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Risk Measures

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1. Introduction

The literature on risk is vast[1]. So is the literature on risk measures. This contribution concentrates[2] on financial risks and on financial risk measures. We assume that the financial consequences of economic activities can be quantified on the basis of a random variable X. This random variable may represent for instance the absolute or relative change (return) of the market or book value of an investment, the periodic profit or the return on capital of a company (bank, insurance, industry), the accumulated claims over a period for a collective of insureds or the accumulated loss for a portfolio of credit risks. In general, we look at financial positions where the random variable X can have positive (gains) as well as negative (losses) realizations. Pure loss situations can be analysed by considering the random variable $S := -X \geq 0$. Usually only process risk is considered ([4, p. 207], speak of model-dependent measures of risk in this context), in some cases also parameter risk (model-free measures of risk)[3].

2. Risk as a Primitive[4]

Measuring risk and measuring preferences is not the same. When ordering preferences, acitivities, e.g. alternatives A and B with financial consequences X_A and X_B, are compared with respect to preferability under conditions of risk. A preference order $A \succ B$ means that A is preferred to B. This order is represented by a preference function Φ with $A \succ B \Leftrightarrow \Phi(X_A) > \Phi(X_B)$. In contrast, a risk order $A \succ_R B$ means, that A is riskier than B and is represented by a function R with $A \succ_R B \Leftrightarrow R(X_A) > R(X_B)$. Every such function R is called a *risk measurement function* or simply a *risk measure*.

A preference model can be developed without explicitely invoking a notion of risk. This, for instance, is the case for expected (or von Neumann/Morgenstern) utility theory. On the other hand, one can establish a preference order by combining a risk measure R with a measure of value V and specifying a (suitable) trade-off function H between risk and value, i.e. $\Phi(X) = H[R(X),V(X)]$. The traditional example is Markowitz portfolio theory, where $R(X) = Var(X)$, the variance of X, and $V(X) = E(X)$, the expected value of X. For such a risk-value model[5] then it is an important question, whether it is consistent (implies the same preference order) with e.g. expected utility theory. As we focus on risk measures we will *not* discuss such consistency results (e.g. whether Markowitz portfolio theory is consistent with expected utility theory) and refer to the relevant literature[6].

3. Two Types of Risk Measures

The vast number of (financial) risk measures in the literature can be broadly subsumized to two categories:

1) Risk as the magnitude of deviations from a target (risk of the first kind).
2) Risk as necessary capital respectively necessary premium (risk of the second kind).

In many central cases there is an intuitive correspondence between these two types of risk conception. Addition of E(X) to a risk measure of the second kind will induce a risk measure of the first kind and subtraction of E(X) from a risk measure of the first kind will induce a risk measure of the second kind (remember that we are considering profit positions, not claims positions). The formal aspects of this correspondence will be discussed in section 5.3 and some specific examples are given in sections 6 and 7.

4. Risk Measures and Utility Theory

As expected utility theory is the standard theory for decisions under risk, one may ask the question, whether there are conceptions of risk which can be derived from utility theory. Formally, the corresponding preference function is of the form $\Phi(X) = E[u(X)]$, where u denotes the utility function (specific to every decision maker). The form of the preference function implies that there is no separate measurement of risk or value within utility theory, both aspects are considered simultaneously. However, is it possible to derive an explicit risk measure for a specified utility function? An answer is given by the contribution [23] of Jia and Dyer introducing the following *standard measure of risk*:

$$R(X) = - E[u(X-E(X))] \ . \qquad (1)$$

Risk therefore corresponds to the negative expected utility of the transformed random variable $X - E(X)$, which makes risk measurement location-free. Specific risk measures are obtained for specific utility functions. Using for instance the utility function $u(x) = ax - bx^2$, we obtain the *variance*

$$Var(X) = E[(X - E(X))^2] \qquad (2)$$

as the corresponding risk measure. From a cubic utility function $u(x) = ax - bx^2 + cx^3$ we obtain the risk measure

$$Var(X) - cM_3(X) \ , \qquad (3)$$

where the variance is corrected by the magnitude of the third central moment $M_3(X) = E[(X - E(X))^3]$. Applying the utility function $u(x) = ax - |x|$, we obtain the risk measure *mean absolute deviation*

$$MAD(X) = E[|X - E(X)|] \quad . \tag{4}$$

In addition (under a condition of *risk independence*) the approach of Jia and Dyer is compatible with risk-value-models.

