Intertemporal Allocation with Incomplete Markets

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Wolfgang Kuhle

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Dekan: Prof. Tom Krebs Ph.D.
Referent: Prof. Dr. Alexander Ludwig
Korreferent: Prof. Axel Börsch-Supan Ph.D.
Korreferent: Prof. David de la Croix Ph.D.
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Nataliya

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1 Introduction and Summary

Falling birth rates accompanied by increasing levels of public debt have been a common trend among OECD countries over the last five decades. In this context, the theories of optimal population and government debt, with their longstanding tradition in social sciences, are of renewed interest. The current thesis presents five neoclassical parables which emphasize particular aspects of the demographic transition and the associated role of government debt. The natural framework for such an analysis is provided by the non-ricardian overlapping generations model. The first part of this thesis is dedicated to the deterministic overlapping generations model with its consumption loan market failure and the pivotal two-part golden rule relation. The second part is concerned with stochastic OLG models where the consumption loan market failure is complemented by the missing markets for factor-price risks.

Regarding methodology, this thesis intends to favor clarity over complexity. The demographic transition and the theory of public debt are therefore treated in an eclectic manner. While the analysis throughout is conducted in general equilibrium, each chapter contains a setting which is adapted to the particular question at hand. To obtain prominent results, the number of assumptions will be kept to the bare minimum necessary to describe the respective objects of interest. The assumptions chosen tend to be neoclassical. Apart from striking results, this rudimentary approach also allows to see their limitations. In particular, results are so transparent that they can immediately be related to the assumptions upon which they rest. In turn these assumptions can, in principle, be evaluated to whether or not they are appropriate in the respective context.

This thesis studies the scope for government intervention which is associated with the characteristic market failure in overlapping generations economies. This market failure and the related concept of "dynamic (Pareto-) efficiency" will be approached from different angels. Our results from the deterministic OLG models of chapters 2 and 3 suggest that the scope for Pareto-improving government interventions is rather narrow. In particular, we find that in models with intracohort heterogeneity the concept of dynamic efficiency regarding the size of the public debt is less restrictive. Except for special cases it is no-longer possible to judge whether an economy is dynamically efficient by the classical golden rule criterion. That is, competitive growth pathes where the rate of return permanently falls short of the growth rate of the aggregate economy can no longer be characterized as inefficient. This picture changes in Chapter 4 where aggregate risks are introduced into the model. In this case there are two missing markets. Those for consumption loans and those for factor-price risks. This double incompleteness of competitive markets increases the scope for government intervention. Namely, it allows to make a restructuring of the public debt Pareto-improving. This suggests that the restructuring of the public debt may be a field where the government can take an active role without the adoption of a strong welfare criterion.

1.1 Organization

This thesis can be divided into two parts. The first one deals with the consumption loan market failure in the deterministic overlapping generations model.[1] In this setting, the two-part golden rule is of pivotal importance as it serves as the watershed between steady states that are efficient and those which are inefficient. In Chapter 2, we study the role of the golden rule in the context of the problem of optimal population growth. Interestingly, it turns out that the growth rate for population which leads the economy to a golden rule path may minimize utility. Moreover, the growth rate for population associated with a golden rule path is never optimal in an economy with government debt. Equipped with these doubts on the golden rule relation, we introduce intracohort heterogeneity in Chapter 3. In this setting we find that, except for one special case, the golden rule ceases to serve as a demarcation line between Pareto-efficient and inefficient steady states.

In the second part of the thesis we introduce aggregate risks into our framework. This gives rise to a second type of market failure. Households can trade neither consumption loans nor factor-price risks. In this setting we analyze whether or not the analytical equivalence of government bonds and pension debt known from the deterministic Diamond (1965) model carries over. While the breakdown of this

[1] The deterministic OLG model is due to Allais (1947), Malinvaud (1953), Samuelson (1958) and Diamond (1965).

strong equivalence/irrelevance result is hardly surprising, the analysis gives rise to an interesting relevance result. Evaluated from an ex-ante expected utility perspective, we show that there exists an optimal composition for the public debt. The fact that this structure can be reached in a Pareto-improving manner makes it attractive. Finally, in the last chapter we revisit the demographic transition in a stochastic overlapping generations model. In this chapter we ask a positive question. Namely, whether the risk free rate on government bonds will react more sensitive to the demographic transition than the rate of profit to risky capital.

1.2 Results

In the first chapter we analyze the role of the two-part golden rule by varying the growth rate for population as in Samuelson (1975a). However, unlike Samuelson (1975a), we discuss a competitive economy rather than a pure planning framework. Via the Serendipity Theorem, we approach the two-part golden rule relation from the side of a competitive economy.[2] The intention with the current approach is to obtain a better understanding for the paradoxical interior minima that were found by Deardorff (1976) and Michel and Pestieau (1993). The results can be summarized as:

1. The growth rate for population under which the competitive economy without government debt obtains a golden rule steady state may either maximize steady state utility or minimize it. Moreover, the growth rate for population which yields a golden rule allocation in an economy with debt is never optimal and differs from the one obtained in the planned economy. Consequently, the Serendipity Theorem does not hold in a model with debt.

2. If the growth rate for population, that yields a competitive golden rule steady state, maximizes utility when compared to the other steady state equilibria, it also maximizes the utility of all planned golden rule steady states and viceversa. That is, the necessary and sufficient conditions for an interior optimum

[2]The Serendipity theorem of Samuelson (1975a) can be stated as follows: provided that there exists only one stable steady state equilibrium, the competitive economy will automatically evolve into the most golden golden rule steady state once the optimum growth rate n^* is imposed. It was later shown that this n^* may also be a minimum. A prominent example for an interior minimum is the case where production and utility are of the Cobb-Douglas type.

are identical.

3. The optimum growth rate for population in 2. exists if and only if high (low) growth rates for population yield efficient (inefficient) steady states.

4. A lower growth rate for population increases (decreases) steady state utility if and only if the original steady state was efficient (inefficient).

5. Finally we show that the growth rate for population that maximizes steady state utility in an economy with debt implies a capital intensity that falls short of the golden rule level.

The results $1-5$ are of interest in the following sense. The pure planning framework discussed by Samuelson (1975a, 1976), Deardorff (1976), Arthur and McNicoll (1977, 1978), Michel and Pestieau (1993) and Bommier and Lee (2003) indicates that the existence of an interior optimum hinges on unobservable parameters. The current approach relates the existence of an interior optimum growth rate for population to observable variables instead. Namely, the growth rate for population and the marginal product of capital. Moreover, we find that simulations based on Cobb-Douglas production functions tend to yield watershed results. For elasticities of capital-labor-substitution smaller (larger) than one, an increase in population growth by one percent will increase the interest rate by more (less) than one percent in the long-run.

In the second chapter, we approach the two-part golden rule relation from another angle by introducing intracohort heterogeneity. In such a setting, it becomes apparent that the two-part golden rule differs substantially from the golden rule of accumulation. The golden rule of accumulation originates purely from the Solow (1956), Swan (1956) models of capital and growth and maximizes per capita consumption only. The two-part golden rule, on the contrary, is a composition of the golden rule of accumulation and the Samuelson (1958) golden rule for consumption loan interest. This composite character becomes visible once households differ regarding their preferences or their labor endowment. More specifically, we obtain the following results:

1. If agents differ with regard to their labor endowment only, the two-part golden

rule continues to maximize steady state utility if preferences are homothetic.[3] In all other cases, however, the two-part golden rule relation ceases to separate efficient from inefficient steady states. There will always exist households whose steady state utility is maximized at a capital intensity exceeding the golden rule level and vice versa. Hence, these steady states are no longer inefficient in a competitive economy.

Taking the perspective of Abel et al. (1989), an increment in capital acts as a source (sink) to society as a whole, i.e. increases aggregate consumption in each period, if $r > n$ ($r < n$). If society consisted of a representative agent, $r = n$ would therefore describe the steady state optimum. However, in a society which is fragmented into different groups this conclusion does not apply. Even if capital is already a sink to society as a whole, it may still act as a source to some groups of that society.

2. If heterogeneity is introduced on the preference side, the two-part golden rule ceases to serve as a demarcation line between efficient and inefficient steady states in general. The classic result of Stein (1969) is therefore not warranted.

3. The two-part golden rule, however, continues to serve as a watershed in the following sense: it separates agents whose present value of savings exceeds (falls short of) the amount of capital absorbed by their labor supply. Those agents with a relatively large (small) supply of savings prefer interest rates exceeding (falling short of) the golden rule level. One may therefore interpret the utility loss of thrifty households which occurs once the economy moves towards the golden rule steady state as a case of the Bhagwati (1958) result on “immiserizing growth”. While societies consumption rises, falling profits and rising wages worsen the “terms of trade” for thrifty agents.

Put differently, agents would unanimously subscribe to the golden rule optimum if they where “representative”. In this case, preferences and production are separable. The Phelps (1961) golden rule maximizes consumption and the Samuelson (1958) golden rule ensures efficient consumption patterns. Taken together, they maximize utility. In a competitive economy with heterogeneous agents, the same golden rule

[3] This condition is equivalent to the requirement that all agents have linear Engel-curves with identical slopes. That is, the propensity to save must be constant and identical for all households.

allocation is still available. However, this time it is dominated by other non-golden rule allocations. Despite their lower level of aggregate consumption. That is, the competitive mechanism brings-about intragenerational transfers which are so strong that they allow some members of society to reach a higher utility than they would reach at the golden rule.

Regarding policy, these transfers force us to think about intragenerational trade-offs. That is, changes in the size of a Bismarckian pension scheme with "intragenerational fairness" induce intragenerational transfers through their effect on factor-prices. In particular, if the propensity to save increases with income, the Bismarck pension scheme reallocates resources from the poor to the wealthy. In the case of a Beveridge scheme, these indirect transfers will thwart some of the direct redistribution. Put differently, this result complements earlier studies, e.g. Börsch-Supan and Reil-Held (2001), on the intragenerational redistribution brought about by Pay-Go pension systems. If one thinks of the propensity to safe as an observable variable the conditions which we derive from our theoretical model are accessible to empirical evaluation.

In the fourth chapter, aggregate factor-price risks are introduced into the overlapping generations model. Now there are two types of market failure as households can neither trade consumption loans nor factor-price risks privately. It is well known, that this second type of market failure introduces a second role for the government to improve upon market allocations.[4] In particular Green (1977), Krüger and Kubler (2006) and Gottardi and Kubler (2008) have compared the risk sharing benefits associated with government debt to the long-run utility losses that stem from the associated crowding-out of capital. *Starting from a situation without debt*, they show that even the introduction of a small social security scheme is not Pareto-improving, i.e. the crowding-out effect dominates the risk sharing benefits.

In Chapter 4 of this thesis we argue that these previous papers have dealt with a specific question where the consumption loan problem is mixed up with the risk sharing properties of government debt. Rather than starting from a situation without debt, we discuss an initial value problem where the government can either issue safe bonds or claims to wage indexed social security to service a given initial liability. In this setting we can separate the crowding-out effect from the risk sharing benefits

[4] See e.g. Diamond (1977), Merton (1983), Gordon and Varian (1988), Shiller (1999) and Ball and Mankiw (2007).

of fiscal policy. In a different interpretation we ask whether or not it is possible to change the composition of the public debt in a Pareto-improving manner. Tracing out this question yields four results.

1. If the government can service a given initial debt by issuing new bonds or by introducing a social security system with a linear contribution rate, there is a set of efficient debt schemes and another set of inefficient debt schemes. This set is characterized by the conflicting interests of the current young agents and the yet unborn generations regarding the allocation of factor-price risks.

2. Unlike deterministic economies, however, intertemporal compensation is possible. By varying the size and the composition of the governments debt scheme, it is possible to shift risks and resources *simultaneously* and *independently* between different generations. Consequently, the government can intermediate between the generations until only one optimal structure for the public debt is left.[5] This structure for government debt is optimal in the following sense: maintaining any other debt structure *permanently*, is (ex-ante) Pareto-inefficient.[6]

3. If society is fragmented into agents who undertake risky investments and others who do not, both of these groups require different debt schemes. If the amount of debt rolled over on the shoulders of those agents who do not undertake risky investment, is too small to transport a sufficient amount of wage income risk into the retirement period, it is Pareto-improving to inject some of the debt from the "capitalists" debt scheme into the pension schemes of "workers".

The results $1-3$ are of particular interest with regard to the current discussion concerning the reform of unfunded social security schemes. While there are many numerical studies available that quantify the effects stemming from "a transition" to a "funded" pension scheme, these studies do not start from an optimization

[5] More precisely, the government can use its two instruments, i.e. the size and the composition of the debt, to steer the economy towards a point on the contract curve.

[6] Note that this concept of Pareto-efficiency is also implicit in the golden rule result. Capital-intensities exceeding the golden rule are only inefficient if the excess capital is maintained permanently. That is, the excessive capital may never be consumed.

problem.[7] It is unclear whether or not the proposed allocations are actually efficient. Regarding this open problem, the current analysis suggests that the prospects of a Pareto-improving reshuffling of the debt are rather good. Consequently, the set of efficient rollover schemes tends to be small. Put differently, our results reconfirm that a change in the *size* of the debt *alone* requires a welfare criterion if $r > n$. There is a continuum of efficient debt sizes. However, if the government can change both, the *size* and *composition* of the debt Pareto-improvements are possible. In the current case, we obtain the strong result that there is only one Pareto-efficient composition of the public debt. While we certainly cannot take this result literally, it still indicates that the restructuring of the public debt may be a field where the government can take an active role without a strong welfare criterion.

Chapter 5 generalizes the results of Chapter 4. It analyzes how the scope of government intervention increases with the number of missing markets: if there are N missing markets and the government commands M different intertemporal budget constraints intertemporal compensation is possible iff $N, M \geqq 2$. If this condition is satisfied, the government can use its budget constraints to open "surrogate markets" for the respective goods, i.e. shift capital, consumption, natural resources and various risks between the generations. Moreover, the efficiency gains associated with the opening of markets can be recovered in a Pareto-improving manner. The resulting new efficiency conditions differ qualitatively from those obtained in a setting where $N = M = 1$ as in the classic Diamond (1965) model.

The last chapter considers whether or not there is a link between the growth rate for population and the equity premium in a stochastic version of the Diamond (1965) model. Put differently, we ask whether the demographic transition will affect the risky or the risk-free rate more severely. We develop a tractable model, that intends to complement previous studies which were based exclusively on numerical examples and yielded conflicting evidence. The present setting emphasizes the portfolio choice behavior of risk avers agents with von Neumann-Morgenstern preferences. We find that:

1. A lower birth rate lowers the overall level of interest. Both, the risky return

[7] See Campbell and Feldstein (1999) for a collection of papers with such reform proposals. See Merton (1983) for a theoretical approach that suffers from a similar difficulty. Merton (1983) does not consider wether a transition towards the steady state "optimum" is Pareto-improving. Moreover, Merton (1983) implements an incomplete markets allocation which may even be inefficient.

to capital and the safe rate earned on government bonds fall. This lower level of interest rates will be associated with a lower equity premium. That is, the risky rate will react more sensitive to changes in the growth rate of population.

2. The falling equity premium originates from an asymmetry in the portfolio adjustment behavior of the households. The portfolio share invested in the risky asset reacts more sensitive to a one percent change in the risk free rate than to a one percent change in the risky rate.

3. In a model where households hold unrealized wage-income, the level effect on the equity premium described in 1 and 2 is thwarted by a "human capital effect". While both rates of return will still fall during the demographic transition, the resulting change in the equity premium depends on the size of the implicit human capital holdings.

2 The Optimum Growth Rate for Population Reconsidered

In this chapter[8], we derive sharp conditions for the existence of an interior optimum growth rate for population in the neoclassical two-generations-overlapping model. In an economy where high (low) growth rates of population lead to a growth path which is efficient (inefficient) there always exists an interior optimum growth rate for population. In all other cases there exists no interior optimum. The Serendipity Theorem, however, does in general not hold in an economy with government debt. Moreover, the growth rate for population which leads an economy with debt to a golden rule allocation can never be optimal.

2.1 Introduction

It was Phelps (1966a) who brought up the idea that there might exist a "golden rule of procreation" in the neoclassical overlapping generations framework. In a subsequent article on "the optimum growth rate for population" Samuelson (1975a) proved − within the basic Diamond (1965) model without government debt − the so-called Serendipity Theorem: provided that there exists only one stable steady state equilibrium, the competitive economy will automatically evolve into the most golden golden rule steady state once the optimum growth rate for population n^* is imposed.

However, Deardorff (1976) pointed out that the optimum growth rate for population n^* of Samuelson (1975a) is not optimal in general. In the special case where both the utility and the production function are of the Cobb-Douglas type utility takes on a global minimum at the n^* of Samuelson.[9] Deardorff also proved that, in an economy with depreciation δ, there always exists an optimal corner solution where $n^* = -\delta$ as long as the elasticity of substitution between capital and labor remains bound above unity. This discussion has been supplemented by Michel and Pestieau (1993), who study the special case of a CES/CIES framework.

[8]This chapter is a revised version of the paper Jaeger and Kuhle (2009).

[9]Recently Abio et al. (2004) considered the problem of Samuelson (1975a) and Samuelson (1975b) in an endogenous fertility setting. They derive general sufficient conditions for the existence of an interior optimum and show that, in such a framework, there may exist an interior optimum growth rate for population even within a double Cobb-Douglas economy.

After all, the debate can be summarized as follows: granted that the respective elasticities of substitution (in consumption and more importantly production) are not "too large" there does exist an interior optimum growth rate for population $n^* > -\delta$ in the *planned* economy. The greatest deficiency in this discussion appears to be the fact that it was necessary to resort to a multitude of special cases in order to examine the significance of the Serendipity Theorem. Especially since Samuelson (1976) points out, that the respective elasticities of substitution are hard to estimate and are prone to change once the growth rate for population is altered.

With the exception of Abio et al. (2004), who discuss the Samuelson (1975a) and Samuelson (1975b) problem in a certain endogenous fertility setting, the recent literature, e.g. Golosov et al. (2007), has not taken up the Samuelson approach to the problem of optimal population. Thus the fundamental question for the exact general structure of the problem of optimal population in the basic Diamond (1965) model where population is exogenous remains, as Cigno and Luporini (2006) note, still unresolved.
The intention with this chapter is twofold: In Section 2.2, we use, contrary to the foregoing essays, a *laissez faire* framework to derive *exact* general sufficient conditions for the existence of an interior optimum growth rate for population in the Diamond (1965) model without government debt. In this framework it is our primary intention to understand why some of the solutions to the Samuelson (1975a) problem are optima while others constitute pessima. Using the concept of dynamic efficiency we will develop a typology which allows to subsume and interpret all special cases which have been discussed so far. In Section 2.3 we reconsider the validity of the results of Samuelson (1975a) in the general Diamond (1965) model with government debt.

