selling price twice in each order cycle, i.e., $N = 2$.

As a base case, we consider $b = 100$ and $r_1 = 15\%$ if $Q \geq 4000 =: \bar{Q}_1$ units. Furthermore, we distinguish $F = 0$ and $F = 500$. Tables 3.3 and 3.4 show the optimal price and lot-sizing decision as well as the optimal cycle length and the optimal average profit for the three frameworks and both purchasing options regular ($l = 0$) and discounted ($l = 1$). Column "Loss" illustrates the loss of profit compared to the coordinated-dynamic framework.

Framework		$F = 0$				
	l	Q_l^*	T_l^*	P_l^*	Π_l^*	Loss
decentral	0	0	0.0	7.5	**625**	0.8%
	1	4000	16.0	7.5	600	
coordinated-	0	0	0.0	7.5	625	
constant	1	4000	13.9	7.1	**627**	0.5%
coordinated-	0	0	0.0	7.5	625	
dynamic	1	4000	(7.0,14.0)	(6.9,7.3)	**630**	

Table 3.3: Results of the base case ($F = 0$, $b = 100$, $\bar{Q}_1 = 4000$, $r_1 = 0.15$)

Framework		$F = 500$				
	l	Q_l^*	T_l^*	P_l^*	Π_l^*	Loss
decentral	0	1414	5.6	7.5	448	
	1	4000	16.0	7.5	**568**	3.9%
coordinated-	0	1361	5.8	7.7	451	
constant		1 4000	14.2	7.1	**588**	0.5%
coordinated-	0	1374	(2.9,5.9)	(7.6,7.8)	452	
dynamic	1	4000	(7.1,14.2)	(7.0.7.4)	**591**	

Table 3.4: Results of the base case ($F = 500$, $b = 100$, $\bar{Q}_1 = 4000$, $r_1 = 0.15$)

If $F = 500$ and the retailer does not use the quantity discount ($l = 0$), we verify the findings of Eliashberg and Steinberg (1993) that decentralized decision-making leads to an underestimation of price and an overestimation of lot-size compared to coordinated decision-making ($7.5 < 7.7$ and $1414 > 1361$). However, this is not necessarily true if the supplier offers a quantity discount. If the retailer uses the quantity discount ($l = 1$), for both $F = 0$ and $F = 500$ it can be observed that the optimal selling price decreases from the decentralized to a coordinated-constant framework ($7.5 > 7.1$) whereas the order-quantity remains the same. In case of decentralized decisions and $F = 0$, the optimal purchasing strategy is a just-in-time strategy with the regular purchasing cost

c_0 ($\Pi_0^* = 625 > 600 = \Pi_1^*$) and an optimal selling price $P_0^* = 7.5$. However, if the retailer optimizes price and purchasing policy simultaneously, the discount strategy becomes more profitable. That is, the *discount* effect dominates the *overhead* effect. It is optimal to accept additional holding costs in order to save variable purchasing costs. Due to the larger lot-size which would result in higher holding costs, it becomes more beneficial to decrease the selling pricing in order to increase the demand rate. In case of $F = 500$, the lot-sizing decision is equal for the decentralized and the coordinated framework. However, in the coordinated framework a lower selling price leads to a higher order frequency (a higher demand rate leads to a lower cycle length) which results in reduced holding costs. Furthermore, it can be seen that the benefit of dynamic pricing compared to coordinated constant pricing for both $F = 0$ and $F = 500$ is lower than 1%. The same observation can be made when we compare the dynamic-coordinated framework with the decentralized framework for the case $F = 0$. If $F = 0$, the benefit of coordinated-dynamic pricing increases to 3.9% per time unit.

Framework	QD	$F = 0$				
	l	Q_l^*	T_l^*	P_l^*	Π_l^*	Loss
decentral	0	0	0.0	7.5	625	
	1	3000	12.0	7.5	**653**	3.4%
coordinated-	0	0	0.0	7.5	625	
constant	1	3000	10.4	7.1	**674**	0.3%
coordinated-	0	0	0.0	(7.5.7.5)	625	
dynamic	1	3000	(5.2,10.5)	(6.9,7.3)	**676**	

Table 3.5: Impact of a lower breakpoint quantity Q_1 ($F = 0$, $b = 100$, $\bar{Q}_1 = 3000$, $r_1 = 0.15$)

Framework	QD	$F = 500$				
	l	Q_l^*	T_l^*	P_l^*	Π_l^*	Loss
decentral	0	1414	5.6	7.5	448	
	1	3000	12.0	7.5	**611**	2.2%
coordinated-	0	1361	5.8	7.7	451	
constant	1	3000	10.7	7.2	**624**	0.2%
coordinated-	0	1374	(2.9,5.9)	(7.6,7.7)	452	
dynamic	1	3000	(5.3,10.7)	(7.0,7.4)	**625**	

Table 3.6: Impact of a lower breakpoint quantity Q_1 ($F = 500$, $b = 100$, $\bar{Q}_1 = 3000$, $r_1 = 0.15$)

Tables 3.5, 3.6, 3.7, and 3.8 illustrate the impact of the discount policy on the optimal solution. Tables 3.5 and 3.6 show the results of a decreasing breakpoint

Framework	QD	$F = 0$				
	l	Q_l^*	T_l^*	P_l^*	Π_l^*	Loss
decentral	0	0	0.0	7.5	625	
	1	4000	16.0	7.5	**670**	6.0%
coordinated-	0	0	0.0	7.5	625	
constant	1	4000	13.3	7.0	**711**	0.3%
coordinated-	0	0	0.0	(7.5,7.5)	625	
dynamic	1	4000	(7.6,13.4)	(6.9,7.1)	**713**	

Table 3.7: Impact of a higher discount rate r_1 ($F = 0$, $b = 100$, $\bar{Q}_1 = 4000$, $r_1 = 0.2$)

Framework	QD	$F = 500$				
	l	Q_l^*	T_l^*	P_l^*	Π_l^*	Loss
decentral	0	1414	5.6	7.5	448	
	1	4000	16.0	7.5	**643**	4.5%
coordinated-	0	1361	5.8	7.7	451	
constant	1	4000	13.6	7.0	**670**	0.4%
coordinated-	0	1374	(2.9,5.9)	(7.6,7.7)	452	
dynamic	1	4000	(6.8.13.6)	(6.9,7.3)	**673**	

Table 3.8: Impact of a higher discount rate r_1 ($F = 500$, $b = 100$, $\bar{Q}_1 = 4000$, $r_1 = 0.2$)

quantity and Tables 3.7 and 3.8 the results of an increasing discount rate. Both changes yield an increasing desirability of the quantity discount. While the benefit of the coordinated-dynamic framework compared to the coordinated-constant framework remains low, the benefit compared to the decentralized framework does increase. Compared to the base case $F = 0$ where a JIT policy was optimal, now the quantity discount is beneficial such that the retailer orders the breakpoint quantity.

Tables 3.9 and 3.10 illustrate the impact of price-sensitivity on the optimal decision and the average profit. Compared to the base case, an increasing price-sensitivity leads to a decreasing selling price among all decision frameworks. For $F = 0$, across all decision frameworks, it is optimal to follow a JIT strategy. Therefore, decentralized decision-making yields the same performance as coordinated decision-making. But in case of $F = 500$ the application of coordinated-dynamic pricing is highly beneficial compared to decentralized and coordinated-constant decision-making.

Framework	QD	$F = 0$				
	l	Q_l^*	T_l^*	P_l^*	Π_l^*	Loss
decentral	0	0	0.0	6.0	**161**	0.0%
	1	4000	26.0	6.0	61	
coordinated-	0	0	0.0	6.0	**161**	0.0%
constant	1	4000	19.0	5.7	118	—
coordinated-	0	0	0.0	(6.0,6.0)	**161**	
dynamic	1	4000	(10.0,20.0)	(5.5,6.0)	124	

Table 3.9: Impact of a larger price-sensitivity of customers b ($F = 0$, $b = 140$, $\bar{Q}_1 = 4000$, $r_1 = 0.15$)

Framework	QD	$F = 500$				
	l	Q_l^*	T_l^*	P_l^*	Π_l^*	Loss
decentral	0	1095	7.3	6.0	23	
	1	4000	26.0	6.0	**42**	55.8%
coordinated-	0	951	8.4	6.3	32	
constant	1	4000	20.0	5.7	**88**	7.4%
coordinated-	0	987	(4.5,9.0)	(6.2,6.5)	35	
dynamic	1	4000	(11.0,21.0)	(5.5,6.1)	**95**	

Table 3.10: Impact of a larger price-sensitivity of customers b ($F = 500$, $b = 140$, $\bar{Q}_1 = 4000$, $r_1 = 0.15$)

3.4.7 Summary and implications

This section considered an economic order quantity (EOQ) model where the supplier offers an all-units quantity discount (AQD) and the retailer faces a price-dependent demand rate. Three different decision frameworks were analyzed, a decentralized decision-making strategy, a coordinated strategy with a constant selling price, and a coordinated strategy where the retailer is allowed to implement a finite number of price adjustments within each order cycle. For the coordinated frameworks, we derived analytical properties and developed efficient algorithms to determine the optimal price and lot-sizing strategy. These algorithms are applicable to all concave, quasi-concave, and concave-convex profit functions. In a numerical example, we showed the difference in decision-making between the decentralized, the coordinated-constant, and the coordinated-dynamic framework. It could be observed that in the case of low price-sensitivity of the customers, the benefit of dynamic pricing compared to constant pricing is rather low. However, with increasing price-sensitivity, coordinated decision-making and, in particular, dynamic pricing becomes more beneficial.

Without a supplier quantity discount, the coordinated framework with a constant pricing strategy yields a higher price, a lower demand rate, and a lower order quantity compared to sequential pricing and replenishment (see Eliashberg and Steinberg (1993)). If the supplier offers an all-units quantity discount, these properties do not necessarily hold. In the discount case, two effects influence the outcome of decision-making. The first effect is the *overhead cost effect*. In contrast to decentralized decision-making, a central decision maker takes into account all relevant costs (setup cost and inventory holding cost). This yields an increasing selling price, a decreasing demand rate, and a decreasing lot-size. The second effect is the *discount effect*. Here, under decentralized decision-making, marketing does not consider the quantity discount schedule of the supplier and optimizes the selling price only using the undiscounted purchasing price. Thus, marketing overestimates operations cost which yields a higher selling price than in the case of coordinated decision-making. However, by taking into account the supplier quantity discount, it might be optimal to order a larger lot-size at a reduced purchasing price. Therefore, an increase or decrease of the selling price, the demand rate, and the order quantity depends on which effect dominates. The overhead cost effect yields an underestimation of the average cost in the decentralized system, whereas the discount effect overestimates the average cost.

A natural extension of the model would be to consider a joint dynamic pricing and lot-sizing determination problem where the supplier offers an incremental units discount. An extension regarding supply chain coordination is to the design of a quantity discount schedule by the supplier if it is anticipated that the retailer employs decentralized decision-making, coordinated decision-making with a constant price, or coordinated decision-making with a dynamic pricing framework.

The following section studies a multi-product dynamic pricing and replenishment problem with limited storage capacity. The objective of this problem is to coordinate pricing strategy, order size, and delivery schedule of multiple products in order to maximize the average profit. Like in this section, a comparison of a decentralized decision-making strategy where marketing and operations optimize selling prices and the replenishment policy sequentially, and a coordinated decision-making where constant and dynamic pricing is distinguished, is made.

3.5 Dynamic pricing and replenishment in the warehouse scheduling problem

A fundamental problem in inventory management is the planning and scheduling of bottleneck operations. Many industries face replenishment problems of multiple products that share a warehouse with limited storage capacity. This section studies the interplay between pricing and procurement decisions of multiple products that share a common warehouse with limited storage capacity.

The capacitated multi-product replenishment problem with the objective to determine the replenishment and staggering policy of all products and where the selling price is exogenous is widely investigated in academia. Gallego et al. (1992) show that such problems are NP-complete. Several heuristics were developed to overcome this complexity. In the *common cycle* or *rotation cycle* approach, it is assumed that all products have the same order cycle length and are ordered once in the order cycle. This heuristic is called common cycle policy, because all products are replenished for the same cycle length.

The objective in the following is to maximize the average profit by choosing the optimal pricing strategy, the optimal lot-size for each product, and the optimal staggering of the order-releases under the common-cycle assumption. We compare differences in performance between a decentralized and a coordinated decision framework. In the decentralized framework, marketing and operations optimize sequentially. Here, marketing decides first on the sales price for each product and then, operations optimizes the replenishment of multiple products. In the coordinated framework, the firm decides on the selling price and the replenishment strategy for all products simultaneously. Hereby, we distinguish between two pricing strategies. In the case of a constant pricing strategy, the retailer determines the optimal selling price that is constant over the entire planning horizon. In the case of a dynamic pricing strategy, it is assumed that the retailer is allowed to adjust the selling price continuously.

For single-product inventory problems without capacity constraints it has been shown in Section 3.3 that the retailer achieves operational efficiency by dynamic pricing. In the case of multiple products and limited storage capacity, it is expected that the retailer can benefit further from such a dynamic pricing strategy.

3.5.1 Model formulation

Let K denote the number of products $k = 1, \ldots, K$. For each product and each time t, the retailer faces a known demand rate $D^k(t)$ that depends solely on the current price $P^k(t)$, i.e., the customers are myopic and the effects of forward buying in expectation of rising prices or postponement in expectation of declining prices are not incorporated. The demand rate for each product k is modeled by a price-response function $D^k := D(P^k)$. The reservation price is denoted by $\bar{P}^k$. For simplicity, we assume that the demand of one product does not depend on the pricing strategy of the other products. This simplification allows us to derive some structural properties analytically.

Following the assumptions of the EOQ model, for each k the retailer places replenishment orders in batches of size Q^k. With the release of any single batch there is an associated fixed ordering cost F^k and a variable procurement cost c^k per unit. Products delivered but not yet sold are kept in inventory subject to a holding cost h^k per unit and unit of time. Backorders are not permitted. Assuming that the order intervals of all products have a common length, this assumption of a common-cycle strategy implies that each product is replenished exactly once in each order cycle. Further, we assume that the sequence of replenishments for the K products has been predetermined. Note that in the classical WSP with given demand rates the sequence is irrelevant which might not necessarily hold here. Furthermore, we consider that the products share a common warehouse with limited and constant storage capacity level S. The storage requirement of each product is defined as the volume s^k displaced by one unit of product k that is held in inventory. The inventory level at time t is denoted by $y^k(t)$.

3.5.2 Decentralized decision-making

Consider the following decentralized framework where pricing and replenishment decisions occur sequentially. First, marketing decides on the sales price for each product only on the basis of direct cost without consideration of overhead costs (inventory holding and fixed ordering costs). It solves a monopolist pricing problem with a given cost structure. Then, operations solves a WSP with given demand rates on the basis of overhead costs and takes into account the limited storage capacity. For all products $k = 1, \dots, K$, marketing maximizes the following independent profit functions:

$$\max_{P^k} \quad \Pi^k(P^k) = (P^k - c^k)D^k, \tag{3.103}$$

$$s.t. \quad 0 \leq P^k \leq \bar{P}^k. \tag{3.104}$$

For marketing, the operations decisions are unknown and there is no anticipation of the relevant setup and holding costs. For every product k, the optimal sales price P^{k*} is obtained from the first-order condition

$$\frac{D^k}{(D^k)'} + P^k = c^k \tag{3.105}$$

where $(D^k)'$ is defined as $\frac{\partial D^k}{\partial P^k}$.

Based on the optimal price P^{k*} resulting from (3.105), the associated demand rate D^{k*} is the input parameter for the operations optimization that decides on lot-sizes for each product such that it minimizes the average cost consisting of ordering costs and inventory holding costs under a limited storage capacity.

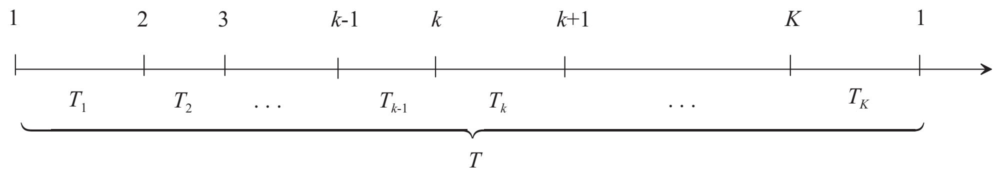

Figure 3.13: Replenishment sequence

Figure 3.13 illustrates a replenishment sequence. Besides the order quantities and the selling prices, in the common cycle approach additional decision variables are the time intervals between two replenishments. T_j denotes the time between the replenishment of product j and $j+1$ and the cycle length is determined by $T = T_1 + \cdots + T_K$. Let y_j^k define the inventory level of product k when product j is ordered. We obtain the following constrained optimization problem:

$$\min_{T,T_1,\cdots,T_K} \mathrm{C} = \sum_{k=1}^{K} \left(\frac{h^k}{2} T D^{k*} + \frac{F^k}{T} \right) \tag{3.106}$$

$$s.t. \quad \sum_{k=1}^{K} s^k y_j^k \leq S \quad j = 1, \ldots, K. \tag{3.107}$$

Constraint (3.107) ensures that the required storage capacity does not exceed the capacity limit S at that time where a product is replenished. That is, when we consider a fixed order cycle with the replenishment sequence $1, \ldots, K$, the current inventory level for all products k is the demand until the next order or, in other words, the last batch size minus the demand since the order arrived. That is,

$$y_j^k = D^{k*} \left(T - \sum_{i=k}^{j-1} T_i \right) \qquad k = 1, \ldots, j-1, \tag{3.108}$$

$$y_j^j = D^{j*} T = Q^j \qquad k = j, \tag{3.109}$$

$$y_j^k = D^{k*} \sum_{i=j}^{k-1} T_i \qquad k = j+1, \ldots, K. \tag{3.110}$$

This problem is analyzed among others by Page and Paul (1976) and Hall (1988) with the result that if an optimal replenishment and staggering policy is used, the peak storage requirement over an order cycle is equal for all replenishment points and the minimum required storage capacity S_{min} that is needed to follow a replenishment strategy with a common cycle length T is given by

$$S_{min} = \frac{\sum_{k=1}^{K} \sum_{l=1}^{k} s^k s^l D^{k*} D^{l*}}{\sum_{k=1}^{K} s^k D^{k*}} T. \tag{3.111}$$

If the capacity constraint is not binding, the optimal common cycle length and the optimal order quantities are

$$T^*_{uncon} = \sqrt{\frac{2 \sum_{k=1}^{K} F^k}{\sum_{k=1}^{K} h^k D^k}} \qquad \text{and} \qquad Q^{k*} = D^{k*} T^*_{uncon} \tag{3.112}$$

and in the case of a binding constraint

$$T^*_{con} = \frac{\sum_{k=1}^{K} s^k D^{k*}}{\sum_{k=1}^{K} \sum_{l=1}^{k} s^k s^l D^{k*} D^{l*}} S \qquad \text{and} \qquad Q^{k*} = D^{k*} T^*_{con}. \tag{3.113}$$

3.5.3 Coordinated decision-making - constant pricing

In the case of coordinated decision-making, we optimize the lot-sizes Q^k, the selling prices P^k, and time intervals between two orders simultaneously. For each product, the retailer charges a constant selling price over the entire planning horizon, e.g., $P^k(t) = P^k$. Therefore, $D^k(t) = D^k$ is constant for $k = 1, \ldots, K$.

This can be modeled by the following constrained optimization problem:

$$\max_{\vec{P},\vec{T},T} \Pi = \sum_{k=1}^{K} \left((P^k - c^k)D^k - \frac{h^k}{2} D^k T - \frac{F^k}{T} \right) \tag{3.114}$$

$$s.t. \quad \left[\sum_{k=1}^{j-1} s^k D^k (T - \sum_{i=k}^{j-1} T_i) + s^j D^j T + \sum_{k=j+1}^{K} s^k D^k \sum_{i=j}^{k-1} T_i \right] \leq S \qquad j = 1, \ldots, K \tag{3.115}$$

$$\sum_{j=1}^{K} T_j = T \tag{3.116}$$

$$0 \leq P^k \leq \bar{P}^k \qquad k = 1, \ldots, K. \tag{3.117}$$

$\vec{P}$ and $\vec{T}$ denote the vector of the selling prices and the vector of the time intervals between order releases, respectively. The objective function (3.114) contains the objective functions (3.103) and from the decentralized framework (3.106). Constraints (3.115) and (3.116) are the same as in the cost-minimization model for the decentralized framework. The difference between (3.107) and (3.115) is that the inventory level does not only depend on the time intervals T_i, but also depends on the charged selling price through the demand rates. Constraint (3.117) ensures that the charged selling prices and the demand rates are nonnegative.

For solving this problem, the *Lagrangian Multiplier* method and the *Karush-Kuhn-Tucker* conditions provide an efficient solution approach. We assume that all products are profitable, e.g., it follows $0 < P^k < \bar{P}^k$ for $k = 1, \ldots, K$. The Lagrangian function is given by

$$L(\vec{P},\vec{T},T) = \Pi(\vec{P},\vec{T},T) + \sum_{j=1}^{K} \mu_j \left(S - \left[\sum_{k=1}^{j-1} s^k D^k (T - \sum_{i=k}^{j-1} T_i) + s^j D^j T + \sum_{k=j+1}^{K} s^k D^k \sum_{i=j}^{k-1} T_i \right] \right). \tag{3.118}$$

The KKT conditions are $\frac{\partial L}{\partial P^k} = 0$, $\frac{\partial L}{\partial T_k} = 0$ for all $k = 1, \ldots, K$,

$$\mu_j \left(S - \left[\sum_{k=1}^{j-1} s^k D^k (T - \sum_{i=k}^{j-1} T_i) + s^j D^j T + \sum_{k=j+1}^{K} s^k D^k \sum_{i=j}^{k-1} T_i \right] \right) = 0$$

for $j = 1, \ldots, K$, which represent the *complementary slackness* conditions, and $\mu_j \geq 0$ for $j = 1, \ldots, K$. Moreover, $T = \sum_{k=1}^{K} T_k$.

The first-order condition of L with respect to a particular P^k gives:

$$\frac{D^k}{(D^k)'} + P^k = c^k + \frac{h^k}{2}T + s^k \Psi(\mu_1, \ldots, \mu_K, T_1, \ldots, T_K) \tag{3.119}$$

where

$$\Psi(\mu_1, \ldots, \mu_K, T_1, \ldots, T_K) := \left(\sum_{j=1}^{k-1} \mu_j \left(\sum_{i=j}^{k-1} T_i \right) + \mu_k T + \sum_{j=k+1}^{K} \mu_j \left(T - \sum_{i=k}^{j-1} T_i \right) \right) \geq 0. \tag{3.120}$$

The derivation of (3.119) is shown in Appendix B.3. It is easy to see that the function $\Psi(\mu_1, \ldots, \mu_K, T_1, \ldots, T_K)$ is an increasing function in μ_j for $j = 1, \ldots, K$.

A comparison of (3.105) and (3.119) gives that coordinated decision-making leads to a higher selling price than the decentralized framework. This results from the underestimation of the relevant cost in the decentralized system when marketing neglects setup costs and inventory holding costs. Furthermore, the capacity constraint yields a further increasing selling price. If the capacity constraint is not binding for all replenishment points j, the shadow prices are zero and the optimal selling prices are equal to the optimal selling prices in the single-product optimization problem, see Eliashberg and Steinberg (1993). If the capacity constraint is binding at least at one replenishment time, the function $\Psi(\mu_1, \ldots, \mu_K, T_1, \ldots, T_K)$ is strictly positive. It follows that $P^{l*}_{(decentral)} \leq P^{l*}_{(coordination)}$ and $P^{l*}_{(uncapacitated)} \leq P^{l*}_{(capacitated)}$.

Proposition 14. *If the retailer follows an optimal replenishment and pricing policy under the common cycle assumption, the inventory immediately after a replenishment is filled up to an equal capacity level.*

For the proof, see Appendix A.10. That is, the property that the peak storage requirement at any replenishment point is equally also holds for the joint pricing and replenishment problem. This property yields that the capacity constraints (3.115) reduce to a single constraint as in the decentralized model. Therefore, we can develop a solution procedure similar to that in Page and Paul (1976) and Hall (1988).

Solution procedure

In a first step, this procedure checks whether the storage capacity constraint is binding, i.e., we check the unconstrained solution for feasibility. If the unconstrained solution is infeasible, the optimal solution is given by the cycle length that provides a binding capacity constraint.

- Determine the optimal unconstrained solution. From (3.112) follows

$$T^*_{uncon} = \sqrt{\frac{2\sum_{k=1}^{K} F^k}{\sum_{k=1}^{K} h^k D(P^{k*})}} \iff (T^*_{uncon})^2 \sum_{k=1}^{K} h^k D(P^{k*}) - 2\sum_{k=1}^{K} F^k = 0. \tag{3.121}$$

  Due to the unconstrained case, $\Psi(\mu_1, \ldots, \mu_K, T_1, \ldots, T_K)$ (from (3.119)) is zero. The unconstrained solution is feasible if

$$T^*_{uncon} \leq \frac{\sum_{k=1}^{K} s^k D^{k*}}{\sum_{k=1}^{K}\sum_{l=1}^{k} s^k s^l D^{k*} D^{l*}} S \tag{3.122}$$

  where $D^{k*} = D(P^{k*}(T^*_{uncon}))$.

- If (3.122) is not satisfied, determine the constrained optimal solution T_{con} that provides a binding capacity constraint (3.122) where $D^{k*} = D(P^{k*}(T^*_{con}))$. The Lagrangian function is given by

$$L(T, \vec{P}) = \Pi(T, \vec{P}) + \mu \left(S \sum_{k=1}^{K} s^k D^k - T \sum_{k=1}^{K}\sum_{l=1}^{k} s^k s^l D^k D^l \right). \tag{3.123}$$

  The KKT conditions are

$$\frac{\partial L}{\partial P^k} = 0 \Leftrightarrow \frac{D^k}{(D^k)'} + P^k = c^k + \frac{h^k}{2} T + \mu \left((s^k)^2 D^k + \sum_{l=1}^{K} s^k s^l D^l \right) \tag{3.124}$$

  for all $k = 1, \ldots, K$,

$$\frac{\partial L}{\partial T} = 0 \Leftrightarrow \sum_{k=1}^{K} \left(\frac{F^k}{T^2} - \frac{h^k}{2} D^k \right) = \mu \sum_{k=1}^{K}\sum_{l=1}^{k} s^k s^l D^k D^l, \tag{3.125}$$

  and

$$\frac{\partial L}{\partial \mu} = 0 \Leftrightarrow S \sum_{k=1}^{K} s^k D^k = T \sum_{k=1}^{K}\sum_{l=1}^{k} s^k s^l D^k D^l. \tag{3.126}$$

From (3.124) it can be seen that the selling price of product k is linear in T and depends on the prices of all other products. Therefore, the solution can only be obtained by solving this system of non-linear equations (3.124) - (3.126) numerically.

Eliashberg and Steinberg (1993) show that in the single-product inventory problem the correct anticipation of the operations cost function ($C_{EOQ} = \sqrt{2FDh}$) in the marketing decision provides the centrally optimal decision even under sequential decision-making. This nice result does no longer hold under a joint capacity constraint since the optimal pricing decision also includes the opportunity cost of using scarce warehouse capacity.

3.5.4 Coordinated decision-making - dynamic pricing

The retailer is allowed to change the selling price continuously over an order cycle. The resulting continuous time optimization problem is:

$$\max_{\vec{P},\vec{T},T} \Pi = \frac{1}{T}\sum_{k=1}^{K}\left[\int_0^T \{(P^k(t) - c^k)D^k(t) - h^k y^k(t)\}\mathrm{d}t - F^k\right] \tag{3.127}$$

$$s.t.\, \dot{y}^k(t) = -D^k(t) \qquad k = 1,\ldots,K \tag{3.128}$$

$$y^k(0) = \int_0^T D^k(t)\mathrm{d}t,\ y^k(T) = 0,\ y^k(t) \geq 0 \qquad k = 1,\ldots,K \tag{3.129}$$

$$\left[\sum_{k=1}^{j-1} s^k \int_{\sum_{i=k}^{j-1} T_i}^{T} D^k(t)\mathrm{d}t + s^j \int_0^T D^j(t) + \sum_{k=j+1}^{K} s^k \int_{T-\sum_{i=j}^{k-1} T_i}^{T} D^k(t)\mathrm{d}t \leq S\right.$$

$$j = 1,\ldots,K \tag{3.130}$$

$$\sum_{j=1}^{K} T_j = T \tag{3.131}$$

$$0 \leq P^k(t) \leq \bar{P}^k \qquad k = 1,\ldots,K. \tag{3.132}$$

The objective (3.127) is to maximize the average profit over all products where for each product the individual profit is the revenue minus the procurement costs, the inventory holding costs, and the ordering costs over the order cycle divided by the cycle length. The optimal selling prices for each product k and each point in time t represent the control variables which are piecewise continuously differentiable functions of t. The price trajectories $P^k(t)$ for $k = 1, \ldots, K$ influence the objective both directly through revenues and indirectly through the impact on the state variables. The state variables are the inventories $y^k(t)$ of each k. The state transition at time t for each state variable $y^k(t)$ is given by (3.128). This constraint is called *state transition equation*. Constraints (3.129) give the initial and the terminal condition for the state variables, e.g., at the beginning of an order cycle the inventory level is equal to the order quantity and at the end, the inventory level is zero (*zero-inventory-property*). Furthermore, they prohibit backordering. Constraints (3.130), (3.131), and (3.132) have the same interpretation as (3.115) - (3.117) in the constant pricing approach in Section 3.5.3.

The problem is solved by a two-stage solution procedure that consists of a non-linear master-problem (first-stage) that uses the results of several continuous time optimization problems (second-stage).

