SONDERFORSCHUNGSBEREICH 504
Rationalitätskonzepte,
Entscheidungsverhalten und
ökonomische Modellierung

No. 05-34

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October 2005

Financial support from the Deutsche Forschungsgemeinschaft, SFB 504, at the University of Mannheim, is gratefully acknowledged. This paper was written while Hans Haller was visiting the SFB 504, University of Mannheim.

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August 10, 2005

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Abstract
Interaction on hypergraphs generalizes interaction on graphs, also known as pairwise local interaction. For games played on a hypergraph which are supermodular potential games, logit-perturbed best-response dynamics are studied. We find that the associated stochastically stable states form a sublattice of the lattice of Nash equilibria and derive comparative statics results for the smallest and the largest stochastically stable state. In the special case of networking games, we obtain comparative statics results with respect to investment costs, for Nash equilibria of supermodular games as well as for Nash equilibria of submodular games.

Key Words: Network Games, Potential Games, Submodular Games, Supermodular Games

JEL Classification: C72, D85

*Section 7 reports in condensed form on the original results of Durieu, Haller, and Solal (2004) on nonspecific networking, presented at SING I. The analysis is extended to the more general framework of interaction on hypergraphs. Comments from SING I participants are appreciated. Support by the French Ministry for Youth, Education and Research, through project SCSHS-2004-04, and the German Science Foundation (DFG), through SFB 504, is gratefully acknowledged.

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1 Introduction

A hypergraph consists of a set of nodes and a collection of non-singleton groups of nodes. In our formal model of interaction on a hypergraph or of a game played on a hypergraph, the players constitute the nodes of the hypergraph. A player participates in a collection of strategic games associated with the groups he belongs to and chooses an action which affects his partial payoff from each of these constituent games. A special case of a hypergraph is an undirected graph, given by a set of nodes and a collection of unordered pairs of nodes. Interaction on an undirected graph is better known as pairwise local interaction. The corresponding class of games are spatial or local interaction games. We extend earlier results by Young (1998) and Baron *et al.* (2002b) for local interaction games and show that if each constituent game is a potential game, then a game played on a hypergraph is a potential game. For supermodular games played on a hypergraph, we demonstrate that the Nash equilibria form a non-empty lattice. For supermodular potential games, we find that the set of maximizers of the potential constitute a sublattice of the joint strategy space. Moreover, for suitably parametrized families of such games with common player and strategy sets, a comparative statics result for the smallest and the largest maximizer of the potential is derived.

Finite potential games and finite supermodular games have in common that a Nash equilibrium in pure strategies exists. Ours is one of the few papers that consider both properties. Dubey, Haimanko, and Zapechelnyuk (2005) show that games of strategic substitutes or complements with aggregation are "pseudo-potential" games, a generalization of best-response potential games introduced by Voorneveld (2000). As a consequence, they obtain existence of a Nash equilibrium and convergence to Nash equilibrium of certain deterministic best-response processes. Brânzei, Mallozzi and Tijs (2003) investigate the relationship between the class of potential games and the class of supermodular games. They essentially focus on two-person zero-sum games (and a special case of Cournot duopoly).

Dubey, Haimanko, and Zapechelnyuk (2005) do not consider stochastic perturbations or "noise" and stochastic stability. In contrast, one of the original contributions of the current paper is the investigation of logit perturbed best response dynamics for supermodular games with potentials and the associated set of stochastically stable states. We follow Blume (1993, 1997), Young (1998), Baron *et al.* (2002b), and others who have shown that for finite potential games, perturbed best response dynamics with logit trembles yield the maximizers of the potential as the stochastically stable states. As a consequence, we obtain that for finite supermodular potential games, the stochastically stable states with respect to logit perturbations form a sublattice and admit specific comparative statics.

To our knowledge only two earlier papers, Kandori and Rob (1995) and Kaarboe and Tieman (1999), combine stochastic stability and supermodularity in a general setting.[1] Both papers focus on a class of global interaction games based on two-player and symmetric strict supermodular games. Players gradually adjust their behavior taking into account a summary statistic. The adjustment process is perturbed by Bernoulli or uniform trembles or slight generalizations thereof. All authors obtain monotonicity results for best responses and show that the set of stochastically stable states is contained in the set of (strict) Nash equilibria of the recurrent game. Hence supermodular games exposed to uniform trembles and potential games exposed to logit trembles both induce perturbed dynamics under which the stochastically stable states form a subset of the set of Nash equilibria of the constituent game. Unlike the present paper, that earlier literature does not examine the structure of the set of stochastically stable states and its variation in response to parametric changes.

In section 7, we focus on networking games, a particular class of local interaction games where each agent chooses an effort level or intensity of networking. In the simplest case, the agent faces a binary choice: to network or not to network. The agent does not choose a specific set of links, in contrast to previous models of strategic network formation (utilization) where each agent selects which direct links to other agents to form (utilize). Our hitherto unexplored model of "nonspecific networking" proves very promising for three reasons. First, it covers a broad spectrum of applications as exemplified in subsection 7.2. Second, the model comprises a large class of games which are both potential and supermodular games. Third, we obtain comparative statics results for Nash equilibria with respect to networking costs in two opposite subcases, networking games with strategic complements and networking games with strategic substitutes.

[1] Schipper (2003) and Alós-Ferrer and Ania (2005) exclusively deal with symmetric aggregative games that are either submodular or supermodular.

2 Preliminaries

Let $N = \{1, 2, \ldots, n\}$ be a finite set with $n \geq 2$, $\mathcal{F}(N) = \{E \in 2^N | \#E \geq 2\}$ the family of subsets of N with at least two elements and $\mathcal{F}'(N) = \{E \in 2^N | \#E = 2\}$ the family of subsets of N with exactly two elements. A pair $(N, \mathcal{E})$ with $\mathcal{E} \subseteq \mathcal{F}(N)$ is called a **hypergraph** on N. $(N, \mathcal{E})$ is an undirected graph in case $\mathcal{E} \subseteq \mathcal{F}'(N)$. An n-**player game in normal form played on a hypergraph** is a list $\mathcal{G} = (N, \mathcal{E}, (G_E)_{E \in \mathcal{E}}, (X^i)_{i \in N}, (C^i)_{i \in N})$. N is the finite set of players. $(N, \mathcal{E})$ is a hypergraph on N describing the interaction possibilities of the players. Each $G_E, E \in \mathcal{E}$, is a game in normal form with player set E. X^i is player i's action set. Player i chooses an action $x^i \in X^i$ which determines i's partial payoff in each of the games G_E in which i participates. Regardless of his interaction with other players, player i incurs a cost (or benefit) $C^i(x^i)$ which depends only on his own action x^i.

The interpretation of the interaction hypergraph is that interaction is only possible within groups of players $E \in \mathcal{E}$. All players in E are called to play a **local game** in normal form $G_E = (E, (X^i)_{i \in E}, (\pi_E^i)_{i \in E})$, where E is the set of players, X^i is the set of actions of player $i \in E$, and $\pi_E^i : \prod_{j \in E} X^j \to I\!R$ is player i's partial payoff function. For convenience, we define $X_E = \prod_{j \in E} X^j$ with generic elements x_E, $X_E^{-i} = \prod_{j \in E \setminus \{i\}} X^j$ with generic elements x_E^{-i}, $X = X_N$ with generic elements x, and $X^{-i} = X_N^{-i}$ with generic elements x^{-i}.

Let $\mathcal{E}_i = \{E \in \mathcal{E} | \, i \in E\}$ denote the family of groups player i belongs to — which determines the local games $G_E, E \in \mathcal{E}_i$, in which i participates. It is assumed that no player is isolated: $\mathcal{E}_i \neq \emptyset$ for all i or, equivalently, $\mathcal{E}$ is a covering of N. Hence each player i participates in at least one local game. Whenever the hypergraph contains only one group of players, that is $\mathcal{E} = \{N\}$, the interaction is global. Otherwise the interaction is local. In the special case $\mathcal{E} \subseteq \mathcal{F}'(N)$, the game $\mathcal{G}$ is a game played on a undirected graph as in Young (1998, chap. 6) and Baron *et al.* (2002b).

