Three Essays on Sovereign Default and Collateral Constraints

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Contents

Chapter 1

General Introduction

This thesis consists of three self-contained chapters. Chapter two introduces a novel solution method for dynamic general equilibrium models. This solution method is tailor-made for models where the optimization problem of agents involves inequality constraints (e.g. borrowing or collateral constraints).
Chapter three explores the effect of collateral requirements on asset prices. We consider a Lucas tree economy with heterogeneous agents that face collateral constraints. The methods used to compute equilibria for this model rely on the solution method developed in Chapter two. We obtain our main results in a setting with two assets where we show that changes in the collateral requirement for one asset have a strong impact on the volatility of the other asset.
Finally, Chapter four develops a new theory about sovereign debt defaults. In a small open economy setting we show that default on government debt can be optimal under full commitment of the government because it allows for increased risk diversification.
A more detailed summary of each chapter is provided below.

1.1 Computing Equilibria in Models with Occasionally Binding Constraints

In the second chapter a method is proposed to compute equilibria in dynamic models with several continuous state variables and occasionally binding constraints, e.g. borrowing or collateral constraints. These constraints induce non-differentiabilities in policy functions.

We develop an interpolation technique that addresses this problem directly: It locates the non-differentiabilities and adds interpolation nodes there. To handle this flexible grid, it uses simplicial interpolation. Hence, we call this method Adaptive Simplicial Interpolation (ASI). We embed ASI into a time iteration algorithm to compute recursive equilibria in an infinite horizon endowment economy where heterogeneous agents trade in a bond and a stock subject to various trading constraints. We show that this method computes equilibria accurately and outperforms other grid schemes by far.

This chapter is based on the paper 'Computing Equilibria in Dynamic Models with Occasionally Binding Constraints' which is joint work with Johannes Brumm. The paper is available as Brumm and Grill (2010).

1.2 Collateral Requirements and Asset Prices

In the third chapter we examine the effect of collateral requirements on the prices of longlived assets. We consider a Lucas-style infinite-horizon exchange economy with heterogeneous agents and collateral constraints. There are two trees in the economy which can be used as collateral for short-term loans. For the first tree the collateral requirement is determined endogenously while the collateral requirement for loans on the second tree is exogenously regulated. We show that the presence of collateral constraints and the endogenous margin requirements for the first tree lead to large excess price-volatility of the second tree. Changes in the regulated margin requirements for the second tree have large effects on the volatility of both trees. While tightening margins for loans on the second tree always decreases the price volatility of the first tree, price volatility of the second tree might very well increase with this change. In our calibration we allow for the possibility of disaster states. This leads to very large quantitative effects of collateral requirements and to realistic equity risk premia. We show that our qualitative results are robust to the actual parametrization of the economy.

This chapter is based on the paper 'Collateral Requirements and Asset Prices' which is joint work with Johannes Brumm, Felix Kubler and Karl Schmedders. The paper is available as Brumm, Kubler, Grill, and Schmedders (2011).

1.3 Optimal Sovereign Debt Default

In the final chapter we determine optimal government default policies for a small open economy in which a domestic government can borrow internationally by issuing non-contingent debt contracts. Unlike earlier work, we consider optimal default policies under full government commitment and treat repayment of international debt as a decision variable. Default can be optimal under commitment because it allows for increased international diversification of domestic output and consumption risk when government bond markets are incomplete. In the absence of default costs, default optimally occurs very frequently and independently of the country's net foreign asset position. Optimal default policies, however, change drastically when a government default entails small but positive dead weight costs: default is then optimal only in response to disaster-like shocks to domestic output, or when a small adverse shock pushes international debt levels sufficiently close to the country's borrowing limit. Optimal default policies increase welfare significantly compared to a situation where default is ruled out by assumption, even for sizable default costs. For sufficiently low levels of default costs the optimal default policies can approximately be replicated by issuing a simple equity-like government bond.

Chapter 4 is based on the paper 'Optimal Sovereign Debt Default' which is joint work with Klaus Adam. The paper is available as Adam and Grill (2011).

Chapter 2

Computing Equilibria in Models with Occasionally Binding Constraints

2.1 Introduction

In many applications of dynamic stochastic (general) equilibrium models, it is a natural modeling choice to include constraints that are occasionally binding. Examples are models with borrowing constraints, liquidity constraints, a zero bound on the nominal interest rate, or irreversible investments. These constraints induce non-differentiabilities in the policy functions, which make it challenging to compute equilibria. In particular, standard interpolation techniques using non-adaptive grids perform poorly both in terms of accuracy and shape of the computed policy function (see, e.g. Judd, Kubler, and Schmedders (2003), pp.270-1). This paper proposes a method that overcomes these problems, even for models with several continuous state variables. We call this method Adaptive Simplicial Interpolation (ASI). Its working principle is to locate the non-differentiablilites that are induced by occasionally binding constraints, and to put additional interpolation nodes there.

We present our algorithm in the setting of a dynamic endowment economy where three or four (types of) agents face aggregate and idiosyncratic risk. To explain the main features of ASI we first compute equilibria in a simple two period version where agents trade in a bond subject to an ad-hoc borrowing constraint. Second, we embed ASI into a time iteration algorithm to solve an infinite horizon version of the model. Finally, we add a Lucas tree-type stock, which is subject to a short sale constraint, and we replace the ad-hoc

borrowing constraint by a collateral constraint. Consequently, short positions in the bond need to be collateralized by stock holdings, while the stock may not be shorted.
Compared to earlier papers using a similar setup, such as Heaton and Lucas (1996), den Haan (2001) or Kubler and Schmedders (2003), the models we consider differ in two respects, which both make it harder to compute equilibria: First, we solve models with more agents, which results in a continuous state space of higher dimension. As the kinks[1] naturally form hypersurfaces in the state space, they are of higher dimension as well. Second, in our extension, the trading constraints that agents face depend on tomorrow's equilibrium price of the stock, which is endogenously determined. Consequently, it is much harder to locate the kink and ad hoc methods fail.

Figure 2.1 illustrates the working principle of ASI. The dashed line displays a simple one-dimensional policy function with a kink. Suppose this function is approximated by linear interpolation between equidistant grid points. The resulting interpolated policy is displayed as a solid line in the left hand side of Figure 1. Clearly, the approximation error is comparatively large around the kink, and this is just because there is no interpolation node near the kink. If we knew the location of the kink and put a node there, then the approximation would be much better, as the right hand side of Figure 2.1 shows. This is the motivation for ASI, which directly addresses the problem of kinks in policy functions by placing additional grid points, called *adapted points*, at these non-differentiabilities. Clearly, in higher dimensional state spaces and with complex constraints, this approach is not as simple as Figure 2.1 suggests. Hence, we need a flexible interpolation technique and a systematic adaptation procedure.
To be able to place grid points wherever needed, we use *Delaunay interpolation*, which consists of two steps. First, the convex hull of the set of grid points is covered with simplices, which results in a so-called *tessellation*. Then we linearly interpolate locally on each simplex[2].
We adapt the grid as follows: First, we solve the system of equilibrium conditions on an

[1] In our terminology, a *kink* associated with a certain constraint is the set of points at which the policy function fails to be differentiable because the constraint is *just* binding, i.e. the constraint is binding *and* the associated multiplier is zero.

[2] Clearly, linear simplicial interpolation is only C^0 at the boundaries. For our purposes, this is desirable, because it provides a better fit at the kinks, and it ensures stability of the time iteration algorithm.

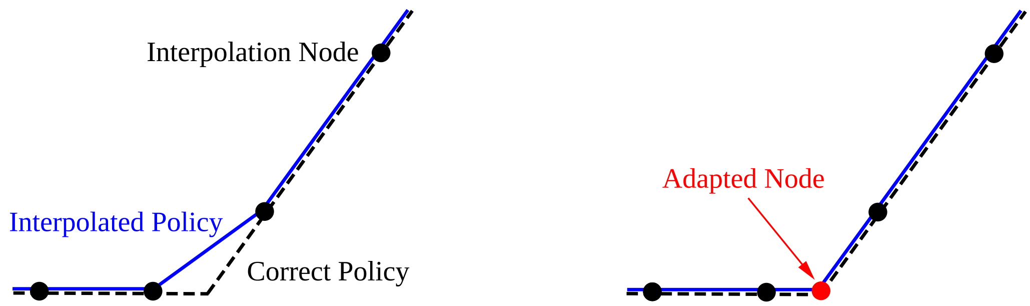


Figure 2.1: Non-Adaptive (lhs) and Adaptive (rhs) Linear Interpolation in 1D

initial grid. Second, we use these solutions to determine which edges of the tessellation cross kinks. Third, on each of these edges, we solve a modified system of equilibrium conditions to determine the point of intersection with the kink. Finally, we place a new grid point there. Using this procedure with state spaces of more than one dimension, we get several adapted grid points for each kink. Delaunay tessellation connects these points by edges, such that the kinks are matched very accurately.

To solve the above described infinite horizon models, we embed adaptive simplicial interpolation in a standard time iteration algorithm (see, e.g. Judd (1998)). To assess the accuracy of the computed equilibria, we follow Judd (1992) in calculating relative errors in Euler equations, subsequently called *Euler errors*. Concerning the measured Euler errors, we find that our method accurately computes equilibria for the two economies considered, both for reasonable and extreme calibrations of our model. Furthermore, we assess the relative performance of the adaptive grid scheme by comparing it to a standard equidistant grid scheme using the same interpolation technique. We find that the adaptive grid scheme dominates by far: One needs to increase the number of equidistant grid points, and thereby CPU time, by more than two orders of magnitude in order to reach the high accuracy of the adaptive grid scheme. Finally, we demonstrate that ad hoc update procedures that place additional points near the kinks are much less efficient than ASI.

In the literature, many algorithms have been applied to dynamic models with occasionally binding constraints. However, none of the existing algorithms addresses the problems of non-differentiabilities directly. Christiano and Fisher (2000) compare how several algorithms compute equilibria in a one sector growth model with irreversible investment, which has only one continuous state variable. None of the applied algorithms uses an adaptive grid scheme. A grid structure which is not adaptive, but endogenous, is proposed by Carrol (2006) and extended by Barillas and Fernández-Villaverde (2007), Rendahl (2007), and Hintermaier and Koeniger (2010). This so called endogenous grid method defines a grid on tomorrow's variables, resulting in an endogenous grid on today's variables. Its major advantage is that it avoids the root finding step. However, as it exploits the specific mapping from next period's variables to today's variables, the applicability as well as the concrete implementation of this method depends very much on details of the model. Maybe most related to our paper, Gruene and Semmler (2004) propose an adaptive grid scheme for solving dynamic programming problems. However, this method is designed for value function iteration, it interpolates on rectangular elements, and uses estimated local errors of the value function to update the grid. Along all these dimensions their method is orthogonal to our algorithm. The sparse grid Smolyak (1963) algorithm is a well known approach to high-dimensional interpolation in economics. Krueger and Kubler (2004) use it to compute equilibria in OLG models with state spaces that have up to 30 dimensions. Certainly, this cannot be achieved in feasible time with our algorithm. However, the Smolyak algorithm requires policy functions to be smooth, which is not the case in models with occasionally binding constraints.

Section 2.2 presents adaptive simplicial interpolation, which consists of two components: Delaunay interpolation and an adaptive grid scheme. The example used to explain ASI is a two period exchange economy where several types of agents trade in a bond subject to ad-hoc borrowing constraints. Section 2.3 shows how the infinite horizon version of this economy is solved by embedding ASI in a time iteration setup. In Section 2.4, ASI is applied to a model where trade in a bond and a stock is subject to collateral constraints and short-selling constraints. Sections 2.3 and 2.4 examine carefully the computational performance of ASI as to the respective models. Section 2.5 concludes.

2.2 Adaptive Simplicial Interpolation

The main innovation of this paper is ASI, which is tailor-made for interpolating policy functions in models with occasionally binding constraints. Section 2.2.1 gives a simple example of such a model: An exchange economy where heterogeneous agents trade in a one-period bond subject to ad-hoc borrowing constraints. Section 2.2.2 provides a formal characterization of the problems we are considering. Section 2.2.3 outlines the adaptive simplicial interpolation algorithm we propose, while Sections 2.2.4 and 2.2.5 describe the two essential ingredients of the method: a simplicial interpolation technique based on Delaunay tessellation, and an adaptive grid scheme. Finally, Section 2.2.6 illustrates the workings of ASI with the help of the simple example from Section 2.2.1.

2.2.1 Simple Example: Borrowing Constraints

The Bond Economy

The economy is populated by H types of agents h $\in \mathbb{H} = \{1, \dots, H\}$ living for T periods. Agents have identical preferences[3], but differ with respect to endowment realizations. They maximize expected time-separable lifetime utility

$$\mathbb{E}\left[\sum_{t=1}^{T} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}\right],$$

where c_t denotes consumption at t, β is the time discount factor, and γ is the coefficient of relative risk aversion.

Uncertainty is captured by a first-order Markov process with domain $X = \{1, ..., K\}$. Aggregate endowment of the single consumption good is given by a time invariant function $\bar{e} : X \to \mathbb{R}^{++}$, which depends on the current shock only. Similarly, agent h's individual endowment is given by $e^h : X \to \mathbb{R}^{++}$.

Each period, agents trade in a one-period bond, which is in zero net supply. Hence, agents face the following budget constraints:

$$c_t^h + b_t^h p_t \leq e_t^h + b_{t-1}^h \;\; \forall t = 1, \dots, T \;\; \forall h \in \mathbb{H},$$

[3]Allowing for heterogeneous preferences does not impede using ASI.

where b_t^h denotes the bond holding that agent h acquires at time t, and p_t denotes the respective price. Moreover, agents face an ad-hoc borrowing constraint:

$$b_t \geq \underline{b} \;\; \forall t = 1, \ldots, T,$$

where $\underline{b} \in \mathbb{R}^-$.

State Space

The state of the economy at the beginning of a period is characterized by the exogenous shock and the asset distribution among agents. Because of bond market clearing, we may use the bond holdings of $H - 1$ agents as the endogenous state variable:

$$y_t = \left(b_{t-1}^1, \ldots, b_{t-1}^{H-1}\right).$$

Assuming that last period's constraints of all agents were satisfied, agent h enters period t with bond holding restricted by

$$b_{t-1}^h \in \left[\underline{b}, -(H-1)\underline{b}\right].$$

Hence, we take the endogenous state space to be

$$Y \equiv \left\{ y \in \left[\underline{b}, -(H-1)\underline{b}\right]^{H-1} \middle| \sum_{i=1}^{H-1} y_i \in \left[\underline{b}, -(H-1)\underline{b}\right] \right\}.$$

The whole state space S is then given by the product of the exogenous part and the endogenous part, i.e.

$$S = X \times Y.$$

Equilibrium Conditions

The endogenous choices and prices in period t are:

$$z_t \equiv \left(\left\{ c_t^h, b_t^h \right\}_{h \in \mathbb{H}}, p_t \right).$$

We call the collection of these endogenous variables *policies*, and denote the space of policies by Z.

The definition of competitive equilibrium is standard and given in Appendix 2.A, where we also derive the first order necessary conditions for equilibrium. Here, we just state these conditions. Along an equilibrium path, policies satisfy market clearing in the bond market, budget constraints, Euler equations, borrowing constraints and complementary slackness conditions[4]:

$$\begin{aligned}
&\sum_{h \in \mathbb{H}} b_t^h = 0, \\
&c_t^h + b_t^h p_t - e_t^h - b_{t-1}^h = 0 \;\; \forall h \in \mathbb{H}, \\
&-u'(c_t^h)p + \mu_t^h + \mathbb{E}\left[\beta u'(c_{t+1}^h)\right] = 0 \;\; \forall h \in \mathbb{H}, \\
&0 \leq b_t^h - \underline{b} \;\perp\; \mu_t^h \geq 0 \;\; \forall h \in \mathbb{H},
\end{aligned}$$

where μ^h denotes the Kuhn-Tucker multiplier on the borrowing constraint of agent h.

Two Period Version

Now consider the simplest dynamic setting: $T = 2$. In this case there is no trade in the second period and agents simply consume all their funds:

$$c_2^h = e_2^h + b_1^h.$$

Consequently, in period one, equilibrium conditions for given initial bond holdings $\{b_0^h\}_{h \in \mathbb{H}}$

[4]The sign $\perp$ denotes orthogonality of two vectors. Hence, for $a, b \in \mathbb{R}^n$:

$$a \perp b :\Leftrightarrow \sum_{k=1}^{n} a_k b_k = 0.$$

If $a, b \geq 0$, then $a \perp b$ implies that for each coordinate $k = 1, \ldots, n$ either $a_k = 0$, $b_k = 0$, or both. Hence, $0 \leq a \;\perp\; b \geq 0$ is equivalent to

$$\forall k = 1, \ldots, n : \quad 0 \leq a_k \;\wedge\; 0 \leq b_k \;\wedge\; (\; a_k = 0 \;\vee\; b_k = 0 \;).$$

simplify to:

$$\sum_{h\in\mathbb{H}} b_1^h = 0,$$
$$c_1^h + b_1^h p_1 - e_1^h - b_0^h = 0 \;\; \forall h \in \mathbb{H},$$
$$-u'(c_t^h)p_1 + \mu_1^h + \mathbb{E}\left[\beta u'(e_2^h + b_1^h)\right] = 0 \;\; \forall h \in \mathbb{H},$$
$$0 \le b_1^h - \underline{b} \;\perp\; \mu_1^h \ge 0 \;\; \forall h \in \mathbb{H}.$$

2.2.2 The General Problem

The above problem of finding an equilibrium policy for the two period bond economy with given initial bond holdings has the following structure:

Equilibrium Problem:

Given a state $s \in S$, and functions

$\phi : S \times \mathbb{R}^{m+n} \to \mathbb{R}^m, \;\; \psi : S \times \mathbb{R}^m \to \mathbb{R}^n,$

find policies and multipliers $(z, \mu) \in \mathbb{R}^m \times \mathbb{R}^n$,

s.t. $\phi(s, z, \mu) = 0, \quad 0 \le \psi(s, z) \;\perp\; \mu \ge 0.$

In the case of our example, the equations $\phi = 0$ contain market clearing, budget constraints, and Euler equations. The inequalities $0 \le \psi$ contain the borrowing constrains, and μ contains the respective Kuhn-Tucker multipliers. To solve such a problem for a given state s, there are many well established procedures. Either one applies solvers that accept complementarity conditions, or one transforms these conditions into equations—as explained in Appendix 2.C—and applies standard non-linear equation solvers.
However, things get more involved, if one is interested in the mapping from the state of the economy, s, into choices and prices, $f(s)$. Then, one faces a parametric problem, with

the state of the economy, s, being the parameter.

Parametric Equilibrium Problem:

Given $\phi : S \times \mathbb{R}^{m+n} \to \mathbb{R}^m,\ \ \psi : S \times \mathbb{R}^m \to \mathbb{R}^n,$

find $f : S \to \mathbb{R}^m,\ \ \mu : S \to \mathbb{R}^n,$

s.t. $\forall s \in S$: $\phi(s, f(s), \mu(s)) = 0, \quad 0 \leq \psi(s, f(s)) \ \perp\ \mu(s) \geq 0.$

One way to compute functions (f, μ) that approximately satisfy these conditions is collocation (see, e.g. Judd (1998)): choose a finite grid $G \subset S$, on which the above conditions have to be satisfied precisely, i.e. require

$$\forall g \in G: \quad \phi(g, f(g), \mu(g)) = 0, \quad 0 \leq \psi(g, f(g)) \ \perp\ \mu(g) \geq 0.$$

For each point on the grid, $g \in G$, the solution $f(g)$ is determined by solving a complementarity problem. Aside from the grid G, collocation determines f by interpolating the solutions $\{f(g)\}_{g \in G}$ found on the grid. Clearly, this does not result in a perfect fit, and more importantly, the quality of the fit depends crucially on the location of the grid points $g \in G$. In particular, if there are kinks in the function f, it is desirable to put grid points there, because any method that interpolates over the kink has no chance to match it exactly.

In general, f is non-differentiable at the points k where for some j both $\psi_j(k, f(k))$ and $\mu_j(k)$ are equal to zero. The reason is as follows: $\psi_j(k, f(k)) = 0$ means that this constraint is binding, and $\mu_j(k) = 0$ means that the associated multiplier is zero though. Loosely speaking, the constraint is binding at one side and non-binding at the other side of the point. In general, this implies that the optimal solution is determined by different sets of equations on the two sides of the point, resulting in different slopes of the policy function. All in all, the above reasoning suggests that we should put interpolation nodes at points where constraints are just binding. We achieve this using the algorithm presented in Sections 2.2.3 to 2.2.5.

2.2.3 The Algorithm

To solve the parametric equilibrium problem presented above, we propose Adaptive Simplicial Interpolation. An overview of this procedure is given below. Steps two and three are black boxes for now. Sections 2.2.4 and 2.2.5 explain these steps in detail. We will explain Delaunay interpolation first, as it includes the concept of tessellation, which we use in the grid adaptation procedure.

Adaptive Simplicial Interpolation:

1. *Initialization:*
   Start with an initial grid G_{init} and solve for the solutions $\{f(g)\}_{g \in G_{init}}$ using standard numerical procedures.

2. *Grid Adaptation:*
   Use the solutions $\{f(g)\}_{g \in G_{init}}$, as explained in *Section 2.2.5*, to solve jointly for adapted grid points G_{adapt} that lie directly on the kinks and for the solutions $\{f(k)\}_{k \in G_{adapt}}$ at these points.

3. *Simplicial Interpolation:*
   Interpolate f on $G = G_{init} \cup G_{adapt}$. To interpolate on a grid with such an irregular shape, use simplicial interpolation, namely Delaunay interpolation, which is explained in *Section 2.2.4*.

2.2.4 Delaunay Interpolation

To get as much flexibility as possible in adapting the collocation grid, we need to have a method that is able to interpolate between points from any arbitrary set of scattered points. In addition, we require the method to work in arbitrary dimensions. Delaunay interpolation fulfills both criteria. This interpolation technique consists of two main steps: First, the state space is divided into simplices, which is done by Delaunay tessellation. Second, simplicial interpolation interpolates locally on these simplices.

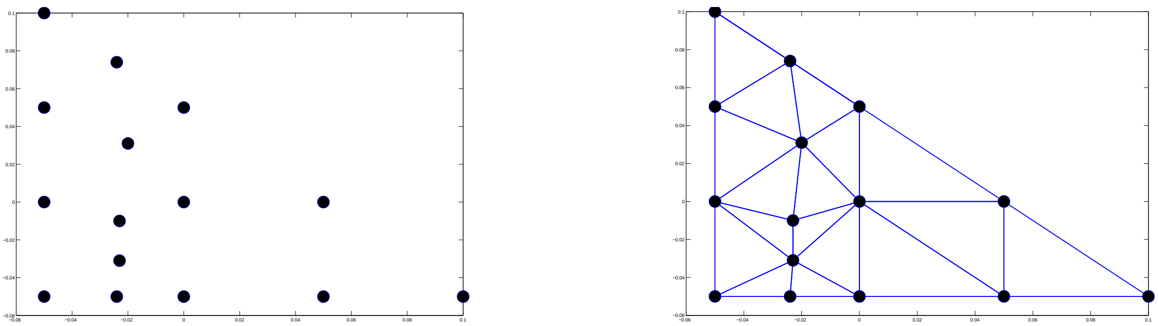


Figure 2.2: Set of Grid Points (lhs) and its Delaunay Triangulation (rhs)

Delaunay Tessellation[5]

In computational geometry, Delaunay tessellation is a well established method to cover the convex hull of an arbitrary set of points with simplices. For the sake of simplicity, we explain Delaunay tessellation for the two dimensional case. In this case, the simplices are just triangles and the method is called triangulation. In Figure 2.2, the left hand side picture shows a set of scattered grid points. The right hand side picture shows the Delaunay triangulation of this set of grid points. Delaunay triangulation is just one possible way to triangulate a set of grid points. However, it imposes discipline on the triangulation by satisfying the following property: inside the circumcircle of any triangle there is no point from the set of points. To make sense of this requirement, note that: by definition, the vertices of a triangle lie on its circumcircle, and in a Delaunay triangulation other points might as well lie on this circumcircle but not inside. Simpson (1978) shows that this procedure maximizes the minimum angle among all angles within the triangulation. Hence, it avoids pointed triangles. From a numerical perspective, this is a convenient property, since it implies that the information used to interpolate at a particular point stems from points that are relatively nearby. For a more extensive discussion of Delaunay Tessellation, see de Berg, Cheong, van Kreveld, and Overmars (2008).

Simplicial Interpolation

Having the tessellation of a set of points at hand, linear simplicial interpolation is straightforward: For any arbitrary point, find the simplex it is contained in. Then calculate its barycentric coordinates within this simplex. Finally, the interpolation value is just a linear

[5] Delaunay tessellation was introduced by Delaunay (1934) and is well known in engineering. However, up to our knowledge, it has never been used to compute equilibria in dynamic models.

combination of the values at the corners of the simplex. The weights are given by the barycentric coordinates of the corners.

2.2.5 An Adaptive Grid Scheme

Let us now turn to the process of adapting the grid. Our aim is to detect kinks and place points on these kinks in order to match them precisely. In terms of the notation of Section 2.2.2, we want to determine points that lie on

$$K = \{s| \ \exists j \ \ \psi_j(s, f(s)) = 0 \ \text{ and } \ \mu_j(s) = 0\}.$$

Hence, we are looking for points where a constraint holds with equality but the respective multiplier is zero, i.e. where the constraint is just binding. To determine such points we proceed as follows.

How to Determine Which Edges Cross Kinks
To determine the location of kinks, we use the solutions $\{f(g)\}$ computed on the initial grid G_{init}. Clearly, if $\psi_j(g, f(g)) = 0$, we know that this constraint, which we call constraint j, is binding at g. Otherwise it is not binding. Furthermore, we make use of the tessellation of the initial grid. We consider each edge of the tessellation and check whether constraint j is binding at one corner and non-binding at the other corner of this edge. If this is the case, we conclude that the associated kink, which we call kink j, crosses this edge. In this way, we find sets of edges $\{\mathcal{E}_j\}$ crossing the kinks $j = 1, \ldots, m$.

How to Put Points Exactly on the Kink
Given the sets of edges $\{\mathcal{E}_j\}$ crossing the kinks $j = 1, \ldots, m$, we need to determine where exactly to put points on these edges. For each individual edge $E \in \mathcal{E}_j$ this is done by solving a modified version of the equation system that characterizes equilibrium. The key conceptional difference is that we let the state variable vary on the edge and do not solve the equation system at a given point in the state space. To pin down the one point that lies on the kink, we force that both ψ_j and μ_j are equal to zero. Hence, we solve jointly for the equilibrium solution *and* for a point in the state space on which the equilibrium solution fulfills a certain requirement, namely that the considered constraint is just binding. More

formally, we solve for the point k, policies z, and multipliers μ such that:

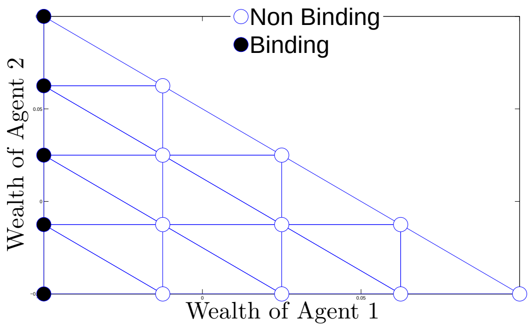
$$\begin{aligned}
&\phi(k, z, \mu) = 0, \quad 0 \leq \psi_{-j}(k, z) \perp \mu_{-j} \geq 0, \\
&\psi_j(k, z) = 0, \quad \mu_j = 0, \\
&k \in E.
\end{aligned}$$

By demanding $\psi_j(k, z) = 0$, $\mu_j = 0$ instead of $0 \leq \psi_j(k, z) \perp \mu_j \geq 0$, we reduce the degrees of freedom by one. But letting the state variable k vary on the one-dimensional object E, in contrast to fixing a point in the state space, increases the degrees of freedom by one. Hence, the modified equation system has a (locally unique) solution (k, z, μ), if (z, μ) is a (locally unique) solution to the original equation system at k. This solution does not only provide the point k that lies on the kink, but at the same time it provides the optimal policy at this point, namely $f(k) = z$.
In this way—for all edges E in all sets $\mathcal{E}_j$—we compute points k and policies $f(k)$. We call these points adaptive, and denote the set containing them by G_{adapt}. Finally, we add them to the initial points to generate the adapted grid: $G = G_{init} \cup G_{adapt}$.[6]

2.2.6 ASI at Work

Figure 2.3 visualizes the working principle of ASI. The left hand side displays an initial grid for a given exogenous state of the 2-period bond economy. On the x-axis we have wealth of agent 1, on the y-axis wealth of agent 2—remember that the wealth of agent 3 is given by market clearing. We place 15 equidistant grid points on this state space, and we solve the equilibrium problem on this initial grid.
Knowing the optimal policies at these points, we now consider each constraint at a time. We start with the borrowing constraint of agent 1. In the left picture, black dots indicate that the constraint of agent 1 is binding, while white dots indicate that it is not binding. Hence, we know on which edges of the triangulation the constraint change from binding to non-binding. On these edges, we apply the second part of our adaptation scheme: we solve the modified equation system that allows us to find the particular point on the edge

[6] Instead of this fine tuned adaptation procedure, one could also use a rather mechanical update of the grid. Instead of locating the kink exactly, one could just add arbitrary points into the triangles of interest, e.g. the center point of the triangle or say 5 randomly distributed points. This is easier to program, but comes at the cost of a less accurate result.



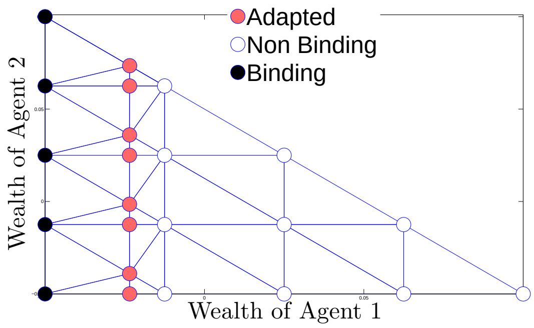


Figure 2.3: Initial Grid (lhs) and Adapted Grid (rhs) Using ASI in 2D

where the constraint is just binding (e.g. where the kink crosses the edge). Doing this for all relevant edges, we end up with 8 adapted points in this example, which are displayed in the right picture in Figure 2.3. Finally, a new triangulation is computed for the set of all grid points, initial and adapted. After this, we consider the next constraint. However, all other constraints are always non binding in this simple example. Hence, there are no further points to be added. Note that the new triangulation connects the adapted points by edges, thus kinks are matched very accurately. This can also be seen in Figure 2.4, where the left graph shows the equilibrium bond demand function of agent 1. The range where agent 1 is constrained by the borrowing limit is displayed by the dark shaded area. The kink induced by the inequality constraint is well approximated by the adapted points. The solid line in the right graph displays a slice of the bond demand function of agent 1. The dashed line represents the policy one gets if an equidistant grid is used. Clearly, this policy is quite inaccurate at the kink.