Solution approach

- Second-stage: For a given first-stage decision, divide the order cycle into $j = 1, \ldots, K$ subintervals such that a particular subinterval j corresponds to the time between ordering product j and $j + 1$. By this partition we can decompose the optimal control problem into several subproblems that can be solved by standard methods of optimal control theory. For each product k in each subinterval j of length T_j we find the price and the associated inventory trajectory with initial and final inventory levels $y^k_{0_j}$ and $y^k_{T_j}$. Subsequently, we solve K^2 optimal control problems that give us the total profit for product k in subinterval j, $\Pi^k_j(T_j,\, y^k_{0_j},\, y^k_{T_j})$, as a function of T_j, $y^k_{0_j}$ and $y^k_{T_j}$.
- First-stage: The master problem connects the subproblems and optimizes the lengths T_j and the initial and final inventories $y^k_{0_j}$ and $y^k_{T_j}$ for each $j, k = 1, \ldots, K$, anticipating the profit impact by using the functions Π^k_j from the second-stage optimal control problems. We solve the master problem with respect to inventory levels and the lengths of the subintervals such that the order cycle length is the sum of all subinterval lengths and at the beginning of each subinterval where a product is ordered the required storage volume does not exceed the available storage capacity.

Optimal control problems $j, k = 1, \cdots, K$

The optimal control problem for subinterval j and product k is given by

$$\max_{P_j^k} \quad \Pi_j^k = \int_0^{T_j} \{(P_j^k(t) - c^k) D_j^k(t) - h^k y_j^k(t)\} \mathrm{d}t \tag{3.133}$$

$$s.t. \quad \dot{y}_j^k(t) = -D_j^k(t), \tag{3.134}$$

$$y_j^k(0) = y_{0_j}^k, \; y_j^k(T_j) = y_{T_j}^k, \tag{3.135}$$

$$0 \leq P_j^k(t) \leq \bar{P}^k. \tag{3.136}$$

The control variables (the selling prices and the associated demand rates) are assumed to be piecewise continuously differentiable functions of time. Constraint (3.135) gives the initial and terminal state condition. As in the case of static pricing, constraint (3.136) ensures the non-negativity of the demand rate. The optimal control problems are solved by *Pontryagin's Maximum Principle.* The Hamiltonian function using the adjoint variable $\lambda_j^k(t)$ to (3.134) is

$$H_j^k = (P_j^k(t) - c^k) D_j^k(t) - h^k y_j^k(t) - \lambda_j^k(t) D_j^k(t).$$

The necessary conditions for optimality are

$$\frac{\partial H_j^k}{\partial P_j^k} = \frac{D_j^k(t)}{(D_j^k)'(t)} + P_j^k(t) - c^k - \lambda_j^k(t) \stackrel{!}{=} 0 \quad \text{and} \tag{3.137}$$

$$\dot{\lambda}_j^k = -\frac{\partial H_j^k}{\partial y_j^k} = h^k. \tag{3.138}$$

The solution of the first-order differential equation (3.138) is

$$\lambda_j^k(t) = h^k t + \lambda_{0j}^k. \tag{3.139}$$

The adjoint variable $\lambda_j^k(t)$ represents the marginal valuation of the state variable y_j^k at time t in the subinterval j. If an exogenous influence would increase the state variable by one unit at time t, the objective value would change by $\lambda_j^k(t)$. The value λ_{0j}^k, that is, $\lambda_j^k(0)$, characterizes the marginal valuation of the state variable at time $t = 0$. The optimal price trajectory is represented as a function $P_j^{k*}(\lambda_{0j}^k, t)$ and the optimal demand trajectory $D_j^{k*}(\lambda_{0j}^k, t)$ is given by the price-response function.

Conditions (3.134) and (3.135) give $\int_0^{T_j} D_j^{k*}(\lambda_{0j}^k, t)\mathrm{d}t = y_{0_j}^k - y_{T_j}^k$ and the initial value for the adjoint variable is a function $\lambda_{0j}^k(T_j, y_{0_j}^k, y_{T_j}^k)$. For the most common price-response functions, linear and iso-elastic, we can obtain $\lambda_{0j}^k(T_j,\, y_{0_j}^k,\, y_{T_j}^k)$ in closed form. Therefore, the optimal selling price trajectory becomes a function $P_j^{k*}(T_j,\, y_{0_j}^k,\, y_{T_j}^k,\, t)$ and the optimal demand trajectory follows from the price-response function $D_j^{k*}(T_j,\, y_{0_j}^k,\, y_{T_j}^k,\, t)$. The inventory level at a particular time t is determined as a function $y_j^{k*}(T_j,\, y_{0_j}^k,\, y_{T_j}^k, t) = y_{0_j}^k - \int_0^t D_j^{k*}(T_j,\, y_{0_j}^k,\, y_{T_j}^k,\, s)\mathrm{d}s$. Substitution of P_j^{k*}, D_j^{k*}, and y_j^{k*} into (3.133) and integration gives the optimal profit of a particular product k and a particular subinterval j as $\Pi_j^k(T_j,\, y_{0_j}^k,\, y_{T_j}^k)$, a function depending on T_j, $y_{0_j}^k$, and $y_{T_j}^k$.

Master problem

Using the optimal control results, the master problem is

$$\max_{T,\vec{T},\vec{y_0},\vec{y_T}} \quad \Pi \;=\; \frac{1}{T}\sum_{k=1}^{K}\left(\sum_{j=1}^{K}\Pi_j^k(T_j,\, y_{0_j}^k,\, y_{T_j}^k) - F^k\right) \tag{3.140}$$

$$s.t. \quad \sum_{k=1}^{K} s^k y_{0_j}^k \leq S, \qquad j = 1,\ldots,K, \tag{3.141}$$

$$\sum_{j=1}^{K} T_j = T, \tag{3.142}$$

$$y_{T_K}^1 = y_{T_{k-1}}^k = 0, \qquad k = 2,\ldots,K, \tag{3.143}$$

$$y_{T_{j-1}}^k = y_{0_j}^k, \qquad j = 2,\ldots,K,\; k = 1,\ldots,K,\; j \neq k, \tag{3.144}$$

$$y_{T_K}^k = y_{0_1}^k, \qquad k = 1,\ldots,K, \tag{3.145}$$

with $\vec{y_0} = \{y_{0_j}^k |\; j,k = 1,\ldots,K\}$ and $\vec{y_T} = \{y_{T_j}^k |\; j,k = 1,\ldots,K\}$, respectively.

Resulting from the given order sequence, the initial inventory at the beginning of subinterval k where product k is ordered ($y_{0_k}^k = Q^k$) and the inventory level at the end of subinterval $k-1$ must be zero (constraint (3.143)). Constraints (3.144) and (3.145) ensure that the ending inventory of a subinterval equals the initial inventory of the following interval.

This problem can also be solved by analyzing the *Lagrangian function* and the KKT conditions. The Lagrangian is

$$L(T,\vec{T},\vec{y_0},\vec{y_T}) = \Pi(T,\vec{T},\vec{y_0},\vec{y_T}) + \sum_{j=1}^{K}\mu_j\left(S - \sum_{k=1}^{K} s^k y_{0_j}^k\right) \tag{3.146}$$

with the KKT conditions $\frac{\partial L}{\partial T} = 0$, $\frac{\partial L}{\partial T_j} = 0$ and $\frac{\partial L}{\partial \mu_j} = 0$ for all $j = 1, \ldots, K$, $\frac{\partial L}{\partial y^k_{0_j}} = 0$ and $\frac{\partial L}{\partial y^k_{T_j}} = 0$ for all $j, k = 1, \cdots, K$, and conditions (3.142) - (3.145). Due to the complexity, this non-linear equation system has to be solved numerically.

3.5.5 Numerical Example

The data that we use are similar to the numerical example of Rajan et al. (1992) without any consideration of decay. Consider a supermarket that is planning the replenishment and pricing for two storable products with linear price-response functions $D^k = a^k - b^k P^k$. Let $k = 1$ and $k = 2$ be identical products with the parameters $a^k = 576$, $b^k = 226$, $c^k = 1.45$, $h^k = 0.29$, $s^k = 1$, and $F^k = 50$. Furthermore, we assume a storage capacity $S = 150$.

	uncapacitated			capacitated		
	decentral	constant	dynamic	decentral	constant	dynamic
Q^{k*}	207	179	188	100	99	115
T^*	1.66	1.92	2.09	0.80	1.47	1.80
Π^*	16.37	24.72	27.46	-16.54	11.72	17.21
S^*_{min}	379	268	283	150	150	150

Table 3.11: Two-product replenishment problem for uncapacitated as well as capacitated storage space

Table 3.11 illustrates the numerical results of the three presented frameworks, first for the case of unlimited storage capacity and second, where we restrict the storage capacity to 150 units. In both the uncapacitated and the capacitated case, the optimal order quantity decreases if the firm decides simultaneously on pricing and replenishment compared to the decentralized framework. This effect comes from the underestimation of inventory and ordering costs. If the firm adjusts the selling price continuously, the optimal order quantity increases again. This effect results from a better balancing of ordering and inventory holding costs which can also be seen in the behavior of S^*_{min}. Even though the optimal order quantity decreases from the decentralized to the coordinated/constant case and then increases in the coordinated/dynamic case, the optimal cycle length increases consistently from decentralized to coordinated/dynamic decision-making. In this example, the results show that if the firm optimizes the profit locally, the replenishment of all products is not profitable and the loss per time unit is 16.54.

Figure 3.14 illustrates the optimal prices of the simultaneous optimization for both constant and dynamic pricing. The numerical example indicates that the price trajectory in the dynamic case is not continuous over the order cycle of a

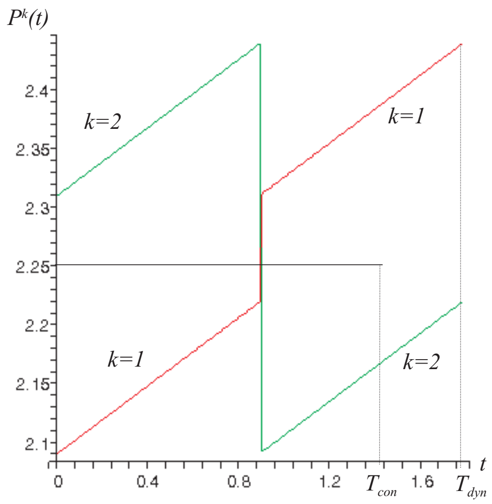

Figure 3.14: Optimal constant and dynamic pricing

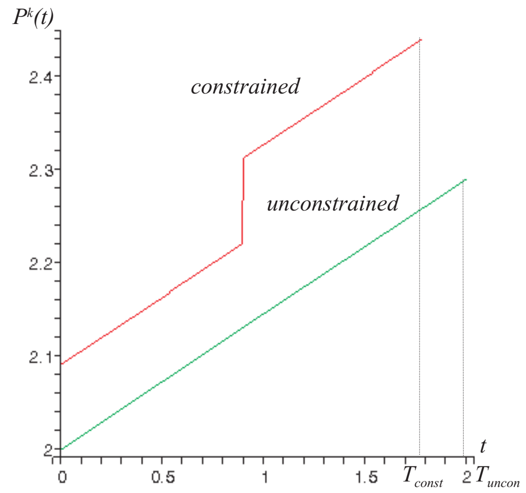

Figure 3.15: Optimal dynamic pricing for the unconstrained and constrained case

product as in the single product case but it is discontinuous at the replenishment point of the other product. This discontinuity technically results from individually optimized adjoint variables λ_{0j}^k for each subinterval j and product k. This variable characterizes the marginal valuation of the state variable (inventory level of product k) at the beginning of subinterval j. Due to the fact that at the beginning of subinterval j product j is replenished and therefore the required storage capacity increases immediately, also the marginal valuation of the state variables of all products changes at this time. In this example, directly after the replenishment of product 1 $k = 1$ at $t = 0$, the selling price of product 1 is set to 2.09. Then the selling price increases continuously to 2.21 until time 0.95. At this time, product $k = 2$ is replenished with an initial selling price of 2.09 (both products are identical) and the selling price of product 1 increases instantaneously to 2.31. Then, from time 0.95 to the end of the order cycle the selling price increases continuously to 2.45. Because both products are identical, they have the same optimal pricing pattern only shifted by half of the cycle length. The optimal price of the constant pricing strategy is 2.25 and is interleaved between the minimum price and the maximum price of the continuous pricing strategy (see *inter-leaved property* in Section 3.3 and Chen et al. (2006)).

Figure 3.15 compares the single-product (unconstrained) case and the multiple-product (constrained) case. As the analytical results have shown, the charged selling price in the constrained case is higher than the selling price in the unconstrained case for all t. Furthermore, in the case with storage capacity constraint the order frequency is larger (see in Table 3.11).

If the retailer charges a constant selling price, the inventory-development is a linear decreasing function in time and is differentiable at each time within an order cycle and only non-differentiable at the boundaries. In case of dynamic pricing, the selling price is not continuously differentiable within the order cycle but it is non-differentiable at the point in time where the other product is replenished. This yields that the inventory level is a non-linear function of time.

Thus, if multiple products are sharing a warehouse with limited storage capacity, a dynamic pricing strategy with the possibility to adjust the selling price continuously and having to regard the replenishment points of all other products yields a better operational efficiency than in the case of constant pricing and without consideration of price jumps within the order cycle. Due to the fact that selling prices can be adjusted at each replenishment point, the retailer can charge a lower selling price at the beginning of an order cycle where the inventory is high (that results in a higher demand) and he increases selling prices during the order cycle both continuously between replenishments and immediately (impulse) at a particular replenishment point. This leads to a better utilization of the limited capacity and to an increasing profit.

3.5.6 Summary and implications

This section examined a warehouse scheduling problem where a retailer simultaneously optimizes pricing and replenishment. In order to illustrate the benefits of coordinated decision-making, dynamic pricing, and the interaction of price and replenishment strategies, a decentralized decision-making strategy where marketing and operations sequentially optimize selling prices and replenishment strategies was compared to a coordinated decision-making where price and replenishment strategies are optimized simultaneously. Moreover, constant and dynamic pricing strategies were distinguished.

It was analytically shown that coordinated decision-making yields a higher selling price than a system with decentralized decision-making. Furthermore, the optimal selling price increases from an uncapacitated to a capacitated framework. We presented a solution procedure for solving the coordinated optimization model with continuous price adjustments. This model was solved in a two-stage approach where several optimal control problems have to be solved at the second stage and a non-linear optimization problem is solved at the first stage. This formulation allows us to determine the price jumps at the replenishment points optimally. In a numerical example, we presented differences in decision-making on price and replenishment. It was indicated that in a warehouse scheduling problem with limited storage capacity where the retailer simultaneously optimizes the replenishment strategy and the dynamic and continuous pricing strategy, the retailer adjusts the prices discontinuously during the order cycle. Both adjustments influence the cycle length and therefore the order frequency of the retailer.

Furthermore, in the case of coordinated decision-making with continuous price adjustment, the price trajectory is not continuous over the entire order cycle but it is discontinuous at the replenishment point of the other products.

The next section extends the traditional EOQ problem to a continuous time Cournot competition problem. Two firms sell substitutable products and compete in quantities on a market whose market price is a commonly known decreasing function of total output. Both firms choose their quantities simultaneously but they differ in their respective cost structure. This section investigates the performance of an EOQ and a JIT replenishment strategy in a competitive environment.

3.6 Economic lot-sizing and dynamic quantity competition

This section studies a problem of dynamic quantity competition in continuous time with two competing retailers facing different cost structures. Retailer 1 replenishes subject to fixed ordering costs and variable procurement costs, and all inventory kept in stock is subject to holding costs. Retailer 2 faces no fixed ordering costs but only variable procurement costs. Both retailers are allowed to change their sales quantities dynamically over time. Following the structure of the economic order quantity (EOQ) model, retailer 1 places replenishment orders in batches whereas retailer 2 follows a just-in-time (JIT) policy. The objective of both retailers is to maximize their individual average profit per time unit being aware of the competitor's replenishment and output decisions.

The problem is modeled by a two-stage hierarchical optimization approach. The second-stage model is a differential game for a given cycle length. An open-loop Nash equilibrium is derived and the equilibrium output strategies of both retailers are compared. At the first stage, the optimal cycle length of the EOQ retailer is determined, anticipating the optimal output trajectories at the second stage. Furthermore, the existence of a unique optimal solution is shown. One issue assuming an open-loop strategy is the required information at any time t. In an open-loop strategy, the retailers does not need to observe the current state at time t except the initial state. A second issue is the degree of commitment. An open-loop strategy does not give the retailers any flexibility to react to signals because both retailers commit their entire output strategy at the beginning of the game. One can argue that this is not a reasonable strategy. However, if the state of the system is not observable, then open-loop strategies are a realistic assumption (Dockner et al., 2000).

3.6.1 Model formulation

Consider two retailers competing over an infinite planning horizon where each of them attempts to maximize its average profit. The average profit equals revenue minus costs where both firms differ in their respective cost structure. A replenishment of retailer 1 is subject to fixed ordering costs F_1 and variable procurement costs c_1 per unit. All inventory that is kept in stock is subject to holding costs h_1 per unit and time unit. Retailer 2 faces only variable procurement costs c_2 per unit. We assume that $c_1 \leq c_2$. This assumption is reasonable because JIT replenishment which is characterized by a higher flexibility and no fixed costs usually results in higher variable procurement costs. Following the assumptions of the EOQ model, retailer 1 places replenishment orders in batches of size Q

every T periods. Thus, the game can be partitioned into identical intervals of length T.

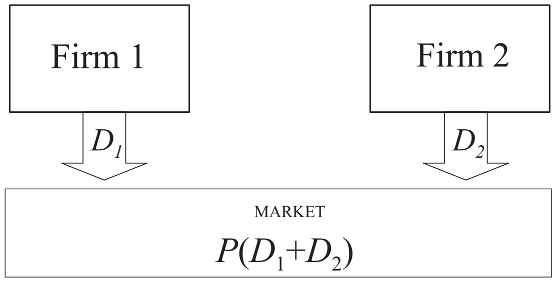

Figure 3.16: Competition model

We assume that the supply source of both retailers is uncapacitated, the order quantity is delivered in one shipment with lead time zero, and backorders are not permitted. Let t denote the time after retailer 1 placed an order and $D_i(t)$ the output that retailer i puts on the market at time t. It is assumed that for all t, the market price is a commonly known, decreasing, concave, and twice-continuously differentiable function of the total output, i.e., $P(D(t))$ with $D(t) = D_1(t)+D_2(t)$ satisfies $\frac{\partial P}{\partial D} < 0$ and $\frac{\partial^2 P}{\partial D^2} \leq 0$. Furthermore, we assume that $P(0) > c_2$, i.e., the unit procurement cost of retailer 2 is lower than the maximum market price. Otherwise, retailer 2 would never participate in this game. Since the market price is a decreasing function of the total output, there exists a critical output level $\bar{D} \in (0, \infty)$ such that $P(D(t)) > 0$ for $D(t) \in [0, \bar{D})$ and $P(D(t)) = 0$ for $D(t) \in [\bar{D}, \infty)$. The critical level can be interpreted as the market potential. That is, if the output of both retailers is equal to or exceeds the level $\bar{D} \leq D_1 + D_2$, the market price drops to zero. The inventory level of retailer 1 at time t is denoted by $y_1(t)$.

The objective functions of both retailers can be formulated as follows. The average profit of retailer 1 depends on the cycle length of an order and the output trajectory over the order cycle, i.e.,

$$\Pi_1 = \max_{T, D_1(t)} \frac{1}{T} \left[\int_0^T ((P(D_1(t) + D_2(t)) - c_1)D_1(t) - h_1 y_1(t))dt - F_1 \right]. \quad (3.147)$$

Retailer 2 determines his optimal output at any time t, i.e.,

$$\Pi_2(t) = \max_{D_2(t)} ((P(D_1(t) + D_2(t)) - c_2)D_2(t)). \quad (3.148)$$

Retailer 1 first determines the optimal order frequency anticipating the optimal output trajectory. Therefore, we decompose the problem into a two-stage hierarchical optimization problem. At the second stage, given a fixed cycle-length T, we solve a differential game and determine a *Nash equilibrium* strategy where both retailers choose their respective output quantities simultaneously and each of them has complete information about the profit function of the other. Furthermore, we assume that the output strategies D_1 and D_2 only depend on time, i.e., $D_i(t)$ with $i = 1, 2$. This strategy is called an *open-loop* strategy. At the first stage, retailer 1 determines T^* and Q^* anticipating the optimal quantity competition from the second stage.

3.6.2 Model analysis

Second stage - quantity competition

The second-stage problem is a *differential game* played for a given cycle length T on a fixed and finite time interval $[0, T]$ where each retailer commits his entire strategy at the beginning of the game. Based on the considerations above, we obtain the following optimal control formulation for the EOQ retailer.

Problem 1: EOQ retailer

$$\Pi_1^T = \max_{D_1(t)} \int_0^T \left[(P(D_1(t) + D_2(t)) - c_1) D_1(t) - h_1 y_1(t) \right] dt, \tag{3.149}$$

$$s.t. \quad \dot{y}_1(t) = -D_1(t), \tag{3.150}$$

$$y_1(0) = Q, \; y_1(T) = 0, \tag{3.151}$$

$$D_1(t) \geq 0. \tag{3.152}$$

(3.149) represents the total profit of a single replenishment cycle $[0, T]$. Since the fixed ordering costs are sunk, F_1 is not relevant for the decision. (3.150) represents the *state equation* which is the instantaneous change rate of inventory at t where the *state variable* is represented by $y_1(t)$. At any t in $[0, T]$, the inventory level decreases by the output $D_1(t)$. The replenishment cycle is repeated over an infinite planning horizon with the *initial* condition $y_1(0) = Q$ and *terminal* condition $y_1(T) = 0$, i.e., the inventory level at the beginning of a replenishment cycle is equal to the lot-size and at the end of a replenishment cycle the inventory level is equal to zero (eq. (3.151)). When the inventory level drops to zero, the next order arrives immediately. (3.152) ensures the non-negativity of the output.

Problem 2: JIT retailer

Since the output of retailer 2 depends on the output of retailer 1 which, in turn, depends on the cycle length, retailer 2 anticipates the entire order cycle of retailer

1.

$$\Pi_2^T = \max_{D_2(t)} \int_0^T \left[(P(D_1(t) + D_2(t)) - c_2) D_2(t) \right] dt, \tag{3.153}$$

$$s.t. \quad D_2(t) \geq 0. \tag{3.154}$$

(3.153) represents the objective of the JIT retailer optimizing his output strategy being aware of the optimal output and replenishment strategy of retailer 1. (3.154) ensures non-negativity of the output of retailer 2.

Let $f_1(y_1(t), D_1(t), D_2(t))$ and $f_2(D_1(t), D_2(t))$ denote the profit functions of both retailers at a particular time t with

$$f_1(y_1(t), D_1(t), D_2(t)) = (P(D_1(t) + D_2(t)) - c_1) D_1(t) - h_1 y_1(t), \tag{3.155}$$

$$f_2(D_1(t), D_2(t)) = (P(D_1(t) + D_2(t)) - c_2) D_2(t). \tag{3.156}$$

Both retailers choose their output strategy simultaneously. An open-loop *Nash equilibrium* that defines $D_1^*(t)$ and $D_2^*(t)$ satisfies

$$\Pi_1^T(D_1^*(t), D_2^*(t)) \geq \Pi_1^T(D_1(t), D_2^*(t)) \text{ and } \Pi_2^T(D_1^*(t), D_2^*(t)) \geq \Pi_1^T(D_1^*(t), D_2(t))$$

for all t. That is, neither of the retailers has an incentive to deviate from this initially chosen strategy. To find the open-loop Nash equilibrium for this game, Problems 1 and 2 have to be solved simultaneously. The *Hamiltonians* corresponding to (3.149) and (3.153) are

$$H_1(t) = f_1(y_1(t), D_1(t), D_2(t)) - \lambda(t) D_1(t) \text{ and } H_2(t) = f_2(D_1(t), D_2(t)) \tag{3.157}$$

with the costate variable $\lambda(t)$ which represents the *shadow price* of the state variable y_1 at time t. It is assumed that $D_1^*(t) + D_2^*(t) < \bar{D}$. Otherwise both retailers would generate a loss at this particular time. Taking constraints (3.152) and (3.154) into account, the *Hamilton-Lagrange functions* are

$$L_i(t) = H_i(t) + \mu_1(t) D_1(t) + \mu_2(t) D_2(t) \quad i = 1, 2 \tag{3.158}$$

with the *Lagrangian multipliers* $\mu_1(t)$ and $\mu_2(t)$. For notational simplicity, in the following we omit the argument t. Applying the standard necessary conditions from differential game theory to (3.158) (Dockner et al., 2000) gives that

$$\frac{\partial L_1}{\partial D_1} = P'(D_1 + D_2) D_1 + P(D_1 + D_2) - c_1 - \lambda + \mu_1 \stackrel{!}{=} 0, \tag{3.159}$$

$$\frac{\partial L_2}{\partial D_2} = P'(D_1 + D_2) D_2 + P(D_1 + D_2) - c_2 + \mu_2 \stackrel{!}{=} 0 \tag{3.160}$$

and the conditions which define the costate variable λ as a function of t

$$\dot{\lambda} = -\frac{\partial L_1}{\partial y_1} = h_1 \Rightarrow \lambda(t) = h_1 t + \lambda_0. \tag{3.161}$$

$\lambda(t)$ measures the value of an additional unit of inventory along the optimal path (Feichtinger and Hartl, 1985). As represented in Section 3.3.1 we get that $\lambda_0 = 0$ so that

$$\lambda(t) = h_1 t. \tag{3.162}$$

Furthermore, μ_1 and μ_2 have to satisfy the *complementary slackness conditions*

$$\mu_1 \geq 0 \quad \text{and} \quad \mu_1 D_1 = 0, \tag{3.163}$$
$$\mu_2 \geq 0 \quad \text{and} \quad \mu_2 D_2 = 0. \tag{3.164}$$

Proposition 15. *For any t, there exists a unique Cournot Nash equilibrium.*

Proof. There exits a unique Cournot equilibrium if $\frac{\partial^2 L_i}{\partial D_i^2} + \frac{\partial^2 L_i}{\partial D_i \partial D_j} < 0$ (e.g., see Tanaka (2001)). From the second derivatives and second cross derivatives, it follows

$$\frac{\partial^2 L_i}{\partial D_i^2} + \frac{\partial^2 L_i}{\partial D_i \partial D_j} =$$
$$P''(D_1 + D_2)D_i + 2P'(D_1 + D_2) + P''(D_1 + D_2)D_i + P'(D_1 + D_2) < 0 \tag{3.165}$$

for $i = 1, 2$ and $j \neq i$. Therefore, the effect of changing the own output on the marginal profit at time t is larger than the effect of a change of the competitor's output. □

The optimal solution has to satisfy the *Karush-Kuhn-Tucker* conditions (3.159), (3.160) (by substituting (3.162)), (3.163) and (3.164) (Sydsæter and Hammond, 2002). The case $\mu_2^* > 0$, $\mu_3^* > 0$ never occurs because it yields that $P(0) < c_2$ which gives that retailer 2 would never enter the market. Thus, we get the following three cases where either retailer 1 or retailer 2 operates as a monopolist or both retailers are in direct competition:

1. $\mu_1^* = 0$ and $\mu_2^* > 0$,
   In this case, retailer 1 is a monopolist. From (3.160) and (3.164), it follows that $\mu_2^* = -(P(D_1) - c_2)$ and $P(D_1) < c_2$. Hence, the market price resulting from the output of retailer 1 is lower than the variable procurement cost of retailer 2. As a consequence, retailer 2 does not offer a positive quantity. Let $D_1^M(t)$ denote the monopoly quantity offered to the market at t. Therefore, (3.159) gives that $D_1^{M^*}$ solves

$$P(D_1) + P'(D_1)D_1 = c_1 + h_1 t. \tag{3.166}$$

(3.166) reflects the well-known optimality condition of marginal revenue equals marginal cost. Since, $\frac{\partial}{\partial D_1}(P(D_1) + P'(D_1)D_1) = 2P'(D_1) + P''(D_1) < 0$, together with (3.166) it follows that $D_1^{M^*}$ decreases in t and $P(D_1^{M^*})$ increases in t. Therefore, there exists a time $\bar{t}_1$ where the monopoly market price is equal to the variable procurement cost of retailer 2. At this time, retailer 2 offers a positive quantity. Substituting $P(D_1) = c_2$ into (3.166) and solving for t, $\bar{t}_1$ is obtained from

$$t - \frac{1}{h_1}(P'(D_1^{M^*})D_1^{M^*} + (c_2 - c_1)) = 0. \tag{3.167}$$

(3.167) determines the point in time where retailer 2 enters the market and offers a positive quantity. Therefore, for all $t \leq \bar{t}_1$, $D_2^* = 0$ and $D_1^{M^*}$ solves (3.166). However, if $P'(D_1^{M^*})D_1^{M^*} + (c_2 - c_1) < 0$, then $\bar{t}_1 < 0$ and the situation where retailer 1 operates as monopolist does not occur.

2. $\mu_1^* > 0$ and $\mu_2^* = 0$,
In this case retailer 2 is a monopolist. From (3.159) and (3.163), it follows that $\mu_1^* = -(P(D_2) - c_1 - h_1 t)$ with $P(D_2) < c_1 + h_1 t$. That is, the market price resulting from the output of retailer 2 is lower than the variable procurement costs plus inventory holding cost until t of retailer 1, i.e., this case only occurs later in an order cycle where the value of a unit of inventory is high. Since we assume at the second stage that the order cycle length is exogenous, this case cannot be eliminated in advance. However, we will show later that retailer 1 who optimizes the cycle length T will set T^* such that this case will not occur. (3.160) gives that the monopoly output of retailer 2, $D_2^{M^*}$ solves

$$P(D_2) + P'(D_2)D_2 = c_2, \tag{3.168}$$

where marginal revenue equals marginal costs. Let $\bar{t}_2$ be the time where $P(D_2^{M^*}) = c_1 + h_1 t$, i.e., where the monopoly price of retailer 2 is equal to marginal cost of retailer 1. Substituting $P(D_2^{M^*}) = c_1 + h_1 t$ into (3.168) it follows that $\bar{t}_2$ solves

$$t - \frac{1}{h_1}(P(D_2^{M^*}) - c_1) = 0 \tag{3.169}$$

and for all $t \geq \bar{t}_2$ it follows that $D_1^* = 0$ and $D_2^{M^*}$ solves (3.168).