2.1.1 Organization

In Section 2.2 we proceed along the following lines: our theoretical starting point is the planning problem of Samuelson (1975a) where an imaginary authority can set all quantities to their respective optimal level. In a second step we discuss a laissez faire framework where the imaginary authority can only vary the growth rate for population. In this competitive framework we utilize the stability condition to derive a relation between the rate of profit r and the growth rate for population n. This crucial r-n relation will then allow to draw the following conclusions:

1. The necessary *and* sufficient conditions for the existence of an interior optimum growth rate for population in a *planned* economy and in a *laissez faire* economy are identical.

2. The existence of an interior optimum growth rate for population hinges solely on the change in efficiency, which occurs in the laissez faire economy once the growth rate for population is changed (increased or decreased) from the optimal/worst level, where $n = n^* = r$. Along these lines we find that it is necessary to distinguish four cases in order to give a complete assessment of the problem of optimal population. Only one of these four cases has been analyzed by Samuelson (1975a).

3. The *exact* sufficient condition for the existence of an optimum growth rate for population is given by $\frac{dr}{dn}_{|n=n^*} > 1$.

As previously mentioned, we will then generalize the foregoing discussion in Section 2.3 by introducing government debt into the framework of analysis. In such a framework we find that:

1. The Serendipity Theorem does not hold in an economy with government debt.

2. In an economy with debt there typically still exists a growth rate $\tilde{n} \gtrless n^*$ for population which leads the laissez faire economy to a golden rule allocation. However, this growth rate will never be optimal. Instead, the optimum growth rate for population n^{**} in a laissez faire economy with government debt will lead to an allocation where $r > n$.

2.2 The Optimum Growth Rate for Population without Debt

2.2.1 The Planning Problem

The planning problem in the conventional Diamond (1965) model for given growth rates of population, can be stated as:[10]

$$\max_{c^1,c^2,k} U(c^1, c^2) \quad s.t. \quad f(k) - nk = c^1 + \frac{c^2}{(1+n)}; \quad f'(k) > 0, \quad f''(k) < 0. \quad (1)$$

[10]People live for two periods, one period of work is followed by one period of retirement. Ordinal wellbeing is described by a quasi-concave utility criterion $U(c^1, c^2)$, where c^1 and c^2 are per capita consumption in the first and second period respectively. Population grows according to: $N_t = (1+n)N_{t-1}$ and each young individual supplies one unit of labor inelastically. Capital and labor inputs, K and N, produce aggregate net output $F(K, N)$. Where $F(K, N)$ is concave and first-degree-

With the familiar first order conditions:

$$\frac{U_{c^1}}{U_{c^2}} = 1+n, \tag{2}$$

$$f'(k) = n, \tag{3}$$

$$f(k)-nk = c^1 + \frac{c^2}{(1+n)}. \tag{4}$$

Condition (2) describes the optimal distribution of income between the generations. Condition (3) describes the optimal accumulation pattern. Taken together conditions (2) and (3) constitute the two-part golden rule. Condition (4) is the social availability/budget constraint. These three conditions define (truly) optimal values c_n^1, c_n^2 and k_n for every given growth rate of population.

By varying the growth rate for population, as in Samuelson (1975a), it is now possible to choose the best among all golden rule paths, i.e. the optimum optimorum:

$$\max_n U(n) = U\Big(f(k_n) - nk_n - \frac{c_n^2}{(1+n)}, c_n^2\Big), \tag{5}$$

where $U(n)$ is the indirect utility function for the planned economy. The first order condition to this problem is:

$$-k_n + \frac{c_n^2}{(1+n)^2} = 0. \tag{6}$$

The corresponding sufficient condition for a maximum is given by:

$$\frac{d^2U}{dn^2}_{|n=n^*} = U_{c^1}\Big(-\frac{dk_n}{dn} + \frac{(1+n)^2\frac{dc_n^2}{dn} - 2(1+n)c_n^2}{(1+n)^4}\Big) < 0. \tag{7}$$

Condition (6) describes the tradeoff between the negative capital widening ($-k_n$) and the positive intergenerational transfer effect ($\frac{c_n^2}{(1+n)^2}$), and implicitly defines the optimum growth rate for population.

Together conditions (2)-(4) and (6) implicitly define optimal values c^{1*}, c^{2*}, k^*, n^* which characterize the social optimum optimorum.[11] However, as previously noted, the first order condition (6) might locate the growth rate for population where the indirect utility function $U(n)$ takes on a global minimum rather than a maximum, i.e. we might actually have $\frac{d^2U}{dn^2}_{|n=n^*} > 0$.

homogenous. Per capita output is thus $f(k) := \frac{F(K,N)}{N}$ with $k := \frac{K}{N}$. The real wage w payed for one unit of labor is defined as $w := f(k) - f'(k)k$. The rental rate r for one unit of capital is defined as $r := f'(k)$. Output can either be consumed by the young generation, the old generation or invested; the resource constraint for the economy is thus given by: $F(K_t, N_t) + K_t = K_{t+1} + c_t^1 N_t + c_t^2 N_{t-1}$. In the following we compare different steady state equilibria only; hence, the time index will be omitted where no misunderstanding is expected.

[11] Deardorff (1976), Samuelson (1976) and especially Michel and Pestieau (1993) show that unique

2.2.2 The Serendipity Theorem

The representative individual is driven by the following maximization problem:

$$\max_{c^1,c^2} U(c^1,c^2) \quad s.t. \quad w = c^1 + \frac{c^2}{(1+r)}; \quad w = f(k) - f'(k)k, \quad r = f'(k). \quad (8)$$

With the corresponding first order conditions:

$$\frac{U_{c^1}}{U_{c^2}} = 1 + r, \quad (9)$$

$$f(k) - rk = c^1 + \frac{c^2}{(1+r)}. \quad (10)$$

Once we set $k = k^*$ and $n = n^*$ so that conditions (3) and (6) hold, we find that the individual behavior, which is described by conditions (9) and (10), is compatible with the remaining conditions (2) and (4) for the social optimum. Since condition (6), with $r = n^*$, is identical with the steady state life-cycle savings condition, we find that the values c^{1*}, c^{2*}, k^*, n^* describe a feasible laissez faire steady state equilibrium. This is the Serendipity Theorem of Samuelson (1975a): provided that there exists only one stable steady state equilibrium, the competitive economy will automatically evolve into the most golden golden rule steady state once the optimum growth rate n^* is imposed.

2.2.3 The Optimum Growth Rate for Population in a Laissez Faire Economy

In order to analyze the welfare implications of changes in the growth rate for population in the laissez faire economy, we assume that consumption in each period is a normal good, and use the life-cycle savings condition which is given by:

$$(1+n)k_{t+1} = s(w_t, r_{t+1}); \quad 0 < s_w < 1. \quad (11)$$

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interior solutions to the first order conditions (2)-(4) and (6) exist for a wide range of parameters (Michel and Pestieau (1993) report only one special instance of multiple solutions). From now on we will assume that there exists one unique interior solution in order to focus on the important question why some of these solutions constitute minima rather than maxima. In other words we are trying to find the unifying economic characteristics of those cases for which we have a planned minimum (maximum). We will also show (Proposition 1) that the results on the existence of interior solutions for the planning framework of Michel and Pestieau (1993) remain fully valid for a laissez faire economy.

Furthermore, we assume the existence of one unique and stable steady state equilibrium with a capital intensity $k = \tilde{k} > 0$:[12]

$$0 < \frac{dk_{t+1}}{dk_t} = \frac{-s_w \tilde{k} f''(\tilde{k})}{(1+n) - s_r f''(\tilde{k})} < 1. \tag{12}$$

Differentiation of (11) allows, by virtue of (12), to derive that an increase in the growth rate for population decreases the steady state capital intensity:

$$\frac{dk}{dn} = \frac{-k}{(1+n) - s_r f''(k) + s_w k f''(k)} < 0. \tag{13}$$

From the life-cycle savings condition (11), the respective factor-prices, and the individual budget constraint, one obtains the following maximization problem for the laissez faire economy:

$$\max_n U(n) = U\Big(f(k) - f'(k)k - (1+n)k, (1+f'(k))(1+n)k\Big); \quad k = k(n). \tag{14}$$

Condition (9), which is always satisfied in a laissez faire economy, allows to rewrite the first order condition for the optimum growth rate for population so that we have:

$$\frac{dU}{dn} = U_{c^1}\left[\frac{n - f'(k)}{1+f'(k)} f''(k)k\right]\frac{dk}{dn} = 0. \tag{15}$$

According to the Serendipity Theorem, condition (15) holds only for $n = n^*$. The sufficient condition for an optimum at n^* is given by:

$$\frac{d^2U}{dn^2}_{|n=n^*} = U_{c^1}\left[\frac{(1 - f''(k)\frac{dk}{dn})}{(1+f'(k))} f''(k)k\right]\frac{dk}{dn} < 0. \tag{16}$$

Condition (16) reveals that the existence of an optimum or a minimum or an inflection point at n^* hinges solely on:

$$\frac{dr}{dn}_{|n=n^*} = f''(k)\frac{dk}{dn} = \frac{-k}{\frac{1}{f''}(1+n) + s_w k - s_r} \gtreqqless 1. \tag{17}$$

However, a priori we can only say that $\frac{dr}{dn} > 0$, if the steady state equilibrium is stable. Hence it is necessary to distinguish four cases:

[12]As in Diamond (1965), we relegate the case of oscillatory stability to Appendix 2.5.3, where we show that for $-1 < \frac{dk_{t+1}}{dk_t} < 0$, we have $\frac{dk}{dn} > 0$. In such an economy, we have $\frac{d^2U}{dn^2} < 0$, i.e. the sufficient condition for an optimum is always satisfied.

1. The economy is growing on a dynamically inefficient (efficient) steady state path where $r < n$ ($r > n$) for low (high) growth rates of population $n < n^*$ ($n > n^*$). In this case we have $\frac{dr}{dn}_{|n=n^*} > 1$, and the sufficient condition for an interior maximum is satisfied.

2. The economy is growing on an efficient (inefficient) steady state path for low (high) growth rates $n < n^*$ ($n > n^*$). In this case we have $\frac{dr}{dn}_{|n=n^*} < 1$, that is, an interior minimum.

3. The economy is growing on an inefficient path for all $n \neq n^*$. In this case we have $\frac{dr}{dn}_{|n=n^*} = 1$ and population should grow as fast as possible. There is an inflection point in the $U(n)$ curve at $n = n^*$.

4. All steady states are efficient and the lowest possible growth rate for population is best. We have, once again, an inflection point in the $U(n)$ curve at $n = n^*$ and $\frac{dr}{dn}_{|n=n^*} = 1$. Similar to Case 3 this is a second special case.

With respect to Case 3 and Case 4 we can note that these cases have not been explored so far. However, as the condition $\frac{dr}{dn}_{|n=n^*} = 1$ indicates and Quadrant II in Diagram 1 illustrates, they seem to be rather special, and in our opinion they are most likely of no relevance.

After these preparations it is now possible to give a complete diagrammatic representation of the problem of optimal population in Diagram 1 (the formal aspects to Diagram 1 are given in Appendix 2.5.1):

At this point we can note that the factor-prices which are associated with the two-part golden rule allocation $-$ for all given growth rates $n \neq n^*$ $-$ allow in general to reach a higher indifference curve in Quadrant III than the set of factor-prices which is generated in the laissez faire framework.

More interesting, however, is a related point which can be found in Quadrant III of Diagram 1: the conditions for the existence of an interior optimum growth rate n^* in a planned economy, where the central authority forces $r = n$ as in Samuelson (1975b) are identical with those in a laissez faire economy: in both cases it is necessary that the indifference curve in the w, r plane is a tangent to the factor-price frontier, i.e. $\frac{dw}{dr}_{|dU=0} = \phi'(r)$, and it is sufficient that the curvature of the indifference curve is algebraically larger than the curvature of the factor-price frontier, i.e. $\frac{d^2w}{dr^2}_{|dU=0} > \phi''(r)$.

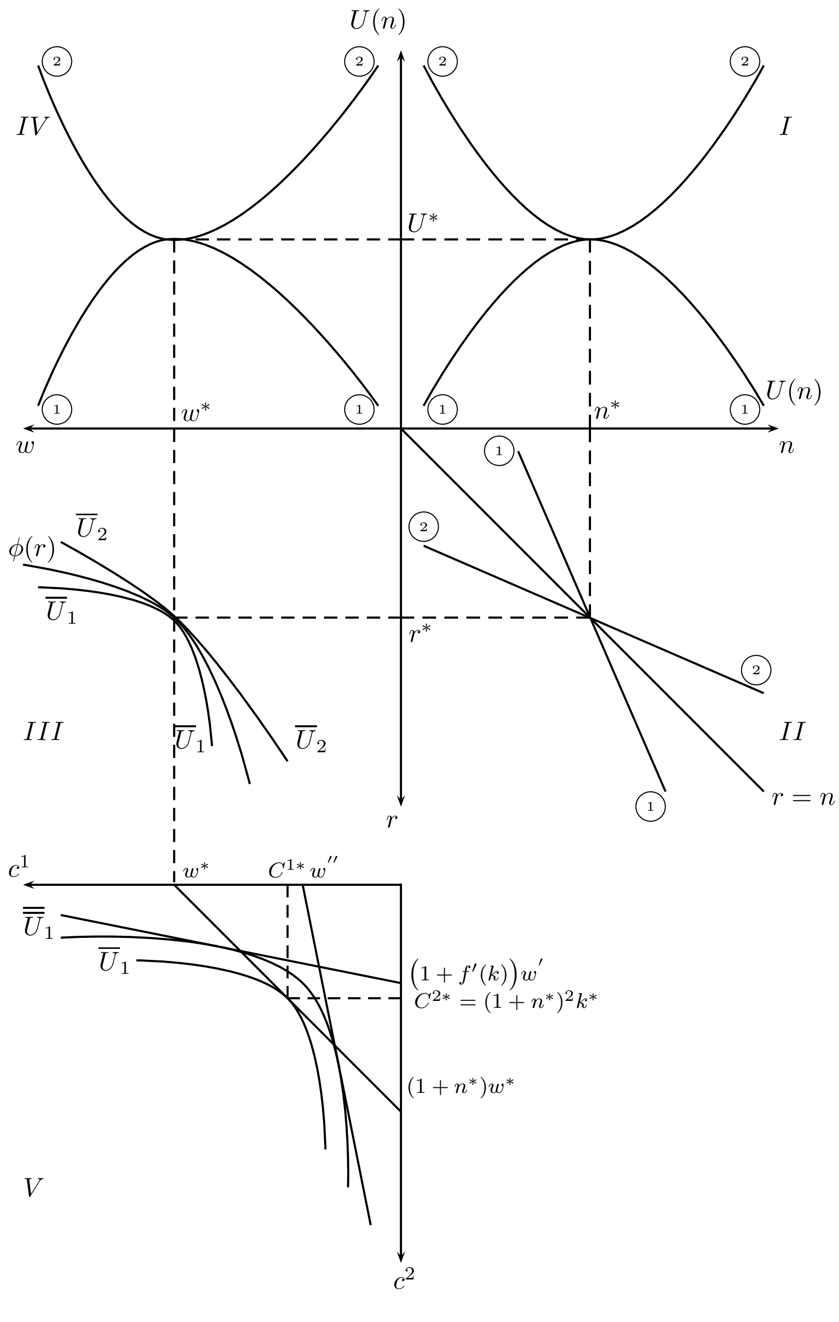


Diagram 1: *Population growth and welfare without debt.*
Quadrant I is the familiar U, n diagram which contains the respective utility contours for the laissez faire economy. Quadrant II is the decisive n, r diagram where all planned equilibria are located along the 45° line. The locus of the laissez faire steady state curve with $\frac{dr}{dn} = f''(k)\frac{dk}{dn} > 0$ is ambiguous and four cases have to be distinguished: Case 1: 1-1, Case 2: 2-2, Case 3: 1-2, Case 4: 2-1. Quadrant III is a w, r diagram which contains the convex factor-price frontier ϕ and the respective indifference curves indicating an optimum (pessimum). Quadrant IV gives the wage utility relation. Quadrant V illustrates the respective individual consumption patterns for different growth rates (Case 1 only).

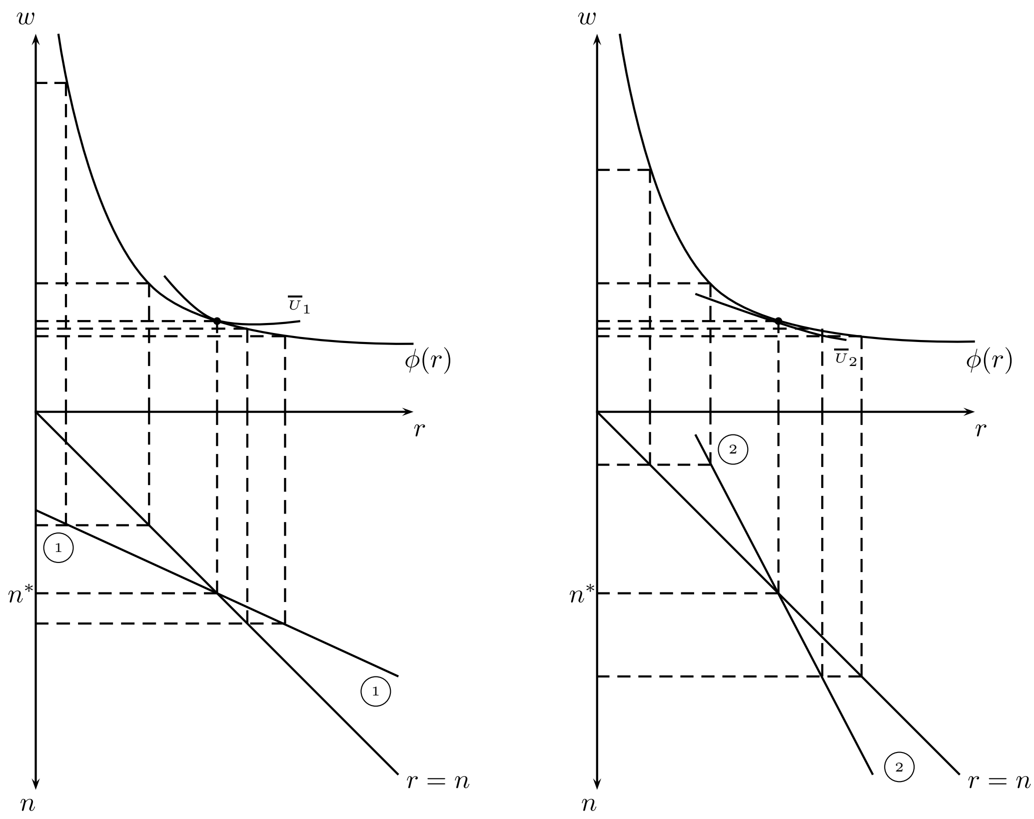


Diagram 2: *The factor-price frontier as a surrogate budget constraint.*

This means that regardless of whether we are in a planned economy or a laissez faire economy: choosing the growth rate for population means choosing a set of factor-prices *on the same* factor-price frontier. The convex factor-price frontier, which in general defines a concave set of feasible allocations, should be interpreted as a surrogate social budget constraint as illustrated in Diagram 2.

