The First-Best Sharing Rule in the Continuous-Time Principal-Agent Problem with Exponential Utility

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Abstract

The continuous-time principal-agent model with exponential utility developed by Holmström and Milgrom (1987) admits a simple closed-form solution: The second-best sharing rule is a linear function of aggregated output. Here, we show that the first-best sharing rule is also linear in aggregated output. The result follows immediately from the separability of the problem and the fact that principal and agent both have CARA utility.

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1 Introduction

The continuous-time principal-agent model with exponential utility was introduced by Holmström and Milgrom (1987) and later generalized by Schättler and Sung (1993, 1997) and Sung (1995). In this model, an agent continuously controls the drift of a Brownian motion during a fixed time interval. Unlike the static principal-agent model, the continuous-time version admits a simple closed-form solution: The second-best sharing rule is a linear function of aggregated output. In this paper, we show that the first-best sharing rule is also linear in aggregated output.

The result follows from the fact that in the absence of incentive constraints the principal's problem is separable. Given the optimal action, efficient risk-sharing dictates the usual linear sharing rule implied by CARA utility due to the equality of marginal rates of substitution across states. Incidentally, this has nothing to do with the agent's control or the form of the technology. By contrast, the second-best linearity result derived by Holmström and Milgrom depends critically on the stationarity of the technology and the constancy of the optimal control. The production technology can be thought of as having a riskless output (as a function of the action choice) plus independent noise. Therefore, the optimal action is obtained by dominance, maximizing net output. The only place where the technology and preferences interact is in the computation of the exact sharing of gains to trade. This is reflected in the specific parameters of the optimal sharing rule which are determined by efficiency and the agent's individual rationality constraint.

2 The Model and Second-Best Sharing Rule

For ease of exposition, we confine ourselves to the case of one-dimensional Brownian motion. The notation is primarily adopted from Schättler and Sung (1993). At time 0, principal and agent agree on a sharing rule which specifies a payment from the principal to the agent at time 1. The sharing rule may depend on a stochastic outcome process X defined on the interval $[0, 1]$ which satisfies $X_0 = 0$ and which is publicly observable. Formally, X is governed by a stochastic differential equation of the form

$$dX_t = f(u_t)\, dt + \sigma dB_t, \tag{1}$$

where $f(u_t)$ is the instantaneous mean, $u_t = u_t(t, X)$ is the agent's control at time t, σ is the diffusion rate, and B is a standard Brownian motion. The principal receives the end-of-period output X_1. Besides, the principal can observe the process X, but not the agent's control. Let $(\Omega, \mathcal{F}, P)$ denote the underlying probability space. The control u is an $\mathcal{F}_t-$predictable process with values in some open bounded control set $U \subseteq \mathcal{R}_+$. That is, the agent's control can be revised continuously during the time interval $[0,1]$ and may depend on the history of X in $[0,t]$, but not on the future $(t,1]$. Denote the class of all such processes by $\mathcal{U}$. The "production function" $f(\cdot)$ is bounded with derivatives $f'(\cdot) > 0$ and $f''(\cdot) < 0$, and the diffusion rate lies in some bounded subset of $\mathcal{R}_{++}$. The agent incurs effort cost $c(u_t)$, where $c(\cdot)$ is bounded with derivatives $c'(\cdot) > 0$ and $c''(\cdot) \geq 0$. Finally, principal and agent both have negative exponential von Neumann-Morgenstern utility with constant coefficient of risk aversion R and r, respectively.

