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package-utilities**

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Equilibria with indivisible goods and *package*-utilities

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Abstract

We revisit the issue of existence of equilibrium in economies with indivisible goods and money, in which agents may trade *many units* of items. In [5] it was shown that the existence issue is related to discrete convexity. Classes of discrete convexity are characterized by the unimodularity of the allowable directions of one-dimensional demand sets.

The class of graphical unimodular system can be put in relation with a nicely interpretable economic property of utility functions, the Gross Substitutability property. The question is still open as to what could be the possible, challenging economic interpretations and relevant examples of demand structures that correspond to other classes of discrete convexity. We consider here an economy populated with agents having a taste for complementarity; their utilities are generated by compounds of specific items grouped in "*packages*". Simple package-utilities translate in a straightforward fashion the fact that the items forming a package are complements. General package-utilities are obtained as the convolution (or aggregation) of simple package-utilities. We prove that if the collection of packages of items, that generates the utilities of agents in the economy, is unimodular then there exists a competitive equilibrium. Since any unimodular set of vectors can be implemented as a collection of 0-1 vectors ([3]), we get examples of demands for each class of discrete convexity.

Keywords: unimodular sets, laminar families, interval collections, indivisible goods, complementarity.

1 Introduction

Economies with indivisible goods may fail to exhibit competitive equilibria. However, the situation is not hopeless and there are cases in which they do. We consider here an economy in which there is only one perfectly divisible good (numeraire or money) and all other goods are indivisible. The most general existence result is due to Danilov et al. [5]. It states that existence of a competitive equilibrium revolves around the issue of whether certain sets namely those formed from all possible demands (and supplies) of agents are discretely convex. Danilov and Koshevoy [4] characterized the classes of discrete convexity. It was shown that unimodularity plays an important role in such a characterization.

A special class of discrete convexity, the class of polymatroidal sets, attracted a great deal of attention in economics for two main reasons. The first reason is that in all known models investigating the issue of equilibria in economies with indivisibilities in the literature, demands turn out to be sets of the class of discrete convexity associated with integral generalized polymatroids (see [5]). The second reason, related to the first, is that utility functions that generate such kinds of polymatroidal demand sets have an immediate interpretability in terms of *gross substitutability* a well known economic property. The following classics of the literature on the topic (i.e. Kelso and Crawford [12], for details see Gul and Stachetti [11], Danilov et al [7], and Murota [13]) document various aspects of how this property comes into play in varied set-ups.

We propose here to the reader to understand and interpret unimodularity in economic terms, namely in terms of *complementarity*. Suppose a consumer is interested only in a specific compound A (we shall use the word *package*) of items, where A is a subset of the set of available indivisible goods in the economy. More precisely, suppose his utility function has the form $U(x) = v$ if $x \geq 1_A$ and $= 0$ otherwise for some $v \geq 0$ (here 1_A represents the characteristic vector of the subset A of the item set). By construction, this utility function expresses that the items in the package A are complementary. The demand of a consumer with such preferences is very simple: it reduces to either A or nothing. We call such utility functions *elementary A-package utility* functions.

To generate richer and more interesting demand patterns we can revert to the aggregation of elementary package functions. Suppose, that we have a collection, \mathcal{T} , of packages formed from the goods available in the economy. Suppose that the consumer is interested only in packages belonging to that collection \mathcal{F} . In this case, we say that the consumer has a \mathcal{T} -package utility function.

Theorem 1 gives a necessary (and almost sufficient) existence condition for a pure exchange economy in which all agents have \mathcal{T} -package utilities. Namely, it states that a competitive equilibrium exists (at any initial endowment) in such an economy if the collection \mathcal{T} is *unimodular*. The collection \mathcal{T} is unimodular when its associated incidence matrix is totally unimodular, that is as soon as anyone of its minors takes any of the values 0, 1 or -1 .

Let us note that any unimodular system can be implemented as a collection of packages (see [3]). Thus, for any class of discrete convexity, we get examples of utility functions with demands from that class.

We consider further two classes of unimodular collections. The first class is that formed by collections of intervals with respect to some ordering of the set of indivisible goods (Proposition 2). The second is that formed by collections obtained as the union of two laminar families (Proposition 4). A family of packages is called laminar if any two members of this family, whose intersection is not the empty set, are such that either the first is a subset of the second or vice versa. The corresponding results about equilibria are formulated in Theorems 2 and 3.

We give generalizations of Theorems 2 and 3. The generalization of Theorem 2 (Theorem 2') accounts for more general forms of demand sets, whereas that of Theorem 3 (Theorem 3') rests on the fact (see Proposition 6) that any laminar-package utility function has the complementarity property.

