You Play (an Action) Only Once[1]

Karl H. Schlag
Economics Dept. III
University of Bonn
Adenauerallee 24-26
53113 Bonn, Germany

Aner Sela
(author to whom proofs are to be sent)
Economics Dept.
University of Mannheim
Seminargebäude A5
68131 Mannheim, Germany

October, 1997

[1]We would like to thank Barry O'Neal for helpful comments. This paper was written during Aner Sela's stay at the University of Bonn. Financial support from the Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 303 at the University of Bonn is gratefully acknowledged.

Abstract

Consider an N-player normal form game played repeatedly in which each player should choose each strategy exactly one time (payoffs are aggregated). Such "play only once" situations occur naturally in the context of scheduling. Assume that each player has the same number of strategies. Then, regardless of the players' preferences, for each player to mix uniformly in each round over his remaining strategies consitutes a subgame perfect equilibrium.

1 Introduction

We begin our discussion by the following stories.

Congestion Games

There are N performers, k fair grounds and k seasons in the year. At the beginning of each season the performers decide at which fair ground they want to give their next show. Demand for any given show at a given fairground is one, hence performers will not visit the same fairground twice. Expected success of a show is decreasing in the number of other shows that take place at the same fairground in the same season. Total success is measured by the sum of the successes in each season. A similar story can be told about N candidates for a given political position who are each organizing their campaign in which k locations should be visited on k dates. Preferences of the candidates may vary as to whether they wish to be in the same location at the same time with a given other candidate or not; e.g., this depends on the ability of a candidate to confront his opponents.

Card Games

Consider a 2 person card game. A deck of cards is distributed evenly among the players such that each of the players has, as his hand, 26 cards. The procedure of play is as follows: in each turn, each player selects a card from his hand, these are shown simultaneously, and the player who showed the higher value takes the trick (when there are ties each player gets his own card). Every card can be used only once. This process continues until all cards are exhausted. The winner is the player who has collected the highest sum of cards values. The loser pays the winner the difference between the winner's and the loser's sum.

Horse Racing

At a given race day at a given race course, N races are to be held sequentially. M individuals each hold at most N race-horses. Each race-horse has a different skill, individual skill is common knowledge. The winner in a given race gets a monetary prize, prizes differ between race meetings. The race-horse owners allocate simultaneously race-horses for the next race meeting whereby each race-horse may only race once in each period. The aim of every race-horse owner is to maximize his expected overall profits (or utility over these prizes).

The common feature of the games described above is that each strategy may only be played once. In the following we will investigate sub-game perfect equilibria (Selten, 1965) of such "play only once" games. Especially we will be interested in the role of previous observations and in the role of the underlying payoff functions. More specifically, we will investigate whether there is an advantage in the card game to remembering which cards have already been played? Will the initial hand dealt influence later play? Should we expect different behavior from political candidates who prefer to confront their opponents as compared to ones who wish to avoid confrontations? Are there skill configurations where there is an optimal order of racing?

As we will demonstrate in the following analysis, the answers to the above questions are somewhat surprising.

2 The POO Game

Let G be an N player normal form game in which player i has the finite set of pure strategies (or actions) S_i, $\Delta(S_i)$ denotes the set of player i's mixed strategies and player i's payoff (utility) function is $U^i : \times_{j=1}^{N} S_j \to \mathbb{R}$, $i = 1, .., N$. In the following we will be investigating the repeated play of G with the restriction that no player may use an action of the one shot game G twice. More specifically, consider the following game, denoted by G^p, that takes place in a finite number of stages. In each stage each player chooses a one shot action from G that he has not used in the previous stages. At each stage players observe the play in the previous stages. The game is over when one of the players has used each of his one shot actions. A player's payoff in this game is given by the sum of the stage game payoffs. We call G^p a *POO* (*Play Only Once*) *game*. The card game, the congestion games and the racing belong to the class of POO games.

If the players have the same number of actions in the stage game, i.e., if $|S_i| = |S_j|$ for all $i, j \in \{1, .., N\}$ we obtain the following result:

Theorem 1 *Consider a POO game G^p based on a N-player normal form game G in which each player has the* same *number of pure strategies (or actions). If each player randomizes uniformly over his remaining one shot actions in each stage, then the players are playing a sub-game perfect equilibrium. In this equilibrium of G^p, player i's expected payoff is equal to $\frac{1}{|S_i|^{N-1}} \sum_{s \in S} U_i(s)$.*

Proof. Given a normal form game G, let $m = |S_i|$ for some i and let G^p be the induced POO game. We will prove the claim by induction. If $m = 1$ then the claim follows immediately. Assume that the claim holds for $m - 1$. In order to prove the claim for m it will be enough to show that player i is indifferent between each of his one shot actions in the first stage provided he randomizes uniformly over his remaining actions in all later stages $2, .., m$ and provided all other players $j \neq i$ follow the proposed equilibrium strategy in all stages. Let $g_i(s)$ denote the expected payoff obtained by player i given that s is the profile that was chosen in stage one ($s = (s_i)_{i=1,..,N} \in S = \times_{i=1}^{N} S_i$). By induction, $g_i(s) - U_i(s) = \frac{1}{(m-1)^{N-1}} \sum_{s' \in S: s'_j \neq s_j \forall j} U_i(s') =: f_i(s)$. Consequently the expected payoff of player i to choosing action r in the first stage is equal to h_r where

$$\begin{aligned} h_r &= \sum_{s \in S: s_i = r} \frac{1}{m^{N-1}} \left[U_i(s) + f(s) \right] \\ &= \frac{1}{m^{N-1}} \sum_{s \in S: s_i = r} \left[U_i(s) + \frac{1}{(m-1)^{N-1}} \sum_{s' \in S: s'_j \neq s_j \forall j} U_i(s') \right]. \end{aligned} \quad (1)$$