3. $\mu_1^* = \mu_2^* = 0$,
In this case, both retailers are in direct competition and both outputs are strictly positive. (3.159) and (3.160) give that the optimal solution has to satisfy

$$P'(D_1 + D_2)(D_1 - D_2) + (c_2 - c_1) = h_1 t. \tag{3.170}$$

(3.170) and $c_2 - c_1 > 0$ give that for $t = 0$, it follows that $D_1^* > D_2^*$. If $c_2 - c_1 = 0$, then $D_1^* = D_2^*$ and if $c_2 - c_1 < 0$, then $D_1^* < D_2^*$. However, for $t > 0$ and $c_2 - c_1 > 0$, it follows that D_1^* might be larger, equal to, or lower than D_2^* depending on t.

In the following, we show the characteristics of the optimal output strategies of both retailers.

Proposition 16. *In an open-loop Nash equilibrium,* $\frac{\partial D_1^*}{\partial t} \leq 0$ *,* $\frac{\partial D_2^*}{\partial t} \geq 0$*, and* $\frac{\partial D^*}{\partial t} \leq 0$*.*

Proof. In order to prove this proposition the three previous cases have to be analyzed. The proof for Case 1 where retailer 1 is a monopolist, i.e., $\mu_1^* = 0$ and $\mu_2^* > 0$, is shown in Section 3.3 and by Rajan et al. (1992). In Case 2 where retailer 2 is a monopolist, i.e., $\mu_2^* = 0$ and $\mu_1^* > 0$, it is obvious that the output remains constant. Therefore, only Case 3 where both retailers are in direct competition needs to be shown in detail.

To determine $\frac{\partial D_1^*}{\partial t} \leq 0$, we apply the *Implicit Function Theorem* (see Sydsæter and Hammond (2002)) to (3.159):

$$P''(D_1 + D_2)\left(\frac{\partial D_1}{\partial t} + \frac{\partial D_2}{\partial D_1}\frac{\partial D_1}{\partial t}\right) + P'(D_1 + D_2)\left(2\frac{\partial D_1}{\partial t} + \frac{\partial D_2}{\partial D_1}\frac{\partial D_1}{\partial t}\right) = 0$$

$$\Leftrightarrow \frac{\partial D_1}{\partial t} = \frac{h_1}{\left(P''(D_1 + D_2)\left(1 + \frac{\partial D_2}{\partial D_1}\right) + P'(D_1 + D_2)\left(2 + \frac{\partial D_2}{\partial D_1}\right)\right)}. \tag{3.171}$$

Applying the *Implicit Function Theorem* to (3.159) and (3.160), respectively, we get

$$\frac{\partial D_i}{\partial D_j} = -\frac{P''(D_1 + D_2)D_i + P'(D_1 + D_2)}{P''(D_1 + D_2)D_i + 2P'(D_1 + D_2)} \in (-1, 0). \tag{3.172}$$

Thus, from (3.171) and (3.172) it follows that $\frac{\partial D_1^*}{\partial t} \leq 0$ and $\frac{\partial D_2^*}{\partial t} = \frac{\partial D_2^*}{\partial D_1}\frac{\partial D_1}{\partial t} \geq 0$. Moreover, we get $\left|\frac{\partial D_1^*}{\partial t}\right| \geq \left|\frac{\partial D_2^*}{\partial t}\right|$. Thus, $\frac{\partial D^*}{\partial t} \leq 0$. □

A graphical illustration for a special linear price-response function is as follows. The best response of one retailer is a linear function of the competitor's decision. Figure 3.17 illustrates the best-response function of both retailers depending on the time within the order cycle. Let t_1 and t_2 be two points in time within the order cycle $[0, T]$ with $t_1 < t_2$. The best-response function of the JIT retailer is not influenced by the point in time, i.e., it does not change over an order cycle. However, for the EOQ-retailer an increasing t yields that the best-response function shifts downwards. This shifting yields that the equilibrium is shifted to the top left, i.e., the optimal output of the EOQ retailer decreases while the optimal output of the JIT retailer increases.

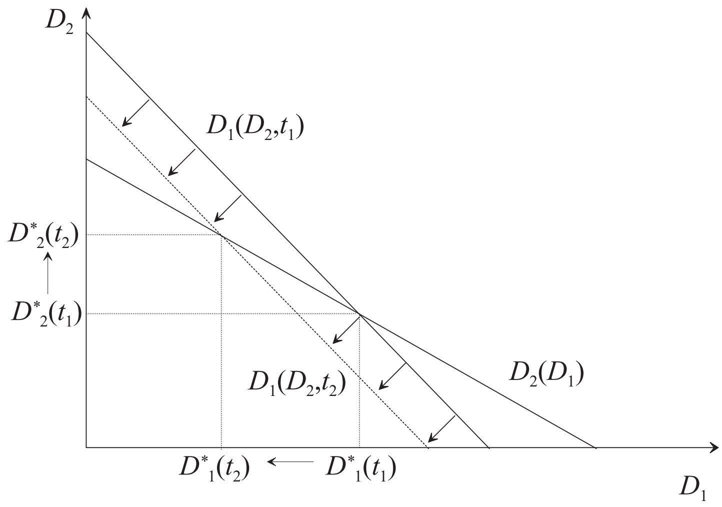

Figure 3.17: Optimal response function with respect to competitor's output depending on t

The intuition behind the output strategies of both retailers is as follows. For retailer 1, it is beneficial to reduce the output over an order cycle. Because at the beginning of an order cycle where inventory is high, retailer 1 offers a larger output in order to reduce the stock and thus, inventory holding costs. With a lower inventory level, the output level is reduced. On the other hand, retailer 2 follows a contrary output strategy where it is beneficial to offer a lower output at the beginning of an order cycle where retailer 1's output is high and to enhance the output over the order cycle. If at time $t = 0$, the procurement costs of retailer 2 exceed the monopoly price of retailer 1, then retailer 2 is deterred from the market, i.e., $D_2^* = 0$. However, over the order cycle, each unit of retailer 1 becomes more expensive because of additional holding costs and therefore, he reduces his output which leads to an increasing market price. Later in the order cycle, there exists a time $\bar{t}_1$ where the monopoly price of retailer 1 is equal to the procurement costs of retailer 2 and it becomes beneficial for retailer 2 to enter the market. From this point in time, both retailers are in direct competition. If procurement plus inventory holding costs of retailer 1 exceed the monopoly price of retailer 2, retailer 1 is displaced out of the market. That is, depending on the cost structure of both retailers, there may exist phases within the order cycle where one retailer operates as a monopolist and phases where both retailers are in direct competition. Figures 3.18 and 3.19 illustrate the three output strategies described above under the assumption that $T > \bar{t}_2$. In case that $\bar{t}_1 \leq 0$ and $c_2 \leq c_1$, the output of retailer 2 is larger than retailer 1's output over the entire order cycle, i.e., $D_2^* > D_1^*$.

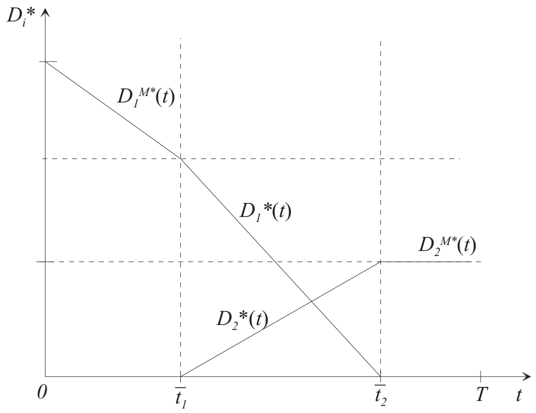

Figure 3.18: Optimal output strategies for the case $\bar{t}_1 > 0$

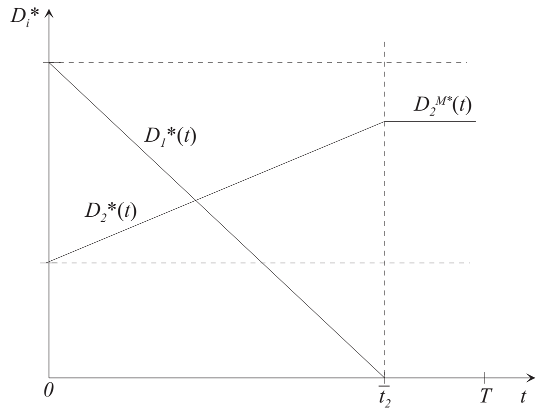

Figure 3.19: Optimal output strategies for the case $\bar{t}_1 \leq 0$

First stage - order cycle decision

At the first stage, retailer 1 determines the optimal cycle length anticipating the optimal output strategies at the second stage by maximizing the average profit

$$\Pi_1(T) = \frac{1}{T}\left(\Pi_1^T(T) - F_1\right) \tag{3.173}$$

where $\Pi_1^T(T)$ is the optimal cycle profit given the cycle length T. By choosing the cycle length, retailer 1 determines indirectly the lot-size which is the total output over the entire order cycle. Ordering a large amount at one time will reduce the order frequency and thus the fixed cost per unit, but it will increase holding costs. However, larger lot-sizes also influence the output trajectories and the market price over an order cycle which has to be taken into account. Since the output of retailer 1 decreases over time, there exists a time t where $D_1^*(t) = 0$. This time determines an upper bound for T^*. From the previous section, we get that for all $t > \bar{t}_2$ the unit procurement costs plus the inventory holding costs of retailer 1 exceed the monopoly price of retailer 2 so that retailer 1 is deterred from the market ($\mu_1^* > 0$ and $\mu_2^* = 0$). Due to the infinite planning horizon and the repeated equal order cycles, it is obvious that retailer 1 never chooses T^* larger than $\bar{t}_2$. Otherwise, he would loose market share within the period $[\bar{t}_2, T]$. That is, the upper bound is

$$T^* \leq \bar{t}_2. \tag{3.174}$$

Furthermore, we get from the previous results that $\Pi_1^T(T)$ is not necessarily a continuous function of T. Depending on the parameter setting, the three cases A, B, and C can occur. If $\bar{t}_1 \leq 0$, then both retailers directly compete over the entire order cycle (Case A). If $\bar{t}_1 > 0$ and using (3.166) and (3.169), we get that if $T < \bar{t}_1$, then retailer 1 is a monopolist over the entire order cycle (Case B). If

$\bar{t}_1 > 0$ and $T > \bar{t}_1$, then there exist both cases, monopoly and competition. For all $t \in [0, \bar{t}_1]$, retailer 1 is a monopolist and for all $t \in [\bar{t}_1, T]$, both retailers are in direct competition (Case C). Given these three cases, the cycle profit is as follows

$$\Pi_1^T(T) = \begin{cases} \Pi_{1A}^T(T), & \text{if } \bar{t}_1 \leq 0 \\ \Pi_{1B}^T(T), & \text{if } \bar{t}_1 > 0 \text{ and } T \leq \bar{t}_1 \\ \Pi_{1C}^T(T), & \text{if } \bar{t}_1 > 0 \text{ and } T > \bar{t}_1. \end{cases} \tag{3.175}$$

with

$$\Pi_{1A}^T(T) = \int_0^T \left[(P(D_1^*(t) + D_2^*(t)) - c_1)D_1^*(t) - h_1 \int_t^T D_1^*(s)ds \right] dt, \tag{3.176}$$

$$\Pi_{1B}^T(T) = \int_0^T \left[(P(D_1^{M^*}(t) - c_1))D_1^{M^*}(t) - h_1 \int_t^T D_1^{M^*}(s)ds \right] dt, \tag{3.177}$$

$$\begin{aligned} \Pi_{1C}^T(T) = & \int_0^{\bar{t}_1} \left[(P(D_1^{M^*}(t)) - c_1)D_1^{M^*}(t) - h_1 \left(\int_t^{\bar{t}_1} D_1^{M^*}(s)ds + \int_{\bar{t}_1}^T D_1^*(s)ds \right) \right] dt \\ & + \int_{\bar{t}_1}^T \left[(P(D_1^*(t) + D_2^*(t)) - c_1)D_1^*(t) - h_1 \int_t^T D_1^*(s)ds \right] dt. \end{aligned} \tag{3.178}$$

Proposition 17. *There exists a unique T^* which satisfies the condition that the marginal profit at T^* is equal to the average profit, i.e.,*

$$\Pi_1(T^*) = (P(D_1^*(T^*) + D_2^*(T^*)) - c_1 - h_1 T^*)D_1^*(T^*).$$

Proof. This proposition has to be shown for the functions $\Pi_{1A}^T(T)$, $\Pi_{1B}^T(T)$, and $\Pi_{1C}^T(T)$. We give the proof for $\Pi_{1A}^T(T)$, the proofs for $\Pi_{1B}^T(T)$ and $\Pi_{1C}^T(T)$ follow in the same manner.

If $\bar{t_1} \leq 0$, then the first-order condition of (3.173) is as follows

$$\begin{aligned}
\frac{\partial \Pi_1}{\partial T} &= -\frac{1}{T^2}\Pi_{1A}^T(T) + \frac{1}{T}\frac{\partial \Pi_{1A}^T(T)}{\partial T} + \frac{F_1}{T^2} \\
&= -\frac{1}{T^2}\int_0^T \left[(P(D_1^*(t) + D_2^*(t)) - c_1)D_1^*(t) - h_1 \int_t^T D_1^*(s)ds \right] dt \\
&\quad + \frac{1}{T}\left(\int_0^T -h_1 D_1^*(T)dt + (P(D_1^*(T) + D_2^*(T)) - c_1)D_1^*(T) \right) + \frac{F_1}{T^2} \\
&= -\frac{\Pi_1(T)}{T} + \frac{1}{T}\left[(P(D_1^*(T) + D_2^*(T)) - c_1 - h_1 T)D_1^*(T) \right] \stackrel{!}{=} 0 \\
&\Leftrightarrow \ \Pi_1(T) = (P(D_1^*(T) + D_2^*(T)) - c_1 - h_1 T)D_1^*(T). \qquad (3.179)
\end{aligned}$$

(3.179) gives that the optimal cycle length is the value where the marginal profit is equal to the average profit. Let

$$MP(T) := (P(D_1^*(T) + D_2^*(T)) - c_1 - h_1 T)D_1^*(T)$$

define the marginal profit at time T. From (3.179) the first-order derivative is

$$\begin{aligned}
\frac{\partial MP(T)}{\partial T} = \left(\frac{\partial P(D_1^*(T) + D_2^*(T))}{\partial T} - h_1 \right) D_1^*(T) \\
+ (P(D_1^*(T) + D_2^*(T)) - c_1 - h_1 T)\frac{\partial D_1^*(T)}{\partial T}. \qquad (3.180)
\end{aligned}$$

Proposition 16 gives that the second term of (3.180) is negative. Furthermore, from (3.159) and (3.162) it follows that

$$P'(D_1^*(T) + D_2^*(T))D_1^*(T) + P(D_1^*(T) + D_2^*(T)) - c_1 - h_1 T = 0, \qquad (3.181)$$

(note that, since $\bar{t_1} \leq 0, \mu_1 = \mu_2 = 0$) and using the *implicit function theorem*, it follows that

$$\left(\frac{\partial P(D_1^*(T) + D_2^*(T))}{\partial T} - h_1 \right) = -\frac{\partial (P'(D_1^*(T) + D_2^*(T))D_1^*(T)}{\partial T}$$

$$= - \Bigg[\underbrace{P''(D_1^*(T) + D_2^*(T))}_{\leq 0} \underbrace{\left(\frac{\partial (D_1^*(T) + D_2^*(T))}{\partial T} \right) D_1^*(T)}_{\leq 0} +$$

$$\underbrace{P'(D_1^*(T) + D_2^*(T))}_{<0} \underbrace{\frac{\partial D_1^*(T)}{\partial T}}_{\leq 0} \Bigg] \leq 0.$$

That is, the marginal profit is a non-increasing function of T, i.e, $\frac{\partial MP(T)}{\partial T} \leq 0$. Since $MP(T)$ is a non-increasing function of T and all values of T which satisfy the first-order condition and thus $MP(T) = \Pi_1(T)$, it follows that there exists only a single intersection point of $MP(T)$ and $\Pi_1(T)$. Figure 3.20 illustrates the behavior of $MP(T)$ and $\Pi_1(T)$. □

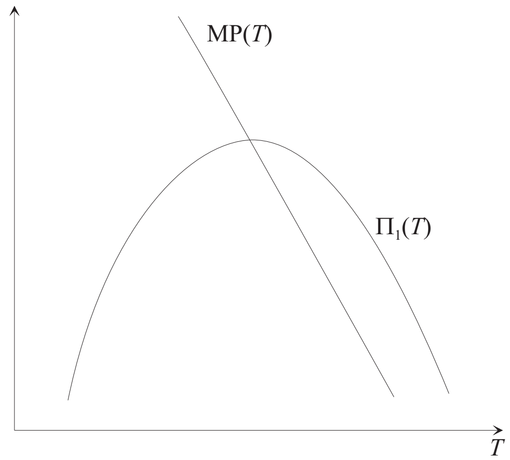

Figure 3.20: Average and marginal profit as functions of T

The intuition behind the optimality condition is the use of scale economies due to fixed ordering costs. If the last increment of profit is larger than the average profit of all previous units, an additional unit of inventory leads to larger economies of scales and thus to an increasing average profit. That is, as long as the marginal profit is larger than the average profit, it is optimal to increase the order cycle.

3.6.3 Solution procedure

Based on the previous results we suggest the following algorithm. Starting with $T = \varepsilon$ when ε is a sufficiently low value and larger than zero, the algorithm calculates the average profit $\frac{1}{T}\left(\Pi_{1k}^T(T) - F_1\right)$ where $k \in \{A, B, C\}$ and the marginal profit $MP(T)$ for increasing values for T and seeks the particular value for T where $\Pi_{1k}^{T}(T) = MP(T)$.

Algorithm

BEGIN

  Set δ as the step size and AP>MP

  Determine $\bar{t}_1$ from (3.167)

  IF $\bar{t}_1 \leq 0$ THEN

    FOR $T = \varepsilon$ WHILE $AP < MP$ DO

      $AP := \frac{1}{T}\left(\Pi_{1A}^{T}(T) - F_1\right)$

      $MP := (P(D_1^*(T) + D_2^*(T)) - c_1 - h_1 T)D_1^*(T)$

      $T := T + \delta$

    END DO

  ELSE

    FOR $T = 0$ WHILE $AP < MP$ DO

      IF $T \leq \bar{t}_1$ THEN

        $AP := \frac{1}{T}\left(\Pi_{1B}^{T}(T) - F_1\right)$

        $MP := (P(D_1^{M^*}(T)) - c_1 - h_1 T)D_1^{M^*}(T)$

        $T := T + \delta$

      ELSE

        $AP := \frac{1}{T}\left(\Pi_{1C}^{T}(T) - F_1\right)$

        $MP := (P(D_1^*(T) + D_2^*(T)) - c_1 - h_1 T)D_1^*(T)$

        $T := T + \delta$

      END IF

    END DO

  END IF

END

3.6.4 Numerical example

This section presents a numerical example illustrating the impact of the replenishment strategy on the average performance in monopoly and a competitive environment. Suppose a linear price-response function

$$P(D_1 + D_2) = \begin{cases} \frac{a}{b} - \frac{D_1+D_2}{b} & : \quad 0 \leq D_1 + D_2 \leq a \\ 0 & : \quad D_1 + D_2 > a \end{cases} . \tag{3.182}$$

The parameter $a = 1000$ units is the market potential and $b = 10$ represents the price-sensitivity. Let the costs parameter of both retailers be as follows: The

Cost parameter	Retailer 1 (EOQ)	Retailer 2 (JIT)
F_i	500	-
c_i	5.00	5.67

Table 3.12: Cost parameters of both retailers

inventory holding cost of retailer 1 is $h_1 = 0.1$ per unit and unit of time. The parameter setting is chosen so that in a monopoly both replenishment strategies (JIT and EOQ) generate identical average profits with $\Pi_1^M = \Pi_2^M = 470$. While retailer 2 follows a JIT strategy, retailer 1 orders in batches of size $Q^M = 1,546$ units which cover an order cycle of length $T^M = 6.6$.

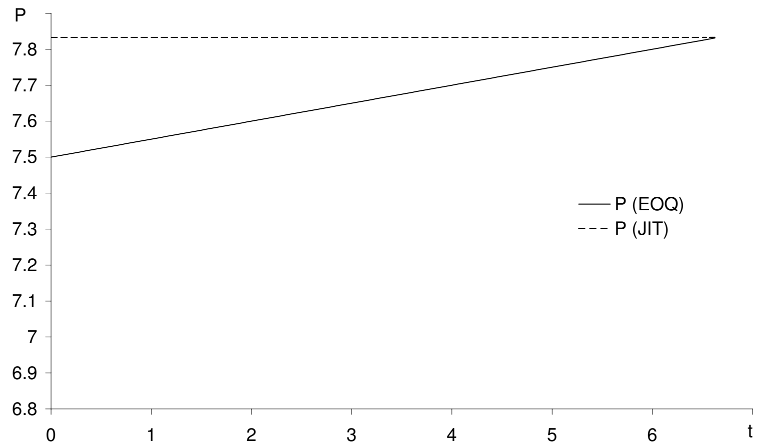

Figure 3.21: Optimal monopoly market price of retailer 1 and retailer 2

The optimal monopoly outputs as well as the respective optimal monopoly prices are illustrated in Figures 3.21 and 3.22. Retailer 2 has no incentive to change the output level over time. His output is constantly 217 units with a resulting market price of 7.83. Retailer 1, however, follows a dynamic output strategy. At

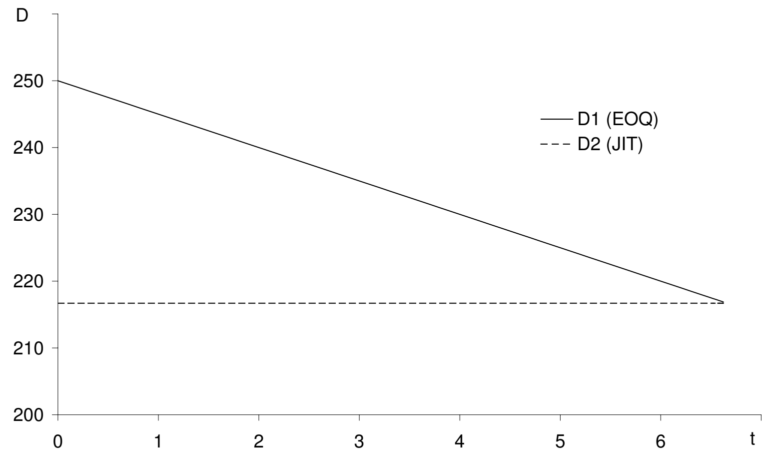

Figure 3.22: Optimal monopoly output trajectories of retailer 1 and retailer 2

the beginning of an order cycle, he offers a higher amount than later in the order cycle. The strategy results from the fact that the inventory level decreases faster so that inventory holding costs are reduced (see Section 3.3 and Rajan et al. (1992)). Although both retailers make an identical average profit, the average output deviates between EOQ and JIT policy. At any t, the optimal output of an EOQ strategy is larger than the output under JIT policy. Consequently, the average market price in an EOQ setting is lower than in a JIT environment.

The results change when EOQ and JIT replenishment are compared in a competitive environment. We consider the same parameter setting as noted above. From the previous section it is known that both retailers do not necessarily compete over the entire order cycle. (3.167) gives that $\bar{t}_1 = -36.6$, hence, under this parameter setting both retailers directly compete over the entire order cycle. Contrary to the monopoly where both replenishment strategies led to identical average profits, in a competitive environment the average profits of EOQ and JIT deviate:

$$\Pi_1^* = 205 \quad \text{and} \quad \Pi_2^* = 179.$$

It is not surprising that the average profit decreases compared to monopoly profit. However, the profit decrease for the JIT retailer is larger than for the EOQ retailer. The optimal order quantity and the optimal cycle length of the EOQ retailer are

$$Q^* = 1,142 \quad \text{and} \quad T^* = 7.0.$$

Figure 3.24 and 3.23 illustrate the optimal output trajectories of retailer 1 and retailer 2, respectively, as well the corresponding market price trajectory.

While the average profit of the EOQ retailer decreases from 470 to 205, the average profit of the JIT retailer decreases to 179. The intuition behind this effect

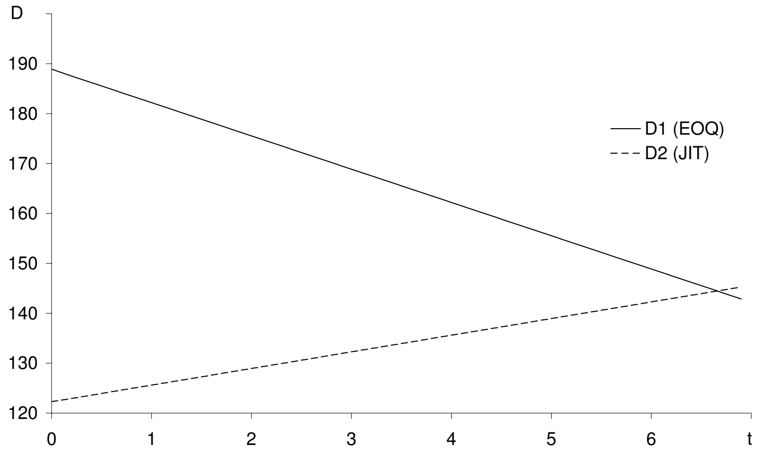

Figure 3.23: Optimal output trajectories under competition

is a commitment advantage of the EOQ retailer. By ordering the output for the entire order cycle at the beginning of an order cycle, the EOQ retailer incurs a commitment of selling this amount. Hence, fixed ordering costs have sunk. At any t, the output of both retailers depends on their respective marginal costs. While marginal costs of the JIT retailer remain constant over time, marginal costs of the EOQ retailer increase over the order cycle.

The results of this example indicate that while a firm can be indifferent in the choice of EOQ or JIT replenishment in monopoly, EOQ replenishment is preferred under competition (given that the competitor follows a JIT strategy). The intuition is that the EOQ retailer achieves a competitive advantage by ordering in batches by which the fixed ordering costs have sunk. Since the marginal costs of the EOQ retailer are lower than the marginal costs of the JIT retailer until t, the time where $c_1 + h_1 t = c_2$, the EOQ retailer offers more on the market than the JIT retailer (the EOQ retailer holds a higher market share than the JIT retailer).

3.6.5 Summary and implications

This section analyzed the interaction of a dynamically changing sales quantity and the replenishment policy in a duopoly. The retailers differed in their replenishment costs. While a replenishment of retailer 1 was subject to fixed ordering costs and variable procurement costs, retailer 2 faced only variable procurement costs. Inventories were subject to holding costs. Hence, retailer 1 places replenishment orders in batches whereas retailer 2 follows a just-in-time policy. Both retailers maximized average profits taking the competitor's decision into account.

This problem was formulated as a two-stage hierarchical optimization problem.

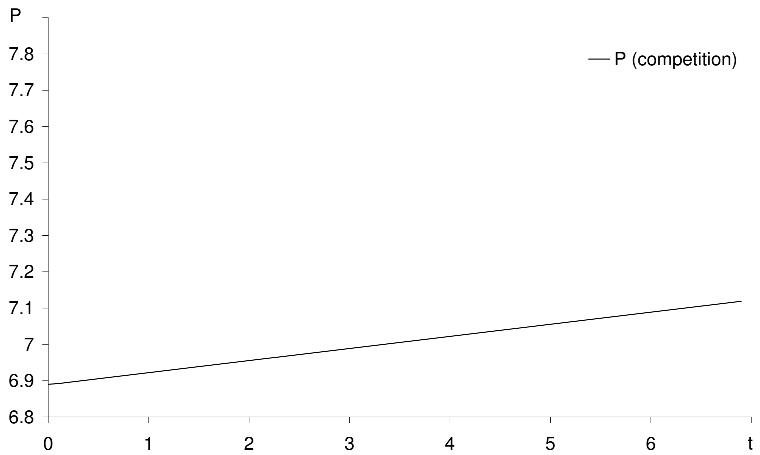

Figure 3.24: Optimal market price trajectory under competition

At the first stage, the optimal cycle length of retailer 1 was determined anticipating the optimal output decisions over the order cycle at the second stage. At the second stage, a differential game was considered where both retailers repeatedly interact over the order cycle of a given length. We derived an open-loop Nash equilibrium and structural properties concerning the existence and uniqueness of an open-loop Nash equilibrium. Furthermore, we analyzed the optimal output strategies of both retailers.

We have shown that, independent of the cycle length, both retailers follow contrary output strategies. While retailer 1 decreases the output over an order cycle, retailer 2 enhances his output. However, the decreasing rate of retailer 1 is larger than the enhancing rate of retailer 2 such that the total output decreases over the order cycle. Given the second-stage solution, we have shown that there exists a uniquely optimal cycle length at the first stage. Since for general price-response functions no closed-form solution can be derived, we developed a solution algorithm to determine T^*. In a numerical example, we showed that while EOQ and JIT generate identical average profits in monopoly, in a competitive environment EOQ replenishment yields a better performance than JIT replenishment.

The competition model assumed that both retailers followed an open-loop strategy, i.e., they commit to pricing and replenishment decision at the beginning of an order cycle. It is questionable if the results hold if both retailers follow a closed-loop strategy, i.e., they adapt future strategies based on the current state of the system. Moreover, the research question whether the open-loop strategy is a good approximation of the closed-loop strategy is of considerable interest.