5. Axiomatic Characterizations of Risk Measures

5.1 The System of Pedersen and Satchell (PS)

In the literature there are a number of axiomatic systems for risk measures. We begin with the system of PS [31]. The axioms are:

(PS 1)	(nonnegativity)	$R(X) \geq 0$
(PS 2)	(positive homogeneity)	$R(cX) = cR(X)$ for $c \geq 0$
(PS 3)	(subadditivity)	$R(X_1 + X_2) \leq R(X_1) + R(X_2)$
(PS 4)	(shift-invariance)	$R(X + c) \leq R(X)$ for all c .

PS understand risk as deviation from a location measure, so $R(X) \geq 0$ is a natural requirement. Homogeneity (PS 2) implies that the risk of a certain multiple of a basic financial position is identical with the corresponding multiple of the risk of the basic position. Subadditivity (PS 3) requires that the risk of a combined position is as a rule less than the sum of the risks of the separate positions[7]. This allows for diversification effects in the investment context and for pooling-of-risks-effects in the insurance context. Shift-invariance (PS 4) makes the measure invariant to the addition of a constant to the random variable, which corresponds to the location-free conception of risk. (PS 2) and (PS 3) together imply that zero risk is assigned to constant random variables. Also, (PS 2) and (PS 4) together imply that the risk measure is convex, which assures for instance compatibility to second order stochastic dominance. As risk is understood to be location-free by PS, their system of axioms is ideally suited for examining risk measures of the first kind for attributes of goodness, cf. section 6.4.

5.2 The System of Artzner/Delbaen/Eber/Heath (ADEH)

The system of ADEH [4] has been a very influential approach. In addition to subadditivity (ADEH 1) and positive homogeneity (ADEH 2) they postulate the axioms[8] (remember that profit positions, not claims positions, are considered):

(ADEH 3) (translation invariance) $R(X + c) = R(X) - c$ for all c

(ADEH 4) (monotonicity) $X \leq Y \Rightarrow R(Y) \leq R(X)$.

A risk measure satisfying these four axioms is called *coherent*. In case of $R(X) \geq 0$ we can understand R(X) as (minimal) additional capital necessary to be added to the risky position X in order to establish a ”riskless position” (and to satisfy regulatory standards for instance). Indeed, (ADEH 3) results in $R(X + R(X)) = 0$. In case $R(X) < 0$ the amount $|R(X)|$ can be withdrawn without endangering safety or violating the regulatory standards respectively. In general, (ADEH 3) implies that adding a sure amount to the initial position decreases risk to that amount. Monotonicity means that if $X(\omega) \leq Y(\omega)$ for every state of nature then X is riskier because of the higher loss potential.

[4] do not only consider[9] the distribution-based case, but also the situation of uncertainty, where no probability measure is given a priori. They are able to provide a representation theorem[10] for coherent risk measures in this case.

Obviously the system of axioms of ADEH is ideally suited for examining risk measures of the second kind for attributes of goodness. In general, the risk measures R(X) = E(-X) and R(X) = max(-X), the maximal loss, are coherent risk measures. This implies that even if the coherency conditions may[11] impose relevant conditions for "reasonable" risk measures, not all coherent risk measures must be reasonable[12].