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2 A pure exchange model

We consider a pure exchange economy¹ with one (perfectly) divisible good (numéraire or money) and with many indivisible items. Let I represent this finite set of indivisible goods, each of which can be consumed in any integer amount. Consider the following set \mathbb{Z}^I of formal linear combinations $\sum_{i \in I} x_i [i]$, where x_i are integers. A bundle x consisting of various integer amounts of items in I can be adequately represented by an element of \mathbb{Z}^I . The bundle consisting of one unit of item i is denoted $[i]$. Finally, \mathbb{Z}_+^I denotes the set of bundles containing non-negative x_i .

¹We confine ourselves to a pure exchange model for reasons of simplicity; production can be adapted in a fully straightforward manner.

The utility functions of the consumers (buyers) in this pure exchange economy are given by monotone functions on “the positive orthant” \mathbb{Z}_+^I . This implies, in particular, that we focus on preferences that are quasi-linear with respect to money. Any linear (and monotone) function $p : \mathbb{Z}^I \rightarrow \mathbb{R}$ can be considered as a price. Alternatively a price is defined as a function $p : I \rightarrow \mathbb{R}_+$, where $p(i)$ denotes the price of a unit of good $[i]$.

Given a price p , the demand of a buyer, whose utility function is $u : \mathbb{Z}_+^I \rightarrow \mathbb{R}$, is a bundle of items $x \in \mathbb{Z}_+^I$ from the set defined as follows

$$D(u, p) = \text{Argmax}(u - p).$$

Suppose we have a set B of buyers (each endowed with a utility function u_b) and a total initial endowment X^2 . A price p is a (competitive) *equilibrium price* of this pure exchange model if

$$X \in \sum_b D(u_b, p),$$

that is if there exist bundles $x_b \in D(u_b, p)$ for all buyers $b \in B$ such that $\sum_b x_b = X$.

It is convenient to aggregate the utilities of all buyers by performing a convolution. Let f and g be two functions given on \mathbb{Z}_+^I , then the (supremal) convolution of these functions, denoted $f * g$, is a function on \mathbb{Z}_+^I , defined as

$$(f * g)(x) = \max(f(y) + g(z), y, z \in \mathbb{Z}_+^I, y + z = x).$$

The convolution $*_b u_b$ of an arbitrary finite family of functions is obtained by associativity. The following equality between the demand associated to the aggregate utility and the individual demands holds true

$$D(*_b u_b, p) = \sum_b D(u_b, p).$$

We shall now formulate the general condition for the existence of competitive equilibria in terms of this aggregate utility function $U = *_b u_b$.

Definition. A function f on \mathbb{Z}_+^I is *pseudo-concave* if it has a super-differential at any point $X \in \mathbb{Z}_+^I$.

Recall that a linear function p on \mathbb{Z}^I is a *super-differential* of a function f at a point X if

$$p - p(X) \geq f - f(X).$$

²We may reason in terms of an aggregate endowment, since we assume that preferences are quasi-linear.

One can show that a function f defined on \mathbb{Z}_+^I is pseudo-concave if and only if it is the restriction on \mathbb{Z}_+^I of some concave function given on \mathbb{R}_+^I .

Lemma 1³. *The following statements are equivalent:*

- 1) *the aggregate utility function $U = *_{b}u_b$ is pseudo-concave,*
- 2) *there exist competitive equilibria for any initial endowment X .*

It is immediate to see that 1 implies 2: let p be a super-differential of U at the point X . Then $X \in D(U, p) = \sum_b D(u_b, p)$. Therefore $X = \sum x_s$, where $x_b \in D(u_b, p)$ and p is an equilibrium price. The converse is even easier to prove. \square

In this context, we see that the argument invoked to warrant the existence of equilibrium in the set-up of indivisible goods economies is very similar to the argument one would invoke in the case of economies with divisible goods and concave utilities. One needs only to substitute the word "pseudo-concave" by the word "concave". In the divisible context, as soon as the individual utilities functions of all the buyers are concave (they are in \mathbb{R}_+^I), then so is the aggregate utility. Existence then follows. However in the indivisible case, the convolution of pseudo-concave functions needs not to be pseudo-concave. This is, indeed, the main cause for non-existence in the context of economies with indivisible goods and transferable utilities.

In their study of existence in production economies with indivisible goods and money, [5] considered a particular sub-class of pseudo-concave functions (*IGP-concave functions*) that was closed with respect to convolution. It turned out that *IGP-concave functions* were meaningful in terms of preferences in that they allowed for substitutability between the goods, and what more, precisely in the sense proposed by Kelso and Crawford [12]. The properties of this class of utility functions have now been exhaustively described, see Danilov, Koshevoy and Lang [6], Gul and Stacchetti [11] or else Murota and Tamura [14]. We proceed to investigate sub-classes of pseudo-concave functions for other classes of discrete convexity.