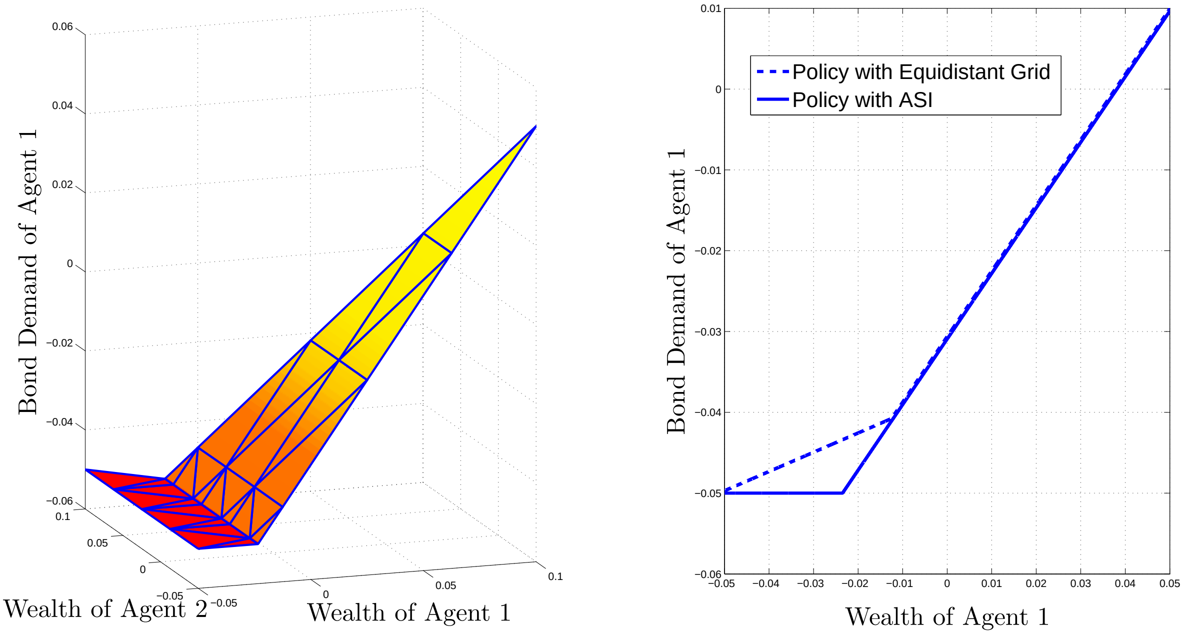


Figure 2.4: 2D Policy with ASI (lhs) and 1D Slice with and without ASI (rhs)

2.3 Time Iteration with ASI

We now consider the infinite horizon version of the bond economy from section 2.2.1. Section 2.3.1 characterizes recursive equilibrium policies for this model. Section 2.3.2 shows how such policies may be computed by embedding ASI into a standard time iteration setup. Details of how we implement this algorithm are given in Section 2.3.3. Finally, Section 2.3.4 analyzes the computational performance of time iteration with ASI.

2.3.1 The Infinite Horizon Bond Economy

Consider the bond economy of Section 2.2.1 with $T = \infty$. We want to describe equilibrium in terms of policy functions that map the current state into current policies:

$$f_t : S \to Z, \;\; f_t : \left(x_t, \left(b^1_{t-1}, \ldots, b^{H-1}_{t-1}\right)\right) \mapsto \left(\left\{c^h_t, b^h_t\right\}_{h\in\mathbb{H}}, p_t\right).$$

For the components of the policy function, we use the same notation as for their values, hence

$$f_t = \left(\left\{c^h_t, b^h_t\right\}_{h\in\mathbb{H}}, p_t\right).$$

For all states, these functions $\{f_t\}$ have to satisfy the period-to-period first order equilibrium conditions (see Appendix 2.A):

$$\begin{aligned}
\forall s : \qquad & \sum_{h\in\mathbb{H}} b^h(s) = 0, \\
& c^h_t(s) + b^h_t(s)p_t(s) - e^h_t(s) - b^h_{t-1}(s) = 0, \;\; \forall h \in \mathbb{H}, \\
& -u'(c^h_t(s))p_t(s) + \mu^h_t(s) + \mathbb{E}\left[\beta u'\left(c^h_{t+1}\left(s_{t+1}\right)\right)\right] = 0, \;\; \forall h \in \mathbb{H}, \\
& 0 \le b^h_t(s) - \underline{b} \;\perp\; \mu^h_t(s) \ge 0, \;\; \forall h \in \mathbb{H},
\end{aligned}$$

where $s_{t+1} = \left(x_{t+1}, \left(b^1_t, \ldots, b^{H-1}_t\right)\right)$.

A recursive equilibrium policy function of this economy is a time invariant policy function f that satisfies these conditions, i.e. the sequence $\{f_t\}$ with $f_t = f \;\; \forall t$ satisfies the above conditions.

2.3.2 The Algorithm

The above period-to-period equilibrium conditions have the following structure:

$$\forall s: \quad \phi[f^{next}]\,(s, f(s), \mu(s)) = 0, \quad 0 \leq \psi\,(s, f(s)) \;\perp\; \mu(s) \geq 0,$$

where time t variables have no index, and the policy in $t+1$ is denoted by f^{next}. The equations $\phi[f^{next}] = 0$, which depend on f^{next}, contain market clearing, budget constraints and Euler equations. Only the latter depend on f^{next}—in this case on the consumption policies only. The inequalities $0 \leq \psi$ contain the borrowing constraints, and μ contains the respective Kuhn-Tucker multipliers. A recursive equilibrium policy function f satisfies:

$$\forall s: \quad \phi[f]\,(s, f(s), \mu(s)) = 0, \quad 0 \leq \psi\,(s, f(s)) \;\perp\; \mu(s) \geq 0.$$

The problem of finding a policy function that (approximately) satisfies this condition is very hard to address directly. In a time iteration procedure, the recursive equilibrium policy function is approximated iteratively: in each step, a simpler problem is solved, where next period's policy, f^{next}, is taken as given. This brings us back to the period-to-period equilibrium conditions:

$$\forall s: \quad \phi[f^{next}]\,(s, f(s), \mu(s)) = 0, \quad 0 \leq \psi\,(s, f(s)) \;\perp\; \mu(s) \geq 0.$$

This problem takes exactly the form of the parametric equilibrium problem discussed in Section 2.2.2. Hence, we may use adaptive simplicial interpolation for this essential step in the time iteration algorithm. The formal structure of the full algorithm is given below. We deviate from a standard time iteration procedure only with regard to the interpolation procedure, which is contained in the inner box.

Time Iteration with Adaptive Simplicial Interpolation:

1. Select a grid G_{init}, an initial policy function f^{init}, and an error tolerance ϵ.
   Set $f^{next} \equiv f^{init}$.

2. Make one time iteration step: For all $g \in G_{init}$, find $f(g)$ that solves

$$\phi[f^{next}]\,(s, f(s), \mu(s)) = 0, \quad 0 \leq \psi\,(s, f(s)) \;\perp\; \mu(s) \geq 0.$$

   Interpolate f by *adaptive simplicial interpolation*:

   First, use the solutions $\{f(g)\}_{g \in G_{init}}$ to solve jointly for adapted points G_{adapt} that lie directly on kinks and for the optimal policy $\{f(g)\}_{g \in G_{adapt}}$ at these points.

   Second, use solutions at all grid points $G = G_{init} \cup G_{adapt}$ to interpolate f by *simplicial interpolation.*

   If $\|f - f^{next}\|_\infty < \epsilon$, go to step 3.
   Else set $f^{next} \equiv f$ and repeat step 2.

3. Set the numerical solution to the infinite horizon optimization problem: $\tilde{f} = f$.

2.3.3 Implementation of the Algorithm

To demonstrate that our algorithm works well with standard equipment, we use Matlab on an Intel Core 2 Duo 2.40 GHz computer to implement our algorithm.

Solving the System of Equilibrium Conditions

To solve the complementarity problem at each grid point, one could use a solver that directly applies to complementarity problems. However, we prefer to transform the complementarity problem into a system of equations (see Appendix 2.C) and then apply a standard non-linear equation solver, e.g. Matlab's fsolve or Ziena's Knitro. We are able to solve our models with both solvers. However, we find that the more equations the equlibrium system involves the better the performance of Knitro compared to fsolve.

Adaptive Simplicial Interpolation

Our method of choice for interpolation is Delaunay interpolation as described in Section 2.2.4. Delaunay Interpolation is widely used in many areas, and hence code in several languages like C++ or Fortran is available on the web. In Matlab, routines for computing Delaunay tessellations and simplicial interpolation come with the standard version.

Time Iteration

For the computation exercise presented below we set the error tolerance $\epsilon = 10^{-5}$. We set the initial policy function f^{init} such that agents consume all their wealth and the price of all assets is equal to zero. Hence, f^{init} corresponds to the policy function in the final period of a finite horizon economy. This is not an efficient starting guess, but it makes the computing times of our examples comparable. As a starting guess for solving the equilibrium problem at a given point, we use the solution from the previous iteration. In case the solver cannot find a root we use the solution from neighboring points as new starting guesses. In this way we always find solutions that satisfy the error tolerance.

To decrease CPU time, we start the time iteration procedure with a relatively coarse equidistant grid, and increase the density of the grid as the error in $\|f - f^{next}\|_\infty$ falls below $\epsilon \cdot 10$. We repeat this several times until we reach a grid of certain predefined size. In the comparison studies below, this refinement of the equidistant grid is done in exactly the same way for the adaptive grid method and the equidistant benchmark.

To further decrease CPU time, we do not use adaptive simplicial interpolation at each iteration step. The first step is not done until all refinements of the equidistant grid are carried out and the error in $\|f - f^{next}\|_\infty$ falls below $\epsilon \cdot 10$ again. Note that kinks in policy functions change their location along the time iteration procedure. Hence, it is important to use a sufficient number of adaptation steps. Furthermore, note that at each adaptation step, we compute new adapted nodes and do not use the adapted nodes from the last step any more.

2.3.4 Computational Performance

To evaluate the computational performance of *time iteration with adaptive simplicial interpolation*, we first report the accuracy of the computed equilibria for various examples. Second, we compare time iteration with ASI to two other grid structures: an equidistant grid, and an ad hoc update scheme that places additional grid points randomly into simplices that are cut by kinks.

Measuring Accuracy
Following Judd (1992) we evaluate the accuracy of a computed equilibrium by calculating *relative errors in Euler equations* (EEs). An EE measures the error that an agent would make in terms of his period-to-period consumption decision, if he used the computed policy function. The unit of measure is the relative deviation of computed (i.e. interpolated) consumption, c_t^{int}, from the one that is optimal, c_t^{opt}, given next periods interpolated consumption, c_{t+1}^{int}. To derive c_t^{opt} from c_{t+1}^{int} one uses an Euler equation. For instance, in the Bond economy of Section 2.2.1 the Euler error $EE^h(\cdot)$ for agent h at a particular point s in the state space is given by

$$EE^h(s) = \left| \frac{c_t^{opt}}{c_t^{int}} - 1 \right| = \left| \frac{u'^{-1}\left(\beta \mathbb{E}_t \left[\frac{u'(c_{t+1}^{int})}{p_{int}}\right]\right)}{c_t^{int}(s)} - 1 \right|,$$

where p_{int} is the interpolated price of the bond today. However, it is possible to back out c_t^{opt} from c_{t+1}^{int} only if the Kuhn-Tucker multiplier entering the Euler equation is zero, i.e. if the respective constraint is non-binding. If it is binding, we set the Euler error equal to zero. Because of this problem with computing Euler errors when constraints are occasionally binding we also report an alternative error measure in Appendix 2.D.
To evaluate the accuracy of computed equilibria, we calculate the Euler errors of all agents at many points in the state space. Concerning the choice of points, we make two alternative choices. First, we draw 10.000 random points from a uniform distribution over the whole state space (EE state space), and compute Euler errors for all agents at these points. Second, we take the points reached along the equilibrium path, when the economy is simulated for 5.000 periods (EE equilibrium path). In both cases, we report both the maximum over all agents and points (max EE) as well as the average across points of the maximum

across agents (∅ EE). This results in four different statistics, which we all report in $\log_{10}$ scale.

The examples that we consider have three or four agents and a borrowing limit of $\underline{b} = 0.1$ or 1.0, i.e. borrowing is restricted to 10% or 100% of average individual yearly income. Concerning all other parameters, we choose values that are considered standard in the literature, which we report in Appendix 2.E. Tables 2.1 and 2.2 report the accuracy measures for the three and four agent examples respectively. Maximal Euler errors over the state space range from -3 (for three agents and $\underline{b} = 0.1$) to -1.7 (for four agents and $\underline{b} = 1.0$). All errors are reasonably low, but could be improved much further by increasing the number of initial grid points, which would in turn also increase the number of adapted points. Generally speaking, a looser borrowing limit $\underline{b}$ and/or a greater number of agents—which both enlarge the state space—result in higher Euler errors. In the case of four agents, we are dealing with a three-dimensional state space, and kinks become two dimensional objects. This is illustrated in Figure 2.5, which displays a three dimensional grid that is adapted to a kink that lies approximately orthogonal to the horizontal axis.

Bond Economy with Three Agents

			EE state space		EE equilibrium path	
$\underline{b}$	points	time(min)	max EE	∅ EE	max EE	∅ EE
0.1	**40**(45)	**0.5**(0.4)	**-3.0**(−1.2)	**-3.8**(−2.1)	**-2.4**(−1.2)	**-4.4**(−2.1)
0.1	**113**(120)	**1.1**(1.0)	**-3.2**(−1.6)	**-4.2**(−2.8)	**-3.2**(−1.6)	**-4.8**(−3, 4)
1.0	**185**(190)	**6.5**(4.5)	**-2.1**(−1.1)	**-3.1**(−2.6)	**-2.2**(−1.1)	**-3.1**(−1.8)
1.0	**941**(946)	**13**(11)	**-3.2**(−1.2)	**-4.2**(−2.9)	**-3.2**(−1.6)	**-4.8**(−3.4)

Table 2.1: Accuracy of Adaptive Grid (Equidistant Grid in Brackets)

Bond Economy with Four Agents

			EE state space		EE equilibrium path	
$\underline{b}$	points	time(min)	max EE	∅ EE	max EE	∅ EE
0.1	**112**(120)	**4.5**(4)	**-2.7**(−1.3)	**-3.3**(−2.0)	**-2.7**(−1.3)	**-3.9**(−1.7)
1.0	**914**(969)	**60**(51)	**-1.7**(−1.1)	**-2.6**(−2.4)	**-1.8**(−1.1)	**-2.6**(−3.9)

Table 2.2: Accuracy of Adaptive Grid (Equidistant Grid in Brackets)

Comparison to Equidistant Grid

In order to assess the relative performance of ASI, we also compute equilibria on a standard

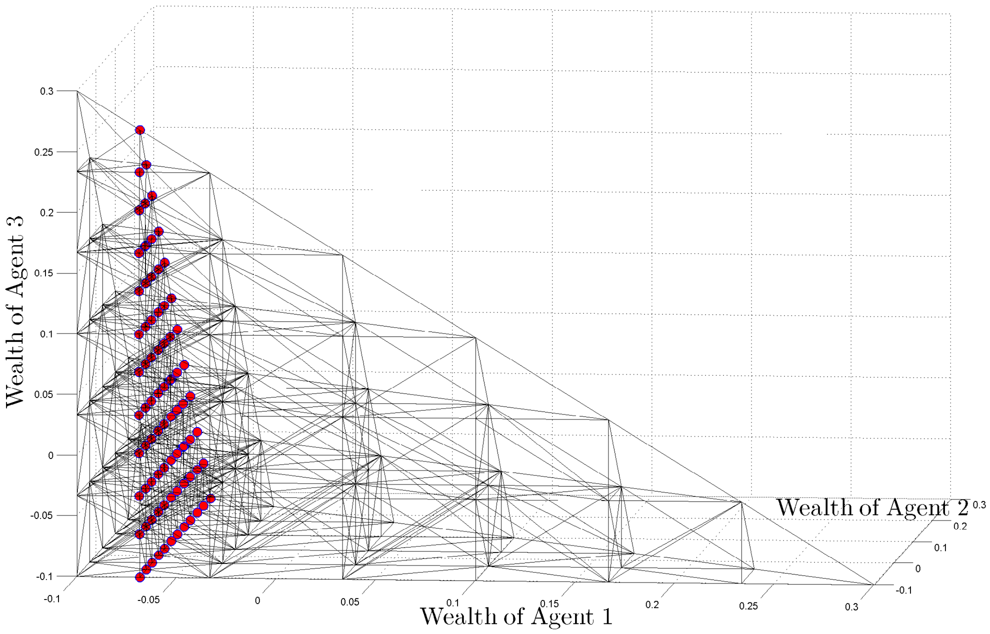


Figure 2.5: Adapted Grid with Three Continuous State Variables

equidistant grid, but still use Delaunay interpolation. To assess the gains from using an adaptive grid scheme, we ask the following questions: First, how do solutions on equidistant grids compare to solutions on adaptive grids, if the same number of grid points is used? Second, how many equidistant grid points are needed to match the accuracy of ASI?
When using the same number or slightly more points, the equidistant grid scheme is slightly faster. However, the difference is quite small, reinforcing our claim that adapting the grid takes very little time compared to overall computing time. More importantly, in all examples our algorithm outperforms the standard grid scheme between one and two orders of magnitude in terms of maximum Euler errors. This holds both for Euler errors drawn over the whole state space and along the equilibrium path. In the first example of Table 2.1, where we compare our results to an equidistant grid with about the same number of points, the adaptive grid yields maximum Euler errors that are about 70 times lower both on the state space and along the equilibrium path. Regarding the average Euler error, these factors are slightly lower but still substantial. We get these lower factors for average Euler errors, because the adaptive grid scheme rather targets the maximum Euler error by placing grid points on kinks, and not elsewhere in the state space. However, for two reasons the impact on average errors is also quite substantial. First, errors at the kinks

Bond Economy with Three Agents: Match Accuracy

			EE state space		EE equilibrium path	
$\underline{b}$	points	time(min)	max EE	∅ EE	max EE	∅ EE
0.1	**40**(20301)	**0.5**(79)	**-3.0**(-2.8)	**-4.1**(-5.3)	**-3.0**(-3.1)	**-4.4**(-6.2)
1.0	**185**(21945)	**6.5**(300)	**-2.1**(-1.9)	**-3.1**(-4.4)	**-2.2**(-1.9)	**-3.1**($-4, 6$)

Table 2.3: Accuracy of Adaptive Grid (Equidistant Grid in Brackets)

are lowered dramatically, having a sizable effect on the average error. And second, even at a point located elsewhere, kinks may still play a role, because agents potentially end up near a kink tomorrow. [7]

As a second exercise, we ask how many equidistant grid points are needed to get the same maximum Euler error as with a given adapted grid. Instead of targeting the number of grid points as above, we therefore target the maximum Euler error over the state space. For the first example with $\underline{b} = 0.1$, we increase the grid size by a factor of 500. Interestingly, adaptive simplicial interpolation still outperforms the equidistant grid in terms of maximum Euler as reported in Table 2.3. Obviously, in terms of average Euler errors, taking 500 times more points makes a big difference, resulting in a lower error for the equidistant grid. For $\underline{b} = 1.0$, due to memory constraints, we cannot multiply the number of grid points by 500. We therefore increase the grid size by a factor of 120, which yields maximum errors that are still higher than with adaptive simplicial interpolation.

When it comes to four agents, we also find that ASI outperforms equidistant grid points by far, as the results in Table 2.4 suggest. Trying to match the maximum Euler Error from the ASI example, we increase the amount of grid points by a factor of 200 for $\underline{b} = 0.1$ and 20 for $\underline{b} = 1.0$. For both cases we find that the maximum Euler Error on the equidistant grid is still far higher.

Comparison to ad hoc Update

Finally we compare the accuracy of equilibria computed with ASI to the accuracy of equilibria computed with an ad hoc update scheme. Using the solution from the initial grid this scheme detects which simplices are cut by a kink. Instead of adding points exactly on the kink as done by ASI, the ad hoc update randomly places additional grid points into

[7] At these points, the equilibrium policy functions also exhibit non-differentiabilities. These are induced by the kinks in next period's policy. An extension of ASI could identify these non-differentiablities as well.

Bond Economy with Four Agents: Match Accuracy

			EE state space		EE equilibrium path	
$\underline{b}$	points	time(min)	max EE	∅ EE	max EE	∅ EE
0.1	**112**(20825)	**4.5**(895)	**-2.7**(−2.0)	**-3.3**(−3.6)	**-2.7**(−2.1)	**-3.9**(−4.0)
1.0	**914**(20825)	**90**(3655)	**-1.7**(−1.1)	**-2.6**(−3.0)	**-1.8**(−1.1)	**-2.6**(−1.9)

Table 2.4: Accuracy of Adaptive Grid (Equidistant Grid in Brackets)

these simplices. To compare this ad hoc update scheme with ASI we now compute equilibria for the examples considered above using the same initial grid as with ASI. As the results in Table 2.5 and 2.6 suggest ASI outperforms such an ad hoc update, even if we use up to 200 times more grid points.

Bond Economy with Three Agents: Comparison to ad hoc Update

			EE state space		EE equilibrium path	
$\underline{b}$	points	time(min)	max EE	∅ EE	max EE	∅ EE
0.1	**40**(8000)	**0.5**(40)	**-3.0**(−1.3)	**-4.1**(−2.7)	**-3.0**(−2.7)	**-4.4**(−4.5)
1.0	**185**(8000)	**6.5**(122)	**-2.1**(−2.0)	**-3.1**(−3, 1)	**-2.2**(−1.9)	**-3.1**(−4, 6)

Table 2.5: Accuracy of Adaptive grid (Grid with ad hoc Update in Brackets)

Bond Economy with Four Agents: Comparison to ad hoc Update

			EE state space		EE equilibrium path	
$\underline{b}$	points	time(min)	max EE	∅ EE	max EE	∅ EE
0.1	**112**(8000)	**4.5**(333)	**-2.7**(−2.3)	**-3.3**(−3.9)	**-2.7**(−2.4)	**-3.9**(−4.3)
1.0	**914**(20825)	**90**(3655)	**-1.7**(−1.3)	**-2.6**(−3.0)	**-1.8**(−1.1)	**-2.6**(−1.9)

Table 2.6: Accuracy of Adaptive Grid (Grid with ad hoc Update in Brackets)

2.4 Extension: Endogenous Collateral Constraints

2.4.1 The Bond and Stock Economy

Setup

We extend the bond economy of Section 2.2.1 by introducing a Lucas tree-type stock which is in unit net supply. It pays out a fixed fraction δ of aggregate endowment each period, i.e. stock holders receive dividends $d(x) = \delta \cdot \bar{e}(x)$ per unit of the stock. Hence, aggregate endowment is given by the sum of individual endowments and dividends, i.e.

$$\bar{e}(x) = \sum_{h \in \mathbb{H}} e^h(x) + d(x) \;\; \forall x \in X.$$

The Lucas tree is traded each period after dividends are paid. Each agent h buys l^h shares of the stock at a price q. Hence, agents face the following budget constraints:

$$c_t^h + b_t^h p_t + l_t^h q_t \leq e_t^h + b_{t-1}^h + l_{t-1}^h \left(q_t + d_t\right) \;\; \forall t = 1, \ldots, T \;\; \forall h \in \mathbb{H}.$$

Moreover, trade in the bond and the stock is subject to constraints. First, we impose a *short-selling constraint* on the stock, i.e.

$$l_t^h \geq 0 \;\; \forall t = 1, \ldots, T \;\; \forall h \in \mathbb{H}.$$

In contrast to the stock, the bond may be shorted. However, only if the stock is used as collateral. More precisely, the short position in the bond may not exceed the minimal value—in terms of resale value plus dividends—that the stock has next period:

$$-b_t^h \leq \min_{x_{t+1} \in X} \left\{ l_t^h \left(q(s_{t+1}) + d(x_{t+1}) \right) \right\}, \forall t = 1, \ldots, T \;\; \forall h \in \mathbb{H},$$

where tomorrow's state is $s_{t+1} = (x_{t+1}, y_{t+1})$. The endogenous part of the state, y_{t+1}, will be specified below. This constraint is motivated by a bankruptcy law which makes it possible to seize an agents' stock holding, but not his income. To put it differently, all future income is exempted. As there is no further punishment for default, an agent will default on his asset position, if and only if his portfolio has a negative value. As this behavior is anticipated—and we assume that default premia may not be charged—no agent will be allowed to acquire such a portfolio, which imposes the above constraint.

State Space

With the above collateral constraint, financial wealth,

$$w_t^h \equiv l_{t-1}^h \left(q(s_t) + d(x_t)\right) + b_{t-1}^h,$$

cannot go below zero. Hence, the fraction of total financial wealth that an agent holds,

$$y^h = \frac{w^h}{\sum_{j \in \mathbb{H}} w^j},$$

is bounded between zero and one. By market clearing, we may use the fractions of financial wealth of the first $H-1$ agents as the endogenous state space:

$$y = \left(y^1, \ldots, y^{H-1}\right) \in Y \equiv \left\{ y \in \mathbb{R}_+^{H-1} \middle| \sum_{i=1}^{H-1} y^i \leq 1 \right\} \subset \mathbb{R}_+^{H-1}.$$

Finally, we define the whole state space S as the product of the exogenous part and the endogenous part, i.e.

$$S = X \times Y.$$

With this definition of the state space, reconsider the collateral constraint above, and note that: Todays choice of any agent, through its impact on tomorrows state, influences tomorrows price of the stock, and hence today's collateral constraint of agent h. In this sense, the collateral constraint is endogenous, which complicates the model considerably.

Equilibrium Conditions

The endogenous choices and prices in period t are

$$z_t \equiv \left(\left(c_t^h, b_t^h, l_t^h\right)_{h \in \mathbb{H}}, p_t, q_t \right).$$

In Appendix 2.B we define competitive equilibrium and derive the first-order equilibrium conditions of this model. Along an equilibrium path, policies have to satisfy market clearing on both asset markets, budget constraints, Euler equations for both assets, and complementary slackness conditions for both kinds of multipliers:

$$
\begin{aligned}
&\sum_{h\in\mathbb{H}} b_t^h = 0, \quad \sum_{h\in\mathbb{H}} l_t^h = 1, \\
&c_t^h + b_t^h p_t + l_t^h q_t - e_t^h - b_{t-1}^h - l_{t-1}^h \left(q_t + d_t\right) = 0, \ \ \forall h \in \mathbb{H}, \\
&-u'(c_t^h)p_t + \mu^h + \mathbb{E}\left[\beta u'(c_{t+1}^h)\right] = 0, \ \ \forall h \in \mathbb{H}, \\
&-u'(c_t^h)q_t + \mu^h \min_{x_{t+1}\in X} \left\{q(s_{t+1}) + d(x_{t+1})\right\} + \nu_t^h + \mathbb{E}\left[\beta u'(c_{t+1}^h)\left(q_{t+1} + d_{t+1}\right)\right] = 0, \ \ \forall h \in \mathbb{H}, \\
&0 \leq \min_{x_{t+1}\in X} \left\{l_t^h \left(q(s_{t+1}) + d(x_{t+1})\right) + b_t^h\right\} \ \perp \ \mu_t^h \geq 0, \ \ \forall h \in \mathbb{H}, \\
&0 \leq l_t^h \ \perp \ \nu_t^h \geq 0, \ \ \forall h \in \mathbb{H},
\end{aligned}
$$

where μ^h and ν^h denote the Kuhn-Tucker multipliers on the collateral and the short-selling constraint of agent h.

2.4.2 Computational Performance

Before we look at errors in Euler equations, we first discuss how the kinks induced by the short selling and collateral constraints are located within the state space. Figure 2.6 shows the adapted grid for an exogenous state where the first agent is hit by a bad idiosyncratic shock. To clearly visualize the kinks, we highlight the edges that connect adapted points. The short selling constraint of the first agent induces a kink which has two components, the one which lies almost on the y-axis and the curved one to the very right. Furthermore, each of the collateral constraints induces one kink, where the kink from the first agent's constraint runs approximately parallel to the y-axis at about 0.08 fraction of wealth of agent 1. In Figure 2.7 one can see how these kinks shape equilibrium an equilibrium policy function. The left hand picture displays the stock demand over the full state space, whereas the picture on the right hand side displays a slice at 0.1 wealth fraction of agent 2. The distinct peak at 0.08 wealth fraction of agent 1 corresponds to the kink induced by his collateral constraint. To the left, the collateral constraint is binding. At higher levels of wealth his demand for the stock goes down until the short selling constraint becomes binding again.

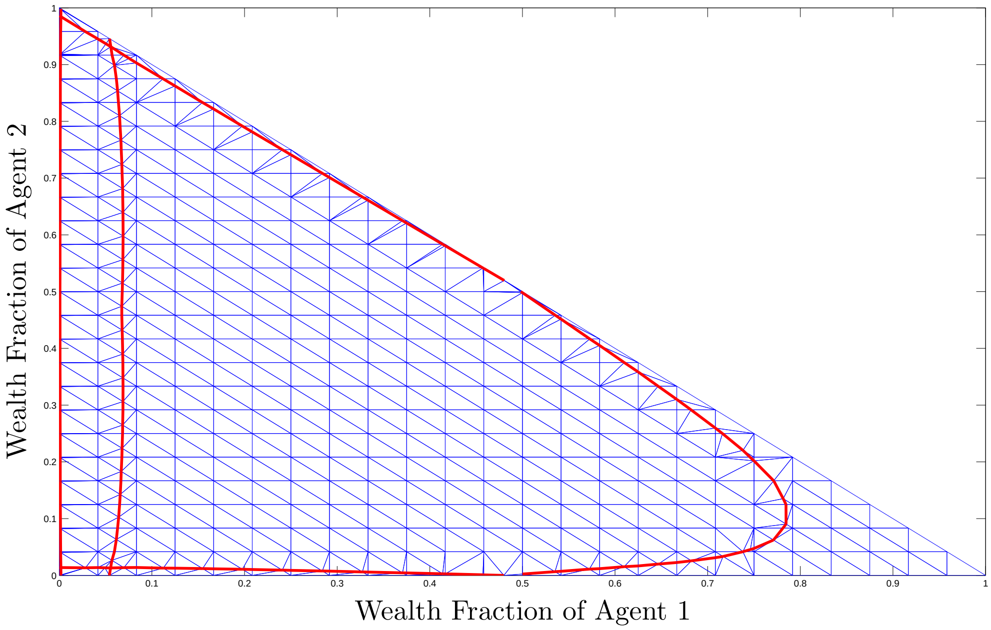


Figure 2.6: Bond and Stock Economy: Adapted grid with Several Identified Kinks

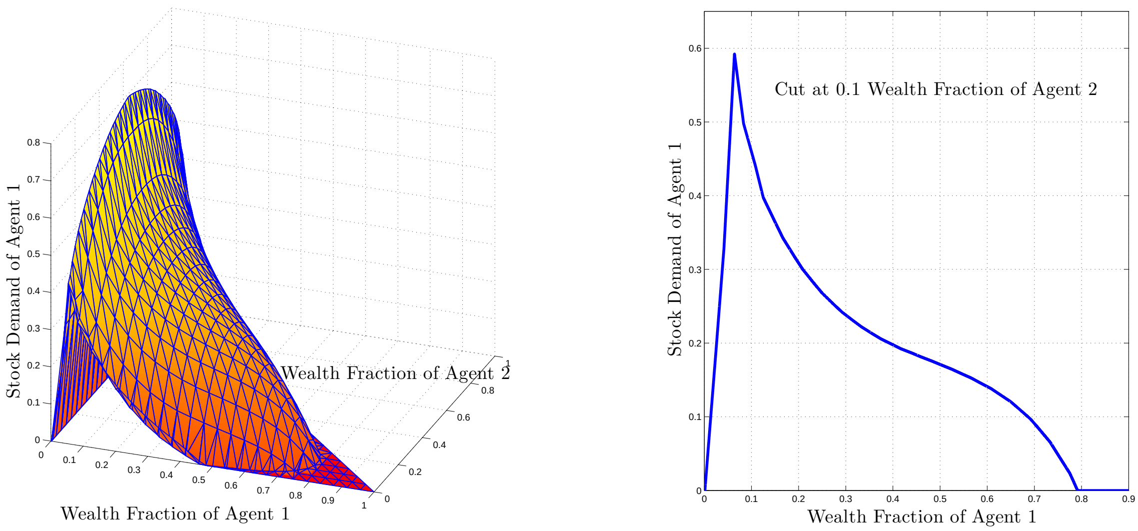


Figure 2.7: Bond and Stock Economy: 2D Stock Demand (lhs) and 1D Slice (rhs)

As in Section 2.3.4, we evaluate the performance of our algorithm by computing relative errors in Euler equations. In Table 2.7, we show results for equilbria computed with ASI using two different values for the dividend parameter δ. For all other parameters, we use the same calibration as for the Bond economy (see Appendix 2.E). Obviously, as the figures above suggest, more points are needed than in the bond model to bring Euler errors down to reasonable values. Comparing the results from ASI with results on equidistant grids, we find that for the same number of grid points, ASI outperforms equidistant grids by approximately one order of magnitude in terms of maximum Euler Error. Again, we ask how many points are needed to match the accuracy of ASI. Increasing the number of points up to a factor of 20 yields almost the same maximum Euler Error, as the results in Table 2.8 show. This factor is still substantial, however, not as high as for the Bond model. The reason are non-linearities away from the kink, as can be seen in Figure 2.7. We have developed an adaptation scheme that adapts the grid to non-linearities, which further improves the relative performance of our algorithm. However, as this is not the focus of this paper, we do not elaborate more on this.