Player i's payoff function in $\mathcal{G}$, $u^i : X \to I\!R$ is defined by

$$u^i(x) = \sum_{E \in \mathcal{E}_i} \pi_E^i(x_E) - C^i(x^i), \tag{1}$$

with the interpretation that the payoff to player i depends on the joint strategy $x \in X$ and consists of i's total payoff from interacting with various groups of players minus i's cost. The equilibrium concept is Nash equilibrium. We denote by $\mathcal{N}(\mathcal{G}) \subseteq X$ the set of Nash equilibria of $\mathcal{G}$.

Formally, the model restricts a player i to choose the same action x^i across all games $G_E, E \in \mathcal{E}_i$, in which the player participates. On the one hand, this can reflect a severe restriction. For instance, suppose that $X^i = \{1, 2\}$ for all $i \in N$ and each game $G_E, E \in \mathcal{E}$, is a pure coordination game. That is there exist numbers $a_E^1, a_E^2 > 0$ such that for every $i \in E$: $\pi_E^i(x_E) = a_E^k$ if $k \in \{1, 2\}$ and $x^j = k$ for all $j \in E$, and $\pi_E^i(x_E) = 0$ otherwise. Here a player makes a single binary choice which affects his payoff in each of the local games $G_E, E \in \mathcal{E}_i$. Given a best response $x^i \in X^i$ against $x^{-i} \in X^{-i}$ in $\mathcal{G}$, x^i is not necessarily a best response against x_E^{-i} in G_E for all $E \in \mathcal{E}_i$, even if $C^i \equiv 0$.

On the other hand, the model is flexible enough to incorporate de facto different actions in different local games. Namely, consider a family of strategically unrelated strategic games in normal form $G_E^* = (E, (S_E^i)_{i \in E}, (\nu_E^i)_{i \in E})$, $E \in \mathcal{E}$, with $S_E^i \cap S_F^i = \emptyset$ for $E, F \in \mathcal{E}, E \neq F, i \in E \cap F$. This situation can be reduced to a game in normal form played on a hypergraph with local games $G_E = (E, (X^i)_{i \in E}, (\pi_E^i)_{i \in E}), E \in \mathcal{E}$, as follows. For each player i, define $X^i = \prod_{E \in \mathcal{E}_i} S_E^i$. For $i \in N$, $E \in \mathcal{E}_i$, and $x_E = (x^j)_{j \in E} \in X_E$ where $x^j = (s_F^j)_{F \in \mathcal{E}_j}$ for $j \in E$, put $\pi_E^i(x_E) = \nu_E^i((s_E^j)_{j \in E})$. Finally, set $C^i \equiv 0$ for all i. Now a strategy $x^i \in X^i$ is of the form $x^i = (s_F^i)_{F \in \mathcal{E}_i}$ and encodes i's play in each of the games G_E^*. Given a best response $x^i \in X^i$ against $x^{-i} \in X^{-i}$ in $\mathcal{G}$, x_E^i is a best response against x_E^{-i} in G_E for all $E \in \mathcal{E}_i$. In fact, a joint strategy $x \in X$ is a Nash equilibrium of $\mathcal{G}$ if and only if for each $E \in \mathcal{E}$, x_E is a Nash equilibrium of G_E. Moreover, a joint strategy $x = ((s_F^1)_{F \in \mathcal{E}_1}, \ldots, (s_F^n)_{F \in \mathcal{E}_n}) \in X$ is a Nash equilibrium of $\mathcal{G}$ if and only if for each $E \in \mathcal{E}$, $s_E = (s_E^i)_{i \in E}$ is a Nash equilibrium of G_E^*.

3 Potential Games

We shall employ the concept of potential P for $\mathcal{G}$ pioneered by Monderer and Shapley (1996), i.e. a function $P : X \to I\!R$ such that

$$u^i(z^i, x^{-i}) - u^i(x^i, x^{-i}) = P(z^i, x^{-i}) - P(x^i, x^{-i})$$

for each $i \in N$, $x^i, z^i \in X^i, x^{-i} \in X^{-i}$. It is clear from the definition of the potential function that $\mathcal{N}(\mathcal{G})$ coincides with the equilibrium set of the game with common payoff function P. Consequently, $\arg\max_{x \in X} P(x) \subseteq \mathcal{N}(\mathcal{G})$. Furthermore, if X is a countably compact topological space and $P : X \to I\!R$ is upper semi-continuous, then P attains a maximum on X which in turn implies $\mathcal{N}(G) \neq \emptyset$.

Young (1998, chap. 6) shows that symmetric games played on graphs admit a potential if each game $G_{\{i,j\}}$ between two neighbors i and j on the graph has a potential. Baron *et al.* (2002b, Proposition 1 and Proposition 3, pp. 548-550) extend this result to the class of games played on quasi-symmetric weighted graphs. The next proposition extends these results to games played on hypergraphs.

Proposition 1 *Let $\mathcal{G} = (N, \mathcal{E}, (G_E)_{E \in \mathcal{E}}, (C^i)_{i \in N})$ be a n-player game played on a hypergraph. If for each group of players $E \in \mathcal{E}$, G_E admits a potential $\phi_E : X_E \to I\!R$ then $\mathcal{G}$ is a game with potential*

$$P(x) = \sum_{E \in \mathcal{E}} \phi_E(x_E) - \sum_{j \in N} C^j(x^j). \tag{2}$$

PROOF. Define the function P as in (2) and pick any $i \in N$, any $x^i, z^i \in X^i$, and any $x^{-i} \in X^{-i}$. We have

$$\begin{aligned}
u^i(z^i, x^{-i}) - u^i(x^i, x^{-i}) &= \sum_{E \in \mathcal{E}_i} \Big(\pi_E^i((z^i, x^{-i})_E) - \pi_E^i(x_E)\Big) - C^i(z^i) + C^i(x^i) \\
&= \sum_{E \in \mathcal{E}_i} \Big(\phi_E((z^i, x^{-i})_E) - \phi_E(x_E)\Big) - C^i(z^i) + C^i(x^i) \\
&= \sum_{E \in \mathcal{E}_i} \phi_E((z^i, x^{-i})_E) + \sum_{E \notin \mathcal{E}_i} \phi_E(x_E) - \sum_{j \in N \setminus \{i\}} C^j(x^j) \\
&\quad -C^i(z^i) - \left(\sum_{E \in \mathcal{E}_i} \phi_E(x_E) + \sum_{E \notin \mathcal{E}_i} \phi_E(x_E) - \sum_{j \in N} C^j(x^j)\right) \\
&= P(z^i, x^{-i}) - P(x^i, x^{-i}).
\end{aligned}$$

■■

Example 1 (Network Formation)

Let $N = \{1, \ldots, n\}$ be a finite set of agents of size $n > 1$ and let $(N, \mathcal{E})$ be a hypergraph structure describing the collection of subsets of agents who can form links. Denote by $G(N, \mathcal{E})$ the set of graphs or networks that can be formed under the hypergraph structure $(N, \mathcal{E})$. To be more precise, links can only be formed between pairs of players i and j both belonging to some group $E \in \mathcal{E}$. Thus, $\mathcal{H} = \{\{i, j\} \in \mathcal{F}'(N) \mid \exists E \in \mathcal{E} : \{i, j\} \subseteq E\}$ is the set of links which can be formed and $G(N, \mathcal{E}) = 2^{\mathcal{H}}$. For $g \in G(N, \mathcal{E})$, let g_E denote the network one obtains from g by eliminating all links involving players outside of E.