Proposition 1. *(Extended Serendipity Theorem): The necessary and sufficient conditions for the existence of an interior optimum growth rate for population in a planned and in a laissez faire economy are identical. The exact general sufficient condition for an interior optimum growth rate for population is given by* $\frac{dr}{dn}_{|n=n^*} > 1$. *In all other cases where the structure of the economy is such that we have* $\frac{dr}{dn}_{|n=n^*} \leqq 1$ *in the laissez faire framework, no interior optimum exists.*

Proof. See Appendix 2.5.2. □

Corollary 1. *The qualitative findings of Michel and Pestieau (1993) on the exis-*

tence of an interior optimum growth rate for population in the CES/CIES planning framework remain fully valid for a laissez faire economy.

Hence, all specifications, most notably the Cobb-Douglas case, where there is an interior planned minimum are consistent with Case 2 and the counterintuitive change in efficiency at $n = n^*$. In our opinion it is this counterintuitive behavior of economies with high elasticities of substitution that should be criticized and not the behavior in the two "corners" where $k \to \infty$ or $n \to \infty$ as in Samuelson (1976).[13]

We can now conclude that the reasoning of Samuelson (1975a) and Samuelson (1975b) only remains valid as long as the economy behaves according to Case 1. However, the assertion of Samuelson (1975a), (p. 535) and Samuelson (1975b), (p. 542) that all economies behave according to Case 1 $-$ which was never questioned by Deardorff (1976) or Michel and Pestieau (1993) $-$ is wrong.

However, Case 1 is obviously the most plausible scenario. Using the data in Marquetti (2004) for the years 1963-2000, Kuhle (2007) shows that real world economies tend to behave according to Case 1. Estimates of the r-n relation for Japan, the USA and a group of 17 mostly developed countries allow to refute the null hypothesis $\frac{dr}{dn} < 1$ with a probability of error (t-test) of less than 2.5 percent.

2.3 The Optimum Growth Rate for Population in an Economy with Government Debt

We will now proceed along the following lines: in a first step the Diamond (1965) model with internal government debt and the corresponding government budget constraint will be restated. In a second step we will show that the Serendipity Theorem is in general not valid in an economy with government debt. The third step is to derive the welfare implications which stem from a change in the growth rate of population in a laissez faire economy where the government runs a constant per capita debt policy.

[13] At this point we shall note that Phelps (1968) shows for a laissez faire economy that the Cobb-Douglas case is consistent with what we have called Case 2, i.e. an interior minimum at $n = n^*$. Hence, in the light of the Serendipity Theorem, it should have been no surprise to Deardorff (1976) and Samuelson (1975a) that the "most golden golden rule steady state" must be a minimum in that case.

2.3.1 The Model

The Diamond (1965) model with debt differs from the one which was discussed in the foregoing section only with respect to the government budget constraint and the steady state life-cycle savings condition. Government debt has a one-period maturity and yields the same interest as real capital and there is no risk of default. In each period the government has to service the matured debt B_{t-1} and it has to pay interest amounting to $f'(k_t)B_{t-1}$. The government can use two tools to meet these obligations: it can raise a lump-sum tax $N_t\tau_t^1$ from the young generation, or it can issue new debt B_t. Hence we have:

$$B_t + N_t\tau_t^1 = (1 + f'(k_t))B_{t-1}. \tag{18}$$

In the following the government will simply pursue a constant per capita debt policy defined by:[14]

$$\frac{B_{t-1}}{N_t} = b \quad \forall t. \tag{19}$$

Thus (18) simplifies to:

$$\tau^1 = \Big[(1 + f'(k_t)) - (1 + n)\Big]b = (f'(k_t) - n)b = \tau^1(k_t). \tag{20}$$

Equation (20) reveals that taxes can be either positive or negative depending on $b \gtrless 0$ and the sign of $(f'(k) - n)$, i.e. on whether the economy is growing on an efficient or inefficient path.

2.3.2 The Serendipity Theorem with Debt

From the perspective of the social planner the problem remains unaltered: the relevant tradeoff is still between capital widening and the intergenerational transfer effect, and conditions (2)-(4) and (6) still describe the social optimum.

The Competitive Economy with Government Debt The individual utility maximization problem is given by:

$$\max_{c^1,c^2} U(c_t^1, c_{t+1}^2) \quad s.t. \quad w(k_t) - \tau_t^1(k_t) = c_t^1 + s_t; \quad c_{t+1}^2 = (1 + f'(k_{t+1}))s_t. \tag{21}$$

[14] Persson and Tabellini (2000) argue why an elected government might rather run such a debt policy than use its budget constraint to steer the economy towards the long run optimum as discussed in De La Croix and Michel (2002).

Thus the representative individual behaves according to:

$$\frac{U_{c^1}}{U_{c^2}} = 1 + f'(k_{t+1}), \tag{22}$$

$$s_t = w(k_t) - \tau_t^1(k_t) - c_t^1, \tag{23}$$

$$c_{t+1}^2 = (1 + f'(k_{t+1}))s_t. \tag{24}$$

Attainability of the Social Optimum In a steady state equilibrium the life-cycle savings condition is given by:

$$s(\tilde{w}(k), f'(k)) = (1+n)(b+k); \quad s > 0; \quad \tilde{w}(k) := w(k) - \tau^1(k), \tag{25}$$

where $s > 0$ is an obvious restriction since negative savings would lead to negative old age consumption. We will now examine whether the social optimum $(c^{1*}, c^{2*}, n^*, k^*)$, which is characterized by (2)-(4) and (6), is a feasible laissez faire steady state equilibrium: once we set $k = k^*$ and $n = n^*$, conditions (3) and (6) hold. According to (20) we have $\tau^1(k^*) = 0$ and the individual budget constraint becomes the same as the availability constraint. In this case the individual will voluntarily choose c^{1*} and c^{2*}. Finally we have to check the steady state life-cycle savings condition:

$$s^* = (1+n^*)k^* = \frac{c^{2*}}{(1+n^*)} \neq (1+n^*)(k^*+b); \quad \forall b \neq 0. \tag{26}$$

This means that since internal debt leads to the substitution of capital with debt (paper) in the portfolio of the representative individual, the Serendipity Theorem does not hold. Thus the only way to decentralize the social optimum is to reduce per capita debt to zero.

2.3.3 The Optimum Growth Rate for Population in a Laissez Faire Economy with Debt

Comparison of the social optimum and the individual behavior revealed that the Serendipity Theorem does not hold in the Diamond model with internally held debt. We will now assess the question of optimal population in a competitive economy. Two related points will be discussed:

1. A change in the constant debt policy for a given growth rate for population.

2. A change in the growth rate for population for a given debt policy.

Temporary Equilibrium As De La Croix and Michel (2002) point out, there are several conditions which have to be met in each period to allow for a meaningful temporary equilibrium:

$$\begin{aligned} s_{t-1} &> 0, & (27) \\ \tilde{w}(k_t, b) = w(k_t) - \tau^1(k_t) &= w(k_t) - b(f'(k_t) - n) > 0, & (28) \\ s(\tilde{w}(k_t, b), f'(k_{t+1})) &= (1+n)(k_{t+1} + b) > (1+n)b. & (29) \end{aligned}$$

While (27) ensures positive consumption of the old generation, $\tilde{w}$ in (28) describes that the income after taxes of the current young individuals must be positive. Condition (29) must hold to allow for a positive capital intensity.

Steady State Equilibrium In order to carry out the following comparative static (in per capita terms) analysis, it is necessary to determine the signs of $\frac{dk}{dn}$ and $\frac{dk}{db}$. As in Diamond (1965), we will assume that there exists a unique stable steady state at $k = \tilde{k}$:

$$0 < \frac{dk_{t+1}}{dk_t} = \frac{-s_{\tilde{w}}(\tilde{k} + b) f''(\tilde{k})}{(1+n) - s_r f''(\tilde{k})} < 1; \quad 0 < s_{\tilde{w}} < 1; \quad \tilde{k} > 0. \tag{30}$$

Total differentiation of the life-cycle savings condition (25) with $db = 0$ leads to:

$$\frac{dk}{dn}_{|db=0} = \frac{k + (1 - s_{\tilde{w}})b}{s_r f'' - (1+n) - s_{\tilde{w}}(k+b) f''} < 0. \tag{31}$$

The sign in the denominator of the expression (31) is negative by virtue of the stability condition (30). The assumption of normality ($0 < s_{\tilde{w}} < 1$) and conditions (27) and (29) reveal that the sign of the numerator is positive. Total differentiation of (25) with $dn = 0$ yields:

$$\frac{dk}{db}_{|dn=0} = \frac{(1+n) + s_{\tilde{w}}(f' - n)}{s_r f'' - (1+n) - s_{\tilde{w}} f''(k+b)} < 0. \tag{32}$$

With $0 < s_{\tilde{w}} < 1$, the sign in the numerator of (32) must be positive. The sign of the denominator is negative according to (30).

Once the signs of $\frac{dk}{dn}$ and $\frac{dk}{db}$ are known to be negative, the key elements to our question can be displayed in Diagram 3.

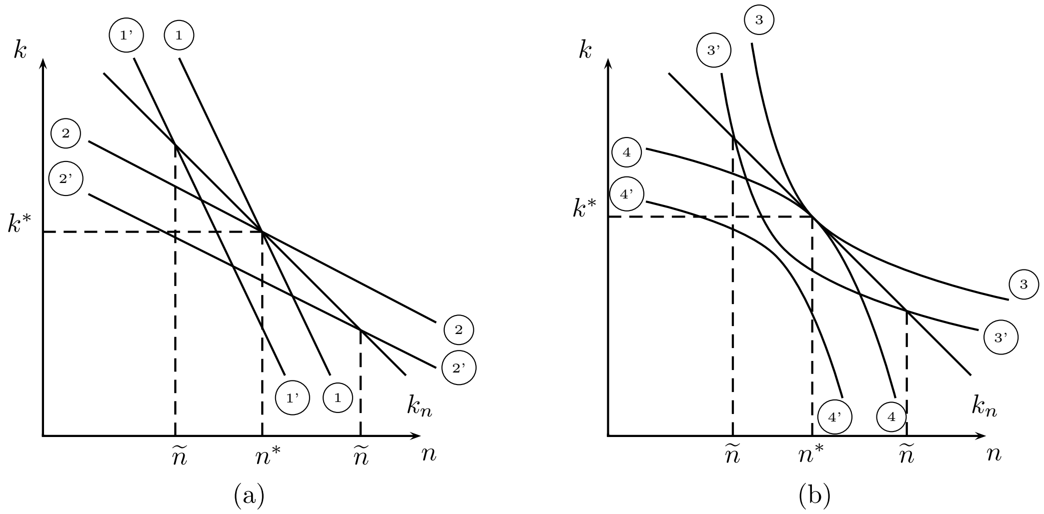


Diagram 3: *The golden rule and government debt.*
The k_n line gives the respective golden rule capital intensities and separates the efficient from the inefficient equilibria. For the laissez faire steady state curves, it is once again necessary to distinguish Cases 1, 2, 3 and 4. Once the government issues debt (the debt loci have an apostrophe) these loci shift according to $\frac{dk}{db} < 0$ and the growth rate of population which leads to a golden rule allocation changes from n^* to $\tilde{n}$. The Serendipity Theorem does not hold in this case.

Debt and Welfare After these preparations, the Diamond (1965) result concerning the welfare implications of a change in the constant per capita internal debt policy can be reproduced: from the life-cycle savings condition (25) and the respective factor-prices one obtains the following indirect utility function:

$$U = U\Big(f(k) - kf'(k) - (1+n)(k+b) - \tau^1(k), (1+f')(1+n)(k+b)\Big). \tag{33}$$

Using (20) allows to rewrite (33) as:

$$U = U\Big(f(k) - kf'(k) - (1+n)k - (1+f'(k))b, (1+f'(k))(1+n)(k+b)\Big). \tag{34}$$

The first order condition for the optimum debt policy is given by:

$$\frac{dU}{db} = U_{c^1}(n - f')\Big(1 + \frac{(k+b)}{(1+f')}f''\frac{dk}{db}\Big) \gtreqless 0. \tag{35}$$

Equation (35) reveals that the sign of $\frac{dU}{dn}$ depends solely on the sign of $(n - f')$. Hence an increase of per capita debt increases (decreases) per capita utility if the economy is experiencing over-accumulation (under-accumulation) in the steady state equilibrium. Thus, debt should be issued (recovered) up to the point where golden rule growth is attained.

Population Growth and Welfare The same indirect utility function (34) can now be used to derive the welfare implications which originate from changes in the growth rate for population. Hence, the first derivative with respect to the growth rate of population is:

$$\begin{aligned} \frac{dU}{dn} &= -U_{c^1}\Big([(k+b)f'' + (1+n)]\frac{dk}{dn} + k\Big) \\ &+ U_{c^2}\Big([(1+n)(1+f') + f''(1+n)(b+k)]\frac{dk}{dn} + (1+f')(k+b)\Big). \end{aligned} \quad (36)$$

Using (22), we obtain:

$$\frac{dU}{dn} = U_{c^1}b + U_{c^1}\frac{(n-f')(k+b)}{1+f'}f''\frac{dk}{dn} \gtreqqless 0; \quad \frac{dk}{dn} < 0. \quad (37)$$

The first order derivative (37) contains two elements: the first element $U_{c^1}b > 0$ (for $b > 0$) is the biological interest rate effect, which suggests that population should grow as fast as possible. The reason for the appearance of the biological interest argument is the following: each young individual buys government debt amounting to $(1+n)b$ and pays taxes $(f'(k) - n)b$. Hence the young individual hands over a total amount of $(1 + f'(k))b$ to the government. In the retirement period the government serves its obligations and pays $(1 + f'(k))(1 + n)b$.

Thus the individual receives the biological rate of interest $(1 + n)$ on its total payments. This also reveals that the total amount of resources which is transferred into the retirement period, at the biological rate of interest, depends on the rate of interest $(1 + f'(k))$ and hence, via the capital intensity, on the growth rate of population.

The second element $U_{c^1}\frac{(n-f')(k+b)}{1+f'}f''\frac{dk}{dn}$ describes the factor-price effects which originate from a change in the growth rate of population. An increase in n leads to a fall in k, which increases the interest rate payed on capital and debt, and decreases wages.

In the special case $b = 0$, (37) degenerates into (15) where $\frac{dU}{dn} = 0$ for $n = n^*$, and at n^* the pair of factor-prices $w(k(n^*)), r(k(n^*))$ ensure maximum (minimum) lifetime utility. The tradeoff is solely between wages and interest.

In the case $b \neq 0$ the situation differs fundamentally: as (37) indicates, the tradeoff is now between what we will call the aggregate factor-price effects and the biological interest rate. The growth rate which maximizes (minimizes) laissez faire utility in an economy with government debt will be referred to as n^{**}. We can note that n^{**} is larger (for Case 1, $b > 0$) than the growth rate $\tilde{n}$ which causes a golden rule allocation, and it may or may not be larger than n^*. The conditions which have to be met to allow for a laissez faire optimum at n^{**} remain, compared to the case without debt, basically unaltered with $\frac{dr}{dn} > 1$; the only additional condition is that the difference $(n - f'(k(n)))$ must increase sufficiently to allow for an interior optimum at n^{**}.

Optimal Population vs. Optimal Debt Conditions (35), and (37) indicate that there is no symmetry in the respective optimal debt and population policies with respect to the golden rule allocation. This gives rise to the following Proposition:

Proposition 2. *In a laissez faire economy with constant per capita government debt, the growth rate of population, which leads to a golden rule allocation, can never be optimal.*

Corollary 2. *If the government pursues an optimal debt policy according to condition (35), the growth rate for population cannot be optimal simultaneously.*

Corollary 3. *: If the government imposes an optimum growth rate for population according to (37), the debt policy cannot be optimal simultaneously.*

Only by setting the per capita level of debt to zero and the growth rate for population to n^*, the two optimality conditions (35) and (37) can be satisfied simultaneously:

Proposition 3. *If the social planner can choose both: the optimum growth rate for population and the optimal amount of debt, the only optimal debt policy is zero debt.*

Illustration Using Case 1 with $b > 0$ as an example (the reader can experiment with (35) and (37), which allow to evaluate the remaining three cases; Cases 2 and 3 may contain multiple solutions), the foregoing discussion concerning the optimum growth rate of population in an economy with government debt can be summarized in Diagram 4.[15]

Diagram 4 illustrates that the optimum growth rate for population n^{**} is larger than $\tilde{n}$. Compared to the case without debt, the preference ordering in the w, r quadrant is changed since the interest rate is not only determining the relative price

[15]In Appendix 2.5.4 we develop the corresponding slope of the households indifference curves displayed in Diagram 4. In Appendix 2.5.5, we show that the quality of our results remains unaltered in a model with pay-go social security.

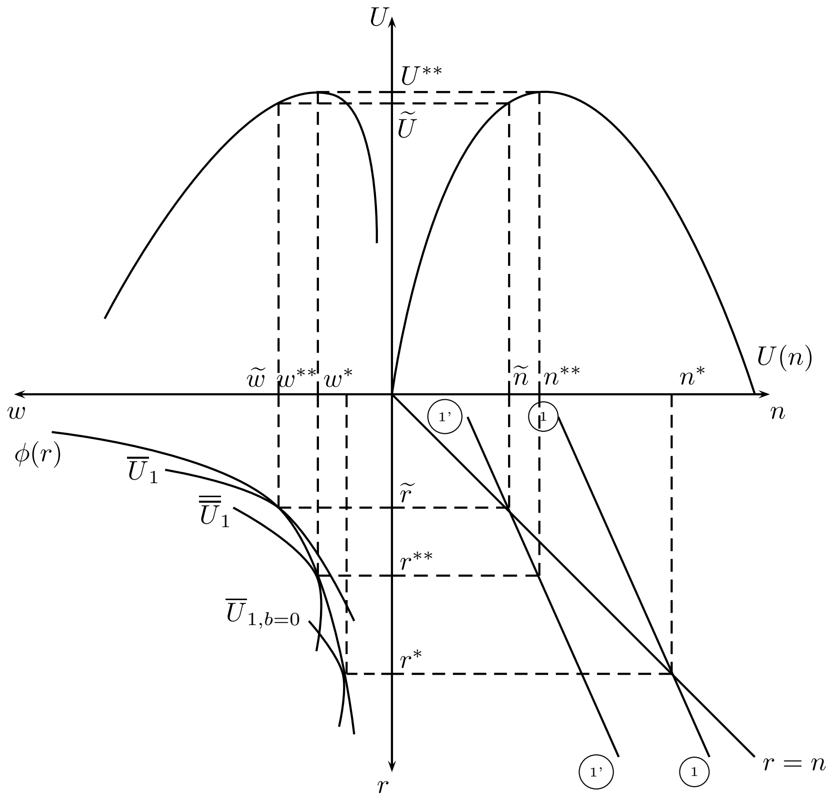


Diagram 4: *The optimum growth rate for population in a laissez faire economy. Case 1, with positive government debt.*

of future consumption; it also determines the total amount of resources which go through the hands of the government and yield the biological interest rate. Thus the indifference curves with debt $\bar{U}_1$ and $\bar{\bar{U}}_1$ may intersect the indifference curve $\bar{U}_{1,b=0}$. At the optimum growth rate for population n^{**}, which might be larger, smaller or equal to the optimal n^* of Samuelson (1975a), we have $(n^{**} - f'(k(n^{**}))) < 0$. Hence, according to (35), the government can always improve steady state utility through a reduction of per capita debt. The (social) optimum optimorum would once again be reached at n^* with $b = 0$ (the $U(n)$ curve for the planned economy is not included in Diagram 4).