3 Package functions and unimodularity property

A *package* is a non-empty subset of the set I and we shall identify it with the bundle $1_A = [A] = \sum_{i \in A} [i]$.

³Actually, this is an alternative formulation of the criterion proposed by Bikhchandani and Mamer [1].

Definition. Let $A \subset I$ be a package. An *elementary A -package function* is a function $u : \mathbb{Z}_+^I \rightarrow \mathbb{R}$ taking the following form

$$u(x) = v \min(1, x_i, i \in A) = \begin{cases} v, & \text{if } x \geq [A] \\ 0, & \text{otherwise,} \end{cases}$$

where v is a non-negative real number ('a reservation value').

A consumer endowed with an A -package utility views the items from A as strict complements. As a consequence the consumer derives a utility amount equal to v out of the consumption of the unitary bundle $[A]$ (or, for the matter, from any bundle with larger amounts of each item than in $[A]$). And consequently, this consumer does not derive any satisfaction from a bundle in which some item i from A would be missing.

The demand of this consumer is easy to figure out. The consumer demands package $[A]$ as soon as its cost $p(A) = \sum_{i \in A} p(i)$ is smaller than v , and demands no package $\{0\}$ when $p(A) > v$. In the boundary case $p(A) = v$, the consumer's demand is the set $D(u, p) = \{0, [A]\}$. Note that the consumer might be inclined to demand any amount of an item i , when the latter's price is 0.

We now move on to utility functions obtained as convolutions of elementary package functions. Let \mathcal{T} be a collection of packages.

Definition. The function $f : \mathbb{Z}_+^I \rightarrow \mathbb{R}$ is a *\mathcal{T} -package function* (or *is adapted to \mathcal{T}*) if f is the convolution of a family of elementary A -package functions with $A \in \mathcal{T}$.

For example, the function $f(x) = \varphi(\min(x_i, i \in I))$, where $\varphi : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ is a pseudo-concave function of one variable and $\varphi(0) = 0$, is compatible with the singleton family $\{I\}$. Indeed, it is the convolution of the following family $(v_n \min(1, x_i), n \in \mathbb{Z}_+)$ of I -package functions, where $v_n = \varphi(n+1) - \varphi(n)$. Conversely, an elementary I -package function has the form $\varphi(\min(x_i))$, where $\varphi(t) = v \min(1, t)$ for $t \geq 0$.

It is clear that if every buyer b , in an economy, is equipped with a utility function u_b adapted to \mathcal{T} , then the aggregate utility function $U = *_b u_b$ is adapted to \mathcal{T} as well.

Elementary package functions are pseudo-concave. However, taking any arbitrary collection of packages \mathcal{T} , and computing the associated \mathcal{T} -package function, we often enough end-up with a function that is not pseudo-concave. Hence, in general, a pure exchange economy with \mathcal{T} -package preferences will fail to exhibit equilibria. Here is a simple, but instructive example.

Example. Consider a pure exchange economy with three consumers a, b, c . Let I consist of three items, 1, 2 and 3. Now consider the following collection $\mathcal{T} := \{(1, 2), (1, 3), (2, 3)\}$ of elementary packages. Assume that the consumers are endowed with the three elementary package utility functions: $u_a = 2 \min(1, x_1, x_2)$, $u_b = 2 \min(1, x_1, x_3)$ and $u_c = 2 \min(1, x_2, x_3)$. Suppose that the initial endowment consists in a unique exemplar of each item, $[1] + [2] + [3]$. This economy has no competitive equilibria.

Here is the reason. By symmetry arguments, we may without loss of generality assume that $p(1) = p(2) = p(3) = p$. Let us now analyze the behavior of the aggregate demand in terms of p . If $p < 1$, then every buyer requests his/her elementary package; the aggregate demand consists in two units of each item and this is larger than the initial endowment. If $p > 1$, each individual's demand is equal to 0, and this will not yield an equilibrium either. Thus the only possible candidate to an equilibrium price is $p = 1$. At this price vector, each buyer is indifferent between buying his package or buying nothing. Computing the aggregate demand for all possible configurations, we easily notice that it never contains the initial endowment. Indeed, the demand of each buyer is limited to an even number of items: 2 or 0. Thus the aggregate demand will also consist of an even number of items and on the other hand the initial endowment encompasses an odd number of items. Thus there is no price for which the aggregate demand matches the aggregate endowment.