A **value function** is a real-valued function $v : G(N, \mathcal{E}) \to I\!R$ which specifies for each network $g \in G(N, \mathcal{E})$ the total value $v(g)$ generated by g. Let V be the set of all value functions v defined on $G(N, \mathcal{E})$. An interesting subclass of value functions are those which preclude externalities across groups of players. We will say that a value function v is **group additive** if there exists a collection of functions $v_E : 2^{\mathcal{F}'(E)} \to I\!R$, $E \in \mathcal{E}$, such that for all $g \in G(N, \mathcal{E})$,

$$v(g) = \sum_{E \in \mathcal{E}} v_E(g_E).$$

An **allocation rule** is a mapping $f : G(N, \mathcal{E}) \times V \to I\!R^n$ such that

$$\sum_{i \in N} f_i(g, v) = v(g).$$

We say that the rule f is **group-wise egalitarian** if for each $i \in N$

$$f_i(g, v) = \sum_{E \in \mathcal{E}_i} \frac{v_E(g_E)}{\#E}.$$

We now model network formation by means of a strategic game played on the hypergraph $(N, \mathcal{E})$ as follows. $\mathcal{H}_i = \{j \in N : j \neq i \text{ and } \{i, j\} \in \mathcal{H}\}$ is the set of agents with whom $i \in N$ can form links. Set $X^i = \{0, 1\}^{\mathcal{H}_i}$ with generic elements $x^i = (x^i_j)_{j \in \mathcal{H}_i}$. The interpretation of $x^i_j = 1$ for $\{i, j\} \subseteq E$ is that player $i \in E$ seeks contact with player $j \in E$. A link is formed between i and j in E if $x^i_j = x^j_i = 1$. Thus the network resulting from a joint strategy x is given as

$$g(x) = \{\{i, j\} \in \mathcal{H} \mid x^i_j = x^j_i = 1\}.$$

Each player $i \in N$ incurs a cost $C^i(x^i)$ when choosing $x^i \in X^i$. The payoff function for player $i \in N$ is then defined as the group-wise egalitarian allocation rule minus i's cost: For all $i \in N$, $x \in X$,

$$u^i(x) = f_i(g(x), v) - C^i(x^i).$$

For each group of players E, a potential for the associated game G_E is given by

$$\phi_E(x_E) = \frac{v_E(g_E(x))}{\#E}.$$

Notice that the formation of a link $\{i,j\}$ can affect the payoffs of i and j in several local games G_E, because in general, $\mathcal{E}$ is not a partition of N and $\{i,j\}$ can be contained in several of the groups $E \in \mathcal{E}$. □□

Example 2 (Collaboration in Multi-Oligopoly)

Consider a finite set of firms N and a hypergraph $(N, \mathcal{E})$ on N. The set N represents an entire industry whereas each $E \in \mathcal{E}$ constitutes a special branch or product group within the industry. Here $\mathcal{H} = \{\{i,j\} \in \mathcal{F}'(N) \mid \exists\, E \in \mathcal{E} : \{i,j\} \subseteq E\}$ is the set of pairs of firms which belong to at least one common branch and $\mathcal{H}_i = \{j \in N : j \neq i \,\text{and}\, \{i,j\} \in \mathcal{H}\}$ is the set of firms with whom firm i has a branch in common. Every firm $i \in N$ is engaged in activities with respect to the branches $E \in \mathcal{E}_i$. Firm i produces a quantity $q^i_E \in I\!R_+$ of a homogeneous good for each group $E \in \mathcal{E}_i$ and has the option to invest an amount of resources $s^i_j \in I\!R_+$ with each firm $j \in \mathcal{H}_i$. An interfirm collaboration is established between i and j only if $s^i_j s^j_i > 0$. (Observe that $j \in \mathcal{H}_i$ if and only if $i \in \mathcal{H}_j$.) A strategy for firm $i \in N$ is a pair $x^i = (q^i, s^i)$ where $q^i = (q^i_E)_{E \in \mathcal{E}_i}$ and $s^i = (s^i_j)_{j \in \mathcal{H}_i}$. A collaboration between i and j in $E \in \mathcal{E}$ yields lower fixed costs of production for the two firms. Precisely, the cost of production for firm $i \in E$ is given by

$$c^i_E(x_E) = \beta^i_E \cdot q^i_E - \sum_{j \in E} \gamma^{ij}_E s^i_j s^j_i$$

where $\beta^i_E > 0$ and $\gamma^{ij}_E = \gamma^{ji}_E > 0$ for each pair of firms i, j in E. Departing from the standard oligopoly model, without affecting equilibrium outcomes, we allow for negative prices p_E and assume a linear inverse demand at all output combinations for all $E \in \mathcal{E}$ as follows:

$$p_E = \alpha_E - \sum_{j \in E} q_E^j, \text{ with } \alpha_E > 0,$$

so that for each $i \in E$ the payoff in G_E is

$$\pi_E^i(x_E) = \left(\alpha_E - \sum_{j \in E} q_E^j \right) q_E^i - c_E^i(x_E).$$

Firm i's payoff in $\mathcal{G}$ becomes

$$u^i(x) = \sum_{E \in \mathcal{E}_i} \pi_E^i(x_E) - C^i(x^i)$$

where $C^i(x^i)$ is a supplementary cost incurred by firm i when choosing $x^i = (q^i, s^i)$. This cost embodies the cost of investment s^i and the joint diseconomies across markets induced by q^i.

We are going to show that each oligopoly with interfirm collaboration, G_E, admits a potential ϕ_E. Consequently, by Proposition 1, the multi-oligopoly game with interfirm collaboration is a potential game. To this end, let us introduce the notion of interaction potential defined by Ui (2000). For $E \in \mathcal{E}$, let $\mathcal{S}_E$ be the collection of all nonempty subsets of E and $\mathcal{S}_E^i$ be the collection of all subsets of E containing i. For each $C \in \mathcal{S}_E$ denote by $X_C = \prod_{i \in C} X^i$, the set of joint strategies restricted to players in C. For $x_E = (x^i)_{i \in E} \in X_E$ and $C \in \mathcal{S}_E$, $x_C = (x^i)_{i \in C} \in X_C$ denotes the restriction of x_E to players in C. A collection of functions $\{\rho^C : X_C \to I\!R \,|\, C \in \mathcal{S}_E\}$ is an **interaction potential** of the game G_E if for every $i \in E$ and every joint strategy $x \in X_E$ it holds that

$$\pi_E^i(x_E) = \sum_{C \in \mathcal{S}_E^i} \rho^C(x_C).$$

Ui (2000, Theorem 3) shows that if G_E has an interaction potential then it is a potential game with potential

$$\phi_E(x_E) = \sum_{C \in \mathcal{S}_E} \rho^C(x_C).$$

Next, for any $E \in \mathcal{E}$ and any $C \in \mathcal{S}_E$ define $\rho_E^C : X_C \to I\!R$ as follows:

$$\rho_E^C(x_C) = \begin{cases} (\alpha_E q_E^i - \beta_E^i - q_E^i) q_E^i & \text{if } C = \{i\} \\ -q_E^i q_E^j + \gamma^{ij} s_j^i s_i^j & \text{if } C = \{i, j\} \\ 0 & \text{otherwise} \end{cases}$$

Clearly $\{\rho_E^C : X_C \to I\!R \,|\, C \in \mathcal{S}_E\}$ is an interaction potential due to the symmetry of γ_E^{ij}. It follows that G_E is potential game, and by Proposition 1 the multi-oligopoly game with interfirm collaboration is a potential game. A potential is given by

$$P(x) = \sum_{E \in \mathcal{E}} \left(\sum_{C \in \mathcal{S}_E} \rho_E^C(x_C) \right) - \sum_{i \in N} C^i(x^i).$$

□□

Example 3 (Nonspecific Networking)

This class of examples will be discussed in more detail in section 7. □□

4 Supermodular Games

Let X be a partially ordered set, with the reflexive, antisymmetric and transitive binary relation $\geq$. Given elements x and z in X, denote by $x \vee z$ the least upper bound or **join** of x and z in X, provided it exists, and by $x \wedge z$ the greatest lower bound or **meet** of x and z in X, provided it exists. A partially ordered set X that contains the join and the meet of each pair of its elements is called a **lattice**. A lattice in which each non-empty subset has a supremum and an infimum is **complete**. A finite lattice is complete. If Y is a subset of a lattice X and Y contains the join and the meet with respect to X of each pair of elements of Y, then Y is a **sublattice** of X. A sublattice Y of a lattice X in which each non-empty subset has a supremum and an infimum with respect to X that are contained in Y is a **subcomplete sublattice** of X. Any finite sublattice of a lattice is subcomplete.

We now define an order on the subsets of a lattice. We use the **strong set order** $\geq_s$ used in Milgrom and Shannon (1994). Let X be a lattice and let Y and Z be two subsets of X. We say that $Y \geq_s Z$ if for every $y \in Y$ and every $z \in Z$, $y \vee z \in Y$ and $y \wedge z \in Z$. Finally, we say that a correspondence $\rho : X \twoheadrightarrow Y$ from a lattice X to a lattice Y is **increasing** in x on X if $\rho(x)$ is a sublattice of Y for every $x \in X$ and if $x \geq z$ implies $\rho(x) \geq_s \rho(z)$ for all $x, z \in X$.