2.4 Concluding Remarks

In the first section we discussed the problem of the optimum growth rate for population in a laissez faire economy. In the course of this discussion we developed a general typology for the problem of optimal population in the Diamond (1965) model without government debt. This led to the conclusion that:

1. The qualitative necessary and sufficient conditions for the existence of an interior optimum growth rate for population in a planned and in a laissez faire economy are identical. In both cases it is the convex factor-price frontier which can be interpreted as the social budget constraint. Hence we have shown that the findings of Michel and Pestieau (1993) for the planned economy remain also valid in the more realistic case of a laissez faire framework.

2. There always exists an interior optimum in an economy where low (high) growth rates for population lead to over-accumulation (under-accumulation). The general sufficient condition for an interior optimum in a laissez faire as well as in a planned economy is hence given by $\frac{dr}{dn}_{|n=n^*} > 1$. All cases where there exists an interior minimum, like the Cobb-Douglas case, are consistent with an economy, in which rapid population growth leads to over-accumulation and low or negative growth rates for population lead to under-accumulation.

3. An increase in the growth rate for population increases (decreases) steady state welfare only if the economy is growing on an inefficient (efficient) steady state path.

In a second step we generalized the discussion by introducing government debt. In such a framework we find that:

1. Due to the substitution between debt and capital in the portfolios of the representative individuals, the Serendipity Theorem does not hold anymore. However, except for the case of permanent efficiency there still exists at least one growth rate for population $\tilde{n}$, which leads the laissez faire economy to (two-part) golden rule growth.

2. In a laissez faire economy with constant per capita debt, the growth rate for population $\tilde{n}$, which leads to a golden rule allocation, cannot be optimal since it only balances the wage-interest tradeoff. The optimum growth rate for population balances the tradeoff between factor-prices and the internal rate of return of the pension/debt scheme. Such an optimum growth rate leads the competitive economy to an allocation where the marginal productivity of capital exceeds the optimum growth rate for population.

2.5 Appendix

2.5.1 Construction of Diagram 1

In this appendix we substantiate our claim that the qualitative conditions for an interior optimum are properly represented in Quadrant III of Diagram 1. Hence we have to show that the necessary condition for an optimum at n^* requires that the indifference curve in the w, r plane is a tangent to the factor-price frontier, i.e. $\frac{dw}{dr}_{|dU=0} = \phi'(r)$. Analogous we show that the sufficient condition is satisfied only if the curvature of the indifference curve is larger than the curvature of the factor-price frontier, i.e. $\frac{d^2w}{dr^2}_{|dU=0} > \phi''(r)$. The factor-price frontier is given by:

$$w = \phi(r); \quad \frac{dw}{dr} = \phi'(r) = -k; \quad \frac{d^2w}{dr^2} = \phi''(r) = \frac{-1}{f''}.$$

The indifference curve of the representative individual in the w, r plane is:

$$U = U(w, r); \quad \frac{dw}{dr}_{|dU=0} = \frac{-s(w, r)}{(1 + r)}; \quad \frac{d^2w}{dr^2}_{|dU=0} = \frac{s_w s(w, r) - s_r(1 + r) + s(w, r)}{(1 + r)^2}.$$

Using the Serendipity Theorem we can show that the first order condition for a laissez faire/planned optimum at an interior n^* is satisfied if $\phi'(r) = \frac{dw}{dr}_{|dU=0}$ at n^*:

$$-k^* + \frac{c^{2*}}{(1 + n^*)^2} = 0 \Leftrightarrow -k^* = \frac{-s^*}{(1 + f'(k(n^*)))}; \quad f'(k(n^*)) = n^*; \quad c^{2*} = (1 + n^*)s^*.$$

Now we will show that the sufficient condition $\frac{d^2w}{dr^2}_{|dU=0} > \phi''(r)$ can be transformed into $f''(k)\frac{dk}{dn} > 1$, which was our sufficient condition (compare with (16) and (17)) for a laissez faire optimum at n^*:

$$\frac{s_w s(w, r) - s_r(1 + r) + s(w, r)}{(1 + r)^2} > \frac{-1}{f''};$$

at the stationary point we have $s = (1 + n)k$ and $n = n^* = r$, and hence:

$$\frac{s_w(1 + n)k - s_r(1 + n) + (1 + n)k}{(1 + n)} \quad > \quad \frac{-1}{f''}(1 + n),$$

this can be rearranged such that:

$$-k < \frac{1}{f''}(1 + n) + s_w k - s_r,$$

with $\frac{1}{f''}(1 + n) + s_w k - s_r < 0$ by virtue of the stability condition (12). Thus we obtain:

$$\frac{-k}{\frac{1}{f''}(1 + n) + s_w k - s_r} \quad > \quad 1 \quad \Leftrightarrow \quad f''(k)\frac{dk}{dn} > 1.$$

2.5.2 Proof of Proposition 1

It follows directly from the Serendipity Theorem that the first order conditions for the existence of an interior n^* in the planned economy and the laissez faire economy both identify the same stationary point; for $n = n^* = r^*$, conditions (6) and (15) are both satisfied.

We will now extend the Serendipity Theorem by proving that the same is also true for the sufficient conditions. Thus we have to show that the sufficient condition for an optimal interior n^* in the planned economy is only satisfied if $\frac{-k}{\frac{1}{f''}(1+n)+s_w k - s_r} > 1$ at the stationary point.

The second order derivative of the indirect utility function (5) for the planned economy was given by:

$$\frac{d^2U}{dn^2}_{|n=n^*} = U_{c^1}\Big(-\frac{dk_n}{dn} + \frac{(1+n)^2\frac{dc_n^2}{dn} - 2(1+n)c_n^2}{(1+n)^4}\Big) \gtreqless 0. \tag{38}$$

The sign of this second order derivative hinges on two distinct elements: the first element $\frac{dk_n}{dn}$ is the aspect of optimal capital accumulation. The second element $\frac{(1+n)^2\frac{dc_n^2}{dn} - 2(1+n)c_n^2}{(1+n)^4}$ is concerned with the optimal consumption pattern.

From the first order condition for the optimal capital accumulation pattern we have:

$$r_n = f'(k_n) = n, \quad \frac{dk_n}{dn} = \frac{1}{f''(k_n)}. \tag{39}$$

For the second element, which is concerned with the optimal consumption pattern, we find that in a planned economy we have:

$$\frac{U_{c^1}(c_n^1, c_n^2)}{U_{c^2}(c_n^1, c_n^2)} = 1 + n,$$

$$w_n = f(k_n) - nk_n = c_n^1 + \frac{c_n^2}{(1+n)^2}.$$

These two equations clearly define an optimal consumption pattern c_n^1 and c_n^2, where $c_n^2 = (1+n)s(w_n, r_n)$; once the individual faces the biological rate of interest it will voluntarily (for all given real wages w_n) choose the optimal (biological) consumption pattern (Samuelson (1958) and Cass and Yaari (1966)). Hence:

$$\frac{dc_n^2}{dn} = \frac{d[(1+n)s(w_n, r_n)]}{dn} = s(w_n, r_n) + \Big(s_w\frac{dw_n}{dn} + s_r\frac{dr_n}{dn}\Big)(1+n), \tag{40}$$

with:

$$\frac{dr_n}{dn} = 1; \quad \frac{dw_n}{dn} = f'(k_n)\frac{dk_n}{dn} - k_n - n\frac{dk_n}{dn} = -k_n.$$

We can now substitute the expressions in (39) and (40) into (38) to evaluate the sign of $\frac{d^2U}{dn^2}$ at the stationary point, where we have $c_n^{2*} = (1+n^*)s(w^*, r^*) = (1+n^*)^2k^*$:

$$\frac{d^2U}{dn^2}_{|n=n^*} = U_{c^1}\Big(-\frac{1}{f''(k^*)} + \frac{(1+n^*)^3k^* + (-s_wk^* + s_r)(1+n^*)^3}{(1+n^*)^4} - \frac{2(1+n^*)^3k^*}{(1+n^*)^4}\Big).$$

Hence $\frac{d^2U}{dn^2}_{|n=n^*}$ is negative if:

$$-k^* < (1+n^*)\frac{1}{f''(k^*)} + s_wk^* - s_r. \tag{41}$$

According to the stability condition (12) we have $(1+n^*)\frac{1}{f''(k^*)} + s_wk^* - s_r < 0$ and we find that $\frac{d^2U}{dn^2}_{|n=n^*} < 0$ if and only if:

$$\frac{-k^*}{(1+n^*)\frac{1}{f''(k^*)} + s_wk^* - s_r} > 1. \tag{42}$$

This sufficient condition for a social optimum (42) is identical with the sufficient condition (17) for a laissez faire optimum at n^*.

2.5.3 Oscillatory Stability

In this appendix we will discuss the case of one unique oscillatory steady state equilibrium. The corresponding stability condition is:

$$-1 < \frac{dk_{t+1}}{dk_t} = \frac{-s_wf''k}{(1+n) - s_rf''} < 0; \quad s_w > 0. \tag{43}$$

Since the numerator is positive s_r must be algebraically large and negative, which is only possible for $\sigma < 1$. From the life cycle savings condition (11) we obtain once again:

$$\frac{dk}{dn} = \frac{-k}{s_wf''k + (1+n) - s_rf''}. \tag{44}$$

Utilizing (43) reveals that $\frac{dk}{dn} > 0$ and hence we have $\frac{dr}{dn} < 0$. It is now easy to show that the sufficient condition for an interior optimum growth rate for population is always satisfied:

$$\frac{d^2U}{dn^2}_{|n=n^*} = U_{c^1}\frac{(1-\frac{dr}{dn})}{(1+r)}k\frac{dr}{dn} < 0. \tag{45}$$

Hence we find that an interior minimum is only possible for
$0 < \frac{dr}{dn}_{|n=n^*} < 1$, in all other instances we have an interior optimum. However, the oscillatory case with $\frac{dk}{dn} > 0$ appears to be rather unrealistic. In addition we note that the claim, that the "stability condition" together with the assumption of normality allows to derive the sign of $\frac{dk}{dn}$, is not accurate. Instead, it is necessary to distinguish two cases, in the same manner as in Diamond (1965).

2.5.4 Formal aspects to Diagram 4

Individual utility is given by:

$$U = U\Big(w(n) - s(\tilde{w}(n), r(n)) - (r(n) - n)b, (1 + r(n))s(\tilde{w}(n), r(n))\Big),$$

by varying the growth rate for population only, we obtain the following slope for the indifference curve:

$$\frac{dw}{dr}_{|dU=0, \frac{U_{c1}}{U_{c2}}=1+r, dn\neq 0} = b - \frac{s}{1+r} - \frac{dn}{dr}b.$$

We can now reproduce the first order condition (37) by setting
$\frac{dw}{dr}_{|dU=0, \frac{U_{c1}}{U_{c2}}=(1+r)} = \phi'(r) = -k$:

$$b - \frac{s}{(1+r)} - \frac{dn}{dr}b = -k \quad \Leftrightarrow \quad \frac{n-r}{1+r}(k+b)\frac{dr}{dn} + b = 0.$$

For $n = \tilde{n} = r$ we have:

$$\frac{dw}{dr} = b - \frac{s}{(1+r)} - b\frac{dn}{dr} < -k,$$

since,

$$-b\frac{dn}{dr} < 0.$$

Hence at $\tilde{n}$ the slope of the indifference curve is algebraically larger than that of the factor-price frontier.

2.5.5 Appendix: Pay-as-you-go Social Security and optimal population

In this appendix we will briefly substantiate the claim that the qualitative conditions for an interior optimum in an economy with debt are similar to those for an economy with a pay-as-you-go social security system. Once we denote the per capita contributions by α and the old age benefits by β, the budget constraint for the social security can be written as:

$$N_t\alpha = N_{t-1}\beta \quad \Leftrightarrow \quad (1+n)\alpha = \beta. \tag{46}$$

Hence the representative individual living in a steady state equilibrium is affected by demographic change according to:

$$U(n) = U(w(k) - \alpha - (1+n)k, \beta + (1+n)(1+f'(k))k).$$

Utilizing (46) and the respective factor-prices gives:

$$U(n) = U(f(k) - f'k - \alpha - (1+n)k, (1+n)\alpha + (1+n)(1+f')k).$$

Hence the first order condition, after some cancelling of terms, for the optimum growth rate is given by:

$$\frac{dU}{dn} = U_{c^1}\frac{\alpha}{(1+f')} + U_{c^1}\frac{(n-f')k}{(1+f')}\frac{dr}{dn} = 0. \tag{47}$$

Comparison between (47) and (37) reveals that once again, the tradeoff is between the increased internal rate of return of the pension system $U_{c^1}\frac{\alpha}{(1+f')}$ and the two factor-prices $U_{c^1}\frac{(n-f')k}{(1+f')}\frac{dr}{dn}$.

The key difference between the constant per capita debt policy and the pay-as-you go social security system is that the contribution rate α does not enter the second term. The reason for this is the following: In an economy with pay-as-you-go pension scheme, the (constant) contribution rate to the system is independent from the growth rate for population. In the economy with government debt there is a link between the amount of resources, which are distributed by the government and the growth rate for population: The tax on the young generation is given by $\tau^1(k) = (f'(k(n)) - n)b)$. Hence, once the growth rate for population is changed, the tax rate also changes and thus the total amount of resources which goes through the hands of the government, which yield the biological rate of return.

3 Dynamic Efficiency and the Two-Part Golden Rule with Heterogeneous Agents

This chapter is concerned with the role of the two-part golden rule as the watershed between equilibria which are dynamically efficient and those, which are inefficient. In an economy where agents differ regarding their labor endowment, the golden rule allocation ceases to serve as such a demarcation line. Except for the special case where all agents possess a linear Engel-curve with identical slope, some agents' maximum steady state utility will always be associated with a capital intensity exceeding (falling short of) the golden rule level. This result stems from the fact that the competitive markets entail an intra-generational redistribution of resources once the capital intensity is altered. If heterogeneity is introduced on the preference side, we find that the golden rule is never optimal for all agents. Consequently, earlier results in the literature (e.g. Stein (1969)) on the two-part golden rule with heterogeneous agents are not warranted.

3.1 Introduction

Having less of something useful may improve welfare. This is one of the key results obtained from the normative evaluation of the Solow (1956), Swan (1956) models of capital and growth: maintaining a capital intensity that *permanently* exceeds the golden rule level is known to be inefficient.[16] At the same time we know from the pure consumption loan economy of the Samuelson (1958) type that a given endowment should be distributed between adjacent cohorts such that the rate of return on consumption loans is equated to the growth rate of population. Taken together, these two optimality conditions constitute the *two-part* golden rule which maximizes steady state utility in the Diamond (1965) model. The golden rule of accumulation ensures maximum consumption in each period. In turn, the golden rule of interest on consumption loans ensures efficient intergenerational distribution of the consumption available.

In more recent studies, this *two-part* golden rule result was shown to be robust with regard to several changes in the underlying assumptions. In particular, Abel et al. (1989) and Zilcha (1990) show that the golden rule criterion carries over to a

[16] See Phelps (1961, 1966b), von Weizsäcker (1962), Cass (1972). See Burmeister and Dobell (1970) and Jones (1975) for more references.

setting with aggregate and idiosyncratic risk. Angel and Garcia (2008) extend the result to a model with endogenous labor supply.

Against this background, the present chapter is concerned with the role of the two-part golden rule in a Diamond (1965) economy with heterogeneous agents. To illustrate our results, two forms of heterogeneity which are widely employed in the overlapping generations literature will be discussed: (i) heterogeneous labor endowments, with homogeneous preferences and (ii) homogeneous labor endowments with heterogeneous preferences. In these settings, the Phelps (1961) golden rule continues to maximize *society's* consumption opportunities, and the Samuelson (1958) golden rule ensures efficient intergenerational consumption patterns. The two-part golden rule, however, no longer describes the demarcation line between *competitive* equilibria which are efficient and those which are inefficient. That is, even if the two-part golden rule equilibrium could be reached at no (transition) cost, there will always be individuals who prefer a lower and others who prefer a higher steady state capital intensity. This result originates from an *intra-generational* reallocation of resources, which operates through the competitive markets and occurs once the capital intensity is altered. Compared to the golden rule level, this transfer effect will allow agents whose present value of savings $\frac{s}{1+r}$ exceeds (falls short of) the capital stock lk, which is absorbed by their labor supply l, to reach higher steady state utility once the capital intensity is reduced (increased). If heterogeneity is introduced on the preference side, the corresponding condition is given by $\frac{s^i}{1+r} \gtreqless k$, where thrifty agents ($\frac{s^i}{1+r} > k$) will once again prefer steady states with interest rates exceeding the golden rule level.

Despite its simplicity, this result differs from what appears to be the consensus in the literature. In particular, the often cited[17] paper by Stein (1969) p. 144 analyzes a competitive Diamond (1965) model where agents differ with regard to their rate of time preference. As we show in Section 3.2.3, his conclusion that the golden rule allocation is optimal for all members of such a society is not warranted.

[17]See Gale (1973), Ihori (1978), Krohn (1981) and Crettez et al. (2002). See De La Croix and Michel (2002) for a recent textbook. De La Croix and Michel (2002) p. 81 criticize Stein (1969) by pointing out that he fails to notice the double infinity aspects emphasized by Shell (1971). However, they do not point out that the golden rule no-longer serves as the watershed between efficient and inefficient steady states under the assumptions made by Stein (1969), where households are heterogeneous. In Section 3.2.3 we complement their criticism and point out that the golden rule does not separate efficient from inefficient steady states in a competitive economy with heterogeneous preferences.

Given the pivotal role played by the golden rule, our results may also provide useful inference with regard to the social security literature (e.g. Persson and Tabellini (2000) and in particular Pestieau et al. (2006)), where it is often argued that Beveridge schemes are redistributive while Bismarck schemes are not.[18] In the current case, we find that both schemes redistribute resources through their induced factor-price changes. In particular, if the propensity to save increases with income, the crowding-out of capital brought about by both social security schemes favors the rich at the expense of the poor. In the case of a Beveridge pension scheme, some of the direct intra-generational redistribution is therefore thwarted by the associated factor-price changes.

The rest of the chapter is organized as follows: In Sections 3.1.1 and 3.1.2, we recall the two golden rule relations for the representative agent economy. In Section 3.1.3 we present a dissection of the overlapping generations structure of incomplete markets. With these elements in place, we isolate the main result in a partial equilibrium setting. In Section 3.2, we prove the result in general equilibrium. Section 3.3 offers concluding remarks.