Another extension of the model is to consider a problem where both retailers face fixed ordering and holding costs. In this context, both retailers have to decide about the output, order quantity, and replenishment cycle. Additionally, they have to anticipate the point in time when the competitor places an order. This problem is rather complex because beyond the output and replenishment policy, the retailers have to decide the optimal time displacement of the orders of both retailers. As in all EOQ-based models, a weakness of the prescribed model lies in the myopic and deterministic environment. However, continuous time models in a competitive environment and uncertain demand are rather complex and achieving structural findings is in the majority of cases impossible.

4 Joint pricing and capacity planning under demand uncertainty

4.1 Introduction

This chapter investigates integrated decision-making of price strategy and capacity acquisition in the presence of uncertain demand.

Capacity management issues are relevant on every stage of the supply chain and every hierarchy level (strategic, tactical, and operational). It greatly influences a firm's ability to match supply with demand. Capacity, in this context, is defined as *the maximal sustainable output rate of a limited resource* (van Mighem, 2008). Firms have a bundle of resources that perform their activities which can be divided into two groups: tangible and intangible. This thesis basically considers tangible resources, e.g., plants, manufacturing equipment, and human resources.

On the other hand, pricing is one of the most important elements of the marketing mix, which generates demand. However, marketing effort can be waste if the capacity decision is suboptimal. In particular, when both price and capacity decisions have to be made in the presence of demand uncertainty, a coordination of these two decisions is essential to optimally match supply with demand.

From traditional RM and economic theory it is known that customer differentiation that segments the market according to customers' buying behavior and their willingness to pay is a considerable instrument resulting in a better capacity utilization, increasing sales, and increasing revenues (Talluri and van Ryzin, 2004). However, a fundamental assumption in traditional RM is that the initial capacity is exogenously given and fixed. This restrictive assumption might lead to suboptimal solutions if the capacity decision did not anticipate the correct customer characteristics.

The central research question that is addressed in this chapter is how firms can benefit from coordinating customer segmentation either by product differentiation (customization) or price discrimination and capacity acquisition in the presence of demand uncertainty. Customization has several benefits. For example, a focused selection of capacity dedicated to products that are directly targeted to specific customers' needs. This decreases costs due to a better capacity utilization and increases sales and revenues (Elmaghraby and Keskinocak, 2003). In particular, this chapter answers the question what the gain of coordinated decision-making of prices that are charged for individual customer classes and capacity acquisition is. Moreover, it is clarified how pricing strategy and capacity decision interact with each other with a special focus on the impact of capacity investment and demand uncertainty.

First, the interaction of price and capacity decisions of a single-product problem is analyzed. This problem is known as the price-setting newsvendor problem and is frequently discussed in literature. Since the price-setting newsvendor problem is an extension of the classical newsvendor problem where it is assumed that the selling price is exogenously given, major differences in capacity decision are identified. Moreover, it is investigated how demand uncertainty affects price decision and capacity acquisition.

After that, the impact of customization is investigated, in particular, the interaction of capacity acquisition and price decisions when the firm is able to segment the market into two customer classes. In order to clarify the benefit of coordinated decision-making, we compare the coordinated framework (i) to the case where the firm is not able to discriminate prices and (ii) to the case of a decentralized decision-making framework where independent sales managers are responsible for price management and capacity acquisition. Moreover, we analyze the impact of capacity investment and demand uncertainty.

Concluding, the specific research questions addressed in this chapter are:

1. How does customer segmentation, in particular price discrimination influence price decision and capacity acquisition in an integrated planning approach?
2. What is the impact of capacity costs and demand uncertainty on price and capacity decisions?
3. What is the benefit of price discrimination compared to a single-pricing strategy?
4. What is the benefit of centralized decision-making compared to decentralized decisions where two product managers decide separately on price and capacity?

The chapter is structured as follows. Section 4.2 introduces the *price-setting newsvendor problem* for a single-product, single-period capacity and price decision problem. Structural properties concerning the existence and characterization of the optimal solution are derived. Moreover, a solution algorithm is presented to obtain the optimal price and capacity decision for the special class of additive demand functions. Section 4.3 extends the model of Section 4.2 to the case where market demand is segmented into two customer classes. Section 4.3.1

develops a stochastic model and Section 4.3.2 analyzes the model and derives structural properties of the optimal solution. Section 4.3.3 presents a solution algorithm. Section 4.3.4 provides a numerical example that illustrates the impact of customer segmentation and price discrimination on price and capacity decisions compared to a single-pricing strategy. Furthermore, in Section 4.3.5 a decentralized decision-making strategy where two sales managers are responsible to decide independently on price and capacity acquisition is compared to central decision-making where both prices and capacity acquisition are decided simultaneously. A brief summary and conclusions are given in Section 4.3.6.

4.2 The price-setting newsvendor problem

This section introduces the price-setting newsvendor problem. In the price-setting newsvendor problem, a decision maker facing random demand for a single product has to decide on a single selling price and the capacity acquisition for a single period. Single period, in this context, does not necessarily mean that the product can only be sold in a single selling period. But, this model is also relevant for multi-period problems when the capacity decision is irreversible, i.e., it cannot be increased or decreased in the short term and, excess capacity cannot be stored and excess demand cannot be produced in advance or in future periods. In the traditional newsvendor problem it is assumed that market parameters such as demand and selling price are exogenous. Incorporating these factors into the model can provide an excellent vehicle for examining how operational problems interact with marketing issues to influence decision-making at the firm level (Petruzzi and Dada, 1999).

4.2.1 Model formulation

Consider a firm facing a price-dependent and uncertain demand has to make three decisions characterized by a two-stage hierarchical structure. At the first stage, the firm has to decide the selling price P for a single product and about capacity acquisition C. These decisions have to be made in the presence of demand uncertainty. At the second stage, i.e., when the firm observes the actual demand, it determines the production quantity.

Capacity acquisition is subject to one-time investment costs c per unit capacity. The demand is represented by an additive, linear, and downward sloping function of price

$$D(P, \Psi) = \Psi - bP \tag{4.1}$$

where Ψ is a non-negative random variable defined on an interval $[A, B]$ with $A \geq 0$, mean μ, and standard deviation σ, and the parameter b describes the price-sensitivity of the customers. CV denotes the coefficient of variation. In order to assure that the demand is positive for some ranges of P, it is assumed that $A - b(c + c_p) \geq 0$ where c_p denotes the unit production cost. Additive demand functions are commonly used in operations and economics literature. This formulation is equivalent to the structure $D(P, \bar{\Psi}) = d(P) + \bar{\Psi}$ where $d(P) = \mu - bP$ is a deterministic and linear price-response function and $\bar{\Psi}$ is a random shock with $\mathbb{E}\{\bar{\Psi}\} = 0$ and $Var\{\bar{\Psi}\} = \sigma^2$, e.g., see Petruzzi and Dada (1999) or Li and Atkins (2002).

We assume that Ψ has a continuous distribution function $F(z)$ with density function $f(z)$. Let $r(z)$ be the *Hazard* (or failure) rate (FR) defined by

$$r(z) = \frac{f(z)}{1 - F(z)}$$

and let $g(z)$ be the generalized failure rate (GFR) defined by

$$g(z) = zr(z).$$

The GFR is the percentage change in the excess demand with respect to the stocking level, which can be interpreted as the elasticity of the excess demand with respect to capacity. For a detailed analysis between FR and GFR, see Lariviere and Porteus (2001) and Lariviere (2006).

Although the pricing issue is only a minor extension of the traditional newsvendor problem, the analytical tractability becomes more difficult. Kocabiyikoglu and Popescu (2007) introduce the concept of the *elasticity of lost sales*. The elasticity of lost sales for a given capacity level C and price P denoted by $\varepsilon_s(P,C)$ is the percentage change in the rate of lost sales with respect to price, i.e.,

$$\varepsilon_s(P,C) = \left| \frac{\%\text{ change in lost sales}}{\%\text{ change in price}} \right|.$$

Definition 2. *The elasticity of lost sales corresponding to the stochastic and price sensitive demand $D(P,\Psi)$ and the capacity level C is defined as*

$$\varepsilon_s(P,C) = -\frac{(P - c_p)}{(1 - F_{D(P)}(C))} \frac{\partial(1 - F_{D(P)}(C))}{\partial P} = \frac{(P - c_p)\frac{\partial F_{D(P)}(C))}{\partial P}}{(1 - F_{D(P)}(C))}$$

where $F_{D(P)}(C)$ defines the distribution function of the random demand.

Kocabiyikoglu and Popescu (2007) show that this concept allows to derive structural properties of the objective function and the optimal solution. The objective is to maximize the expected profit which is

$$\Pi^* = \max_{C,P} \left\{ \int\limits_{A}^{C+bP} (P - c_p)(z - bP)f(z)dz + \int\limits_{C+bP}^{B} (P - c_p)Cf(z)dz - cC \right\}. \tag{4.2}$$

If the demand during the period is lower than the capacity, i.e., $z - bP \leq C$, then the profit is $(P - c_p)(z - bP)$ where z denotes a realization of the random variable Ψ. It is assumed that excess capacity $C - (z - bP)$ does not cause any costs and cannot be used otherwise. If demand exceeds the capacity, i.e., $z - bP > C$, then the profit is $(P - c_p)C$ and all units $(z - bP) - C$ are lost.

For mathematical convenience, we use the variable transformation $K = C +$

bP which simplifies the computation. An intuition of K can be obtained by a rearrangement of the transformation to

$$K - \mu = C - (\mu + bP). \tag{4.3}$$

$K - \mu$ can be interpreted as the safety capacity, which is defined as the deviation of the capacity acquisition from the expected demand, so that K is the expected market potential plus a safety capacity (Petruzzi and Dada, 1999). By executing some algebraic transformation, the expected profit is

$$\Pi(K, P) = \Lambda(P) - L(K, P) \tag{4.4}$$

with

$$\Lambda(P) = (P - c - c_p)(\mu - bP) \tag{4.5}$$

and

$$L(K, P) = c \int_A^K (K - z) f(z) dz + (P - c - c_p) \int_K^B (z - K) f(z) dz. \tag{4.6}$$

$\Lambda(P)$ represents the profit if there is no variation of demand from its mean. $L(K, P)$ represents the expected loss function which is the sum of the expected overage and expected underage costs (Silver et al., 1998). For the case that z is lower than K, leftovers $(K - z)$ are assessed with overage costs c. If z exceeds K, then shortages $(z - K)$ are assessed with underage costs $(P - c - c_p)$.

4.2.2 Structural properties

Necessary conditions for a maximum are $\frac{\partial \Pi}{\partial P} = 0$ and $\frac{\partial \Pi}{\partial K} = 0$. The first-order condition of (4.4) w.r.t. K and P give

$$\frac{\partial \Pi}{\partial K} = 0 \Longleftrightarrow K(P) = F^{-1}\left(\frac{P - c_p - c}{P - c_p}\right) \tag{4.7}$$

and

$$\frac{\partial \Pi}{\partial P} = 0 \Longleftrightarrow P(K) = \frac{\mu + b(c_p + c)}{2b} - \frac{1}{2b} \int_K^B (z - K) f(z) dz. \tag{4.8}$$

Therefore, K^* can be expressed as a function of P and P^* can be expressed as a function of K.

From the second derivatives of (4.4) w.r.t. P and K

$$\frac{\partial^2 \Pi}{\partial K^2} = -(P - c_p) f(z) \tag{4.9}$$

and

$$\frac{\partial^2 \Pi}{\partial P^2} = -2b \tag{4.10}$$

we get that for a given K (4.4) is concave in P and for a given P (4.4) is concave in K. However, it can be easily shown that for general distribution functions (4.4) is not necessarily jointly concave in P and K. We will show later which conditions lead to a unique optimal solution.

The intuition of (4.7) is obtained by a rearrangement to

$$cF(K) = (P - c - c_p)(1 - F(K)).$$

The objective is to find the optimal K^* where the expected loss of having one unit unused capacity is equal to the expected gain of having one additional capacity unit. The expected loss of having one unit unused capacity is equal to the unit capacity costs times the probability that the demand is lower than the capacity level. The expected gain of an additional capacity unit is the benefit of using this unit in order to produce and sell an additional product, i.e., the selling price minus capacity and production costs times the probability that the demand exceeds the capacity. From (4.3) it follows that

$$F^{-1}\left(\frac{P - c_p - c}{P - c_p}\right) - \mu = C - (\mu - bP),$$

which represents the optimal safety capacity. Resubstitution $C = K - bP$ yields that the optimal capacity level is determined by

$$C^* = F^{-1}\left(\frac{P - c - c_p}{P - c_p}\right) - bP.$$

The term $\left(\frac{P-c-c_p}{P-c_p}\right)$ represents the ratio between underage cost c_u and overage cost c_o, i.e., $\frac{c_u}{c_u+c_o}$ where $c_u = P - c - c_p$ and $c_o = c$ (see (4.6)). This ratio is also called *newsvendor ratio.*

The following proposition states the impact of demand uncertainty on the selling price (e.g., see Mills (1959) and Petruzzi and Dada (1999)).

Proposition 18. *If $D(P, \Psi)$ is an additive price-response function, then the optimal selling price P^* is lower than the price P_0 that maximizes the profit*

$\Lambda(P) = (P - c - c_p)(\mu - bP)$, *i.e.*,

$$P^* \leq P_0$$

with

$$P_0 = \frac{\mu + b(c_p + c)}{2b}. \tag{4.11}$$

The proof directly follows from (4.8) since the expected shortfall $\Theta(K) := \int_K^B (z - K) f(z) dz$ is nonnegative.

The intuition of Proposition 18 is the incentive of the firm to reduce uncertainty (Petruzzi and Dada, 1999). With rising uncertainty there is a rising risk of overcapacity and capacity shortage. In case of a fixed price, this risk can only be hedged with an adjustment of capacity investment (safety capacity). However, if the firm simultaneously optimizes price and capacity, uncertainty is balanced by price and capacity adjustments. In case of symmetric distributions it follows that if $P \geq 2(c + c_p) \Leftrightarrow \left(\frac{P-c-c_p}{P-c_p}\right) \geq \frac{1}{2}$, which equivalently means that underage costs exceed overage costs, then increasing uncertainty yields an increasing investment in safety capacity. On the other hand, if $P < 2(c + c_p) \Leftrightarrow \left(\frac{P-c-c_p}{P-c_p}\right) < \frac{1}{2}$, then overage costs exceed underage costs. In this case, it is more expensive to have one unit leftover than one unit shortage which yields that an increasing demand uncertainty leads to decreasing capacity acquisition.

In how far price optimization affects expected overage and underage costs depends on the characteristic of the price-response. If the price-response function is additively connected with the random variable, then for a given K the selling price does not influence expected overcapacity and expected shortfall but it just influences the expected demand. Thus, while a price increase yields a decreasing CV, the variance of the demand remains constant because

$$Var(D(\Psi, P) = \Psi - bP) = Var(\Psi) \text{ and CV} = \frac{\sqrt{Var(D(\Psi, P))}}{\mu - bP}.$$

For the sake of completeness the multiplicative case is compared. If a deterministic price-response function $d(P)$ is multiplicatively connected with the random variable Ψ, i.e., $D(P) = d(P)\Psi$ with $\mathbb{E}(\Psi) = 1$ (all multiplicative demand functions can be normalized to $\mathbb{E}(\Psi) = 1$)(Petruzzi and Dada, 1999), then

$$Var(D(\Psi, P) = \Psi \cdot d(P)) = d(P)^2 Var(\Psi) \text{ and CV} = \sigma.$$

In this case, it can be seen that while price changes affect the variance of demand, the CV is unaffected. Therefore, in order to reduce uncertainty, i.e., to decrease the CV, the selling price has to be increased (Karlin and Carr, 1962).

Figures 4.1 and 4.2 illustrate the impact of increasing demand uncertainty on price and capacity decisions in the traditional newsvendor problem and in the price-setting newsvendor problem. In both cases, increasing demand uncertainty changes the characteristic of the density function. In the traditional newsvendor problem (see Figure 4.1) it can be observed that an increasing uncertainty (σ increases to σ') only affects the shape of the curve but not the position. That is, increasing uncertainty does not influence the mean of demand but the only impact of increasing uncertainty is an increasing C.

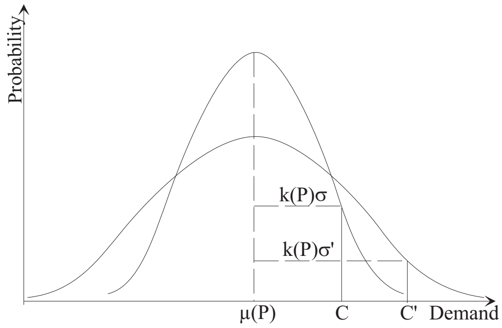

Figure 4.1: Impact of increasing uncertainty in the traditional newsvendor problem

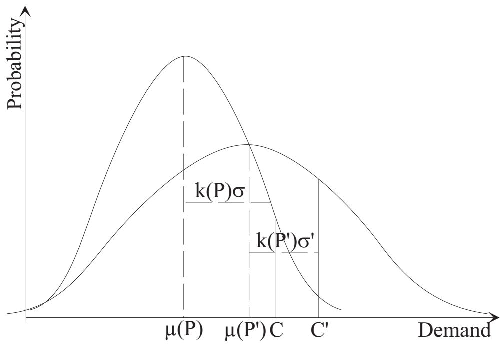

Figure 4.2: Impact of increasing uncertainty in the price-setting newsvendor problem

Figure 4.2 demonstrates that increasing demand uncertainty affects both shape and position of the density function. Since the price decreases with increasing uncertainty, the expected demand increases. This yields that the density function shifts to the right. As consequence, the additional safety capacity which is acquired to hedge increasing uncertainty is lower in the price-setting newsvendor problem than in the traditional newsvendor problem with an exogenous price. The following proposition shows the monotonicity of the optimal price $P^*(K)$ and the optimal capacity $K^*(P)$.

Proposition 19. *The optimal price and the optimal capacity are concave increasing in their respective arguments. That is,*

$$\frac{\partial P^*(K)}{\partial K} \geq 0, \; \frac{\partial K^*(P)}{\partial P} \geq 0 \; \textit{and} \; \frac{\partial^2 P^*(K)}{\partial K^2} \leq 0, \; \frac{\partial^2 K^*(P)}{\partial P^2} \leq 0.$$

The proof is presented in Appendix A.11.

For general distribution functions, (4.4) is not necessarily jointly concave in P and K. However, for specific functional forms of price-response and distribution functions the existence and uniqueness of an optimal capacity level and optimal selling price has been shown. Petruzzi and Dada (1999) show that if the failure rate $r(z)$ satisfies $2r(z)^2 + r'(z) > 0$, then there exists a unique pair (K^*, P^*) which maximizes (4.4). Yao et al. (2006) provide a more general condition. They show that if the distribution function F has an increasing GFR, then $\Pi(P, K^*(P))$ is a quasi-concave function in P which guarantees the existence and uniqueness of an optimal solution. Kocabiyikoglu and Popescu (2007) show that if $\varepsilon_s(P, K) > \frac{1}{2}$, then (4.4) is jointly concave in P and K, which ensures that a unique (K^*, P^*) exists.

Proposition 20. *There exists a unique solution (K^*, P^*) which solves (4.8) and (4.7).*

The proof is represented in Appendix A.12. In the following section, an iterative algorithm for computing the optimal price and capacity decision is presented.

4.2.3 Solution procedure

The following algorithm, generally formalized by Zhan and Shen (2005), can be used to determine the optimal solution. This algorithm starts with the price P_0 which represents an upper bound for P. Then it calculates the value $K^0 = K(P_0)$ by solving (4.7). Given K^0, the algorithm calculates a new price $P^1 = P(K^0)$ which is the input to update K. The iteration is repeated until the absolute difference $|P^n - P^{n-1}|$ is lower than a precision criterion ε.

BEGIN

Set $\delta > \varepsilon$,

Calculate P^0 from (4.11),

REPEAT

Calculate $K^0 := K(P^0)$ using (4.7),

Calculate $P^1 := P(K^0)$ using (4.8),

Calculate $\delta := |P^1 - P^0|$,

$P^0 := P^1$,

UNTIL $\delta \leq \varepsilon$

$P^* := P^0$ and $K^* := K^0$

END

Thus, the optimal capacity level is determined by $C^* = K^* - bP^*$.

4.2.4 Summary and implications

This section presented a single-product single-period capacity decision and pricing problem of a firm facing uncertain demand. This problem is known as price-setting newsvendor problem. While the traditional newsvendor problem assumes that the selling price is exogenous, in the price-setting newsvendor problem the selling price is, beside the initial capacity, an additional decision variable. Incorporating pricing issues into capacity decision problems provides an important vehicle for examining how operational problems interact with marketing issues.

The focus of this section was two-fold: First, structural properties of an optimal pricing strategy were analyzed and it was investigated how the pricing issue affects the firm's optimal capacity decision. Second, since the problem is rather complex so that no closed-form solutions exist, an algorithm for computing the optimal price and capacity decision was presented. Differences between the traditional and the price-setting newsvendor problem were revealed on how increasing demand uncertainty is hedged in decision-making

The results of this section indicated how demand uncertainty is balanced by price and capacity adjustment. While in the traditional newsvendor problem increasing uncertainty is solely balanced from the supply side, i.e., capacity adjustment, it was shown that this is not necessarily optimal. In order to optimally match supply with demand both the supply and demand side have to be adjusted in an appropriate ratio. From the supply-side, increasing uncertainty is balanced by capacity adjustment in order to reduce expected underage and overage costs. From the demand perspective, increasing uncertainty is balanced by a decreasing price. The lower selling price has two effects. First, it decreases underage costs and second the expected demand increases. This, in turn, decreases the expected overcapacity. Consequently, from a company point of view that maximizes the total profit, a simultaneous optimization yields a better match of supply with demand than an independent optimization of capacity, given a fixed selling price.

The following section investigates the impact of customer segmentation on price decision and capacity acquisition in an integrated planning approach.

4.3 The price-setting Newsvendor model with customer segmentation

This section studies an extension of the price-setting newsvendor problem where the firm is able to segment the market into two customer classes called H and L, respectively (*high-class* and *low-class* customers). Market segmentation either results from product differentiation or from price discrimination. If price discrimination is not possible because of arbitrage and cannibalization, the firm can use product differentiation, e.g., different brands or different quality levels in order to avoid these effects. In case of product differentiation, both products are produced with the same resource. The decision-making structure is hierarchical. At the first stage the decision maker has to establish the capacity acquisition and the two selling prices that are charged for each customer class. These decisions are made in the presence of demand uncertainty. At the second stage when the demand uncertainty is resolved, the decision maker establishes production quantities.

The majority of papers that integrate pricing issues into capacity decision problems with two demand classes (or two products) assume responsive pricing, i.e., the capacity decision has to be made under demand uncertainty whereas the pricing decision can be postponed after uncertainty is resolved (Chod and Rudi (2005) and Bish and Wang (2004)). In this section it is assumed that both pricing and capacity decisions have to be made under demand uncertainty.

The contribution of this section is to investigate the interaction of the selling prices that are charged for each customer class and the capacity investment with respect to different parameter values, e.g., capacity costs and uncertainty. A stochastic model is developed and structural properties of the objective function and the optimal solution are derived. Then, a numerical example is presented that demonstrates the benefits of customer segmentation and price differentiation and derives properties on the interaction prices and capacity. Additionally, coordinated decision-making is compared to decentralized decision-making where independent sales managers are responsible for price management and capacity acquisition.

4.3.1 Model formulation

It is considered that the demand can be segmented into two demand classes, H and L, which are characterized by independent price-response functions. For both demand classes an individual price is charged. P_H is the price that is charged to customer class H while P_L is the price that is charged to customer class L. It is assumed that the firm is able to perfectly segment the market. There are no substitution effects between the demand classes, i.e., the demand of both classes is only influenced by their respective price and not by the price that is charged to the other class.

The demand of H is modeled by an additive random, downward-sloping, and linear function

$$D_H := D(P_H, \Psi) = \Psi - bP_H$$

where the market potential Ψ is a random variable defined on the interval $[A, B]$ with $A \geq 0$, mean μ, standard deviation σ, and parameter b that describes the price-sensitivity of the H-class customers. A realization of Ψ is represented by z such that $d_H = z - bP_H$ denotes a realization of D_H. $f(z)$ and $F(z)$ represent the probability density function and the cumulative density function of Ψ, respectively.

Demand for L is modeled by a deterministic, downward-sloping, and linear price-response function with

$$d_L := d(P_L) = \alpha - \beta P_L$$

for $0 \leq P_L \leq \frac{\alpha}{\beta}$ and 0 otherwise.

Customer segmentation can be achieved either by product differentiation or price discrimination. Product differentiation is characterized by different production costs where production of one unit of H and L is subject to variable production cost c_H and c_L, respectively. It is assumed that $c_H > c_L$ for product differentiation and $c_H = c_L$ if price discrimination is applied. Capacity acquisition is subject to capacity cost of c per unit capacity. For the sake of simplicity, it is assumed that production of H and L consumes the same amount of capacity of one unit per product.

As in the previous section, the firm faces a two-stage decision-making process. At the first stage, the capacity level C and the selling prices P_H and P_L are decided under demand uncertainty. At the second stage when the demand is known, the firm decides on production quantities. If the total demand H plus L exceeds the capacity, it is assumed that H-class customers have a higher priority than L-class customers, which is reasonable, for instance, when customers have a prioritized customer support (Anderson and Dana, 2006). In particular, the initial capacity C is allocated by a priority rule which states that H-class demand is satisfied before L-class demand. All unmet demand is lost. An implication following this priority rule is that despite of L-class demand being deterministic, since the uncertain H-class demand is satisfied first, the available capacity to produce L is uncertain. Consequently, L-class sales are uncertain as well.

As in the single-product problem, we define the measure of the elasticity of lost sales for both products H and L (Kocabiyikoglu and Popescu, 2007).

Definition 3. *The elasticity of lost sales of H corresponding to its stochastic and price sensitive demand $D(P_H,\Psi)$ and the capacity level C defines the percentage change in the rate of lost sales of H with respect to a change in $(P_H - c_H)$:*

$$\varepsilon_s(P_H, C) = -\left(\frac{P_H - c_H}{1 - F_{D(P_H)}(C)}\right)\frac{\partial(1 - F_{D(P_H)}(C))}{\partial P_H} = \frac{(P_H - c_H)\frac{\partial F_{D(P_H)}(C)}{\partial P_H}}{(1 - F_{D(P_H)}(C))}$$

where $(1 - F_{D(P_H)}(C))$ defines the probability that demand $D(P_H,\Psi)$ exceeds the capacity level C which results in lost sales of H.

The definition of the elasticity of lost sales for L is slightly different. In order to determine the elasticity of lost sales of L with respect to a change in $(P_L - c_L)$, the lost sales probability of L has to be determined by

$$Pr(d(P_L) > C - \Psi + bP_H) = 1 - F_{D(P_H)}(C - d(P_L)).$$

If $D(P_H) \geq C$, then regardless of P_L the entire L-class demand is lost. Therefore, if $D(P_H) \geq C$, the percentage change in the rate of lost sales of L with respect to a change in $(P_L - c_L)$ is equal to zero because a change of P_L does not affect the rate of lost sales. Formally written, this can be expressed as

$$\frac{\partial(1 - F_{D(P_H)}(C))}{\partial P_L} = 0.$$

It follows that

$$\frac{\partial(1 - F_{D(P_H)}(C - d(P_L)))}{\partial P_L} = \frac{\partial(F_{D(P_H)}(C) - F_{D(P_H)}(C - d(P_L)))}{\partial P_L}.$$

Definition 4. *The lost sales elasticity of L corresponding to the stochastic demand $D(P_H,\Psi)$ and the capacity level C is defined as the percentage change in the rate of lost sales of L, with respect to a change in $(P_L - c_L)$:*

$$\varepsilon_s(P_L, C) =$$
$$-\left(\frac{P_L - c_L}{F_{D(P_H)}(C) - F_{D(P_H)}(C - d(P_L))}\right)\frac{\partial(F_{D(P_H)}(C) - F_{D(P_H)}(C - d(P_L)))}{\partial P_L}$$

$$\Leftrightarrow \varepsilon_s(P_L, C) = \frac{(P_L - c_L)\frac{\partial F_{D(P_H)}(C - d(P_L))}{\partial P_L}}{(F_{D(P_H)}(C) - F_{D(P_H)}(C - d(P_L)))} \tag{4.12}$$

where $(F_{D(P_H)}(C) - F_{D(P_H)}(C - d(P_L)))$ defines the probability that $d(P_L) < C - D(P_H,\Psi)$ and $D(P_H,\Psi) \leq C$.

The profit for a given demand realization z is the difference between revenue and total costs (variable manufacturing and capacity costs):

$$\begin{aligned}\pi(C, P_H, P_L) = (P_H - c_H)\min\{z - bP_H, C\} \\ + (P_L - c_L)\min\{d_L, \max\{0, C - (z - bP_H)\}\} - cC. \quad (4.13)\end{aligned}$$

If the H-class demand does not exceed C, then the profit resulting from H is $(P_H - c_H)(z - bP_H)$ and the excess capacity is available to satisfy L-class demand so that L-class sales depend on the remaining capacity. If the L-class demand is lower than the remaining capacity, then L-class sales are equal to L-class demand and the profit is $(P_L - c_L)d_L$. Otherwise, L-class sales are $C - (z - bP_H)$ and the profit is $(P_L - c_L)(C - (z - bP_H))$. If the H-class demand during a period exceeds C, then the profit resulting from H is $(P_H - c_H)C$ and there is no remaining capacity to satisfy L-class demand.

