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Abstract

A population of players is considered in which each player may select his neighbors in order to play a 2×2 coordination game with each of them. We analyze how the payoffs in the underlying coordination game effect the resulting equilibrium neighborhood resp. network structure. Depending on the size of the communication costs the resulting equilibrium networks may be characterized by bipartite graphs if the coordination game is of the Hawk/Dove type while networks show a tendency to build complete or disconnected graphs if neighbors play a pure coordination game.

JEL classification: C72, C92

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1 Introduction

Coordination games attracted many theoretically and experimentally oriented economists during the past decade (see, for example, van Huyck/Battalio/Beil, 1990, Cooper et al., 1992, Berninghaus/Schwalbe, 1996a, Young, 1998). In our paper we consider simple symmetric normal form 2×2 games which are characterized by having two equilibria in pure strategies. If such a 2×2 game is played in large populations with players who are pairwise randomly matched a serious equilibrium selection problem may arise. We know from the theoretical (e.g., Boyer/Orleans, 1992) and the experimental literature (e.g., Cooper et al., 1992) that in case of pure coordination games both symmetric equilibria may be candidates for strategy selection. In coordination games with two asymmetric equilibria still less is known about strategy choice in experimental games. *Conventions* might sometimes help to solve these problems (Lewis, 1969, Young, 1993, Berninghaus 2003). By conventions players are guided to select a particular equilibrium and, therefore, avoid coordination failures. In real world societies conventions will not arise spontaneously but rather result from a long run evolutionary process.

The problem of the evolution of conventions in large populations has often been considered under a particular assumption concerning *neighborhood structures* or *local interaction structures* in the populations (see, for example, Berninghaus/Schwalbe, 1996a, 1996b, Eshel et al., 1998, Blume, 1993). In such a framework a member of the population is not supposed to be randomly matched with any other member of the population but he is only matched with members of his neighborhood which is a proper subset of the whole population. The neighborhoods of the players constitute a local interaction structure or sometimes called a *network structure* on the population. Much of research in this field has been devoted to populations with exogenously fixed local interaction structures imposed on the population.[1] In recent research this restrictive assumption has been relaxed and players were allowed to choose their neighbors in each period by themselves (see, Bala/Goyal, 2000, Goyal/Vega-Redondo, 2002, Jackson/Wolinsky, 1996). In these models local interaction structures are regarded as a result of individual equilibrium decisions and not as being imposed exogeneously.

In our paper we consider two types of 2×2 coordinations games which are played in a population, that is, *pure coordination games* and *Hawk/Dove games* as well. In pure coordinations games equilibria in pure strategies are characterized by the requirement that players choose the same strategy, while Hawk/Dove games equilibria in pure strategies are characterized by the requirement that both players choose different strategies (aymmetric equilibria). Hawk/Dove games have a long tradition in evolutionary game theory. Maynard Smith and Price

[1]For example, players have been supposed to be located at a circle or at a two-dimensional grid.

(1973) developed their famous equilibrium concept, the *evolutionary stable state* (ESS), for this type of games. In an evolutionary framework one has the following interpretation in mind. Two members of a species are randomly matched to compete for the same territory. If both members choose the Hawk strategy this results in territory fighting with serious wounds for both. If they choose the Dove strategy they share the territory after some kind of ritual fighting. The only Nash equilibria in pure strategies are the asymmetric strategy configurations (Hawk, Dove) resp. (Dove, Hawk). The only symmetric equilibrium is the mixed strategy equilibrium which can be shown to be the unique ESS of the game.

Coordination in large populations with pure coordination games has been extensively studied during the past decade (see, for example, van Huyck/Battalio/Beil, 1990, Cooper, 1999). We do not know of comparable studies for Hawk/Dove games. We argue that successfull coordination in large populations is much more interesting in Hawk/Dove games than in pure coordination games since each player wants to be matched only with players who employ just the opposite strategy. First experimental results (see Berninghaus/Vogt, 2003a) support our view.