5.3 The System of Rockafellar/Uryasev/Zabarankin (RUZ) for Expectation-Bounded Risk Measures

RUZ [34] develop a second system of axioms for risks of the second kind. They impose the conditions (ADEH 1-3) and the additional condition

(RUZ) (expectation-boundedness) R(X) > E(-X) for all non-constant X and R(X) = E(-X) for all constant X.

Risk measures satisfying these conditions are called *expectation-bounded*. If, in addition monotonicity (ADEH 4) is satisfied, then we have an expectation-bounded coherent risk measure.

The basic idea of RUZ is, that applying a risk measure of the second kind not to X, but to X – E(X) will induce[13] a risk measure of the first kind and vice versa. Formally there is a one-to-one correspondence between expectation-bounded risk measures and risk measures (of the first kind) satisfying a slightly sharper version of the PS-system of axioms. A standard example[14] for this correspondence[15] is $R_I(X) = b\sigma(X)$ for $b > 0$ and $R_{II}(X) = b\sigma(X) - E(X)$.

5.4 Axioms for Premium Principles and the System of Wang/Young/Panjer (WYP)

In the following we concentrate on the insurance context (disregarding operative expenses and investment income). Two central tasks of insurance risk management are the calculation of risk premiums π and the calculation of the risk capital C (solvency). In contrast to the banking or investment case respectively, both premiums and capital can be used to finance the accumulated claims $S \geq 0$. So we have to consider two cases. If we assume that the premium π is given (e.g. determined by market forces) we only have to determine the (additional) risk capital necessary and we are in the situation of section 5.3 when we look at $X := C_0 + \pi - S$, where C_0 is an initial capital.

A second application is the calculation of the risk premium. Here the capital available is typically not taken into consideration. This leads to the topic of *premium principles* π, which assign a risk premium $\pi(S) \geq 0$ to every claim variable $S \geq 0$. Considering risk of the first kind, especially deviations from E(S), then one systematically obtains premium principles of the form $\pi(S) := E(S) + aR(S)$, where aR(S) is the *risk loading*. Considering on the other hand risk of the second kind and interpreting R(X) as the (minimal) premium necessary to cover the risk $X := -S$, then one systematically obtains premium principles of the form $\pi(S) := R(-S)$.

In mathematical risk theory a number of requirements (axioms) for reasonable premium principles exist[16]. Elementary requirements are $\pi(S) > E(S)$ and $\pi(S) < \max(S)$, the no-ripoff conditon. Translation invariance is treated as well as positive homogeneity. Finally, subadditivity is regarded, although with the important difference[17], that subadditivity is required only for independent risks.

A closed system of axioms for premiums in a competitive insurance market is introduced by WYP [45]. They require monotonicity, certain continuity properties and finally

(WYP) (comonotone additivity) X, Y comonotone[18] $\Rightarrow \pi(X + Y) = \pi(X) + \pi(Y)$.

Under certain additional conditions WYP are able to prove the validity of the following representation for π :

$$p\,(X) = \int_0^\infty g(1 - F(x))\,dx \ . \tag{5}$$

Here F is the distribution function of X and g is an increasing function (*distortion function*) with $g(0) = 0$ and $g(1) = 1$.

Another central result is, that for any concave function the resulting premium principle $\pi(X)$ will be a coherent risk measure[19]. This means that in addition we have obtained an explicit method of constructing coherent risk measures[20].

6. Risk as the Magnitude of Deviation from a Target

6.1 Two-Sided Risk Measures

In the following the expected value will be considered as the relevant target. Two-sided risk measures measure the magnitude of the distance (in both directions) from the realizations of X to E(X). Different functions of distance lead to different risk measures. Looking for instance at quadratic deviations (volatility) this leads to the risk measure variance according to (2) or to the risk measure standard deviation respectively by taking the root, i.e.

$$\sigma(X) = + \sqrt{Var(X)} \quad . \tag{6}$$

Variance or standard deviation respectively have been the traditional risk measures in economics and finance since the pioneering work of Markowitz [28], [29]. These risk measures exhibit a number of nice technical properties. For instance, the variance of a portfolio return is the sum of the variances and covariances of the individual returns. Furthermore, the variance is used as a standard optimization function (quadratic optimization). Finally, there is a well established statistical toolkit for estimating variance and the variance/covariance-matrix respectively.