In case $X = X' \times X''$ is the product of two partially ordered sets, we say that a function $f : X \to \mathbb{R}$ has, exhibits or satisfies **increasing differences** on $X' \times X''$ if for all pairs $(x', x'') \in X' \times X''$ and $(z', z'') \in X' \times X''$, the relations $x' \geq' z'$ and $x'' \geq'' z''$ imply

$$f(x', x'') - f(z', x'') \geq f(x', z'') - f(z', z''). \tag{3}$$

f has **decreasing differences** on $X' \times X''$ if the inequality in (3) is reversed.

A game in normal form played on a hypergraph $\mathcal{G} = (N, \mathcal{E}, (G_E)_{E \in \mathcal{E}}, (C^i)_{i \in N})$ is **supermodular** if the following four conditions are met:

1. For each $i \in N$, X^i is the Cartesian product of $m_i \in \mathbb{N}$ compact subsets X^i_k of $\mathbb{R}$. The set X^i is equipped with the usual partial order relation $\geq$ where $x^i \geq z^i$ in $\mathbb{R}^{m_i}$ if $x^i_k \geq z^i_k$ for $k = 1, 2, \ldots, m_i$, which makes X^i a compact sublattice of $\mathbb{R}^{m_i}$.

2. For each $E \in \mathcal{E}$ and $i \in E$, $\pi^i_E : X_E \to \mathbb{R}$ is upper semicontinuous and **supermodular** in x^i. The latter means that

$$\pi^i_E(x^i \vee z^i, x^{-i}_E) + \pi^i_E(x^i \wedge z^i, x^{-i}_E) \geq \pi^i_E(x^i, x^{-i}_E) + \pi^i_E(z^i, x^{-i}_E)$$

   for any $x^i, z^i \in X^i, x^{-i}_E \in X^{-i}_E$.

3. For each $E \in \mathcal{E}$ and $i \in E$, $\pi^i_E : X_E \to \mathbb{R}$ has **increasing differences** on $X^i \times X^{-i}_E$.

4. For each $i \in N$, C^i is **submodular**, that is for any $x^i, z^i \in X^i$,

$$C^i(x^i \vee z^i) + C^i(x^i \wedge z^i) \leq C^i(x^i) + C^i(z^i),$$

   and C^i is lower semicontinuous.

For details, discussion and further references, see Topkis (1998) and Chapter 2 of Vives (1999). As a first result, we obtain:

Proposition 2 *Let* $\mathcal{G} = (N, \mathcal{E}, (G_E)_{E\in\mathcal{E}}, (C_i)_{i\in N})$ *be a supermodular game played on a hypergraph. Then the set* $\mathcal{N}(\mathcal{G})$ *is a non-empty complete lattice.*

PROOF. As the Cartesian product of compact sublattices, $X = \prod_{i\in N} X^i$ is a compact sublattice of $I\!R^m$, $m = \sum_{i\in N} m_i$. For each $i \in N$, $u^i(x^i, x^{-i})$ is supermodular in x^i on X^i for each x^{-i}. To see this, pick any $x^i, z^i \in X^i$. We obtain

$$\begin{aligned} u^i(x^i \vee z^i, x^{-i}) + u^i(x^i \wedge z^i, x^{-i}) &= \sum_{E\in\mathcal{E}_i} \Big(\pi^i_E(x^i \vee z^i, x^{-i}_E) + \pi^i_E(x^i \wedge z^i, x^{-i}_E)\Big) \\ &\quad -C^i(x^i \vee z^i) - C^i(x^i \wedge z^i) \\ &\geq \sum_{E\in\mathcal{E}_i} \Big(\pi^i_E(x^i, x^{-i}_E) + \pi^i_E(z^i, x^{-i}_E)\Big) \\ &\quad -C^i(x^i) - C^i(z^i) \\ &= u^i(x^i, x^{-i}) + u^i(z^i, x^{-i}) \end{aligned}$$

where the inequality follows from supermodularity of the functions π^i_E on X^i and from submodularity of the cost function C^i on X^i. For each $i \in N$, $u^i(x^i, x^{-i})$ has increasing differences in $(x^i, x^{-i}) \in X^i \times X^{-i}$. To see this, pick $x^i, z^i \in X^i$ with $x^i \geq z^i$ and $x^{-i}, z^{-i} \in X^{-i}$ with $x^{-i} \geq z^{-i}$. Then

$$\begin{aligned} u^i(x^i, x^{-i}) - u^i(z^i, x^{-i}) &= \sum_{E\in\mathcal{E}_i} \Big(\pi^i_E(x^i, x^{-i}_E) - \pi^i_E(z^i, x^{-i}_E)\Big) - C^i(x^i) + C^i(z^i) \\ &\geq \sum_{E\in\mathcal{E}_i} \Big(\pi^i_E(x^i, z^{-i}_E) - \pi^i_E(z^i, z^{-i}_E)\Big) - C^i(x^i) + C^i(z^i) \\ &= u^i(x^i, z^{-i}) - u^i(z^i, z^{-i}) \end{aligned}$$

where the inequality follows from the fact that the functions π^i_E have increasing differences on $X^i \times X^{-i}_E$. Next, for each $i \in N$ and $x^{-i} \in X^{-i}$, the payoff function $u^i(x^i, x^{-i})$ is upper semicontinuous in $x^i \in X^i$ as the finite sum of upper semicontinuous functions on X^i. Hence $(N, (X^i)_{i\in N}, (u^i)_{i\in N})$ is a supermodular game in the sense of Zhou (1994, section 3). The assertion follows from Zhou's Theorem (1994, p. 299). ■■

5 Supermodular Potential Games

Here we explore the relationships between the class of potential games and the class of supermodular games.

Proposition 3 *Let $\mathcal{G} = (N, \mathcal{E}, (G_E)_{E\in\mathcal{E}}, (C^i)_{i\in N})$ be a supermodular game played on a hypergraph. Suppose $\mathcal{G}$ has a potential $P : X \longrightarrow I\!R$. Then* $\arg\max_{x\in X} P(x)$ *is a sublattice of X.*

PROOF. Pick any $i \in N$. First note that $u^i(x^i, x^{-i})$ has increasing differences on X^i for each $x^{-i} \in X^{-i}$ by Theorem 2.6.1 in Topkis (1998) i.e. $u^i(x^i, x^{-i})$ has increasing differences in (x^i_k, x^i_l) on $X^i_k \times X^i_l$ for each distinct $k, l \in \{1, \ldots, m_i\}$ and for each $(x^i_{-kl}, x^{-i}) \in \left(\prod_{r\neq k,l} X^i_r\right) \times X^{-i}$. It follows that P has increasing differences on X^i for each $x^{-i} \in X^{-i}$ by definition of a potential P. Now, pick any $j \in N$, $j \neq i$. For all $x \in X$ and $z^i_k \in X^i_k, z^j_l \in X^j_l$, $k \in \{1, \ldots, m_i\}, l \in \{1, \ldots, m_j\}$ such that $x^i_k \geq z^i_k$ and $x^j_l \geq z^j_l$, we have

$$\begin{aligned}
& P(x^i_k, x^i_{-k}, x^j_l, x^j_{-l}, x^{-ij}) - P(z^i_k, x^i_{-k}, x^j_l, x^j_{-l}, x^{-ij}) \\
= \;& u^i(x^i_k, x^i_{-k}, x^j_l, x^j_{-l}, x^{-ij}) - u^i(z^i_k, x^i_{-k}, x^j_l, x^j_{-l}, x^{-ij}) \\
\geq \;& u^i(x^i_k, x^i_{-k}, z^j_l, , x^j_{-l}, x^{-ij}) - u^i(z^i_k, x^i_{-k}, z^j_l, x^j_{-l}, x^{-ij}) \\
= \;& P(x^i_k, x^i_{-k}, z^j_l, x^j_{-l}, x^{-ij}) - P(z^i_k, x^i_{-k}, z^j_l, x^j_{-l}, x^{-ij})
\end{aligned}$$