In our paper we analyze which types of equilibrium networks prove to be compatible with the underlying coordination game when players are allowed to select simultaneously their neighbors in the population **and** the strategy in the base game. Decision making is supposed to be deterministic. Opening a new link to a member of the population is supposed to generate constant connection costs per link for the agent who initiates the link. It is easy to see that the relative size of linking costs compared with the payoffs of the 2×2 game has a big impact on the resulting equilibrium network in the population. Similar work on this topic has been done by Goyal/Vega-Redondo (2002). They concentrate on pure coordination games and, furthermore, analyze the *stochastic stable states* of the process of network formation for pure coordination games by introducing mutation at the individual level of strategy and partner choice. Our model is purely static. We consider simultaneous network linking choice and action choice in the coordination game to be elements of a one shot game. It is the main aim of our study to study the impact of the particular coordination base game on the resulting equilibrium network structure. The resulting equilibrium networks are characterized by non directed graphs. Depending on the particular value of linking costs we obtain different graphs for pure coordination and Hawk/Dove games. Our work is an extension of pure network formation approaches (e.g., Bala/Goyal, 2000) in which only network decisions are considered abstracting from any other strategic decisions in 2×2 base games. And it is also in some sense an extension of Goy0al/Vega-Redondo's results (2002) since we also consider Hawk/Dove games as coordination base games in the process of network formation.[2]

[2]Note, however, that the goal of Goyal/Vega-Tedondo's paper to deduce stochastically stable states is different from ours.

2 Model description and results

2.1 Hawk/Dove games

We consider a set $I = \{1, \ldots, n\}$ of n agents who are engaged in playing a Hawk/Dove game with each of his neighbors who are linked via a network of players (=local interaction structure). If two players i and j are linked with each other they play the Hawk/Dove game for one period. The Hawk/Dove game is a symmetric 2×2 normal form game $\Pi_{HD} = \{\Sigma, H(\cdot)\}$ with $\Sigma := \{X, Y\}$ which is characterized by the payoff table

	X	Y
X	d,d	a,c
Y	c,a	b,b

with $a > b > c > d > 0$, i.e. Y is called the "dove strategy" and X is called the "hawk strategy".

We do not impose a fixed network structure on the population of players but assume that networks can be built up by individual decision making. More precisely, we assume that all players participate in a *network game*. An individual strategy in the network game of player i is a vector of ones and zeros $g_i \in \{0, 1\}^{n-1}$. We say that player i wants to establish a link to player j if $g_{ij} = 1$, otherwise it is equal to zero. A link between two players allows both players to play the simple Hawk/Dove game Π_{HD}. Note, that a bilateral connection between two players is supposed to be already established if at least **one** player wants to open it.[3]

Each strategy configuration $g = (g_1, \ldots, g_n)$ generates a directed graph denoted by $\mathcal{G}_g$, where the vertices represent players and a directed edge between i and j, i.e. $g_{ij} = 1$, signals that i plans to open a link with j. In the following we will ignore the orientation of the edges and regard the graph as non directed graph. The neighbors of player i given a network $\mathcal{G}_g$ is defined to be the set of players to whom i wants to open a link (active neighbors, $g_{ij} = 1$) and the players who want to open a link with i (passive neighbors, $g_{ji} = 1$). By defining $\bar{g}_{ij} := \max\{g_{ij}, g_{ji}\}$ we sinply define the set of neighbors as follows

$$N_i(\mathcal{G}_g) := \{j \mid \bar{g}_{ij} = 1\}.$$

[3]This assumption and its extensions have been extensively discussed in the literature on network formation (see Bala/Goyal, 2000, and Jackson/Wolinsky, 1996). At a first glance it seems to be strange that a player has to accept the offer of any other player to play with him. However, this assumption simplifies the model considerably and, moreover, Berninghaus/Vogt (2003b) show that by this assumption one obtains results which do not significantly differ from a model in which both players have to agree befor a link is opened. This is at least true in the model framework we consider in this paper. Moreover, Goyal/Vega-Redondo (2002) argue that *positive payoffs* in the base game suffice to induce a rational player to agree to play with a partner who just opened a link to him. By refusing to play with him he would actually lose payoff.

The cardinality of this set is given by $n_i(\mathcal{G}_g) := |N_i(\mathcal{G}_g)|$.

Obviously, the set of neighbors need not concide with the set of active neighbors which only depends on i's strategy network strategy vector g_i. We define set of active neigbors of i as follows

$$N_i^a(g_i) := \{j \mid g_{ij} = 1\}.$$

The cardinality of the set of direct neighbors is defined by $n_i^a(g_i) := |N_i^a(g_i)|$.