Using the same substitution as in Section 4.2, i.e., $K = C + bP_H$ and taking expectation of (4.13) yields

$$\begin{aligned}\Pi(C, P_H, P_L) = \int_{A}^{C+bP_H-d_L} ((P_H - c_H)(z - bP_H) + (P_L - c_L)d_L)\, f(z)dz \\ + \int_{C+bP_H-d_L}^{C+bP_H} ((P_H - c_H)(z - bP_H) + (P_L - c_L)(C - z + bP_H))\, f(z)dz \\ + \int_{C+bP_H}^{B} (P_H - c_H)Cf(z)dz - cC.\end{aligned}$$

By some algebraic transformations illustrated in Appendix B.4, it follows that

$$\Pi(K, P_H, P_L) = \Lambda(P_H, P_L) - L(K, P_H, P_L) \quad (4.14)$$

with

$$\Lambda(P_H, P_L) = (P_H - c - c_H)(\mu - bP_H) + (P_L - c - c_L)d_L \quad (4.15)$$

and

$$\begin{aligned}L(K, P_H, P_L) = c\int_{A}^{K-d_L} (K - z - d_L)f(z)dz + (P_L - c - c_L)\int_{K-d_L}^{K} (d_L + z - K)f(z)dz, \\ + (P_H - c - c_H)\int_{K}^{B} (z - K)f(z)dz + (P_L - c - c_L)\int_{K}^{B} d_L f(z)dz. \quad (4.16)\end{aligned}$$

The substitution $K = C + bP_H$ provides a mathematical convenience to analyze the price-sensitive newsvendor problem such that K can be interpreted as safety factor or safety capacity (see, Petruzzi and Dada (1999) and Li and Atkins (2002)). $\Lambda(P_H, P_L)$ and $L(K, P_H, P_L)$ represent the expected revenue minus direct costs and the expected loss. As in (4.5), $\Lambda(P_H, P_L)$ is the profit if there is no variation of H-demand from its mean and P_L is set such that the complete L-demand can be satisfied. The expected loss function $L(K, P_H, P_L)$ can be interpreted as in Silver et al. (1998) by expected overage and underage costs. For the case that $z + d_L$ is lower than K, leftovers $(K - z - d_L)$ are assessed with overage costs c. If $z + d_L$ exceeds K and z is lower than K, then shortages for L $(d_L - (K - z))$ are assessed by underage costs $(P_L - c - c_L)$. If z exceeds K, then shortages for H and L occur where H-shortages $(z - K)$ cost $(P_H - c - c_H)$ and L-shortages (d_L) cost $(P_L - c - c_L)$.

4.3.2 Structural properties

This section analyzes structural properties of the objective function and the optimal solution. The first and second derivatives w.r.t. P_H give

$$\frac{\partial \Pi}{\partial P_H} = \mu + b(c + c_H) - \int_K^B (z - K) f(z) dz - 2bP_H \tag{4.17}$$

and

$$\frac{\partial^2 \Pi}{\partial P_H^2} = -2b. \tag{4.18}$$

From (4.18), it follows that for a given K and P_L, (4.14) is concave in P_H.

Proposition 21. *The optimal P_H can be represented as a closed-form function of K:*

$$P_H^*(K) = P_{H_0} - \frac{\Theta(K)}{2b} \tag{4.19}$$

where $P_{H_0} = \frac{\mu + b(c + c_H)}{2b}$ and $\Theta(K) = \int_K^B (z - K) f(z) dz$.

Proposition 21 follows directly from rearranging (4.17). As in the single-product problem, P_{H_0} denotes the optimal price which optimizes the expected profit $\Lambda(P_H, P_L)$ if there is no variation of H and $\Theta(K)$ is the expected shortfall of H (Petruzzi and Dada, 1999). It can be easily shown that $P_H^*(K)$ is increasing in K and lower than P_{H_0}.

The first and second derivatives of (4.14) w.r.t. P_L are

$$\frac{\partial \Pi}{\partial P_L} = F(K - d_L)\left(\alpha - 2\beta P_L + \beta c_L\right) + \int_{K-d_L}^{K} (K - z) f(z) dz \tag{4.20}$$

and

$$\frac{\partial^2 \Pi}{\partial P_L^2} = -2\beta F(K - d_L) - \beta^2 (P_L - c_L) f(K - d_L) \leq 0. \tag{4.21}$$

From (4.21) we get that for a given K, $\Pi(K, P_H^*(K), P_L)$ is a concave function of P_L.

Theorem 2. *For a given K, (4.14) is jointly concave in P_H and P_L.*

Since the cross derivatives $\frac{\partial^2 \Pi}{\partial P_H \partial P_L}$ and $\frac{\partial^2 \Pi}{\partial P_L \partial P_H}$ are equal to zero, the proof of Theorem 2 follows directly from the second derivatives (4.18) and (4.21).

Proposition 22. *For a fixed K, the optimal P_L can be represented as*

$$P_L^*(K) = P_{L_0} + \frac{1}{2\beta F(K - d_L)} \int_{K-d_L}^{K} (K - z) f(z) dz \tag{4.22}$$

where $P_{L_0} = \frac{1}{2\beta}\left(\alpha + \beta c_L\right)$.

Proposition 22 follows directly from (4.20). P_{L_0} is the optimal price given that the marginal cost only consists of unit production cost c_L and sunk capacity costs. However, P_L^* cannot be given as a closed-form function of K.

Conjecture 2. *The optimal selling price for L satisfies the following condition:*

$$\frac{1}{2\beta}\left(\alpha + \beta c_L\right) = P_L^0 \leq P_L^* \leq P_L^1 = \frac{1}{2\beta}\left(\alpha + \beta(c + c_L)\right).$$

The intuition of this conjecture is that a decision maker who simultaneously optimizes prices and capacity takes into account that potential excess capacity of H can be used to produce L. The inequality $P_L^0 \leq P_L^*$ directly follows from Proposition 22. However, a proof of the inequality $P_L^* \leq P_L^1$ remains outstanding and is subject to future research.

The first-order derivative of (4.14) w.r.t. K gives

$$\frac{\partial \Pi}{\partial K} = (P_H - c_H)(1 - F(K)) + (P_L - c_L)(F(K) - F(K - d_L)) - c. \tag{4.23}$$

The interpretation of this equation is obtained by rearranging the first-order condition $\frac{\partial \Pi}{\partial K} = 0$ into

$$(P_H - c_H - c)(1 - F(K)) + (P_L - c_L - c)(F(K) - F(K - d_L)) = cF(K - d_L). \quad (4.24)$$

which states that the optimal K has to be set such that expected underage costs (left-hand side) are equal to expected overage costs (right-hand side).

Even for the simple single-product price and capacity decision problem a unified solution framework and a general understanding of what drives the tractability of this problem is lacking (Kocabiyikoglu and Popescu, 2007) and the effort required to compute P_H^*, P_L^* and K^* depends on structural properties of the demand function. The following theorems provide properties that guarantee the existence and uniqueness of an optimal solution.

Theorem 3. *If the distribution $F(z)$ satisfies that its failure rate $h(z) = \frac{f(z)}{1-F(z)}$, is increasing and concave in z, then for a given P_L, $\Pi(K, P_H^*(K), P_L)$ is quasi-concave and has a unique maximum $K^*(P_L)$.*

The proof is presented in Appendix A.13. The next results show that the existence of a unique solution can be guaranteed by a lower bound of the lost sales elasticity.

Theorem 4. *If $\varepsilon_s(P_L, K) \geq 1$, then for a given P_H which is sufficiently large, $\Pi(K, P_H, P_L)$ is jointly concave in K and P_L. Since $P_H^*(K)$ is concavely increasing in K, there exists a unique optimal solution (K^*, P_H^*, P_L^*).*

The proof is presented in Appendix A.14. If the problem satisfies the conditions of Theorem 3 and 4, the optimal solution can be determined efficiently. Otherwise, an exhaustive numerical search is needed.

The following section presents an algorithm in order to determine the optimal prices P_H^* and P_L^* as well as the optimal capacity C^* efficiently for all profit functions satisfying these conditions.

4.3.3 Solution procedure

The algorithm starts with the initial solution (K^0, P_H^0) obtained from the single-product problem of Section 4.2 that considers H as the single product. The initial values K^0 and P_H^0 are determined by using the algorithm from Section 4.2.3. Given the initial solution (K^0, P_H^0), the algorithm calculates P_L^0 based on equation (4.22) which is the input to update K^0 to K^1. The algorithm stops when the absolute difference $|K^0 - K^1|$ falls below a precision criterion ε. Termination of the algorithm is assured by Theorem 4.

Algorithm

Calculate the initial values P_H^0 and K^0 from (4.7) and (4.8) by using the algorithm from Section 4.2,

Calculate $P_L^0 := P_L(K^0)$ by using (4.22),

REPEAT

Calculate $K^1 := K(P_L^0)$ using (4.24),

Calculate $P_H^1 := P_H^*(K^0)$ using (4.19),

Calculate $P_L^1 := P_L^*(K^0)$ using (4.22),

Calculate $\delta := |K^0 - K^1|$,

$K^0 := K^1$, $P_H^0 := P_H^1$ and $P_L^0 := P_L^1$

UNTIL $\delta \leq \varepsilon$

$P_H^* := P_H^0$, $P_L^* := P_L^0$, and $K^* := K^0$

END

4.3.4 The benefits of customer segmentation

This section presents a numerical example that illustrates the impact of customer segmentation on price and capacity decisions as well as its benefit compared to a single-price strategy. Two strategies are compared. The first strategy denoted as CS (customer segmentation) presumes that the firm is able to segment the market and can differentiate selling prices between the two customer classes. The second strategy denoted as NCS (non-customer segmentation) presumes that the firm is able to segment the market but cannot discriminate prices so that a single price is charged for both customer classes. Note that NCS is a special case of CS with the assumption that $P_H = P_L$.

Consider H-class demand to be characterized by Ψ which is uniformly distributed on the interval $[500, 1500]$ and sensitivity coefficient $b = 20$. L-class demand is deterministic with $d_L = \alpha - \beta P_L$, $\alpha = 1000$, and $\beta = 40$. It is assumed that the expected market potential of H and the market potential of L are identical whereas the price sensitivities of both customer classes deviate. In this example, the price-sensitivity of L-class customers is larger than the price-sensitivity of H-class customers. This assumption is reasonable when L-class customers are more price-oriented and have (on average) a lower willingness to pay. The unit production costs are identical for both products with $c_H = c_L = 2$, hence, price discrimination is considered.

To demonstrate the impact of capacity costs, c is varied between 1 and 19. In particular, a value of $c = 1$ corresponds to inexpensive capacities while a value of $c = 19$ corresponds to expensive capacities. The firm simultaneously decides on selling prices P_H and P_L charged for each customer class as well as capacity acquisition C.

Impact of customer segmentation on selling prices

Figures 4.3 illustrates the impact of price discrimination on selling prices for different capacity costs c. It can be observed that P^*, the optimal single price under NCS, is interleaved between P_H^* and P_L^* which are the optimal prices under CS. In case of NCS, the firm faces a lost gain of $P_H^* - P^*$ for each H-class customer served. On the other hand, the CS-price that is charged to L-class customers is lower than P^*. Although the firm additionally gains an extra payment of $P^* - P_L^*$ per unit sold, it indirectly faces lost sales from any customers who is not willing to pay the higher price.

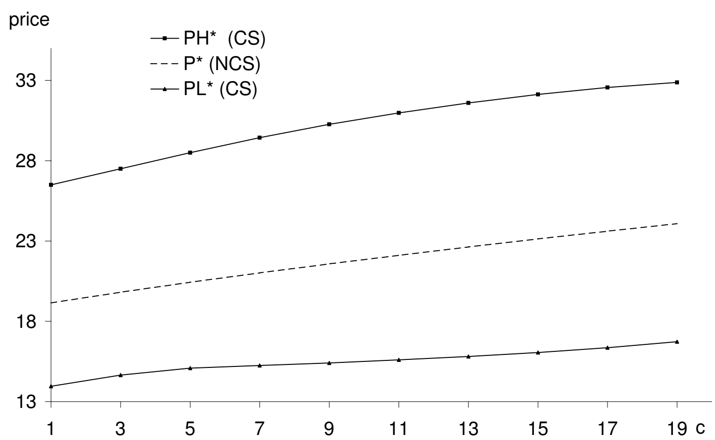

Figure 4.3: Impact of customer segmentation on price

For both strategies CS and NCS, the selling prices increase with increasing capacity costs c. At first sight, this is an intuitive result because increasing costs have to be balanced by increasing prices. However, we will see in case of higher uncertainty this result does not hold in general.

Another effect which can be observed from Figure 4.3 is that increasing capacity costs have different impacts on P_H^* and P_L^*. The price curves indicate that P_H^* increases more in c than P_L^*. This effect is intuitive because of the higher price-sensitivity of L-class customers. For example, if both prices increase by one unit, then the demand decrease of L-class customers is larger than the one of H-class customers.

Another finding is that for expensive capacities L appears to be unprofitable. When c is larger than 15, the selling price is lower than the total costs, i.e.,

$P_L^* < c + c_L$. At first sight, it appears to be unprofitable to serve L-class customers. However, the deterministic nature of L-class demand reduces the risk of overcapacity of the entire system, such that L-class demand serves as a safety buffer for the higher priority H-class demand and as a real option, which is anticipated by the decision maker.

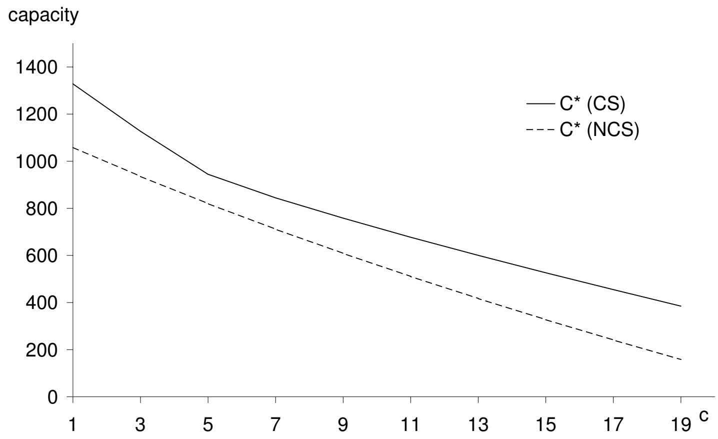

Figure 4.4: Impact of customer segmentation on capacity

Impact of customer segmentation on capacity acquisition

Figure 4.4 illustrates the capacity effect. Two results can be observed: first, capacity acquisition decreases in capacity cost and second, CS leads to higher capacity acquisition than NCS. From economic theory it is known that price discrimination enables a firm to capture more consumer surplus which, in turn, leads to increasing demand. In order to satisfy this demand the firm has to increase capacities.

The effect that the capacity gap between CS and NCS decreases for $c = 1, \ldots, 5$ and then increases from $c = 5, \ldots, 19$ results from the positive lower bound of the uniform distribution. If c is lower than 5, capacity acquisition is sufficiently cheap such that it is beneficial to acquire a capacity level which is able to satisfy all H-class demand, independent of demand realization, i.e., $C^* \geq B - bP_H$. In this case, expected underage costs of H are equal to zero. However, if $c \geq 5$, then $C^* < B - bP_H$, i.e., there is a positive probability of H having a capacity shortfall such that the capacity gap between CS and NCS increases in c.

Benefit of customer segmentation

Figure 4.5 illustrates the performance improvement of CS compared to NCS. The results indicate that the difference in profit between CS and NCS is roughly

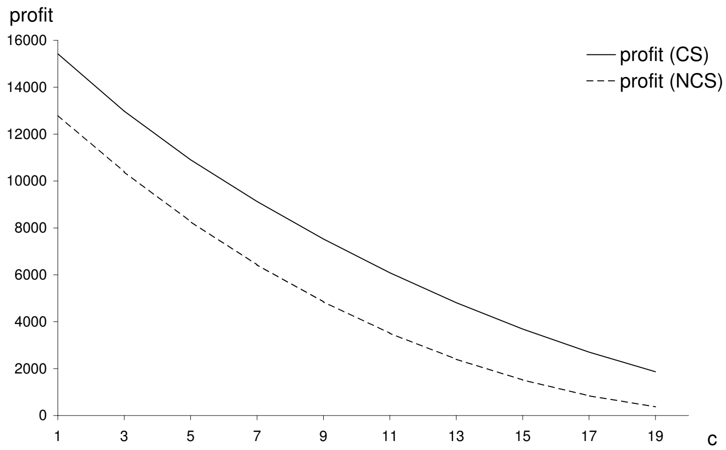

Figure 4.5: Impact of customer segmentation on profit

constant for different capacity costs, and it appears that capacity costs do not influence the performance improvement of CS compared to NCS.

Impact of demand uncertainty

In order to analyze the impact of demand uncertainty on price and capacity decisions, the CV is varied. For uniformly distributed random variables on the interval $[A, B]$, it follows that $\mu = \frac{A+B}{2}$ and $\sigma = \sqrt{\frac{(B-A)^2}{12}}$. Up to now, Ψ has been uniformly distributed on the interval $[500, 1500]$, which corresponds to a CV $= 0.28$. If we increase the interval to $[200, 1800]$ the mean remains unchanged but the CV increases to CV $= 0.46$.

Impact of demand uncertainty on selling prices

Figure 4.6 illustrates the impact of demand uncertainty and capacity costs on selling prices for both CS and NCS. The dashed curves represent the optimal prices given a moderate demand uncertainty with CV$=0.28$, while the solid curves represent the optimal prices given that demand uncertainty is high with CV$=0.46$. Similar to Figure 4.3, the two curves in the middle represent the optimal single price under NCS whereas the two upper and lower curves represent the prices under CS.

First, the most obvious result is that all selling prices, in case of CS and NCS decrease with increasing demand uncertainty. The intuition of this effect is the same as analytically shown in Section 4.2. Increasing demand uncertainty implies a higher risk of overage and underage costs. Let (C^*, P_H^*, P_L^*) be the optimal solution for a particular CV$_1$. If the CV increases to a larger value CV$_2$, it follows

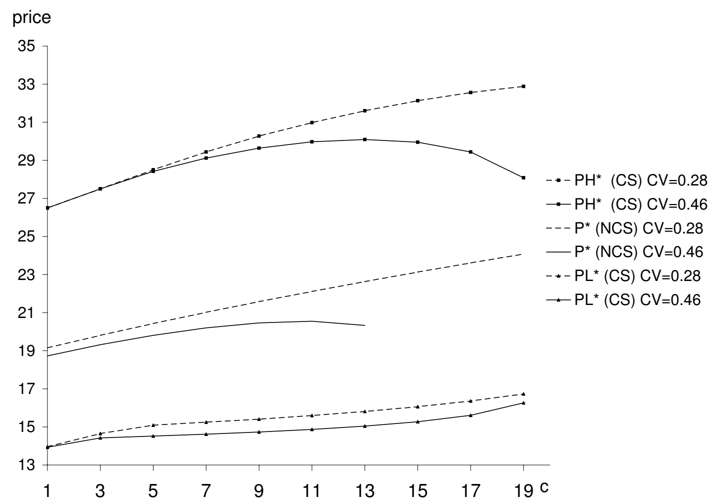

Figure 4.6: Impact of demand uncertainty on prices of CS and NCS

that the expected sales decrease. In order to decrease this risk, it is optimal to decrease the price, which, in turn, leads to an increase in expected demand and, additionally, to an increase in expected sales.

Another result is that in case of NCS and high demand uncertainty the price curve stops at $c = 13$ which implies that production is not beneficial when c exceeds 13. This also can be seen from Figure 4.8, if capacity investment costs are larger than 13, then the expected profit of a single-pricing strategy drops to zero. Consequently, a product which is unprofitable under a single-pricing strategy can become profitable when the firm is able to discriminate prices.

An effect that could not be observed in case of moderate demand uncertainty, i.e., CV = 0.28 is that selling prices are not necessarily monotonous in c. For CV = 0.28, it appears that prices are monotonously increasing in c. For CV = 0.46 and low values of c, however, P_H^* increases in c, and for large values of c, P_H^* decreases. The cause of these rather counter-intuitive results can be explained by two covering effects influencing P_H^*. Recall Proposition 21 states that

$$P_H^*(K) = \frac{1}{2b}\left(\mu + b(c + c_H) - \int_K^B (z - K) f(z) dz\right). \tag{4.25}$$

To gain an intuition of these two effects, we take a look at the framework of a decentralized decision-making where price and capacity are optimized independently. Consider that P_H is optimized without taking capacity restrictions into account and for simplification let $P_L = c_L$, therefore, $P_H^* = \frac{1}{2b}(\mu + b(c + c_H))$. It is easy to see that P_H^* increases in c. On the other hand, for a given P_H it follows from (4.23) that $K = F^{-1}\left(\frac{P_H - c - c_H}{P_H - c_H}\right)$ decreases in c which implies that the expected shortfall $\Theta(K) = \int_K^B (z-K)f(z)dz$ (see Proposition 21) is increasing. Therefore, from (4.25) it follows that the two covering effects have a direct and an indirect effect on P_H^*. The direct effect leads to an increase of P_H^* and is linear in c whereas the indirect effect leads to a decrease of P_H^*. From Figure 4.6 it can be seen that with increasing c the indirect impact is stronger than the direct impact which means that the selling price decreases in c.

Impact of demand uncertainty on capacity acquisition

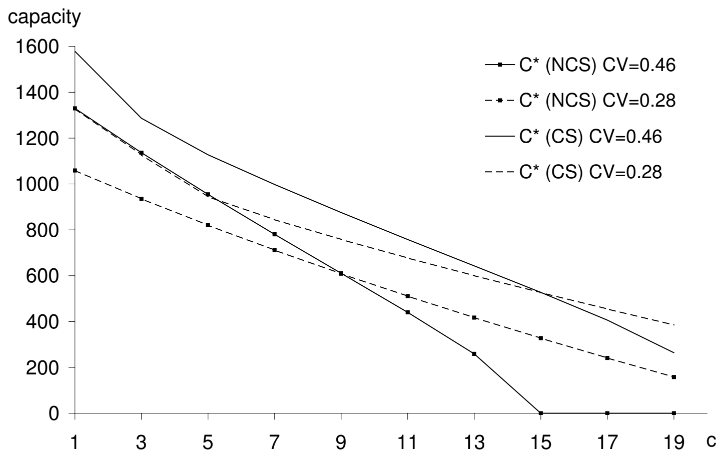

Figure 4.7: Impact of demand uncertainty on capacity of CS and NCS

Figure 4.7 illustrates the impact of demand uncertainty on capacity investments for both CS and NCS. The solid curves represent optimal capacity acquisition when the demand uncertainty is high (CV = 0.46) and the two dashed curves represent optimal capacity decisions under a moderate demand uncertainty (CV = 0.28). The entirely solid and dashed curves represent the CS-strategy, while the dotted solid and dotted dashed curves represent the NCS-strategy.

As with moderate demand uncertainty, capacity acquisition decreases in capacity costs. Moreover, it can be observed that for inexpensive capacities, increasing demand uncertainty leads to increasing capacity acquisition, contrary to expensive capacities where increasing uncertainty leads to a capacity decrease.

The intuition behind this effect is a different ratio of underage and overage costs between inexpensive and expensive capacities. For inexpensive capacities costs of having a capacity shortfall are larger than costs of having overcapacities. Therefore, an increasing demand uncertainty increases the risk of not being able to satisfy demand, which, in turn, leads to a higher capacity buffer. However, this ratio changes for expensive capacities where the costs of a having overcapacities exceed the costs of having a capacity shortage, such that an increase in uncertainty reduces the capacity buffer.

For both CS and NCS, the point where the solid curve intersects the dashed curve represents the ratio where overage costs are equal to underage costs. While this point is reached at $c = 15$ for CS, it is already reached at $c = 9$ for NCS, which implies that a firm applying CS becomes more flexible in its demand management and achieves a higher profitability than a firm applying NCS.

Impact of demand uncertainty on the benefit of customer segmentation

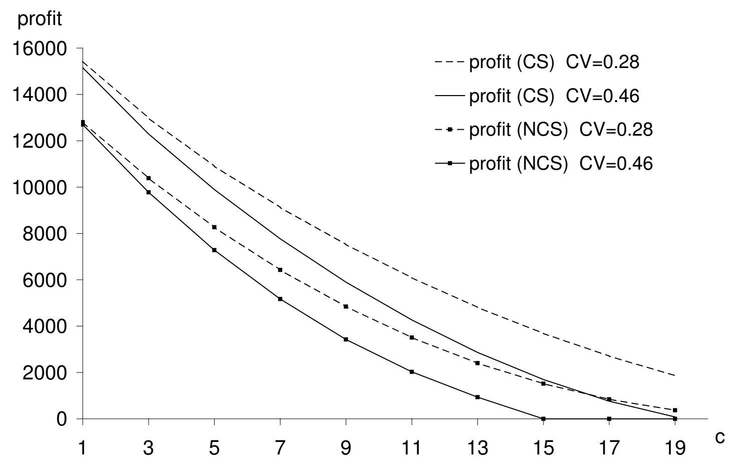

Figure 4.8: Impact of demand uncertainty on the performance of CS and NCS

Figure 4.8 illustrates the performance improvement of CS compared to NCS and the impact of demand uncertainty. The results indicate that the performance improvement of CS compared to NCS is roughly identical for inexpensive and expensive capacities regardless of the degree of uncertainty.

In summary, this example illustrated the impact of CS on price and capacity decisions and the benefit compared to a NCS strategy. CS does not only influence the demand but also affects capacity decisions. By applying CS a firm achieves a higher flexibility in supply and demand management than by NCS. In particular, customized pricing entails a different price for each customer class according to their respective willingness to pay. Resulting flexibility is higher and yields higher profits for both products, and it changes the ratio of underage and overage costs in such a way that it becomes more expensive not to serve a customer than to have excess capacities (underage costs exceed overage costs).

4.3.5 The benefits of resource centralization

This section considers that all business activities concerning different customer classes are managed by independent sales managers. These sales managers are responsible for setting the selling prices and reserving capacities. Traditionally, sales managers plan their activities independent. It is known that decentralized planning yields only suboptimal performance compared to an integrated planning approach. It is yet unknown, however, how far decentralized decisions deviate from simultaneous decisions, especially, how pricing and capacity decisions depend on each other, and how much performance improvement can be achieved by simultaneous decision-making.

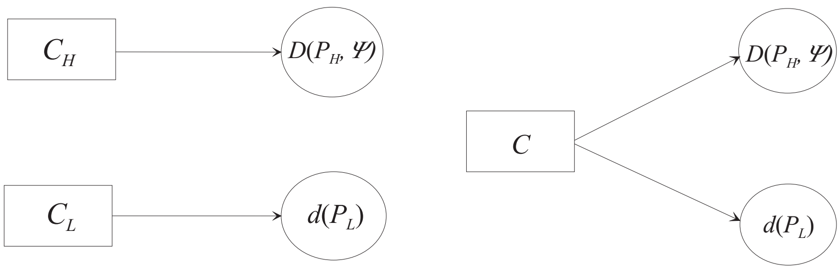

Figure 4.9: Decentralized versus simultaneous planning

This section compares decentralized decision-making where two sales managers, which are either responsible for H or L independently determine required capacities (dedicated capacities) and selling prices to simultaneous decision-making. Figure 4.9 illustrates the relationship between capacity and demand class allocation in case of decentralized and simultaneous decision-making.

In decentralized planning it is considered that sales manager H maximizes the expected profit by optimizing selling price P_H and capacity C_H. This decision problem corresponds to the single-product price-setting newsvendor problem presented in Section 4.2. Sales manager L, on the other hand, faces a deterministic price-sensitive demand. By setting a particular selling price P_L, he is able to determine the accurate capacity C_L required to produce adequate supply.

Table 4.1 shows the profit functions as well as the optimal price and capacity decisions of sales manager H and L, respectively. The results for L are easy to verify, the results for H are obtained using the results from Section 4.2.

As in the previous section, the following numerical example is analyzed. The H-class demand is characterized by Ψ which is uniformly distributed. In the first case, Ψ is uniformly distributed on the interval $[500, 1500]$ which corresponds to a CV$= 0.28$ and in the second case, Ψ is uniformly distributed on the interval $[200, 1800]$ which corresponds to a CV$= 0.46$. The sensitivity coefficient is $b = 20$. L-class demand is deterministic with $\alpha = 1000$ and $\beta = 40$ and the unit

Sales manager H	Sales manager L
$\Pi_H = (P_H - c_H - c)(\mu - bP_H)$ $-c\int_A^K (K-z)f(z)dz$ $-(P_H - c_H - c)\int_K^B (z-K)f(z)dz$	$\Pi_L = (P_L - c_L - c)(\alpha - \beta P_L)$
$P_H^* = \frac{\mu + b(c+c_H)}{2b} - \frac{1}{2b}\int_K^B (z-K)f(z)dz$	$P_L^* = \frac{\alpha+\beta(c+c_L)}{2\beta}$
$K^* = F^{-1}\left(\frac{P_H - c - c_H}{P_H - c_H}\right)$	$d_L^* = \frac{\alpha - \beta(c+c_L)}{2}$
$C_H^* = K^* - bP_H^*$	$C_L^* = d_L^*$

Table 4.1: Profit function, optimal price, and capacity decision of H and L under decentralized planning

production cost are $c_H = c_L = 2$. In order to show the impact of capacity investment costs, c is varied between 1 and 19.

Impact of demand uncertainty on selling prices

Figures 4.10 and 4.11 illustrate the impact of capacity investment costs ($c = 1, \ldots, 19$) and uncertainty (CV$= 0.28$ and CV$= 0.46$) on P_H and P_L in case of decentralized decision-making (in the figures P_{H_d} and P_{L_d} are denoted by PHd and PLd) and in case of simultaneous planning (in the figures P_H^* and P_L^* are denoted by PH* and PL*).

It can be observed that the impact of simultaneous decision-making compared to decentralized decision-making is opposed between P_H and P_L. While P_{L_d} is lower than P_L^*, P_{H_d} is larger than P_H^*. The intuition of these price effects is risk reduction by demand adjustments. A central decision maker anticipates that the production of the less prioritized product L can use excess capacity of H, which decreases the risk and thus the costs of overcapacity. As a consequence P_L decreases because capacity costs are partially sunk. On the other hand, reducing the risk of overcapacity decreases overage costs for H which implies that P_H increases. Since in case of decentralized decision-making uncertainty does not influence P_L, the price curve PLd for CV $= 0.28$ is equal to the price curve PLd for CV $= 0.46$.