We now provide a criterion to assess the pseudo-concavity of a \mathcal{T} -package function. This criterion rests upon unimodularity, as follows from [4]. We associate to any family \mathcal{T} the incidence matrix $M(\mathcal{T}) = (m_{i,A})$, rows correspond to elements of I , whereas columns correspond to sets from \mathcal{T} , defined as follows. For $i \in I$ and $A \in \mathcal{T}$, $m_{i,A}$ is equal to 1, if $i \in A$, and is equal to 0 otherwise. For instance, in the Example above M looks like this

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Definition. A family \mathcal{T} is said to be *unimodular* if the matrix $M(\mathcal{T})$ is totally unimodular, that is if its minors take values equal to 0 or ± 1 .

Note, that the family of packages considered in the preceding Example is not unimodular, because $\det(M) = -2$. Interested readers can revert to ([16], Chapter 19) to read about totally unimodular matrices.

Proposition 1. *Let \mathcal{T} be a collection of packages. The following two assertions are equivalent:*

- 1) the collection \mathcal{T} is unimodular;
- 2) \mathcal{T} -package functions are pseudo-concave.

The proof is given in the Appendix.

Taking Lemma 1 and Proposition 1 together, we obtain the following theorem.

Theorem 1. *Let \mathcal{T} be a unimodular collection of packages. Suppose that buyers have utilities, that are compatible with \mathcal{T} . Then the economy has competitive equilibria at any initial endowment.*

In [3], it was shown that any unimodular system can be implemented as a collection of Boolean vectors.

Below we present two examples of unimodular collections. These are, namely, the collection formed by *intervals of goods* and that formed by the *union of two laminar families* on the set of goods. The first one provides an implementation of the graphical system with a complete graph. Some cographical systems can be considered in the framework of the second example. For remark that the cographic systems $A(G_1)$, $A(G_3)$ and $A(G_4)$ (see section 9 in [2]) take the form of the union of two laminar families.

4 Interval packages

4.1 Intervals

Suppose now that the set of goods I is endowed with a linear order. We can then identify I with the set $\{1, \dots, n\}$ and thus write \mathbb{Z}_+^n instead of \mathbb{Z}_+^I . An *interval* in I is a subset of the form $\{i, i+1, \dots, j\}$, where $1 \leq i \leq j \leq n$. Let \mathcal{I} be the set of all intervals in I . A consumer with \mathcal{I} -package preferences, will value intervals rather than individual items.

Suppose that the indivisible goods are fields, or land areas. Then one could very well conceive situations in which a buyer might precisely value the fact that his fields are adjacent one to another. Therefore such a buyer would not only value the total area of land he possesses, but also the fact that his fields are connected one to another (see also examples in [15]).

The following Proposition is well known, but we shall give a proof in the Appendix.

Proposition 2. *The collection \mathcal{I} of intervals is unimodular.*

Equipped with this first and economically meaningful example of a unimodular collection, we can now state the following theorem.

Theorem 2. *A pure exchange economy with indivisible goods has competitive equilibria if its buyers are endowed with \mathcal{I} -package utilities.*

4.2 Interval-concave functions

We present here a generalization of the Theorem 2, based upon the requiring that utility function satisfy certain properties, and not upon the requiring that they take a specific functional form. This enlarges the class of interesting functions with respect to the previously defined " \mathcal{I} -package" functions. We introduce a few new notions in order to define this class.

Let f be a pseudo-concave function defined on \mathbb{Z}_+^I and p be a linear functional. The convex hull $\text{co}D(f, p)$ of the set $D(f, p) = \text{Argmax}(f - p)$ is called an *affinity domain* of f . The affinity domains form a polyhedral decomposition⁴ of the positive orthant \mathbb{R}_+^I as p varies; this decomposition is called the *parquet* of f and the polyhedra $\text{co}D(f, p)$ are called *cells* of the parquet.

Definition. A function f on $\mathbb{Z}_+^I \rightarrow \mathbb{R}$ is called *\mathcal{I} -concave* (interval-concave) if it is pseudo-concave and if every one-dimensional cell of its parquet is parallel to some vector 1_A , where A is an interval in I ⁵.

Remark 1: It is possible to give another (equivalent) definition of \mathcal{I} -concavity called *Interval Package Improvement*. It is inspired by the *Single Improvement* property due to Gul and Stacchetti [11].

A function f is \mathcal{I} -concave if and only if it satisfies the property of *Interval Package Improvement*, that is if at any point $x \in \mathbb{Z}_+^I$, that is not a maximum for the function $f - p$, there exists an interval A in I such that $\max[(f - p)(x \pm 1_A)] > (f - p)(x)$.

The class of \mathcal{I} -concave functions has two important properties.

⁴A polyhedral decomposition is a collection of polyhedra, such that anyone of their faces belongs to the collection as well and, such that any pair of polyhedra of the collection has a common separating face if the polyhedra intersect.