We suppose that it is not costless to establish a link with another player. Therefore, total payoff of player i in the network game is composed of the aggregate payoff player i can extract from playing with her neighbors and the costs of establishing links to her active neighbors. Let k denote the constant linking costs. Since the payoff player i can extract from playing the Hawk/Dove game depends on his own strategy choice, the strategy choice of his (direct) neighbors and the network generated by g we define

$$\Pi_i^X(\sigma_{-i}, g) := a \sum_{j \in N_i(\mathcal{G}_g)} 1_{\sigma_j = X} + b \sum_{j \in N_i(\mathcal{G}_g)} 1_{\sigma_j = Y} - k n_i^a(g_i),$$

$$\Pi_i^Y(\sigma_{-i}, g) := c \sum_{j \in N_i(\mathcal{G}_g)} 1_{\sigma_j = X} + d \sum_{j \in N_i(\mathcal{G}_g)} 1_{\sigma_j = Y} - k n_i^a(g_i),$$

where $\Pi_i^X(\cdot)$ resp. $\Pi_i^Y(\cdot)$ denotes the payoff a player choosing X resp. Y can gain and $\sigma = (\sigma_{-i}, \sigma_i)$ denotes the vector of actions $\sigma_i \in \{X, Y\}$ for the H/D game. An important consequence of our payoff definition is player i may benefit from a connection to j although she has not to pay for it (that is, $g_{ij} = 0$, but $g_{ji} = 1$).

For the following we use the convention concerning payoff:

> $\Pi_i^{(\cdot)}(\cdot, g) \equiv 0$ if the set of i's neighbors $N_i(\mathcal{G}_g)$ is empty. More generally, if a strategy configuration g generates the *empty network*, i.e. a graph $\mathcal{G}_g$ in which all vertices are isolated then the payoff of each player is equal to zero.

We model the strategic situation of a player in a population as a non-cooperative game in which individual strategies are composed of the simultaneous choice of neighbors $i \in I$ and actions $\sigma_i \in \{X, Y\}$ in the bilateral H/D game. That is, we consider a non-cooperative game in normal form

$$\Gamma = \{S_1, \ldots, S_n, P_1(\cdot), \ldots, P_n(\cdot)\}$$

with $S_i := \{0, 1\}^{n-1} \times \{X, Y\}$ and

$$P_i : S_1 \times \ldots \times S_n \longrightarrow I\!R$$

where

$$P_i(s) := \Pi_i^{\sigma_i}(\sigma_{-i}, g).$$

Each strategy configuration $s = (s_1, \ldots, s_n)$ in Γ induces a network represented by a non directed graph $\mathcal{G}_{\}}$.

It remains to consider which network structures $\mathcal{G}_g$ and action configurations σ in Γ will prove to be stable ? Our notion of stability is purely non-cooperative. Therefore, we use a canonical extension of the Nash concept to our model of endogenous network formation.

Definition 1 *The strategy configuration $s^* = (g^*, \sigma^*)$ in Γ is an equilibrium if*

$$\forall i: \quad P_i(s^*_{-i}, s^*_i) \geq P_i(s^*_{-i}, s_i) \text{ for } s_i \in S_i.$$

In an equilibrium no player has an incentive either to change his neighbors or to change his action choice σ^*_i unilaterally.

It follows immediately from the equilibrium definition that we need not consider configurations $g = (g_1, \ldots, g_n)$ in which two players simultaneously want to open a bilateral link with each other, that is, if the relations $g_{ij} = g_{ji} = 1$ hold for two players $i, j \in I$. In this case either i or j could improve her payoff by dropping the link and thereby saving linking costs k. Such networks can never be equilibrium candidates. Therefore, we concentrate on so called *simple networks* as equilibrium candidates.

Definition 2 *A strategy configuration $s = (g, \sigma)$ is called simple if the following relation holds*

$$\forall i, j: \ \bar{g}_{ij} = 1 \implies g_{ij} \cdot g_{ji} = 0.$$

A simple strategy configuration s always generates a simple non directed graph.[4]

In the following theorem we characterize equilibrium network structures and action configurations in a population playing the Hawk/Dove game. We will see that in most cases it suffices to consider networks represented by non directed graphs where the orientation of the underlying bilateral links can be in either direction.