On the other hand, a two-sided measure contradicts the intuitive notion of risk that only negative deviations are dangerous, it is *downside risk* that matters. In addition, variance does not account for fat tails of the underlying distribution and for the corresponding tail risk. This leads to the proposition[21] to include higher (normalized) central moments as e.g. skewness and kurtosis into the analysis to assess risk more properly. An example is the risk measure (3) considered by Jia and Dyer.

Considering the absolute deviation as a measure of distance, we obtain the MAD-measure according to (4) or the more general risk measures

$$R(X) = E\left[|X - E(X)|^k\right] \qquad \text{resp.} \tag{7a}$$

$$R(X) = E\left[|X - E(X)|^k\right]^{1/k} \quad . \tag{7b}$$

The latter risk measure is considered by Kijima and Ohnishi [25].

More generally, RUZ [34] define a function $f(x) = ax$ for $x \geq 0$ and $f(x) = b|x|$ for $x \leq 0$ respectively with coefficients $a, b \geq 0$ and consider the risk measure[22] ($k \geq 1$)

$$R(X) = E[f(X - E(X))^k]^{1/k} \quad . \tag{8}$$

This risk measure of the first kind allows for a different weighting of positive and negative deviations from the expected value and was already considered by Kijima and Ohnishi [25], too.

6.2 Measures of Shortfall Risk

Measures of shortfall risk are one-sided risk measures and measure the shortfall risk (downside risk) relative to a target variable. This may be the expected value, but in general it is an arbitrary deterministic target z (target gain, target return, minimal acceptable return) or even a stochastic benchmark[23].

A general class of risk measures is the class of *lower partial moments* of degree k (k = 0, 1, 2, …)

$$LPM_k(z; X) := E\left[\max(z - X, 0)^k\right] \quad , \tag{9a}$$

or, in normalized form (k ≥ 2)

$$R(X) = LPM_k(z; X)^{1/k}. \tag{9b}$$

Risk measures of type (9a) are studied by Fishburn [12].

Basic cases, playing an important role in applications, are obtained for k = 0, 1 and 2. These are the *shortfall probability*

$$SP_z(X) = P(X \le z) = F(z), \tag{10}$$

the *shortfall expectation*

$$SE_z(X) = E[\max(z - X, 0)] \tag{11}$$

and the *shortfall variance*

$$SV_z(X) = E\left[\max(z - X, 0)^2\right] \tag{12a}$$

as well as the *shortfall standard deviation*

$$SSD_z(X) = E\left[\max(z - X, 0)^2\right]^{1/2} . \tag{12b}$$

Variations are obtained for z = E(X), for example *lower-semi-absolute deviation* (LSAD), as considered by Ogryczak and Ruszczynski [30] and Gotoh and Konno [19], i.e.

$$\mathrm{R}(\mathrm{X}) = \mathrm{E}[\max\{\mathrm{E}(\mathrm{X}) - \mathrm{X}, 0\}], \tag{13}$$

the *semivariance*

$$R(X) = E\left[\max\{E(X) - X, 0\}^2\right] \tag{14}$$

and the *semi-standard deviation*

$$R(X) = E\left[\max\{E(X) - X, 0\}^2\right]^{1/2} . \tag{15}$$

Another variation of interest is to consider conditional measures of shortfall risk. An important example is the *mean excess loss* (conditional shortfall expectation)

$$MEL_z(X) = E(z - X \mid X \le z) = \frac{SE_z(X)}{SP_z(X)} , \tag{16}$$

the average shortfall under the condition, that a shortfall occurs. The MEL can be considered[24] as a kind of worst-case risk measure.