⁵Interestingly, the class of \mathcal{I} -concave functions is reverted into the class of polymatroidal functions provided we make an appropriate unimodular transformation of variables. The system \mathcal{I} is isomorphic to the system \mathbb{A}_n discussed among others in [4].

Proposition 3. *The convolution of \mathcal{I} -concave functions yields an \mathcal{I} -concave function.*

This derives from the Theorem 2 proved in [4] (see also Theorem 4 in [5]).

Proposition 4. *An \mathcal{I} -package functions is \mathcal{I} -concave.*

Proof. It suffices (according to Proposition 3) to check that an elementary A -package function is \mathcal{I} -concave for any arbitrary interval A . To see this revert to the description of the demand sets generated by an elementary A -package function that appears at the beginning of section 3. \square

The following generalization of Theorem 2 is obtained as a corollary to Proposition 3.

Theorem 2'. *A pure exchange economy with indivisible goods for which all buyers have \mathcal{I} -concave utilities has competitive equilibria.*

Remark 2. The following three conditions⁶: *Complementarity between adjacent items (SMa), Neutrality between non-adjacent items (SMb) and Decreasing marginal utility (DMU)* define a large subclass of \mathcal{I} -concave functions that differs from the class of \mathcal{I} -package functions.

- *SMa.* For any i and j , such that $|i - j| = 1$,

$$f(x) + f(x + [i] + [j]) \geq f(x + [i]) + f(x + [j]).$$

This requirement formalizes the idea that adjacent items are complements. Indeed rephrasing the requirement, *SMa*, as follows

$$f(x + [i] + [j]) - f(x + [j]) \geq f(x + [i]) - f(x)$$

for i and $j = i \pm 1$, adjacent items, we see that *SMa* expresses that the “marginal utility” of item i increases when we add to x one unit of the adjacent item j ,

$$\Delta_i f(x) \leq \Delta_i f(x + [j]).$$

where $\Delta_i f$ represents the difference operator: $\Delta_i f(x) = f(x + [i]) - f(x)$ and is the discrete analogue of the partial derivative (at the point x) in the direction of i .

⁶Compare with the definition of *CC*-functions in Section 5.2.

- *SMb*. For any i and j , such that $|i - j| > 1$,

$$f(x) + f(x + [i] + [j]) = f(x + [i]) + f(x + [j]).$$

SMb, also conveniently rewritten $\Delta_i f(x) = \Delta_i f(x + [j])$, requires that the marginal utility of item i does not change when we add to x one (or many) unit of non-adjacent items ($|i - j| > 1$). We can summarize this requirement by saying: non-adjacent items are mutually neutral.

- *DMU*. For any $i \in I$

$$f(x) + f(x + [i] + [I]) \leq f(x + [i]) + f(x + [I]).$$

Recombined in this form $\Delta_i f(x) \geq \Delta_i f(x + [I])$, the property *DMU* states that the marginal utility of any item i decreases as soon as the “full package” $[I]$ is added to the initial bundle x . Now it also requires that the marginal utility $\Delta_I f$ with respect to the complete package I at a bundle x decreases with an increase of this bundle x . For indeed it means that $\Delta_I f(x) \geq \Delta_I f(x')$, if $x \leq x'$, which amounts to requiring the “concavity” of f .

Functions taking the form $\sum_{i=1}^{n-1} \phi_i(x_i - x_{i+1}) + \sum_{i=1}^n \psi_i(x_i)$, where ϕ_i and ψ_i are concave functions of one variable, belong to this subclass.

5 Laminar families and complementary-concave functions

5.1 Laminar families

Definition. A family \mathcal{L} of packages is *laminar* (or is a hierarchy or a tree family) if, for any $A, B \in \mathcal{L}$, we have either $A \cap B = \emptyset$, or $A \subset B$ or $B \subset A$.

The notion of laminar family is discussed in [15] (where it actually appears under the name of nested system). It is well known that any laminar family is unimodular, however we can find the following stronger assertion in [8].

Proposition 5. *The union of two laminar families is a unimodular family.*

We give a proof of this in the Appendix. Let us stress however that the union of three laminar families can fail to be unimodular; this was shown in the Example from Section 3.

As a consequence we obtain.

Theorem 3. *Suppose that the utilities of the buyers of some economy are adapted to a package family \mathcal{T} , obtained as the union of two laminar families. Then the economy has competitive equilibria.*

5.2 Complementary-concave functions

Suppose that the buyers in the economy under study can be divided in two sub-groups. The buyers belonging to the first sub-group have utility functions that are compatible with \mathcal{L}_1 a laminar family, and those of the second sub-group are compatible with \mathcal{L}_2 a second laminar family. This economy has equilibria by Theorem 3.