Bond and Stock Economy

			EE state space		EE equilibrium path	
δ	points	time(min)	max EE	∅ EE	max EE	∅ EE
0.10	**1235**(1250)	**310**(260)	**-2.5**(−1.4)	**-3.8**(−3.2)	**-3.1**(−1.6)	**-4.1**(−3.5)
0.25	**1160**(1225)	**302**(251)	**-2.2**(−1.4)	**-3.3**(−2.9)	**-2.2**(−1.4)	**-3.4**(−2.7)

Table 2.7: Accuracy of Adaptive grid (Equidistant Grid in Brackets)

Bond and Stock Economy: Match Accuracy

			EE state space		EE equilibrium path	
δ	points	time(min)	max EE	∅ EE	max EE	∅ EE
0.10	**1235**(25425)	**310**(4500)	**-2.5**(−2.4)	**-3.8**(−4.0)	**-3.1**(−2.6)	**-4.1**(−4.2)
0.25	**1160**(25425)	**302**(4812)	**-2.2**(−2.1)	**-3.3**(−4.2)	**-2.2**(−2.3)	**-3.4**(−4.4)

Table 2.8: Accuracy of Adaptive Grid (Equidistant Grid in Brackets)

2.5 Conclusion

This paper presents an algorithm that is tailor-made for computing equilibria in dynamic models with occasionally binding constraints. To directly address the problem of kinks in such models, we develop a new interpolation technique based on adaptive grids and simplicial interpolation. We show that Adaptive Simplicial Interpolation accurately computes equilibria in dynamic models with several continuous state variables and various inequality constraints. Comparison studies show that our method outperforms standard grid techniques by up to two orders of magnitude in terms of maximum errors in Euler equations. Clearly, occasionally binding constraints become more and more important in quantitative economics, e.g. in modeling financial frictions. Hence, we hope that ASI will help economists in solving such models.

Appendix

2.A Details Bond Economy

In this appendix, we define competitive equilibrium and derive first-order equilibrium conditions for the bond economy presented in Section 2.2.1. For this purpose, some additional notation is needed. We denote the shock at time t by x_t, but the history of shocks that occurred up to period t by x^t. The set of histories up to period t is denoted by X^t, and the set of all possible histories by $\mathbb{X} \equiv \bigcup_{t=1}^{T} X^t$. For x^{t+1} being a possible successor of x^t we write $x^{t+1} \geq x^t$. Finally, the probability of history x^t is denoted by $\pi(x^t)$ and the conditional transition probability by $\pi(x^{t+1}|x^t)$

Competitive Equilibrium

A competitive equilibrium for an economy with agents' initial bond holdings

$$\left(b_0^h\right)_{h\in\mathbb{H}}$$

is a collection

$$\{z(x^t)\}_{x^t\in\mathbb{X}} \equiv \left\{\left(c^h(x^t), b^h(x^t)\right)_{h\in\mathbb{H}}, p(x^t)\right\}_{x^t\in\mathbb{X}}$$

of consumption allocations, bond holdings, and bond prices that satisfy the following conditions:

1. Markets clear[8]:

$$\sum_{h\in\mathbb{H}} b^h(x^t) = 0 \quad \forall x^t \in \mathbb{X}.$$

2. Given prices $(p(x^t))_{x^t\in\mathbb{X}}$, each agent chooses

$$\left(c^h(x^t), b^h(x^t)\right)\right)_{x^t\in\mathbb{X}}$$

[8] By Walras' Law market clearing in the asset market(s) implies market clearing in the consumption goods market.

to maximize lifetime utility such that $\forall x^t \in \mathbb{X}$ the following constraints hold:

$$\begin{aligned} \textit{budget constraint} \qquad & c^h(x^t) + b^h(x^t)p(x^t) \leq e^h(x^t) + b^h(x^{t-1}), \\ \textit{borrowing constraint} \qquad & b^h(x^t) \geq \underline{b}. \end{aligned}$$

First-Order Equilibrium Conditions

Each individual agent faces the following optimization problem:

$$\begin{aligned} & \max_{(c(x^t),b(x^t))_{x^t\in\mathbb{X}}} \quad \mathbb{E}\left[\sum_{t=1}^{T} \beta^t u(c(x^t))\right] \\ & \text{s.t. } \forall x^t \in \mathbb{X}: \\ \textit{budget constraint} \qquad & c^h(x^t) + b^h(x^t)p(x^t) \leq e^h(x^t) + b^h(x^{t-1}), \\ \textit{borrowing constraint} \qquad & b^h(x^t) \geq \underline{b}. \end{aligned}$$

Denote the multiplier associated with these constraints by $\lambda(x^t)$ and $\mu(x^t)$. Differentiating the Lagrangian with respect to the different choice variables gives

$$\begin{aligned} c(x^t): \qquad & \pi(x^t)\beta^t u'(c(x^t)) - \lambda(x^t) = 0 \\ c(x^{t+1}): \qquad & \pi(x^{t+1})\beta^{t+1} u'(c(x^{t+1})) - \lambda(x^{t+1}) = 0 \\ b(x^t): \qquad & -\lambda(x^t)p(x^t) + \mu(x^t) + \sum_{x^{t+1}\geq x^t} \left(\lambda(x^{t+1})\right) = 0 \end{aligned}$$

Substituting the first two FOCs into the last one, we get the following Euler equation for the bond:

$$-u'(c(x^t))p(x^t) + \mu(x^t) + \sum_{x^{t+1}\geq x^t} \beta\pi(x^{t+1}|x^t)u'(c(x^{t+1})) = 0.$$

In addition, the Kuhn-Tucker FOCs include the following complementarity condition:

$$0 \leq b(x^t) - \underline{b} \perp \mu(x^t) \geq 0.$$

Combined with market clearing conditions and budget constraints, these are the equilibrium conditions stated in Section 2.2.1.

2.B Details Bond and Stock Economy

In this appendix, we define competitive equilibrium and derive first-order equilibrium conditions for the economy presented in Section 2.4. The notation is as introduced in the beginning of Appendix 2.A.

Competitive Equilibrium

A competitive equilibrium for an economy with agents' initial portfolios

$$\left(b_0^h, l_0^h\right)_{h\in\mathbb{H}}$$

is a collection

$$\{z(x^t)\}_{x^t\in\mathbb{X}} \equiv \left\{\left(c^h(x^t), b^h(x^t), l^h(x^t)\right)_{h\in\mathbb{H}}, p(x^t), q(x^t)\right\}_{x^t\in\mathbb{X}}$$

of consumption allocations, bond and stock holdings, and prices that satisfy the following conditions:

1. Markets clear:

$$\sum_{h\in\mathbb{H}} b^h(x^t) = 0, \quad \sum_{h\in\mathbb{H}} l^h(x^t) = 1 \quad \forall x^t \in \mathbb{X}.$$

2. Given prices $(p(x^t), q(x^t))_{x^t\in\mathbb{X}}$, each agent chooses

$$\left(c^h(x^t), b^h(x^t), l^h(x^t))\right)_{x^t\in\mathbb{X}}$$

to maximize lifetime utility such that $\forall x^t \in \mathbb{X}$ the following constraints hold:

$$\begin{aligned}
\textit{budget constraint} \qquad & c^h(x^t) + b^h(x^t)p(x^t) + l^h(x^t)q(x^t) \leq \\
& e^h(x^t) + b^h(x^{t-1}) + l^h(x^{t-1})\left(q_t(x^t) + d_t(x^t)\right), \\
\textit{short selling constraint} \qquad & l^h(x^t) \geq 0 \text{ and} \\
\textit{collateral constraints} \qquad & \min_{x^{t+1}\geq x^t} \left\{l^h(x^t)\left(q(x^{t+1}) + d(x^{t+1})\right) + b^h(x^t)\right\} \geq 0.
\end{aligned}$$

First-Order Equilibrium Conditions

Each individual agent faces the following optimization problem:

$$
\begin{aligned}
& \max_{(c(x^t),b(x^t),l(x^t))_{x^t\in\mathbb{X}}} \quad \mathbb{E}\left[\sum_{t=1}^{T}\beta^t u(c(x^t))\right] \\
& \text{s.t. } \forall x^t \in \mathbb{X}: \\
\textit{budget constraint} \quad & c^h(x^t) + b^h(x^t)p(x^t) + l^h(x^t)q(x^t) \le \\
& e^h(x^t) + b^h(x^{t-1}) + l^h(x^{t-1})\left(q_t(x^t) + d_t(x^t)\right), \\
\textit{short selling constraint} \quad & l^h(x^t) \ge 0 \ \text{ and} \\
\textit{collateral constraints} \quad & \min_{x^{t+1}\ge x^t}\left\{l^h(x^t)\left(q(x^{t+1}) + d(x^{t+1})\right) + b^h(x^t)\right\} \ge 0.
\end{aligned}
$$

Denote the multipliers associated with these constraints by $\lambda(x^t)$, $\nu(x^t)$, and $\mu(x^t)$. Differentiating the Lagrangian gives

$$
\begin{aligned}
c(x^t): \quad & \pi(x^t)\beta^t u'(c(x^t)) - \lambda(x^t) = 0 \\
c(x^{t+1}): \quad & \pi(x^{t+1})\beta^{t+1} u'(c(x^{t+1})) - \lambda(x^{t+1}) = 0 \\
b(x^t): \quad & -\lambda(x^t)p(x^t) + \mu(x^t) + \sum_{x^{t+1}\ge x^t}\left(\lambda(x^{t+1})\right) = 0 \\
l(x^t): \quad & \nu(x^t) - \lambda(x^t)q(x^t) + \mu(x^t)\min_{x^{t+1}\ge x^t}\left\{q(x^{t+1}) + d(x^{t+1})\right\} \\
& + \sum_{x^{t+1}\ge x^t}\left(\lambda(x^{t+1})\right)\left(q(x^{t+1}) + d(x^{t+1})\right) = 0.
\end{aligned}
$$

Substituting the first two FOCs into the last two, we get the following Euler equations for the bond and the stock:

$$
\begin{aligned}
& -u'(c(x^t))p(x^t) + \mu(x^t) + \sum_{x^t+\ge x^t}\left(\beta\pi(x^{t+1}|x^t)u'(c(x^{t+1}))\right) = 0, \\
& \nu(x^t) - u'(c(x^t))q(x^t) + \mu(x^t)\min_{x^{t+1}\ge x^t}\left\{q(x^{t+1}) + d(x^{t+1})\right\} \\
& + \sum_{x^{t+1}\ge x^t}\left(\beta\pi(x^{t+1}|x^t)u'(c(x^{t+1}))\right)\left(q(x^{t+1}) + d(x^{t+1})\right) = 0.
\end{aligned}
$$

In addition, the Kuhn-Tucker FOCs include the following complementarity conditions:

$$
\begin{aligned}
&0 \leq \min_{x^{t+1} \geq x^t} \left\{ l^h(x^t) \left(q(x^{t+1}) + d(x^{t+1}) \right) + b^h(x^t) \right\} \perp \mu(x^t) \geq 0 \\
&0 \leq l(x^t) \perp \nu(x^t) \geq 0.
\end{aligned}
$$

Combined with market clearing conditions and budget constraints, these are the equilibrium conditions stated in Section 2.4.

2.C Transforming Complementarities into Equations

At the initial gridpoints, ASI solves the follwing complementarity problem:

$$
\begin{aligned}
&\text{Given a state } s \in S \text{ , and functions} \\
&\phi : S \times \mathbb{R}^{m+n} \to \mathbb{R}^m, \;\; \psi : S \times \mathbb{R}^m \to \mathbb{R}^n, \\
&\text{find policies and multipliers } (z, \mu) \in \mathbb{R}^m \times \mathbb{R}^n, \\
&\;\text{s.t. } \phi(s, z, \mu) = 0, \quad 0 \leq \psi(s, z) \perp \mu \geq 0.
\end{aligned}
$$

Following Garcia and Zangwill (1981), we transform this complementarity problem into a system of equations, to be able to apply a standard non-linear equation solver. Key to the transformation are the following definitions:

$$
\alpha \equiv \begin{cases} \mu & \text{for} \quad \mu \geq 0, \;\; \psi(s,z) = 0 \\ -\psi(s,z) & \text{for} \quad \mu = 0, \;\; \psi(s,z) > 0 \end{cases}
$$

and

$$
\begin{aligned}
\alpha^+ &= (\max(0, \alpha))^k \\
\alpha^- &= (\max(0, -\alpha))^k,
\end{aligned}
$$

where $k \in \mathbb{N}^+$. Using these definitions, the problem reads:

$$\begin{aligned}
&\text{Given a state } s \in S \text{ , and functions} \\
&\phi : S \times \mathbb{R}^{m+n} \to \mathbb{R}^m, \ \ \psi : \mathbb{R}^m \to S \times \mathbb{R}^n, \\
&\text{find policies and alphas } (z, \alpha) \in \mathbb{R}^m \times \mathbb{R}^n, \\
&\ \text{s.t. } \phi(s, z, \alpha^+) = 0, \ \ \psi(s, z) - \alpha^- = 0.
\end{aligned}$$

2.D Alternative Error Measure

As explained in Section 2.3.4, measuring accuracy in models with occasionally binding constraints using EEs is not unproblematic. We therefore suggest an alternative error measure. To apply this measure we need to solve the equilibrium system (given the solution from the time iteration algorithm as tomorrow's policies) at the point in the state space where we want to measure accuracy. From that solution, we get consumption values c_{opt} for all agents. Then, we compare these values to the interpolated consumption values c_{int}. In spirit of the Euler Error we compute the relative deviation of the interpolated policy from the optimal solution. Hence, the error is given by $E = \left|\frac{c_{int}}{c_{opt}} - 1\right|$. In the tables below we report the maximum and average errors over the state space and along the equilibrium path for the same examples as in Section 2.3.4. With respect to the alternative error measure, ASI still outperforms standard equidistant grid schemes by far. However, the difference in accuracy is not as extreme as with EEs.

Bond Economy with Three Agents

			E state space		E equilibrium path	
$\underline{b}$	points	time(min)	max E	∅ E	max E	∅ E
0.1	**40**(45)	**0.5**(0.4)	**-3.2**(−2.6)	**-4.1**(−3.5)	**-3.2**(−2.6)	**-4.7**(−4.0)
0.1	**113**(120)	**1.1**(1.0)	**-3.3**(−2.5)	**-4.5**(−3.6)	**-3.3**(−2.5)	**-5.1**(−4.2)
1.0	**185**(190)	**6.5**(4.5)	**-2.3**(−1.7)	**-3.2**(−3.0)	**-2.5**(−1.7)	**-3.2**(−2.7)
1.0	**941**(946)	**13**(11)	**-3.3**(−2.3)	**-4.5**(−3.6)	**-3.3**(−2.5)	**-5.1**(−3.3)

Table 2.9: Accuracy of Adaptive Grid (Equidistant Grid in Brackets)

Bond Economy with Four Agents

			E state space		E equilibrium path	
$\underline{b}$	points	time(min)	max E	∅ E	max E	∅ E
0.1	**112**(120)	**4.5**(4)	**-2.9**(−1.9)	**-3.5**(−2.7)	**-2.9**(−2.0)	**-4.0**(−3.0)
1.0	**914**(969)	**60**(51)	**-1.9**(−1.5)	**-2.7**(−2.6)	**-1.9**(−1.5)	**-2.8**(−3.4)

Table 2.10: Accuracy of Adaptive Grid (Equidistant Grid in Brackets)

Bond and Stock Economy

			E state space		E equilibrium path	
δ	points	time(min)	max E	∅ E	max E	∅ E
0.10	**1235**(1250)	**310**(260)	**-2.1**(−1.8)	**-3.5**(−3.4)	**-2.1**(−1.8)	**-3.6**(−3.4)
0.25	**1160**(1225)	**302**(251)	**-2.2**(−1.8)	**-3.3**(−2.2)	**-2.2**(−1.9)	**-3.4**(−3.2)

Table 2.11: Accuracy of Adaptive Grid (Equidistant Grid in Brackets)

2.E Parameterization

We set the discount factor $\beta = 0.95$ and the risk aversion parameter $\gamma = 1.5$ for all agents. Concerning the exogenous shock process, we make the following choices: We assume that agents may either receive a good or a bad idiosyncratic shock. One agent always gets the bad shock and all others get the good one. This results in three or four states per aggregate shock, depending on the number of agents. Allowing for two aggregate shocks the exogenous part of the state space comprises six or eight states respectively. We denote the ratios of good to bad idiosyncratic and aggregate shocks by ν_{idio} and ν_{agg}. We finally denote the persistence of idiosyncratic and aggregate shocks by ρ_{idio} and ρ_{agg}. We compute equilibria for two values of the borrowing limit $\underline{b}$, namely $\underline{b} = 0.1$ and 1, i.e. borrowing up to 10% or 100% of average individual yearly income. All parameter values can be found in Table 2.12.

γ	ν_{idio}	ν_{agg}	ρ_{idio}	ρ_{agg}	β	$\underline{b}$
1.5	1.6	1.06	0.9	0.65	0.95	0.1/1.0

Table 2.12: Parameter Values

Chapter 3

Collateral Requirements and Asset Prices

3.1 Introduction

The vast majority of debt, especially if it extends over a long period of time, is guaranteed by tangible assets called collateral. For example, residential homes serve as collateral for short- and long-term loans to households, and investors can borrow money to establish a position in stocks, using these as collateral. The margin requirement dictates how much collateral one has to hold in order to borrow one dollar. Clearly these margin requirements will have important implications for the price of collateral. In the recent financial crisis it was argued that excessively low margin requirements were part of the cause of the crisis. In this paper, we conduct a quantitative study on the effect of margins requirements on asset prices.

Many previous papers have formalized the idea that borrowing on collateral might give rise to cyclical fluctuations in real activity and enhance volatility of prices (see e.g. Geanakoplos (1997), Kiyotaki and Moore (1997) and Aiyagari and Gertler (1999)). In these models, it is possible to have substantial departures of the market price from the corresponding price under frictionless markets. These results have led researchers to suggest that by managing leverage (or the amount of collateralized borrowing), a central bank can reduce aggregate fluctuations (see e.g. Ashcraft, Gârleanu, and Pedersen (2010) or Geanakoplos (2009)). However, establishing the quantitative importance of collateral requirements as a source of excess volatility has been a challenge in the literature (see Kocherlakota (2000) or Cordoba and Ripoll (2004)). Moreover, so far, there have been few quantitative studies that take into account that a household can use several different assets as collateral, and that regulated margin requirements for loans on one asset might have important effects on the volatility of other assets in the economy.

In this paper we consider a Lucas (1978) style exchange economy with heterogeneous agents and collateral constraints. We assume that agents can only take short positions if they hold an infinitely-lived asset (a Lucas tree) as a long position. This model was first analyzed by Kubler and Schmedders (2003) and subsequently used by Cao (2010) and Brumm and Grill (2010). As in Kubler and Schmedders (2003) we assume that agents can default on a negative bond position at any time without any utility penalties or loss of reputation. Financial securities are therefore only traded if the promises associated with these securities are backed by collateral. Our main focus is on an economy with two trees which can be used as collateral for short-term loans. For the first tree the collateral requirement is determined endogenously while the collateral requirement for loans on the second tree is exogenously regulated. We show that the presence of collateral constraints and the endogenous margin requirements for the first tree lead to large excess price-volatility of the second tree. Changes in the regulated margin requirements for the second tree have large effects on the volatility of both trees. While tightening margins for loans on the second tree always decreases the price volatility of the first tree, price volatility of the second tree might very well increase with this change. In our calibration we allow for the possibility of disaster states. This leads to very large quantitative effects of collateral requirements and to realistic equity risk premia.

Margin requirements are a crucial feature of our model. They determine with how much leverage agents can invest in risky assets. Following Geanakoplos (1997) and Geanakoplos and Zame (2002), we endogenize the margin requirements by introducing a menu of financial securities. All securities promise the same payoff, but they distinguish themselves by their respective margin requirement. In equilibrium only some of them are traded, thereby determining an endogenous margin requirement. This implies, of course, that for many bonds and many next period's shocks, the face value of the debt falls below the value of the collateral. As a result there is default in equilibrium. However, in an extension of the model we allow for costly default by introducing a real cost to the lender. We examine the impact of such default costs on equilibrium trading volume and prices. As an alternative to endogenous margin requirements, we also consider regulated margin requirements. In particular, our two-tree economy allows us to compare a tree with endogenous margins to a tree with regulated margins.[1]

In our calibration of the model there are two heterogeneous agents with Epstein-Zin utility. They have identical elasticities of substitution (IES) but distinguish themselves by their risk-aversion (RA). The agent with the low risk aversion is the natural buyer of risky assets and takes on leverage to finance these investments. The agent with the high risk aversion has a strong insurance motive against bad shocks and, therefore, is a natural buyer of safe bonds and a natural seller of risky assets. The idea behind this model setup is as follows. When the economy is hit with a negative shock, the collateral constraint forces the leveraged agent to reduce consumption or to even sell risky assets to the risk-averse agent, thereby resulting in substantial changes in the wealth distribution which in turn affect agents' portfolios and asset prices.

We start our analysis with an economy with a single Lucas tree that can be used as collateral. In this baseline model we exogenously assume that collateral requirements are set to the lowest possible level that still ensures that there is never default in equilibrium. To obtain a sizable market price of risk, we follow the specification in Barro and Jin (2011) and introduce the possibility of 'disaster shocks' into the otherwise standard calibration. In this model, the effect of scarce collateral on the volatility of the tree is quantitatively large. We then allow agents to choose from a menu of bonds with different margin requirements which are determined in equilibrium. Agents do trade bonds that have a positive probability of default. However, as soon as we introduce moderate default cost, trade in these default bonds is shut down.

The main contribution of the paper is the analysis of an economy with two trees which have identical cash-flows but distinguish themselves by their 'collateralizability'. We first

[1]Depending on the asset that is used as collateral, market forces might play an important role in establishing margin requirements. For stocks the situation is not obvious: The Federal Reserve Board sets minimum margin requirements for broker-dealer loans, using what is called Regulation T. In fact, until 1974, the Fed considered initial margin percentages as an active component of monetary policy and changed them fairly often (see Willen and Kubler (2006)). In the US housing market, there are no such regulations and margins can be arbitrarily small.

analyze a specification of the model in which only the first tree can be used as collateral. In this specification, the return volatility of the collateralizable tree is significantly smaller than that of the single tree in the baseline model. However, the volatility of the second tree, which cannot be used as collateral, is comparable. A possible interpretation of these findings is to identify the collateralizable tree with housing and the non-collateralizable tree with the aggregate stock market. Using stocks as collateral is subject to many regulations and often very costly, while individuals can easily use houses. Volatility and excess returns for houses is much smaller than for stocks, which is in line with our findings.

We then relax the assumption of the non-collaterizability of the second tree. We assume that a regulating agency sets an exogenous margin requirement for this tree. We find that regulation of the second tree has a strong impact on the volatility of the first tree. In particular, a tightening of margin requirements for the regulated tree uniformly decreases volatility of the unregulated tree. For the regulated tree, tighter margins initially increase the price volatility but then decrease it once margins become very large. We further show how the regulation of margin requirements only in times when the economy exhibits strong growth can substantially decrease volatility compared to the case of uniform regulation of margin requirements. This result holds true both for the baseline model with a single tree as well as the two-tree economy and suggests a strong policy recommendation for counter-cyclical margin requirements.

Finally, we conduct a thorough sensitivity analysis and show that our qualitative results are robust to the actual parametrization of the economy. In particular, we document that the key effects for the two-tree economy are robust to changes in the magnitude of the disaster shocks.

The remainder of this paper is organized as follows. We introduce the model in Section 3.2. In Section 3.3 we discuss results for economies with a single tree. Section 3.4 focuses on economies with two trees. In Section 3.5 we consider extensions and sensitivity analysis. Section 3.6 concludes.

3.2 The Economic Model

We examine a model of an exchange economy that extends over an infinite time horizon and is populated by infinitely-lived heterogeneous agents.

3.2.1 Infinite-Horizon Economy

This section describes the details of the infinite-horizon economy.

The Physical Economy

Time is indexed by $t = 0, 1, 2, \ldots$. A time-homogeneous Markov chain of exogenous shocks (s_t) takes values in the finite set $\mathcal{S} = \{1, \ldots, S\}$. The $S \times S$ Markov transition matrix is denoted by π. We represent the evolution of time and shocks in the economy by a countably infinite event tree Σ. The root node of the tree represents the initial shock s_0. Each node of the tree, $\sigma \in \Sigma$, describes a finite history of shocks $\sigma = s^t = (s_0, s_1, \ldots, s_t)$ and is also called date-event. We use the symbols σ and s^t interchangeably. To indicate that $s^{t'}$ is a successor of s^t (or s^t itself) we write $s^{t'} \succeq s^t$. We use the notation s^{-1} to refer to the initial conditions of the economy prior to $t = 0$.

At each date-event $\sigma \in \Sigma$ there is a single perishable consumption good. The economy is populated by H agents, $h \in \mathcal{H} = \{1, 2, \ldots, H\}$. Agent h receives an individual endowment in the consumption good, $e^h(\sigma) > 0$, at each node. In addition, at $t = 0$ the agent owns shares in Lucas trees. We interpret these Lucas trees to be physical assets such as firms, machines, land or houses. There are A different such assets, $a \in \mathcal{A} = \{1, 2, \ldots, A\}$. At the beginning of period 0, each agent h owns initial holdings $\theta_a^h(s^{-1}) \geq 0$ of tree a. We normalize aggregate holdings in each Lucas tree, that is, $\sum_{h \in \mathcal{H}} \theta_a^h(s^{-1}) = 1$ for all $a \in \mathcal{A}$. At date-event σ, we denote agent h's (end-of-period) holding of Lucas tree a by $\theta_a^h(\sigma)$.

The Lucas trees pay positive dividends $d_a(\sigma)$ in units of the consumption good at all date-events. We denote aggregate endowments in the economy by

$$\bar{e}(\sigma) = \sum_{h \in \mathcal{H}} e^h(\sigma) + \sum_{a \in \mathcal{A}} d_a(\sigma).$$

The agents have preferences over consumption streams representable by the following recursive utility function, see Epstein and Zin (1989),

$$U^h(c, s^t) = \left\{ \left[c^h(s^t)\right]^{\rho^h} + \beta \left[\sum_{s_{t+1}} \pi(s_{t+1}|s_t) \left(U^h(c, s^{t+1})\right)^{\alpha^h} \right]^{\frac{\rho^h}{\alpha^h}} \right\}^{\frac{1}{\rho^h}},$$

where $\frac{1}{1-\rho^h}$ is the intertemporal elasticity of substitution (IES) and $1 - \alpha^h$ is the relative risk aversion of the agent.

Security Markets

At each date-event agents can engage in security trading. Agent h can buy $\theta_a^h(\sigma) \geq 0$ shares of tree a at node σ for a price $q_a(\sigma)$. Agents cannot assume short positions of the Lucas trees. Therefore, the agents make no promises of future payments when they trade shares of physical assets and thus there is no possibility of default.

In addition to the physical assets, there are J financial securities, $j \in \mathcal{J} = \{1, 2, \ldots, J\}$, available for trade. These assets are one-period securities in zero-net supply. Security j traded at node s^t promises a payoff of one unit of the consumption good at each immediate successor node s^{t+1}. We denote agent h's (end-of-period) portfolio of financial securities at date-event σ by $\phi^h(\sigma) \in \mathbb{R}^J$ and denote the price of security j at this date-event by $p_j(\sigma)$. Whenever an agent assumes a short position in a financial security j, $\phi_j^h(\sigma) < 0$, she promises a payment in the next period. In our economy such promises must be backed up by collateral holdings.

Collateral and Default

At each node σ, we associate with each financial security $j \in \mathcal{J}$ a tree $a(j) \in \mathcal{A}$ and a collateral requirement $k_{a(j)}^j(\sigma) > 0$. If an agent sells one unit of security j, then she is required to hold $k_{a(j)}^j(\sigma)$ units of tree $a(j)$ as collateral. If an asset a can be used as collateral for different financial securities, the agent is required to buy $k_{a(j)}^j(\sigma)$ shares for each security $j \in \mathcal{J}_a$, where $\mathcal{J}_a \subset \mathcal{J}$ denotes the set of financial securities collateralized by the same tree a. In the next period, the agent can default on her earlier promise. In this case the agent loses the collateral she had to put up. In turn, the buyer of the financial security receives this collateral associated with the initial promise.[2]

[2]Following Geanakoplos and Zame (2002) we make the strong assumption that an agent can default

Since there are no penalties for default, a seller of security j at date-event s^{t-1} defaults on her promise at a successor node s^t whenever the initial promise exceeds the current value of the collateral, that is, whenever

$$1 > k^j_{a(j)}(s^{t-1})\left(q_{a(j)}(s^t) + d_{a(j)}(s^t)\right).$$

The payment by a borrower of security j at node s^t is, therefore, always given by

$$f_j(s^t) = \min\left\{1, k^j_{a(j)}(s^{t-1})\left(q_{a(j)}(s^t) + d_{a(j)}(s^t)\right)\right\}.$$

Our model includes the possibility of costly default. This feature of the model is meant to capture default costs such as legal cost or the physical deterioration of the collateral asset. For example, it is well known that housing properties in foreclosure deteriorate because of moral hazard, destruction, or simple neglect. We model such costs by assuming that part of the collateral value is lost and thus the payment received by the lender is smaller than the value of the borrower's collateral. Specifically, the loss is proportional to the difference between the face value of the debt and the value of collateral, that is, the loss is

$$l_j(s^t) = \lambda\left(1 - k^j_{a(j)}(s^{t-1})\left(q_{a(j)}(s^t) + d_{a(j)}(s^t)\right)\right)$$

for some parameter $\lambda \geq 0$. The resulting payment to the lender of the loan in security j when $f_j(s^t) < 1$ is thus given by

$$r_j(s^t) = \max\left\{0, f_j(s^t) - l_j(s^t)\right\} = \max\left\{0, (1+\lambda)k^j_{a(j)}(s^{t-1})(q_{a(j)}(s^t) + d_{a(j)}(s^t)) - \lambda\right\}.$$

on individual promises without declaring personal bankruptcy and giving up all the assets he owns. There are no penalties for default and a borrower always defaults once the value of the debt is above the value of the collateral. Since this implies that the decision to default on a promise is independent of the debtor, we do not need to consider pooling of contracts as in Dubey, Geanakoplos, and Shubik (2000), even though there may be default in equilibrium. This treatment of default is somewhat unconvincing since default does not affect a household's ability to borrow in the future and it does not lead to any direct reduction in consumption at the time of default. Moreover, declaring personal bankruptcy typically results in a loss of all assets, and it is rarely possible to default on some loans while keeping the collateral for others. However, there do exist laws for collateralizable borrowing where default is possible without declaring bankruptcy. Examples include pawn shops and the housing market in many US states, in which households are allowed to default on their mortgages without defaulting on other debt. It is certainly true that the recent 2008 housing crises makes this assumption look much better.

If $f_j(s^t) = 1$ then $r_j(s^t) = f_j(s^t) = 1$. This repayment function does not capture all costs associated with default. For example, it does not allow for fixed costs which are independent of how much the collateral value falls short of the repayment obligation. However, our functional form offers the advantage that the resulting model remains tractable since the repayment function is continuous in the value of the collateral.

The specification of the collateral requirements $k_a^j(s^t)$ for bond j, tree a and across date-events s^t has important implications for equilibrium prices and allocations. The collateral levels $k_a^j(s^t)$ are endogenously determined in equilibrium. In this paper we examine two different rules for the endogenous determination of collateral levels. The first rule determines endogenous collateral requirements along the lines of Geanakoplos and Zame (2002). The second rule assumes exogenously regulated capital-to-value ratios which in turn lead to endogenous collateral requirements.