The two equalities follow from the definition of a potential P. The inequality follows from the assumption that $u^i(x^i, x^{-i})$ has increasing differences in (x^i, x^{-i}) on $X^i \times X^{-i}$. This means that P has increasing differences in (x^i_k, x^j_l) in $X^i_k \times X^j_l$ for each $j \neq i$, $k \in \{1, \ldots, m_i\}$, $l \in \{1, \ldots, m_j\}$ and for each $(x^i_{-k}, x^j_{-l}, x^{-ij})$. As this property holds for all $i \neq j$ and since for each $i \in N$, P has increasing differences on X^i for each $x^{-i} \in X^{-i}$, it follows that P has increasing differences on X. The potential P is supermodular on X by Corollary 2.6.1 in Topkis (1998). By Theorem 2.7.1 of Topkis (1998) the set of maximizers of P is a sublattice of X. This completes the proof. ■■

Remark If each set X^k_i, $i \in N$, $k \in \{1, \ldots, m_i\}$ is a finite set or if P is upper semicontinuous on X, then $\arg\max_{x\in X} P(x)$ is a non-empty compact sublattice of X. Since a potential is unique up to an additive constant, $\arg\max_{x\in X} P(x)$ is independent of the particular potential P.

Further notice that if each payoff function u_i is supermodular on X, then $W = \sum_i u_i$ is supermodular as the finite sum of supermodular functions, by Lemma 2.6.1 in Topkis (1998). In that case $\arg\max_{x\in X} W(x)$, the set of "efficient" strategy profiles is a sublattice of X as well. Moreover, then the strategy profiles which maximize both P and W form also a (possibly empty) sublattice of X, because the intersection of sublattices is a sublattice by Lemma 2.2.2 in Topkis (1998).

In the sequel, let Θ denote a non-empty subset of some Euclidean space $\mathbb{R}^p$, $p \in \mathbb{N}$, with generic elements θ, ϑ.

Proposition 4 *Let $\mathcal{G}^\theta, \theta \in \Theta$, be a collection of supermodular games played on hypergraphs, with common player set N and common strategy sets $X^i, i \in N$. Suppose each game $\mathcal{G}^\theta$ has a potential P^θ, $\theta \in \Theta$. Further suppose that for each $i \in N$ and each $x^{-i} \in X^{-i}$, the payoff function $u^{i,\theta}(x^i, x^{-i})$ has increasing differences in (x^i, θ) on $X^i \times \Theta$. Then $\sup_X \left(\arg\max_{x\in X} P^\theta(x)\right)$ $\left[\inf_X \left(\arg\max_{x\in X} P^\theta(x)\right)\right]$, if it exists, is weakly increasing in θ on Θ.*

PROOF. Pick any $x, z \in X$ with $x \geq z$ and any $\theta, \vartheta \in \Theta$ with $\theta \geq \vartheta$. Define $x(0), x(1), \ldots, x(n) \in X$ as follows: $x(0) = x$, $x^i(j) = z^i$ for $i, j \in N$, $i \leq j$, and $x^i(j) = x^i$ for $i, j \in N$, $i > j$. By construction, $x(j) \geq x(j+1)$ for $j = 0, 1, 2, \ldots, n-1$. Because $\mathcal{G}^\theta$ and $\mathcal{G}^\vartheta$ are potential games and the payoff function of each player i has increasing differences on $X^i \times \Theta$, it is the case that $x \geq z$ and $\theta \geq \vartheta$ implies

$$\begin{aligned} P^\theta(x) - P^\theta(z) &= \sum_{i=1}^n \Big(P^\theta(x(i-1)) - P^\theta(x(i))\Big) \\ &= \sum_{i=1}^n \Big(u^{i,\theta}(x(i-1)) - u^{i,\theta}(x(i))\Big) \\ &\geq \sum_{i=1}^n \Big(u^{i,\vartheta}(x(i-1)) - u^{i,\vartheta}(x(i))\Big) \\ &= \sum_{i=1}^n \Big(P^\vartheta(x(i-1)) - P^\vartheta(x(i))\Big) \\ &= P^\vartheta(x) - P^\vartheta(z). \end{aligned}$$

This means that $P^\theta(x)$ has increasing differences in (x, θ) on $X \times \Theta$. For each $\theta \in \Theta$, $P^\theta(x)$ is supermodular in x on X by Proposition 3. Then the correspondence $S : \Theta \twoheadrightarrow X$, $\theta \mapsto S(\theta) = \arg\max_{x\in X} P^\theta(x)$ is increasing with respect to the strong set order by Theorem 2.8.1 of Topkis (1998). The assertion follows by Lemma 2.4.2 of Topkis (1998). ■■

6 Stochastic Stability

Let $\mathcal{G}$ be a finite game played on a hypergraph. Our concept of stochastic stability is based upon best response dynamics with logit perturbations. Throughout, we consider dynamics with asynchronous updating and persistent noise, with discrete time $t = 0, 1, \ldots$ and states $x \in X$. Let $q = (q^1, \ldots, q^n) \gg 0$ be an n-dimensional probability vector. The recurrent game $\mathcal{G}$ on a hypergraph is played once in each period. In each period t, one player, say i, is drawn with probability $q^i > 0$ from this population to adjust his strategy and does so according to a perturbed adaptive rule. The draws are i.i.d. across time. The non-selected players repeat the strategies they have played in the previous period.

The perturbed adaptive rule is a logit rule. Suppose the current state is $x = (x^j)_{j \in N}$. In principle, the updating player i wants to play a best reply against $x^{-i} = (x^j)_{j \neq i}$. But with some small probability, the player trembles and plays a non-best reply. If the player follows a logit rule, then for all $y^i \in X^i$, the probability that i chooses y^i in state x is given by

$$p^i(y^i|x) = \frac{\exp[u^i(y^i, x^{-i})/\epsilon]}{\sum_{z^i} \exp[u^i(z^i, x^{-i})/\epsilon]}, \tag{4}$$

where $\epsilon > 0$ is a noise parameter. For given ϵ, two choices that yield the same payoff to i are equally likely. If one of them yields a higher payoff, it will be chosen with a higher probability. In particular, any best reply to x^{-i} is more likely to be chosen than a non-best reply. As $\epsilon \to 0$, the probability that a best reply is chosen goes to 1. For given $\epsilon > 0$, one obtains a stationary Markov process on X with transition matrix $M(\epsilon)$. $M(\epsilon)$ has entries $m_{x,x'}(\epsilon)$ with the following properties. If x and x' differ in more than one component, then $m_{x,x'}(\epsilon) = 0$. If x and x' differ only in the ith coordinate and $x' = (y^i, x^{-i})$, then $m_{x,x'}(\epsilon) = q^i \cdot p^i(y^i|x)$. If $x = x'$, then $m_{x,x}(\epsilon) = \sum_{j \in N} q^j \cdot p^j(x^j|x)$. The process is irreducible and aperiodic, hence it is ergodic and has a unique stationary distribution, represented by a row probability vector $\mu(\epsilon)$. Like in many prior studies of perturbed evolutionary games we want to determine the behavior of the system when $\epsilon \to 0$, that is when the noise becomes arbitrarily small. If the limit stationary distribution $\mu^* = \lim_{\epsilon \to 0} \mu(\epsilon)$ exists, we write X^* for its support:

$$X^* \equiv \{x \in X : \mu^*_x > 0\}$$

The joint strategies in X^* will be referred to as **stochastically stable states**. These are the states in which the system stays most of the time when very little, but still some noise remains. Baron *et al.* (2002a) show that X^* can be partitioned into minimal sets closed under asynchronous best replies. It turns out that the limit stationary distribution exists and the stochastically stable states are the maximizers of the potential, if the underlying game $\mathcal{G}$ has a potential.