An interesting feature of functions compatible with a single laminar family is that they imply demand complementarity in a similar fashion as elementary package function do. We present below a class of functions \mathcal{C} that entails demand complementarity, contains all laminar functions and is such that the convolution of any two functions from \mathcal{C} is a pseudo-concave function. The existence of this class \mathcal{C} , enables us to provide a more general context in which an economy populated with agents that can be divided into two distinct groups where in each group agents are ‘similar’ with respect to their intra group preferences exhibits equilibria.

Definition. A function $f : \mathbb{Z}_+^I \rightarrow \mathbb{R}$ is said to be *complementary concave* (or a *CC-function*) if it satisfies the following two requirements (SM) and (DMU):

SM. For any i and j , such that $i \neq j$,

$$f(x) + f(x + [i] + [j]) \geq f(x + [i]) + f(x + [j]).$$

DMU. For any $i \in I$

$$f(x) + f(x + [i] + [I]) \leq f(x + [i]) + f(x + [I]).$$

Note that the notion of *CC-function*, we propose here, is equivalent to that of L^\sharp -function introduced by [9], see also [13].

The notion of *CC-function* embodies the idea of complementarity between items, see the comments to properties *SMa* and *DMU* in Section 4. Note, moreover, that the *SM* and *DMU* requirements have important and interesting implications on the behavior of the agents’ demand sets $D(f, p)$

as functions of prices. For instance, assume that the price of the i -eth item increases (while the prices of all other items remain unchanged), then the quantities demanded of anyone of the items will decrease (albeit not necessarily strictly). We neither prove this implication here, nor use it, but it helps acquire a better understanding of what CC -functions and their corresponding demand sets are.

A CC -function is always pseudo-concave and more the convolution of any *two* CC -functions is pseudo-concave, see [7].

Theorem 3'. *Suppose we have two groups of buyers such that the aggregate utility function of each group is a CC -function. Then there exist equilibria.*

The convolution of CC -functions does not yield a CC -function, in general. In the sequel, we provide examples of CC -functions. We also importantly discuss the conditions under which the convolution of two CC -functions yields a CC -function.

5.3 Properties of CC -functions

The present subsection is devoted to improving our understanding of CC -functions. In the course of action, we shall also introduce various construction principles for CC -functions.

Let us start with generalities about the structure of the set of CC -functions. The set of CC -functions is a convex cone; the finite sum of CC -functions is a CC -function and the product of a CC -function by a non-negative real is a CC -function. Moreover any affine function is a CC -function.

Let J be a subset of I and let f be a CC -function of the variables in J , then the same function f viewed with respect to variables in I is a CC -function.

Functions of a single variable. Let $f : \mathbb{Z} \rightarrow \mathbb{R}$ be a function of a single variable ($|I| = 1$). Then f is a CC -function if and only if f satisfies *DMU*, that is if and only if f exhibits "decreasing increments". This is equivalent to saying that f is the restriction to \mathbb{Z} of some concave function $\varphi : \mathbb{R} \mapsto \mathbb{R}$.

Thanks to this simple remark, we already dispose of a rather wide class of separable CC -functions. Let φ_i be a concave function of a single variable, for $i \in I$, then

$$f(x) = \sum_{i \in I} \varphi_i(x_i)$$

is a CC -function.

Note, however, that those items for which the utility function exhibits separability should be viewed as *independent* one of another, rather than *complementary* one to another. Therefore separable functions are certainly not the most exciting examples of CC -functions. Nevertheless, they form a neutral class with respect to the convolution operator, in the sense that if one takes any CC -function and convolutes it with a separable CC -function, it will result in a CC -function.

Convolutions of CC -functions. In principle, it is difficult to expect that the convolution of CC -functions will yield a CC -function. We saw this happen in the Example of section 3. As we aggregated our agents, we noticed that the preferences of the resulting "collective" agent started to exhibit something akin to "substitutability".

Assume, for example, that the first consumer was interested in the package $\{1, 2\}$, whereas the second consumer was interested in the package $\{2, 3\}$. Assume moreover that each buyer viewed the items, he was interested in, as complementary. As soon as we aggregate these two agents, we notice that the resulting aggregate agent views 1 and 2 as "substitutes". And, in fact, if the latter has already put hands on one unit of item 3, it now appears that he is pretty indifferent between completing his draw with a unit of 1 or a unit of 2. The aggregation process produces a subtle and complex combination of complementary and substitution between items, which is typically responsible of the non-existence problem.

Nevertheless, the CC property is conserved under convolution in a few specific instances. We present now the few cases (at least to our knowledge) for which this happens.

The first simple instance for which the CC property is conserved is in some sense when it is inactive. That is if f and g are CC -functions and if f and g depend on two subsets *non intersecting* of items, then the convoluted function $f * g$ is a CC -function. In this case the convolution operator just simply reduces to the summation operator.