Default and Endogenous Collateral Requirements

One of the contributions of this paper is to endogenize collateral requirements in an infinite-horizon dynamic general equilibrium model. For this purpose, our first collateral rule follows Geanakoplos (1997) and Geanakoplos and Zame (2002) who suggest a simple and tractable way to endogenize collateral requirements. They assume that, in principle, financial securities with any collateral requirement could be traded in equilibrium. Only the scarcity of available collateral leads to equilibrium trade in only a small number of such securities. Our first rule follows this approach.

Recall that the S direct successors of a node s^t are denoted $(s^t, 1), \ldots, (s^t, S)$ and that $\mathcal{J}_a$ denotes the set of bonds collateralized by the same tree a. We define endogenous margin requirements for bonds $j \in \mathcal{J}_a$ collateralized by the same tree $a \in \mathcal{A}$ as follows. For each shock next period, $s' \in \mathcal{S}$, there is at at least one bond which satisfies $k_{a(j)}^j(s^t)\left(q_{a(j)}(s^t, s') + d_{a(j)}(s^t, s')\right) = 1$. For each bond in the set $\mathcal{J}_a$ the promised payoff is equal to the collateral in (generically) exactly a single state. Generically the set $\mathcal{J}_a$ thus contain exactly S bonds, however the bond with the lowest collateral requirement is redundant in our model because its payoff vector is collinear with the tree's dividend vector. (Therefore, we consider only models with at most $S - 1$ bonds in our numerical analysis of the model.) The arguments in Araújo, Kubler, and Schommer (2010) show that adding additional bonds with other collateral requirements (also only using tree a as collateral) do not change the equilibrium allocation. In the presence of S bonds as specified above, any bond with an intermediate collateral requirement can be replicated by holding a portfolio of the existing bonds using the same amount of collateral.

We begin our model examinations always with economies with a single bond, $J = 1$, on which agents cannot default. That is, the collateral requirements are endogenously set to the lowest possible value which still ensures no default in the subsequent period (this specification is similar to the collateral requirements in Kiyotaki and Moore (1997)). Formally, the resulting condition for the collateral requirement $k^1_{a(1)}(s^t)$ of this bond is

$$k^1_{a(1)}(s^t)\left(\min_{s^{t+1}\succ s^t}\left(q_{a(1)}(s^{t+1}) + d_{a(1)}(s^{t+1})\right)\right) = 1.$$

We refer to this bond as the 'risk-free' or 'no-default' bond.

To simplify the discussion of models with several bonds, it is useful to refer to the different bonds by the number of states in which they default, respectively. In our model specifications below, the set $\mathcal{J}_a$ always contains a no-default bond. In models with several bonds, the second bond defaults in precisely one state, the third bond in precisely two states, and so on. Hence we refer to these additional bonds as the 1-default bond, the 2-default bond etc. In the absence of default costs, some of these bonds will typically be traded in equilibrium. However, we see below that, in our calibration, rather moderate default costs generally suffice to shut down trade in these bonds.

Financial Markets Equilibrium with Collateral

We are now in the position to formally define the notion of a financial markets equilibrium. To simplify the statement of the definition, we assume that for a set of trees $\hat{\mathcal{A}} \subset \mathcal{A}$ collateral requirements are endogenous, that is for each $\hat{a} \in \hat{\mathcal{A}}$, there exist a set $\mathcal{J}_{\hat{a}}$ of S bonds for which this tree can be used as collateral. It is helpful to define the terms $[\phi^h_j]^+ = \max(0, \phi^h_j)$ and $[\phi^h_j]^- = \min(0, \phi^h_j)$. We denote equilibrium values of a variable x by $\bar{x}$.

Definition 3.1. *A financial markets equilibrium for an economy with initial tree holdings* $(\theta^h(s^{-1}))_{h\in\mathcal{H}}$ *and initial shock* s_0 *is a collection of agents' portfolio holdings and consump-*

tion allocations as well as security prices and collateral requirements for all trees $\hat{a} \in \hat{\mathcal{A}} \subset \mathcal{A}$

$$\left(\left(\bar{\theta}^h(\sigma), \bar{\phi}^h(\sigma), \bar{c}^h(\sigma)\right)_{h\in\mathcal{H}} ; (\bar{q}_a(\sigma))_{a\in\mathcal{A}} , (\bar{p}_1(\sigma))_{j\in\mathcal{J}} , \left(\bar{k}^j_{\hat{a}}(\sigma)\right)_{j\in\mathcal{J}_{\hat{a}}, \hat{a}\in\hat{\mathcal{A}}} \right)_{\sigma\in\Sigma}$$

satisfying the following conditions:

(1) Markets clear:

$$\sum_{h\in\mathcal{H}} \bar{\theta}^h(\sigma) = 1 \quad \text{and} \quad \sum_{h\in\mathcal{H}} \bar{\phi}^h(\sigma) = 0 \quad \text{for all } \sigma \in \Sigma.$$

(2) For each agent h, the choices $\left(\bar{\theta}^h(\sigma), \bar{\phi}^h(\sigma), \bar{c}^h(\sigma)\right)$ *solve the agent's utility maximization problem,*

$$\begin{aligned} &\max_{\theta\geq 0,\phi,c\geq 0} U_h(c) \quad s.t. \quad \text{for all } s^t \in \Sigma \\ c(s^t) &= e^h(s^t) + \sum_{j\in\mathcal{J}} \left([\phi_j(s^{t-1})]^+ r_j(s^t) + [\phi_j(s^{t-1})]^- f_j(s^t) \right) + \\ &\qquad \theta^h(s^{t-1}) \cdot \left(\bar{q}(s^t) + d(s^t)\right) - \theta^h(s^t) \cdot \bar{q}(s^t) - \phi^h(s^t) \cdot \bar{p}(s^t) \\ 0 &\leq \theta^h_{\hat{a}}(s^t) + \sum_{j\in\mathcal{J}_{\hat{a}}} \bar{k}^j_{\hat{a}}(s^t)[\phi^h_j(s^t)]^-, \quad \text{for all } \hat{a} \in \hat{\mathcal{A}}. \end{aligned}$$

(3) For all s^t *and for each* $\hat{a} \in \hat{\mathcal{A}}$, *there exists for each state* $s' \in \mathcal{S}$ *a financial security* j *such that* $\hat{a} = a(j)$ *and*

$$\bar{k}^j_{\hat{a}}(s^t) \left(\bar{q}_{\hat{a}}(s^t, s') + d_{\hat{a}}(s^t, s')\right) = 1.$$

The approach in Kubler and Schmedders (2003) can be used to prove existence. The only non-standard part—besides the assumption of recursive utility, which can be handled easily—is the assumption of default costs. Note, however, that our specification of these costs still leaves us with a convex problem and standard arguments for continuity of best responses go through.

To approximate equilibrium numerically, we use the algorithm in Brumm and Grill (2010). In Appendix 3.A, we describe the computations and the numerical error analysis in detail. For the interpretation of the results to follow it is useful to understand the recursive formulation of the model. The natural endogenous state-space of this economy consists of all agents' beginning of period financial wealth as a fraction of total financial wealth (i.e. value of the trees cum dividends) in the economy. That is, we keep track of the current shock s_t and of

$$\omega^h(s^t) = \frac{\sum_{j \in \mathcal{J}} \left([\phi_j^h(s^{t-1})]^+ r_j(s^t) + [\phi_j^h(s^{t-1})]^- f_j(s^t) \right) + \theta^h(s^{t-1}) \cdot (\bar{q}(s^t) + d(s^t))}{\sum_{a \in \mathcal{A}} q_a(s^t) + d(s^t)},$$

across all agents $h \in \mathcal{H}$. As in Kubler and Schmedders (2003) we assume that a recursive equilibrium on this state space exists and compute prices, portfolios and individual consumptions as a function of the exogenous shock and the distribution of financial wealth. In our calibration we assume that shocks are iid and that these shocks only affect the aggregate growth rate. In this case, policy- and pricing functions are independent of the exogenous shock, thus depend on the wealth distribution only, and our results can be easily interpreted in terms of these functions.

Regulated Collateral Requirements

The second rule for setting collateral requirements relies on regulated capital-to-value ratios. An agent selling one unit of bond j with price $p_j(s^t)$ must hold collateral with a value of at least $k^j_{a(j)}(s^t)q_{a(j)}(s^t)$. We can interpret the difference between the value of the collateral holding and the debt as the amount of capital an agent must put up to obtain the loan in form of a short position in the financial security. A (not further modeled) regulating agency now requires debtors to hold a certain minimal amount of capital relative to the value of the collateral they hold. Put differently, the regulator imposes a lower bound $m^j_{a(j)}(s^t)$ on this capital-to-value ratio,

$$m^j_{a(j)}(s^t) = \frac{k^j_{a(j)}(s^t)q_{a(j)}(s^t) - p_j(s^t)}{k^j_{a(j)}(s^t)q_{a(j)}(s^t)}.$$

Using language from financial markets we also call these bounds margin requirements. If the margin requirement is regulated to be $m^j_{a(j)}(s)$ in shock $s \in \mathcal{S}$ and constant over time, then the collateral requirement at each node s^t is

$$k^j_{a(j)}(s^t) = \frac{p_j(s^t)}{q_{a(j)}(s^t)(1 - m^j_{a(j)}(s_t))}.$$

Note that, contrary to the exogenous margin requirement, the resulting collateral requirement is endogenous since it depends on equilibrium prices. For economies with regulated margins, condition (3) of the definition of a financial markets equilibrium must be replaced by the following condition.

(3') For all s^t and for each $\hat{a} \in \hat{\mathcal{A}}$, the collateral requirement $\bar{k}^j_{\hat{a}}(s^t)$ of the unique bond j with $\hat{a} = a(j)$ and the given margin requirement $m^j_{\hat{a}}(s_t)$ satisfies

$$\bar{k}^j_{\hat{a}}(s^t) = \frac{\bar{p}_j(s^t)}{\bar{q}_{\hat{a}}(s^t)(1 - m^j_{\hat{a}}(s_t))}.$$

Sometimes people use the term margin requirement for the capital-to-loan ratio,

$$\frac{k^j_a q_a(s^t) - p_j(s^t)}{p_j(s^t)},$$

which does not have a natural normalization and can be larger than one. On the contrary, the margin requirement $m^j_{a(j)}(s_t)$ as defined above has a natural normalization since it is bounded above by one.

3.2.2 Calibration

This section discusses the calibration of the model's exogenous parameters. We calibrate our model to yearly data.

Growth Rates

We consider a growth economy with stochastic growth rates. The aggregate endowment at date-event s^t grows at the stochastic rate $g(s_{t+1})$ which (if no default cost are incurred) only depends on the new shock $s_{t+1} \in \mathcal{S}$, that is, if either $\lambda = 0$ or $f_j(s_{t+1}) = 1$ for all

$j \in \mathcal{J}$, then

$$\frac{\bar{e}(s^{t+1})}{\bar{e}(s^t)} = g(s_{t+1})$$

for all date-events $s^t \in \Sigma$. If there is default in s_{t+1}, then the endowment $\bar{e}(s_{t+1})$ is reduced by the costs of default and the growth rate is reduced respectively.
There are $S = 6$ exogenous shocks. We declare the first three of them, $s = 1, 2, 3$, to be "disasters". We calibrate the disaster shocks to match the first three moments of the distribution of disasters in Barro and Jin (2011). Also following Barro and Jin, we choose transition probabilities such that the six exogenous shocks are i.i.d. The non-disaster shocks, $s = 4, 5, 6$, are then calibrated such that their standard deviation matches "normal" business cycle fluctuations with a standard deviation of 2 percent and an average growth rate of 2.5 percent, which results in an overall average growth rate of about 2 percent. We sometimes find it convenient to call shock $s = 4$ a "recession" since $g(4) = 0.966$ indicates a moderate decrease in aggregate endowments. Table 3.1 provides the resulting growth rates and probability distribution for the six exogenous shocks of the economy.

Shock s	1	2	3	4	5	6
$g(s)$	0.566	0.717	0.867	0.966	1.025	1.089
$\pi(s)$	0.005	0.005	0.024	0.065	0.836	0.065

Table 3.1: Growth Rates and Distribution of Exogenous Shocks

In our results sections below we report that collateral requirements have quantitatively strong effects on equilibrium prices. Obviously, the question arises what portion of these effects is due to the large magnitude of the disaster shocks. We address this issue in the discussion of our results. In addition, Section 3.5 examines the equilibrium effects of collateral requirements for an economy with less severe disaster shocks.

Endowments and Dividends

There are $H = 2$ types of agents in the economy, the first type, $h = 1$, being less risk-averse than the second. Each agent h receives a fixed share of aggregate endowments as individual endowments, that is, $e^h(s^t) = \eta^h \bar{e}(s^t)$. We assume that $\eta^1 = 0.092$, $\eta^2 = 0.828$. Agent 1 receives 10 percent of all individual endowments, and agent 2 receives the remaining 90 percent of all individual endowments. The remaining part of aggregate endowments enters the economy as dividends of Lucas trees, that is, $d_a(s^t) = \delta_a(s_t)\bar{e}(s^t)$ and $\sum_a \delta_a(s) = 0.08$ for all $s \in \mathcal{S}$.

Several comments on the distribution of the aggregate endowment are in order. First, we abstract from idiosyncratic income shocks because it is difficult to disentangle idiosyncratic and aggregate shocks for a model with two types of agents. We conjecture that our effects would likely be larger if we considered a model with a continuum of agents receiving i.i.d. idiosyncratic shocks. Second, a dividend share of 8 percent may appear a little too low if one interprets the tree as consisting of both the aggregate stock market as well as housing wealth. However, this number is in line with Chien and Lustig (2010) who base their calibration on NIPA data. We conduct some sensitivity analysis below and, in particular, report results for the case $\sum_a \delta_a(s) = 0.15$ and thus $\eta^1 = 0.085$, $\eta^2 = 0.765$. Third, for simplicity we do not model trees' and other assets' dividends to have different stochastic characteristics as aggregate consumption. Fourth, in Section 3.4 we examine an economy with two Lucas trees. For such economies, we want to interpret the first tree as aggregate housing and its dividends as housing services while we interpret the second tree as the aggregate stock market. Following Cecchetti, Stephen, and Mark (1993), we calibrate dividends to be 4 percent of aggregate consumption which leaves housing services to be of the same size. In order to focus on the effects of collateral and margin requirements, we assume that the two trees have the exact same dividend payments, that is, in the absence of collateral constraints these two trees would be identical assets. Therefore, this calibration allows for a careful examination of the impact of different collateral properties of the two trees.

Utility Parameters

The choice of an appropriate value for the IES is rather difficult. On the one hand, several studies that rely on micro-data find values of about 0.2 – 0.8, see, for example, Attanasio and Weber (1993). On the other hand, Vissing-Jørgensen and Attanasio (2003) use data on stock owners only and conclude that the IES for such investors is likely to be above one. Barro (2009) finds that for a successful calibration of a representative-agent asset-pricing model the IES needs to be larger than one.

In our benchmark calibration both agents have identical IES of 1.5, that is, $\rho^1 = \rho^2 = 1/3$. In our sensitivity analysis we also consider the case of both agents having an IES of 0.5.

For this specification the quantitative results slightly change compared to the benchmark calibration, but the qualitative insights remain intact.

Agent 1 has a risk aversion of 0.5 and so $\alpha^1 = 0.5$ while agent 2's risk aversion is 6 and thus $\alpha^2 = -5$. Recall the weights for the two agents in the benchmark calibration, $\eta^1 = 0.092$ and $\eta^2 = 0.828$. The majority of the population is therefore very risk-averse, while 10 percent of households have low risk aversion. This heterogeneity of the risk aversion among the agents is the main driving force for volatility in the model. (Agent 1 wants to hold the risky assets in the economy and leverages to do so. In a bad shock, his de-leveraging leads to excess volatility.) In the equilibria of our model, the risky assets are mostly held by agent 1, but there are extended periods of time where also agent 2 holds part of the asset. Loosely speaking, we therefore choose the fraction of very risk-averse agents to match observed stock-market participation.

Finally, we set $\beta^h = 0.95$ for both $h = 1, 2$, which turns out to give us a good match for the annual risk-free rate.

3.3 Economies with a Single Lucas Tree

We first consider economies with a single Lucas tree available as collateral. We show that scarce collateral has a large effect on the price volatility of this tree and examine how the magnitude of this effect depends on the specification of margin requirements. This section sets the stage for our analysis of economies with two trees in Section 3.4.

3.3.1 Collateral and Volatility with a Single Risk-Free Bond

The starting point of our analysis is an economy with a single Lucas tree and a single bond. We assume that the collateral requirements on the single bond ensure that there is no default in equilibrium and so the bond is risk-free. We calibrate this baseline model according to the parameters presented above.
For an evaluation of the quantitative effects of scarce collateral, we benchmark our results against those for two much simpler models. The model *B1: No bonds* is an economy with a single tree and no bond. Thus, agents in this economy cannot borrow. The model *B2: Unconstrained* is an economy in which agents can use their entire endowment as collateral. This model is equivalent to a model with natural borrowing constraints. Table 3.2 reports four statistics for each of the three economies. (See Appendix 3.A for a description of the estimation procedure.) Throughout the paper we measure tree-price volatility by the average standard deviation of tree returns over a long horizon. Another meaningful measure is the average one-period-ahead conditional price volatility. These two measures are closely correlated for our models. In Table 3.2 we report both measures but omit the second one in the remainder of the paper. We also report average interest rates and equity premia. While our paper does not focus on an analysis of these measures, we do check them because we want to ensure that our calibration delivers reasonable values for these measures.

Recall that in our calibration agents of type 1 are much less risk averse than type 2 agents. And, therefore, in the long run agent 1 holds the entire Lucas tree in model *B1* with no borrowing and agent 2 effectively lives in autarchy. As a result the tree price is determined entirely by the Euler equation of agent 1, and so the price volatility is as low as in the model with a representative agent whose preferences exhibit very low risk aversion. The wealth

Model	Std returns	1-period price vol.	Risk-free rate	EP
B1: No bonds	5.33	4.98	n/a	n/a
B2: Unconstrained	5.38	5.05	5.88	0.55
Scarce Collateral	8.14	7.54	1.10	3.86

Table 3.2: Three Economies with a Single Tree (all Figures in Percent)

distribution remains constant across all date-events. In the second benchmark model *B2* the less risk-averse agent 1 holds the entire tree during the vast majority of time periods. A bad shock to the economy leads to shifts in the wealth distribution and a decrease of the tree price. However, since in our calibration shocks are iid, these shifts in the wealth distribution have generally small effects on prices (except in the very low-probability case of several consecutive disaster shocks). The resulting price volatility in model *B2* is of similar magnitude as the volatility in *B1*. Moreover, in the model *B2* the risk-free rate is high and the equity premium is very low. Despite the presence of disaster shocks, the market price of risk is low because it is borne almost entirely by agent 1 who has very low risk aversion.

Table 3.2 shows that both first and second moments show substantial differences when we compare models without collateral requirements to a model with tight collateral constraints. The perhaps most striking result reported in Table 3.2 is that volatility in our baseline economy is about 50 percent larger than in the two benchmark models without borrowing (*B1: No bonds*) and with natural borrowing constraints (*B2: Unconstrained*), respectively. The standard deviation of returns is 8.14 percent in the baseline economy but only 5.33 percent and 5.38 percent for the benchmark models *B1* and *B2*, respectively.[3]
Collateral constraints drastically increase the volatility in the standard incomplete markets model. Figure 3.1 shows the typical behavior of four variables in the long run during a

[3]The stock return volatility in our baseline economy is considerably smaller than the volatility in U.S. data. For comparison, Lettau and Uhlig (2002) report that the quarterly standard deviation of returns of S&P-500 stocks in post-war US data is about 7.5 percent. Similarly, Fei, Ding, and Deng (2010) report an annual volatility of about 14.8 percent for the period January 1987 to May 2008. However, it is important to note that we want to interpret the aggregate tree as a mix of stocks and housing assets. The volatility of housing prices is U.S. data is much lower. Fei, Ding, and Deng (2010) report an annual volatility of the Case/Shiller housing price index of less than 3 percent (for January 1987 to May 2008). A similar comment applies to the equity premium. While the average risk-free rate roughly matches U.S. data, the equity premium is substantially lower than in the data. We discuss this point in more detail in Section 3.4 for an economy with two trees.

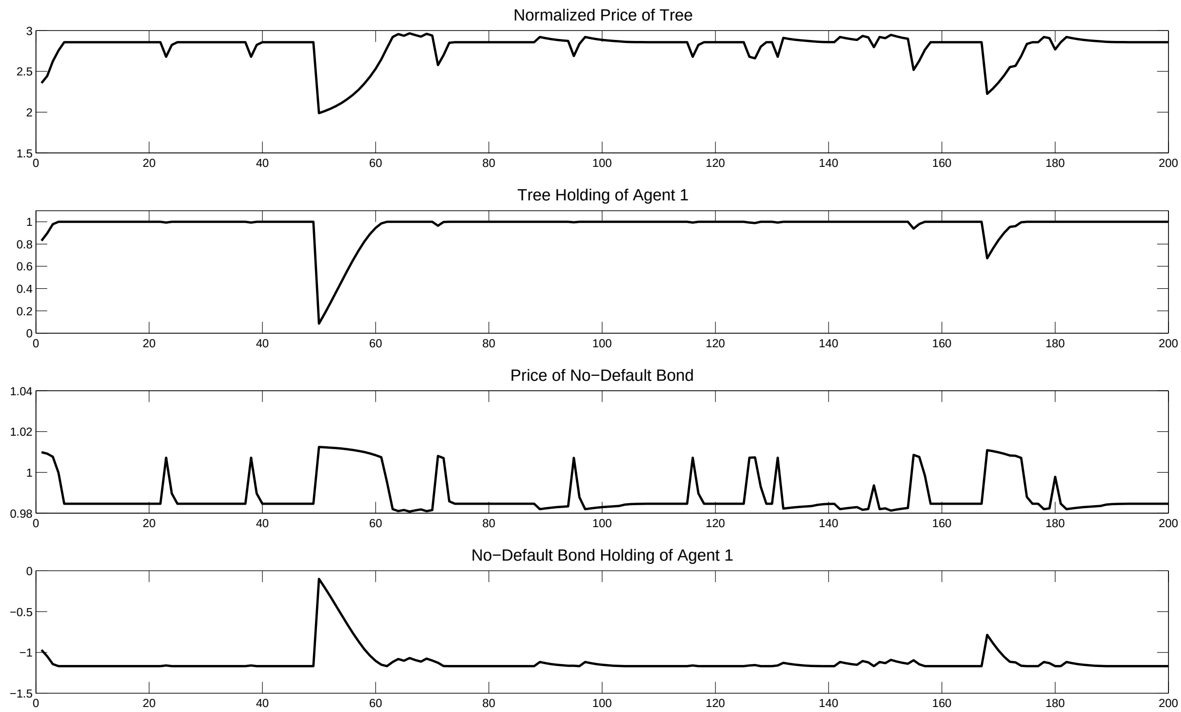


Figure 3.1: Snapshot from a Simulation of the Baseline Model

simulation for a time window of 200 periods. The first graph displays agent 1's holding of the Lucas tree. The second graph shows the normalized tree price, that is, the equilibrium price of the tree divided by aggregate consumption in the economy. The last two graphs show the price and agent 1's holding of the risk-free bond, respectively. In the sample displayed in Figure 3.1, the disaster shock $s = 3$ (smallest disaster with a drop of aggregate consumption of 13.3 percent) occurs in periods 71 and 155 while disaster shock 2 occurs in period 168 and disaster shock 1 (worst disaster) hits the economy in period 50.

When a bad shock occurs, both the current dividend and the expected net present value of all future dividends of the tree decrease. As a result the price of the tree drops, but in the absence of further effects, the normalized price should remain the same since shocks are iid. (That's exactly what happens in the benchmark model *B1*.) In our baseline economy with collateral constraints, however, additional effects occur in equilibrium. First, note that agent 1 is typically leveraged, that is, when a bad shock occurs his beginning-of-period financial wealth falls relative to the financial wealth of agent 2. This effect is the strongest when the worst disaster shock 1 occurs. If agent 1 was fully leveraged in the previous period then her wealth decreases to zero because shock 1 always determines the collateral requirement $k^j_{a(j)}$.

High leverage leads to large changes in the wealth distribution when bad shocks occur. The fact that collateral is scarce in our economy now implies that these changes in the wealth distribution strongly affect equilibrium portfolios and prices. Since agent 1 cannot borrow against her future labor income, she can only afford to buy a small portion of the tree if her financial wealth is low. In equilibrium, therefore, the price has to be sufficiently low to induce the much more risk-averse agent 2 to buy a substantial portion of the tree. On top of that within-period effect, there is a dynamic effect at work. As agent 1 is poorer today, she will also be poorer tomorrow (at least in shocks 2-6) implying that the price of the tree tomorrow is depressed as well. This further reduces the price that agent 2 is willing to pay for the tree today. Clearly, this dynamic effect is active not only for one but for several periods ahead, which is displayed in Figure 3.1 by the slow recovery of the normalized price of the tree after bad shocks. Figure 3.1 shows that the total impact of the above described effects is very strong for shock $s = 1$ but also large for shock 2. Note that the prices are normalized prices, so the drop of the actual tree price is much larger than displayed in the figure. In disaster shock 1, agent 1 is forced to sell almost the entire tree and the normalized price drops by almost 30 percent (the actual price drops by approximately 60 percent). In shock 2 she sells less than half of the tree but the price effect is still substantial. In shock 3 the effect is still clearly visible, although the agent has to sell only very little of her tree.

While the effects of collateral and leverage on volatility are very large, it is important to note that in the baseline specification of our model with a single tree and a single bond there is no *financial accelerator*. Kiyotaki and Moore (1997), Aiyagari and Gertler (1999) and others highlight the idea that in the presence of collateral constraint the fact that the price of collateral decreases might make it more difficult for the borrower to maintain his debt position because collateral requirements increase in anticipation of a value of the collateral in the next period which is now lower than if the shock had not happend. In the baseline case, this effect is absent for two reasons. First, whenever agent 1 is constrained, the collateral requirement $k^j_{a(j)}$ is independent of today's price of the collateral, it is in fact constant. This is because the collateral requirement is determined by tomorrow's tree price (plus dividend) in case of the worst shock. If this shock occurs and agent 1 is constrained today, he has to use his entire tree holding to repay his debt. Hence, no matter how large agent 1's tree holding is today, he ends up with zero financial wealth tomorrow. This implies a specific price for the tree tomorrow in shock 1 which is independent of today's price (as long as agent 1 is constrained) and consequently a specific collateral requirement today[4]. Second, an examination of the bond price in Figure 3.1 reveals an important general equilibrium effect in our economy that counteracts an increase of the margin requirement. When a bad shock occurs and the share of financial wealth of agent 1 decreases, then the demand of the now relatively richer agent 2 for the risk-free asset increases the bond price substantially. In fact, occasionally the interest rate even becomes negative. As a result of the constant collateral requirement, the increase in the bond price and the decrease in the tree price the equilibrium margin requirement actually decreases substantially in a bad shock.

In sum, scarce collateral plays an important role for the volatility of the tree price because it leads to large price drops in bad shocks since agent 1 cannot borrow against future labor income. As we would expect, this effect depends on the amount of available collateral in the economy. Figure 3.2 illustrates this point. The figure depicts the tree's average return volatility and the fraction of times the collateral constraint is binding for agent 1 (i.e. the probability of constraint being binding) as a function of the dividend share δ in the economy.

For very small values of δ, there is only little collateral in the economy and so the collateral constraint is almost always binding. However, the stock is so small that agent 1 does not have to sell the stock even if the economy is hit by an extremely bad aggregate shock. The resulting return volatility is relatively small. As δ increases the probability of the collateral constraint being binding decreases rapidly but the effects of it being binding become larger. There is an interior maximum for the stock-return volatility around $\delta = 0.07$. Although the constraint is much less often binding than for a smaller tree, the trade-off between agent 1 being forced to sell the tree and agent 1 getting into this situation leads to maximal volatility. As δ increases further, the constraint becomes binding much less frequently and eventually at $\delta = 1$ the stock return volatility is very low, simply because the collateral constraint never binds and so collateral plays no role. This situation is identical to the case

[4]If we assume that the tree's dividends cannot be used as collateral, this argument is no longer correct. However, for our calibration the effects of this assumption are quantitatively negligable.

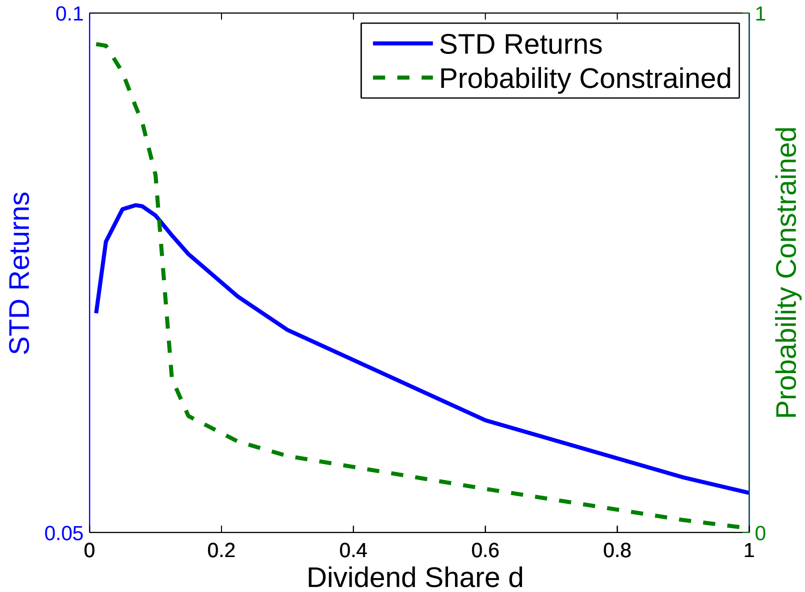


Figure 3.2: Volatility as a Function of the Dividend Share

of natural borrowing constraints where a binding constraint would imply zero consumption for the borrower.

3.3.2 Collateral and Several Bonds

In the economy with a single tree and a single bond, equilibrium margin requirements are sufficiently high to ensure that there is no default. The bond is risk-free and always pays its face value. We now examine whether the observed results are just a consequence of this restrictive assumption. In the enhanced model a menu of bonds is available for trade and the accompanying collateral requirements are endogenously determined in equilibrium.

Full Set of Bonds without Costly Default

Our calibrated model with $S = 6$ exogenous states allows the analysis of economies with five bonds. As explained above, these bonds are characterized by the number of shocks in which they default and so we call them no-default bond, 1-default bond, 2-default bond, etc. Figure 3.3 shows the portfolio holdings of agent 1 as well as the normalized tree price along the same simulated series of shocks as in Figure 3.1 above.