Proposition 5 *Suppose that $\mathcal{G}$ is a finite game and has potential P. Then $X^* = \arg\max_{x \in X} P(x)$ and all stochastically stable states have equal probability.*

PROOF. See Blume (1993, 1997), Young (1998), Baron *et al.* (2002a,b) for the key argument. ■■

As an immediate consequence of Propositions 2, 3, and 5, we obtain

Corollary 1 *If $\mathcal{G}$ is a finite supermodular game played on a hypergraph and has potential P, then X^* is a non-empty sublattice of X and of $\mathcal{N}(\mathcal{G})$.*

Moreover, the comparative statics of Proposition 4 apply to the smallest (largest) stochastically stable state of the games $\mathcal{G}^\theta, \theta \in \Theta$. For example, suppose that for some integer $m > 1$, $\Theta = \{1, 2, \ldots, m\}$. Moreover, $X^i = \Theta$ for each $i \in N$ and $u^{i,\theta}(x) = \min\{\theta, x^1, \ldots, x^n\}$ for all $i \in N, \theta \in \Theta, x \in X$. Then the game has the potential $P^\theta(x) = \min\{\theta, x^1, \ldots, x^n\}$. For any $\theta \in \Theta$, the smallest stochastically stable state is $(\theta, \ldots, \theta)$ and the largest stochastically stable state is $(m, \ldots, m)$.

Observe that if in addition, $\mathcal{G}$ is a symmetric game, then the corollary further implies that $\mathcal{G}$ has at least one symmetric stochastically stable state. The result that the set X^* of stochastically stable states forms a non-empty sublattice of X (rather than merely a lattice), is also of some practical interest. Namely, then one can easily find a new stochastically stable state knowing that two joint strategies (equilibria) are stochastically stable: If

$x = (x^1, \ldots, x^n)$ and $y = (y^1, \ldots, y^n)$ are in X^*, then so are $\sup_X\{x, y\} = (\max\{x^1, y^1\}, \ldots, \max\{x^N, y^N\})$ and $\inf_X\{x, y\} = (\min\{x^1, y^1\}, \ldots, \min\{x^n, y^n\})$. One cannot necessarily proceed this way within the equilibrium set $\mathcal{N}(\mathcal{G})$. For the conclusion of Proposition 2 that the set of Nash equilibria $\mathcal{N}(\mathcal{G})$ is a non-empty lattice can be hardly replaced by the stronger assertion that $\mathcal{N}(\mathcal{G})$ is a sublattice of the set of joint strategies X. The reason is that Zhou's Fixed-Point Theorem (1994, p. 297) cannot be generalized to show the set of fixed points of an increasing correspondence from a nonempty complete lattice X into itself to be a sublattice of X; see Zhou (1994, p. 298) and Example 2.5.1 of Topkis (1998, p. 40). For the special case of a two-player supermodular game where players' strategy sets are totally ordered, Echenique (2003) establishes that the set of Nash equilibria is a sublattice of the set of joint strategies. But he observes that a supermodular game with more than two players need not have an equilibrium set that is a sublattice even if players' strategy sets are totally ordered.

7 Application to Nonspecific Networking

In contrast to the literature on network formation and network utilization, the focus of this application lies on nonspecific networking, meaning that an agent cannot select a specific subset of feasible links which he wants to establish, strengthen or utilize. Rather, each agent chooses an effort level or intensity of networking. In the simplest case, the agent faces a binary choice: to network or not to network. If an agent increases his networking effort, all direct links to other agents are strengthened to various degrees. We assume that benefits accrue only from direct links. Each agent has a finite strategy set consisting of the networking levels to choose from. For any pair of agents, their networking levels determine the individual benefits which they obtain from interacting with each other. An agent derives an aggregate benefit from the pairwise interactions with all others. This aggregate benefit is a function of the chosen joint strategy of networking levels. In addition, the agent incurs networking costs, which are a function of the agent's own networking level. The agent's payoff is his aggregate benefit minus his cost.

We model nonspecific networking by means of networking games, a special class of games played on a hypergraph. A game $\mathcal{G}$ will be called a **networking game**, if it satisfies two restrictions, one on strategy sets and one on interactions:

1. `Restriction on strategy sets:` Every player $i \in N$ has strategy set $X^i = K = \{k_0, k_1, \ldots, k_T\} \subseteq \mathbb{R}_+$, with $T \geq 1$ and $0 = k_0 < k_1 < \ldots < k_T$. The $T+1$ individual strategies $0, k_1, \ldots, k_T$ constitute the **networking levels** a player can choose from and for simplicity are assumed to be the same for all players. Depending on the context, a higher networking level may mean more effort in socializing, more investment in networking skills, more investment in communication and information hardware or software, subscription to better network services. Player $i \in N$ incurs a **cost** $c_i(x^i)$ when choosing $x^i \in X^i$. As a rule, the choice of a higher networking level is more costly: $0 = c_i(0) < c_i(k_1) < \ldots < c_i(k_T)$.

2. `Restriction on interactions:` Players receive **benefits from pairwise interaction** with others, that is $\mathcal{E} \subseteq \mathcal{F}'(N)$. It follows that $\mathcal{H} = \mathcal{E}$. For $i \in N$, $\mathcal{H}_i$ is the set of i's neighbors in the undirected network $\mathcal{E}$.

We use ij as short-hand for an ordered pair of players $(i,j) \in N \times N$. We define benefit functions $b_{ij} : K \times K \to I\!R$ as follows. For $i \in N, \{i,j\} \in \mathcal{E}$, and $x_{\{i,j\}} = (x^k)_{k \in \{i,j\}} \in X_{\{i,j\}}$, we set $b_{ij}(x^i, x^j) = \pi^i_{\{i,j\}}(x_{\{i,j\}})$. Player i receives the benefit $b_{ij}(x^i, x^j) \in \mathbb{R}$ from interacting with j, if i chooses $x^i \in X^i$ and j chooses $x^j \in X^j$. $\{i,j\} \notin \mathcal{E}$ means that interaction between i and j is infeasible because of geographic distance, language barriers, lack of physical infrastructure, etc. At times, it is convenient to pretend that interaction between i and j is feasible but ineffective, by setting $b_{ij}(x^i, x^j) = 0$ for $\{i,j\} \notin \mathcal{E}$, $(x^i, x^j) \in X^i \times X^j$. We shall proceed this way. Then the payoff function (1) becomes

$$u^i(x) = \sum_{j \neq i} b_{ij}(x^i, x^j) - c_i(x^i). \tag{5}$$

All our previous results are directly applicable to instances of nonspecific networking. First of all, Propositions 1 and 5 apply accordingly: If each of the two-player games $G_{\{i,j\}}$ with player set $\{i,j\}$, strategy sets X^i, X^j, and payoff functions $b_{ij}(x^i, x^j)$ and $b_{ji}(x^j, x^i)$ has a potential $\phi_{\{i,j\}}$, then $\mathcal{G}$ has a potential given by (2). If $\mathcal{G}$ has a potential and is finite, then the assertion of Proposition 5 holds for the stochastically stable states of the best response dynamics with logit perturbations. In case b_{ij} satisfies **increasing differences** in $(x^i, x^j) \in X^i \times X^j$ for all $i \neq j$, the game $\mathcal{G}$ is supermodular and the assertion of Proposition 2 holds. In case $\mathcal{G}$ is supermodular and has a potential, then the set of maximizers of the potential has the structural properties of Proposition 3. Finally, Proposition 4 has implications for the comparative statics with respect to networking costs, a fact on which we are going to elaborate in more detail in the next subsection.

7.1 Comparative Statics in Networking Costs

Intuitively, one would expect that networking activities intensify if networking costs decline. This conjecture proves at least partially true in the presence of strategic complements in pairwise interactions on the one hand and in the presence of strategic substitutes in pairwise interactions on the other hand. To be precise, consider the following four assumptions:

(A) For all $i \neq j$, b_{ij} satisfies increasing differences in $(x^i, x^j) \in X^i \times X^j$.

(B) For all $i \neq j$, b_{ij} satisfies decreasing differences in $(x^i, x^j) \in X^i \times X^j$.

(C) There exist $C_1 \geq 0, \ldots, C_n \geq 0$ such that $c_i(x^i) = C_i \cdot x^i$ for $i \in N$, $x^i \in X^i$.

(D) For $i \in N$, there exists a unique best response against each $x^{-i} \in X^{-i}$.

We commence with two comparative statics results for submodular networking games.