3.1.1 Consumption Maximizing Growth

In the standard one-sector growth models of Solow (1956) and Swan (1956), where production is homogeneous of degree one, output can be consumed or saved. With flexible factor-prices, savings always equal investment and at each point in time we have:

$$F(K_t, L_t) = K_{t+1} - K_t + C_t = L_t f(k_t); \quad k_t = \frac{K_t}{L_t}, \quad f'() > 0, \quad f''() < 0. \tag{48}$$

With population growing at a constant proportional rate n, per capita steady state consumption $c := \frac{C}{L}$ is then:

$$c = f(k) - nk. \tag{49}$$

[18] Pestieau et al. (2006), p. 591 discuss an open economy with general non-homothetic preferences with heterogeneous labor endowment. However, they conclude their analysis by remarking "If instead the pension system were purely contributive (Bismarckian), there would be no redistribution across individuals, and thus no tax competition per se. Then capital mobility would not have any effect on the choice of the payroll tax, and each country would use it to achieve dynamic efficiency just as in autarky" (p. 595). The current analysis shows that the usual notion of dynamic efficiency, i.e. the golden rule result does not apply to the framework analyzed by Pestieau et al. (2006). That is, Bismarckian schemes are not only contributive but also redistributive.

As shown by Phelps (1961), von Weizsäcker (1962) and others, consumption is maximized if the rate of return on capital investment is equal to the economy's growth rate:

$$\max_k c(k) = \max_k \Big(f(k) - nk\Big) = f(k_n) - nk_n, \quad f'(k_n) = n, \tag{50}$$

where k_n denotes the golden rule capital intensity. Once reached, this consumption maximizing steady state can be sustained by investing profits, $s_n f(k_n) = nk_n = f'(k_n)k_n$, and the consumption of wages, $c(k_n) = f(k_n) - nk_n = w_n$.

3.1.2 Utility Maximizing Growth

In the Diamond (1965) life-cycle model, steady state utility is at its long-run optimum once the two-part golden rule is implemented. In each period t, aggregate consumption C_t can now be allocated between the old and the young cohort, i.e. $C_t = C_t^1 + C_t^2$. Recalling the resource constraint (48), the social planner chooses the consumption maximizing capital intensity k_n defined in (50). In turn, consumption is allocated to the old and young according to the second biological interest rate relation of Samuelson (1958). With utility concave in first and second period consumption, we have the two-part golden rule optimum:

$$\max_{k,c^2} U(c^1, c^2) \quad s.t. \quad f(k) - nk = c^1 + \frac{c^2}{1+n}; \tag{51}$$

$$f'(k_n) = n, \tag{52}$$

$$\frac{U_{c^1}(c_n^1, c_n^2)}{U_{c^2}(c_n^1, c_n^2)} = 1 + n, \tag{53}$$

where (52) maximizes consumption and (53) ensures efficient intergenerational distribution. As Diamond (1965), Samuelson (1975b) and Ihori (1978) show, this utility maximizing allocation can be decentralized by an appropriate intergenerational transfer that forces $k = k_n$ such that (52) is satisfied. In turn, consumers will voluntarily choose consumption according to (53). A striking property of (52) and (53) is their asymmetry. A change in the utility function affects the consumption profile c_n^1, c_n^2 but does not affect the optimal capital intensity k_n. However, a change in the production function affects all three quantities k_n, c_n^1, c_n^2. If preferences are not homothetic, it also affects the ratio $\frac{c_n^1}{c_n^2}$.

3.1.3 Competitive Incomplete Markets

In this section, we dissect the competitive apparatus of maximizing behavior and market clearing, which restricts the set of feasible steady state allocations. This dissection will provide the background to interpret our results in Section 3.2. In a first step we summarize the OLG market structure in the c^1, c^2 plane. In a second step, we approach the same question in the w, r plane to emphasize the role of the factor-price frontier as a resource constraint. The key insight in this section is that the golden rule result vanishes for the representative agent economy once one of the equations which describe the competitive equilibrium is not taken into account. In Section 3.2, where we introduce heterogeneity, it will turn out that some equations, most notably the life-cycle savings condition, are less restrictive: in a model with heterogeneous agents, *average* savings rather than *each* agent's savings have to be sufficient to support the steady state capital stock. That is, in what follows we show that dropping one of the equations of the representative agent model will make the golden rule allocation suboptimal. In turn, we show in Section 3.2 that heterogeneity will have an effect which is similar to the dropping of one equation: the life-cycle savings condition will only require that *average* savings support the steady state. *It is therefore less restrictive and the golden rule result vanishes in the same manner as it does in the representative agent economy where the life-cycle savings condition is not taken into account.*

The Golden Rule Consumption Profile The equations describing the competitive Diamond (1965) model, where taxes are raised in each period to keep the debt to labor ratio $b_t = \frac{B_t}{L_t}$ constant over time, may be summarized as follows:

$$L_{t+1} = (1+n)L_t, \tag{54}$$

$$s_t = (1+n)(k_{t+1} + b), \tag{55}$$

$$s_t = w_t - c_t^1 - (r_t - n)b, \tag{56}$$

$$c_t^1 + \frac{c_{t+1}^2}{(1 + r_{t+1})} = w_t - (r_t - n)b, \tag{57}$$

$$w_t = f(k_t) - f'(k_t)k_t, \tag{58}$$

$$r_t = f'(k_t), \tag{59}$$

$$f(k_t) = (1+n)k_{t+1} - k_t + c_t^1 + \frac{c_t^2}{1+n}. \tag{60}$$

Taken together, (54)-(59) describe market clearing, the households' budget constraint and profit maximizing firms, leaving open the households' savings decision. Equation (60) is the aggregate resource constraint. It is straightforward to show that (54)-(60) have one linearly dependent equation. We can therefore drop (60) and work with the system (54)-(59). Along a steady state path, disregarding the households' savings decision for the moment, first and second period consumption can now be described as functions of the capital intensity:

$$c^1 = f(k) - \big(f'(k) + (1+n)\big)k - (1 + f'(k))b, \tag{61}$$

$$c^2 = (1 + f'(k))(1+n)(k+b). \tag{62}$$

Differentiation of (61)-(62) yields the locus OT of all feasible consumption bundles:

$$\frac{dc^2}{dc^1}_{|OT} = -\frac{(1+n)[f''(k)(k+b) + (1+f'(k))]}{f''(k)(k+b) + (1+n)}. \tag{63}$$

Comparison of the slope of the market constrained consumption curve OT with that of the aggregate resource constraint (51), where $\frac{dc^2}{dc^1}_{|dk=0} = -(1+n)$, yields:

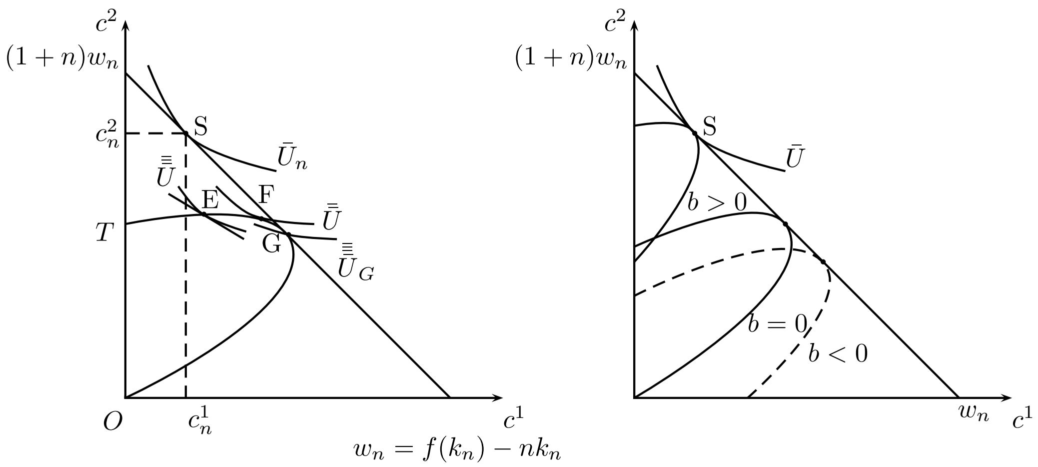
$$-\frac{dc^2}{dc^1}_{|OT} - (1+n) = \frac{(1+n)(f'(k)-n)}{(1+n)+(k+b)f''(k)}, \tag{64}$$

where (64) indicates that the OT curve is a tangent to society's resource constraint once $k = k_n$. Diagram 5a depicts the market process in the "pure case" without government debt, where $b = 0$ in the c^1, c^2 plane, as a curve ranging from O to T. In particular, diagram 5a indicates that the utility maximizing social optimum S is typically incompatible with the market process without debt, i.e. the OT-line. Moreover, even if the market process is capable of golden rule growth G, market constrained utility would be higher at F. If, however, individuals choose savings privately, the model is closed and a specific equilibrium E can be located on the OT curve. This equilibrium E, where the marginal rate of substitution is equal to the slope of the households' budget constraint, may or may not be inferior to F and G. Adding the households' savings decision explicitly, we have:

$$\max_s U = U(c^1, c^2), \quad s.t. \quad c^1 = w - (r-n)b - s; \quad c^2 = (1+r)s. \tag{65}$$

Hence, optimal savings are:

$$s = s(w - (r-n)b, r), \quad s_r \gtreqless 0, \quad 0 < s_w < 1, \tag{66}$$



$5a$ $5b$

Diagram 5: *Competitive incomplete markets.*
The OT curve in Diagram 5a depicts the market constrained $c^1 - c^2$ pairs. The equilibrium E obtained on the OT curve may be either efficient, i.e. located on the OG section, or, as in the present case, in the inefficient section GT. Note that G is associated with a capital intensity which satisfies (52), but the consumption bundle at G violates (53); hence, we have $\bar{\bar{\bar{U}}}_G < \bar{U}_S$. Diagram 5b illustrates the social optimum S which can be decentralized by an appropriate over/under-funded social security/debt scheme.

where $0 < s_w < 1$ implies that consumption in each period is assumed to be a normal good. Once we combine the savings function defined in (66) with the market process (54)-(59), our model is closed and an equilibrium E as in Diagram $5a$ will be realized. Diagram $5b$ now depicts how the government can vary per capita debt b as in Diamond (1965), in order to decentralize the social optimum S.

The Wage Interest Tradeoff From a different perspective, we may disregard the life-cycle savings condition (55) for the moment and consider the conditions under which an increase in the capital intensity increases individual utility. From a partial equilibrium perspective, it is easy to trace out individual preferences for the optimal capital intensity:

$$\max_k U\Big(w - (r-n)b - s(w,r), (1+r)s(w,r)\Big), \tag{67}$$

where savings are defined as in (66). Hence, recalling (58) and (59), the optimal capital intensity is the root of:

$$\frac{dU}{dk} = -U_{c^1}\Big(k + b - \frac{s(w,r)}{(1+r)}\Big)f''(k) = 0 \quad \Leftrightarrow \quad -k = b - \frac{s(w(k),r(k))}{(1+r(k))}. \quad (68)$$

In a partial equilibrium context, we therefore have the following proposition:[19]

Proposition 4. *An increase in the capital intensity increases (decreases) utility iff the present value of individual savings falls short of (exceeds) the per capita stock of assets in the economy. Put differently, an increase in the capital intensity increases (decreases) utility iff the slope of the individual indifference curve* $b - \frac{s}{(1+r)}$ *in the* $w-r$ *plane is algebraically smaller (larger) than the slope of the factor-price frontier* $-k$.

Proof. Follows from (68). See Appendix 3.4.1 for the slopes of the factor-price frontier and the households' indifference curves and a Cobb-Douglas example showing that (68) is prone to corner solutions. □

Hence, from a partial equilibrium point of view, households would (if they could) choose the capital intensity such that their indifference curve in the $w-r$ plane is a tangent to the factor-price frontier as depicted by point F in Diagram 6. The notable property of (68) is that it is by no means related to a golden rule condition. In particular, we find that changes either on the preference side or on the labor endowments side will also change the optimal capital intensity, a behavior which did not occur in the command optimum (52)-(53), where $f'(k_n) = n$ was defining a unique capital intensity k_n − independent of the preference ordering or (as is easy to check) the households' labor endowment. However, taking into account the life-cycle savings condition (55), the competitive economy will settle in some point E. As in Diamond (1965), the government may now choose a negative level of per-capita debt to decentralize the golden rule optimum S, where, for $r = n$, (54)-(59) and (68) hold simultaneously and the golden rule describes an optimum.

In the following Section 3.2, we show how the golden rule result changes once we introduce intra-cohort heterogeneity into the model. While the government can still decentralize the golden rule allocation (52)-(53), described in this section, it is no longer the best allocation for each group in society. Depending on the propensity to save, some households will attain maximum utility in a steady state where $r > n$

[19] In an economy with homogeneous agents it is now clear from the life-cycle savings condition (55) that the golden rule capital intensity k_n solves (68).

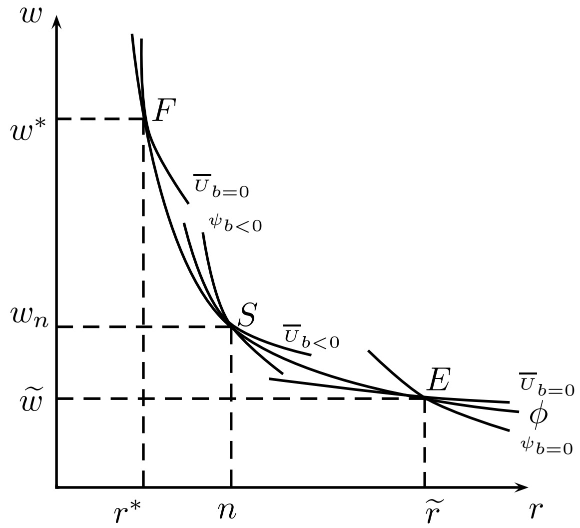


Diagram 6: *The wage-interest tradeoff.*
The factor-price frontier ϕ describes the wage-interest tradeoff implied by the neoclassical production function. The pair (w^*, r^*) maximizes unconstrained utility in partial equilibrium without debt. Taking the life-cycle savings condition ψ into account, the equilibrium E, where $r > n$, is obtained. Lowering per capita debt to a negative level pivots ψ such that the two-part golden rule optimum S is reached (see Appendix 3.4.1 for a derivation of ψ and the corresponding shifts in ψ and $\overline{U}$).

and vice versa. That is, in a competitive economy where $r > n$, some members of society will reach a consumption profile located "north-east" of the $w_n, (1+n)w_n$ golden rule constraint described in Diagram 5a. Equivalently, thinking in terms of Diagram 6, heterogeneity pivots the households' indifference curves in the $w - r$ plane such that some agents prefer a wage-interest pair different from n, w_n.

3.2 Competitive Markets with Heterogeneous Agents

In this section we derive the main results. We proceed in two steps:

1. We analyze a setting where heterogeneity is introduced with regard to the amount of efficient labor. Again, we do so in two steps:

   (a) First in a setting with government debt/paygo pensions, we analyze how changes in the level of debt influence utility through the induced changes in the capital intensity.

(b) In a second step, we show that the break-down of the golden rule result depends on the structure of the competitive markets. That is, we show that it is independent from the particular debt scheme which we employ.

2. Heterogeneity is introduced on the preference side as in Stein (1969). Once again we study a setting with debt and in turn show that the result carries over to the case without debt.

The intention with this approach is to have one stylized setting where changes in the size of the public debt are evaluated. To this end we discuss the basic Diamond (1965) setting where the changes in the capital intensity are caused by changes in per capita debt. Subsequently we add the setting without debt as a robustness check to show that the golden rule result does not vanish because of the specific tax-scheme which the government runs.

Section 3.2.4 shows that the results will also carry over to a setting where the change in the capital intensity is caused by a change in total factor productivity. In all three cases, we find that the result isolated in Proposition 4 will carry over seamlessly into general equilibrium, i.e. heterogeneity will weaken the life-cycle savings condition and the golden rule ceases to maximize steady state utility.

3.2.1 Heterogeneous Labor Endowment with Debt

The economy in this section is inhabited by a continuum of agents who differ regarding their innate labor endowment. Normalizing period 0 labor supply to unity, aggregate labor evolves according to:

$$L_t = (1+n)^t \int_{\check{l}}^{\hat{l}} l dF(l); \quad \int_{\check{l}}^{\hat{l}} l dF(l) = \mu = 1, \quad \hat{l} > \check{l} > 0, \tag{69}$$

where l denotes the type of agent and $dF(l)$ the size of the group of type l agents. As in Diamond (1965), the government is assumed to collect the taxes needed to finance interest payments by raising a lump-sum tax on wages:[20]

$$\tau(l) = (r-n)lb; \quad \int_{\check{l}}^{\hat{l}} \tau(l) dF(l) = (r-n)b. \tag{70}$$

[20] The government budget constraint reads $B_{t+1} - B_t(1+r_t) = -L_t \int_{\check{l}}^{\hat{l}} \tau(l)dF(l)$. Defining $b_t \equiv \frac{B_t}{L_t}$, we have the steady state relation $\int_{\check{l}}^{\hat{l}} \tau(l)dF(l) = (r-n)b$.

We choose the specific tax scheme in (70) for three reasons: (i) it allows to extend the golden rule result to a setting with homothetic preferences, (ii) it closely resembles the formulation used in Diamond (1965), and (iii), as we show at the end of Appendix 3.4.2, it gives the same qualitative first order conditions that would be obtained from a setting with a Bismarckian pension scheme with a linear contribution rate on wages.