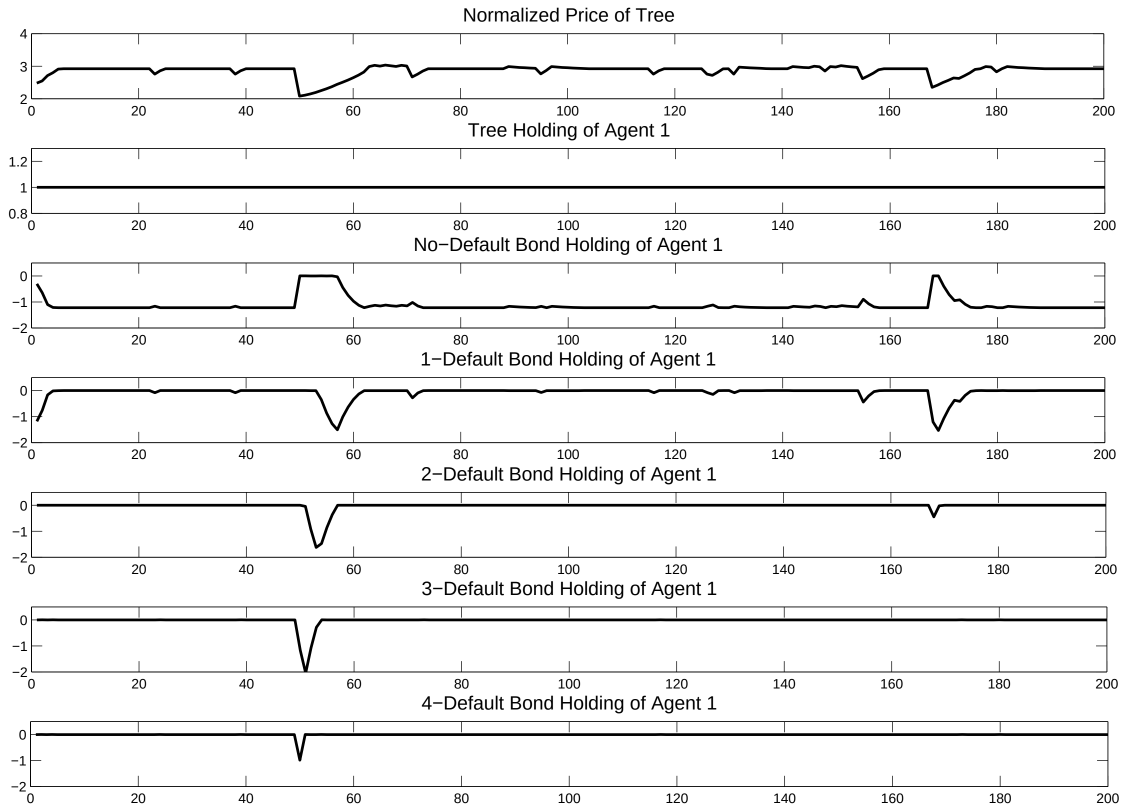


Figure 3.3: Snapshot from a Simulation of the Model with 1 Tree and 5 Bonds

During "normal times" (that is, if the last disaster shock occurred sufficiently long ago) only the no-default bond is traded in equilibrium. (There is a tiny amount of trade in the 1-default bond in recessions, shock 4, which is quantitatively negligible.) In normal times the agents' portfolios resemble those in an economy with a single risk-free bond. The risk-averse agent 2 holds the risk-free bond while agent 1 holds the risky tree and is short in the bond.

Disaster shocks are the only reason for equilibrium trade in default bonds. In our economy, the risk-averse agent 2 always seeks to buy an asset that insures him against bad aggregate shocks — only the risk-free bond can play this role. However, the risky default bonds play an important role once a disaster shock occurs. Agent 1 no longer needs to sell the stock but is now able to raise additional funds by selling default bonds to agent 2. Such a trade shifts some of the tree's risk to agent 2 who demands a high interest rate for assuming such risk. But the default bonds are still less risky than the tree and thus preferred by the risk-averse agent. In fact, the presence of the default bonds enables agent 1 to always hold the entire tree. Figure 3.3 shows that after an occurrence of the worst disaster shock 1, which happens in period 50, agent 1 is able to hold on to the entire tree and to sell the 4-default and the 3-default bond to agent 2. As the economy recovers, agent 1 sells the 1-default bond to agent 2 and holds a short-position in this bond for approximately 10 periods until her wealth has recovered sufficiently so that she is able to leverage exclusively in the default free bond.

Despite the fact that the leveraged agent 1 no longer has to sell the tree after bad shocks, such shocks continue to have a strong impact on asset prices. Figure 3.3 shows that the normalized tree price decreases in all three disaster shocks as well as in recessions, just as in an economy with a single risk-free bond, see Figure 3.1. By selling the default bonds to the risk-averse agent 2, agent 1 shifts the tree's (tail) risk to agent 2. This circumstance must be reflected in the equilibrium price. This reasoning becomes clear if we considered the case of identical dividends in shocks 5 and 6. Under this scenario, the tree and the 4-default bond have identical payoffs and hence it should be irrelevant for the price of the tree who holds it, that is, whether agent 1 holds it financed by a short position in the 4-default bond or agent 2 holds it directly.

Moreover, unlike in the previous model with one bond, the financial accelerator now plays a role. A lower tree holding of agent 1 in this period reduces the price of the tree in the next period in shocks 2-6 and hence makes it more difficult for agent 1 to hold default bonds.

Table 3.3 reports the tree-return volatility for economies with $1, 2, \ldots, 5$ bonds, respectively. The presence of a bond that defaults only in shock 1 (when the economy shrinks by 43.4 percent) leads to a decrease in the volatility of the tree price. A third bond that defaults in shocks 1 and 2 leads to an additional small reduction of volatility. The impact of additional bonds is negligible. This fact is not surprising since we observed that these bonds are rarely traded.

Unfortunately, the fact that investors only trade bonds with a high probability of default during bad times seems counterfactual. Several features of our model may lead to this

	One bond	Two bonds	Three bonds	Four bonds	All bonds
Std returns	8.14	7.87	7.84	7.84	7.84

Table 3.3: The Effect of Endogenous Margins on Return Volatility

result. Clearly bad times are often persistent and not iid as in our calibration. More importantly, default is typically costly. We next show that fairly small default costs eliminate trade in default bonds.

Costly Default

Until now our treatment of default is somewhat unsatisfactory since it neglects both private and social costs of default. We now introduce default costs as described in Section 3.2.1 above. Table 3.4 shows how the trading volume of the default bonds changes as a function of the cost parameter λ. The reported trading volume is the average absolute bond holding of agent 1 (which is the same as that of agent 2) over the simulation path.

	$\lambda = 0$	$\lambda = 0.01$	$\lambda = 0.05$	$\lambda = 0.10$	$\lambda = 0.2$	$\lambda = 0.25$
Std dev tree return	7.84	7.87	7.98	8.12	8.15	8.14
Total trading	1.260	1.236	1.183	1.161	1.126	1.123
No-default bond	1.110	1.099	1.076	1.076	1.099	1.123
1-default bond	0.084	0.080	0.075	0.085	0.027	0
2-default bond	0.034	0.034	0.032	0	0	0
3-default bond	0.026	0.023	0	0	0	0
4-default bond	0.006	0	0	0	0	0

Table 3.4: The Effect of Default Costs on Tree-Return Volatility and Bond Trading Volume

In the absence of default costs ($\lambda = 0$), the average trading volume of all bonds is nonzero. As we observed in the previous section, it is substantial for the no-default and 1-default bond and rather small for the remaining bonds. Proportional default cost of as low as 10 percent ($\lambda = 0.1$) result in zero trade for the bonds defaulting in two or more states. For default costs of 25 percent, trade in any type of default bond ceases to exist. Only the risk-free bond is traded and the resulting equilibrium prices and allocations are identical to our baseline economy above.

Recall from the description in Section 3.2.1 that the cost is proportional to the difference of the face value of the bond and the value of the underlying collateral. Therefore, a proportional cost of 25 percent means a much smaller cost as a fraction of the underlying collateral. Campbell, Giglio, and Pathak (2011) find an average 'foreclosure discount' of 27 percent for foreclosures in Massachusetts from 1988 until 2008. This discount is measured as a percentage of the total value of the house. As a percentage of the difference between the house value and face value of the debt this figure would be substantially larger. A value of $\lambda = 0.25$, therefore, seems certainly realistic and is, if anything, too small when we compare it to figures from the U.S. housing market.

Table 3.4 also reveals that the trading volume of the 1-default bond remains stable up to default costs of around 10 percent when other default bonds are no longer traded. The 1-default bond remains an attractive asset in this economy even for moderate default costs. It is traded when agent 1 is poor. Compared to the no-default bond, it allows to take on more debt for a given amount of collateral. Compared to bonds that default in more states, the expected default costs are much lower. For these reasons, the 1-default bond is the preferred choice in this situation.

Table 3.4 also shows that the volatility of the tree return increases as cost of default increases, and for sufficiently high default cost the economy is the same as the baseline economy with a single risk-free bond. It appears that an economy with default costs of 20 percent and trade in the 1-default bond exhibits slightly higher return volatility than the baseline economy. This feature is due to the fact that default implies real losses in our economy which make the economic impact of the worst disaster shock even worse since default leads to a further drop in aggregate endowment.

3.3.3 Volatility with Regulated Margin Requirements

As a final step in the analysis of economies with a single collateralizable tree, we consider the case of regulated collateral requirements as described in Section 3.2.1. We assume that there is a regulatory agency setting minimal margin requirements (just as in stock markets). We first consider margin requirements that are constant across all shocks, so $m^j_{a(j)}(s^t)$ does not depend on the current date-event s^t. As margin requirements become larger, we observe two opposing effects. On the one hand, the amount of leverage decreases in equilibrium which leads to less de-leveraging in disaster shocks which in turn leads to smaller price changes. On the other hand, the collateral constraint is more likely to become binding in equilibrium which increases the probability of de-leveraging episodes which in turn should lead to a higher volatility of the tree return. The solid line in Figure 3.4 displays the resulting tree return volatility.

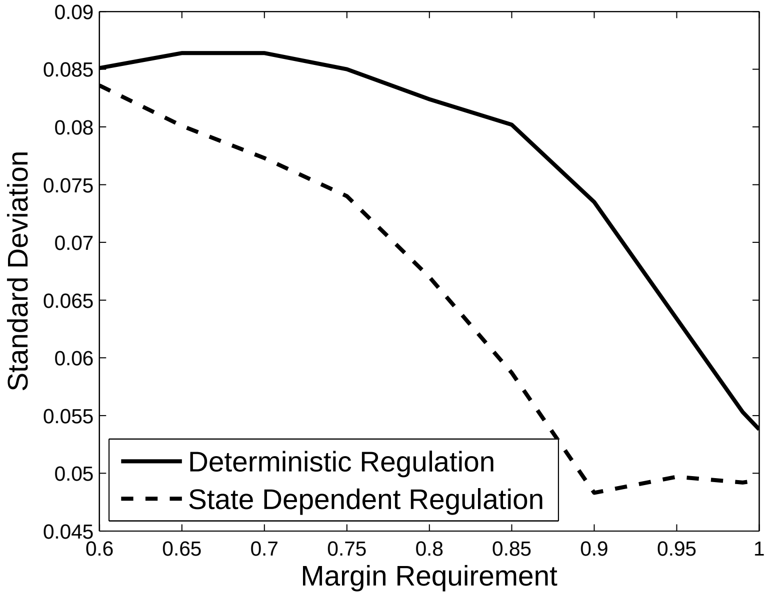


Figure 3.4: Volatility as a Function of the Margin Requirement

Initially, volatility increases as margin requirements increase. At a margin level of about 70 percent, the volatility reaches its maximum. A further tightening of margins then decreases volatility substantially. Of course, as the margin level approaches one the economy approaches the benchmark model *(B1: No bonds)* without borrowing and so volatility becomes very small.

At a margin level of 60 percent, the implied collateral requirement uniformly exceeds the corresponding varying levels for the no-default bond under the rule of endogenous collateral requirements in our baseline economy analyzed above. Therefore, the regulated bond is default-free for all possible values of $m^j_{a(j)}$ in Figure 3.4. Interestingly, for values of the margin level between 60 and 80 percent, the regulated bond leads to higher tree return volatility than the no-default bond under the rule of endogenous collateral requirements.

As a last exercise, we examine an economy in which margins are only regulated in booms while in recessions and disasters they are left to the market. In particular, we assume that in shocks 1 through 4 collateral requirements are endogenously determined at the level of the risk-free bond as in our baseline economy, while a regulating agency sets margin requirements in the shocks with positive growth. We assume that the margin levels are set to the same level in both shocks 5 and 6. The dashed line in Figure 3.4 shows the resulting tree return volatility.

It is readily apparent that limiting the regulation of margin requirements to boom times reduces the tree return volatility substantially if margin levels are sufficiently high. For example, boom-time margin levels of 80 percent lead to a return volatility of 6.5 percent as compared to values exceeding 8 percent when collateral requirements are determined endogenously or margin regulation is state-independent.

Why is state-dependent regulation so much better in reducing volatility? As with state-independent margins, agent 1 holds less leverage in good times, which leaves him with more financial wealth if a bad shock hits. In addition, collateral constraints are now looser in case of a bad shock and agent 1 may retain an even larger portion of the tree. In the extreme, if margin requirements in booms are well above 80 percent, agent 1 even increases its tree holding in case of a bad shock. This increases the relative price of the tree and thus dampens the drop in the absolute price. All in all, setting conservative margins in good times turns out to be a powerful tool to dampen the negative impact of bad shocks.

3.4 Two Trees

Up to this point our analysis focused on an economy with a single tree representing aggregate collateralizable wealth in the economy. However, households trade in various assets and durable goods. Some of them, e.g. houses, can be used as collateral very easily and at comparatively low interest rates, others assets, e.g. stocks, can only be used as collateral for loans with high margin requirements and typically very high interest rates (see Willen and Kubler (2006)), and still others, like works of art, cannot be used as collateral at all. These observations motivate us to examine a model with two Lucas trees. For simplicity, we assume that the two trees have identical cash-flows and distinguish themselves only by the extent to which they can be used as collateral. This model feature allows for a clean analysis of the effect of collateral. We consider two different cases. First, we assume that tree 1 can be used as collateral with endogenous margin requirements, while tree 2 cannot be used as collateral. We then allow the second tree to serve as collateral, but we assume that the collateral requirements on loans backed by tree 2 are exogenously regulated. In both cases we find that the two assets' price dynamics are substantially different, despite the fact that they have identical cash-flows. Furthermore, we show that tightening the margin requirements on the regulated tree has a strong impact on the return volatility of the non-regulated tree. This effect proves to be quantitatively important. Our analysis suggests that this effect should be carefully considered in any policy discussion on the regulation of margin requirements.

3.4.1 Only one Tree can be Used as Collateral

We first consider the case where the second tree cannot be used as collateral. As before in an economy with a single tree, default costs of $\lambda = 0.25$ suffice to shut down all trade in default bonds. We therefore restrict attention to an economy in which only the no-default bond is traded. We conclude the analysis in this section below with a brief discussion of an economy with costless default and argue that it produces similar quantitative results.

Table 3.5 reports moments of the two trees' returns as well as the interest rate and aggregate moments. Observe that the two trees exhibit substantially different returns despite the fact that the two trees have identical cash-flows. The tree that can be used as collateral, tree 1, now exhibits much lower return volatility and a slightly lower expected excess return than

	Std returns	EP agg	Std returns agg	Risk-free rate	Equity-premium
Tree 1	6.64	3.69	7.04	0.38	4.50
Tree 2	8.05	6.31			

Table 3.5: Moments of Trees' Returns (only Tree 1 Collateralizable)

the single tree in the baseline economy in Section 3.3. The standard deviation of returns of the second tree is much higher than that of tree 1. In fact, it is comparable to the corresponding value (8.14) of the single tree in the baseline economy. Turning to equity premia, the excess return of tree 2 — the tree that cannot be used as collateral — is now almost twice as large as it is for the single tree in the baseline economy and is similar to figures observed in the data.

To understand the price dynamics of the two trees, we consider the analogue of Figure 3.1. Figure 3.5 shows the time series of eight variables along 'our' sample path. The first two graphs show the (normalized) price and the first agent's holding of tree 1, respectively. The next two graphs display the corresponding values for tree 2. The fifth and sixth graph show the corresponding values for the no-default bond. The price and holding graphs reveal three features of the equilibrium. First, the price volatility of tree 1 is much lower than that of the single tree in the baseline economy. Secondly, the price volatility for tree 2 is larger than for tree 1 and its average price is much smaller. Lastly, agent 1 holds tree 1 the entire time (except for a tiny blip in disaster shock 1) but frequently sells tree 2. The second-to-last graph in the figure shows the endogenous margin requirement and the last graph depicts the *collateral premium* for tree 1. This quantity is the difference between the actual price of the tree and next period's payoff, normalized with agent 1's marginal utilities. Whenever agent 1 is unconstrained then this value is zero. However, when agent 1 becomes constrained, the collateral premium is significant.

Our observations lead us to a simple explanation of the first moments for the two tree prices. Tree 1 is more valuable to agent 1 because of its collateral value — when agent 1 is fully leveraged the value of the tree exceeds next period's discounted (with agent 1's state prices) cash-flows since it provides value for agent 1 as collateral. Since both trees have identical cash-flows, an agent can only be induced to hold tree 2 if it pays a higher average return. The specific magnitude of the difference between the two tree prices is, of course, a quantitative issue. In our calibration with a reasonable market price of risk, the effect

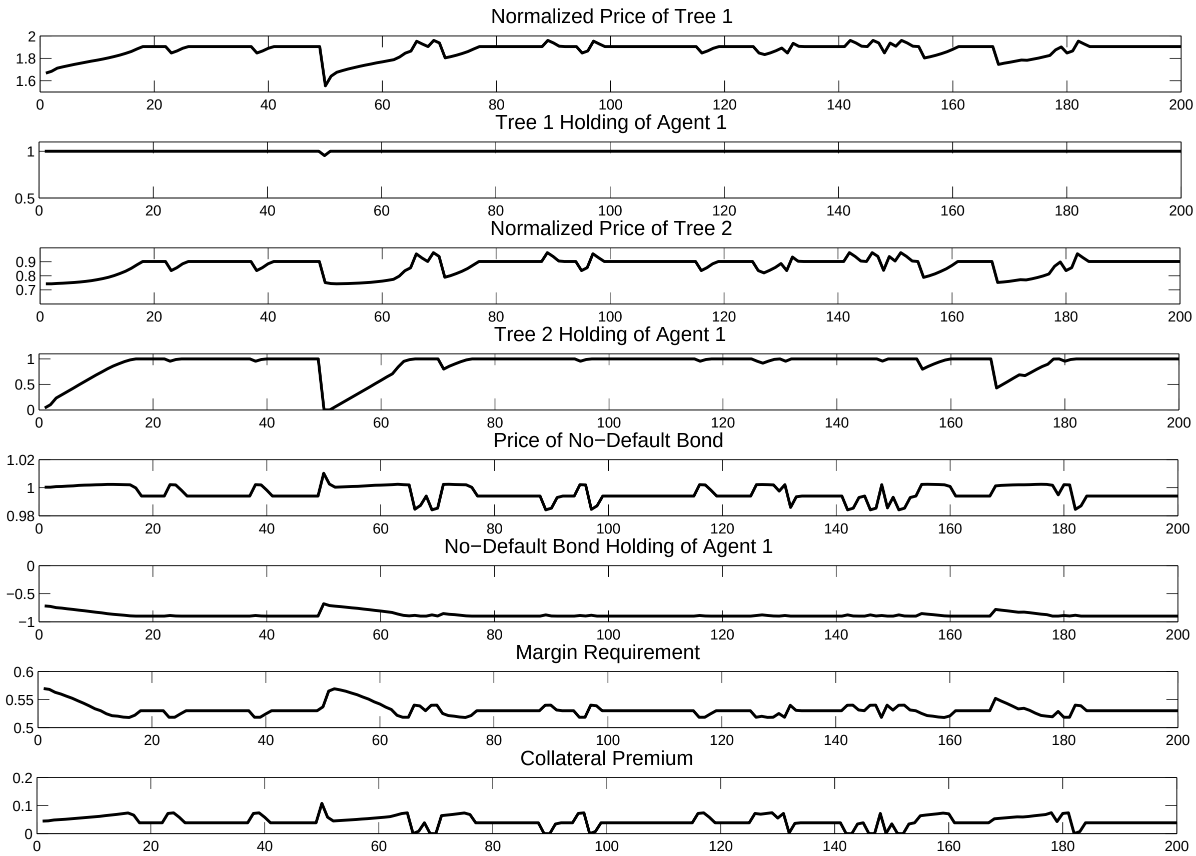


Figure 3.5: Snapshot from a Simulation of the Model with 2 Trees and 5 Bonds

is indeed large — the average excess return of the second tree is now comparable to that observed in U.S. stock market data.

There are several key factors that play a role for asset price volatility in the two-tree economy. For a discussion of these factors it is helpful to consider the policy and price functions in Figure 3.6. When faced with financial difficulties after a bad shock, agent 1 holds on to tree 1 for as long as possible, because this tree allows her to hold a short-position in the bond. (In fact, as the bond-holding function of agent 1 in Figure 3.6 shows, agent 2 never goes short in the bond. Therefore, the collateral value is one of the reasons why tree 1 is much more valuable to agent 1.) So, after suffering a reduction in financial wealth, agent 1 first sells tree 2. In fact, in our calibration agent 1 only sells a portion of tree 1 after she sold off the entire tree 2. In our sample path this happens only after the

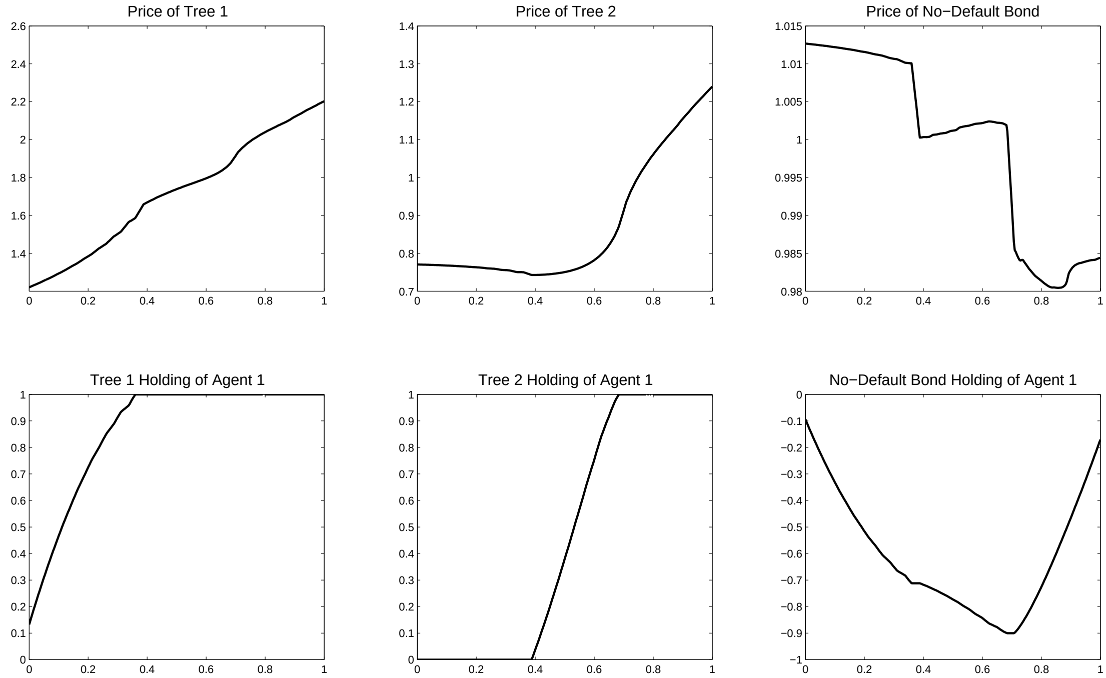


Figure 3.6: Price and Policy Functions of the Model with 2 Trees and 5 Bonds

worst disaster shock in period 50. (Of course, the policy functions in Figure 3.6 show that it would happen in a more pronounced way after two or more consecutive disaster shocks but such a sequence has extremely low probability.) Whenever agent 1 sells a portion of a risky tree to agent 2 its price must fall, just as in the single-tree baseline economy. And so one key factor contributing to the different volatility levels of the two trees is that tree 2 is traded much more often and in larger quantities than tree 1.

Furthermore, since tree 2 is not collateralizable, only half of the aggregate tree can be used as collateral. This constraint limits the ability of agent 1 to leverage and consequently makes her less vulnerable to negative aggregate shocks. This factor reduces the return volatility of both trees.

If agent 1 holds both trees and then becomes poorer after a bad shock, the prices of both trees fall. But since the agent first sells tree 2, the price of tree 2 falls much faster than the price of tree 1. In fact, the price drop for tree 1 is dampened by the onset of the collateral premium. This effect also contributes to the difference in the return volatilities of the two trees.

Finally, there is another key effect that was not present in the one-tree baseline economy. Now the financial accelerator plays an important role! In 'normal times' agent 1 holds both trees but is fully leveraged. In a bad shock, agent 1 must sell part of tree 2 which makes him poorer in the subsequent period. This in turn increases the collateral requirement this period, leading to an increase in the margin requirement despite the fact that the interest rate decreases. This effect is clearly visible in the second-to-last graph of Figure 3.5. Whenever a bad shock occurs the margin requirement increases.

In sum, the fact that only 4 percent of aggregate output are collateralizable in this economy leads to a decrease in leverage and to much smaller movements in the wealth distribution than in the baseline economy. This effect reduces the return volatility of tree 1. For tree 2 such a reduction effect is strongly counteracted through two channels. First, the price of tree 2 is not stabilized by a collateral premium since this tree cannot be used as collateral. Secondly, a decrease in the holdings of tree 2 leads to an increase in the margin requirements for loans on tree 1 which in terms forces agent 1 to sell more of tree 2 (recall that initially he does not sell tree 1, since only this tree can be used as collateral).

While we do not want to push the interpretation of our results too far, it is worthwhile to note that a natural interpretation of the two trees is the aggregate stock market versus the aggregate housing market. As Willen and Kubler (2006) report, it is much more difficult to use stocks instead of a house as collateral. The data clearly shows that volatility in the stock market is much higher than in the housing market, see Fei, Ding, and Deng (2010). This interpretation clearly should be taken with some caution, since we do not really have a good model of the housing market — such a model would need to include transaction costs, non-divisibilities, and certainly different cash-flow dynamics. Nevertheless it is worthwhile to point out that the equity premium for tree 2 is similar to what can be observed in the data for stock returns. Moreover, volatility of "housing returns" (tree 1) is much smaller than that of stock returns.

We complete our discussion of the economy in which the second tree cannot be used as collateral with a robustness check and consider the case of costless default, $\lambda = 0$. Just as in the economy with a single tree, the default bonds are traded if the economy experiences a disaster shock. However, trade in these bonds is typically much smaller because agent 1's financial wealth remains larger, as we discussed above. Overall, zero default costs lead to very small changes in the first and second moments. Without default costs, the standard deviation of tree 1's return drops from 6.64% to 6.56% while the standard deviation of tree 2's return drops from 8.05% to 7.98%.

3.4.2 One Tree is Regulated

Until now we have assumed that tree 2 cannot be used as collateral. This assumption is rather restrictive if not unrealistic. Stocks can be used as collateral, however, margins are regulated and large, and interest rates are much higher than mortgage rates. Therefore, we assume now that margins for tree 2 are set exogenously while collateral requirements for tree 1 are endogenous. Throughout this section, we assume default costs of $\lambda = 0.25$ which suffice to shut down all trade in default bonds.

State-Independent Regulation of Tree 2

As before, we first consider margin requirements that are constant across states. The effect of an exogenous margin requirement is obvious in the limit as the requirement m_2 for tree 2 approaches one. In this case the resulting collateral requirement k_2 diverges to infinity and so the model tends to the economy of Section 3.4.1 in which this tree cannot be used as collateral. Figure 3.7 display the volatility of both trees' returns as a function of the margin m_2 set for tree 2. Observe that as m_2 tends to one, the return volatilities for the two trees approach the values from Table 3.5, namely 6.64% and 8.05%, respectively.
Figure 3.7 shows the return volatilities for values of m_2 between 0.6 and 1. The lowest value of 0.6 of the margin requirement exceeds the endogenously determined (unregulated) margin requirement of tree 1 in all states. As a result, the return volatility of tree 2 is higher that that of tree 1. If margin requirements on tree 2 are now increased, the volatility of this tree's return initially increases, while the volatility of the freely collateralizable tree 1 substantially decreases. The volatility of tree 2 is largest when its exogenous margin requirement is quite high (about 75 percent). After this peak, the volatility of tree 2 decreases until the boundary value of one has been reached. At this point tree 2 can no longer be used as collateral. The quantitatively most interesting case is a regulated margin requirement of 75 percent. At this point, the volatility of tree 2 is above 8.6 percent while the volatility of tree 2 is below 7.5 percent. Aggregated volatility is still high, but it is readily apparent that the regulation of tree 2 has substantial effects on its own volatility

as well as on the volatility of the other, unregulated, tree 1.

For an interpretation of the observed volatility variation, note that an increase of the margin requirement m_2 of tree 2 has two immediate effects. This tree becomes less attractive as collateral and the agents' (aggregated) ability to leverage decreases. These two effects influence agent 1's portfolio decisions after a bad shock occurs. First, when agent 1 must de-leverage her position, then she first sells tree 2. In equilibrium, this effect occurs more often as m_2 increases. Initially this effect leads to an increase in the return volatility of tree 2. The second effect, a reduced ability to leverage, decreases the return volatility of tree 1. Similar to the effect we observed in the one-tree economy in Section 3.3, the return volatility of tree 1 decreases as agent 1's ability to leverage decreases. The reason for this effect is the increased probability with which she can hold onto the tree after a bad shock. Observe that the two described effects counteract each other for tree 2. For small increases of m_2 above 0.6, the first effect dominates the second and the tree's return volatility increases. As m_2 increases further, the second effect eventually dominates and the return volatility of tree 2 starts to decrease. Moreover, as the margin requirement on tree 2 becomes large, price effects as a result of agent 1 de-leveraging her positions become smaller. Recall that whenever agent 1 is collateral constrained, then the price of the underlying collateralized tree reflects a collateral premium. Since agent 2 never enters leveraged positions, this price impact is never present when agent 2 holds tree 2. As a result the collateral premium affects the price volatility of tree 2. This effect is greatly diminished as m_2 becomes sufficiently large. Put differently, the impact of the collateral premium on the return volatility fades as m_2 gets large.

To support this interpretation of our results, it is interesting to consider the excess returns of the two trees as a function of the margin requirement m_2 on tree 2. Figure 3.8 shows that the relation between excess return and m_2 is monotone for tree 2. As its margin requirement increases, the collateral premium and the price of the tree decrease and the average return increases. For tree 1, average excess returns remain more or less constant. They initially decrease slightly, then increase slightly. Aggregate excess returns increase, but clearly the quantitatively most striking effect is on the returns of tree 2. Collateral constraints and regulated margins clearly have a quantitatively significant impact on asset prices in this economy.

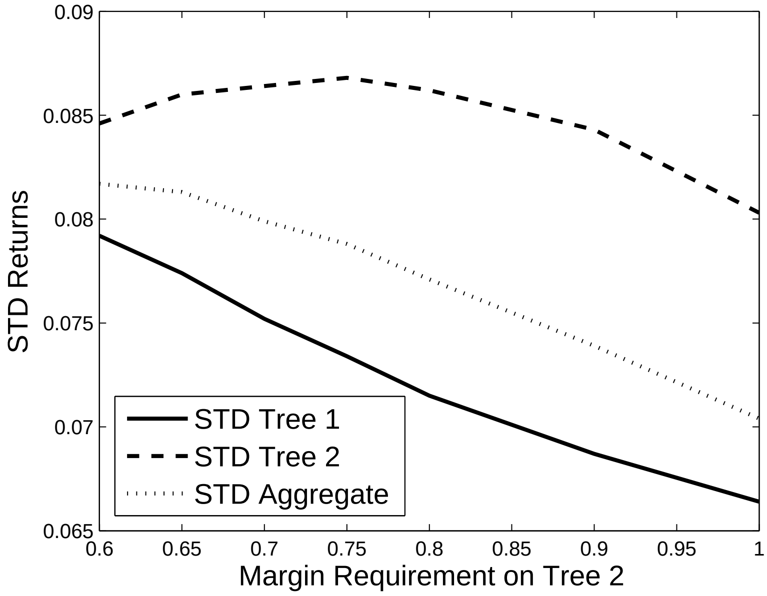


Figure 3.7: Volatility of Tree 1 and 2 as a Function of the Margin Requirement on Tree 2

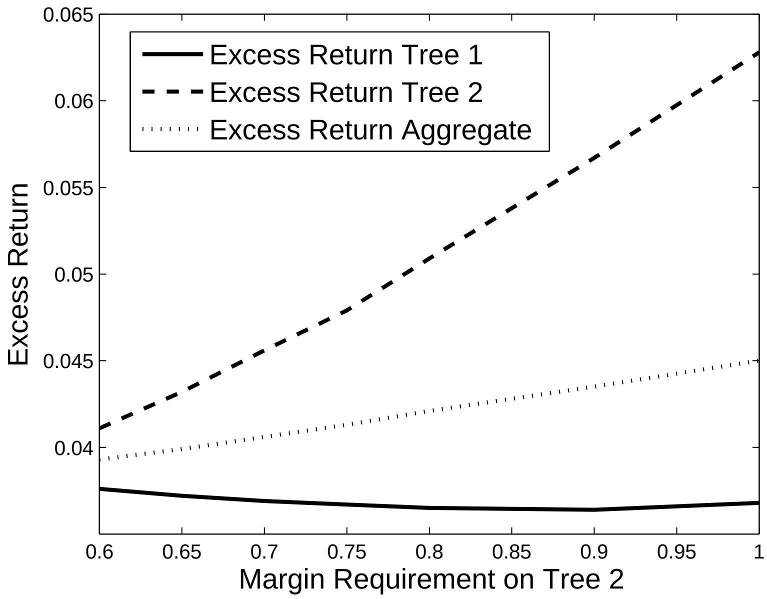


Figure 3.8: Excess Returns of Tree 1 and 2 as a Function of the Requirement on Tree 2

State-Dependent Regulation of Tree 2

Our results so far have shown that, for moderate margin requirements between 0.6 and 0.75, it is impossible to reduce volatility for both trees by adjusting the regulated margin of tree 2. A small change of the margin requirement always reduces volatility of one tree at the expense of the other. We now analyze whether a state-dependent regulation of the second tree can solve this dilemma.