Using the fact that

$$\sum_{s \in S: s_i = r} \sum_{s' \in S: s'_j \neq s_j \forall j} U_i(s') = (m-1)^{N-1} \sum_{s \in S: s_i \neq r} U_i(s) \ ,$$

(1) simplifies to

$$h_r = \frac{1}{m^{N-1}} \sum_{s \in S} U_i(s).$$

Since h_r does not depend on r, player i is indifferent between each of his options at the first stage and the proof is complete. ■

Consequently, in the card game, the congestion games and the racing, uniform randomization constitutes a sub-game perfect equilibrium. Notice that equilibrium play does not depend on whether a player has a lucky draw in the card game nor on the utility of performing at a given fairground. Moreover, in this equilibrium there is no advantage to deriving conclusions from past play of others. Of course, the equilibrium described in Theorem 1 is not necessarily the unique sub-game perfect equilibrium. This will be illustrated in the following example.

Let G be the 2×2 (two person) game described by the following matrix

$$\begin{array}{ccc} & l & r \\ t & a, a' & c, b' \\ b & b, c' & d, d' \end{array} .$$

The normal form of the POO game G^p is then given by

$$\begin{array}{ccc} & lr & rl \\ tb & a+d, a'+d' & b+c, b'+c' \\ bt & b+c, b'+c' & a+d, a'+d' \end{array} , \tag{2}$$

where tb denotes the strategy to play t in the first round. Direct calculation reveals the following observation: when G has multiple Nash equilibria then so does (2), each of which in turn are subgame perfect equilibria of the POO game G^P.

Notice that the equilibrium we characterize is the unique equilibrium that does not depend on the underlying payoffs. For instance, if G is a $2x2$ coordination game, the equilibrium we characterize involves the off-diagonal payoffs, whereas there are other equilibria that do not. In fact, for two person games, there is no lower equilibrium payoff for either player. This is a direct consequence of the following corollary.

Corollary 2 *If G is a two person game in which both players have the same number of different actions, then uniform randomization over the remaining actions in each stage is a maxmin strategy of the POO game G^p derived* *from G, and the individually rational payoff of a player in G^p is equal to this player's equilibrium payoff given in Theorem 1.*[1]

Proof. For a given two person game G we will consider the claim from the standpoint of player one. Consider the two person game G' based on the same strategy set for either player and payoff functions $U_1' \equiv U_1$ and $U_2' \equiv -U_1$. Apply Theorem 1 to this game and we obtain that uniform randomization constitutes play of a Nash equilibrium in $(G')^p$. Since G' is a zero-sum game, so is $(G')^p$. Applying the minimax theorem of von Neumann (1928) we obtain $\frac{1}{|S_i|^{N-1}} \sum_{s \in S} U_1'(s)$ is the individual rational payoff level of player one in $(G')^p$ and hence also for G^p. Similarly it follows that uniform randomization is a maxmin strategy of G^p. ■

In the rest of the paper we wish to demonstrate why the statement of Theorem 1 can not be generalized in two aspects.

In the first example we show that Theorem 1 does not necessarily hold when the players' strategy sets are not all of the same size.

	l	m	r
u	$0,1$	$0,1$	$1,0$
d	$0,1$	$1,0$	$0,1$

The POO game of this normal form game has two stages. The weakly dominant strategy of the column player in this POO game is to play l in the first stage and after that to play m in stage 2 if the row player played d in the first stage and to play r otherwise. Consequently, the column player will not randomize if the row player does.

Finally, we want to investigate the asymmetric setting in which the row player is a POO player whereas the column player is unrestricted, i.e., the column player is allowed to play the same action of G more than once during

[1] $x_1 \in \Delta(S_1)$ is a *maxmin* strategy for player 1 with the corresponding *individually rational payoff level* $v \in \mathbb{R}$ of a given N player normal form game G' if there exist $x_i \in \Delta(S_i)$, $i > 1$ such that

$$v = U_1(x) = \max_{x_1' \in \Delta(S_1)} \min_{x_i' \in \Delta(S_i), i>1} U_1(x').$$

the repeated game. The number of rounds this game is played is equal to the number of the actions of the POO player in the underlying one shot normal form game G. The natural question is whether Theorem 1 holds for this asymmetric case too. The following example shows that this is not the case.

	a	b
a	$1, 0$	$0, 1$
b	$0, 3$	$1, 0$

If the row player randomizes uniformly over his actions in the first stage, then the best reply of the column player is to play a at the first stage and to copy the play of the row player from the first stage in the second stage.

3 References

Selten R (1965) Spieltheoretische Behandlung eines Oligopolmodells mit Nachfrageträgheit. Zeitschrift für die gesamte Staatswissenschaft 12: 301-324.

von Neumann J (1928) Zur Theorie der Gesellschaftsspiele. Mathematische Annalen 100: 295-320.