Proposition 6 *Let $\mathcal{G}$ be a networking game satisfying (B)-(D) and let $\mathcal{G}'$ be a second such networking game that differs from $\mathcal{G}$ only in the marginal networking costs, which are $C'_1 \geq 0, \ldots, C'_n \geq 0$ in $\mathcal{G}'$. Further, let $x \in X$ be an equilibrium of $\mathcal{G}$ and $y \in X$ be an equilibrium of $\mathcal{G}'$. Suppose $C'_i \leq C_i$ for all i and $y \neq x$. Then $y^i > x^i$ for some i.*

PROOF. Let $\mathcal{G}, \mathcal{G}', C_1, \ldots, C_n, C'_1, \ldots, C'_n, x, y$ be as hypothesized. Since $x \neq y$, there is $i \in N$ such that $x^i \neq y^i$. Consider this player i and suppose the conclusion is false, that is $y^j \leq x^j$ for all $j \in N$. We have:

$$\begin{aligned}
0 &< \sum_j b_{ij}(x^i, x^j) - C_i \cdot x^i - \left(\sum_j b_{ij}(y^i, x^j) - C_i \cdot y^i \right) \\
&= \sum_j \Big(b_{ij}(x^i, x^j) - b_{ij}(y^i, x^j) \Big) - C_i \cdot x^i + C_i \cdot y^i \\
&\leq \sum_j \Big(b_{ij}(x^i, y^j) - b_{ij}(y^i, y^j) \Big) - C_i \cdot x^i + C_i \cdot y^i \\
&\leq \sum_j \Big(b_{ij}(x^i, y^j) - b_{ij}(y^i, y^j) \Big) - C'_i \cdot x^i + C'_i \cdot y^i \\
&= \sum_j b_{ij}(x^i, y^j) - C'_i \cdot x^i - \left(\sum_j b_{ij}(y^i, y^j) - C'_i \cdot y^i \right) < 0,
\end{aligned}$$

a contradiction. The first inequality follows from optimality of x^i at x^{-i}, $x^i \neq y^i$, and (D). The second inequality follows from (B). The third inequality is a consequence of $C'_i \leq C_i$ and $y^i \leq x^i$. The last inequality follows from optimality of y^i at y^{-i}, $x^i \neq y^i$, and (D). Hence, to the contrary, the conclusion has to be true. ■■

The assumption (D) of unique best responses can be disposed of if one postulates strict cost reductions instead:

Proposition 7 *Let $\mathcal{G}$ be a networking game that satisfies (B) and (C) and let $\mathcal{G}'$ be a second such networking game that differs from $\mathcal{G}$ only in the marginal networking costs, which are $C'_1 \geq 0, \dots, C'_n \geq 0$ in $\mathcal{G}'$. Further, let $x \in X$ be an equilibrium of $\mathcal{G}$ and $y \in X$ be an equilibrium of $\mathcal{G}'$. Suppose $C'_i < C_i$ for all i and $y \neq x$. Then $y^i > x^i$ for some i.*

PROOF. Let $\mathcal{G}, \mathcal{G}', C_1, \dots, C_n, C'_1, \dots, C'_n, x, y$ be as hypothesized. Suppose the conclusion is false, that is $y^i \leq x^i$ for all $i \in N$. Now take any $i \in N$. By assumption, x^i is a best response of i against x^{-i} in $\mathcal{G}$. Since $y^j \leq x^j$ for all $j \neq i$ and (B) and (C) hold, the largest best response $\widehat{x}^i$ of i against y^{-i} in $\mathcal{G}$ satisfies $\widehat{x}^i \geq x^i$. Since $C'_i < C_i$, (B) and (C) hold, and $\mathcal{G}$ and $\mathcal{G}'$ differ only in marginal networking costs, one obtains $\widetilde{y}^i \geq \widehat{x}^i$ for any best response $\widetilde{y}^i$ of i against y^{-i} in $\mathcal{G}'$ and any best response $\widehat{x}^i$ of i against y^{-i} in $\mathcal{G}$. It follows that $y^i \geq x^i$ because y^i is a best response of i against y^{-i} in $\mathcal{G}'$. But $y^i \geq x^i$ and $y^i \leq x^i$ imply $y^i = x^i$. Since i was arbitrary, $y = x$, which contradicts the hypothesis of the proposition. Hence, to the contrary, the conclusion has to be true. ■■

We now turn to comparative statics for supermodular networking games. If a networking game $\mathcal{G}$ satisfies (A), then it is supermodular, and by Proposition 2, it has a smallest and a largest Nash equilibrium. If in addition, the game has a potential, then by Proposition 3, there exist a smallest and a largest maximizer of the potential — which are also Nash equilibria and the smallest and the largest stochastically stable state, respectively, under best response dynamics with logit perturbations. We obtain weak monotonicity results for the smallest and largest Nash equilibria of supermodular networking games by applying an earlier result of Milgrom and Roberts (1990): These distinct equilibria will never decrease in response to a cost reduction. By Proposition 4, the comparative statics à la Milgrom and Roberts for supermodular games extend to the smallest and largest stochastically stable states of supermodular potential games.

Proposition 8 *Let $\mathcal{G}$ be a networking game that satisfies (A) and (C) and let $\mathcal{G}'$ be a second such networking game that differs from $\mathcal{G}$ only in the marginal networking costs, which are $C'_1 \geq 0, \dots, C'_n \geq 0$ in $\mathcal{G}'$. Suppose $C'_i \leq C_i$ for all i.*

(i) *If $x \in X$ is the smallest (largest) equilibrium of $\mathcal{G}$ and $y \in X$ is the smallest (largest) equilibrium of $\mathcal{G}'$, then $y^i \geq x^i$ for all i.*

(ii) *If $\mathcal{G}$ and $\mathcal{G}'$ have respective potentials P and P', $x \in X$ is the smallest (largest) maximizer of P and $y \in X$ is the smallest (largest) maximizer of P', then $y^i \geq x^i$ for all i.*

PROOF. Let $\mathcal{G}, \mathcal{G}', C_1, \dots, C_n, C'_1, \dots, C'_n$ be as hypothesized.

(i) Consider a family of networking games $\mathcal{G}^\tau$ satisfying (A) and (C) which only differ in the marginal cost parameters $\tau = (C_1, \dots, C_n) \in \mathbb{R}^n_+$. Then the smallest and the largest equilibrium of $\mathcal{G}^\tau$ are non-increasing functions of τ. Namely, endow the parameter space $\mathbb{R}^n_+$ with the reverse of its canonical partial order, that is for $\tau, \tau' \in \mathbb{R}^n_+$, $\tau \leq \tau'$ if and only if $\tau_i \geq \tau'_i$ for all i. Then the payoff functions given by (5) satisfy condition (A5) of Milgrom and Roberts (1990). (A) and (C) imply that each game $\mathcal{G}^\tau$ is supermodular. Therefore, by Theorem 6 of Milgrom and Roberts, the smallest and the largest equilibrium of $\mathcal{G}^\tau$ are non-decreasing in τ with respect to the reverse canonical partial order of $\mathbb{R}^n_+$. Hence the assertion.

(ii) Consider a family of networking games $\mathcal{G}^\tau$ which are satisfying (A) and (C), have respective potentials P^τ, and differ only in the marginal cost parameters $\tau = (C_1, \ldots, C_n) \in \mathbb{R}^n_+$. With $\theta = -\tau$, individual payoffs satisfy increasing differences in (x^i, θ). By Proposition 4, the smallest and the largest maximizer of the potential are weakly increasing in $\theta = (-C_1, \ldots, -C_n)$. Hence the assertion. ■■

Notice that the conclusion of Propositions 6 and 7 cannot be substantially strengthened for two reasons. For one, $\mathcal{G}$ and $\mathcal{G}'$ may have the same equilibria, even if $C'_i < C_i$ for all i. This follows from the discreteness of the model. Secondly, let $\mathcal{G}$ be given by $n = 6$, the circular graph $\mathcal{E} = \{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}\}$, the set of available networking levels $K = \{h/2 : h = 0,1,\ldots,10\}$, pairwise benefit functions $b_{ij}(x^i, x^j) = \sqrt{x^i + x^j}$ for $\{i,j\} \in \mathcal{E}$; $b_{ij}(x^i, x^j) = 0$ for $\{i,j\} \notin \mathcal{E}$, and the marginal cost parameters $C_i = 1$ for all i. Let $\mathcal{G}'$ be a game that differs from G only with respect to marginal networking costs. Specifically, set $C'_i = 1/2$ for i odd and $C'_j = C' < 1$ for j even. $x^* = (1/2, 1/2, 1/2, 1/2, 1/2, 1/2)$ is an equilibrium of $\mathcal{G}$. If C' is sufficiently close to 1, then $x^{**} = (4,0,4,0,4,0)$ is an equilibrium of $\mathcal{G}'$. Obviously $x^{**} \neq x^*$. But despite the cost reduction, some players have lowered their efforts in x^{**} relative to x^*.