We examine an economy in which the margins of tree 2 are only regulated for positive-growth shocks 5 and 6 while they are endogenously determined for the remaining four shocks. Figure 3.9 shows that the return volatilities of both trees are monotonically decreasing in the margin requirement imposed on the regulated tree 2 in good shocks. Not only does increasing the regulated margin now reduce the volatilities of both assets, but, in fact, it does reduce aggregate market volatility much more than in the economy with state-independent regulation. For instance, an increase of state-dependent margin requirements from 0.6 to 0.7 on tree 2 decreases aggregate volatility by about 4.5% (see Figure 3.9), while such an increase would bring about a reduction of only 2% in the case of state-independent regulation (see Figure 3.7). Therefore, concerning the regulation of margin requirements, the result from the single-tree economy is strongly confirmed by the analysis of the two-tree economy: regulation is much more efficient at reducing price volatility, if it is state-dependent.

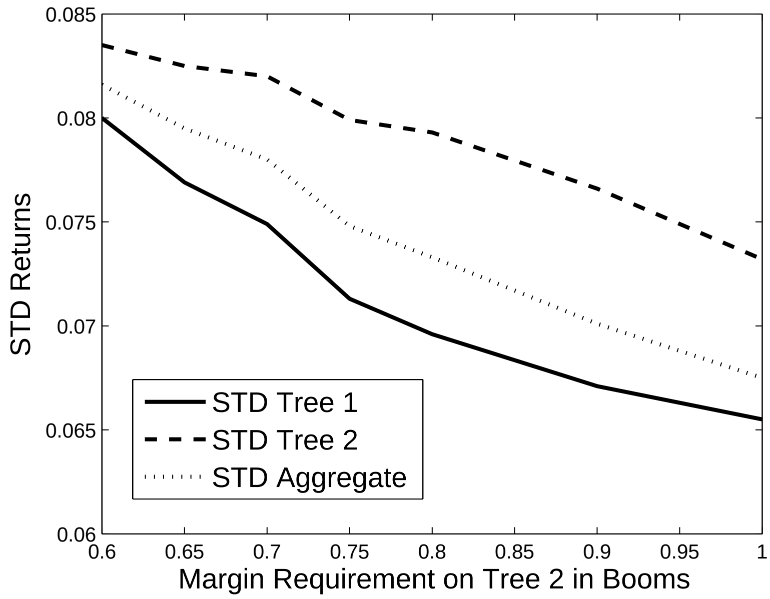


Figure 3.9: Volatility as a Function of the Margin Requirement on Tree 2 in Booms

3.5 Sensitivity Analysis and Extensions

As in any quantitative study, our results above hinge on the parametrization of the economy. In this section, we first discuss how our results change with other preference parameters. Then we highlight the important role of the disaster shocks for our quantitative results. Finally, we present an example which has less severe disaster shocks but nevertheless exhibits strong quantitative effects of collateral constraints.

3.5.1 Different Preferences in the Baseline Model

As a robustness check for the results in our baseline model (with one tree and one bond) from Section 3, we consider different specifications for the IES, the coefficients of risk aversion, and the discount factor, β. Obviously, changes in the IES and the risk aversion coefficients affect the risk-free rate. For these cases, we also examine specifications with an adjusted β so that the risk-free rates remain comparable. Table 3.6 reports asset-price moments for several different combinations of these parameters. For convenience, we repeat the results for our baseline model, $(IES, RA, \beta) = ((1.5, 1.5), (0.5, 6), (0.95, 0.95))$, and report them as the case (P1). For each model specification, we also report the standard deviation of returns for the benchmark case *B1: No bonds*.

In case (P2), a model in which both agents have an IES of 0.5, the tree return volatility is considerably lower than in the baseline case (P1). However, it is still much higher than in an economy with the same preferences but without borrowing, see column B1 of (P2). We checked this result for other values of the IES below 1.5 and always observed

$(IES^1, IES^2), (RA^1, RA^2), (\beta^1, \beta^2)$	Std returns	Risk-free rate	EP	Std in B1
(P1): (1.5,1.5),(0.5,6),(0.95,0.95)	8.14	1.10	3.86	5.33
(P2): (0.5,0.5),(0.5,6),(0.95,0.95)	7.20	1.75	4.18	5.33
(P3): (1.5,1.5),(0.5,6),(0.92,0.92)	7.70	4.07	3.77	5.51
(P4): (1.5,1.5),(0.5,6),(0.98,0.98)	8.57	-1.17	3.95	5.23
(P5): (1.5,1.5),(0.5,10),(0.95,0.95)	10.79	-8.58	12.55	5.34
(P6): (1.5,1.5),(0.5,10),(0.81,0.81)	8.50	1.25	13.36	6.24
(P7): (1.5,1.5),(0.5,4),(0.95,0.95)	6.58	1.59	4.22	5.34
(P8): (1.5,1.5),(0.5,4),(0.98,0.98)	6.97	1.18	1.73	5.22

Table 3.6: Sensitivity Analysis for Preferences (all Reported Figures in Percent)

the same phenomenon: Volatility effects are qualitatively similar but quantitatively less pronounced.[5]

Next we consider a change in the discount factor β. For the benchmark case *B1*, a higher β decreases return volatility simply because it decreases levels of returns and we report absolute volatility as opposed to the coefficient of variation. The effects in our model with one tree and one bond are quite different. As β increases from 0.95 in our baseline case (P1) to 0.98 in (P4), the return volatility increases from 8.14 to 8.47. The reason for this increase is simple. As β increases and the stock becomes more expensive, it is more difficult for agent 1 to buy a significant portion of the stock when he is in financial difficulties. This fact depresses the price of the stock when agent 1 is poor. Changes in the wealth distribution are large when agent 1 is fully leveraged and lead now to larger swings in the tree price.

In light of the intuition that we developed for the baseline case in Section 3.3, we expect an increase in the risk aversion of agent 2 to lead to both a higher price volatility and a higher equity premium. This intuition is strongly confirmed by the comparison of (P1) and (P5). However, the increase in the second agent's risk aversion also leads to a large reduction of the interest rate to unrealistically low levels. In (P6) we recalibrate the model to obtain a positive interest rate and we find that the previously described effect of a smaller β dampens the impact of a higher risk aversion. But still, overall volatility increases substantially once the risk aversion and β are changed simultaneously: For risk aversions of 4, 6, and 10, (cases (P8), (P1) and (P6)) the return volatility is 6.97, 8.14, and 8.50 respectively.

3.5.2 Endowments

As we have seen repeatedly in our analysis, our model produces asset pricing moments that are comparable to observed values in the data. Clearly, this nice feature of our model depends on the magnitude of the disaster shocks. We now report results for models with less severe disaster shocks and demonstrate that the results remain qualitatively the same. We conduct two different types of sensitivity analysis for our shock process. First, in the

[5] For low values of the IES, there is an additional unwanted effect. As one agent holds most of the wealth (that is, as the other agent becomes poor), asset prices increase because of the desire of the rich agent to save. This effect on the boundary of the state space is absent when the IES is set to 1.5 which we, therefore, do for the remainder of our analysis.

case (E1) we hold the magnitude of the disaster shocks constant, but reduce the overall probability of a disaster by 50 percent. Instead of setting the probabilities of shocks 1, 2, and 3 to 0.005, 0.005, and 0.024, respectively, we set them at 0.0025, 0.0025, and 0.012, respectively, and increase the probability of shock 5 accordingly. Secondly, in the case (E2) we leave the probabilities of the shocks unchanged but shift their support. In particular, we replace the growth rates in shocks 1, 2, and 3 of 0.566, 0.717, and 0.867, respectively, by the new values of 0.783, 0.8585, and 0.9335, respectively. Table 3.7 shows the analogue of Table 3.6 for these two cases.

	Std returns	Risk-free rate	Equity-premium	Std in B1
Case (E1)	5.95	3.44	2.17	4.15
Case (E2)	3.92	5.97	0.36	3.51

Table 3.7: Sensitivity Analysis for Endowments (all Figures are in Percent)

The table shows that a decrease in the probability of disaster has a relatively small effect on volatility while a change in the support has quite a large effect. As we explained above, the disaster states play two roles in our model. First, they lead to high excess returns of the tree, in particular whenever the risk-averse agent 2 must hold the tree. Secondly, they lead to endogenously high margin requirements. As we decrease the probability of disaster, the second effect remains unchanged. In contrast, the change in the support of the disaster shocks mitigates both effects above.

3.5.3 Large Effects with Smaller Shocks

The results for the case (E2) above show that the quantitative impact of collateral constraints depends heavily on the size of the disaster shocks. However, we now demonstrate that even with halved disaster shocks as in (E2), there are still substantial effects. For this purpose, we consider the model with two trees where tree 1 is collateralizable and tree 2 is not, and assume that agent 1's risk aversion is 10. We recalibrate the discount factor β to be 0.98, which results in a risk-free rate of 1.94. Table 3.8 shows that aggregate volatility with collateral constraints is now 48% higher than in the benchmark B1. This increase is of similar magnitude as in the baseline model. The high aggregate volatility is mostly driven by the volatility of tree 2, which increases by 95% compared to this benchmark.

	Std returns	EP	Std returns agg	Risk-free rate	EP agg	Std in B1
Tree 1	4.41	0.77	5.05	1.94	1.02	3.42
Tree 2	6.68	1.65				

Table 3.8: Moments of Trees' Returns (Tree 1 Collateralizable, Tree 2 not)

3.6 Conclusion

In this paper we show that collateral and margin requirements play a quantitatively important role for prices of long-lived assets. This is true even for assets that cannot be used as collateral. In fact, somewhat surprisingly, we show that the presence of collateral constraints has a larger effect on the volatility of non-collateralizable assets than on the underlying collateral.

The recent financial crisis has lead researchers to suggest that central banks should regulate collateral requirements, see, for example, Ashcraft, Gârleanu, and Pedersen (2010) or Geanakoplos (2009). We show that tightening margins uniformly over the business cycle can increase the price volatility of the underlying collateral but typically decreases price volatility of other long-lived assets in the economy that are not directly affected by the regulation. The only policy to achieve a decrease of the price volatility of all assets is to tighten margins only in boom times but leave them to market forces in recessions or crises. Our calibration assumes the presence of disaster shocks as in Barro (2009). We provide alternative parameterizations of preferences and endowments under which our main qualitative results continue to hold.

Appendix

3.A Details on Computations

The algorithm used to solve all versions of the model is based on Brumm and Grill (2010). Equilibrium policy functions are computed by iterating on the per-period equilibrium conditions, which are transformed into a system of equations. We use KNITRO to solve this system of equations for each grid point. Policy functions are approximated by piecewise linear functions. By using fractions of financial wealth as the endogenous state variables, the dimension of the state space is equal to the number of agents minus one. Hence with two agents, the model has an endogenous state space of one dimension only. This makes computations much easier than in Brumm and Grill (2010), where two and three dimensional problems are solved. In particular, in one dimension reasonable accuracy may be achieved without adapting the grid to the kinks. For the reported results we used 320 or 640 grid points depending on the complexity of the version of the model, which results in average (relative) Euler errors with order of magnitude 10^{-4}, while maximal errors are about ten times higher. If the number of gridpoints is increased to a few thousands, then Euler errors fall about one order of magnitude. However, the considered moments only change by about 0.1 percent. Hence, using 320 or 640 points provides a solution which is precise enough for our purposes. Compared to other models the ratio of Euler errors to the number of grid points used might seem large. However, note that due to the number of assets and inequality constraints our model is numerically much harder to handle than standard models. For example, in the version with one tree and five bonds, eleven assets are needed (as long and short positions in bonds have to the treated as separate assets) and we have to impose eleven inequality constraints per agent.

3.B Equilibrium Conditions

We state the equilibrium equations as we implemented them in Matlab for economies with a single tree and a single bond. For our computation of financial markets equilibria we normalized all variables by the aggregate endowment $\bar{e}$. To simplify the notation, we drop the dependence on the date-event s^t and, in an abuse of notation, denote the normalized parameters and variables by e_t, d_t and c_t, q_t, p_t, r_t, f_t, respectively. Similarly, we normalize both the objective function and the budget constraint of agents' utility maximization problem. The resulting maximization problem is then as follows (index h is dropped).

$$\begin{array}{llcl}
\max & u_t(c_t) = & \left\{(c_t)^\rho + \beta\left[E\left(u_{t+1}g_{t+1}\right)^\alpha\right]^{\frac{\rho}{\alpha}}\right\}^{\frac{1}{\rho}} \\
\text{s.t.} & 0 \;= & c_t + \phi_t p_t + \theta_t q_t - e_t - [\phi_{t-1}]^+ \dfrac{r_t}{g_t} + [\phi_{t-1}]^- \dfrac{f_t}{g_t} - \theta_{t-1}\left(q_t + d_t\right) \\
& 0 \;\le & \theta_t + k_t[\phi_t]^-, \quad 0 \;\le\; [\phi_t]^+, \quad [\phi_t]^- \;\le\; 0,
\end{array}$$

The latter two inequalities are imposed because, for the computations, we treat the long and short position in the bond, $[\phi_t]^+$ and $[\phi_t]^-$, as separate assets. Note that $\phi_t = [\phi_t]^+ + [\phi_t]^-$. Let λ_t denote the Lagrange multiplier on the budget constraint. The first-order condition with respect to c_t is as follows,

$$0 = (u_t)^{1-\rho}(c_t)^{\rho-1} - \lambda_t.$$

Next we state the first-order condition with respect to c_{t+1}.

$$0 = \beta u_t^{1-\rho}\left[E\left(u_{t+1}g_{t+1}\right)^\alpha\right]^{\frac{\rho-\alpha}{\alpha}}\left(u_{t+1}g_{t+1}\right)^{\alpha-1} g_{t+1}(u_{t+1})^{1-\rho}(c_{t+1})^{\rho-1} - \lambda_{t+1}.$$

Below we need the ratio of the Lagrange multipliers,

$$\frac{\lambda_{t+1}}{\lambda_t} = \beta\left[E\left(u_{t+1}g_{t+1}\right)^\alpha\right]^{\frac{\rho-\alpha}{\alpha}}(u_{t+1})^{\alpha-\rho}(g_{t+1})^\alpha\left(\frac{c_{t+1}}{c_t}\right)^{\rho-1}$$

Let μ_t denote the multiplier for the collateral constraint and let $\hat{\mu}_t = \frac{\mu_t}{\lambda_t}$. We divide the first-order condition with respect to θ_t,

$$0 = -\lambda_t q_t + \mu_t + E\left(\lambda_{t+1}\left(q_{t+1} + d_{t+1}\right)\right)$$

by λ_t and obtain the equation

$$0 = -q_t + \hat{\mu}_t + \beta\left[E\left(u_{t+1}g_{t+1}\right)^\alpha\right]^{\frac{\rho-\alpha}{\alpha}} E\left((u_{t+1})^{\alpha-\rho}(g_{t+1})^\alpha\left(\frac{c_{t+1}}{c_t}\right)^{\rho-1}\left(q_{t+1} + d_{t+1}\right)\right)$$

Similarly, the first-order conditions for $[\phi_t]^+$ and $[\phi_t]^-$ are as follows,

$$
\begin{aligned}
0 &= -p_t + \nu_t^+ + \beta \left[E\left(u_{t+1}g_{t+1}\right)^{\alpha}\right]^{\frac{\rho-\alpha}{\alpha}} E\left((u_{t+1})^{\alpha-\rho}(g_{t+1})^{\alpha}\left(\frac{c_{t+1}}{c_t}\right)^{\rho-1}\left(\frac{r_{t+1}}{g_{t+1}}\right)\right) \\
0 &= -p_t + \hat{\mu}_t k_t - \nu_t^- + \beta \left[E\left(u_{t+1}g_{t+1}\right)^{\alpha}\right]^{\frac{\rho-\alpha}{\alpha}} E\left((u_{t+1})^{\alpha-\rho}(g_{t+1})^{\alpha}\left(\frac{c_{t+1}}{c_t}\right)^{\rho-1}\left(\frac{f_{t+1}}{g_{t+1}}\right)\right),
\end{aligned}
$$

where ν_t^+ and ν_t^- denote the multipliers on $0 \leq [\phi_t]^+$ and $[\phi_t]^- \leq 0$, respectively.

Chapter 4

Optimal Sovereign Debt Default

4.1 Introduction

Sovereign debt crises are by no means rare events in history.[1] These crises and the subsequent debt defaults are widely believed to occur because governments are simply unwilling to honor initially promised payment streams and because there exist insufficient incentives making repayment optimal ex-post from the country's perspective. The weakness of ex-post incentives is thereby routinely attributed to 'sovereign immunity' which presumably protects governments from being sued in courts.[2] Viewed through this lens, the option of a sovereign to default is inefficient from an ex-ante welfare perspective, as anticipation of a possible default constrains international borrowing to suboptimally low levels.

In this paper we propose to interpret sovereign default events in a fundamentally different way. Instead of being the result of insufficient ex-post incentives in a situation without commitment, we propose to interpret sovereign defaults as an opportunity for more efficient international risk sharing in a situation where government debt is non-contingent. This interpretation has previously been advanced in Grossman and Van Huyck (1988) who distinguish between 'excusable' and 'non-excusable' default, with the first being part of

[1] Over the past decade governments in Argentina, Uruguay, Moldova and the Dominican Republic partially defaulted on their debt, with the Argentinian default being in dollar terms the largest ever recorded in history. The governments in Greece, Ireland and Portugal have recently been forced to apply for foreign assistance.

[2] Following this view, the main economic puzzle is then to explain how government debt can exist at all, if debt repudiation is an option available to sovereign debtors. An important literature, starting with the classic paper by Eaton and Gersovitz (1981) and ranging all the way to recent contribution by Broner, Martin, and Ventura (2010), has examined this view.

an ex-ante anticipated risk-sharing arrangement between the borrower and the lender, and the latter being the result of debt repudiation in the presence of weak ex-post incentives for repayment. Unlike Grossmann and Van Huyck, however, we consider a situation where the government possesses full commitment, thus discuss optimal borrowing and default from a purely normative perspective. And as we show, it has profound implications for the optimal default patterns. While in models with limited commitment the incentives to default are strongest in good times[3], the present model predicts default to be optimal in low output states.

The assumption of committed sovereigns is more plausible than generally recognized in the economics literature. First, as argued in Panizza, Sturzenegger, and Zettelmeyer (2009), legal changes in a range of countries in the late 1970's and early 1980's eliminated the legal principle of 'sovereign immunity' when it comes to sovereign borrowing. Specifically, in the U.S. and the U.K. private parties can sue foreign governments in courts, if the complaint relates to a commercial activity, amongst which courts regularly count the issuance of sovereign bonds. Second, although there now exists a voluminous literature on potential mechanisms supporting sovereign debt in the absence of commitment, these mechanisms have received limited empirical support.[4] In the light of these facts, it appears natural to deduce that governments can issue debt simply because they can in fact credibly commit to repay debt in some future states of the world, although they might actually choose not to repay in some states in which repayment turns out to be excessively costly.

To analyze the role of sovereign debt default as a vehicle for international risk shifting in a setting with a committed government, we construct a small open economy with production in which a domestic government can internationally borrow by issuing non-contingent bonds. The government can also accumulate international reserves by investing in (riskless) bonds issued by foreign lenders. The domestic economy is subject to shocks that affect the productivity of the domestic capital stock and the government can smooth the consumption implications of such shocks either via borrowing and lending in international

[3]Grossman and Van Huyck (1988), for example, state: 'the incentive to repudiate is largest in the good state' (p.1095). Recent work by Mendoza and Yue (2008) overcomes this problem and generates counterycyclical default by incorporating the effects of sovereign default on the default of domestic firms and the availability of foreign imports as inputs into domestic production.

[4]In the words of Panizza, Sturzenegger, and Zettelmeyer (2009): *'Almost three decades after Eaton and Gersovitz' pathbreaking contribution there still exists no fully satisfactory answer to how sovereign debt can exist in the first place. None of the default punishments that the classic theory of sovereign debt has focused on appears to enjoy much empirical backing' (p. 692).*

capital markets or via defaulting on its debt. The paper is concerned with the question of which channel the government should rely on to smooth domestic consumption, and specifically with the question: when is it optimal to (partially) default on government debt in a setting with a fully committed government?

In a first step, we analytically show that in the absence of default costs, optimal government default decisions can implement the first best consumption allocation and achieve full domestic consumption smoothing. The level of default is then generally decreasing with aggregate productivity and (partial) debt default occurs frequently and for all but the best productivity realization. In the absence of default costs, allowing the government to choose whether or not to repay government debt is thus a way to achieve the same consumption allocation as in a setting with a complete set of contingent government debt instruments.

In a second step, we introduce default costs. These costs feature prominently in political discussions and we model them as a simple dead weight cost that is proportional to the size of the government debt default. We show how low levels for the default costs make it generally optimal for the government *not* to default following business cycle sized shocks. Only when the country's net foreign debt position approaches the maximum level implied by the (marginally binding) natural borrowing limits, is a sovereign default still optimal after an adverse shock. With positive default costs, the optimal default policy thus depends on whether or not the country is close to its maximally sustainable net foreign debt position.

Given that small amounts of default costs largely eliminate government debt default, we introduce economic 'disaster' risk into the aggregate productivity process, following Barro and Jin (2011). Default then reemerges as part of optimal government policy, following the occurrence of a disaster shock. This is the case even for sizable default costs and even when the country's net foreign asset position is far from its maximally sustainable level. It continues to be optimal, however, not to default following business cycle sized shocks to aggregate productivity, as long as the net foreign debt position is not too close to its maximal level.

Finally, we evaluate the utility consequences of using the government default option as a way to insure domestic consumption against aggregate productivity shocks, comparing it to a situation where the government is *assumed* to repay debt unconditionally. In the latter case the government can use international wealth adjustments only to smooth domestic consumption. We show that the consumption equivalent welfare gain from considering default is in the order of one to two percentage points of consumption each period, even when there are sizable dead weight cost associated with a government debt default. If the default costs are sufficiently low, a large share of this welfare increase can be captured if instead of defaulting, the government optimally issues a combination on non-defaultable bonds and equity-like bonds that do not repay when one of the economic disaster states materializes. We thereby assume that non-repayment on the equity bond generates the same dead weight costs as an outright default. For higher levels of the dead weight costs, we show that outright default dominates the issuance of a combination of non-defaultable and equity bonds.

Sims (2001) discusses insurance in the context of whether or not Mexico should dollarize its economy, showing that giving up the domestic currency allows for less insurance in the presence of non-contingent nominal debt because the government is deprived of using the price level as a shock absorber. Unlike in the work of Sims, who considers non-contingent nominal bonds, the present paper considers a setting with non-contingent real bonds and considers optimal outright default policies. In the light of Sims' discussion, one could interpret the setting analyzed in the present paper as one in which the government issues (non-contingent) nominal bonds but has given up control over monetary policy and the price level, e.g., via joining a monetary union. As we show, the default option then still provides the country with a mechanism to make bond repayments contingent.

Angeletos (2002) explores an alternative insurance channel in a closed economy setting, showing that a government can use the maturity structure of domestic government bonds to insure against domestic shocks. This is achieved by exploiting the fact that bond yields of different maturities react differently to domestic shocks. This channel is unavailable, however, in our small open economy setting: in the absence of domestic default, the domestic yield curve is identical to the foreign yield curve for risk free assets and thus also independent of domestic shocks.

Juessen, Linnemann, and Schabert (2010) also analyze government default and the behavior of government bond premia. Considering a setting in which government behavior is characterized by simple rules, they show that multiple equilibria with different risk premia and default probabilities exist.

The present paper is structured as follows. Section 4.2 introduces the economic model and derives the optimal policy problem. It also determines an equivalent formulation of the optimal policy problem that facilitates numerical solution of optimal policies. Section 4.3 derives an analytical result for the case with no default costs and section 4.4 evaluates the effects of introducing default costs in a setting with business cycle sized shocks. We then introduce economic disaster shocks in section 4.5 and discuss their quantitative implications for optimal default policies. In section 4.6 we consider the welfare implications of using the default option and show how optimal default policies can approximately be implemented with a simple equity-like government bond instrument. In section 4.7 we discuss the effects of introducing long maturity bonds. A conclusion briefly summarizes. Technical material is contained in a series of appendices.

4.2 The Model

This section introduces a small open production economy and derives the government's optimal policy problem.

4.2.1 Private Sector: Households and Firms

The household side of the domestic economy is described by a representative consumer with utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{4.2.1}$$

where $\beta \in (0,1)$ denotes the discount factor and $u(c)$ the period utility function. The latter is assumed to be twice continuously differentiable, increasing in c and strictly concave, for all values of $c > \bar{c}$ where $\bar{c} \geq 0$ denotes the subsistence level for consumption. We shall assume that $u(c) = -\infty$ for all $c \leq \bar{c}$ and that Inada conditions hold, i.e., $\lim_{c \to \bar{c}^+} u'(c) = +\infty$ and $\lim_{c \to \infty} u'(c) = 0$.

The production side of the economy is described by a representative firm which produces consumption goods using the production function

$$y_t = z_t k_{t-1}^{\alpha},$$

where y_t denotes output in period t, k_{t-1} the capital stock from the previous period, $\alpha \in (0,1)$ the capital share, and $z_t > 0$ an exogenous stochastic productivity disturbance.

Productivity shocks assume values from some finite set $Z = \left\{z^1, ..., z^N\right\}$ with $N \in \mathbb{N}$. The transition probabilities for productivity across periods are described by some measure $\pi(z'|z)$ for $z', z \in Z$. Firms are owned by households and must decide on the capital stock one period in advance, i.e., before future productivity is known. For simplicity we assume that capital depreciates fully after one period.

4.2.2 The Government

The government seeks to maximize the utility of the representative domestic household (4.2.1) and is fully committed to its plans. It can invest in riskless international bonds issued by foreign lenders, issue own non-contingent bonds, and decide on the repayment of its maturing bonds. Unless otherwise stated, we assume that the risk free interest rate r on international bonds satisfies $\frac{1}{1+r} = \beta$. Furthermore, we assume that all bonds are zero coupon bonds and have a maturity of one period. The effects of introducing also longer maturity domestic bonds are discussed in section 4.7.[5]

The government's holdings of international bonds in period t (which mature in period $t+1$) constitutes a long position and is denoted by $G_t^L \geq 0$. The own (potentially risky) bonds issued by the government in period t represents a short position and is denoted by $G_t^S \geq 0$. The government can use adjustments in the long and short positions to insure domestic consumption against domestic productivity shocks. In addition, it can decide in period t to (partially) default on the bonds maturing in period $t+1$. More formally, the default decision is described by a vector of default profiles

$$\Delta_t = (\delta_t^1, ..., \delta_t^N) \in [0,1]^N ,$$

where $\delta_t^n \in [0,1]$ denotes the fraction of outstanding domestic bonds issued in period t that is *not* repaid in period $t+1$ when the bonds mature and when the productivity state is $z_{t+1} = z^n$. Default is thus state-contingent and an entry equal to zero indicates full repayment. Full repayment is typically *assumed* in much of the previous literature dealing with optimal fiscal policy under commitment with incomplete markets, e.g., Angeletos (2002), Buera and Nicolini (2004) or Marcet and Scott (2009). In our setting repayment

[5]The fact that we consider only a single maturity for the international bond is without loss of generality. Since foreign interest rates are independent of domestic conditions, the government cannot use the maturity structure of foreign bonds to insure against domestic shocks.

is treated as a choice variable.

Total repayment on maturing domestic bonds in period $t+1$ when productivity is equal to z_{t+1} is then given by

$$G_t^S \cdot (1-(1-\lambda)\delta_t^{I(z_{t+1})})$$

where $I(z_{t+1})$ denotes the index of the productivity shock, i.e., $I(z_{t+1}) = n$ if and only if $z_{t+1} = z^n$. The parameter $\lambda \geq 0$ captures the possibility that the government's default decision gives rise to dead weight costs. Our specification assumes that these dead weight costs are proportional to the size of the absolute debt default chosen by the government. For $\lambda = 0$ a default does not produce any additional costs: by defaulting the government then gains resources equal to $G_t^S \delta_t^{I(z_{t+1})}$ relative to the case without default and foreign lenders loose the corresponding amount. A setting with $\lambda > 0$ indicates that the government's default produces additional costs for the government that are not accruing to lenders. This is a short-cut to capture costs that are associated with having to defend legal positions in foreign courts or with possible disruptions in the financial system following a sovereign debt default.

We can now define the amount of resources available to the domestic government at the beginning of the period, i.e., before issuing new debt and making investment decisions on international bonds, but after (partial) repayment of maturing bonds.[6] We refer to these resources as beginning-of-period wealth and define them as

$$w_t \equiv z_t k_{t-1}^{\alpha} + G_{t-1}^L - G_{t-1}^S \cdot (1-(1-\lambda)\delta_{t-1}^{I(z_t)})$$

Beginning-of-period wealth will serve as a useful state variable when computing optimal government policies later on. The government can raise additional resources in period t by issuing own government bonds. It can then use the resulting funds to invest in international riskless bonds, to invest in the domestic capital stock, and to finance domestic consumption. The economy's budget constraint is thus given by

$$c_t + k_t + \frac{1}{1+r} G_t^L = w_t + \frac{1}{1+R(z_t,\Delta_t)} G_t^S$$

[6]Below we do not distinguish between the government budget and the household budget, instead consider the economy wide resources that are available. This implicitly assumes that the government can costlessly transfer resources between these two budgets, e.g., via lump sum taxes.

where $\frac{1}{1+r}$ denotes the price of the risk-free international bond and $\frac{1}{1+R(z_t,\Delta_t)}$ the price of the domestic bond. The real interest rate $R(z_t, \Delta_t)$ of the domestic bond depends on the default profile Δ_t chosen by the government and on the current productivity state, as it may affect the likelihood of entering different states tomorrow. Due to the small open economy assumption, the government takes the pricing function $R(\cdot,\cdot)$ as given in its optimization problem. Assuming risk-neutral international lenders, no-arbitrage implies that the pricing function for domestic bonds is given by

$$\frac{1}{1+R(z_t,\Delta_t)} = \frac{1}{1+r}\sum_{n=1}^{N}(1-\delta_t^n)\cdot\pi(z^n|z_t)$$

so that the expected return on the domestic bond is equal to the return on the riskless international bond.

We are now in a position to formulate the government's optimal policy problem (Ramsey allocation problem):

$$\max_{\left\{G_t^L\geq 0, G_t^S\geq 0, \Delta_t\in[0,1]^N, k_t\geq 0, c_t\geq \bar{c}\right\}} \quad E_0\sum_{t=0}^{\infty}\beta^t u(c_t) \tag{4.2.2a}$$

$$s.t. \quad : \quad c_t = w_t - k_t + \frac{G_t^S}{1+R(z_t,\Delta_t)} - \frac{G_t^L}{1+r} \tag{4.2.2b}$$

$$w_{t+1} \geq NBL(z_{t+1}) \quad \forall z_{t+1}\in Z \tag{4.2.2c}$$

$$w_0, z_0 : given$$

We have added the natural borrowing limits (4.2.2c) so as to prevent explosive debt dynamics (Ponzi schemes). In our numerical application we will set the natural borrowing limits (NBLs) to values such that they are just marginally binding, as this facilitates computation. Since these marginally binding limits may depend on the state of the productivity shock, we make this dependence explicit here. Imposing laxer and possibly non-contingent natural borrowing limits would have no implication for the optimal policies. We also assume that initial conditions are such that there exists a solution with $w_t \geq NBL(z_t)$ for all t and all possible realizations $z_t \in Z$.