Theorem 1 *Given a HD-game Γ, then the following statements hold:*

a) If $k > a$ then the unique equilibrium network $\mathcal{G}_{g^*}$ is the empty network and the action choice of each player in the Hawk/Dove game is not determined.

b) If $k < d$ then the unique equilibrium network $\mathcal{G}_{\bar{g}^*}$ is the complete graph. In the complete graph no uniform choice of either X or Y is possible as an equilibrium action choice. Let n^*_X resp. n^*_Y denote the number of players

[4] A non directed graph is called to be simple if it has no loops and each pair of vertices is connected by at most one edge.

choosing X resp. Y as an equilibrium choice in the complete network then these numbers are determined by the relation

$$\frac{n(a-b)-(a-b)}{a-b+c-d} < n_X^* < \frac{n(a-b)+(c-d)}{(a-b+c-d)} \tag{1}$$

and $n_Y^* = n - n_X^*$.

c) *If the relations* $d < k < c$ *hold then an equilibrium network* $\mathcal{G}_{g^*}$ *is a graph whose vertices can be partitioned into two non-empty sets* I_1 *of X-players and* I_2 *of Y-players such that all vertices in* I_1 *are connected with all vertices in* I_2 *but not with each other while all vertices in* I_2 *are also connected with each other. Again uniform action choice is not possible in equilibrium and the number of players choosing X* (n_X^*) *has to satisfy the condition*

$$\frac{n(a-b)-(a-b)}{a-b+c-d} < n_X^* < \frac{n(a-b)+(c-k)}{a-b+c-k} \tag{2}$$

and $n_Y^* = n - n_X^*$.

d) *If the relation* $c < k < b$ *holds then* $\mathcal{G}_{g^*}$ *is a bipartite graph where each vertex in a set* I_1 *(X-players) is connected with all vertices in* I_2 *(Y-players) but not with vertices in* I_1*, while all vertices in* I_2 *are also connected with each other. Furthermore, Y players do not have active links with X players. Again no uniform action choice can be part of an equilibrium.* n_X^* *has to satisfy the condition*

$$n_X^* > \frac{n(a-b)-(a-b)}{a-b+c-d}. \tag{3}$$

e) *If* $b < k < a$ *then an equilibrium network* $\mathcal{G}_{g^*}$ *is characterized either by a bipartite graph with* $n_X^*, n_Y^* > 0$ *which is characterized by the following property: only X players in* I_1 *have direct links to Y players in* I_1*, that is,* $g_{ij}^* = 1$ *for* $i \in I_1$ *and* $j \in I_2$ *while we have* $g_{ji}^* = 0$ *for* $j \in I_2, i \in I_1$ *and, furthermore,* $g_{jm}^* = 0$ *for* $j, m \in I_2$ *resp.* $j, m \in I_1$*. Or the equilibrium network is the empty graph where* $n_X^* = n$*.*

Proof: a) Suppose there exists at least one link between two players i and j, i.e. $\bar{g}_{ij}^* = 1$. Since k is supposed to be larger than the maximum payoff a player can gain from the Hawk/Dove game the net payoff from each link is negative irrespectively of the individual action choices in the Hawk/Dove game. Therefore, establishing no link to any other player results in maximum individual payoff equal to 0 according to our convention on payoffs. Consequently, payoff is independent of a player's action choice and, therefore, action choice is not determined.

b) First let us suppose that $k < d$. Since opening a new connection to either an X or a Y player results in positive net payoffs it pays to open as many links as possible where it has to be taken into account that a player i should only open a link to j if $g_{ji} = 0$, otherwise payoffs do not have the Nash property.

Obviously, no uniform action configuration ($n_X^* = n$ or $n_Y^* = n$) is in equilibrium, since any player i could switch to the opposite action and increase her payoff. Now let us consider the case $n_X^*, n_Y^* > 0$. For a player choosing X the payoff has to be higher than for choosing Y. Then the following condition has to be satisfied[5]

$$(n_X^* - 1)d + n_Y^* a > (n_X^* - 1)c + n_Y^* b \quad \Longleftrightarrow \quad (n_X^* - 1)(c - d) < n_Y^*(a - b).$$