In an insurance context the MEL is considered in the form $MEL_z(S) = E[S - z \mid S \geq z]$ for an accumulated claim variable $S := -X \geq 0$ to obtain an appropriate measure of right-tail risk[25]. Another measure of right-tail risk can be obtained in the context of section 5.5 when applying the distortion function $g(x) = \sqrt{x}$ (and subsequently subtracting E(S) to obtain a risk measure of the first kind):

$$R(S) = \int_0^\infty \sqrt{1 - F(s)}\, ds - E(S) \quad . \tag{17}$$

This is the *right-tail deviation* considered by Wang [43].

Despite the advantage of corresponding closer to an intuitive notion of risk, shortfall measures have the disadvantage that they lead to greater technical problems with respect to the disaggregation of portfolio risk, optimization and statistical identification.

6.3 Classes of Risk Measures

Stone (1993) defines a general three-parameter class of risk measures of the form

$$R(x) = \left[\int_{-\infty}^{z} (| x - c |)^k \; f(x)\, dx \right]^{1/k} \tag{18}$$

with the parameters z, k and c. The class of Stone contains for example the standard deviation, the semi-standard deviation, the mean absolute deviation as well as Kijima and Ohnishi's risk measure (7b).

More generally, Pedersen and Satchell [31] consider the following five-parameter class of risk measures

$$R(X) = \left[\int_{-\infty}^{z} (|\, x - c\,|)^{a} \; w[F(y)] \;\; f(y)\, dy \right]^{b} \quad , \tag{19}$$

containing Stone's class as well as the variance, the semi-variance, the lower partial moments (9a) and (9b) and additional risk measures from literature as well.

6.4 Properties of Goodness

The following risk measures of the first kind satisfy the axioms of Pedersen and Satchell as considered in section 5.1: standard deviation, MAD, $LPM_k\,(z,X)^{1/k}$ for $z=E(X)$, semi-standard deviation and Kijima/Ohnishi's risk measures (7b) and (8). In addition, Pedersen and Satchell give a complete characterization of their family of risk measures according to section 6.3 with respect to their system of axioms.

7. Risk as Necessary Capital or Necessary Premium

7.1 Value-at-Risk

Perhaps the most popular risk measure of the second kind is value-at-risk[26]. In the following we concentrate on the management of market risks[27]. If we define V_t as the market value of a financial position at time t, $L := v_t - V_{t+h}$ is the potential *periodic loss* of the financial position over the time interval [t, t+h]. We then define the value-at-risk $VaR_\alpha = VaR(\alpha; h)$ at confidence level $0 < \alpha < 1$ by the requirement

$$P(L > VaR_\alpha) = \alpha \quad . \tag{20}$$

An intuitive interpretation of the value-at-risk is that of a probable maximum loss (PML) ore more concrete, a $100(1-\alpha)\%$ maximal loss, because $P(L \leq VaR_\alpha) = 1 - \alpha$, which means that in $100(1-\alpha)\%$ of the cases the loss is smaller or equal to VaR_α. Interpreting the value-at-risk as necessary underlying capital to bear risk, relation (20) implies that this capital will on average not be exhausted in $100(1-\alpha)\%$ of the cases. Obviously the value-at-risk is identical to the $(1-\alpha)$-quantile of the loss distribution, i.e. $VaR_\alpha = F^{-1}(1-\alpha)$, where F is the distribution of L. In addition, one can apply[28] the VaR-concept to $L - E(L)$ instead of L, which results in a risk measure of type I.

Value-at-risk satisfies a number of goodness criteria. With respect to the axioms of Artzner et al. it satisfies monotonicity, positive homogeneity and translation invariance. In addition, it possesses the properties of law invariance and comonotone additivity[29].