The principal's second-best problem is to choose a sharing rule S and a control u that maximize her expected utility subject to the usual individual rationality and incentive compatibility constraints. Let W_A denote the agent's certainty equivalent at time 0. The principal solves

$$\max_{S,u} E\left[-\exp\left\{-R\left(X_1 - S\right)\right\}\right] \tag{2}$$

s.t.

$$dX_t = f\left(u_t\right)dt + \sigma dB_t, \tag{3}$$

$$E\left[-\exp\left\{-r\left(S - \int_0^1 c\left(u_t\right)dt\right)\right\}\right] \geq -\exp\left\{-rW_A\right\}, \tag{4}$$

and

$$u \in \arg\max_{\hat{u}\in\mathcal{U}} E\left[-\exp\left\{-r\left(S - \int_0^1 c\left(\hat{u}_t\right)dt\right)\right\}\right] \tag{5}$$

Holmström and Milgrom (1987) and Schättler and Sung (1993) show that the optimal sharing rule takes the form

$$S^* = K + \frac{c'\left(u^*\right)}{f'\left(u^*\right)}X_1, \tag{6}$$

where K is a constant. Hence, the second-best optimal control is constant (i.e. $u_t^* \equiv u^*$) and the agent's remuneration is a linear function of aggregated output X_1. However, Schättler and Sung (1997) have pointed out that the slightest deviation from the model such as the introduction of a time- or state-dependent drift or cost function yields a (stochastic) feedback control and typically destroys the linearity result.

3 The First-Best Sharing Rule

In a first-best world, the agent's control is observable and can be enforced at no cost. The principal's first-best problem is

$$\max_{S,u} E\left[-\exp\left\{-R\left(X_1 - S\right)\right\}\right] \tag{7}$$

s.t.

$$dX_t = f(u_t)\,dt + \sigma dB_t, \tag{8}$$

and

$$E\left[-\exp\left\{-r\left(S - \int_0^1 c(u_t)\,dt\right)\right\}\right] \geq -\exp\{-rW_A\}, \tag{9}$$

In the absence of incentive constraints, this problem is separable and can be solved as a standard static risk-sharing problem. The following proposition shows that the optimal control is constant and the optimal sharing rule is a linear function of aggregated output X_1.

Proposition 3.1: The first-best sharing rule is

$$S_{FB}^* = K + \frac{R}{R+r}X_1, \tag{10}$$

where K is a constant. Moreover, the first-best control is constant, unique, and determined by the equality of marginal productivity and marginal cost

$$f'(u_{FB}^*) = c'(u_{FB}^*). \tag{11}$$

Proof: Define net compensation as $Z = S - \int_0^1 c(u_t)\,dt$. The problem can be simplified further by integrating (8) and inserting the result in (7). The principal's first-best problem can then be expressed as

$$\max_{Z,u} E\left[-\exp\left\{-R\left(\sigma B_1 + \int_0^1 [f(u_t) - c(u_t)]\,dt - Z\right)\right\}\right] \tag{12}$$

s.t.

$$E[-\exp\{-rZ\}] \geq -\exp\{-rW_A\}. \tag{13}$$

Pointwise maximization yields the first-order conditions

$$f'(u_{tFB}^*) = c'(u_{tFB}^*) \tag{14}$$

for all t, and

$$Z^* = \frac{1}{R+r}\left[\ln\left(\frac{\lambda r}{R}\right) - R\int_0^1 c(u_{tFB}^*)\,dt\right] + \frac{R}{R+r}X_1, \tag{15}$$

where $X_1 = \sigma B_1 + \int_0^1 f(u_t)\,dt$, and where λ is the Lagrangean multiplier. Constancy and uniqueness of the first-best control follows from (14) and the fact that $f(\cdot)$ and $c(\cdot)$ are strictly concave and convex, respectively.

Finally, substituting back $Z^* = S^*_{FB} - \int_0^1 c\left(u^*_{tFB}\right) dt$ in (15) and setting $K = \frac{1}{R+r}\left[\ln\left(\frac{\lambda r}{R}\right) + r\int_0^1 c\left(u^*_{tFB}\right) dt\right]$ yields (10). ■

Proposition 3.1 follows immediately from the separability of the problem. Since both parties have CARA utility, efficient risk-sharing dictates that principal and agent share the risk linearly in proportion to their share of total absolute risk tolerance.

4 References

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