Analogously, for a player choosing Y the following inequality holds

$$n_X^* d + (n_Y^* - 1)a < n_X^* c + (n_Y^* - 1)b \quad \Longleftrightarrow \quad n_X^*(c - d) > (n_Y^* - 1)(a - b).$$

By substituting $n_Y^* = n - n_X^*$ we obtain from these inequalities the relations

$$n_X^* < \frac{n(a - b) + (c - d)}{a - b + c - d}$$

and

$$n_X^* > \frac{n(a - b) - (a - b)}{a - b + c - d}$$

which are equivalent to condition (1).

c) Suppose inequality $d < k < c$ holds. Then it will not pay for an X-player to be connected with other X-players since it will give him negative payoff. However, a Y-player may open as many connections as possible provided there is not already another player who opened a link to him. Therefore, the resulting graph can be partitioned into two sets of vertices I_1 (X-players) whose elements are connected with each vertex of I_2 (Y-players) and each element of I_2 is, moreover, connected with each other member of I_2.

Uniform action choice is not possible in equilibrium since any player could benefit from either switching from X to Y resp. from Y to X. To determine the equilibrium number n_Y^* of Y resp. n_X^* of X players we first consider the decision problem of a X player who can switch from X to Y *and* open new connections to the remaining n_X^* X players. This will not be profitable for a X player if the following condition holds

$$n_Y^* a - n_i(g_i^*)k > n_Y^* b - n_i(g_i^*)k + [(n_X^* - 1)c - (n_X^* - 1)k],$$

where the expression in "[...]" brackets denotes the net benefits from opening (as a Y player) links to the remaining X players.[6] This condition is equivalent to

$$n_X^* < \frac{n(a - b) + c - k}{a - b + c - k}.$$

[5]Since the equilibrium network is the complete graph we omit the connection cost in this case.

[6]Note that we required to check the additional payoffs of opening new links to *all* remaining X players. Because of $(c - k) > 0$ this suffices to make it not profitable to open new links to a smaller number of X players than $(n_X^* - 1)$.

Since all Y players are connected with the rest of the population, Y will be an equilibrium action choice when it satisfies the condition

$$n_X^* d + (n_Y^* - 1)a < n_X^* c + (n_Y^* - 1)b \quad \Longleftrightarrow \quad n_X^* > \frac{n(a-b) - (a-b)}{a-b+c-d}.$$

Both inequalities for n_X^* together imply condition 2.

d) Now suppose that $c < k < b$ holds. Then it follows from the arguments of the proof of part c) that each X-player will only connect to a Y-player. Y players, however, will only connect with Y players since any other link will give them negative payoffs. Therefore, the graph generated by g^* is a bipartite graph where the whole set of vertices can be partitioned into two non-empty sets I_1 (X-players) and I_2 (Y-players) such that all elements of I_1 are connected only to elements of I_2 and all elements of I_2 are connected with one another. Obviously, no uniform action choice can be part of an equilibrium.

For a X player it is not profitable to deviate to Y since the relation

$$n_Y^* a - n_i(g_i^*)k > n_Y^* b - n_i(g_i^*)k$$

holds (because of $a > b$). For a Y player the following condition has to be satisfied[7]

$$n_X^* c + (n_Y^* - 1)b - n_i(g_i^*)k > n_X^* d + (n_Y^* - 1)a - n_i(g_i^*)k$$

which can easily be transformed to condition 3.

e) Consider a Y player. His maximum payoff is equal to b which can be reached by playing with other Y players. However, because of $k > b$ a Y player cannot extract positive net payoffs from any connection (with either a X or Y player). Therefore, in an equilibrium network Y players cannot have active neighbors. The maximum payoff of a X player is equal to a which can only be reached by being matched with a Y player. Furthermore, X can extract only a positive net payoff from being linked to a Y player (as his active neighbor). By being linked to another X player he obtains a negative net payoff. Therefore, the only candidate for an equilibrium network is a graph where the vertice set can be partioned into two sets I_1 (of X players) and I_2 (of Y players).

Obviously, uniform choice of Y is not possible in equilibrium since each player can benefit from switching to X and build up links to at least one of the remaining Y players. However, uniform choice of X is only possible when the resulting equilibrium network is the empty graph.

[7]Note that $n_i(g_i^*)k$ denotes the linking costs with the remaining Y players. A Y player need only consider action switching from Y to X. It does not pay to change the links with the remaining players.

q.e.d.