While intuitive, the formulation of the optimization problem (4.2.2) has a number of unattractive features. First, the price of the domestic government bond depends on the chosen default profile, so that constraint (4.2.2b) fails to be linear in the government's choice variables. It is thus unclear whether problem (4.2.2) is concave. Second, the inequality constraints for G_t^L, G_t^S and especially those for Δ_t are difficult to handle computationally, as they will be occasionally binding.[7] Moreover, the optimal default policies Δ_t turn out to be discontinuous. For these reasons, we derive in the next section an equivalent formulation of the problem that can be shown to be concave, that features fewer occasionally binding inequality constraints, and gives rise to continuos optimal policy functions.

4.2.3 Equivalent Formulation of the Government Problem

We now formulate an alternative optimal policy problem with a different asset structure than in problem (4.2.2) and thereafter show that it is equivalent to the original problem (4.2.2).

Specifically, we assume that there exist N Arrow securities and a single riskless bond in which the country can go either long or short. The vector of Arrow security holdings is denoted by $a \in \mathbb{R}^N$ and the n-th Arrow security pays one unit of output tomorrow if productivity state z^n materializes. The associated price vector is denoted by $p \in \mathbb{R}^N$. Given the risk-neutrality of international lenders, the price of the n-th Arrow security in period t is

$$p_t(z^n) = \frac{1}{1+r}\pi(z^n|z_t). \tag{4.2.3}$$

Letting b denote the country's holdings of riskless bonds, beginning-of-period wealth for this asset structure is then given by

$$\widetilde{w}_t \equiv z_t\widetilde{k}_{t-1}^{\alpha} + b_{t-1} + (1-\lambda)a_{t-1}(z_t) \tag{4.2.4}$$

where $a_{t-1}(z_t)$ denotes the amount of Arrow securities purchased for state z_t, $\widetilde{k}_{t-1}$ capital invested in the previous period, and $\lambda \geq 0$ is the parameter capturing potential default costs in the original problem (4.2.2).

Next, consider the following alternative optimization problem:

[7] The fact that marginal utility increases without bound as $c_t \to \overline{c}$ and that marginal productivity of capital increases without bound as $k_t \to 0$ will insure interior solutions for these two choice variables, allowing to ignore the inequality constraints for these variables when computing numerical solutions.

$$\begin{aligned}
\max_{\left\{b_t, a_t \geq 0, \widetilde{k}_t \geq 0, \widetilde{c}_t \geq \bar{c}\right\}} & \quad E_0 \sum_{t=0}^{\infty} \beta^t u(\widetilde{c}_t) && (4.2.5a) \\
s.t.\ \ \forall t: \widetilde{c}_t &= \widetilde{w}_t - \widetilde{k}_t - \frac{1}{1+r} b_t - p_t \cdot a_t && (4.2.5b) \\
\widetilde{w}_{t+1} &\geq NBL(z_{t+1}) \quad \forall z_{t+1} \in Z \\
\widetilde{w}_0 = w_0, z_0 & \quad given.
\end{aligned}$$

Problem (4.2.5) has the same concave objective function as problem (4.2.2), but the constraint (4.2.5b) is now linear in the choices, so that first order conditions (FOCs) are necessary and sufficient. The FOCs can be found in appendix 4.A. Furthermore, problem (4.2.5) reveals that the optimization problem has a recursive structure with the state in period t being described by the vector $(z_t, \widetilde{w}_t)$, allowing us to express optimal policy functions as a function of these two state variables only. Finally, the relevant inequality constraints are given by $a_t \geq 0$and the marginally binding natural borrowing limits.[8]

We now show that if a consumption path $\{c_t\}_{t=0}^{\infty}$ is feasible in problem (4.2.2), then is it also feasible in problem (4.2.5), and vice versa, i.e., the two different asset structures allow to implement the same set of consumption paths. One can thus use the solution to problem (4.2.5), which is easier to compute, to derive the asset structure and default profiles implementing the same consumption path in the original problem (4.2.2).

Consider some state contingent beginning-of-period wealth profile w_t arising from some combination of bond holdings, default decisions and capital investment (G^L_{t-1}, G^S_{t-1}, Δ_{t-1}, k_{t-1}) in problem (4.2.2). We now show that one can generate the same state contingent beginning-of-period wealth profile $\widetilde{w}_t = w_t$ in problem (4.2.5) by choosing $\widetilde{k}_{t-1} = k_{t-1}$ and by choosing an appropriate investment profile (a_{t-1}, b_{t-1}). Moreover, the funds required to purchase (a_{t-1}, b_{t-1}) are the same as those required to purchase $\left(G^L_{t-1}, G^S_{t-1}\right)$ when the default profile is Δ_{t-1}. With the costs of financial investments being the same in both problems, identical physical investments, and identical beginning of period wealth profiles, it then follows from constraints (4.2.2b) and (4.2.5b) that the implied consumption paths are also the same in both problems, establishing the equivalence between the two problems.

[8] As before, the Inada conditions on utility and the fact that marginal productivity of capital increases without bound as $k_t \to 0$ will insure interior solutions for c_t and k_t, allowing to ignore the inequality constraints for these variables when computing numerical solutions.

To keep notation as simple as possible we establish the previous claim for the case with 2 productivity states only. The extension to N states is relatively straightforward. Consider the following state contingent initial wealth profile

$$\begin{pmatrix} w_t(z^1) \\ w_t(z^2) \end{pmatrix} = \begin{pmatrix} z^1 k_{t-1}^{\alpha} + G_{t-1}^L - G_{t-1}^S(1-(1-\lambda)\delta_{t-1}^1) \\ z^2 k_{t-1}^{\alpha} + G_{t-1}^L - G_{t-1}^S(1-(1-\lambda)\delta_{t-1}^2) \end{pmatrix}.$$

One can replicate this beginning-of-period wealth profile in problem (4.2.5) by choosing $\widetilde{k}_{t-1} = k_{t-1}$ and by choosing the portfolio

$$b_{t-1} = G_{t-1}^L - G_{t-1}^S, \tag{4.2.6}$$

$$a_{t-1} = \begin{pmatrix} G_{t-1}^S \delta^1 \\ G_{t-1}^S \delta^2 \end{pmatrix} \tag{4.2.7}$$

The previous equations show that b in problem (4.2.5) has an interpretation as the net foreign asset position in problem (4.2.2) and that a in problem (4.2.5) can be interpreted as the state contingent default on outstanding own bonds. We will make use of this interpretation in the latter part of the paper. The funds f_{t-1} required for (G_{t-1}^L, G_{t-1}^S) under the default profile $(\delta_{t-1}^1, \delta_{t-1}^2)$ are given by

$$f_{t-1} = \frac{1}{1+r} G_{t-1}^L - \frac{1}{1+R(z_{t-1}, (\delta_{t-1}^1, \delta_{t-1}^2))} G_{t-1}^S$$

where the interest rate satisfies

$$\frac{1}{1+R(z_{t-1}, (\delta_{t-1}^1, \delta_{t-1}^2))} = \frac{1}{1+r}\left((1-\delta_{t-1}^1)\pi(z^1|z_{t-1}) + (1-\delta_{t-1}^2)\pi(z^2|z_{t-1})\right).$$

The funds $\widetilde{f}_{t-1}$ required to purchase (b_{t-1}, a_{t-1}) are

$$\widetilde{f}_{t-1} = \frac{1}{1+r}(G_{t-1}^L - G_{t-1}^S) + \frac{1}{1+r}\left(\delta_{t-1}^1 \pi(z^1\,|z_{t-1}) + \delta_{t-1}^2 \pi(z^2\,|z_{t-1})\right) G_{t-1}^S,$$

where we used the price of the Arrow security in (4.2.3). As is easy to see $\widetilde{f}_{t-1} = f_t$, as claimed.

Finally, note that we need to impose the restriction $a \geq 0$ on problem (4.2.5), as otherwise it would follow from equation (4.2.7) that one could implement a consumption path in problem (4.2.5) that cannot be implemented in problem (4.2.2) with values of δ^i satisfying $\delta^i \in [0,1]$ for all i. This completes the equivalence proof.

4.3 Zero Default Costs

In the absence of default costs, the solution to problem (4.2.5) can be analytically determined. The following proposition summarizes the main finding. The proof can be found in appendix 4.B.

Proposition 4.1. *Without default costs ($\lambda = 0$) the solution to problem (4.2.5) involves constant consumption equal to*

$$c = (1-\beta)(\Pi(z_0) + \widetilde{w}_0) \tag{4.3.8}$$

where $\Pi(\cdot)$ denotes the maximized expected profits from future production, defined as

$$\Pi(z_t) \equiv E_t \left[\sum_{j=0}^{\infty} \beta^j \left(-k^*(z_{t+j}) + \beta z_{t+j+1} \left(k^*(z_{t+j}) \right)^{\alpha} \right) \right]$$

with

$$k^*(z_t) = (\alpha \beta E(z_{t+1}|z_t))^{\frac{1}{1-\alpha}} \tag{4.3.9}$$

denoting the profit maximizing capital level. For any period t, the optimal default level satisfies

$$a_t(z_t) \propto -\left(\Pi(z_t) + z_t \left(k^*(z_{t-1}) \right)^{\alpha} \right) \tag{4.3.10}$$

The proposition shows that in the absence of default costs, it is optimal to fully smooth consumption. The option of partial repayment thus allows for complete insurance of domestic production risk, as would be the case in a complete market setting. Equation (4.3.10) thereby reveals that default must occur frequently and for virtually all productivity realizations.[9] Default thereby insures the country against two components: first, against (adverse) news regarding the expected profitability of future investments, as captured by $\Pi(z_t)$; and second, against low output due to a low realization of current productivity,

[9] Default is not required for states z_t achieving the maximal value for $\Pi(z_t) + z_t(k^*(z_{t-1}))^{\alpha}$ across all $z_t \in Z$. For such states default can be set equal to zero, otherwise default levels are strictly positive.

as captured by $z_t\left(k^*(z_{t-1})\right)^{\alpha}$. If expected future profits commove positively with current productivity, e.g. if z_t is a persistent process, or in the special case with iid productivity shocks, where expected future profits are independent of current productivity, it follows from equation (4.3.10) that optimal default levels are inversely related to the current level of productivity. Default is then optimal whenever z_t falls short of its highest possible value and the optimal size of default is increasing in the amount by which productivity falls short if its highest possible level.

4.4 Optimal Default Policies with Default Costs

The previous section abstracted from potential dead weight costs associated with a government debt default decision. The trade-off between insuring consumption via default or via (international) wealth accumulation/decumulation is then resolved fully in favor of using the default option. As is clear from equation (4.2.4), however, it becomes optimal to rely exclusively on self-insurance via international wealth adjustments, i.e., to set $a_t \equiv 0$, if the dead weight costs from default become sufficiently high, e.g., if $\lambda \geq 1$. To evaluate how the trade-off between default and self-insurance is resolved for intermediate levels of default costs, we now consider a quantitative setup with business cycle shocks to aggregate productivity. As we show below, fairly low levels of default costs then make it optimal to almost exclusive rely on self-insurance through international reserve adjustments. Only when the country's net foreign asset position is sufficiently close to the (marginally binding) natural borrowing limits, will it be optimal to default on government debt.

4.4.1 Calibration

We now calibrate the model. A standard parameterization for quarterly productivity is given by a first order autoregressive process with quarterly persistence of 0.9 and a standard deviation of 0.5% for the quarterly innovation.[10] Since we use a yearly model, we annualize these values by choosing an annual persistence of technology equal to $(0.9)^4$ and use an annual standard deviation of the innovation of 1%. We then use Tauchen's 1986 procedure to discretize the shock process into a process with a high and a low productivity

[10] The quantitative results reported below are not very sensitive to the precise numbers used. A similar calibration is employed in Adam (2011).

state. Normalizing average productivity to one, the resulting high productivity state is $z^h = 1.0133$ and the low productivity state $z^2 = 0.9868$. The procedure also yields the following transition matrix for the states

$$\pi = \begin{pmatrix} 0.8077 & 0.1923 \\ 0.1923 & 0.8077 \end{pmatrix}.$$

We set the capital share parameter in the production function to $\alpha = 0.34$. The annual discount factor is $\beta = 0.97$ and we consider households with a flow utility function given by

$$u(c) = \frac{(c - \bar{c})^{1-\sigma}}{1 - \sigma}$$

where $\bar{c} \geq 0$ denotes the subsistence level of consumption and σ parameterizes risk aversion. We choose $\sigma = 2$ and calibrate the subsistence level of consumption $\bar{c}$ such that in an economy where the government is forced to repay debt always, the marginally binding natural borrowing limit implies that the net foreign asset position of the country is not below -100% of average GDP in any productivity state.[11] We thereby seek to capture the fact that industrialized countries do not appear to have net foreign asset positions below -100% of GDP, see figure 10 in Lane and Milesi-Ferretti (2007). Moreover, three out of the five industrialized countries approaching this boundary in the year 2004 later on faced fiscal solvency problems (Greece, Portugal and Iceland). It thus appears plausible to assume that countries cannot sustain higher external debt levels without running the risk of a government default.

Positive default costs and the small open economy assumption imply that the equilibrium outcomes are non-stationary, unless we choose $1 + r < 1/\beta$. To insure that the equilibrium process is ergodic, we set the annual international interest rate five basis points below the rate implied by the inverse of the discount factor. Optimal default policies are rather robust to the precise number chosen.[12]

[11] Appendix 4.D explains how one can compute the marginally binding NBL for each productivity state. Average GDP is defined as the average output level associated with efficient investment, i.e., when $k_t = k^*(z_t)$ each period, and where we average over the ergodic distribution of the z process. For our parameterization this yields an average output level of 0.5661. Furthermore, the net foreign asset position of the country is independent of government policy at the marginally binding NBL, instead exlusivley determined by the desire to prevent debt from exploding, so that this measure can be used to calibrate the model. The resulting level for subsistence onsumption is $\bar{c} = 0.357$.

[12] We also experimented with larger gaps of 50 basis points.

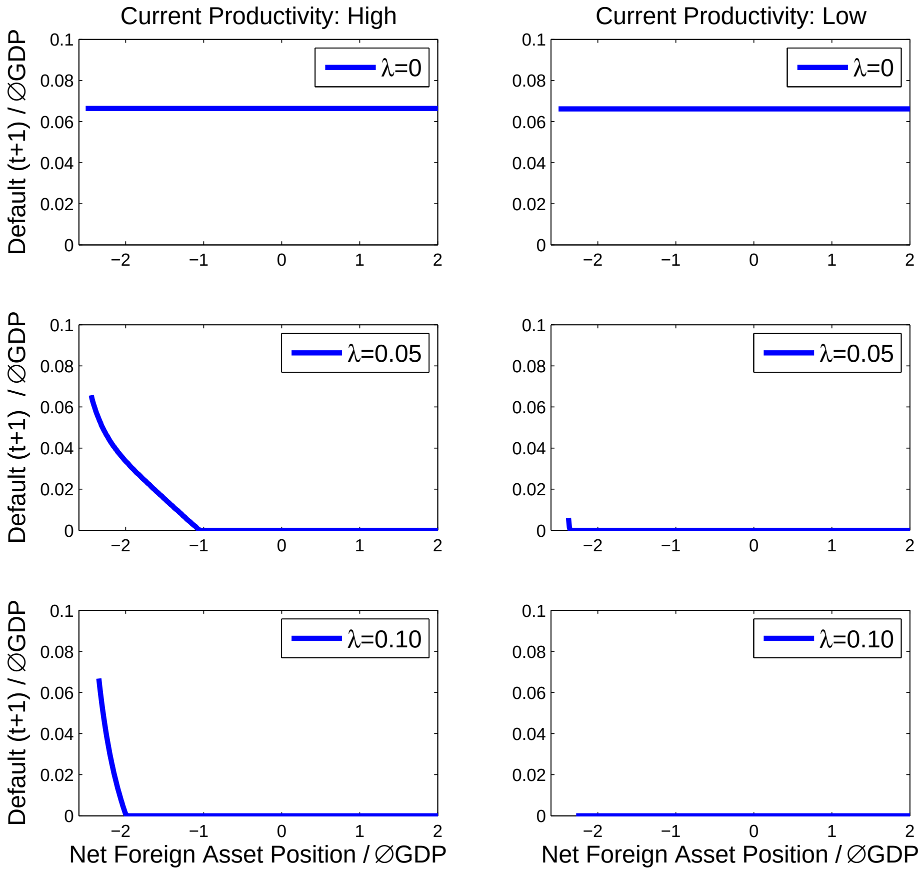


Figure 4.1: Optimal Default Policies: The Effect of Default Costs

4.4.2 Evaluating the Effect of Default Costs

Figure 4.1 reports the optimal default policies for the next period as a function of the current (end-of period) net foreign asset position and the current productivity state.[13] Each row in the figure thereby corresponds to a different default cost parameterization (λ). To simplify the interpretation of results, the default policies and the net foreign asset positions are normalized by average GDP. The panels on the left thereby depicts the optimal default policy in the high productivity state (z^h) and the panels on the right policy for the low productivity state (z^l). Appendix 4.C explains how the optimal policies can be

[13] As explained in section 4.2.3, the net foreign asset position is given by the optimal value of b in the corresponding period.

determined numerically.

The graphs shown in the first row of Figure 4.1 report the outcome for the case when default costs are zero.[14] Specifically, they depict the optimal amount of default in the next period, when the future productivity state happens to be low (z^l). Note that there will never be default if the productivity state z^h realizes in the next period. Interestingly, the optimal amount of default is independent of the country's net foreign asset position and almost independent of the current state of productivity.[15] As is clear from proposition 4.1, these default policies fully insure future consumption against fluctuations in productivity.

The middle and bottom rows in Figure 4.1 report the optimal default policies when default costs equal 5% and 10% of the defaulted amount, respectively. For large parts of the state space default then ceases to be optimal. Moreover, there is less default in the future if the current productivity state is low already. This is optimal because insurance against a future low state is more costly when current productivity state is low already, due to the persistence of productivity. Default continues to be optimal, however, if the net foreign asset position is sufficiently negative. Marginal utility of consumption is then very sensitive to further consumption fluctuations, because consumption approaches its subsistence level as the net foreign asset position approaches the limits implied by the (marginally binding) naturally borrowing limits.

Overall, Figure 4.1 shows that moderate levels of default costs shift optimal policy strongly towards using adjustments in international wealth to insure domestic consumption. Only if the country's net foreign asset position approaches the borrowing limit will a government debt default still be optimal.

4.5 Optimal Default and Economic Disasters

The previous section showed that with moderate levels of default costs it becomes suboptimal to default on government debt, provided the country is not too close to its borrowing limit. In this section we evaluate whether this conclusion continues to be true for a setting with much larger economic shocks. This is motivated by the observation that countries

[14]Since there exists a multiplicity of optimal default policies when $\lambda = 0$, the first row shows the outcome in the limiting case $\lambda \to 0$

[15]From equation (4.3.10) follows that the default in the next period does depend on the current state because the optimal investment $k^*(z_t)$ depends on the current productivity.

occasionally experience very large negative shocks, as previously argued by Rietz (1988) and Barro (2006), and that such shocks tend to be associated with a government default.[16] To capture the possibility of large shocks, we augment the model by including disaster like shocks to aggregate productivity and then explore the quantitative implications of disaster risk on optimal government debt default decisions.

4.5.1 Calibrating Economic Disasters

To capture economic disasters we introduce two disaster sized productivity levels to our aggregate productivity process. We add two disaster states rather than a single one to capture the idea that the size of economic disasters is uncertain ex-ante. This will become important in section 4.6, when we discuss how well simple financial instruments can approximate optimal default policies.

We calibrate the disaster shocks to match the mean and variance of GDP disasters, as documented in Barro and Jin (2011). Using a sample of 157 GDP disasters, they report a mean reduction in GDP of 20.4% and a standard deviation of 12.64%. Assuming that it is equally likely to enter both disaster states, this yields the productivity states $z^d = 0.9224$ and $z^{dd} = 0.6696$. Our vector of possible productivity realizations thus takes the form $Z = \left\{z^h, z^l, z^d, z^{dd}\right\}$ where the parameterization of the business cycle states $\left(z^h, z^l\right)$ is the same as in the previous section. The state transition matrix for the shock process is given by

$$\pi = \begin{pmatrix} 0.7770 & 0.1850 & 0.019 & 0.019 \\ 0.1850 & 0.7770 & 0.019 & 0.019 \\ 0.1429 & 0.1429 & 0.3571 & 0.3571 \\ 0.1429 & 0.1429 & 0.3571 & 0.3571 \end{pmatrix},$$

The transition probability from the business cycle states into the disaster states is chosen so as to match the unconditional disaster probability of 0.038, as reported in Barro and Jin (2011). We thereby assume that it is equally likely to reach both disaster states. The persistence of the disaster states is set to match the average duration of GDP disasters, which equals 3.5 years, see Barro and Jin. Finally, the transition probabilities of the business cycle states are adjusted to reflect the presence of disaster risk.

[16]Barro (2006) and Gourio (2010) also allow for default on government bonds in disaster states. Since the focus of their analysis is different, they use exogenous probabilities and default rates.

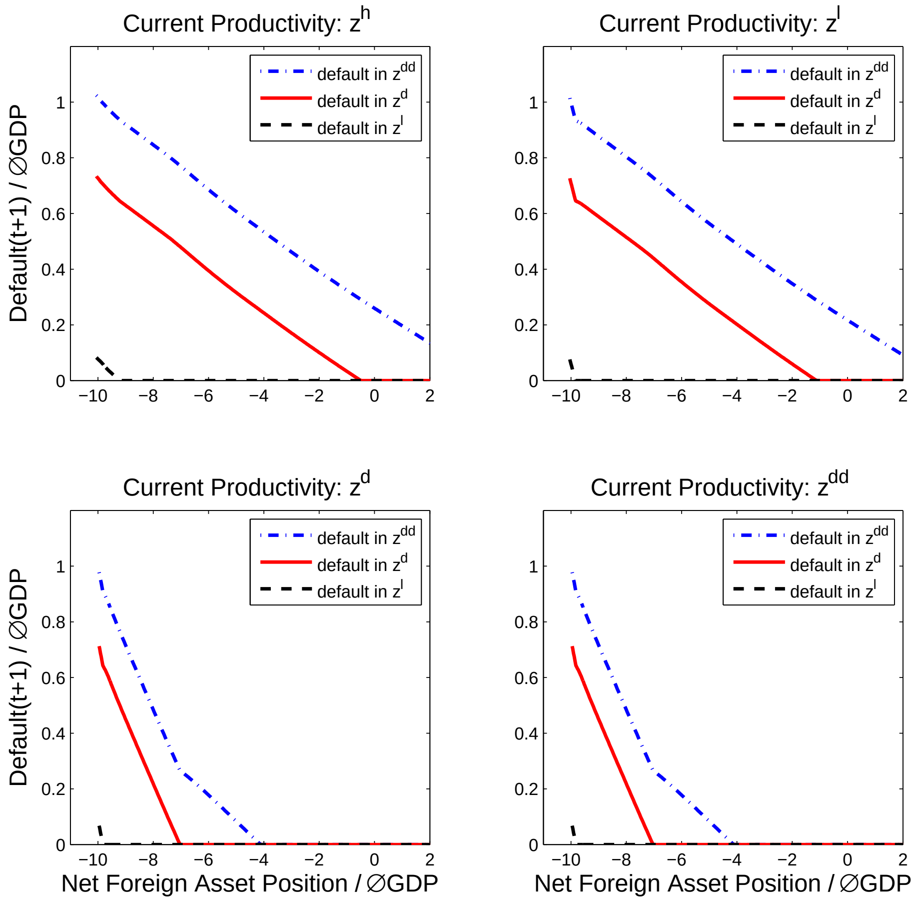


Figure 4.2: Optimal Default Policies with Disaster States

Since the presence of disaster risk strongly affects the marginally binding NBLs (they become much tighter and potentially require even positive net foreign asset positions in all states), we recalibrate the subsistence level for consumption $\bar{c}$. As in section 4.4 before, we choose $\bar{c}$ such that in an economy where bonds must be repaid always, the economy can sustain a maximum net foreign asset of -100% of average GDP in the business cycle states (z^h, z^l).[17] Choosing tighter limits does not affect the shape of the optimal default policies but only shifts the policies reported in the next subsection 'further to the right'.

[17]This yields an adjusted value of $\bar{c} = 0.198$.

4.5.2 Optimal Default with Disasters: Quantitative Analysis

Figure 4.2 reports the optimal default policies for the economy with disaster shocks. Each panel in the figure corresponds to a different productivity state today and reports the intended amount of default in tomorrow's states z^l, z^d and z^{dd} as a function of the country's net foreign asset position today.[18] We thereby assume that the dead weight costs of default equal 10% of the defaulted amount, corresponding to the default cost value used for computing the lowest row in Figure 4.1.

Figure 4.2 shows that it is virtually never optimal to default in the low business cycle state (z^l), unless the net foreign debt position is very close to its maximally sustainable level, similar to section 4.4 where we considered business cycle shocks only. Furthermore, for a wide range of net foreign asset positions, it is optimal to default if the economy makes a transition from a business cycle state to a disaster state, see the top panels in the figure. Default is optimal for a transition to the severe disaster state (z^{dd}), even when the country's net foreign asset position is positive before the disaster. Overall, the optimal amount of default is increasing as the country's net foreign asset position worsens. Yet, once the economy is in a disaster state, a further default in the event that the economy remains in the disaster state is optimal only if the net foreign asset position is very low, see the bottom panels of Figure 4.2. Since the likelihood of staying in a disaster state is quite high, choosing not to repay if the disaster persists would have very high effects on interest rate costs ex-ante. As a result, serial default in case of a persistent disaster will not necessarily be part of the optimal default policy.

The overall shape of the optimal default policies is fairly robust to assuming different values for the default costs λ. Larger costs shift the default policies towards the left, i.e., default occurs only for more negative net foreign asset positions. However, higher costs also tighten the maximally sustainable net foreign asset positions, thereby reducing the range of net foreign asset positions over which default occurs. Lower cost have the opposite effect, i.e., they induce a rightward shift and allow to sustain more negative net foreign asset positions.

Figure 4.3 reports a typical sample path for the net foreign asset position and the amount of default implied by optimal policy for $\lambda = 0.1$. We start the path at a zero net foreign asset position and each model period corresponds to one year. The figure shows that it is optimal to improve the net foreign asset position when the economy is in the business

[18] Recall that default is never optimal if z^h realizes in the next period.

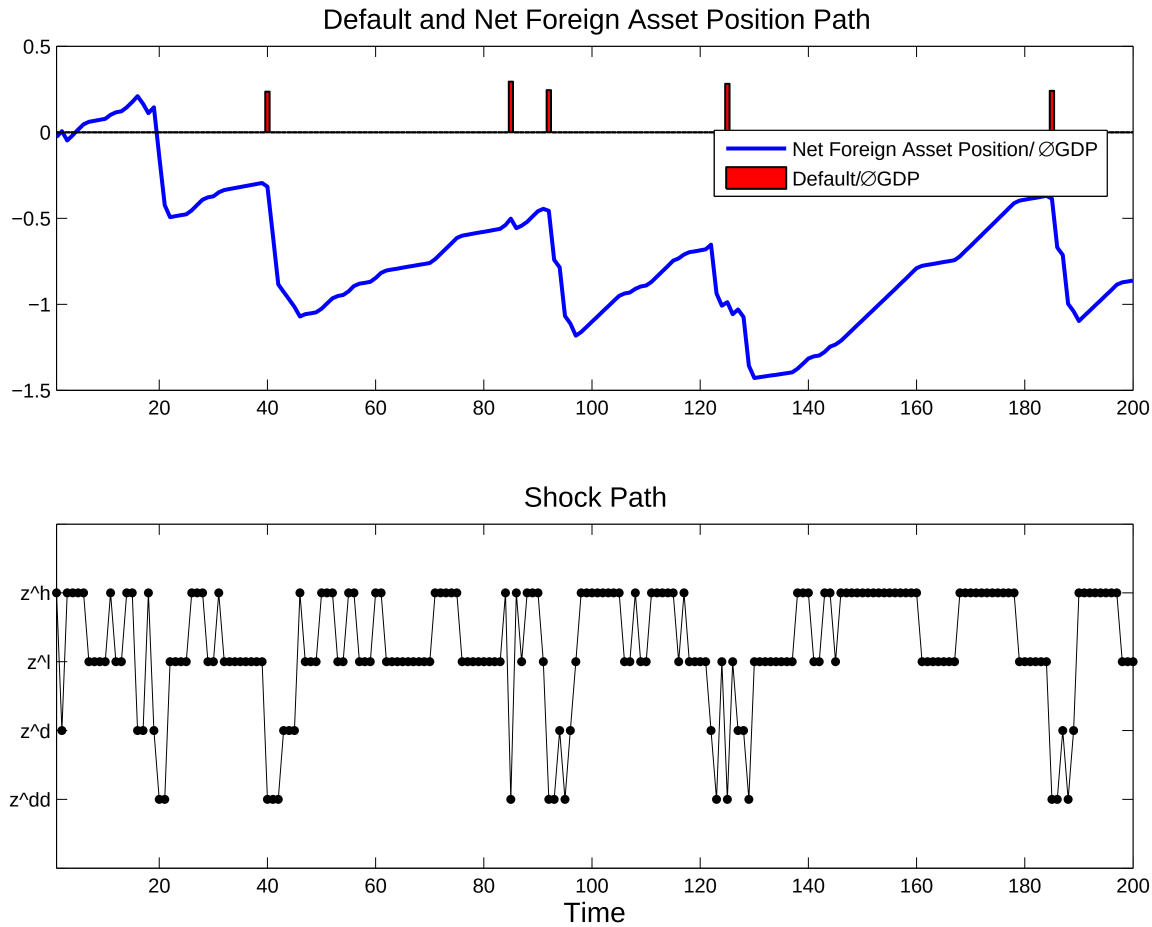


Figure 4.3: The Evolution of Net Foreign Assets and Default under Optimal Policy ($\lambda = 0.1$)

cycle states, with faster improvements in the high state. This is the case even though the international risk free rate is 5 basis points below the inverse of the domestic discount factor. A transition to a disaster state leads to a default provided the economy's net foreign asset position is not too high (unlike in year 16). Also, following a disaster, the net foreign asset position deteriorates whenever the disaster persists for more than one period (see for example year 40), otherwise the net foreign asset position is largely unaffected or improves even slightly (see year 85). Overall, the net foreign asset dynamics are characterized by rapid deteriorations during persistent disaster periods and gradual improvements during normal times.