As shown in Appendix 3.4.2, all remaining agent specific quantities can be integrated/summed, such that the aggregate economy behaves as characterized in (54)-(59). In particular, the life-cycle savings condition requires that:

$$\int_{\check{l}}^{\hat{l}} s(wl - (r-n)bl, r)dF(l) = (1+n)(k+b). \tag{71}$$

For our purpose it is important that (71) only requires that *average* savings support the steady state. The indirect utility of a type l agent now reads:

$$U(wl - (r-n)bl - s(wl - (r-n)bl, r), (1+r)s(wl - (r-n)bl, r)). \tag{72}$$

Taking into account the equations (58) and (59) and the households' Euler equation, the first order condition for the optimum quantity of debt is:[21]

$$\frac{dU}{db} = -U_{c^1}\left((r-n)l + \left[(k+b)l - \frac{s(wl-(r-n)bl,r)}{1+r}\right]f''(k)\frac{dk}{db}\right) = 0, \quad \frac{dk}{db} < 0. \tag{73}$$

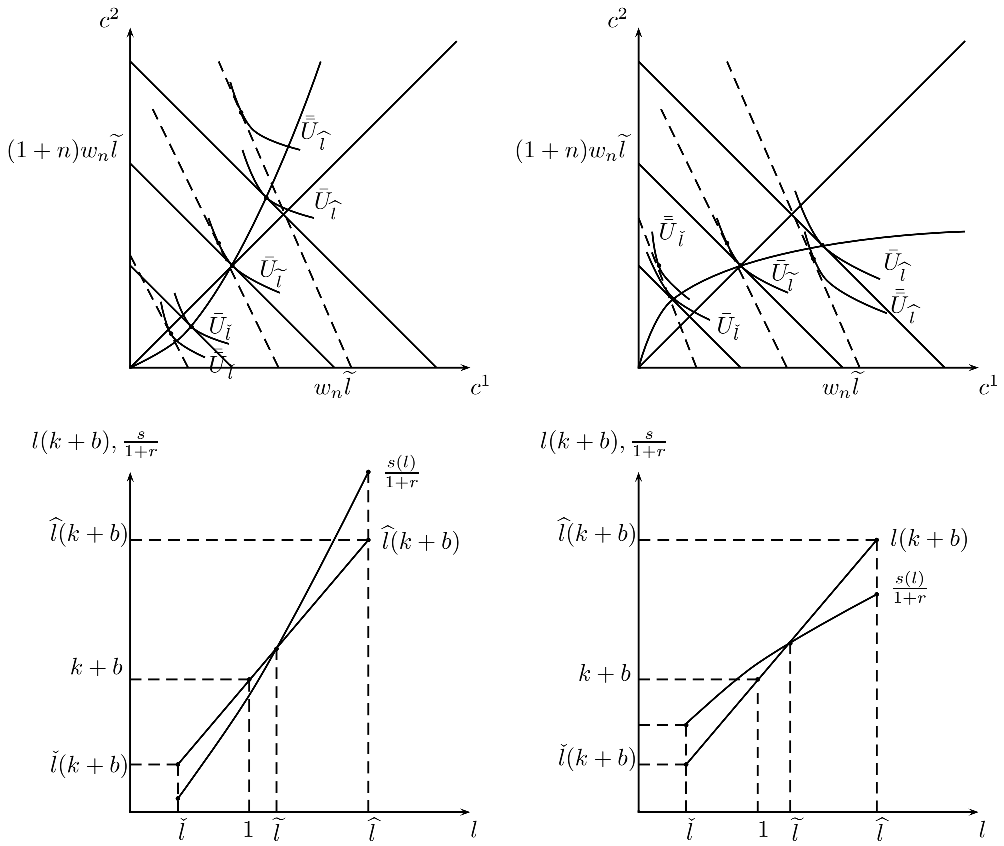
Given $\tilde{k}$, condition (73) identifies the type $\tilde{l} = l(\tilde{k})$ agent whose utility is optimized by the particular capital intensity, or, given $\breve{l}$, the optimal capital intensity $\breve{k} = k(\breve{l})$ which maximizes the respective utility of a type l agent. In the present setting, the golden rule allocation rather separates "savers" from "non savers" than efficient from inefficient steady states. Obviously, the two-part golden rule result is violated for all agents who do not exactly hold their proportional stock of debt and capital in their portfolio. For $r = n$, thrifty agents prefer that debt is issued to a point where $r > n$ and vice versa. Hence, we can graph condition (73) in diagrams 7 and 8. Moreover, we have the corresponding proposition:[22]

Proposition 5. *Agents who hold less than their proportional share of assets $(k+b)l$ benefit (suffer) from a capital intensity which exceeds (falls short of) the level prescribed by the golden rule. If preferences are not homothetic, there are always some agents that prefer a steady state where $n > r$ and vice versa.*

[21] In Appendix 3.4.2, we use the stability condition to establish that $\frac{dk}{db} < 0$.

[22] In Appendix 3.4.2, we show that the same result can be obtained for a setting with a Bismarck pension scheme with a linear contribution rate.

Proof. See Appendix 3.4.2. □



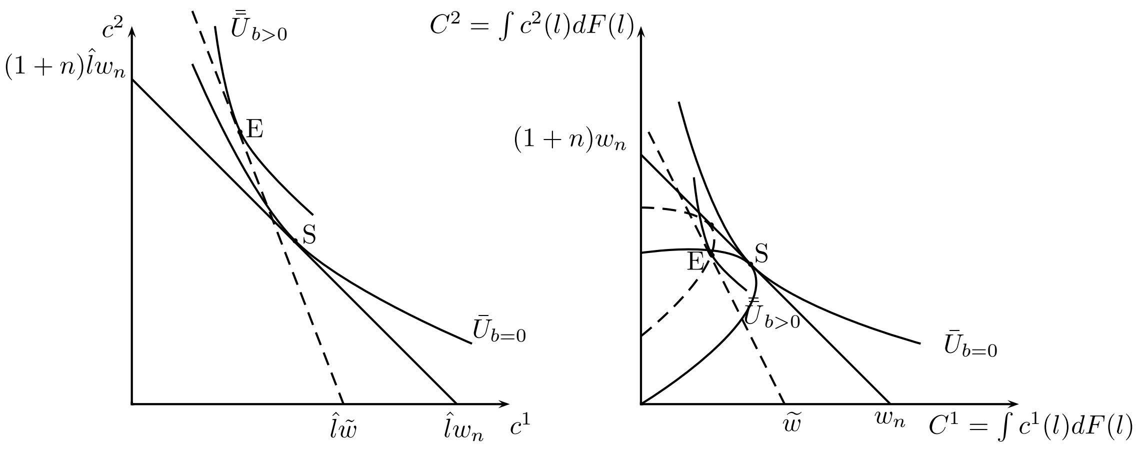
7*a* 7*b*

Diagram 7: *Intragenerational redistribution and the Engel-curve. Intragenerational redistribution via competitive markets with an increasing (decreasing) propensity to save. For the empirically relevant case, where the propensity to save increases with income, a decrease in the capital intensity below golden rule levels increases "capitalists'" utility at the expense of "workers'" utility and vice versa.*

Corollary 4. *If preferences are homothetic, the savings function is of the form* $s = \xi(r)(w-(r-n)b)l$*. In this case, where the Engel-curve is linear and of identical slope for all agents, the golden rule describes the demarcation line between efficient and inefficient steady states.*

Proof. See Appendix 3.4.2. □

Diagrams 7 and 8 depict cases where utility is not homothetic. We now show that the results displayed above are not an artifact of the particular debt scheme.



8*a* 8*b*

Diagram 8: *Intragenerational redistribution with nonhomothetic preferences. Diagram* 8*a compares a thrifty high income household's utility at the two-part golden rule equilibrium S to the market equilibrium E. Diagram* 8*b illustrates the inferior consumption bundle at E, which is allocated to the "average household".*

3.2.2 Heterogeneous Labor Endowment without Debt

In this section there is no government debt to adjust the capital intensity. Instead we ask whether a household of a particular type would prefer to be born into a society where savings support a steady state with a capital intensity that exceeds (falls short of) the golden rule level. Once we recall the indirect utility function (72) and the life-cycle savings condition (71) it suffices to set $b = 0$. The associated first order condition for the optimal capital intensity yields the following analogue to Proposition 5:

Corollary 5. *Agents who hold less than their proportional share of assets kl benefit (suffer) from a capital intensity which exceeds (falls short of) the level prescribed by the golden rule. If preferences are not homothetic, there are always some agents that prefer a steady state where $n > r$ and vice versa. If preferences are homothetic, the golden rule separates efficient from inefficient steady states.*

Proof. Consider the indirect utility function in (72). If we set $b = 0$, changes in the capital intensity affect utility of a type l agent according to:

$$\frac{dU}{dk} = -U_{c^1}\Big(kl - \frac{s(wl,r)}{(1+r)}\Big)f''(k) \gtreqless 0 \quad \Leftrightarrow \quad kl \gtreqless \frac{s(w(k)l, r(k))}{(1+r(k))}. \tag{74}$$

Taking into account the life-cycle savings condition, which once again requires that *average savings* are sufficient to support the steady state, we have:

$$\int_{\check{l}}^{\hat{l}} s(wl, r)dF(l) = (1+n)k. \tag{75}$$

If the economy grows on a golden rule path where we have $r = n$, conditions (75) and (74) yield for a steady state:

$$kl = \frac{\int_{\check{l}}^{\hat{l}} s(wl, n)dF(l)}{1+n} l \gtreqless \frac{s(wl, n)}{(1+n)}. \tag{76}$$

Hence, depending on the propensity to save out of income, a household of type l will prefer a capital intensity that exceeds (falls short of) the golden rule level. Finally, if preferences are homothetic, savings are known to be a positive fraction $\xi(r)$ of the available income. That is, $s(wl, r) = \xi(r)wl$. Plugging this savings function into (76), we find that it holds with equality at the golden rule capital intensity. The golden rule therefore maximizes steady state utility if $U()$ is homothetic. □

3.2.3 Heterogeneous Preferences

Now let us change the perspective slightly and consider an economy inhabited by agents with uniform labor endowment. As in Stein (1969), heterogeneity will now be introduced on the preference side only. Society now consists of a continuum of agents, where each group i has its own preference ordering $U_i(c^1, c^2); i \in [0,1]$. Thus, according to (65), each agent chooses a unique optimal $s_i(w, r)$. Integration over the index set in turn yields the aggregates in the same manner as in the previous section. If agents are taxed equally, as in Diamond (1965), we have the following first order condition with regard to the optimal capital intensity for a type i agent:[23]

$$\frac{dU_i}{db} = -U_{i,c^1}\left((r-n) + \left[(k+b) - \frac{s_i}{(1+r)}\right]\frac{dr}{db}\right) = 0; \quad \frac{dr}{db} = f''(k)\frac{dk}{db} > 0. \tag{77}$$

Condition (77) indicates and Diagram 9 illustrates that thrifty agents benefit from capital-intensities below the golden rule level. Impatient agents prefer capital-intensities exceeding golden rule levels. This is in fact the opposite of the result derived by Stein (1969) p. 141, who considers a competitive economy with heterogeneous time preferences just like ours and concludes his analysis:[24] *"No interpersonal comparisons of utility are involved in this concept of an optimum since* (7)

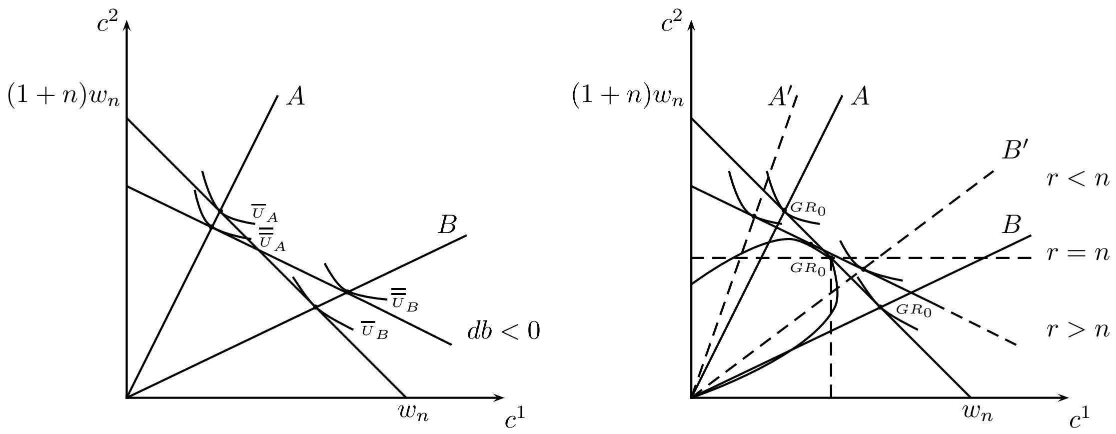
[23]Once again, we obtain $\frac{dk}{db} < 0$ from the stability condition.

[24]See also footnote 2 in Stein (1969), *p*.142, where he assumes "For simplicity we assumed that $\beta_i = \beta$....". The subsequent optimality conditions would therefore not hold without this assumption.

$[r = n]$ is valid for all β_i where $1 > \beta_i > 0$. No other point than $r(k) = n$ would be chosen in a social compact entered into by all generations present and future." Instead, we find the following proposition:

Proposition 6. *In a competitive economy with heterogeneous preferences, there are always agents who prefer a capital intensity exceeding the golden rule level. Moreover, there are also agents who prefer a capital intensity falling short of the golden rule level. Consequently, changes in the capital intensity always require interpersonal utility comparisons.*

Proof. see Appendix 3.4.3. □



$9a$ $9b$

Diagram 9: *Dynamic efficiency and the Engel-curve.*
Households with a steep (flat) income expansion path lose (gain) utility once the rate of return falls short of the golden rule level. Diagram 9*a depicts the income expansion path with an intertemporal elasticity of substitution of unity but heterogeneous time preferences as in Stein (1969). Diagram* 9*b depicts a case where current and future consumption are complements where $s_r < 0$.*

Propositions 5 and 6 may be seen as trivial corollaries to Proposition 4. Once we note that the direct change in the tax rate is $r - n = 0$ at the golden rule equilibrium, we are left with the pure factor-price trade off $(k + b) - \frac{s_i}{(1+r)}$ described in Proposition 4. This trade off, however, is unrelated to a golden rule result. Each change in preferences or labor endowments has an effect on the desired capital intensity, i.e., the optimal wage-interest pair on the factor-price frontier.

In a different interpretation we may think of $(k + b) - \frac{s_i}{(1+r)}$ as a sort of terms of trade. Those agents whose labor supply absorbs many assets compared to their own saving, prefer a capital intensity exceeding the golden rule level and vice versa. Consequently, the decrease in utility incurred by some agents once the capital intensity moves towards the golden rule can be seen as a case of "immiserizing growth". Similar to the result on foreign trade derived by Bhagwati (1958), we find that an increase in the capital intensity towards the golden rule increases per capita consumption in each period. However, it also worsens the terms of trade of the thrifty households.

Taking the current view it is also intuitive that the golden rule result holds in the setting of section 3.2.1, where the labor supply was the sole source of heterogeneity, if preferences are homothetic and homogeneous. In this case, each agent supplies precisely the amount of savings necessary to absorb his proportional share of assets in the economy. Consequently, in our trade interpretation, each group of agents lives in "autarky". In this case, the golden rule result carries over.

Finally, we note that the golden rule once again also ceases to serve as a watershed in an economy without debt. Setting $b = 0$, the first order condition for the optimal steady state capital intensity is given by:

$$\frac{dU_i}{dk} = -U_{i,c^1}\left(k - \frac{s_i}{(1+r)}\right) = 0. \tag{78}$$

Once again it is now easy to show that a household of type i will prefer to be born into an economy where $r \gtreqless n$, depending on whether his propensity to save is above (below) the economy's average propensity to save.

3.2.4 Hicks Neutral Technological Change

To further illustrate how Proposition 4 carries over to different settings, we will now briefly reconsider the results which were derived regarding the adoption of a superior "Hicks neutral" technology. While Matsuyama (1991) and Galor (1988) have shown that such a technological innovation may only decrease utility if the economy is dynamically inefficient, we will now show that this result once again does not carry over to our heterogeneous agent setting.

Augmenting the production function with a technology level α, we define $\tilde{y} \equiv \alpha f(k)$, $\tilde{r} \equiv \alpha f'(k)$ and $\tilde{w} \equiv \alpha[f(k) - f'(k)k]$. Changes in α now affect utility according to:[25]

$$\frac{dU}{d\alpha} = U_{c^1}\frac{1}{\alpha}\left[\tilde{w} + \frac{\tilde{r}}{1+\tilde{r}}s\right] + U_{c^1}\left[\frac{s(\tilde{w}l,\tilde{r})}{1+\tilde{r}} - kl\right]\alpha f''(k)\frac{dk}{d\alpha} = 0. \tag{79}$$

The first term in (79) indicates that utility increases due to the outward shift of the factor-price-frontier. At a given capital intensity, wages and returns to capital rise. The second term represents the movement along the factor-price frontier, resulting from the changed capital intensity. This change may or may not increase the respective agent's utility obtained from his particular factor supplies $(l, s(\tilde{w}l, \tilde{r}))$. Disregarding the heterogeneity in the labor endowment, condition (79) is identical with that derived by Galor (1988). In the current case, however, the second term will always be negative for some agents even if $r > n$. Hence, the results of Galor (1988) and Matsuyama (1991), who show that technological progress may only decrease utility of some agents in an economy which is dynamically inefficient, do not carry over.[26]

If we think of our households as small open economies, the second term in (79) may once again be interpreted as a worsening of the "terms of trade" for thrifty households. In a similar fashion it would now be straightforward to show that the two-part golden rule ceases to serve its watershed role in similar problems, like for example the optimum growth rate for population analyzed in Samuelson (1975a), Michel and Pestieau (1993) and Jaeger and Kuhle (2009).[27] Rather than separating efficient equilibria from inefficient ones, the golden rule separates savers from non-savers as in Propositions 4, 5 and 6 respectively.

3.3 Conclusion

The two-part golden rule balances the wage interest tradeoff faced by the representative agent in the competitive Diamond (1965) model. Competitive paths with a

[25]Note that $\frac{dk}{d\alpha} = \frac{s_w(f(k)-kf'(k))+s_r f'(k)}{(1+n)+s_w\alpha f''(k)k - s_r\alpha f''(k)} \gtreqless 0$. Assuming asymptotic (cyclical) stability, we can show that the denominator is positive (negative). The numerator is positive if $s_r > 0$, but ambiguous if s_r is sufficiently negative.

[26]The notable exception is of course the case with identical homothetic preferences.

[27]The first order condition for the optimum growth rate for population for a type l agent is given by:

$$\frac{dU}{dn} = -U_{c^1}\left(kl - \frac{s(lw,r)}{1+r}\right)f''(k)\frac{dk}{dn} = 0, \tag{80}$$

and has the same structure as the foregoing ones.

capital intensity exceeding the golden rule level are therefore dynamically inefficient. This result carries over to economies where the present value of *each* agent's savings is equal to his proportional share of assets in the economy. This condition is satisfied in economies where all agents have linear Engel-curves with identical slopes, i.e. preferences which are homogenous and homothetic.

In all other cases, it turns out that the golden rule ceases to serve as a demarcation line between efficient and inefficient steady states. There are always some members in society who prefer a capital intensity exceeding the golden rule level. Hence, these steady states can no longer be viewed as inefficient.[28] Taking the perspective of Abel et al. (1989), we find that an increment in capital acts as a source (sink) to society as a whole, i.e. increases aggregate consumption in each period, if $r > n$ ($r < n$). If society consisted of a representative agent, $r = n$ would therefore describe the steady state optimum. However, in the current case society is fragmented into different groups. That is, while capital may already be a sink to society as a whole, it may still act as a source to some groups of that society.[29]

Given the growing interest in models with heterogeneous agents, the current result may provide useful inference in two directions: The simple comparison of interest rate and aggregate growth rate is feasible for economies with heterogeneous agents as long as consumption expansion paths are linear and identical, e.g. CES specifications with homogeneous time discount rates. In another interpretation, the above analysis may be a case against the reliance on CES specifications to the extent that empirical evidence suggests that the savings propensity increases (or at least varies) with income. Consequently, the intra-generational redistribution, which is a byproduct of changes in the capital intensity, is not captured.

[28] Steady states with a capital intensity falling short of the golden rule level are now efficient for a second reason. Even if a golden rule allocation could be reached at no cost, this new path would always bring about *long-run* utility losses for some agents.

[29] Naturally, this result carries over to competitive OLG models where risk avers agents face ideosyncratic risks.