Remarks A) Note that in part b) of the theorem a complete graph may be generated by many different individual Nash configurations g which only have to be simple. As an extreme case one could consider a complete graph in which there is one player who need not open any link with the remaining $(n-1)$ players since they all want to be linked with him. Indeed, one can easily check that this is also an equilibrium network. Similar conclusions hold for the equilibrium networks in part c) and d) of the theorem. When a link between two players exists it does not matter which player opened the link. Consequently, in equilibrium networks payoffs may vary significantly from one player to the next even if they choose the same action.

B) In part c), d) and e) of the theorem we characterize equilibrium constellations s^* by the type of graph which is generated by g^*. Obviously, there exist many ways how to partition the players' set I into two non-empty sets I_1 (X players) and I_2 (Y players). Therefore, there exist finitely many action choice equilibria where the players, however, all are connected by the same type of network.

C) The result of d) shows that all dove players are connected with each other but do not have active links to hawk players. One could call this situation a "dove network" which is stabilized by some hawk "invaders". Note that the doves in this network are playing non-equilibrium strategies of the one shot HD game. This demonstrates drastically that the strategic decision problem in a one shot game changes when considered as a part of a more general network decision problem.

D) Our result in part e) has an interesting economic interpretation. We see that part of the population (X-players in I_1) is subsidized by the remaining part (Y-players in I_2). X players benefit from playing with Y players without bearing the linking costs. This is due to our particular assumption on establishing links and the supposed restriction on linking costs k.

E) It is well known in the theory of network formation (see Bala and Goyal, 2000, Goyal and Vega-Redondo, 2002) that so called *star shaped* network structures may be stable equilibria in a particular model of strategic neighbor's choice. In a star shaped structure one player (the "center" of the star) is connected with the rest of the population and the remaining players are exclusively linked to the "center" player. Our bipartite (directed) graph can be interpreted as a generalization of the star shaped structure such that in our structure we have finitely many (≥ 1) "center" players.

2.2 Pure coordination games

In this section we assume that if two players i and j are linked with each other they play a pure coordination game. The symmetric 2×2 pure coordination game is a symmetric normal form game $\Pi_C = \{\Sigma, H(\cdot)\}$ with $\Sigma := \{X, Y\}$ which is characterized by the payoff table

	X	Y
X	b,b	c,d
Y	d,c	a,a

with $a > b > c > d > 0$ and $(b-d) > (a-c)$, i.e. (Y, Y) is the payoff dominant equilibrium and (X, X) is the risk dominant equilibrium.

As in the previous section we do not impose a fixed network structure on the population of players but assume that networks can be built up by individual decisions. All members of the population are supposed to be players of the *network game* Γ which has already been introduced in the previous section. It has the same formal structure as before only the restrictions concerning the numerical payoffs of the 2×2 base game are changed. It will be demonstrated by the results of the following theorem that the resulting equilibrium network structures will be completely changed by altering the payoff structure of the base game from a Hawk/Dove game to a pure coordination game.

Theorem 2 *Given a network game Γ where the underlying 2×2 game is a pure coordination game.*

a) If $k > a$ then the unique equilibrium network $\mathcal{G}_{g^*}$ is the empty network and the action choice of each player in the pure coordination game is not determined.

b) If $k < d$ holds then the unique equilibrium network $\mathcal{G}_{g^*}$ is the complete graph and σ^* is given either by $\sigma^* = (X, \ldots, X)$ or by $\sigma^* = (Y, \ldots, Y)$. That is, either all players choose the equilibrium strategy X or all players choose the equilibrium strategy Y.

c) If $d < k < c$ holds then either the equilibrium is the one obtained in part b) or there is an equilibrium configuration s^* such that the resulting network is characterized by a graph $\mathcal{G}_{g^*}$ with two sets of vertices I_1 (X players) and I_2 (Y players) such that $g^*_{ij} = 1$ for $i \in I_1$ and $j \in I_2$ and $\bar{g}^*_{jm} = 1$ for $j, m \in I_1$ resp. $\bar{g}^*_{jm} = 1$ for $j, m \in I_2$. In this case the numerical payoffs of the base game and the n^*_X the number of players choosing X and n^*_Y the number of players choosing Y has to satisfy the relation

$$\frac{n(a-c+d-k) + (b-d) + n_i(g^*_i)(k-d)}{a-c+b-k} < n^*_X < \frac{n(a-c) - (a-c)}{a-c+b-d} \tag{4}$$

for all X players $i \in I_1$.