The second simple instance of CC conservation is in the convolution with an elementary A -package function (where A is either single-element set or is equal to I).

Proposition 6. *The convolution of any CC -function with an elementary A -package function is a CC -function if either A is a singleton set or $A = I$.*

Proposition 4 provides a procedure by which one can generate new CC -

functions.

Corollary. *Let A_1, \dots, A_k be disjoint subsets of I , and let f_i be CC -functions of variables in A_i . Suppose that f_I is an elementary I -package function. Then the convolution $f_I * f_1 * \dots * f_k$ is a CC -function.*

Proposition 7. *Any function f adapted to a laminar family \mathcal{L} is a CC -function.*

Proof. Let \mathcal{L} be a laminar family and assume that A_1, \dots, A_k, \dots are maximal elements of $\mathcal{L} \setminus \{I\}$. Maximality implies that the sets A_k are disjoint. Thus any arbitrary package A from \mathcal{L} (and different from I) is contained exactly in one of the A_k , for the A_k are maximal by construction. Therefore, we can write f as follows

$$f = f_I * (*_k f_k),$$

where $f_k = *_{A \in \mathcal{L}, A \subset A_k} f_A$ and f_A is a convolution of elementary A -package functions. This decomposition is performed so as to ensure that each function f_k depends now only on the variables in A_k . The reasoning is by induction and yields that each function f_k is a CC -function. Then applying the previous Corollary, we conclude that f is a CC -function. \square

Remark 3. Given that our main focus is on the "indivisibility" of the goods, we have formulated the definition of a CC -function directly in terms of indivisible quantities (that is elements of the integer space \mathbb{Z}_+^I). Nevertheless we can devise perfectly divisible examples of CC -functions, i.e. functions f of continuous variables x_i , for $i \in I$ (that is functions on the positive orthant \mathbb{R}_+^I).

Under the assumption that the function is of class C^2 , the condition SM takes the following form $\partial^2 f / \partial x_i \partial x_j \geq 0$ for all $i \neq j$, whereas the condition IM takes the form $(\partial / \partial x_1 + \dots + \partial / \partial x_n) \partial f / \partial x_i \leq 0$ for all $i \in I = \{1, \dots, n\}$. The restriction of a 'continuous' CC -function of that form to the integer orthant \mathbb{Z}_+^I yields a 'discrete' CC -function (this was observed in [13]).

For instance, let us consider the following quadratic function:

$$f(x_1, \dots, x_n) = \sum_{ij} a_{ij} x_i x_j,$$

where (a_{ij}) is a symmetric matrix. The f is a CC -function if and only if the following two conditions are met: a) $a_{ij} \geq 0$ for $i \neq j$, and b) $\sum_h a_{hi} \leq 0$ for all $i = 1, \dots, n$.

Appendix

Proof of Proposition 1. We saw previously that the demand set $D(u_A, p)$, associated to an elementary A -package function u_A , is obtained as the (Minkowski) sum of several “segments” - taking either forms $\{0, [A]\}$ or $\{0, [i]\}$, $i \in I$. A function f , compatible with a family \mathcal{T} , yields for its part, demand sets $D(f, p)$, that are obtained as the sum of the demand sets corresponding to elementary package functions. Therefore they may also be decomposed as a sum of “segments” of the form $\{0, [A]\}$, $A \in \mathcal{T}$, or of the form $\{0, [i]\}$, $i \in I$. These sums of segments, if viewed as sets, are pseudo-convex when the family \mathcal{T} is unimodular. The proof of this can be found in Theorem 2 of [4]. A set $D \subset \mathbb{Z}^I$ is pseudo-convex if

$$D = \text{co}(D) \cap (\mathbb{R}^I).$$

It is easy to see thereafter that the pseudo-concavity of all demand sets $D(f, p)$ is equivalent to the pseudo-concavity of the function f . The converse to this assertion is proven in [4] as well. \square

Proof of Proposition 2. Let $I = \{1, \dots, n\}$, and let \mathcal{I} be the family of interval subsets in I . We have to check that any minor of the matrix $M(\mathcal{I})$ takes the values 0 or ± 1 . If A is an interval in I and, if $J \subset I$, then the set $A \cap J$ is an interval in J (endowed with the induced order). We can assume that the size of these minors is $n \times n$.

Thus, let M be an $n \times n$ -matrix, whose columns are formed by the characteristic vectors 1_{A_i} of some intervals A_1, \dots, A_n in I . We have to prove that the determinant of M is 0 or ± 1 .

Suppose first, that $[1]$ does not belong to any of the A_i , $i = 1, \dots, n$. In this case, M 's first row consists only of 0 and then its determinant is 0.