3.4 Appendix

3.4.1 Construction of Diagram 6

In this appendix we develop the properties of the Diamond (1965) model in the $w-r$ space:

Factor-price frontier The firm's first order conditions are given by $r = f'(k)$, $w = f(k) - f'(k)k$. For $k \in R^+$, $f'(k)$ is bijective and defines $k = k(r)$. Differentiation now yields:[30]

$$\phi(r) = w(k(r)), \quad \frac{dw}{dr} = \phi'(r) = -k, \quad \frac{d^2w}{dr^2} = \phi''(r) = -\frac{1}{f''(k)}. \tag{81}$$

Indifference curves Regarding preferences, we recall (65) and (66), which imply $\frac{U_{c1}}{U_{c2}} = 1 + r$, to obtain:

$$\frac{dw}{dr}_{|dU=0} = b - \frac{s}{1+r}, \quad \frac{d^2w}{dr^2}_{|dU=0} = \frac{s_w s - s_r(1+r) + s}{(1+r)^2}. \tag{82}$$

Stability and life-cycle savings Throughout this chapter we assume that the equilibrium is asymptotically stable. Using the life-cycle condition $(1+n)(k_{t+1}+b) = s(w_t - (r_t - n)b, r_{t+1})$, the local stability condition around the steady state, where $k = k_t = k_{t+1}$, reads:

$$0 < \frac{dk_{t+1}}{dk_t} = \frac{-s_w f''(k)(k+b)}{(1+n) - s_r f''(k)} < 1. \tag{83}$$

For a stable economy, the locus ψ of $w-r$ pairs, where life-cycle savings support a steady state, i.e. $(1+n)(k(r)+b) = s(w - (r-n)b, r)$, is now given by:

$$\frac{dw}{dr}_{|\psi} = \frac{(1+n) - s_r f''(k)}{s_w f''(k)} + b <_{|(83)} -(k+b) + b = -k. \tag{84}$$

Hence, the savings locus ψ in the $w-r$ space is steeper than the factor-price frontier ϕ. Moreover, for $r > n$, ψ is also steeper than the indifference curve, i.e., $\frac{dw}{dr}_{|\psi} < -k = \phi'(r) = b - \frac{s}{1+n} <_{|r>n} b - \frac{s}{1+r} = \frac{dw}{dr}_{|dU=0}$. Varying per capita debt will now

[30] See Samuelson (1962) for an exposition of the factor-price frontier.

allow the planner to move both the savings locus and the households' indifference curve to a point where life-cycle savings support a golden rule steady state:

$$\frac{dw}{db}_{|dr=0,\psi} = \frac{1+n+(r-n)s_w}{s_w} > 0, \tag{85}$$

$$\frac{dr}{db}_{|dw=0,\psi} = \frac{-(1+n+s_w r)}{(1+n+s_w b f''(k) - s_r f''(k))\frac{1}{f''(k)}} > 0. \tag{86}$$

Hence, increasing debt shifts the savings locus ψ to the right. The households' indifference curves rotate around the point w_n, n according to:

$$\frac{dw}{db}_{|dU=0} = r-n \gtreqless 0; \quad \frac{dr}{db}_{|dU=0} = \frac{r-n}{b-\frac{s}{1+r}} \gtreqless 0. \tag{87}$$

Example Finally we may note an example which shows that condition (68), with $b=0$, is prone to corner solutions. Assuming a Cobb-Douglas production function, full depreciation and logarithmic utility, with a propensity to save out of wage-income of $0<\xi<1$, we have $y=k^\alpha$, $w=(1-\alpha)k^\alpha$, $1+r=\alpha k^{\alpha-1}$, and thus (68) reads:

$$\frac{dU}{dk} = -\frac{1}{c^1}\Big(1-\xi\frac{(1-\alpha)}{\alpha}\Big)k \gtreqless 0; \quad \frac{\alpha}{1-\alpha} \gtreqless \xi. \tag{88}$$

3.4.2 Comparative Statics

In this section we briefly show how the market clearing apparatus (54)-(59) carries over to an economy with heterogeneous agents. Moreover, we show that the stability condition implies that an increase in per capita debt decreases the capital intensity.

Integrating the per capita quantities yields the respective aggregates:

$$L_{t+1} = \int_{\check{l}}^{\hat{l}} (1+n)^t l dF(l) = (1+n)L_t, \tag{89}$$

$$\int_{\check{l}}^{\hat{l}} ls(w_t l - (r_t - n)bl)dF(l) = (1+n)(k_{t+1}+b), \tag{90}$$

$$\int_{\check{l}}^{\hat{l}} s_t(l)dF(l) = \int_{\check{l}}^{\hat{l}} (w_t l - c_t^1(l) + (r_t-n)bl)dF(l), \tag{91}$$

$$\int_{\check{l}}^{\hat{l}} (c_t^1(l) + \frac{c_{t+1}^2(l)}{1+r_{t+1}})dF(l) = \int_{\check{l}}^{\hat{l}} (w_t l - (r_t-n)bl)dF(l), \tag{92}$$

$$w_t = f(k_t) - f'(k_t)k_t, \tag{93}$$

$$r_t = f'(k_t). \tag{94}$$

Defining $C_t^1 \equiv \int_{\check{l}}^{\hat{l}} c_t^1(l)dF(l)$, $C_{t+1}^2 \equiv \int_{\check{l}}^{\hat{l}} c_{t+1}^2(l)dF(l)$, $W_t \equiv \int_{\check{l}}^{\hat{l}} w_t l dF(l)$ and $S \equiv \int_{\check{l}}^{\hat{l}} s_t(l)dF(l)$, we obtain the same set of equations as in (54)-(59). The geometric exposition in Diagram $5a, 5b$ can now be constructed as before. However, C^1, C^2 now depict *average* consumption rather than the consumption of the *representative* consumer.

Lemma 1: *An increase in per capita debt decreases the steady state capital intensity if the equilibrium exhibits Walrasian stability.*

Proof: In per capita terms the life-cycle savings condition (90) reads $\int_{\check{l}}^{\hat{l}} s\Big(w(k_t)l - (r(k_t)-n)bl, r(k_{t+1})\Big)dF(l) = (1+n)(k_{t+1}+b)$. Differentiation now yields the following (Walrasian) stability condition: $0 < \frac{dk_{t+1}}{dk_t} = \frac{-(k+b)f''(k)\int_{\check{l}}^{\hat{l}} s_w(wl-(r-n)bl,r)ldF(l)}{(1+n)-f''(k)\int_{\check{l}}^{\hat{l}} s_r(wl-(r-n)bl,r)dF(l)} < 1$. Government debt will therefore change the steady state capital intensity according to: $\frac{dk}{db} = \frac{(1+n)+\int_{\check{l}}^{\hat{l}} s_w(f'(k)-n)ldF(l)}{f''(k)\int_{\check{l}}^{\hat{l}} s_r dF(l)-(1+n)-(k+b)f''(k)\int_{\check{l}}^{\hat{l}} s_w ldF(l)} < 0$, where the numerator is positive since $0 < s_w < 1$. By utilizing the stability condition, it is straightforward to show that the denominator is negative.

Proof of Proposition 5 With general non-homothetic preferences, we have $\frac{\partial s(wl,r)}{\partial l} \gtreqless \frac{\partial s(w\tilde{l},r)}{\partial \tilde{l}}$ for some $l, \tilde{l} \in [\check{l}, \hat{l}]$. Consequently, the life-cycle savings condition $\int_{\check{l}}^{\hat{l}} s(wl,r)dF(l) = \int_{\check{l}}^{\hat{l}} (1+n)(k+b)ldF(l)$ is only satisfied if the savings of some agents exceed $(1+n)(k+b)l$ and other agents' savings fall short of what they absorb. Assuming the contrary, $s(wl,r) > (1+n)(k+b)l$, for all $l \in [\check{l}, \hat{l}]$ yields $\int_{\check{l}}^{\hat{l}} s(wl,r)dF(l) > (1+n)(k+b)$, which contradicts the steady state condition $\int_{\check{l}}^{\hat{l}} s(wl,r)dF(l) = (1+n)(k+b)$. Thus, according to (73) there will always be two groups of agents. One group preferring a capital intensity exceeding the golden rule and another preferring a lower capital intensity where $r > n$. Those preferring the golden rule are of course a measure zero set.

Proof of Corollary 4 For homothetic preferences savings are known to be a positive fraction ξ of wealth, i.e. $s = \xi(r)(w-(r-n)b)l$.[31] Consequently, the golden rule steady state life-cycle savings condition in (71) reads $\xi(n)w(k_n) = (1+n)(k_n+b)$. Hence, we have $\frac{s(l)}{1+r} = \frac{s(l)}{1+n} = (k_n+b)l$. Consequently condition (73) is satisfied for

[31] See Mas-Colell et al. (1995) p. 50 or De La Croix and Michel (2002) p. 53-54 for a proof.

all agents at the golden rule capital intensity.

Bismarck Pensions If the debt scheme is Bismarckian, where τ^s denotes the contribution rate, indirect utility can be written as:

$$U = U\Big((1-\tau^s)wl - s, (1+r)s + (1+n)\tau^s wl\Big). \tag{95}$$

Hence, we have the following relation between the contribution rate and welfare:

$$\frac{dU}{d\tau^s} = -U_{c^1}\frac{r-n}{1+r}l\Big(w - \tau^s\frac{dw}{d\tau^s}\Big) - U_{c^1}\Big(kl - \frac{s}{1+r}\Big)f''(k)\frac{dk}{d\tau^s}. \tag{96}$$

Once again the golden rule conclusion would require that each agent saves precisely what he absorbs. Consequently, taking into account that the life-cycle savings condition $(1+n)k = \int_{\underline{l}}^{\hat{l}} s(wl, r, \tau^s)$, Proposition 5 and Corollary 4 carry over.

3.4.3 Proof of Proposition 6

In per capita terms, the steady state equations are given by:

$$\int_0^1 s_i di = (1+n)(k+b), \tag{97}$$

$$\int_0^1 (c_i^1 + \frac{c_i^2}{1+r})di = w - (r-n)b, \tag{98}$$

$$\int_0^1 s_i di = w - \int_0^1 c_i^1 di + (r-n)b, \tag{99}$$

$$w = f(k) - f'(k)k, \tag{100}$$

$$r = f'(k). \tag{101}$$

The proof of Proposition 6 is now straightforward: If savings are heterogeneous, we have $s^i \neq s^j$ for some $i, j \in [0,1]$. Consequently, the life-cycle savings condition $\int_0^1 s^i di = (1+n)(k+b)$ is only satisfied if some agents' savings exceed $(1+n)(k+b)$ and other agents' savings fall short of the average. Assuming on the contrary $s^i > (1+n)(k+b)$ for all $i \in [0,1]$ yields $\int_0^1 s^i di > (1+n)(k+b)$, which contradicts the steady state condition $\int_0^1 s^i di = (1+n)(k+b)$. Thus, according to (77) there will once again be two groups of agents: one group preferring a capital intensity exceeding the golden rule and another one preferring a lower capital intensity where $r > n$. Those preferring the golden rule are, again a measure zero set.

4 The Optimum Structure for Government Debt

This chapter studies the structural differences between implicit and explicit government debt in a two-generations-overlapping model with stochastic factor-prices. If a government can issue safe bonds and new claims to wage-indexed social security to service a given initial obligation, there exists a set of Pareto-efficient ways to do so. This set is characterized by the conflicting interests of the current young and the yet unborn generations regarding the allocation of factor-price risks.

However, it is shown that there will always exist a simple intertemporal compensation mechanism which allows to reconcile these conflicting interests. This compensation mechanism narrows the set of Pareto-efficient debt structures until only one remains. This result hinges on the double-incomplete markets structure of stochastic OLG models where households can neither trade consumption loans nor factor-price risks privately.

4.1 Introduction

Privatizing social security has often been described as a pure "shell game", where an implicit liability is replaced by an explicit liability of equal size.[32] From a different perspective, this equivalence between implicit and explicit government debt, may also be seen as a counterpart to the Modigliani-Miller Theorem in corporate finance. The underlying argument for this irrelevance result has its roots in the consumption loan nature of both debt instruments. A pure reallocation of resources between two adjacent cohorts can at most yield the biological interest rate.[33] For a deterministic economy, which is dynamically efficient in the sense of Diamond (1965), bonds are issued with a rate of return that is, at first sight, superior to the biological return earned on social security contributions. However, to prevent an eventual default, the government has to collect a tax that exactly offsets this return advantage. Taking these taxes into account, both instruments yield identical

[32]See e.g. Breyer (1989), Fenge (1995), Belan and Pestieau (1999), Friedman (1999). See Sinn (2000) for a survey. Samuelson (1975b) proves the related result that fully funded social security is also neutral. More recently, Ludwig and Reiter (2009) have extended the result to a stochastic setting with state dependent taxes.

[33]Samuelson (1958, 1959), Lerner (1959), Aaron (1966) and Cass and Yaari (1966). In the sequel, we abstract from technological progress as it does not change the basic tradeoffs.

allocations.[34] In particular, they reduce long-run utility by crowding-out capital.

In stochastic overlapping generations models Enders and Lapan (1982), Merton (1983), Gordon and Varian (1988), Gale (1990), Krüger and Kubler (2006) and Gottardi and Kubler (2008) have shown that intergenerational transfers via PAYGO pension schemes and safe government debt may serve a second role. They allow to facilitate intergenerational risk sharing.[35] In-turn, these beneficial aspects of government debt have been compared to the negative long-run losses which stem from the crowding-out of capital. In particular Green (1977), Krüger and Kubler (2006) and Gottardi and Kubler (2008) examine this trade-off between risk sharing and worsening factor-prices. Their analysis indicates that even the introduction of a very small social security system tends to decrease long-run utility. That is, the positive risk sharing effect is dominated by the negative crowding-out effect.[36]

The current analysis complements this literature by taking a different perspec-

[34]Both schemes pay the same returns, cause (in absence of intragenerational redistribution) the same excess burdens in the labor market, reallocate the same amount of resources between generations, displace an equal amount of private savings and lower long-run utility.

[35]In particular, Fischer (1983) and Gale (1990), discuss the desirability of safe debt and its maturity structure in an OLG context with rate-of-return risk. Enders and Lapan (1982) examine a mature pay-go scheme in an economy where fiat money is the only alternative store of value. Merton (1983) derives closed-form solutions for a three period OLG model with simultaneous demographic, TFP and income share risks. He shows that a tax and transfer system may replicate an (incomplete markets) equilibrium where agents can trade human capital freely. In the Merton (1983) setting such an intervention is always warranted as young agents would starve under "total market failure". Bohn (1998, 2003) shows that a constant debt to GDP ratio leads to pro-cyclical debt issues, that amplify aggregate risks. Starting from a situation without government debt, Krüger and Kubler (2006) give numerical evidence that the introduction of unfunded social security is unlikely Pareto-improving − despite its risk sharing capacities − due to the crowding-out of capital. Gottardi and Kubler (2008) discuss the prospects of an ex-ante Pareto-improving introduction of unfunded social security in an economy with land. See Diamond (1977, 2000) for a broader assessment of intergenerational and intragenerational insurance aspects of social security, and Shiller (1999) for more references on the sharing of aggregate risks. See Abel (2001a), Diamond and Geanakoplos (2003), and Ball and Mankiw (2007) for different approaches to utilize trust-fund assets − a question somewhat related to the present one. To focus firmly on the unfunded component of social security we will not introduce a trust-fund. Moreover, we leave-out idiosyncratic risks. As Bester (1984) and Abel (1989) show these can be insured within each cohort, i.e. are not essential in the current context.

[36]Intuitively this result is plausible if we think of it in terms of the Finetti (1952), Pratt (1964), Arrow (1970) approximation: $E[U(c_0+\varepsilon)] \approx U(c_0) + U'(c_0)\mu_\varepsilon + \frac{1}{2}U''(c_0)\sigma_\varepsilon^2$. The crowding-out of capital induces first order welfare losses by lowering expected consumption μ_ε. The risk sharing benefits, however, are only of second order. For the above approximation we have used the approximation $\sigma_\varepsilon^2 = E[\varepsilon^2] - E[\varepsilon]^2 \approx E[\varepsilon^2]$, which is accurate if $E[\varepsilon]$ is small. For $E[\varepsilon] = 0$ we have $E[U(c_0+\varepsilon)] \approx U(c_0) + \frac{1}{2}U''(c_0)\sigma_\varepsilon^2$. In this case, the lower consumption would be associated with a reduction in c_0.

tive. We ask whether it is possible to restructure the vast debt which is already present in most countries in a Pareto-improving manner. Following this question, we show that it is possible to separate the crowding-out effect from the risk sharing problem. Changes in the composition of the public debt leave *expected* intergenerational transfers constant over time but alter the allocation of factor-price risks between different cohorts. Changes in the size of the debt change intergenerational transfers but tend to leave the allocation of factor-price risks unaltered. This *separation* of crowding-out and intergenerational risk sharing associated with public debt will in general allow the government to make a restructuring of the debt Pareto-improving. To derive this result, we set up an initial value problem. Each member of an initial old generation holds claims from past pension promises and debt issues amounting to g_0. The government can now raise a share λ of the revenue needed to service these claims through the introduction of a linear social security tax on the current young generations wage income. The remainder share $1 - \lambda$ has to be financed by selling safe debt. Finally, there is the group of yet unborn generations who have to service future pension claims issued to the current young generation.

There are two corollaries to the separation result sketched earlier: (i) if the government can only change the composition of the existing debt, there will be a set of efficient debt structures and another set of inefficient ones. The efficient set is characterized by the conflicting interests of those agents who are currently young and those who are yet unborn. The unborn generations benefit from the ex-ante diversification of their wage risk if a large share λ of the initial debt is injected into social security. The current young, who have already observed their wage income, on the contrary prefer safe debt, i.e. safe retirement benefits. (ii) if the government can also issue/recover additional bonds, i.e. change the size of the expected future intergenerational transfers, the set described in (i) can be narrowed to only one Pareto-optimal debt structure, which maximizes societies (ex-ante) "Marshallian surplus" from intergenerational risk sharing. Put differently, the government can use its two instruments, i.e. the size and the composition of the debt, to steer the economy towards a point on the contract curve.

This second result appears to be of particular interest, when compared to the problem of optimal capital accumulation in a deterministic Diamond (1965) model. In analogy to our result (i), there always exists a set of efficient capital intensities. This means that every change in the capital intensity requires a welfare criterion as we can either shift resources into the future or redirect resources from the future towards current generations.[37] In the present stochastic setting, however, we show that it is possible to compensate intertemporally. We can shift resources and risks between the current young and the yet unborn members of society *simultaneously* and *independently*. As a consequence, the government can compensate intertemporally and narrow the set of efficient debt structures (without compensation) to the set of points on the contract curve (with compensation).