d) If the relation $c < k < b$ holds then either the equilibrium is the one obtained in part b) or an equilibrium configuration s^* induces a disconnected graph $\mathcal{G}_{g^*}$ with two components where each component is a complete graph and players in one component (I_1) choose action X, players in the other component (I_2) choose action Y. The number of X players n_X^* has to satisfy the condition

$$\frac{n(a-k)+(b-d)+n_i(g_i^*)(k-d)}{a+b-k-d} < n_X^* < \frac{n(a-c)-(a-c)+n_j(g_j^*)(c-k)}{a+b-k-c} \tag{5}$$

for all $i \in I_1$ and $j \in I_2$.

e) If the relation $b < k < a$ holds then an equilibrium network $\mathcal{G}_{g^*}$ is the complete graph with all players choosing Y. An alternative equilibrium is the empty graph with all players choosing X.

Proof: a) Same arguments as in theorem 1 hold.

b) The arguments concerning equilibrium network formation for $k < d$ are analogous to theorem 1. Since each connection to another player increases a player's payoff he tries to open as many links as possible, provided the resulting network is a simple one. Therefore, the complete graph is the only candidate for an equilibrium network. If all players choose X or Y this is obviously an equilibrium action choice. It can easily be seen that $n_X > 0$ and $n_Y > 0$ is not compatible with an equilibrium choice. For, suppose $n_X, n_Y > 0$, then for an X resp. Y-player the following relations must hold

$$\begin{aligned}
(n_X^*-1)b + n_Y^*c > (n_X^*-1)d + n_Y^*a \quad &\Longleftrightarrow \quad (n_X^*-1)(b-d) > n_Y^*(a-c) \\
&\Longleftrightarrow \frac{n_X^*-1}{n_Y^*} > \frac{(a-c)}{(b-d)}, \\
n_X^*b + (n_Y^*-1)c < n_X^*d + (n_Y^*-1)a \quad &\Longleftrightarrow \quad n_X^*(b-d) < (n_Y^*-1)(a-c) \\
&\Longleftrightarrow \frac{n_X^*}{n_Y^*-1} < \frac{(a-c)}{(b-d)},
\end{aligned}$$

which implies

$$\frac{n_X^*}{n_Y^*-1} < \frac{n_X^*-1}{n_Y^*},$$

a contradiction.

c) Suppose that $d < k < c$ holds. It is obvious that the equlibria of part b) are also equilibria in this case. If some players select X and some select Y the following holds. A Y player makes positive profits when he is connected with another Y player while he extracts negative payoffs ($(d-k) < 0$) when opening a link to a X player. However, he benefits if a X player wants to open a link with him. On the other hand, a X player benefits from opening as many links as possible (with X and Y players as well).

It only remains to be shown that there is no incentive for an X player or Y player to deviate from his strategy choice. For a X player i it is not profitable for him to switch to Y and dropping his links with X players if the following inequality holds

$$(n_X^* - 1)b + n_Y^* c - (n_i(g_i^*) - n_Y^*)k - n_Y^* k > (n_X^* - 1 - (n_i(g_i^*) - n_Y^*))d + n_Y^* a - k n_Y^*$$

where $(n_i(g_i^*) - n_Y^*)$ is the number of direct links a X player has established to other X players. This inequality is equivalent to the inequality

$$n_X^* > \frac{n(a - c + d - k) + (b - d) + n_i(g_i^*)(k - d)}{(a - c + b - k)}.$$

For a Y player the following condition has to be satisfied

$$n_X^* d + (n_Y^* - 1)a - n_i(g_i^*)k > n_X^* b + (n_Y^* - 1)c - n_i(g_i^*)k$$

which is equivalent to the inequality

$$n_X^* < \frac{n(a - c) - (a - c)}{a - c + b - d}.$$

Both restrictions on n_X^* imply condition 4.