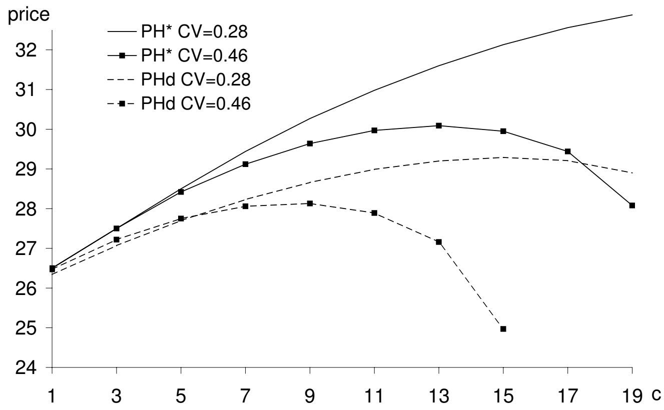

Figure 4.10: Decentralized versus simultaneous planning - impact of increasing demand uncertainty on P_H

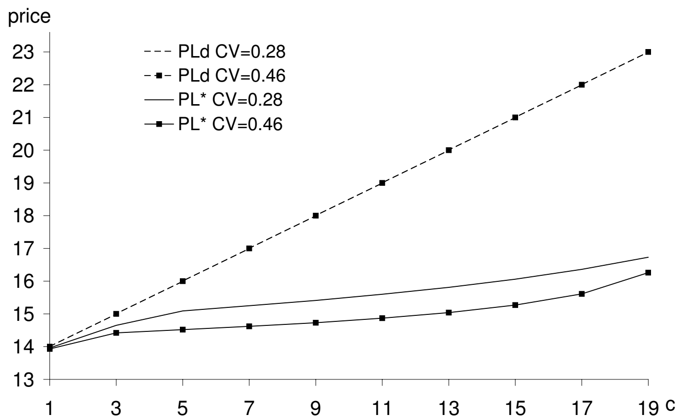

Figure 4.11: Decentralized versus simultaneous planning - impact of increasing demand uncertainty on P_L

P_H is not monotonous in c for the same reason as explained in the previous section. Increasing capacity cost causes two overlapping pricing effects. The first effect is a direct effect which leads to a price increase. The second indirect effect follows from increasing overage costs and expected shortages, and leads to a price decrease. Depending on which effect is stronger, P_H increases or decreases in capacity costs (see explanation to Figure 4.6).

Additionally, the price difference between decentralized and simultaneous decision-making is larger for expensive capacities than for inexpensive capacities. The intuition behind this effect is that for expensive capacities where overage costs are high, simultaneous decision-making and thus a more flexible use of capacity becomes more important than for inexpensive capacities.

Impact of demand uncertainty on capacity acquisition

Figure 4.12 illustrates the impact of capacity costs ($c = 1, \ldots, 19$) and uncertainty (CV = 0.28 and CV = 0.46) on C in case of decentralized and in case of simultaneous decision-making (in the figures C_d and C^* are denoted by Cd and C*, respectively).

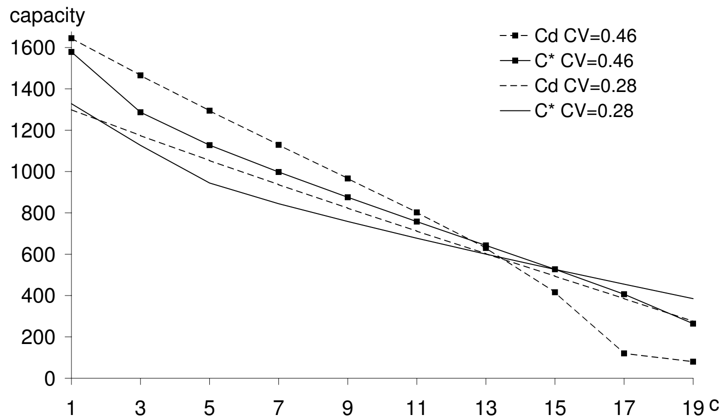

Figure 4.12: Decentralized versus simultaneous planning - impact of increasing demand uncertainty on C

Besides the rather intuitive result that capacity decreases with increasing investment costs, it can be observed that for inexpensive capacities, a decentralized decision-making leads to a higher capacity acquisition compared to simultaneous decision-making, whereas for expensive capacities, the opposite effect is true. As with prices, the intuition behind this effect is a reduction of the risk of overcapacity and capacity shortage. In decentralized decision-making, both sales managers determine dedicated capacity independently before exact demand is known. For inexpensive capacities, where underage costs are larger than overage costs, sales manager H acquires safety capacity in order to buffer possible demand peaks. Since this safety capacity is determined independently from the capacity decision of sales manager L, the total capacity by decentralized decision-making is larger than by simultaneous decision-making. For expensive capacities where overage costs are larger than underage costs, sales manager H acquires a capacity level which is lower than the expected demand such that a simultaneous decision-making reduces the risk of overcapacity.

The reason that C_d declines more for $c \geq 15$ follows from the fact that a decentralized planning of H yields a negative expected profit so that it becomes unprofitable to serve the H-class customer if c exceeds 15. As a consequence $C_H = 0$ and $C_d = C_L$.

Impact of demand uncertainty on the benefit of centralization

Figure 4.13 illustrates the impact of capacity costs ($c = 1, \ldots, 19$) and uncertainty (CV = 0.28 and CV = 0.46) on the difference between profits under decentralized (profit-d) and simultaneous planning (profit*).

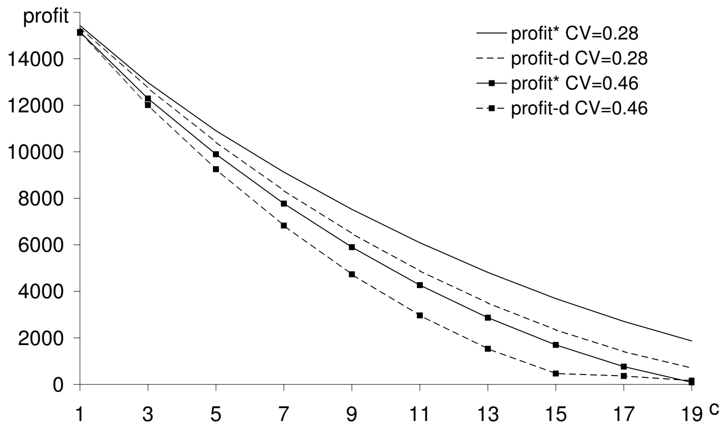

Figure 4.13: Decentralized versus simultaneous planning - impact of increasing demand uncertainty on profit

The results indicate that for a given c the difference in profits between decentralized and simultaneous decision-making is similar under moderate and high demand uncertainty. This conjectures that demand uncertainty does not influence the performance improvement of simultaneous decision-making. However, the difference in profits between decentralized and simultaneous decision-making is larger for expensive capacities than for inexpensive capacities.

4.3.6 Summary and implications

This section investigated an extension of the price-setting newsvendor problem when a firm is able to segment the market into two customer classes with different willingness to pay. The firm simultaneously determines the capacity level and two selling prices charged for each customer class in the presence of demand uncertainty. It was assumed that customer segmentation can be achieved by either price discrimination or product differentiation. If price discrimination is not possible because of arbitrage and cannibalization, the firm can use product differentiation, e.g., different brands or different quality levels in order to avoid these effects. Product differentiation was characterized by different production costs.

A stochastic model was developed and analyzed that simultaneously optimizes capacity acquisition and selling prices for each customer class. Despite a non-convex optimization problem, structural properties were derived which showed that under certain circumstances there exists a unique optimal solution. Based on these results, an algorithm was developed which efficiently determines the prices as well as the optimal capacity investment simultaneously.

The goal was to investigate the interaction of capacity planning and price decisions in the presence of demand uncertainty. In particular, the benefit of price discrimination compared to a single-pricing strategy was analyzed. In order to examine the interaction of pricing and capacity planning, we analyzed the decision problem for different capacity costs and different levels of demand uncertainty. It was examined that price discrimination provides a firm a higher flexibility and thus the ability for a better matching of supply with demand.

On the one hand, customized pricing entails a different price for each customer class according to their willingness to pay. On the other hand, the capacity acquisition is optimally adjusted to demand. When the market environment changes, e.g., a changing cost structure or demand uncertainty, customer segmentation provides the firm a higher flexibility to respond on this. For instance, the firm adjusts prices differently between customer classes by taking their respective price-sensitivity and other demand characteristics into account. The selling price that is charged for the customer class L which has a higher price-sensitivity changes less than the selling price that is charged for the less price sensitive customer class H.

In terms of capacity acquisition, it was observed that CS yields a higher capacity acquisition than in case of NCS. Obviously, by applying CS the firm is able to capture a higher consumer surplus which, in turn, yields a higher demand. However, increasing demand uncertainty affects capacity decisions depending on whether the firm applies CS or NCS. Since price discrimination leads to a higher product profitability (i.e., the ratio of price and cost changes), it also changes the ratio between underage and overage costs. This means that under NCS the costs of having a capacity shortfall are larger than the costs of having excess capacity (overage costs exceed underage costs) whereas under CS it becomes more expensive not being able to serve a customer than to have overcapacity. Hence, in case of NCS, increasing demand uncertainty leads to a capacity reduction whereas in case of CS increasing demand uncertainty leads to a larger capacity investment.

Another result is that an apparent unprofitable demand class (or product in case of product differentiation) which is characterized by a well-predictable demand can become profitable under simultaneous planning. This implies that by applying a two-customer-class (two-product) strategy where a low-positioned demand class (product) is added to a high-positioned demand-class (product), a firm has more flexibility to reduce the risk of underage and overage costs. This leads to a better match of supply and demand than in case of a single customer class (product).

Furthermore, a decentralized decision-making strategy with dedicated capacities where two sales managers decide independently on price and capacity was compared to simultaneous decision-making. One result is that decentralized decision-making has an inverse impact on the prices that are charged for H-class and L-class customers, respectively. While the H-class sales manager underestimates the price for H, the selling price for L is overestimated, or casually spoken, high-class customers pay too little whereas low-class customers pay too much.

Whether capacity acquisition is larger or lower is not unique but it depends on the value of capacity costs. For inexpensive capacities simultaneous decision-making yields lower capacity acquisition whereas for expensive capacities it leads to a larger capacity acquisition. The driving force behind this is a reduction of uncertainty. For inexpensive capacities simultaneous decision-making reduces the risk of a capacity shortage whereas as for expensive capacities simultaneous decision-making reduces the risk of excess capacities.

A weakness of the presented model is that the capacity allocation to both demand classes is exogenously given. However, depending on the parameter setting it might be better to satisfy L-demand first. In order to avoid preallocation of capacity, the capacity allocation has to be determined endogenously based on cost structure and demand characteristics of both products. A further restriction of the model is the assumption of a linear price response function. It is questionable whether the results also hold for arbitrary price response functions. Moreover, in practice it is often difficult to integrate price and operations decisions perfectly because these decisions are made by independent organizational units. It is rather realistic to develop incentive systems which induce organizational units to decide in compliance with the overall company objective.

5 Conclusions and outlook

This chapter summarizes the major findings of this thesis and discusses possible future research.

5.1 Conclusions

This thesis contributed to the emerging field of literature investigating the interaction between revenue management methods and operations management activities. It centered around two major research topics: the interaction between dynamic pricing and inventory replenishment in the presence of fixed ordering costs (Chapter 3) and the impact of customer segmentation on price and capacity decisions in a coordinated planning approach and in the presence of uncertain demand (Chapter 4).

Chapter 3 focused on the interaction of dynamic pricing on inventory replenishment in an EOQ framework. To incorporate price decisions into the EOQ problem, which is concerned with answering the question of how much and equivalently how frequently inventory should be replenished, the demand rate was formulated as a price-response function. Since continuous time was assumed, this formulation basically allowed to change prices at any time t. The objective was to maximize the average profit by simultaneously optimizing the pricing strategy and the replenishment policy.

In Section 3.3, EOQ models incorporating dynamic pricing in a monopoly were developed and analyzed. Section 3.3.1 considered continuous price adjustments and Section 3.3.2 generalized the model to the case that the number of price changes over an order cycle are optimized. Besides providing further evidence for the benefit of dynamic pricing, its impact on order quantity and order frequency was analyzed.

The major finding of this section was that the trade-off between fixed ordering and inventory holding costs yields an increasing selling price over an order cycle. Moreover, it was shown that both the optimal order quantity and the optimal cycle length increase with the number of price changes. The general intuition behind this effect is to achieve operational efficiency. In the presence of fixed ordering costs and inventory holding costs it is beneficial to increase demand when inventories are high in order to reduce inventories earlier and thus reduce holding costs. Furthermore, it was shown that the solution obtained from maximizing the average profit is approximately optimal with respect to maximizing the NPV which is widely accepted as the right framework for valuing inventories.

In Section 3.4 the model of Section 3.3 was extended. It investigated the impact of a supplier quantity discount. Considering that the supplier offers an all-units quantity discount, the benefit of coordinated decision-making was analyzed where the retailer optimizes a dynamic pricing strategy and replenishment policy simultaneously. The results were compared to a decentralized decision framework where selling price and replenishment policy are optimized independently.

While previous findings stated that decentralized decision-making leads to an underestimation of selling prices, the major finding of this section was that this property does not necessarily hold if the supplier offers an all-units quantity discount. In this case, two effects influence the outcome of decision-making: the overhead cost effect and the discount effect. The overhead cost effect results from the fact that a decision maker who decides on the price only by taking into account variable procurement costs and disregards fixed ordering and inventory holding costs underestimates total relevant costs. This leads to a lower selling price, a higher demand rate, and a larger lot-size. The discount effect, on the other hand, results from the fact that a decision maker does not take a supplier quantity discount into account. Therefore, costs are overestimated which, in turn, yields an overestimation of the selling price. Depending on which effect dominates, selling price, demand rate, and order quantity increase or decrease.

Section 3.5 analyzed a multiple product dynamic pricing and replenishment problem with limited storage capacity. The major extension to the previous sections was to determine the optimal staggering of the order-releases. Since this problem is NP-complete already without pricing issues, a heuristic approach was considered assuming that all products are replenished once in an order cycle (common cycle approach). A two-stage optimization model was developed integrating pricing and replenishment considerations. Furthermore, decentralized decision-making where selling prices and replenishment policies are optimized sequentially was compared to coordinated decision-making where a central decision maker simultaneously optimizes selling price and replenishment policy. Moreover, constant and dynamic pricing were distinguished.

The major finding of this section was that, while for both decentralized decision-making and centralized decision-making with a constant price the selling prices are continuous over an order cycle, in case of dynamic pricing there are points of discontinuity. At times when the product is ordered, the price for this product instantaneously decreases while the prices of all other products instantaneously increase. This opposed pricing effect yields a better matching of demand resulting in higher revenues with available inventory and thus an optimal utilization of the limited capacity.

Section 3.6 investigated the interaction of dynamically changing prices and different replenishment strategies in a competitive environment. Two retailers were allowed to change their sales quantity dynamically over time and the market price was sensitive with respect to the total sales quantity. The retailers differed in their replenishment strategy. Retailer 1 followed an EOQ policy whereas retailer 2 ordered just-in-time. The primary goal of this study was to analyze the optimal replenishment policy and the equilibrium output strategy. A differential game was developed where both retailers repeatedly interact over the order cycle and an open-loop Nash equilibrium was derived. Output and replenishment policies were optimized by a two-stage model.

The major findings of this section are that output decisions are not only influenced by the competitor's output but also by the current inventory level of the EOQ retailer. Since the inventory level decreases over an order cycle, the optimality conditions change continuously. It was shown that both retailers follow contrary output strategies over an order cycle. The EOQ retailer, driven by inventory holding costs, decreases his market share over an order cycle. This strategy is caused by a larger inventory reduction right after an order and leads to lower holding costs. Although retailer 2 followed an JIT strategy and both demand rate and cost structure were stationary, the output of retailer 2 increased over an order cycle. Moreover, we showed that depending on the cost parameters, the retailers do not necessarily compete over the entire order cycle. For the JIT retailer, it might be beneficial not to serve the market permanently but only when the output of the EOQ retailer is low. Therefore, the EOQ retailer is partially a monopolist. Furthermore, a numerical example indicated that while EOQ and JIT replenishment might yield identical profits in a monopoly environment, under competition the average profit of the EOQ retailer is larger than the average profit of the JIT retailer. The intuition is that the EOQ retailer has a competitive advantage by ordering all units at the beginning of an order cycle such that he obtains similar advantages like a Stackelberg leader by a sunk cost effect.

The contribution of Chapter 4 was to investigate the benefits of customer segmentation when price and capacity decision are simultaneously made in the presence of demand uncertainty. Customer segmentation was achieved either by price discrimination or product differentiation. In Section 4.2, existing results of the price-setting newsvendor problem with a single demand class were reviewed. The major contribution resulted from Section 4.3. A stochastic model was developed that distinguishes two demand classes differing in their respective price sensitivity. This model integrates price differentiation and capacity decision in a simultaneous optimization approach and is characterized by a non-convex structure. Structural properties were derived which showed that under certain circumstances a unique optimal solution exists. Based on these results, an algorithm was developed which efficiently determines the optimal prices as well as the optimal capacity investment.

The major findings of this chapter can be concluded as follows. Customized pricing which is integrated in capacity planning provides a bilateral capability to match supply with demand. On the one hand, the firm charges different prices to different groups of customers with the ability to charge a higher price to the group with a less price-elastic demand and a relatively lower price to the group with a more elastic demand. Adopting such a strategy, the firm can increase its sales and its revenues. On the other hand, the firm optimally determines capacity and allocates it to the respective customer demand.

It is known that price discrimination leads to an increasing demand, which, in turn, leads to increasing capacity acquisition. However, price discrimination does not only influence the demand side but it also changes the ratio between overage and underage costs, which is relevant for the capacity decision. Due to increasing product profitability (the firm can charge a more appropriate price to each customer class) underage costs increase, which additionally leads to increasing capacity acquisition.

Moreover, customer segmentation might change the influence of increasing demand uncertainty. Since customer segmentation changes the ratio between underage and overage costs in such a way that underage costs increase, there exist parameter settings where increasing uncertainty yields decreasing capacity acquisition in a single-pricing strategy (without customer segmentation) whereas in case of customer segmentation capacity acquisition increases.

Finally, simultaneous decision-making on price and capacity was compared to decentralized decision-making where two sales managers decide independently on price and capacity and capacity was dedicated in advance. Dedicated capacities yield that a firm loses its flexibility to respond to uncertainties across products although all products are produced with the same resource. A consequence is a mismatch of supply and demand: capacity is used to serve a low-class customer while the demand of a high-class customer is lost. This cannot occur in the case of simultaneous decision-making. Another effect of decentralized decision-making is that the sales manager who is responsible for the high-class demand underestimates the price whereas the selling price charged to the low-class demand is overestimated. This mismatch leads to lower profit margins and higher risk of excess capacity and capacity shortfall.

Concluding, the integration of customer segmentation and capacity planning provides a bilateral and thus a higher flexibility to match supply with demand. A two demand-class strategy does not only lead to increasing demand but, in addition, it provides an option to reduce risk of excess capacity and capacity shortfall by a more flexible control of demand.

5.2 Outlook

While there are many remaining challenges in the two research streams operations management and revenue management, there is an increasing awareness of the importance of their integration. A significant portion of papers that investigate joint production/procurement and pricing decisions is based on either EOQ or newsvendor-type frameworks. However, besides more specific research directions provided in Chapter 3 and Chapter 4 based on extensions of the presented models, extensions exist which have not been covered yet.

An important element in terms of dynamic pricing disregarded in this thesis is the consideration of strategic customer behavior. An interesting direction of research is to investigate pricing and replenishment strategies when it is known that customers behave strategically, i.e., they buy earlier in expectation that the selling price will increase or they wait in expectation that the price will decrease. There are a plenty of papers that examine dynamic pricing in the presence of strategic customer behavior. However these papers do not consider inventory or capacity issues (Besanko and Winston, 1990). There are very few papers analyzing joint inventory and dynamic pricing problems with a one-time order, e.g., see Su (2007) and Su and Zhang (2007). However, an interesting but challenging research question is to investigate joint inventory and dynamic pricing problems with replenishment option and the assumption that customers act strategically.

All models analyzed in this thesis assumed that the supplier or manufacturer is completely reliable such that all units ordered are delivered on time. It is known from literature that the integration of yield uncertainty into inventory management has a significant impact on replenishment decisions, e.g., see Henig and Gerchak (1990), Bollapragada and Morton (1999), Inderfurth and Transchel (2007). An interesting direction for research is to analyze the interaction of pricing and replenishment in the presence of uncertain supply, e.g., see Li and Zheng (2006).

For large companies pricing and operations decisions are delegated to independent organizational units. Operations management is involved in purchasing raw materials and components, or in setting up production capacity and is evaluated as a cost center seeking for lower costs and operational efficiency. On the other hand, marketing is making decisions on prices for finished goods or services and is evaluated as revenue center. It is extremely unlikely that these both decision-making units will work together perfectly without the right incentives. However, marketing efforts to create demand can be wasted if supply is suboptimal, and vice versa. An interesting research direction is to analyze contract mechanisms coordinating marketing and operations activities in order to achieve the overall maximal company profit. There is not much research on this topic. Two papers that investigate intra-firm coordination of marketing and operations by coordinating contracts are Li and Atkins (2002) and Jerath et al. (2007).

Ultimately, the systematic integration of supply and demand-oriented activities is still in an emerging stage, both in academia and in business practice. The good news is that researchers and managers recognized the possibilities such that there is enormous research potential to capture a full understanding of this integration.

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A Proofs

A.1 Proof of Proposition 2

Proof. An optimal solution of the optimization problem (3.29)-(3.31) has to satisfy the following *Karush-Kuhn-Tucker* conditions: $\frac{\partial L^{(N)}}{\partial P_i} = 0$, $\lambda_i^*(\bar{P} - P_i) = 0$, $P_i \geq 0$, and $\lambda_i \geq 0$ as well as $\frac{\partial L^{(N)}}{\partial t_i} = 0$, $\mu_i(t_i - t_{i-1}) = 0$, $t_i \geq 0$, and $\mu_i \geq 0$ for all $i = 1, \cdots, N$.

The first partial derivative of (3.33) w.r.t. P_i gives

$$\frac{\partial L^{(N)}}{\partial P_i} = \frac{1}{t_N}\left(D(P_i) + (P_i - c - \frac{h}{2}(t_i + t_{i-1}))D'(P_i)(t_i - t_{i-1})\right) - \lambda_i.$$

If $\lambda_i > 0$ for at least one i, then the associated constraint is binding, i.e., $P_i^* = \bar{P}$. Therefore, we get

$$\frac{\partial L^{(N)}}{\partial P_i} = 0 \Leftrightarrow (\bar{P} - c - \frac{h}{2}(t_i + t_{i-1}))D'(\bar{P})(t_i - t_{i-1}) - t_N\lambda_i = 0. \tag{A.1}$$

From (3.33) it follows that $(\bar{P} - c - \frac{h}{2}(t_i + t_{i-1})) \geq 0$, otherwise L^N is negative. Since $t_i - t_{i-1} \geq 0$ and $D'(\bar{P}) < 0$ it follows that $(\bar{P} - c - \frac{h}{2}(t_i + t_{i-1}))D'(\bar{P})(t_i - t_{i-1}) < 0$. Thus, in order to satisfy (A.1), λ_i has to be negative, which violates the *Karush-Kuhn-Tucker* conditions. Therefore, $P_i^* = \bar{P}$ is not optimal, $\lambda_i^* = 0$, and the optimal selling price is characterized by the first-order condition

$$P_i^* + \frac{D(P_i^*)}{D'(P_i^*)} = c + \frac{h}{2}\left(t_i^* + t_{i-1}^*\right), \quad \text{for } i = 1, \cdots, N. \tag{A.2}$$

Since $t_i^* - t_{i-1}^* \geq 0$ for all $i = 1, \cdots, N$, the right-hand side of (A.2) increases in i so that we need to show that the left-hand side of (A.2) is increasing in P_i. The first derivative of the left-hand side is

$$\frac{\partial}{\partial P_i^*}\left(P_i^* + \frac{D(P_i^*)}{D'(P_i^*)}\right) = 2 - \frac{D''(P_i^*)D(P_i^*)}{(D'(P_i^*))^2}.$$

With the assumption that $2 - \frac{D''(P_i^*)D(P_i^*)}{(D'(P_i^*))^2} \geq 0$, it follows that P_i^* increases in i, i.e., $P_{i-1}^* \leq P_i^*$.

A.1 Proof of Proposition 2

In order to show that the prices are strictly increasing in i, which equivalently gives that $t_i^* < t_{i+1}^*$, we analyze the first partial derivative of (3.33) w.r.t. t_i:

$$\frac{\partial L^{(N)}}{\partial t_i} = 0 \quad \Leftrightarrow$$

$$(P_i - c - ht_i)D(P_i) - (P_{i+1} - c - ht_i)D(P_{i+1}) + t_N(\mu_i - \mu_{i+1}) = 0, \quad \text{(A.3)}$$

$$t_i^* = \frac{(P_i - c)D(P_i) - (P_{i+1} - c)D(P_{i+1})}{h(D(P_i) - D(P_{i+1}))} + \frac{t_N(\mu_i - \mu_{i+1})}{h(D(P_i) - D(P_{i+1}))}. \quad \text{(A.4)}$$

If P_i^* is not strictly increasing in i, then there exists at least one i where $P_i^* = P_{i+1}^*$. Therefore, from (A.2) it follows that $t_{i-1}^* = t_i^* = t_{i+1}^*$, which, in turn, gives that $\mu_i^* > 0$ and $\mu_{i+1}^* > 0$ and from (A.3) we get that $\mu_i^* = \mu_{i+1}^*$. Now, we assume that $P_{i-1}^* < P_i^* = P_{i+1}^* < P_{i+2}^*$ (however, an analogous argumentation holds for the case where $P_{j-1}^* < P_j^* = P_{j+1}^* = \cdots = P_i^* = P_{i+1}^* < P_{i+2}^*$).

From (A.2) it follows that $t_{i-2}^* < t_{i-1}^* = t_i^* = t_{i+1}^* < t_{i+2}^*$ and thus, $\mu_{i-1}^* = \mu_{i+2}^* = 0$. Moreover, from (A.4) we get

$$t_{i-1}^* = \frac{(P_{i-1} - c)D(P_{i-1}) - (P_i - c)D(P_i)}{h(D(P_{i-1}) - D(P_i))} - \frac{t_N\mu_i}{h(D(P_{i-1}) - D(P_i))} \quad \text{(A.5)}$$

and

$$t_{i+1}^* = \frac{(P_{i+1} - c)D(P_{i+1}) - (P_{i+2} - c)D(P_{i+2})}{h(D(P_{i+1}) - D(P_{i+2}))} + \frac{t_N\mu_{i+1}}{h(D(P_{i+1}) - D(P_{i+2}))}. \quad \text{(A.6)}$$

Let $R(P) := (P-c)D(P)$. The second-order derivative is $R''(P) = 2D'(P) + (P-c)D''(P)$. By the assumption that $2 - \frac{D''(P_i^*)D(P_i^*)}{(D'(P_i^*))^2} \geq 0 \Leftrightarrow 2D'(P)^2 - D(P)D''(P) \geq 0$, it follows that $R''(P) \leq 0$, i.e., $R(P)$ is a concave function in P. Using the properties that $D(P)$ is decreasing and convex in P and $R(P)$ is concave in P, from (A.2) it follows that $R'(P_i^*) < 0$ and the following inequalities hold

$$(P_{i-1}^* - c)D(P_{i-1}^*) - (P_i^* - c)D(P_i^*) < (P_{i+1}^* - c)D(P_{i+1}^*) - (P_{i+2}^* - c)D(P_{i+2}^*)$$

and

$$h(D(P_{i-1}^*) - D(P_i^*)) > h(D(P_{i+1}^*) - D(P_{i+2}^*)).$$

Figure A.1 illustrates the functions $D(P)$ and $R(P)$ where $A1 = (P_{i-1} - c)D(P_{i-1}) - (P_i - c)D(P_i)$, $B1 = D(P_{i-1}) - D(P_i)$, $A2 = (P_{i+1} - c)D(P_{i+1}) - (P_{i+2} - c)D(P_{i+2})$, and $B2 = D(P_{i+1}) - D(P_{i+2})$. Therefore, we get

$$\frac{(P_{i-1}^* - c)D(P_{i-1}^*) - (P_i^* - c)D(P_i^*)}{h(D(P_{i-1}^*) - D(P_i^*))} < \frac{(P_{i+1}^* - c)D(P_{i+1}^*) - (P_{i+2}^* - c)D(P_{i+2}^*)}{h(D(P_{i+1}^*) - D(P_{i+2}))}. \quad \text{(A.7)}$$

Since $t_{i-1}^* = t_{i+1}^*$ and $\mu_i^* = \mu_{i+1}^*$ by using (A.7) it follows that (A.5), (A.6),

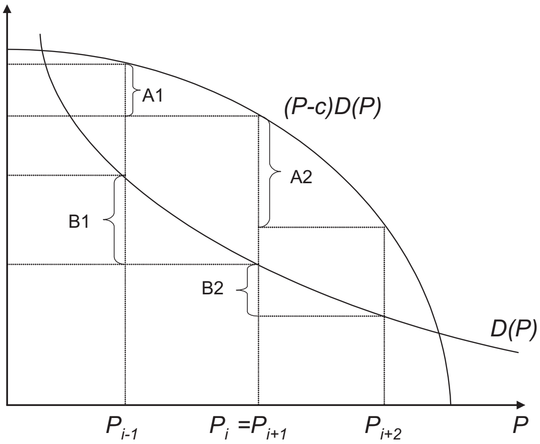

Figure A.1: Functions $D(P)$ and $(P-c)D(P)$

and $\mu_i \geq 0$ cannot be valid at the same time so that the Karush-Kuhn-Tucker conditions are violated. Therefore, $P_i^* < P_{i+1}^*$ for all $i = 1, \cdots, N-1$.