Thus let us assume that $[1]$ belongs to some A_i , say $[1] \in A_1$. Suppose on top that A_1 has the minimal possible size. If some A_i also contains $[1]$, then by minimality of A_1 , we have that $A_1 \subset A_i$. Substitute column A_i by $A_i \setminus A_1$. The determinant remains unchanged, indeed, for we subtracted the first column from the i -th. The set $A_i \setminus A_1$ is also an interval as well. By doing the same for all intervals containing $[1]$, we end up with a new “interval” matrix with the same determinant as the initial one and such that $[1]$ belongs to A_1 only.

Deleting the first row and the first column of that matrix, we obtain a new “interval” matrix, that has the same determinant but is of size $n - 1 \times n - 1$. By induction, its determinant is either 0 or ± 1 and Proposition 2 is proven.

Proof of Proposition 5. The three following simple points below will be useful.

1. A sub-family of a laminar family is laminar.
2. Let \mathcal{L} be a laminar family on a set X and $Y \subset X$. If we replace all elements $A \in \mathcal{L}$ by $A \cap Y$, we obtain a laminar family on the set Y .
3. Let \mathcal{L} be a laminar family and let C be a minimal element in \mathcal{L} . Consider the new family \mathcal{L}' consisting of C and all sets of the form $A \setminus C$, $A \in \mathcal{L}$. Then \mathcal{L}' is laminar as well.

Let us now provide the proof of Proposition 5. We have two laminar families \mathcal{L} and \mathcal{R} on a set X , and $\mathcal{T} = \mathcal{L} \cup \mathcal{R}$. We form the matrix $M(\mathcal{T})$. Recall that the rows of this matrix correspond to elements of X and that the columns have the form 1_A , $A \in \mathcal{T}$. (For convenience let us place on the left those columns corresponding to 1_A , where $A \in \mathcal{L}$ and place on the right, those columns corresponding to 1_A , where $A \in \mathcal{R}$.) We have to check that the minors of this matrix take either of the two values 0 or ± 1 . Recalling points 1 and 2 above, we can assume that the matrix $M(\mathcal{T})$ is square and check that its determinant equals to 0 or ± 1 .

Consider the following characteristic number $\rho(\mathcal{L})$ of a family \mathcal{L}

$$\rho(\mathcal{L}) = \sum_{A \in \mathcal{L}} |A| - \left| \bigcup_{A \in \mathcal{L}} A \right|.$$

This number is a non-negative integer, that equals 0 if and only if the family \mathcal{L} consists of non-intersecting sets.

To begin with, let us assume that $\rho(\mathcal{L})$ is strictly positive. This implies that some elements of \mathcal{L} meet and, by laminarity, that one of those elements is strictly contained in another. Let C be a minimal element of \mathcal{L} , that is strictly contained in another element of \mathcal{L} . Let us form the family \mathcal{L}' as in point 3, then

- a. The determinant does not change because by doing this we indeed subtract the column 1_C to some (left) columns of our matrix.
- b. The family \mathcal{L}' is laminar as well.

- c. The characteristic number $\rho(\mathcal{L}')$ for the new family \mathcal{L}' is strictly inferior to that associated to the initial family, $\rho(\mathcal{L})$.

Reasoning in an analogous fashion for the right side of the matrix, we can assume for the sake of the proof, that the laminar families \mathcal{L} and \mathcal{R} are non-intersecting.

Consider now an arbitrary row of the matrix $M(\mathcal{T})$. Three cases are possible.

First case. The row consists of zeros only. (That is the corresponding element x belongs to no subset from \mathcal{T} .) In this case the determinant of the matrix equals to 0.

Second case. The row consists of a unique 1. (That is the corresponding element x belongs to a unique set A from \mathcal{T} .) If we delete this row and the corresponding column 1_A from the matrix, we notice that the determinant is equal (up to a sign) to the determinant of the matrix for the family $\mathcal{T}' = \mathcal{T} \setminus A$ on the set $X \setminus x$. Point 1, stated above, implies that laminarity is preserved; therefore reasoning by induction, the determinant of $M(\mathcal{T})$ takes one of the required values.

Third case. Every row meets both the left and the right side of the matrix. That is every element $x \in X$ belongs both to some set from \mathcal{L} and to some set from \mathcal{R} . Since the two families \mathcal{L} and \mathcal{R} do not intersect by assumption, it must be that both \mathcal{L} and \mathcal{R} are partitions of the set X . This implies in particular that the sum of all left columns equals to 1_X , as well as the sum of all right columns. But this means that the rows of the matrix $M(\mathcal{T})$ are linearly dependent and thus that its determinant is equal to 0. And this proves Proposition 5.

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