d) As in case b) and d) the complete graph with all players selecting either X or Y is an equilibrium. If some players select X and some select Y the following holds. For X players it is profitable to build up as many links as possible with other X players. The same argument holds for Y players. All other links result in a payoff loss (either $(c - k) < 0$ or $(b - k) < 0$). In order to have no incentive for a X player to deviate it suffices to consider the effects of an action switch from X to Y together with opening links to all Y players. Such a deviation is not profitable when the inequality[8]

$$(n_X^* - 1)b - n_i(g_i^*)k > n_Y^* a - n_Y^* k + (n_X^* - 1 - n_i(g_i^*))d$$

is satisfied which is equivalent to

$$n_X^* > \frac{n(a - k) + (b - d) + n_i(g_i^*)(k - d)}{a + b - k - d}.$$

[8]In calculating the payoff generated by deviation not that a) building up new links to all Y players generates communication costs equal to $n_Y^* k$ and b) a deviating X player has still $(n_X^* - 1 - n_i(g_i^*)$ X players who have active links with him. From each of these players he will extract an individual payoff of equal to d.

For a Y player an analogous restriction holds

$$(n_Y^* - 1)a - n_j(g_j^*)k > n_X^* b - n_X^* k + (n_Y^* - 1 - n_j(g_j^*))c$$

which can equivalently be transformed into

$$n_X^* < \frac{n(a-c) - (a-c) + n_j(g_j^*)(c-k)}{a+b-k-c}.$$

Both restrictions on n_X^* imply condition 5.

e) Since $a > k > b$ it only pays for an individual player to choose Y and to look for as many Y-player connections as possible. However, this argument only works when at least one player in the population selects Y. If all players choose X the best reply of each individual player is to shut down all connections with the remaining players.

q.e.d.

Remarks A The result in part d) and e) of theorem 2 seems to have some features in common with the literature on "equilibrium selection by migration" (e.g., Ely, 1995, Bhaskar/Vega-Redondo, 1996). In these models players can move to different locations where they play a simple 2×2 coordination game with each player at the same location. As a main result it can be demonstrated that all players move to the same location where the payoff dominant equilibrium will be played provided the migration costs are low enough. In our framework, we have the opposite implications of communication costs. Moderate communication costs prevent players from coordination failure and let players build up isolated groups in which they choose the same action. If communication costs are large enough such that they make coordination on X not profitable then players select the payoff dominant equilibrium in one completely connected group.

B) We know from theoretictal and experimental work on equilibrium selection in coordination games that there exist many situations in which players do **not** select the payoff dominant equilibrium in coordination games (see, for example, Kandori/Mailath/Rob, 1993, Berninghaus/Schwalbe, 1996a, Cooper, 1999). In our framework it can be guaranteed that the payoff dominant equilibrium in the coordination game is selected if connection costs are "large enough." This is made precise by the condition on k in part e) of theorem 2.

3 Concluding remarks

Our results in theorems 1 and 2 show drastically the impact of communication costs and the type of the base game on network formation. As a main conclusion we state that one cannot separate the network linking decision from the action decision in a particular base game. Strategy choice in a population of players crucially depends on the communication (=linking) costs **and** the numerical payoff constellations in the base game.

Our framework is still rather simple and completely static. It should be extended in many ways. First, it is certainly not very realistic to assume that players will find an equilibrium in such a complicated one-shot network game Γ in one period. We need to find an extension of our simple static model to a dynamic strategy adaptation process in which players change their decisions (network and actions decisions) according to some well defined adaptation rules. One could also incorporate into such a dynamic process the more or less plausible assumption that players show a lower speed of adaptation in their network decisions than in their action decisions (in the base game). For example, one could model that neighborhood decisions are revised every 5th period while action decisions may be changed in every period.

In communication network games it is often assumed that one player can reach many players by one active link to another player provided this player is connected with many other players via active or passive links. It is not easy to justify such an assumption in game networks, where all relationsships are only bilateral (at least for two person games). Nevertheless, it seems to be interesting to "experiment" with this assumption. One could assume, for example, that opening a link to a player would guarantee access to his "club" of active and passive neighbors. A similar assumption has been made in pure communication networks (Berninghaus/Ott/Vogt, 2003). Many other variants of access to a player's neighborhood is possible.

Finally, one can think of substituting the simple 2×2 symmetric coordination games by more complex ones, either by asymmetric coordination games or by $N \times N$ coordinations games (for $N > 2$). Which equilibrium networks could be expected in such models ?

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