□

A.2 Proof of Proposition 4

We simplify the notation of equation (3.46) by substitution of the coefficients A_3, A_2, A_0.

$$\Pi_N^{'}(t_N) := \underbrace{\frac{4N^2-1}{N^2}}_{A_3} t_N^3 \underbrace{- \frac{6(a-bc)}{hb}}_{A_2} t_N^2 + \underbrace{\frac{24F}{h^2 b}}_{A_0} = 0 \tag{A.8}$$

Using the *Trigonometrical Solution Method* (see Bronshtein et al. (2004)), the number of real roots is established by the sign of the discriminant $\Delta_2 := B_2^2 + B_1^3$ with $B_1 = -\frac{A_2^2}{9A_3^2}$ and $B_2 = \frac{A_2^3}{27A_3^3} + \frac{A_0}{2A_3}$:

$$\Delta_2 := \frac{48F}{h^4 b^2} \frac{N^4}{(4N^2-1)^2} \left[3F - \frac{4(a-bc)^3}{hb^2} \frac{N^4}{(4N^2-1)^2} \right]. \tag{A.9}$$

If $\Delta_2 > 0$ which requires that $F > \frac{4(a-bc)^3}{3hb^2} \frac{N^4}{(4N^2-1)^2}$, (A.8) has only one real root

$$t_N^* := r \left[-2\cosh\left(\frac{\phi}{3}\right) + 1 \right], \tag{A.10}$$

with

$$\phi = \operatorname{arccosh}\left(\frac{B_2}{r^3}\right), \; r = \frac{2(a-bc)}{hb}\frac{N^2}{(4N^2-1)} \geq 0,$$

and

$$B_2 = \frac{4N^2}{h^2 b(4N^2-1)}\left[3F - \frac{2(a-bc)^3}{hb^2}\frac{N^4}{(4N^2-1)^2}\right] > 0.$$

From the characteristics of the function hyperbolic cosine ($\cosh(x) \geq 1$), it follows that $t_N^* < 0$. If $\Delta_2 < 0$ which requires that $F < \frac{4(a-bc)^3}{3b^2h}\frac{N^4}{(4N^2-1)}$, (3.46) has exactly three different real roots:

$$\begin{aligned} t_1 &= -2r\cos\left(\frac{\phi}{3}\right) - \frac{A_2}{3A_3}, \\ t_2 &= 2r\cos\left(\frac{\pi}{3} - \frac{\phi}{3}\right) - \frac{A_2}{3A_3}, \\ t_3 &= 2r\cos\left(\frac{\pi}{3} + \frac{\phi}{3}\right) - \frac{A_2}{3A_3}, \end{aligned} \tag{A.11}$$

where $r = \pm\sqrt{\mid \frac{-A_2^2}{9A_3^2} \mid}$ and $\phi = \arccos\left(\frac{\frac{A_2^3}{27A_3^3}+\frac{A_0}{2A_3}}{r^3}\right)$. The sign of r depends on the sign of $\frac{A_2^3}{27A_3^3} + \frac{A_0}{2A_3}$, i.e.,

$$r = \begin{cases} \frac{A_2}{3A_3} \leq 0 & : \quad \frac{A_2^3}{27A_3^3} + \frac{A_0}{2A_3} \leq 0 \Longleftrightarrow F \leq \frac{2}{3}\frac{(a-bc)^3}{hb^2}\frac{N^4}{(4N^2-1)^2} \\ -\frac{A_2}{3A_3} \geq 0 & : \quad \frac{A_2^3}{27A_3^3} + \frac{A_0}{2A_3} \geq 0 \Longleftrightarrow F > \frac{2}{3}\frac{(a-bc)^3}{hb^2}\frac{N^4}{(4N^2-1)^2}. \end{cases} \tag{A.12}$$

Analyzing the limiting values $\lim\limits_{t_N^* \to -\infty} \Pi_N' = -\infty$ and $\lim\limits_{t_N^* \to \infty} \Pi_N' = \infty$ gives that the slopes at the real roots with the lowest and highest value are strictly positive whereas the slope of the middle real root is strictly negative. Thus, only the middle real root is a local maximum. We show that the roots t_1 and t_2 can be excluded for an optimal solution due to the following condition $t_1 < t_3 < t_2$ or $t_2 < t_3 < t_1$. The smallest value is non-positive and the largest value is larger than the maximum cycle length t_N^{Max} (see (3.45)):

If $F \leq \frac{2}{3}\frac{(a-bc)^3}{hb^2}\frac{N^4}{(4N^2-1)^2} \implies r = \frac{A_2}{3A_3} = -\frac{2(a-bc)}{hb}\frac{N^2}{(4N^2-1)}$ and $F \leq F_{max}$, ϕ

becomes:

$$\phi = \arccos\left(\underbrace{1 - \underbrace{\frac{3}{2}\frac{hb^2}{(a-bc)^3}\frac{(4N^2-1)^2}{N^4}F}_{\in[0,1]}}_{\in[0,1]}\right) \Longrightarrow \phi \in [0, \frac{\pi}{2}] \tag{A.13}$$

which gives that

$$t_1 = \underbrace{-\frac{2(a-bc)}{hb}\frac{N^2}{(4N^2-1)}}_{\leq 0} \cdot \left(\underbrace{-2\cos\left(\frac{\phi}{3}\right) - 1}_{\in[-3,-(\sqrt{3}-1)]}\right) \geq 0,$$

$$t_2 = \underbrace{-\frac{2(a-bc)}{hb}\frac{N^2}{(4N^2-1)}}_{\leq 0} \cdot \left(\underbrace{2\cos\left(\frac{\pi}{3} - \frac{\phi}{3}\right) - 1}_{\in[0,\sqrt{3}-1]}\right) \leq 0,$$

$$t_3 = \underbrace{-\frac{2(a-bc)}{hb}\frac{N^2}{(4N^2-1)}}_{\leq 0} \cdot \left(\underbrace{2\cos\left(\frac{\pi}{3} + \frac{\phi}{3}\right) - 1}_{\in[-1,0]}\right) \geq 0.$$

It is obvious that $t_1 > t_3$ and with the above profit function analysis, the profit function has a local maximum at t_3 and a local minimum at t_1. Further, it can be shown that for $N \geq 2$ the value t_1 exceeds the maximum cycle length t_N^{Max} given in (3.45).

$$t_1 = \underbrace{\frac{2(a-bc)}{hb}\frac{N}{(2N-1)}}_{t_N^{Max}} \cdot \underbrace{\frac{N}{(2N+1)}\left(2\cos\left(\frac{\phi}{3}\right) + 1\right)}_{>1\ for N\geq 2} > t_N^{Max}.$$

An equivalent proof can be made for the case $F > \frac{2}{3}\frac{(a-bc)^3}{hb^2}\frac{N^4}{(4N^2-1)^2}$. Thus, the optimal cycle length is as follows

$$t_N^* = \begin{cases} -2\frac{(a-bc)}{hb}\frac{N^2}{(4N^2-1)}\left(2\cos(\frac{\pi}{3}+\frac{\phi}{3})-1\right) & : F \leq \frac{2}{3}\frac{(a-bc)^3}{hb^2}\frac{N^4}{(4N^2-1)^2} \\ \phi = \arccos\left(1-\frac{3F}{2}\frac{hb^2}{(a-bc)^3}\frac{(4N^2-1)^2}{N^4}\right) & \\ & \\ 2\frac{(a-bc)}{hb}\frac{N^2}{(4N^2-1)}\left(2\cos(\frac{\pi}{3}+\frac{\phi}{3})+1\right) & : F > \frac{2}{3}\frac{(a-bc)^3}{hb^2}\frac{N^4}{(4N^2-1)^2} \\ \phi = \arccos\left(\frac{3F}{2}\frac{hb^2}{(a-bc)^3}\frac{(4N^2-1)^2}{N^4}-1\right). & \end{cases}$$

In the following, we show that t_N^* is increasing in N, i.e., $\frac{\partial t_N^*}{\partial N} \geq 0$. If $F \leq \frac{2}{3}\frac{(a-bc)^3}{hb^2}\frac{N^4}{(4N^2-1)^2}$,

$$t_N^* = t_3 = \underbrace{-2\frac{(a-bc)}{hb}\frac{N^2}{(4N^2-1)}}_{u\leq 0}\underbrace{\left(2\cos(\frac{\pi}{3}+\frac{\phi}{3})-1\right)}_{v\leq 0}$$

$$\frac{\partial t_N^*}{\partial N} = \frac{\partial u}{\partial N}\cdot v + u \cdot \frac{\partial v}{\partial N}.$$

It is easy to show that $\frac{\partial u}{\partial N} = \frac{4(a-bc)}{hb}\frac{N}{(4N^2-1)^2} \geq 0$. The first derivative with respect to v gives

$$\frac{\partial v}{\partial N} = -2\sin\left(\frac{\pi}{3}+\frac{\phi}{3}\right)\cdot\frac{1}{3}\phi'(N), \text{ with } \phi'(N) = \frac{\frac{4}{N(4N^2-1)}}{\sqrt{\frac{4}{3F}\frac{(a-bc)^3}{hb^2}\frac{N^4}{(4N^2-1)^2}-1}}$$

$$\Rightarrow \frac{\partial v}{\partial N} = -\,2\sin\left(\frac{\pi}{3}+\frac{\phi}{3}\right)\cdot\frac{\frac{4}{3}\frac{1}{N(4N^2-1)}}{\sqrt{\frac{4}{3F}\frac{(a-bc)^3}{hb^2}\frac{N^4}{(4N^2-1)^2}-1}}.$$

Therefore,

$$\begin{aligned}
\frac{\partial t_N^*}{\partial N} =& \frac{4(a-bc)}{hb}\frac{N}{(4N^2-1)^2}\cdot\left(2\cos\left(\frac{\pi}{3}+\frac{\phi}{3}\right)-1\right)\\
&+\frac{4(a-bc)}{hb}\frac{N}{(4N^2-1)^2}\frac{\frac{4}{3}\sin\left(\frac{\pi}{3}+\frac{\phi}{3}\right)}{\sqrt{\frac{4}{3F}\frac{(a-bc)^3}{hb^2}\frac{N^4}{(4N^2-1)^2}-1}}\\
=& \underbrace{\frac{4(a-bc)}{hb}\frac{N}{(4N^2-1)^2}}_{\geq 0}\cdot\\
&\left(\underbrace{\left(2\cos\left(\frac{\pi}{3}+\frac{\phi}{3}\right)-1\right)}_{\in[-1,0]}+\underbrace{\frac{\frac{4}{3}\sin\left(\frac{\pi}{3}+\frac{\phi}{3}\right)}{\sqrt{\frac{4}{3F}\frac{(a-bc)^3}{hb^2}\frac{N^4}{(4N^2-1)^2}-1}}}_{\geq 1}\right)\geq 0.
\end{aligned}$$

The proof for $F > \frac{2}{3}\frac{(a-bc)^3}{hb^2}\frac{N^4}{(4N^2-1)^2}$ follows analogous. □

A.3 Proof of Proposition 5

Equation (3.44) gives

$$P_i^*(N, t_N^*) = \frac{1}{2}\left(\frac{a}{b}+c+\frac{h}{2}\frac{(2i-1)}{N}t_N^*\right),$$

thus,

$$\begin{aligned}
P_i^*(N+1, t_{N+1}^*) &\leq P_i^*(N, t_N^*)\\
\Leftrightarrow \frac{1}{2}\left(\frac{a}{b}+c+\frac{h}{2}\frac{(2i-1)}{N+1}t_{N+1}^*\right) &\leq \frac{1}{2}\left(\frac{a}{b}+c+\frac{h}{2}\frac{(2i-1)}{N}t_N^*\right),\\
\Leftrightarrow \frac{t_{N+1}^*}{N+1}\leq\frac{t_N^*}{N} \quad\Leftrightarrow\quad & \frac{N}{N+1}\leq\frac{t_N^*}{t_{N+1}^*}.
\end{aligned} \tag{A.14}$$

An equivalent transformation of the second inequality provides:

$$\begin{aligned}
P_i^*(N, t_N^*) &\leq P_{i+1}^*(N+1, t_{N+1}^*)\\
\Leftrightarrow \frac{1}{2}\left(\frac{a}{b}+c+\frac{h}{2}\frac{(2i-1)}{N}t_N^*\right) &\leq \frac{1}{2}\left(\frac{a}{b}+c+\frac{h}{2}\frac{(2i+1)}{N+1}t_{N+1}^*\right),
\end{aligned}$$

$$\Leftrightarrow \frac{t_N^*}{t_{N+1}^*} \leq \frac{(2i+1)}{(2i-1)} \frac{N}{N+1} \quad \forall i = 1, \cdots, N. \tag{A.15}$$

For proving this proposition, we have to prove (A.14) and (A.15). For $F \leq \frac{2}{3} \frac{(a-bc)^3}{hb^2} \frac{N^4}{(4N^2-1)^2}$, from Proposition 4 and (3.48) we get

$$\frac{t_N^*}{t_{N+1}^*} = \frac{\frac{N^2}{(4N^2-1)}}{\frac{(N+1)^2}{(4(N+1)^2-1)}} \frac{\left(2\cos(\frac{\pi}{3} + \frac{\phi_N}{3}) - 1\right)}{\left(2\cos(\frac{\pi}{3} + \frac{\phi_{N+1}}{3}) - 1\right)},$$

$$\text{with} \quad \phi_N = \arccos\left(1 - \frac{3F}{2} \frac{hb^2}{(a-bc)^3} \frac{(4N^2-1)^2}{N^4}\right).$$

Some algebraic transformations give

$$\frac{t_N^*}{t_{N+1}^*} = \frac{N}{(N+1)} \underbrace{\frac{(2N+3)}{(2N+1-\frac{1}{N})}}_{\theta_1(N)} \underbrace{\frac{\left(2\cos(\frac{\pi}{3} + \frac{\phi_N}{3}) - 1\right)}{\left(2\cos(\frac{\pi}{3} + \frac{\phi_{N+1}}{3}) - 1\right)}}_{\theta_2(N)}. \tag{A.16}$$

First, we analyze the factors $\theta_1(N)$ and $\theta_2(N)$ separately. It is easy to show that $\theta_1(N) \geq 1$ for all $N \geq 1$. For the factor $\theta_2(N)$ we have to analyze ϕ_N. Equation (A.13) indicates $\phi_N \in [0, \frac{\pi}{2}]$ for all N. The factor $\frac{(4N^2-1)^2}{N^4}$ is increasing in N. Therefore,

- $(1 - \frac{3F}{2} \frac{hb^2}{(a-bc)^3} \frac{(4N^2-1)^2}{N^4})$ is decreasing in N,
- $\phi_N = \arccos(1 - \frac{3F}{2} \frac{hb^2}{(a-bc)^3} \frac{(4N^2-1)^2}{N^4})$ is increasing in N,

i.e., $\phi_N \leq \phi_{N+1}$. Using $\phi_N \in [0, \frac{\pi}{2}]$ $\forall N$ it follows $(2\cos(\frac{\pi}{3} + \frac{\phi_N}{3}) - 1) \in [-1, 0]$ $\forall N$ and

$$\phi_N \leq \phi_{N+1} \iff \left|2\cos(\frac{\pi}{3} + \frac{\phi_N}{3}) - 1\right| \leq \left|2\cos(\frac{\pi}{3} + \frac{\phi_{N+1}}{3}) - 1\right|. \tag{A.17}$$

Thus,

$$\theta_2(N) = \frac{\left(2\cos(\frac{\pi}{3} + \frac{\phi_N}{3}) - 1\right)}{\left(2\cos(\frac{\pi}{3} + \frac{\phi_{N+1}}{3}) - 1\right)} \leq 1. \tag{A.18}$$

Using this result, we prove inequality (A.15). It is obvious that $\frac{(2i+1)}{(2i-1)}$ is decreasing in i for all $i = 1, \cdots, N$. Therefore, it is sufficient to show that $\frac{t_N^*}{t_{N+1}^*} \leq \frac{(2N+1)}{(2N-1)} \frac{N}{(N+1)}$. (A.16) and (A.18) give

$$\frac{t_N^*}{t_{N+1}^*} \leq \frac{N}{(N+1)} \frac{(2N+3)}{(2N+1-\frac{1}{N})}. \tag{A.19}$$

Considering inequality (A.15), let $g_1(N) = \frac{(2N+1)}{(2N-1)}$ and $g_2(N) = \frac{(2N+3)}{(2N+1-\frac{1}{N})}$. We will show that $g_1(N) \geq g_2(N)$ for all $N \geq 1$. This is easy to verify for $N = 1$, $g_1(1) = 3 \geq 2.5 = g_2(1)$. Furthermore, we show that there is no intersection point for $N > 1$:

$$g_1(N) = g_2(N) \Leftrightarrow \frac{(2N+1)}{(2N-1)} = \frac{(2N+3)}{(2N+1-\frac{1}{N})} \Leftrightarrow N = \frac{1}{2}.$$

Therefore, for all $N > 1$ and $i = 1, \cdots, N$ $\frac{(2i+1)}{(2i-1)} \geq \frac{(2N+3)}{(2N+1-\frac{1}{N})}$ and thus,

$$\frac{t^*_N}{t^*_{N+1}} \leq \frac{(2i+1)}{(2i-1)} \frac{N}{N+1} \qquad \forall i = 1, \cdots, N.$$

For proving relation (A.14), we have to show that $\theta_1(N) \cdot \theta_2(N) \geq 1$ for all $N \geq 1$. The expression $\theta_1(N)$ depends only on N, however, $\theta_2(N)$ is also influenced by the cost and demand parameters F, h, c, a, and b through ϕ_N. The goal of the proof is to find a parameter configuration such that the numerator and the denominator of $\theta_2(N)$ have the largest difference, i.e., a change from $N \longrightarrow N+1$ has the largest impact on ϕ_N. Then, $\theta_2(N)$ has the lowest value. For this lower-bound configuration we show that $\theta_1(N) \cdot \theta_2(N) \geq 1$.

From (A.17), we get that $\phi_N = \arccos\left(1 - \frac{3F}{2}\frac{hb^2}{(a-bc)^3}\frac{(4N^2-1)^2}{N^4}\right)$ is increasing in N. From $\phi_N \in [0, \frac{\pi}{2}]$ follows $(\frac{\pi}{3} + \frac{\phi_N}{3}) \in [\frac{\pi}{3}, \frac{\pi}{2}] \implies \cos(\frac{\pi}{3} + \frac{\phi_N}{3}) \in [0, \frac{1}{2}]$. An increase in N has the largest impact on the cosine for $\phi_N = \frac{\pi}{2}$, i.e., when $\left(1 - \frac{3F}{2}\frac{hb^2}{(a-bc)^3}\frac{(4N^2-1)^2}{N^4}\right) = 0$. This is achieved when F is close to its maximum value. From $F \leq \frac{2}{3}\frac{(a-bc)^3}{hb^2}\frac{N^4}{(4N^2-1)^2}$ for all $N \geq 1$ follows $F \leq \frac{1}{24}\frac{(a-bc)^3}{hb^2}$ for $N \longrightarrow \infty$. Therefore, the upper bound for F is $F_{max} = \frac{1}{24}\frac{(a-bc)^3}{hb^2} \longrightarrow \phi_N = \arccos(1 - \frac{1}{16}\frac{N^4}{(4N^2-1)^2})$. Due to the characteristics of the cosine-function, $\theta_2(N)$ is a lower bound for $F_{max} = \frac{1}{24}\frac{(a-bc)^3}{hb^2}$, e.g., for $F < F_{max}$, $\theta_2(N)$ will be larger. Therefore, $\theta_1(N) \cdot \theta_2(N) \geq 1$ and it follows

$$\frac{N}{N+1} \leq \frac{t^*_N}{t^*_{N+1}}.$$

The proof for the case $F > \frac{2}{3}\frac{(a-bc)^3}{hb^2}\frac{N^4}{(4N^2-1)^2}$ is similar. □

A.4 Proof of Proposition 7

Let $f_1(P) = P - c - \frac{1}{b}$ and $f_2(P) = \sqrt{\frac{Fh}{2ae^{-bP}}}$ from (3.56). It is easy to verify that the function

$$g(P) = f_2(P) - f_1(P) = \sqrt{\frac{Fh}{2ae^{-bP}}} - P + c + \frac{1}{b}$$

is convex in P, thus, the (sufficient) first-order condition $\frac{\partial g}{\partial P}$ establishes the local minimum,

$$P_{min} = -\frac{2}{b} ln\left(\frac{b}{2}\sqrt{\frac{Fh}{2a}}\right).$$

If $g(P_{min}) \geq 0$, there are no real roots, and thus, no solution of (3.56) exists

$$g(P_{min}) = \sqrt{\frac{Fh}{2a}} e^{-ln(\frac{b}{2}\sqrt{\frac{Fh}{2a}})} + \frac{2}{b} ln\left(\frac{b}{2}\sqrt{\frac{Fh}{2a}}\right) + c + \frac{1}{b} \geq 0 \Leftrightarrow F \geq \frac{8a}{hb^2} e^{-(3+bc)}.$$

If $F < \frac{8a}{hb^2} e^{-(3+bc)}$, then $g(P)$ has two different real roots. By analyzing the profit function (3.6) with an exponential price-response function,

$$\Pi(P) = (P - c)ae^{-bP} - \sqrt{2Fhae^{-bP}}, \tag{A.20}$$

we note that $\lim_{P \to 0} \Pi(P) = -\infty$ and $\lim_{P \to \infty} \Pi(P) = 0$. Under the necessary condition $F < \frac{8a}{hb^2} e^{-(3+bc)}$, P_1^0 is a local maximum and P_2^0 is a local minimum and $P^* = P_1^0$. To determine the optimal price P^* that maximizes the profit function (A.20), the following algorithm can be used.

Algorithm 1

Let $P_1 = 0$, $P_2 > 0$, and ε sufficiently small.
WHILE $|P_1 - P_2| \leq \varepsilon$

1. Evaluate $f_2(P_1)$.
2. Find P_2 such that $f_2(P_1) = f_1(P_2)$.
3. $P_1 := P_2$

END $\square$

A.5 Proof of Proposition 8

The proof is by induction:

- First, we show $\frac{t_1^*}{t_2^*} \leq \frac{1}{2}$. Assume $t_0 = 0$. Then, from (3.59) t_1 is given by

$$t_1^* = \frac{2}{hb} - \frac{t_2^*}{e^{\frac{hb}{2}t_2^*} - 1}. \tag{A.21}$$

  Thus, we obtain

$$\frac{t_1^*}{t_2^*} = \frac{2}{hbt_2^*} - \frac{1}{e^{\frac{hb}{2}t_2^*} - 1}. \tag{A.22}$$

  Expression (A.22) is strictly decreasing in t_2^*. With the conditions $t_1^* \leq t_2^*$ and $t_1^*, t_2^* \geq 0$, the statement holds considering the limiting values $\lim\limits_{t_2^* \to 0} \frac{t_1^*}{t_2^*} = \frac{1}{2}$ and $\lim\limits_{t_2^* \to \infty} \frac{t_1^*}{t_2^*} = 0$.

- Assume the statement holds for $k = i - 1$, i.e., $\frac{t_{i-2}}{t_{i-1}} \leq \frac{i-2}{i-1}$. For $\frac{t_{i-1}}{t_i}$ we obtain

$$\begin{aligned}
\frac{t_{i-1}}{t_i} &= \frac{2}{hbt_i} - \frac{e^{-\frac{hb}{2}t_i}}{(e^{-\frac{hb}{2}t_{i-2}} - e^{-\frac{hb}{2}t_i})} + \frac{t_{i-2}}{t_i}\frac{e^{-\frac{hb}{2}t_{i-2}}}{(e^{-\frac{hb}{2}t_{i-2}} - e^{-\frac{hb}{2}t_i})} \\
&\leq \frac{2}{hbt_i} - \frac{1}{(e^{\frac{hb}{2}(t_i - t_{i-2})} - 1)} + \frac{t_{i-2}}{t_{i-1}}\frac{1}{(1 - e^{-\frac{hb}{2}(t_i - t_{i-2})})}.
\end{aligned}$$

  Due to the assumption $\frac{t_{i-2}}{t_{i-1}} \leq \frac{(i-2)}{(i-1)} \leq \frac{(i-1)}{i}$,

$$\frac{t_{i-1}}{t_i} \leq \frac{2(e^{\frac{hb}{2}(t_i - t_{i-2})} - 1) - hbt_i}{hbt_i(e^{\frac{hb}{2}(t_i - t_{i-2})} - 1)} + \left(\frac{i-1}{i}\right)\frac{1}{(1 - e^{-\frac{hb}{2}(t_i - t_{i-2})})}.$$

  Considering the limiting values $\lim\limits_{t_i^* \to 0} \frac{t_{i-1}^*}{t_i^*} = \frac{1}{2}$ and $\lim\limits_{t_i^* \to \infty} \frac{t_{i-1}^*}{t_i^*} = \frac{i-1}{i}$, and the fact that $\frac{t_{i-1}^*}{t_i^*}$ is strictly decreasing in t_i^*, the proposition holds.

$\square$

A.6 Proof of Proposition 9

Let $C(P) := \sqrt{2Fh_l D(P)}$. $C(P)$ is a monotonically decreasing function in P, i.e., $\frac{\partial C}{\partial P} \leq 0$. Furthermore, $\frac{\partial^2 C}{\partial P^2} = \sqrt{\frac{Fh_l}{2D(P)}} \left(\frac{(D'(P))^2}{2D(P)} - D''(P) \right)$. If $D''(P) \leq 0$, then $C(P)$ is a convex function in P, i.e., $\frac{\partial C}{\partial P} \geq 0$. Therefore, $-C(P)$ is a concave function in P. Because $\tilde{\Pi}_l$ is a concave function in P, $\Pi_l(P)$ is a concave function. However, if $D''(P) > 0$, then $C(P)$ is not uniquely convex or concave. For all P with $\frac{(D'(P))^2}{2D(P)} - D''(P) \geq 0$, $C(P)$ is convex and it follows the same argumentation as above. For all P where $\frac{(D'(P))^2}{2D(P)} - D''(P) \leq 0$, $C(P)$ is concave and thus, $-C(P)$ is convex. Due to the monotonicity, $\Pi_l(P)$ is concave-convex. An analysis of the limiting values gives $\lim\limits_{P \to 0} \Pi_l(P) = -\sqrt{2Fh_l D(0)} \leq 0$ and $\lim\limits_{P \to \bar{P}} \Pi_l(P) = 0$.

□

A.7 Proof of Proposition 11

Given the first-order conditions in (3.83), we have to show that $\frac{\partial}{\partial P} \left(\frac{D(P)}{D'(P)} + P \right) \geq 0$. The optimal P has to satisfy $\frac{D(P)}{D'(P)} + P \geq 0 \Leftrightarrow \varepsilon_P = -\frac{D'(P)}{D(P)} P \geq 1$. The IPE from Definition 1 gives that

$$\begin{aligned} \frac{\partial \varepsilon_P}{\partial P} = -\frac{\partial}{\partial P} \left(\frac{D'(P)}{D(P)} P \right) &\geq 0 \\ \Leftrightarrow -\frac{\partial}{\partial P} \left(\frac{D'(P)}{D(P)} \right) P - \frac{D'(P)}{D(P)} &\geq 0. \end{aligned}$$

Some algebraic transformations give

$$-\frac{D''(P)}{D'(P)} P \leq 1 + \varepsilon_P. \tag{A.23}$$

Assume that $\frac{\partial}{\partial P} \left(\frac{D(P)}{D'(P)} + P \right) < 0$. Thus $\frac{\partial}{\partial P} \left(\frac{D(P)}{D'(P)} \right) + 1 < 0 \Leftrightarrow 2 - \frac{D(P)D''(P)}{(D'(P))^2} < 0$. Then,

$$-\frac{D''(P)}{D'(P)} P > 2\varepsilon_P. \tag{A.24}$$

Because $\varepsilon_P \geq 1$, inequality (A.23) and (A.24) cannot be valid simultaneously. Thus, $\frac{\partial}{\partial P} \left(\frac{D(P)}{D'(P)} + P \right) \geq 0$. From $c_0 < c_0 + \frac{F}{Q_0^*}$ and (3.83) we get that $\tilde{P}^* < P^*$. However, it is not assured that $c_0 < c_l + \frac{F}{Q_l^*}$ for a quantity discount r_l with $l > 0$.