As a main disadvantage, however, the VaR is lacking subadditivity and therefore is not a coherent risk measure in the general case. This was the main motivation for establishing the postulate of coherent risk measures. However, for special classes of distributions, the VaR is coherent, for instance[30] for the class of normal distributions (as long as $\alpha < 0.5$). Moreover, the VaR does not take the severity of potential losses in the $100\alpha\%$ worst cases into account. Beyond this, a number of additional criticisms are to be found in the literature[31].

7.2 Conditional Value at Risk

The risk measure c*onditional value-at-risk* at the confidence level α is defined[32] by

$$CVaR_{\alpha}(L) = E[L \mid L > VaR_{\alpha}] \ . \tag{21}$$

Based on the interpretation of the VaR as a 100(1-α)%-maximum loss, the CVaR can be interpreted as the average maximal loss in the worst 100α% cases. The CvaR as defined in (21) is a coherent risk measure in case of the existence of a density function, but not in general[33]. In the general case therefore one has to consider alternative risk measures like the expected shortfall or equivalent risk measures[34], when coherency is to be ensured.

The CVaR satisfies the decomposition

$$CVaR_{\alpha}(L) = VaR_{\alpha}(L) + E[L - VaR_{\alpha} \mid L > VaR_{\alpha}] \ , \tag{22}$$

i.e., the CVaR is the sum of the VaR and the mean excess over the VaR in case there will be such an excess. This implies that the CVaR always will lead to a risk-level that is at least as high as measured with the VaR.

The CVaR is not lacking criticism, either. For instance, Hürlimann [22, pp. 245 ff.], on the basis of extensive numerical comparisons comes to the conclusion that the CVaR is not consistent with increasing tail-thickness.

7.3 Lower Partial Moments

Fischer [13] proves the fact that the following risk measures are coherent for $0 \leq a \leq 1$, $k \geq 1$:

$$R(X) = -E(X) + aLPM_k\,(E(X); X)^{1/k}\ . \tag{23}$$

This again is an example for the one-to-one correspondence between risk measures of the first and the second kind according to RUZ [34].

7.4 Distorted Risk Measures

We refer to the system of axioms of Wang/Young/Panjer in section 5.4, but now are looking at general gain/loss-distributions. In case $g: [0,1] \to [0,1]$ is an increasing distortion function with $g(0) = 0$ and $g(1) = 1$ the transformation $F^*(x) = g(F(x))$ defines a distorted distribution function. We now consider the following risk measure for a random variable X with distribution function F:

$$E^*(X) = -\int_{-\infty}^{0} g(F(x))\,dx + \int_{0}^{\infty} [1 - g(F(x))]\,dx. \tag{24}$$

The risk measure therefore is the expected value of X under the transformed distribution F*. The TCE corresponds to the distortion function $g(u) = 0$ for $u < \alpha$ and $g(u) = (u - \alpha)/(1-\alpha)$ for $u \geq \alpha$, which is continuous but non-differentiable in $u = \alpha$. Generally, the TCE and as well the VaR only consider information from the distribution function for $u \geq \alpha$, the information in the distribution function for $u < \alpha$ are lost. This is the criticism of Wang [44], who proposes the use of alternative distortion functions, e.g. the Beta family of distortion functions[35] or the Wang transform[36].

8. Selected Additional Approaches

8.1 Capital Market Related Approaches

In the framework of the CAPM the beta factor

$$b(R, R_M) = \frac{Cov(R, R_M)}{Var(R_M)} = \frac{r\,(R, R_M)\, s\,(R)}{s\,(R_M)} \quad , \tag{25}$$

where R_M is the return of the market portfolio and R the return of an arbitrary portfolio, is considered to be the central risk measure, as only the *systematic risk* and not the entire portfolio risk is valued by the market.

8.2 Tracking Error and Active Risk

In the context of passive (tracking) or active portfolio management with respect to a benchmark portfolio with return R_B the quantity $\sigma(R - R_B)$ is defined[37] as tracking error or as active risk of an arbitrary portfolio with return R. Therefore, the risk measure considered is the standard deviation, however, in a specific context.