Without a strategic substitutes (submodularity) assumption, a cost decline is consistent with a universal reduction of networking activities. Next we provide a numerical example with this property, which exhibits strategic complements (supermodularity) and also demonstrates that the conclusion of Proposition 8 cannot be substantially strengthened.

Example. Let $e = \exp(1)$ be the Euler number. Set $K = \{0, e^{1/4} - 1, e - 1\}$, $n = 6$, $\mathcal{E} = \{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}\}$. Define pairwise benefit functions as $b_{ij}(x^i, x^j) = \frac{1}{2}\ln(1+x^i)\cdot\ln(1+x^j)$ for $\{i,j\} \in \mathcal{E}$; $b_{ij}(x^i, x^j) = 0$ for $\{i,j\} \notin \mathcal{E}$. With $C_i = e^{-1}$ for all i, we obtain a game $\mathcal{G}$ which has two symmetric equilibria, $x^0 = (0,\ldots,0)$ and $x^\bullet = (e-1,\ldots,e-1)$. Setting $C'_i = e^{-1/4}/4 < C_i$ for all i defines a game $\mathcal{G}'$ which has three symmetric equilibria, x^0, $x^\bullet$, and $x^{\bullet\bullet} = (e^{1/4}-1,\ldots,e^{1/4}-1)$. Thus, the example has several interesting features. First, there exists the equilibrium x^0, an instance of mutual obstruction where nobody has an incentive to network if nobody else is networking. Next there exists the equilibrium $x^{\bullet}$ where everybody exerts maximum networking effort. Further, a cost reduction leads to the emergence of a third equilibrium, $x^{\bullet\bullet}$ where everyone makes a positive but less than maximal effort. Regarding our original point, the conclusion of Propositions 6 and 7 obviously need not hold if the strategic substitutes assumption of the form (B) is violated. Moreover the conclusion of Proposition 8 manifests itself in its weak form: After a cost reduction for all players, the smallest equilibrium x^0, the largest equilibrium $x^{\bullet}$, and the unique stochastically stable state (maximand of the potential) $x^{\bullet}$ remain unchanged. □□

Echenique and Sabarwal (2003, p. 309) give a condition on a pair of parameters $\theta, \theta' \in \Theta, \theta \leq \theta'$, which implies $\sup \mathcal{N}(\mathcal{G}) \leq \inf \mathcal{N}(\mathcal{G}')$ for the games $\mathcal{G}$ and $\mathcal{G}'$ identified by θ and θ', respectively. The corresponding inequality hold for the maximizers of the respective potentials if applicable.

7.2 Instances of Nonspecific Networking

To illustrate the scope of applications, let us specialize and assume a decomposition

$$b_{ij}(x^i, x^j) = p_{ij}(x^i, x^j) \cdot v_{ij}, \tag{6}$$

where $p_{ij} \geq 0$ is the probability, reliability or strength of the link from i to j or the intensity of i interacting with j. v_{ij} is i's benefit, appreciation or valuation of an interaction with j. If $0 \leq p_{ij} \leq 1$ and p_{ij} is interpreted as a probability, then player i receives benefit v_{ij} with probability p_{ij}, zero benefit with probability $1-p_{ij}$, and expected benefit b_{ij}. It is possible that players are linked without any effort or investment, that is $p_{ij}(0,0) > 0$. It is also possible that the strength or probability of certain links proves irresponsive to effort or investment, that is p_{ij} is constant. Frequently, though not necessarily, the link probabilities satisfy the following regularity conditions, any two of which imply the third one:

(i) *Identity*: $p_{ij}(x^i, x^j) = p_{ji}(x^j, x^i)$ for all $(x^i, x^j) \in K \times K$.

(ii) *Symmetry*: $p_{ij}(x^i, x^j) = p_{ji}(x^i, x^j)$ for all $(x^i, x^j) \in K \times K$.

(iii) *Interchangeability*: $p_{ij}(x^i, x^j) = p_{ij}(x^j, x^i)$, $p_{ji}(x^i, x^j) = p_{ji}(x^j, x^i)$ for all $(x^i, x^j) \in K \times K$.

There exists also a symmetry condition for the valuations v_{ij}, with a host of subcases,

(iv) *Mutual Affinity*: $v_{ij} = v_{ji}$,

or more generally, $\text{sign}(v_{ij}) = \text{sign}(v_{ji})$. Mutual affinity can result, e.g., from similarity (kindred spirits) or from complementarity (attraction of opposites). There can be mutual lack of interest, $v_{ij} = v_{ji} = 0$, mutual antipathy, dislike or disadvantage, $v_{ij} = v_{ji} < 0$ and mutual advantage or sympathy, $v_{ij} = v_{ji} > 0$. Instead of mutual affinity, there can be adversity or antagonism: $v_{ij} = -v_{ji}$ or, more generally, $\text{sign}(v_{ij}) = -\text{sign}(v_{ji})$.

Both in traditional and in electronic interactions, some agents are much more active in networking than others and might be called "networkers". A networker is more eager than others to form and utilize networks because of (actual or perceived) benefit or cost advantages. One possibility is that there exist numbers $v_i > 0$, $i \in N$, such that $v_{ij} = v_i$ for any distinct pair ij. Then *ceteris paribus*, i has a greater incentive to network than j if $v_i > v_j$, since i's benefit from any interaction is higher than j's. Another possibility is that there exist numbers $v_i > 0$, $i \in N$, such that $v_{ij} = v_i v_j$ for any distinct pair ij. Then $v_i > v_j$ and $v_k > 0$ imply $v_{ik} > v_{jk}$ and $v_{ki} > v_{kj}$. Not only has i a higher benefit from any interaction and, therefore, a greater incentive to network than j. It is also the case that any third player k would have a higher benefit from interacting with i than from interacting with j. In some cases this implies that being surrounded by networkers, i.e. players with high v_i, may induce a player to make a large networking effort. Hence, under certain circumstances, the presence of networkers fosters networking by networkers and others. A further possibility is that some individuals have a cost advantage which may induce them to invest more in networks and in turn may cause reduced networking efforts by others.

Finally, one might distinguish between good and bad neighbors when valuations assume the form $v_{ij} = v_j$. In that case, player j with $v_j > 0$ would be considered a good neighbor and $v_k < 0$ would make player k a bad neighbor.

8 Extensions

The formal analysis of interaction on hypergraphs encompasses but is not limited to models of nonspecific networking. The latter provide a major application and the main motivation for our research. The broader framework yields some immediate generalizations of some of the results on nonspecific networking, including comparative statics. For instance, one can consider multi-dimensional effort choices, like choosing software-hardware combinations.

A further alternative could make the set of available efforts a (one- or multi-dimensional) interval or convex set and assume sufficient differentiability of the cost and benefit functions. As Brueckner (2003) demonstrates in the context of specific networking, one arrives at some conclusions very elegantly, if such a continuous model is highly symmetry, but does not get very far otherwise. Most of our subcases and examples can be easily embedded into a larger continuous model. But again, while this might produce some eloquence and quickness of derivations in some cases, it would only render the analysis more complicated in others. An added complication stems from the fact that the concept of stochastic stability developed in the literature so far (based on logit or other perturbations) and employed in the present paper relies on a finite state space.

The idea that the strength or probability of a link might depend on the efforts of both agents involved, is also central to the model of Brueckner (2003). Similarly, Haller and Sarangi (2005) consider the possibility that the reliability of a link depends on the efforts of both agents. Since we allow for negative affinity or attraction, some agents might not only abstain from networking but might take counter-measures against the networking attempts of others and be willing to incur costs in order to weaken or sever links. This eventuality suggests a further extension of the formal model.

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