From the fact that $\frac{\partial}{\partial P}\left(\frac{D(P)}{D'(P)}+P\right)\geq 0$, it follows that the selling price in the coordinated system is higher if and only if

$$c_0 < c_l + \frac{F}{Q}$$

with $Q=\sqrt{\frac{2FD(P)}{h_l}}$. The property follows from

$$c_0 - c_l < \sqrt{\frac{Fh_l}{2D(P)}} \Leftrightarrow D(P) < \frac{Fh_l}{2(c_0-c_l)^2} \Leftrightarrow P_l^* > D^{-1}\left(\frac{Fh_l}{2(c_0-c_l)^2}\right).$$

□

A.8 Proof of Proposition 12

The partial derivation of (3.97) gives

$$\begin{aligned}
\frac{\partial \Pi_l^{(N)}}{\partial c_l} &= -\left(\tfrac{a}{b}-c_l-h_l t_N\right)-\left(\tfrac{a}{b}-c_l\right)\left(1+\tfrac{\partial h_l}{\partial c_l}t_N\right)+\tfrac{h_l}{6}\tfrac{\partial h_l}{\partial c_l}\tfrac{(4N^2-1)}{N^2}t_N^2\\
&= -2\left(\tfrac{a}{b}-c_l-\tfrac{h_l}{2}t_N\right)-\left(\tfrac{a}{b}-c_l\right)\tfrac{\partial h_l}{\partial c_l}t_N+\tfrac{h_l}{6}\tfrac{\partial h_l}{\partial c_l}\tfrac{(4N^2-1)}{N^2}t_N^2\\
&= -2\left(\tfrac{a}{b}-c_l-\tfrac{h_l}{2}t_N\right)-\tfrac{\partial h_l}{\partial c_l}\left(\tfrac{a}{b}-c_l-\tfrac{h_l}{6}\tfrac{(4N^2-1)}{N^2}t_N\right)t_N.
\end{aligned}$$

Using $\frac{\partial h_l}{\partial c_l}\geq 0$ and $D_i\geq 0$ for all $i=1,\cdots,N$ with

$$D_N=\frac{b}{2}\left(\frac{a}{b}-c_l-\frac{h_l}{2}\frac{(2i-1)}{N}t_N\right)\geq 0 \Leftrightarrow \frac{a}{b}-c_l-\frac{h_l}{2}\frac{(2i-1)}{N}t_N\geq 0,$$

it follows

$$\frac{\partial \Pi_l^{(N)}}{\partial c_l} = -2\left(\underbrace{\frac{a}{b}-c_l-\frac{h_l}{2}t_N}_{\geq 0}\right)-\frac{\partial h_l}{\partial c_l}\left(\underbrace{\frac{a}{b}-c_l-\frac{h_l}{6}\frac{(4N^2-1)}{N^2}t_N}_{\geq 0}\right)t_N\leq 0.$$

□

A.9 Proof of Proposition 13

The first-order condition gives that $\Pi_l^{(N)}(t_N)$ has three local extreme values. The limiting values indicate that $\lim\limits_{t_N \to -\infty} \Pi_l^{(N)}(t_N) = +\infty$, $\lim\limits_{t_N \to 0^-} \Pi_l^{(N)}(t_N) = +\infty$. Thus, the function $\Pi_l^{(N)}(t_N)$ has a local minimum for $t_N < 0$ and a vertical asymptote at $t_N = 0$. Furthermore, we have $\lim\limits_{t_N \to +0} \Pi_l^{(N)}(t_N) = -\infty$ and $\lim\limits_{t_N \to +\infty} \Pi_l^{(N)}(t_N) = +\infty$. Therefore, for $t_N \geq 0$ the profit function $\Pi_l^{(N)}(t_N)$ comes from $-\infty$, has first a local maximum, then a local minimum, and goes to $+\infty$, i.e., $\Pi_l^{(N)}(t_N)$ is a concave-convex function for $t_N \geq 0$.

□

A.10 Proof of Proposition 14

The first-order condition of (3.118) with respect to T_k, the conditions (3.108) - (3.110), and substitution $T = \sum\limits_{k=1}^{K} T_k$ gives:

$$\frac{\partial L}{\partial T_k} = 0 \quad \Leftrightarrow \quad \sum_{j=1}^{K} \left(\frac{F^j}{T^2} - \frac{h^j}{2} D^j \right) - \sum_{j=1}^{K} \mu_j \Theta_{kj}(P^1, \cdots, P^K) \stackrel{!}{=} 0 \, k = 1, \cdots, K. \tag{A.25}$$

with

$$\Theta_{kj}(P^1, \cdots, P^k) = \begin{cases} \sum\limits_{i=1}^{j} s^i D^i + \sum\limits_{i=k+1}^{K} s^i D^i & : \quad j \leq k \\ \sum\limits_{i=k+1}^{j} s^i D^i & : \quad j > k \end{cases}. \tag{A.26}$$

Because the first term of (A.25) is equal for any pair (k, k'),$k \neq k'$ of products, it follows

$$\sum_{j=1}^{K} \mu_j \Theta_{kj}(P^1, \cdots, P^k) = \sum_{j=1}^{K} \mu_j \Theta_{k'j}(P^1, \cdots, P^k) \tag{A.27}$$

$$\Leftrightarrow \quad \sum_{j=1}^{K} \mu_j \left(\Theta_{kj}(P^1, \cdots, P^k) - \Theta_{k'j}(P^1, \cdots, P^k) \right) = 0. \tag{A.28}$$

We assume that all products are profitable, i.e., $D^k > 0$ for all $k = 1, \cdots, K$. Therefore, from (A.26) it can be seen that $\Theta_{kj} \neq \Theta_{k'j}$ for all $k \neq k'$ and $\Theta_{kj} \neq \Theta_{kj'}$ for all $j \neq j'$, therefore, $\Theta_{kj} - \Theta_{k'j'} \neq 0$ and each summand of (A.28) must be different from zero.

In the following, we show that either the capacity constraint is binding or it is not binding for all replenishments. We prove this property by a contradiction to the assumption that there is a replenishment where the capacity constraint is not binding.

Consider a pair (k, k') of products. Without loss of generality assume $k < k'$. Using (A.26), (A.28) can be transformed into:

$$\sum_{j=1}^{k} \mu_j \left(\sum_{i=k+1}^{k'} s^i D^i \right) - \sum_{j=k+1}^{k'} \mu_j \left(\sum_{i=1}^{k} s^i D^i + \sum_{i=k'+1}^{K} s^i D^i \right) + \sum_{j=k'+1}^{K} \mu_j \left(\sum_{i=k+1}^{k'} s^i D^i \right) = 0. \quad \text{(A.29)}$$

Suppose there is a finite number of replenishments with $\mu_k = 0$ and a finite number of replenishments with $\mu_j > 0$ for all $j, k \in 1, \cdots, K$. That is, the replenishment of a product k does not fill up the storage volume to the capacity level. Thus, we can omit this particular summand k in (A.29).

$$\sum_{j=1}^{k-1} \mu_j \left(\sum_{i=k+1}^{k'} s^i D^i \right) - \sum_{j=k+1}^{k'} \mu_j \left(\sum_{i=1}^{k} s^i D^i + \sum_{i=k'+1}^{K} s^i D^i \right) + \sum_{j=k'+1}^{K} \mu_j \left(\sum_{i=k+1}^{k'} s^i D^i \right) = 0.$$

Changing the sequence of the summation gives:

$$\sum_{i=k+1}^{k'} s^i D^i \left(\sum_{j=1}^{k-1} \mu_j \right) - \sum_{i=1}^{k} s^i D^i \left(\sum_{j=k+1}^{k'} \mu_j \right) - \sum_{i=k'+1}^{K} s^i D^i \left(\sum_{j=k+1}^{k'} \mu_j \right) + \sum_{i=k+1}^{k'} s^i D^i \left(\sum_{j=k'+1}^{K} \mu_j \right) = 0. \quad \text{(A.30)}$$

Because (A.29) must hold for each $k \neq k'$, it must also hold for $k'' \neq k'$ with $k'' = k - 1$:

$$\sum_{j=1}^{k-1} \mu_j \left(\sum_{i=k}^{k'} s^i D^i \right) - \sum_{j=k+1}^{k'} \mu_j \left(\sum_{i=1}^{k-1} s^i D^i + \sum_{i=k'+1}^{K} s^i D^i \right) + \sum_{j=k'+1}^{K} \mu_j \left(\sum_{i=k}^{k'} s^i D^i \right) = 0.$$

Changing the sequence of the summation gives:

$$\sum_{i=k}^{k'} s^i D^i \left(\sum_{j=1}^{k-1} \mu_j \right) - \sum_{i=1}^{k-1} s^i D^i \left(\sum_{j=k+1}^{k'} \mu_j \right) \\ - \sum_{i=k'+1}^{K} s^i D^i \left(\sum_{j=k+1}^{k'} \mu_j \right) + \sum_{i=k}^{k'} s^i D^i \left(\sum_{j=k'+1}^{K} \mu_j \right) = 0. \quad \text{(A.31)}$$

Equating (A.30) and (A.31) gives

$$s^k D^k \left(\sum_{j=1, j\neq k}^{K} \mu_j \right) = 0.$$

This is a contradiction of the statement $\mu_j \neq 0$ for all $j \neq k$.

Thus, either the given capacity level is binding and $\mu_j > 0$ for all $j = 1, \cdots, K$, or the given capacity level is not binding and $\mu_j = 0$ for all $j = 1, \cdots, K$. It follows implicitly that with each replenishment the storage volume is filled up to the same capacity level.

□

A.11 Proof of Proposition 19

From (4.8) it follows that

$$\frac{\partial P^*(K)}{\partial K} = \frac{1}{2b} \left(K(1 - F(K)) \right) \geq 0$$

and

$$\frac{\partial^2 P^*(K)}{\partial K^2} = \frac{1}{2b} \left((1 - F(K)) - Kf(K) \right) \leq 0 \text{ because } \frac{Kf(K)}{(1 - F(K))} \geq 0.$$

Optimality of $K^*(P)$ and the *implicit function theorem* imply that

$$\frac{\partial K^*(P)}{\partial P} = \frac{1}{f(K)(P - c_p)^2} \geq 0$$

and the second derivative gives

$$\frac{\partial^2 K^*(P)}{\partial P^2} = \frac{-2}{f(K)(P - c_p)^3} \leq 0.$$

□

A.12 Proof of Proposition 20

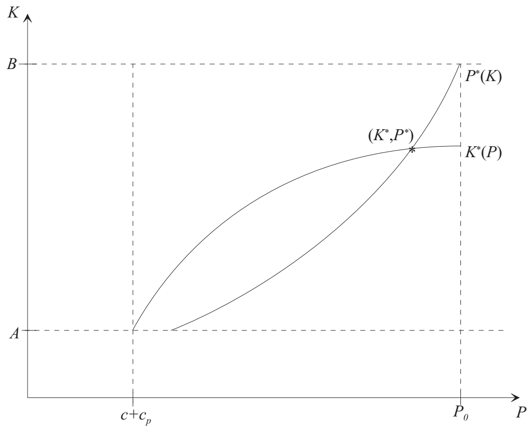

Figure A.2: Response functions $K^*(P)$ and $P^*(K)$

(4.7) gives that for the lowest price $P = c + c_p$, thus $K^*(c + c_p) = F^{-1}(0) = A$. Furthermore, the highest price is P_0 and with (4.11) we get

$$K^*(P_0) = F^{-1}\left(\frac{\mu - b(c_p + c)}{\mu - b(c - c_p)}\right) < B.$$

From (4.8) it follows that $P^*(B) = P_0$ and with the condition $A \geq b(c + c_p)$ that guaranteed nonnegative demand it follows that

$$P^*(A) = \frac{1}{2b}(A + b(c + c_p)) \geq c + c_p.$$

Additionally, by the concavity of $P^*(K)$ and $K^*(P)$ shown in Proposition 19, it follows that there exists a unique optimal solution (K^*, P^*) (see Figure A.2).

□

A.13 Proof of Theorem 3

Denote $R(K) := \frac{\partial \Pi(K, P_H^*(K), P_L)}{\partial K}$ where $P_H^*(K)$ is substituted from (4.19).

$$R(K) = \left(P_H^0 - \frac{\Theta(K)}{2b} - c_H\right)[1 - F(K)] + (P_L - c_L)[F(K) - F(K - d_L)] - c \tag{A.32}$$

The first and second derivatives give

$$R'(K) = -\frac{f(K)}{2b} \cdot \left[2b(P_H^0 - c_H) - \frac{[1 - F(K)]}{h(K)} - \Theta(K) - 2b(P_L - c_L) + 2b(P_L - c_L)\frac{f(K - d_L)}{f(K)}\right]$$

and

$$R''(K) = \frac{f'(K)}{f(K)}R'(K) - \frac{f(K)}{2b}\left[\frac{f(K)h(K) + h'(K)[1 - F(K)]}{h(K)^2}\right.$$
$$\left. + [1 - F(K)] + 2b(P_L - c_L)\left(\frac{f'(K - d_L)f(K) - f(K - d_L)f'(K)}{f(K)^2}\right)\right].$$

The value of $R''(K)$ at K that satisfies $R'(K) = 0$ defines the behavior of Π at the inflection points, that is

$$R''(K)\mid_{R'(K)=0} = -\frac{f(K)[1 - F(K)]}{2bh(K)^2} \cdot \left[\underbrace{2h(K)^2 + h'(K) + \frac{2bP_L}{[1 - F(K)]^3}(f'(K - d_L)f(K) - f(K - d_L)f'(K))}_{(1)}\right]. \tag{A.33}$$

Using the condition that $h(K)$ is increasing in K, i.e.,

$$h'(K) = h(K)\left(\frac{f'(K)}{f(K)} + h(K)\right) \geq 0,$$

it follows that $2h(K)^2 + \frac{\partial h(K)}{\partial K} \geq 0$ and due to $h''(K) \leq 0$ where

$$h''(K) = h'(K)\left(\frac{f'(K)}{f(K)} + h(K)\right) + h(K)\left(\frac{\partial}{\partial K}\left(\frac{f'(K)}{f(K)}\right) + h'(K)\right) \leq 0$$
$$\Leftrightarrow \frac{\partial}{\partial K}\left(\frac{f'(K)}{f(K)}\right) \leq 0.$$

Further $\frac{\partial}{\partial K}\left(\frac{f'(K)}{f(K)}\right) \leq 0$ gives

$$\frac{f'(K-d_L)}{f(K-d_L)} \geq \frac{f'(K)}{f(K)} \Leftrightarrow f'(K-d_L)f(K) - f(K-d_L)f'(K) \geq 0 \tag{A.34}$$

which yields that the term (1) in (A.33) is nonnegative. Consequently,

$$R''(K)\mid_{R'(K)=0} \leq 0$$

and the local extreme point of $R(K)$ is a local maximum, i.e., $R(K)$ is unimodal and has at most two roots where the larger root of $R(K)$ corresponds to a local maximum and the lower root of $R(K)$ corresponds to a local minimum of Π . Figure A.3 illustrates $\Pi(K, P_H(K), P_L)$ and $R(K)$ given P_L.

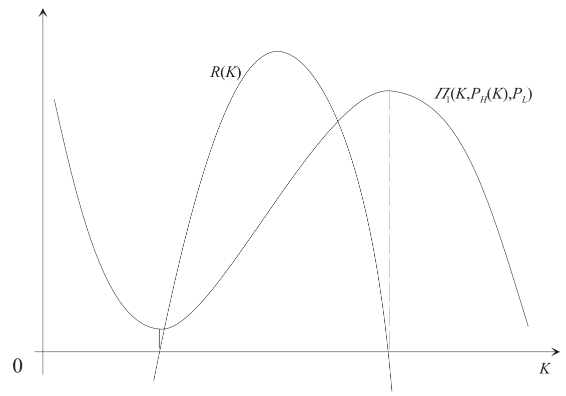

Figure A.3: $\Pi(K, P_H(K), P_L)$ and $R(K)$ for a given P_L

Let K_{NV} be the optimal capacity of the single-product case. We will show $K^* \geq K_{NV}$. Assume that $K^* < K_{NV}$. Then it follows that $F(K^*) < F(K_{NV})$ and by using (4.19) it follows $P_H(K^*) < P_H(K_{NV})$. Since K_{NV} is the optimal solution of the single-product case, K_{NV} solves (4.7), i.e.,

$$(P(K_{NV}) - c_H)(1 - F(K_{NV})) - c = 0.$$

Furthermore, K^* satisfies (4.23). Thus, it follows that

$$\begin{aligned}(P(K_{NV}) - c_H)(1 - F(K_{NV})) &= (P_H^*(K^*) - c_H)(1 - F(K^*)) \\ &\quad +(P_L - c_L)(F(K^*) - F(K^* - d_L)) \\ (P(K_{NV}) - c_H)(1 - F(K_{NV})) &\geq (P_H^*(K^*) - c_H)(1 - F(K^*))\end{aligned}$$

which is equivalent to

$$(P(K_{NV}) - c_H) - (P_H^*(K) - c_H) \geq (P(K_{NV}) - c_H)F(K_{NV}) - (P_H^*(K) - c_H)F(K).$$

Since $F(K_{NV}) > F(K^*)$ it follows that "$\geq$" cannot be true. Thus, $K^* \geq K_{NV}$.

By (4.7) we get that in the single-product case K_{NV} has to satisfy $(P_H^*(K_{NV}) - c_H)(1 - F(K_{NV})) - c = 0$. Therefore, from (A.32) it follows $R(K_{NV}) > 0$ which, in turn, gives that K_{NV} is larger than the local minimum and lower than the local maximum of $\Pi(K, P_H^*(K), P_L)$. Therefore, $\Pi(K, P_H^*(K), P_L)$ is quasi-concave with the unique maximum $K^*(P_L)$.

□

A.14 Proof of Theorem 4

In order to show that for a given P_H, $\Pi(K, P_L)$ is jointly concave in K and P_L, it has to be shown that the *Hessian matrix* of $\Pi(K, P_L)$ is negative semi-definite. The second-order derivatives are

$$\frac{\partial^2 \Pi}{\partial P_L^2} = -(P_L - c_L)\beta^2 f(K - d_L) - 2\beta F(K - d_L),$$

$$\frac{\partial^2 \Pi}{\partial K^2} = -(P_H - c_H)f(K) + (P_L - c_L)(f(K) - f(K - d_L)),$$

$$\frac{\partial^2 \Pi}{\partial K \partial P_L} = (F(K) - F(K - d_L)) - f(K - d_L)\beta(P_L - c_L).$$

It is to check if for a given P_H the determinant of the Hessian of $\Pi(K, P_L)$ is non-negative. Let $\Delta_H(K, P_L)$ be the determinant of the Hessian with

$$\Delta_H(K, P_L) = \frac{\partial^2 \Pi}{\partial P_L^2}\frac{\partial^2 \Pi}{\partial K^2} - \left[\frac{\partial^2 \Pi}{\partial K \partial P_L}\right]^2.$$

That is,

$$\begin{aligned}\Delta_H(K,P_L) = &-((P_L-c_L)\beta^2 f(K-d_L)+2\beta F(K-d_L))\\&\cdot(-(P_H-c_H)f(K)+(P_L-c_L)(f(K)-f(K-d_L)))\\&-[(F(K)-F(K-d_L))-f(K-d_L)\beta(P_L-c_L)]^2\end{aligned}$$

Some algebraic transformations give

$$\begin{aligned}\Delta_H(K,P_L) = &(P_H-c_H-P_L+c_L)[(P_L-c_L)\beta^2 f(K-d_L)f(K)+2\beta F(K-d_L)f(K)]\\&-[(F(K)-F(K-d_L))]^2+(P_L-c_L)2\beta f(K-d_L)F(K). \quad \text{(A.35)}\end{aligned}$$

Using the substitution $K = C + bP_H$ on (4.12) it follows that

$$\begin{aligned}\varepsilon_s(P_L,C)\geq 1 \quad&\Leftrightarrow\quad -[F(K)-F(K-d_L)]+(P_L-c_L)\beta f(K-d_L)\geq 0\\&\Leftrightarrow -[F(K)-F(K-d_L)]^2+(P_L-c_L)2\beta f(K-d_L)F(K)\geq 0. \quad \text{(A.36)}\end{aligned}$$

It is easy to see that the first term in (A.35) is positive for sufficiently large P_H and with (A.36) it follows that for sufficiently large P_H, $\Pi(K,P_L)$ is jointly concave in K and P_L.

□

B Derivations

B.1 Derivation of (3.39)

We derive (3.39) from (3.29). Let $R^{(N)}$ denote the average revenue and $C^{(N)}$ denote the average cost per unit of time. For notational simplification, we use $D_i = D(P_i)$.
Partial derivation of $R^{(N)}$ with respect to t_N gives

$$\frac{\partial R^{(N)}}{\partial t_N} = -\frac{1}{t_N^2}\sum_{i=1}^{N} P_i D_i (t_i - t_{i-1}) + \frac{1}{t_N} P_N D_N,$$

$$\begin{aligned}\frac{\partial C^{(N)}}{\partial t_N} = -\frac{1}{t_N^2}\Bigg[F + c\sum_{i=1}^{N} D_i\,(t_i - t_{i-1}) + \underbrace{\frac{h}{2}\sum_{i=1}^{N} D_i\,(t_i - t_{i-1})^2}_{(1)} \\ \underbrace{+\, h\sum_{i=1}^{N-1}\left((t_i - t_{i-1})\sum_{j=i+1}^{N} D_j\,(t_j - t_{j-1})\right)\Bigg]}_{(2)} \\ + \frac{1}{t_N}\left[cD_N + hD_N \underbrace{\sum_{i=1}^{N}(t_i - t_{i-1})}_{(3)}\right].\end{aligned}$$

It is easy to verify that $(1) + (2) = \frac{h}{2}\sum\limits_{i=1}^{N} D_i\left(t_i^2 - t_{i-1}^2\right)$ and $(3) = t_N$. Therefore, the first-order condition $\frac{\partial R^{(N)}}{\partial t_N} \stackrel{!}{=} \frac{\partial C^{(N)}}{\partial t_N}$ gives

$$\sum_{i=1}^{N-1}(P_i - c)D_i(t_i - t_{i-1}) - (P_N - c)D_N t_{N-1} - \frac{h}{2}\sum_{i=1}^{N} D_i(t_i^2 - t_{i-1}^2) + hD_N t_N^2 - F \stackrel{!}{=} 0.$$

$$\Rightarrow \underbrace{\sum_{i=1}^{N-1}(P_i - c)D_i(t_i - t_{i-1}) - (P_N - c)D_N t_{N-1}}_{(4)}$$

$$\underbrace{-\frac{h}{2}\sum_{i=1}^{N-1} D_i(t_i^2 - t_{i-1}^2) + \frac{h}{2}D_N t_{N-1}^2}_{(5)} - F \stackrel{!}{=} -\frac{h}{2}D_N t_N^2$$

with $(4) = \sum\limits_{i=1}^{N-1}\left[(P_i - c)D_i - (P_{i+1} - c)D_{i+1}\right] t_i$ and $(5) = -\frac{h}{2}\sum\limits_{i=1}^{N-1} t_i^2\left[D_i - D_{i+1}\right]$. The optimal cycle length is determined by

$$t_N^* = \sqrt{\frac{2F}{hD_N} + \frac{1}{D_N}\sum_{i=1}^{N-1} t_i^2\left[D_i - D_{i+1}\right] - \frac{2}{hD_N}\sum_{i=1}^{N-1}\left[(P_i - c)D_i - (P_{i+1} - c)D_{i+1}\right] t_i}.$$

By substitution of (3.36), the optimal cycle length becomes

$$t_N^* = \sqrt{\frac{2F}{hD_N} - \frac{1}{D_N}\sum_{i=1}^{N-1} t_i^2\left[D_i - D_{i+1}\right]}.$$

B.2 Derivation of (3.97)

From (3.90) and (3.94) - (3.96) follows

$$\begin{aligned}
\Pi_l^{(N)} &= \frac{1}{t_N}\sum_{i=1}^{N}\left[\left(\frac{1}{2}\left(\frac{a}{b} - c_l + \frac{h_l}{2}(2i-1)\frac{t_N}{N}\right) - \frac{h_l}{2}(2i-1)\frac{t_N}{N}\right)\right. \\
&\qquad\qquad \left.\cdot\frac{b}{2}\left(\frac{a}{b} - c_l - \frac{h_l}{2}(2i-1)\frac{t_N}{N}\right)\frac{t_N}{N}\right] - \frac{F}{t_N} \\
&= \frac{1}{t_N}\sum_{i=1}^{N}\left[\frac{b}{4}\frac{t_N}{N}\sum_{i=1}^{N}\left(\frac{a}{b} - c_l - \frac{h_l}{2}(2i-1)\frac{t_N}{N}\right)^2\right] - \frac{F}{t_N} \\
&= \frac{1}{t_N}\sum_{i=1}^{N}\left[\frac{b}{4}\frac{t_N}{N}\left[\left(\frac{a}{b} - c_l\right)^2 N - \left(\frac{a}{b} - c_l\right)h_l t_N N + \frac{h_l^2}{4}\frac{t_N^2}{N^2}\sum_{i=1}^{N}(2i-1)^2\right]\right] \\
&\quad - \frac{F}{t_N}.
\end{aligned}$$

Therefore,

$$\Pi_l^{(N)}(t_N) = \frac{b}{4}\left[\left(\frac{a}{b} - c_l\right)\left(\frac{a}{b} - c_l - h_l t_N\right) + \frac{h_l^2 t_N^2}{12}\frac{(4N^2-1)}{N^2}\right] - \frac{F}{t_N}.$$

B.3 Derivation of (3.119)

The first-order condition of L with respect to P^k for all $k = 1, \cdots, K$ gives:

$$\frac{\partial L}{\partial P^k} = \frac{\partial \Pi(\vec{P}, \vec{T}, T)}{\partial P^k}$$
$$+\frac{\partial}{\partial P^k}\sum_{j=1}^{K}\mu_j\left(S - \left[\sum_{l=1}^{j-1} s^l D^l\left(T - \sum_{i=l}^{j-1} T_i\right) + s^j D^j T + \sum_{l=j+1}^{K} s^l D^{k*}\sum_{i=j}^{k-1} T_i\right]\right) \stackrel{!}{=} 0.$$

The partial derivation of the first term follows from (3.114) and the partial derivation of the second term can be determined from (3.108):

$$D^k + (P^k - c^k - \frac{h^k}{2}T)(D^k)' - \sum_{j=1}^{k-1}\mu_j s^k (D^k)'\left(\sum_{i=j}^{k-1} T_i\right)$$
$$- \mu_k s^k (D^k)' T - \sum_{j=k+1}^{K}\mu_j s^k (D^k)'\left(T - \sum_{i=k}^{j-1} T_i\right) \stackrel{!}{=} 0,$$

$$\Leftrightarrow \frac{D^k}{(D^k)'} + P^k = c^k + \frac{h^k}{2}T + s^k \underbrace{\left(\sum_{j=1}^{k-1}\mu_j\left(\sum_{i=j}^{k-1} T_i\right) + \mu_k T + \sum_{j=k+1}^{K}\mu_j\left(T - \sum_{i=k}^{j-1} T_i\right)\right)}_{=:\Psi(\mu_1,\cdots,\mu_K,T_1,\cdots,T_K)\geq 0}.$$

B.4 Derivation of (4.14)

For a simplified notation set $d_L := d(P_L)$. Taking expectation from (4.13) gives

$$\Pi(C,P_H,P_L)=\int_A^{C+bP_H-d_L}\left((P_H-c_H)(z-bP_H)+(P_L-c_L)d_L\right)f(z)dz$$
$$+\int_{C+bP_H-d_L}^{C+bP_H}\left((P_H-c_H)(z-bP_H)+(P_L-c_L)(C-z+bP_H)\right)f(z)dz$$
$$+\int_{C+bP_H}^{B}(P_H-c_H)Cf(z)dz-cC$$

$$=(P_H-c_H)(\mu-bP_H)-(P_H-c_H)\int_{C+bP_H}^{B}(z-bP_H)f(z)dz+(P_H-c_H)\int_{C+bP_H}^{B}Cf(z)dz$$
$$+(P_L-c_L)d_L-(P_L-c_L)\int_{C+bP_H-d_L}^{B}d_Lf(z)dz+(P_L-c_L)\int_{C+bP_H-d_L}^{C+bP_H}(C-z+bP_H)f(z)dz-cC.$$

Adding $0=c(\mu-bP_H+d_L)-c(\mu-bP_H+d_L)$ we have

$$=(P_H-c-c_H)(\mu-bP_H)+(P_L-c-c_L)d_L-(P_L-c_L)\int_{C+bP_H-d_L}^{C+bP_H}(d_L-C+z-bP_H)f(z)dz$$
$$-(P_H-c_H)\int_{C+bP_H}^{B}(z-bP_H-C)f(z)dz-(P_L-c_L)\int_{C+bP_H}^{B}d_Lf(z)dz-c(C-\mu+bP_H-d_L).$$

By

$$c(C-\mu+bP_H-d_L)=c\int_A^{C+bP_H-d_L}(C-z+bP_H-d_L)f(z)dz$$
$$+c\int_{C+bP_H-d_L}^{C+bP_H}(C-z+bP_H-d_L)f(z)dz+c\int_{C+bP_H}^{B}(C-z+bP_H-d_L)f(z)dz$$

and some transformations we get

$$= (P_H - c - c_H)(\mu - bP_H) + (P_L - c - c_L)d_L - c \int\limits_A^{C+bP_H-d_L} (C - z + bP_H - d_L) f(z)dz$$

$$- (P_L - c - c_L) \int\limits_{C+bP_H-d_L}^{C+bP_H} (d_L - C + z - bP_H) f(z)dz$$

$$- (P_H - c - c_H) \int\limits_{C+bP_H}^{B} (z - bP_H - C) f(z)dz - (P_L - c - c_L) \int\limits_{C+bP_H}^{B} d_L f(z)dz.$$

The substitution $K := C + bP_H$ gives

$$= (P_H - c - c_H)(\mu - bP_H) + (P_L - c - c_L)d_L$$

$$- c \int\limits_A^{K-d_L} (K - z - d_L) f(z)dz - (P_H - c - c_H) \int\limits_K^B (z - K) f(z)dz$$

$$- (P_L - c - c_L) \int\limits_{K-d_L}^{K} (d_L + z - K) f(z)dz - (P_L - c - c_L) \int\limits_K^B d_L f(z)dz.$$

Ehrenwörtliche Erklärung

Ich versichere, dass ich diese Arbeit ohne Hilfe Dritter und ohne Benutzung anderer als der angegebenen Quellen und Hilfsmittel angefertigt und die den benutzten Quellen wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht habe. Diese Arbeit hat in gleicher oder ähnlicher Form noch keiner Prüfungsbehörde vorgelegen.

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Sandra Transchel

CURRICULUM VITAE

SANDRA TRANSCHEL
University of Mannheim
Business School
Department of Business Administration and Logistics
Schloss
Email: Sandra.transchel@bwl.uni-mannheim.de

EDUCATION AND ACADEMICE EXPERIENCE

04/2004 – 10/2008 University of Mannheim, Mannheim, Germany

- Ph. D. Candidate and Research Assistant at the Department of Logistics (Prof. Dr. Stefan Minner)

10/2006 – 02/2007 Dartmouth College, Hanover, NH, USA

- Visiting Researcher at the Tuck School of Business, Department of Operations Management & Management Science (Prof. David F. Pyke)

10/1998 – 03/2004 University of Magdeburg, Magdeburg, Germany

- Studies in Business Mathematics (Diploma)