8.3 Ruin Probability

In an insurance context the risk reserve process of an insurance company is given by $R_t = R_0 + P_t - S_t$, where R_0 denotes the initial reserve, P_t the accumulated premium over [0,t] and S_t the accumulated claims over [0,t]. The ruin probability is defined as the probability that during a specific (finite or infinite) time horizon the risk reserve process becomes negative, i.e. the company is (technically) ruined. Therefore, the ruin probability is a dynamic variant of the shortfall probability (relative to target zero).

Endnotes

1 The academic disciplines involved are for instance investment and finance, economics, operations research, management science, decision theory and psychology.

2 This excludes the literature on perceived risk (the riskiness of lotteries perceived by subjects), cf. e.g. [7], the literature dealing with the psychology of risk judgements, cf. e.g. [37] and as well the literature on inequality measurement, cf. e.g. [31, section 2.2].

3 For model risk in general cf. e.g. [8, chapter 15].

4 Cf. [7] for this terminology.

5 Cf. for this terminology e.g. [36].

6 For a survey cf. [36].

7 Cf. [4, p. 209] for a number of additional arguments for the validity of subadditivity.

8 In contrast to [4], we assume a risk-free interest rate $r = 0$, to simplify notation.

9 [4] suppose that the underlying probability space is finite, extensions to general probability measures are given in [9].

10 Giving up the requirement of homogeneity [14] and [15] introduce convex measures of risk and are able to obtain an extended representation theorem.

11 Cf. [17, 18], which are very critical about the uncritical use of coherent risk measures disregarding the concrete practical situation.

12 E.g., in the context of insurance premiums, cf. section 5.4.

13 Only $R(X) \geq E(-X)$ then guarantees (PS 1).

14 $R_{II}(X)$, however, is not coherent, cf. [4, p. 210]. This is the main motivation for RUZ for their distinction between coherent and non-coherent expectation bounded risk measures.

15 In case of pure loss variables $S := -X \geq 0$ this correspondency is $R_{II}(S) = E(S) + a\sigma(S)$ which is more intuitive.

16 Cf. [16].

17 [17, 18] stress this point.

18 I.e., there is a random variable Z and monotone functions f and g, with $X = f(Z)$ and $Y = g(Z)$.

19 Cf. [47, p. 339] and [9, p. 15].

20 For gain/loss-positions X a corresponding result is existing, cf. section 7.4

21 Cf. e.g. [5, p. 56].

22 $R(X - E(X))$ is (only) coherent for $a = 0$ and $b \geq 1$ as well as for $k = 1$, cf. [34].

23 Cf. [5, p. 51].

24 For an application to the worst case risk of a stock investment cf. [2].

[25] Cf. [46, p. 110].

[26] Cf. e.g. [10] and [24].

[27] In the context of credit risks cf. [10, chapter 9] and [24, chapter 13].

[28] Partially VaR is directly defined this way, e.g. [50, p. 252]. [10, pp. 40 - 41] speaks of VaR "relative to the mean" as opposed to VaR in "absolute dollar terms".

[29] Cf. [41, p. 1521].

[30] For the general case of elliptical distributions cf. [11, p. 190].

[31] Cf. e.g. [40, p. 1260].

[32] We define the risk measure in terms of L for a better comparison to VaR.

[33] Cf. [1].

[34] In the literature, a number of closely related risk measures like expected shortfall, conditional tail expectation, tail mean and expected regret have been developed, satisfying in addition a number of different characterisations. We refer to [1], [21], [22], [27], [32], [33], [40], [41], [48] and [49].

[35] Cf. [47, p. 341].

[36] A more complex distortion function is considered by [42], leading to risk measures giving different weight to "upside" and "downside" risk.

[37] Cf. [20, p. 39].

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