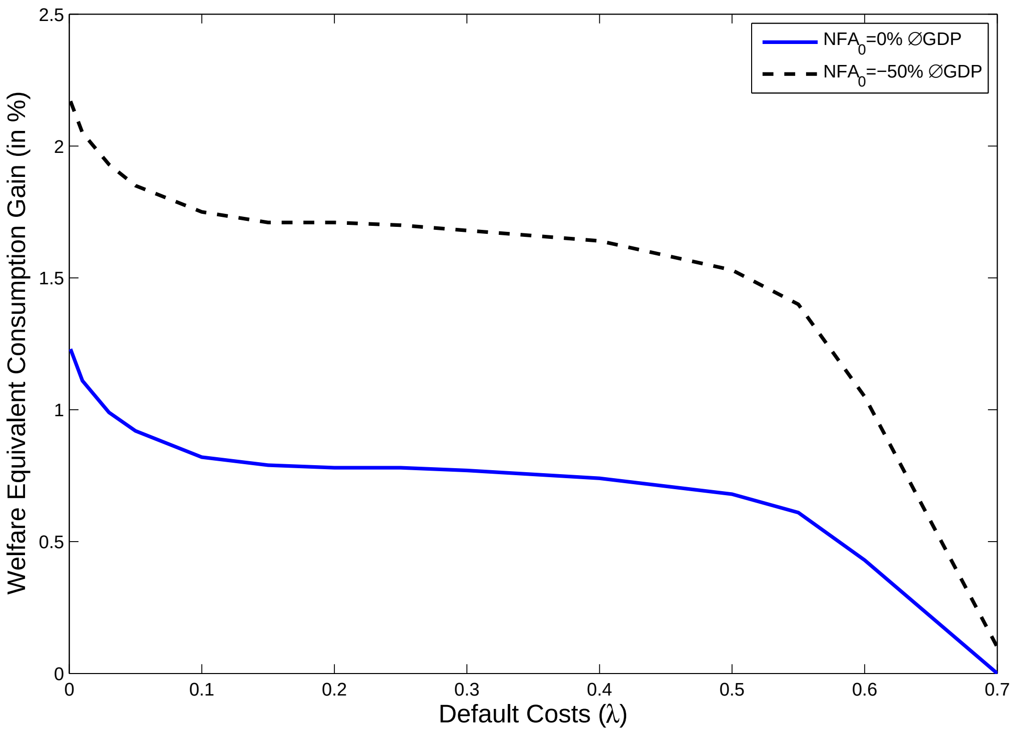


Figure 4.4: Welfare Gains from Using the Default Option

4.6 Welfare Analysis and Approximate Implementation

This section determines the welfare effects of letting the government choose whether or not to repay its debt compared to a situation where repayment is simply forced upon the government (or assumed) in each state. Furthermore, we study the approximate implementation of optimal default policies via a combination of equity-like bonds and non-defaultable bonds.

4.6.1 Welfare Comparison

We now compare the welfare gains associated with optimal default policies to a setting in which repayment of bonds is required to occur in all states. We base our welfare comparison on the model with disaster states from section 4.5 and consider a broad range of default costs. We evaluate the utility consequences in terms of welfare equivalent consumption changes over the first 500 years. Specifically, letting c_t^1 denote the optimal state contingent consumption path in the no-default economy and c_t^2 the corresponding consumption path with (costly) default, we report for each level of default costs the welfare equivalent consumption change ω solving

$$E_0\left[\sum_{t=0}^{500}\beta^t\frac{((c_t^1(1+\omega)-\bar{c}))^{1-\gamma}}{1-\gamma}\right]=E_0\left[\sum_{t=0}^{500}\beta^t\frac{(c_t^2-\bar{c})^{1-\gamma}}{1-\gamma}\right]$$

where the expectations are evaluated by averaging over 10000 sample paths. To highlight the effects of the country's initial international wealth position, we consider two scenarios, one where the initial net foreign asset position is zero and one where it equals -50% of average GDP.[19] The outcome of this procedure is reported in Figure 4.4. It shows that the welfare gains amount to 1-2% of consumption each period for a broad range of default costs. The welfare gains are surprisingly robust to the level of the default costs, instead are more sensitive to the initial net foreign asset position. Yet, for default costs $\lambda \geq 0.5$ the welfare gains from default decrease steeply. This has to do with the fact that for such high levels of the default costs it becomes suboptimal to insure against a future disaster state when the economy is already in a disaster, independently of the country's net foreign asset position. This is shown in the lower panel of Figure 4.5 which reports the optimal default policies when $\lambda = 0.7$.

With these default costs, the government receives only 0.3 units of consumption for each unit of default. Since the likelihood of a specific disaster state (either z^d or z^{dd}) to re-occur is 0.3571, the cost of using the default option for any of these states is $0.3571/(1+r) > 0.3$. Therefore, use of the default option is dominated by using the unconditional bond to transfer resources into a future disaster state. Repayment therefore optimally occurs in all future states, once the economy has hit a disaster state. As a result, the borrowing constraints tighten significantly[20] in the disaster states at this level of default costs and the required amount of insurance in the business cycle state (z^l, z^h) increases strongly as the net foreign asset position deteriorates.

[19] More precisely, we set the initial value of $(1-\lambda)a_{-1}+b_{-1}$ equal to these values and set period zero output equal to $(k^*(1))^\alpha$ in both economies and choose $z_1 = z^h$.

[20] They reach the levels applying in the economy with non-defaultable bonds.

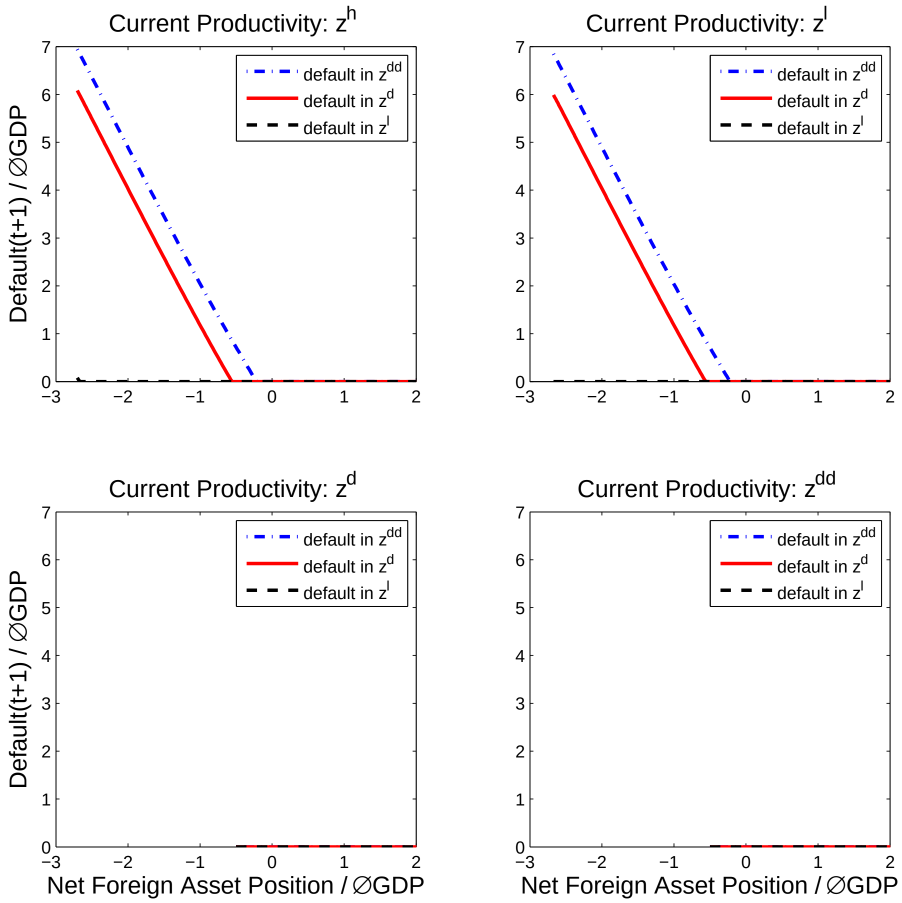


Figure 4.5: Optimal Default Policy with Disaster States and High Default Cost ($\lambda = 0.7$)

4.6.2 Approximate Implementation

We now consider a setting where the government issues two kinds of financial instruments: a simple non-contingent bond that repays in all future contingencies, as well as an equity-like bond that repays one unit of consumption in normal times (z^h, z^l), but zero when a the disaster occurs (either z^d or z^{dd}). The fact that there is only one instrument but two disaster states implies that the government bond market is still far from complete, so that it is unclear to what extent the welfare gains from outright default could approximately be captured by this simple contingent bond structure. To make the setting with a contingent bond comparable to the setting with outright default analyzed in the previous section, we

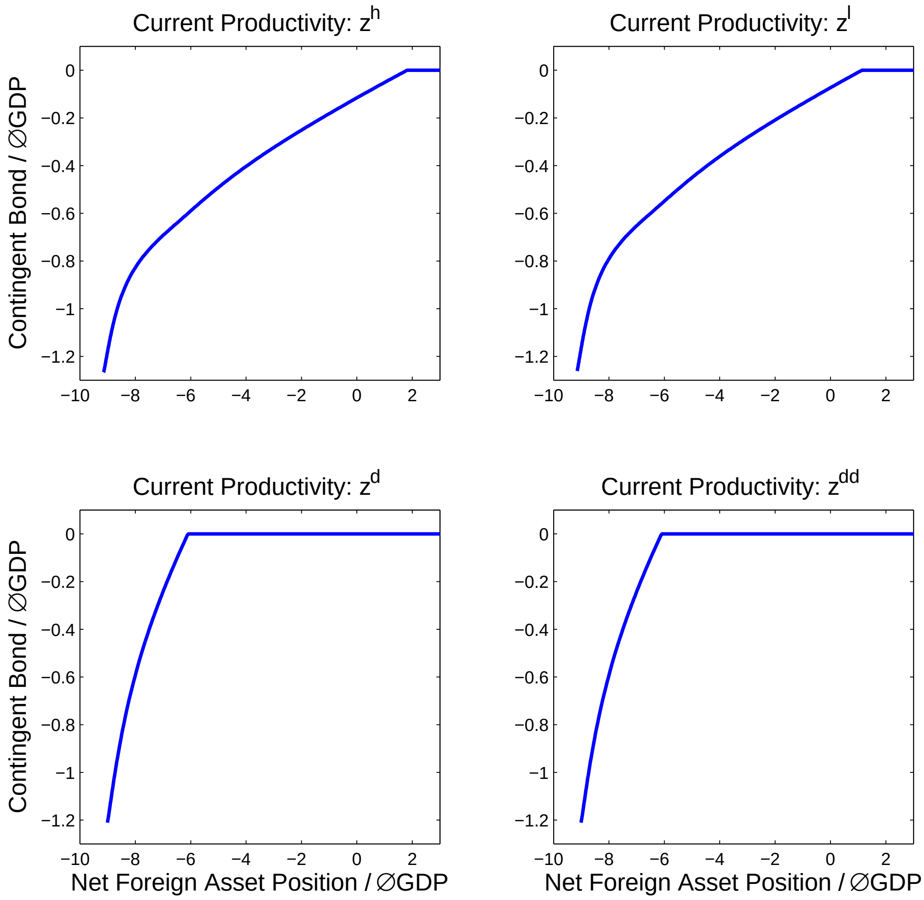


Figure 4.6: Optimal Equity Bond Issuance ($\lambda = 0.1$)

assume that the government must pay a cost λ per unit of equity bond issued in case a disaster state actually materializes. And to facilitate comparison to the results reported in section 4.5, we set $\lambda = 0.1$.

The optimal issuance of equity bonds is reported in Figure 4.6. The figure shows that the equity bond policies are approximately a convex combination of the (negative of the) default policies for states z^d and z^{dd} shown in Figure 4.2. The figure reveals that the government optimally issues the equity bond before an economic disaster actually happens and continues to issue such bonds while being in a disaster only if the net foreign asset position is sufficiently negative.

Figure 4.7 reports how well the two available financial instruments allow to capture the

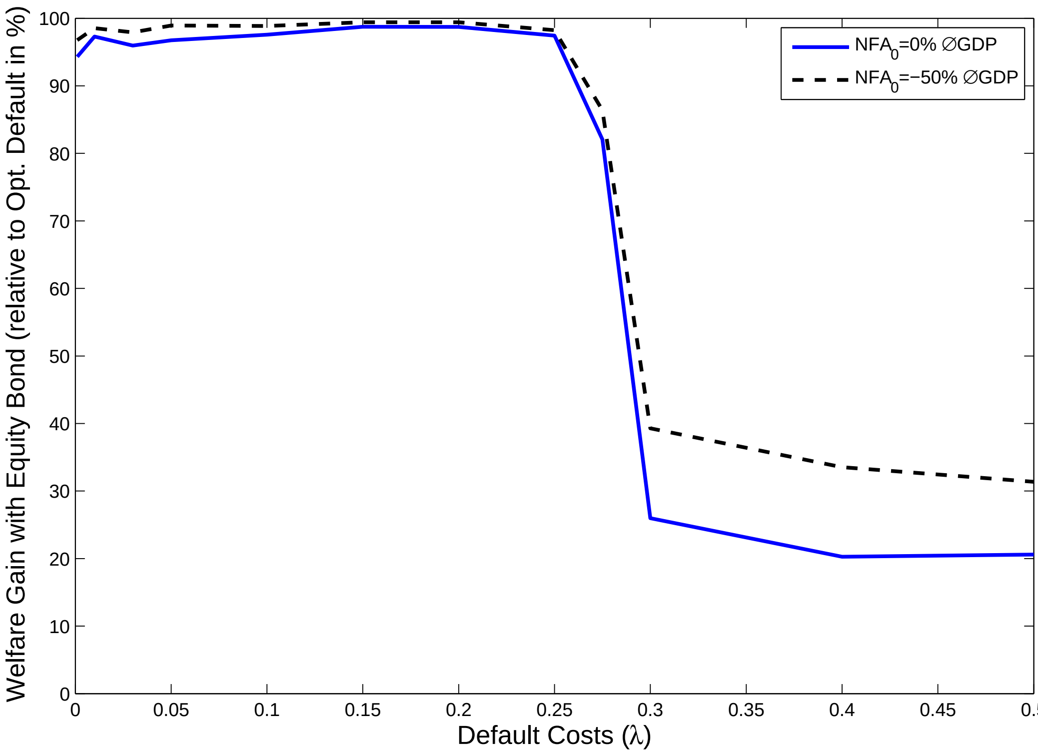


Figure 4.7: Welfare Implications of Approximate Implementation

welfare gains induced by optimal default policies, as reported in Figure 4.4. Specifically, the figure depicts the share of the welfare increase that can be realized with the considered simple asset structure. It shows that for default cost up to about $\lambda = 0.25$ there is virtually no difference between relying on optimal default policies or using the considered simple assets. This holds independently of the initial net foreign asset position. Yet, for sufficiently high levels of λ the simple asset structure cannot capture the achievable welfare gains from optimal default. Whenever $1 - \lambda$ exceeds the combined persistence of the disaster states (z^d and z^{dd}), it becomes suboptimal to issue the equity bond if the economy is already in a disaster. As discussed in section 4.6.1, it is then optimal to issue non-defaultable bonds only. This tightens the (marginally binding) borrowing limits significantly in the disaster states and decreases the opportunities for risk sharing, when compared to a setting with optimal default policies, where one can insure against disaster states individually.

4.7 Long Maturities and Optimal Bond Repurchase Programs

We now discuss the effects of introducing domestic bonds with longer maturity.[21] Long bonds can offer an advantage over one period bonds, as considered in the previous part of the paper, if the market value of long bonds reacts to domestic conditions in a way that allows the government to insure against domestic shocks. It would be desirable, for example, if the market value of outstanding long bonds decreases following a disaster shock. This allows the government to repurchase the outstanding stock of debt at a lower price, thereby realizing a capital gain that lowers the overall debt burden. Unlike in Angeletos (2002), capital gains will not materialize unless the government plans not to repay fully the long bonds in (at least some contingency) in the future. The depreciation of the market value, thus, can only be induced via the anticipation of default in the future when long bonds mature.

Issuing long bonds will offer an advantage against outright default on maturing bonds, whenever the dead-weight costs associated with repurchasing bonds at a devaluated market price is lower than the dead weight costs of an outright default on maturing bonds today. If both costs are identical, i.e., if the capital gains on long bonds resulting from default in the future induce the same costs as a default on maturing bonds, then there will be of no additional value associated with issuing long bonds. Yet, if the repurchase of long bonds at low prices fails to produce dead-weight costs, then the government could fully insure domestic consumption, i.e., achieve the first best allocation, independently of the costs associated with an outright default on maturing bonds. The optimal bond issuance strategy will then have the feature that the government issues each period long bonds that (partially) default at maturity, depending on the productivity realization tomorrow. The default at maturity needs to be calibrated such that the capital gains realized tomorrow fully insure domestic consumption against domestic productivity shocks, i.e., satisfies the proportionality restriction (4.3.10). Tomorrow, the government could then repurchase the existing stock of long bonds and issue a new long bonds with a new contingent repayment profile. In this way outright default on maturing bonds never occurs.

[21] Introducing also longer maturities for the risk-free foreign debt has no consequences for the outcomes.

4.8 Conclusions

In a setting with incomplete government bond markets, debt default is part of the optimal government policy under commitment. The choice whether or not to repay maturing debt allows for increased international risk sharing and significantly relaxes the net foreign debt positions that a country can sustain. Moreover, it considerably increases welfare, even when default costs are sizable. Default in low productivity states can be part of a country's optimal policy in a setting with full commitment, especially if the net foreign asset position is close to the level implied by the country's (marginally binding) natural borrowing limits.

Appendix

4.A First Order Equilibrium Conditions

This appendix derives the first order conditions for problem (4.2.5). We first rewrite the problem replacing beginning-of-period wealth by components (see definition (4.2.4)):

$$\begin{aligned}
\max_{\left\{b_t, a_t \geq 0, \widetilde{k}_t \geq 0, \widetilde{c}_t \geq \bar{c}\right\}} \quad & E_0 \sum_{t=0}^{\infty} \beta^t u(\widetilde{c}_t) \\
s.t. \quad \forall t : \widetilde{c}_t \quad = \quad & z_t \widetilde{k}_{t-1}^{\alpha} + b_{t-1} + (1-\lambda) a_{t-1}(z_t) \\
& - \widetilde{k}_t - \frac{1}{1+r} b_t - p_t \cdot a_t \\
z_{t+1} \widetilde{k}_t^{\alpha} + b_t + (1-\lambda) a_t(z_{t+1}) \quad \geq \quad & NBL(z_{t+1}) \quad \forall z_{t+1} \in Z \\
\widetilde{w}_0 = w_0, z_0 \quad & given,
\end{aligned}$$

Next, we formulate the Lagrangian and let η_t denote the multiplier on the budget constraint in period t, $\nu_t^{z^n}$ the multiplier for the short-selling constraint on the Arrow security that pays off in state z^n in $t+1$, and ω_{t+1} the multiplier associated with the natural borrowing limits. We drop the inequality constraints for $\widetilde{k}_t$ and $\widetilde{c}_t$, as the Inada conditions guarantee an interior solution for these variables. Differentiating the Lagrangian with respect to the choice variables one obtains

$$\begin{aligned}
\widetilde{c}_t : \quad & u'(\widetilde{c}_t) - \eta_t = 0 \\
b_t : \quad & -\eta_t \frac{1}{1+r} + \beta E_t \eta_{t+1} + \beta E_t \omega_{t+1} = 0 \\
a_t(z^n) : \quad & -\eta_t p_t(z^n) + \beta \pi(z^n | z^t) \eta_{t+1}(z^n)(1-\lambda) \\
& + \nu_t^{z^n} + \beta \pi(z^n | z^t) \omega_{t+1}(z^n)(1-\lambda) = 0 \quad \forall n \in N \\
\widetilde{k}_t : \quad & -\eta_t + \alpha \widetilde{k}_t^{\alpha-1} \beta E_t \eta_{t+1} z_{t+1} + \alpha \widetilde{k}_t^{\alpha-1} \beta E_t \omega_{t+1} z_{t+1} = 0
\end{aligned}$$

Using the FOC for consumption to replace η_t in the last three FOCs, one obtains Euler equations for the bond holdings, the Arrow securities and capital investment:

$$\begin{aligned} \textit{Bond} &: -u'(\widetilde{c}_t)\frac{1}{1+r} + \beta E_t u'(\widetilde{c}_{t+1}) + \beta E_t \omega_{t+1} = 0 & \text{(4.A.11a)} \\ \textit{Arrow} &: -u'(\widetilde{c}_t)p_t(z^n) + \beta\pi(z^n|z_t)u'(\widetilde{c}_{t+1}(z^n))(1-\lambda) \\ &\quad +\nu_t^{z^n} + \beta\pi(z^n|z^t)\omega_{t+1}(z^n)(1-\lambda) = 0 \quad \forall n \in N & \text{(4.A.11b)} \\ \textit{Capital} &: -u'(\widetilde{c}_t) + \alpha\widetilde{k}_t^{\alpha-1}\beta E_t u'(\widetilde{c}_{t+1})z_{t+1} + \alpha\widetilde{k}_t^{\alpha-1}\beta E_t\omega_{t+1}z_{t+1} = 0 & \text{(4.A.11c)} \end{aligned}$$

In addition, the Kuhn-Tucker FOCs include the following complementarity conditions:

$$\begin{aligned} 0 &\leq a_t(z^n) \ \perp \ \nu_t^{z^n} \geq 0 \ \forall n \in N & \text{(4.A.11d)} \\ 0 &\leq z^n\widetilde{k}_t^{\alpha} + b_t + (1-\lambda)a_t^{z^n} - NBL(z^n) \ \perp \ \omega_{t+1}(z^n)) \geq 0 \ \forall n \in N. & \text{(4.A.11e)} \end{aligned}$$

Combined with the budget constraint, the Euler equations and the complementarity conditions constitute the optimality conditions for problem (4.2.5).

4.B Proof of Proposition 4.1

We first show that the proposed consumption solution (4.3.8) satisfies the budget constraint, that the inequality constraints $a \geq 0$ are not binding, and that the NBLs are also not binding. Thereafter, we show that the remaining first order conditions of problem (4.2.5), as derived in appendix 4.A, also hold.

We start by showing that the portfolio implementing (4.3.8) in period $t = 1$ is consistent with the flow budget constraint and $a \geq 0$. The result for subsequent periods follows by induction. In period $t = 1$ with productivity state z^n, beginning-of-period wealth under the optimal capital investment strategy (4.3.9) is

$$\widetilde{w}_1^n \equiv z^n \left(k^*(z_0)\right)^{\alpha} + b_0 + a_0(z^n) \qquad \text{(4.B.12)}$$

To insure that consumption can stay constant from $t = 1$ onwards we need again

$$c = (1-\beta)(\Pi(z^n) + \widetilde{w}_1^n) \qquad \text{(4.B.13)}$$

for all possible productivity realizations $n = 1, ..N$. This provides N conditions that can be used to determine the $N+1$ variables b_0 and $a_0(z^n)$ for $n = 1, ..., N$. We also have the condition $a_0(z^n) \geq 0$ for all n and by choosing $\min_n a_0(z^n) = 0$, we get one more condition that allows to pin down a unique portfolio (b_0, a_0). Note that the inequality constraints on a do not bind for the portfolio choice, as we have one degree of freedom, implying that the multipliers $v_1^{z_n}$ in appendix 4.A are all zero. It remains to show that the portfolio achieving (4.B.13) is feasible given the initial wealth $\widetilde{w}_0$. Using (4.B.12) to substitute $\widetilde{w}_1^n$ in equation (4.B.13) we get

$$c = (1-\beta)(\Pi(z^n) + z^n \left(k^*(z_0)\right)^\alpha + b_0 + a_0(z^n)) \forall n = 1, ...N.$$

Combining with (4.3.8) we get

$$\Pi(z^n) + z^n \left(k^*(z_0)\right)^\alpha + b_0 + a_0(z^n) = \Pi(z_0) + \widetilde{w}_0$$

Multiplying the previous equation with $\pi(z^n|z_0)$ and summing over all n one obtains

$$E_0 \left[\Pi(z_1) + z_1 \left(k^*(z_0)\right)^\alpha\right] + b_0 + \sum_{n=1}^{N} \pi(z^n|z_0) a_0(z^n) = \Pi(z_0) + \widetilde{w}_0.$$

Using $\Pi(z_0) = -k^*(z_0) + \beta E_0 \left[z_1 \left(k^*(z_{t+j})\right)^\alpha\right] + \beta E_0 \left[\Pi(z_1)\right]$ and (4.2.3) the previous equation delivers

$$(1-\beta) E_0 \left[\Pi(z_1) + z_1 \left(k^*(z_0)\right)^\alpha\right] + b_0 + (1+r) p_0 a_0 = -k^*(z_0) + \widetilde{w}_0$$

Using $\beta = 1/(1+r)$ this can be written as

$$\begin{aligned} & (1-\beta) E_0 \left[\Pi(z_1) + z_1 \left(k^*(z_0)\right)^\alpha\right] \\ & + \frac{1-\beta}{\beta} p_0 a_0 + (1-\beta) b_0 + \frac{1}{1+r} b_0 + p_0 a_0 \quad = \quad -k^*(z_0) + \widetilde{w}_0 \end{aligned} \tag{4.B.14}$$

From (4.B.13) follows that the first terms in the previous equation are equal to

$$(1-\beta) E_0 \left[\Pi(z_1) + z_1 \left(k^*(z_0)\right)^\alpha\right] + \frac{1-\beta}{\beta} p_0 a_0 + (1-\beta) b_0 = c$$

so that (4.B.14) is just the budget equation for period zero. The portfolio giving rise to (4.B.13) in $t = 1$ thus satisfies the budget constraint of period zero. The results for $t \geq 1$ follow by induction.

It then follows from equation (4.B.13) that $\widetilde{w}_t$ is bounded, as $\Pi(z_t)$ is bounded, so that the process for beginning-of-period wealth does not involve explosive debt. The NBLs are thus not binding so that the multipliers $\omega_{t+1} = 0$ for all t and all contingencies.

Using $v_t^{z_n} \equiv 0$, $\omega_{t+1} \equiv 0$, the fact that capital investment is given by (4.3.9) and that the Arrow security price is (4.2.3), the Euler conditions (4.A.11a) - (4.A.11c) then all hold when consumption is given by (4.3.8). This completes the proof.

4.C Numerical Solution Approach

To compute recursive equilibria for Problem 4.2.5 we apply a global solution method as to account for the non-linear default policies in our model. As endogenous state variable we use beginning-of-period wealth, defined as above. Combined with exogenous productivity shocks we define our state space S to be

$$S = \left\{ z^1 \times \left[NBL(z^1), w_{max} \right], ..., z^N \times \left[NBL(z^N), w_{max} \right] \right\}$$

where we set w_{max} such that in equilibrium optimal policies never imply wealth values above this threshold. The NBLs are set such they are marginally binding. How these values are derived is shown in Appendix 4.D.

We want to describe equilibrium in terms of time-invariant policy functions that map the current state into current policies. Hence, we want to compute policies

$$\widetilde{f} : (z_t, w_t) \rightarrow (\{c_t, k_t, b_t, a_t\}),$$

where their values (approximately) satisfy the equilibrium conditions derived above. We use a time iteration algorithm where equilibrium policy functions are approximated iteratively. In a time iteration procedure, one takes tomorrow's policy (denoted by f^{next}) as given and solves for the optimal policy today (denoted by f) which in turn is used to update the guess for tomorrow's policy. Convergence is achieved once $||f - f^{next}|| < \epsilon$ and we set $\widetilde{f} = f$. In each time iteration step we solve for optimal policies on a sufficient number of grid points distributed over the continuous part of the state space. Between grid points we use linear splines to interpolate tomorrow's consumption policy. Following Garcia and Zangwill (1981), we can transform the complementarity conditions of our first order equilibrium conditions into equations. To solve for a root of the resulting non-linear equation system at a particular grid point we use Ziena's Knitro, an optimization software that can be called from Matlab. For more details on the time iteration procedure and how one transforms complementarity conditions into equations, see for example, Brumm and Grill (2010). To come up with a starting guess for the consumption policy we use the fact that at the NBLs optimal consumption equals the subsistence level. We therefore guess a convex, monotonically increasing function g which satisfies $g(z^i, NBL(z^i)) = \bar{c}\ \forall i$ and use a reasonable value for $g(z^i, w_{max})$.

4.D Natural Borrowing Limits (NBLs)

In this section we derive the NBLs that we use as lower bounds of the state space in our numerical application. For each state we define the NBL as the maximum level of indebtedness that is still consistent with non-explosive debt. To put it differently, we determine the minimum level of beginning-of-period wealth that is necessary to finance the capital stock and portfolio (assuming consumption equal to the subsistence level) such that in all possible states tomorrow beginning-of-period wealth is at or above the respective limit. To compute these bounds we use Problem 4.2.5, simplifying the exhibition substantially and yielding the same solution as for Problem 4.2.2. To derive these state dependent borrowing limits we proceed as follows: we first postulate potential solutions. Then we set up the problem that yields the minimum level of wealth today that can finance a portfolio such that in all possible states tomorrow wealth is above the postulated solution. We use this problem formulation to derive the capital and portfolio decisions requiring the lowest level of wealth today and at the same time satisfy the wealth constraints in all states tomorrow. Using these optimal choices we can then set up the linear equation system to back out the minimum levels of wealth that can just be financed. To simplify exhibition we consider just two possible TFP shocks (N=2), denoted by z^1 and z^2. However, it is straightforward to extend our analysis to the general case of N TFP shocks, as we argue below. Note that we use only one Arrow security, the one for state 2. This choice is without loss of generality as with positive default cost costs will be minimized and therefore the government will not acquire Arrow securities for the state where the least funds are needed. Without default costs, we can as well omit the Arrow security for the best productivity state as we have more assets than states available for trade. Finally note that we omit the short selling constraint on the Arrow security to simplify the exhibition[22].

Step 1:
We denote potential solution by w^1 and w^2.

Step 2:
For a given state $s \in \{1, 2\}$, we now want to determine the minimum level of wealth necessary to ensure that wealth tomorrow $(w(s)_{tom})$is above the postulated bounds:

$$\begin{aligned} \min \; w(s) & \\ s.t. \;\; w(1)_{tom} &\geq w^1 \\ w(2)_{tom} &\geq w^2 \end{aligned}$$

We can now use the budget constraint that is implied by consumption equal to the subsistence level $\bar{c}$: $\bar{c} = w(s) - k - q \cdot b - p_{z^s} \cdot a$, and use it to substitute $w(s)$in the above minimization problem:

$$\begin{aligned} \min_{k,b,a} \bar{c} + k + q \cdot b + p_{z^s} \cdot a, & \\ s.t. \qquad z^1 k^\alpha + b &\geq w^1 \\ z^2 k^\alpha + b + (1-\lambda)a &\geq w^2 \end{aligned}$$

using $w(1)_{tom} = z^1 k^\alpha + b$ and $w(2)_{tom} = z^2 k^\alpha + b + (1-\lambda)a$.
The Lagrangian for this optimization problem has the following form:

$$L = \bar{c} + k + q \cdot b + p^s \cdot a + \lambda_1 (z^1 k^\alpha + b - w^1) + \lambda_2 (z^2 k^\alpha + b + (1-\lambda)a - w^2),$$

[22]When computing the NBLs in our numerical applications, it is important to take the short-selling constraints into account as with positive default costs they may actually be binding at the NBL

We can now derive first order conditions for all states. Solving these conditions for optimal choices in state 1 (state 2 is analogous) we get

$$
\begin{aligned}
k_{1_{opt}} &= \left(\frac{1}{\alpha(z^2 p_1 - z^1(p^1 - q))}\right)^{\frac{1}{\alpha-1}}, \\
b_{1_{opt}} &= w^1 - z^1(k^1_{opt})^\alpha, \\
a_{1_{opt}} &= \frac{w^2 - z^2(k^1_{opt})^\alpha - b^1_{opt}}{(1-\lambda)}.
\end{aligned}
$$

Step 3:

We now come back to the original fixed point problem: we want to determine the minimum values of wealth necessary to ensure that we are not below these values tomorrow. By setting the optimal choices derived above (which are a function of wealth) equal to the postulated wealth levels, the original fixed point problem translates into a linear equation system that yields in general a unique solution:

$$
\begin{aligned}
w^1 &= \bar{c} + k^1_{opt} + q \cdot b^1_{opt} + p^1 \cdot a^1_{opt} \\
w^2 &= \bar{c} + k^2_{opt} + q \cdot b^2_{opt} + p^2 \cdot a^2_{opt}
\end{aligned}
$$

Plugging in the optimal capital and portfolio choices derived above, we are left with an linear equation system containing only the wealth levels and exogenous parameters. The solution of the equation system yields the NBLs that we need for our numerical applications. For the general case of N TFP shocks, the analysis is conceptually equivalent, as the structure of the Lagrangian is preserved.

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Eidesstattliche Erklärung

Hiermit erkläre ich, die vorliegende Dissertation selbständig angefertigt und mich keiner anderen als der in ihr angegebenen Hilfsmittel bedient zu haben. Insbesondere sind sämtliche Zitate aus anderen Quellen als solche gekennzeichnet und mit Quellenangaben versehen.

Mannheim, 31.08.2011

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