Regarding our assumptions, a notable aspect of our analysis is that we rule-out state-contingent lump-sum transfers. Following Merton (1983), Gordon and Varian (1988), Bohn (1998, 2003), Krüger and Kubler (2006) and Gottardi and Kubler (2008) we try to capture the basic features of most real-world pension and debt schemes by limiting the government debt instruments to safe bonds and a linear social security contribution rate on wages. We do so for two reasons: (*i*) while state-contingent lump-sum transfers may allow to reach better allocations than our simplistic debt instruments, they are not observed in actual policy. (*ii*) The optimal allocations which are derived for such state-contingent tax and transfer systems usually imply that the public debt follows a random walk as described in Gordon and Varian (1988) and Ball and Mankiw (2001, 2007).[38] Hence, if the government would actually implement these policies, it would default in finite time with probability one. One may therefore argue that such a risk sharing policy amplifies rather than dampens the small risks faced by each generation as they create a tremendous default risk.

Subsequently, in Section 4.2 we begin by laying out our model. The representative households, are assumed to maximize expected utility. Moreover, first and second period consumption are assumed to be normal goods. Savings can be invested in a risky and a safe production technology. Wages are determined according to a third risky technology. As in Diamond and Geanakoplos (2003), it is assumed that aggregate investment does not affect marginal returns. This tri-linear setting

[37]The lack of such a compensation mechanism led to the turnpike literature, see, e.g., Samuelson (1968) or Blanchard and Fischer (1989). The absence of such an intertemporal compensation mechanism is of course also the reason for the intertemporal efficiency of pay-go schemes that we have been referring to in Footnote 32.

[38]Gordon and Varian (1988), p. 192, and Ball and Mankiw (2001, 2007) (Proposition 2), point out that their debt schemes that reallocate risks “optimally” imply that per capita debt will follow a random walk. Hence per-capita debt will hit any boundary in finite time. Consequently, as Gordon and Varian (1988), p. 192 point out, the economies total assets will eventually be negative, forcing the government to default at some point.

will help us to bring out the underlying economic mechanisms more clearly.[39] In a different interpretation we may think of our model as a small open economy. Subsequently, the budget constraints of the social security system and the treasury are introduced. With the model in place, the two main results (i) and (ii) are derived in Section 4.2. In Section 4.3, we show that our results carry over once some of the restrictive assumptions made in Section 4.2 are relaxed. Namely, the assumption of a constant risk-free rate will be dropped. Moreover, we consider a defined benefit social security system, and briefly touch upon an economy with intra-cohort heterogeneity. Section 4.4 offers concluding remarks.

4.2 The Model

In this chapter we first introduce our assumptions regarding technology and preferences. Subsequently, we trace out the preferences of the current young and the yet unborn generations regarding the composition of the debt. The key results on the efficiency of different debt schemes are derived towards the end of the chapter.

4.2.1 Population and factor-prices

The economy is inhabited by two-period-lived agents that form overlapping generations. During the first period of life each agent supplies one unit of labor inelastically. Population evolves according to:

$$N_{t+1} = (1+n)N_t, \tag{102}$$

where N_t is the size of the cohort born in period t and $1+n$ is the number of children raised by each member of cohort t.

The wage rate w_t and the interest rate to risky capital R_t are both stochastic. They follow an exogenously given, serially i.i.d., distribution. The stochastic wage rate w_t realized in period t has a lower bound $\breve{w} > 0$. Risky investments have the limited liability property, i.e. $\check{R}_t \geqq -1$. Furthermore the rate of return R_t may be correlated with the wage rate w_t, i.e. $cov(w_t, R_t) \gtreqless 0$.[40] In our baseline specification

[39] As the per capita size of *expected* intergenerational transfers will be kept constant over time we do not expect large changes in aggregate savings once implicit debt is replaced by explicit debt (cf. Diamond (1996)). Hence the crowding-out effects along the neoclassical competitive factor-price-frontier, which are so notable when *additional* debt is issued, do not come into play in the current analysis.

[40] In Appendix 4.5.4, we discuss the different types of risks involved. We point out that it is not implausible to assume that $cov(w, R) < 0$.

we assume that the safe rate r is exogenously given; respectively defined by a safe linear technology. In the sequel we also assume that $\check{R} < r < E[R]$, such that both risky and riskfree assets may be held by risk-avers investors. In Section 4.3, we relax the assumption of a constant riskfree rate.

4.2.2 Implicit and Explicit Government Debt

The government can interact with the competitive economy both via an unfunded pay-as-you-go social security system and through the intertemporal budget constraint of the treasury. While both of these schemes may be used to roll over debt, they differ with respect to the way that wage-income is taxed.

An unfunded social security system with a contribution rate τ^s and per capita benefits p is characterized by its budget constraint:

$$\tau_t^s w_t N_t = p_t N_{t-1}. \tag{103}$$

Using the biological interest rate relation (102), constraint (103) can be rewritten, such that per capita pension benefits are given by:

$$p_t = (1+n)\tau_t^s w_t. \tag{104}$$

Equation (104) indicates that an agent born in period t will contribute an amount $\tau^s w_t$ to the pension system in exchange for uncertain future benefits $(1+n)\tau_t^s w_{t+1}$. In terms of expectations, the consumption loan scheme will grow at rate n if the contribution rate is fixed. In this case, it remains constant in per capita terms:

$$E_{w_{t+1}}[p_{t+1}] = (1+n)\tau^s E_{w_{t+1}}[w_{t+1}]. \tag{105}$$

The second channel through which the government can roll over debt is the treasury's budget constraint. Denoting the total amount of outstanding debt by B_t, the amount of claims that are due in period $t+1$ by B_{t+1} and the treasury's tax rate by τ_t^t, the treasury's intertemporal budget constraint for period t is:

$$B_{t+1} = (1+r_{t+1})(B_t - N_t \tau_t^t w_t). \tag{106}$$

Defining debt per worker by $b_t \equiv \frac{B_t}{N_t}$ and substituting (102) into (106) yields:

$$(1+n)b_{t+1} = (1+r_{t+1})(b_t - \tau_t^t w_t). \tag{107}$$

If no taxes were levied, per capita debt would grow at a proportional rate of $\frac{r_{t+1}-n}{(1+n)}$, from period t to period $t+1$. To ensure that in per capita terms no additional debt is passed forward from generation t to generation $t+1$, the treasury has to collect taxes from generation t amounting to:

$$\tau_t^t w_t = \frac{r_{t+1} - n}{(1 + r_{t+1})} b_t. \tag{108}$$

Taxes are either positive or negative depending on whether the returns to intergenerational redistribution dominate market returns, i.e. if $r \gtreqqless n$.[41]

4.2.3 The Structure of Government Debt

At the beginning of time there is an initial generation -1 of retirees and a generation 0 of workers. The generation of retirees holds per capita claims to an existing social security system and/or from past issues of government debt, amounting to g_0. To service these claims the government has to raise a revenue of $\frac{g_0}{1+n}$ from each member of generation 0. A share $\lambda \in [0, 1]$ of the needed revenue can now be raised via the initiation of an unfunded pension scheme with a defined contribution rate τ^s:[42]

$$\tau_0^s w_0 N_0 = \lambda g_0 N_{-1}, \quad \Leftrightarrow \quad \tau^s = \tau_0^s = \frac{\lambda}{w_0} \frac{g_0}{(1+n)}. \tag{109}$$

The remainder share $(1 - \lambda)$ can then be raised by issuing safe government bonds:

$$(1 - \lambda) g_0 N_{-1} = B_0, \quad \Leftrightarrow \quad (1 - \lambda) \frac{g_0}{(1+n)} = b_0. \tag{110}$$

Recalling (108), per capita taxes in period 0 must satisfy:

$$\tau_0^t = (1 - \lambda) \frac{(r_1 - n)}{(1 + r_1) w_0} \frac{g_0}{(1+n)}. \tag{111}$$

Once we do not ask any future generation to redeem the debt, all subsequent generations will be taxed according to:

$$\tau_t^t = (1 - \lambda) \frac{r_{t+1} - n}{(1 + r_{t+1}) w_t} \frac{g_0}{(1+n)}. \tag{112}$$

[41] The taxes needed to keep per capita debt from growing to infinity, will be paid by the young consumers. However, as long as the representative agent invests into the riskfree technology, he will be indifferent between a tax of $\frac{(r_{t+1}-n)}{1+r_{t+1}} b$ when young or a tax of $(r_{t+1} - n)b$ when old.

[42] Note that as with the explicit debt scheme, the amount resources transferred via social security may not permanently outpace the economy. At the same time lowering the contribution rate would amount to a repayment of some debt by the affected generation of retirees. To make both schemes feasible and comparable, we therefore fix τ^s.

Inspection of (109) and (112) immediately yields the equivalence proposition that we have been referring to in the introduction.[43] In what follows, we drop the time index where no misunderstanding is expected.

4.2.4 The Optimum Structure for Government Debt

In this section we start by tracing out the preferences of the current young regarding the structure for government debt λ. Subsequently, we characterize the interests of the yet unborn generations. With these results at hand, the two main results are derived in Section 4.2.5. A representative member of cohort 0 can allocate his net income to first period consumption c^1, invest an amount a_0 into the safe technology and devote h_0 to the risky technology:

$$\begin{aligned} \max_{c^1,c^2} \quad & W = U(c^1) + \beta E_{wR}[U(c^2)]; \quad U'() > 0, \quad U''() < 0, \qquad (113) \\ s.t. \quad & c^1 = w_0(1-\tau_0^t-\tau_0^s) - a_0 - h_0, \\ & c^2 = a_0(1+r) + h_0(1+R_1) + \tau_0^s w_1(1+n). \end{aligned}$$

The corresponding first order conditions, which imply a_0^* and h_0^*, are:

$$\frac{\partial W}{\partial a_0} = -U'(c^1) + \beta(1+r)E_{wR}[U'(c^2)] = 0, \qquad (114)$$

$$\frac{\partial W}{\partial h_0} = -U'(c^1) + \beta E_{wR}[(1+R)U'(c^2)] = 0. \qquad (115)$$

If felicity, $U()$ in (113), is such that first and second period consumption are normal goods we have:[44]

$$s = s(w;\tau^s) = a + h; \quad 0 < \frac{\partial s}{\partial w} < (1-\tau^s). \qquad (116)$$

Equipped with these conditions, the social planner can, disregarding the utility of subsequent generations for the moment, use the two debt instruments by choosing

[43] In the standard Diamond (1965) economy, the steady state budget constraint of the representative agent reads $c^1 + \frac{c^2}{1+r} = w(1-\tau^s-\tau^t) + \frac{\tau^s w}{1+r}(1+n)$. Plugging the two budget constraints of the treasury (112) and the social security administration (109), with $w_0 = w$, into this budget constraint yields for the right-hand-side: $w - \frac{r-n}{1+r}\frac{g_0}{1+n}(1-\lambda) - \frac{g_0}{(1+n)}\lambda + \lambda\frac{g_0}{1+r} = w - \frac{(r-n)g_0}{(1+r)(1+n)}$. The life-cycle savings condition is also independent of λ: $(1+n)(\lambda\frac{g_0}{1+n} + (1-\lambda)\frac{g_0}{1+n} + k) = g_0 + (1+n)k = s$. Hence, changing the debt structure along the steady state, is irrelevant as it neither affects the household's budget constraint nor the life-cycle savings condition.

[44] The increment in income from a high realization of w_t is given by $(1-\tau^s-\tau^t(w_t)) + \frac{\partial\tau^t(w_t)}{\partial w_t}w_t = (1-\tau^s)$.

λ such that the indirect utility of generation 0 is maximized. Taking into account the budget constraints (109) and (111) yields the planning problem:[45]

$$\begin{aligned}\max_{\lambda} V_0 = \quad & U(w_0(1-\tau_0^s-\tau_0^t)-a_0-h_0) \\ & +\beta E_{wR}[U(a_0(1+r)+h_0(1+R)+\tau_0^s w(1+n))], \\ s.t. \quad & (109), \quad (111).\end{aligned} \tag{117}$$

Utilizing the envelope condition (114) and the covariance rule, λ^* is implicitly defined by:

$$\frac{dV_0}{d\lambda} = \frac{U'(c^1)g_0}{1+r}\Big(\frac{E[w]-w_0}{w_0} + \frac{cov_{wR}(U'(c^2),w_1)}{w_0 E_{wR}[U'(c^2)]}\Big) = 0. \tag{118}$$

Condition (118), which is reminiscent of the C-CAPM, indicates that members of generation 0 will benefit from a high fraction of debt that is injected into the social security system as long as the expected excess rate-of-return on this fraction of debt, compared to the after-tax-return on safe bonds, is positive, i.e. $\frac{Ew-w_0}{w_0} > 0$. The other relevant component is the covariance between second period marginal utility and the pension benefit. Depending on $cov(R_1, w_1) \gtreqless 0$, we have $cov(U'(c^2), w_1)_{|\lambda=0} \lesseqgtr 0$, i.e. the wage-indexed social security claims may or may not be a welcome opportunity to diversify stock market risks.

Subsequent Generations The social planner's perspective on the welfare of subsequent generations, which is obviously connected to the current choice of λ, will be an ex-ante perspective. While the social planner knows the distribution over R and w, the realizations are yet unknown. The agents, however, will start to make their consumption savings decisions in period t *after* w_t has been realized. The consumer's behavior is therefore still characterized by conditions (114) and (115) which imply the wage dependent investment decisions $a_t = a_t(w_t;\lambda)$ and $h_t = h_t(w_t;\lambda)$. Put differently, the social planner, who optimizes ex-ante utility, has to take note of the agent's investment decisions conditional on the realization of w_t. Moreover, the budget constraints (109) and (112) have to be satisfied in each period. From the

[45]Note that there is no life-cycle savings condition for bonds and capital in a small open economy, i.e. we only take note of the taxes that are needed to keep per capita debt from growing. In a closed economy with a tri-linear technology, we can also neglect the market clearing condition as long as agents demand safe investments in excess of the debt offered. In the following we assume that agents are equating at the margin, i.e. we omit the prospect of Kuhn-Tucker-type ramifications.

perspective of period 0, the planning problem is therefore given by:

$$\max_{\lambda} V_t = E_{w_t}\Big[U(w_t(1-\lambda\frac{g_0}{w_0(1+n)}) - \frac{r-n}{(1+r)}\frac{(1-\lambda)g_0}{(1+n)} - a_t - h_t)\Big] \tag{119}$$
$$+\beta E_{w_t w_{t+1} R_{t+1}}\Big[U(a_t(1+r) + h_t(1+R) + \lambda\frac{g_0}{w_0}w_{t+1})\Big].$$

The first order condition for an optimum debt structure, taking the envelope conditions (114) and (115) into account (see Appendix 4.5.1), is then given by:[46]

$$\frac{dV_t}{d\lambda} = \frac{g_0}{(1+n)}\Big(\frac{n-r}{1+r}\frac{E[w]-w_0}{w_0}E_{w_t}[U'(c^1)] \tag{120}$$
$$-cov_{w_t}(U'(c^1), \frac{w_t}{w_0}) + \beta(1+n)cov_{w_t w_{t+1} R}(U'(c^2), \frac{w_{t+1}}{w_0})\Big) = 0.$$

Equation (120) characterizes the debt structure λ^{**} which maximizes long-run expected utility. Inspection of (120) indicates that agents who are not yet born will suffer a loss from excessive intergenerational redistribution if the safe returns exceed the biological returns on consumption loans. That is, the expected excess amount of resources − when compared to bonds which are not wage-indexed − that is redistributed via social security is given by $\frac{Ew-w_0}{w_0}$.[47] The second element is the intergenerational diversification of wage-income risk. With $\lambda > 0$ we have a positive social security tax rate τ^s, which transfers some of the risk associated with the realization of w_t into period $t+1$, where w_{t+1}, i.e. the pension benefits are realized. The sufficient condition for an interior optimum requires that $\frac{dV}{d\lambda}$ is downward-sloping in λ. A first inspection of (120) suggests $\frac{dcov(U'(c^1),w_t)}{d\lambda} > 0$, $\frac{dcov(U'(c^2),w_{t+1})}{d\lambda} < 0$, and therefore $\frac{d^2V}{d\lambda^2} < 0$. Hence, as we shift wage-income risk from the first into the second period, we expect the wage related covariance risk to move in the same direction (see Appendix 4.5.2 for the associated conditions). However, as the set of admissible debt structures is closed and bounded, there will always exist a "best" debt structure $\lambda^{**} \in [0,1]$.

The efficiency of the *size* of the debt scheme can be assessed once we ask whether the unborn generations benefit from a larger initial debt. Taking the first derivative

[46] Taking advantage of our assumption that the stochastic wage rate w_t is serially uncorrelated we may rewrite $cov_{w_t w_{t+1} R}(U'(c^2), \frac{w_{t+1}}{w_0}) = cov_{w_{t+1} R}(E_{w_t}U'(c^2), \frac{w_{t+1}}{w_0})$. If such a serial correlation existed, it would affect the location of λ^{**}. If a and h are normal, we have $\frac{da}{dw_t} > 0$ and $\frac{dh}{dw_t} > 0$; thus we would have a smaller λ^{**} if $cov(w_t, w_{t+1}) > 0$, and vice versa.

[47] The expected intergenerational transfer through social security is $E[\tau^s w] = \frac{g_0}{(1+n)w_0}Ew$. Regarding bonds, the transfer is $\frac{g_0}{(1+n)}$. The difference in the expected size of the transfers, which yield the inferior biological return, is therefore given by $\frac{g_0}{(1+n)}\frac{(Ew-w_0)}{w_0}$.

of V_t with respect to g_0 yields:

$$\frac{dV_t}{dg_0}_{|d\lambda=0} = \frac{n-r}{(1+r)(1+n)}\Big(\frac{w_0+\lambda(E[w]-w_0)}{w_0}\Big)E[U'(c^1)] \quad (121)$$
$$+\lambda\frac{1}{(1+n)}\Big((1+n)\beta cov(E_{w_t}[U'(c^2)],\frac{w_{t+1}}{w_0}) - cov(U'(c^1),\frac{w_t}{w_0})\Big) \gtreqless 0.$$

The first element in (121) is the familiar return condition; larger intergenerational reallocation of resources is desirable as long as consumption loans dominate market returns. The second element reflects the benefits from intergenerational risk sharing through the share λ of debt that is injected into the pension system. To see this more clearly, we recall (120) and rearrange (121) such that:

$$\frac{dV_t}{dg_0}_{|d\lambda=0} = \frac{n-r}{(1+r)(1+n)}E[U'(c^1)] + \frac{\lambda}{g_0}\frac{dV_t}{d\lambda} \gtreqless 0. \quad (122)$$

If λ is zero or at its long-run optimum λ^{**}, the second risk sharing related term vanishes and (122) exhibits the pure interest condition.

Furthermore, (122) indicates that safe debt does not reallocate risks, while social security does. This is the opposite of the Bohn (1998, 2003) conclusion, where debt was issued pro-cyclical such that it shifted risks towards future generations. Equation (122) also shows that if the national debt is small, then this debt should be injected entirely into the pension scheme if $\frac{dV_t}{d\lambda}, \frac{dV_0}{d\lambda} > 0$, such that the benefits from risk sharing are maximized with $\lambda = 1$. In a different interpretation, the sign of (122) is the subject studied by Green (1977), Krüger and Kubler (2006) and Gottardi and Kubler (2008).

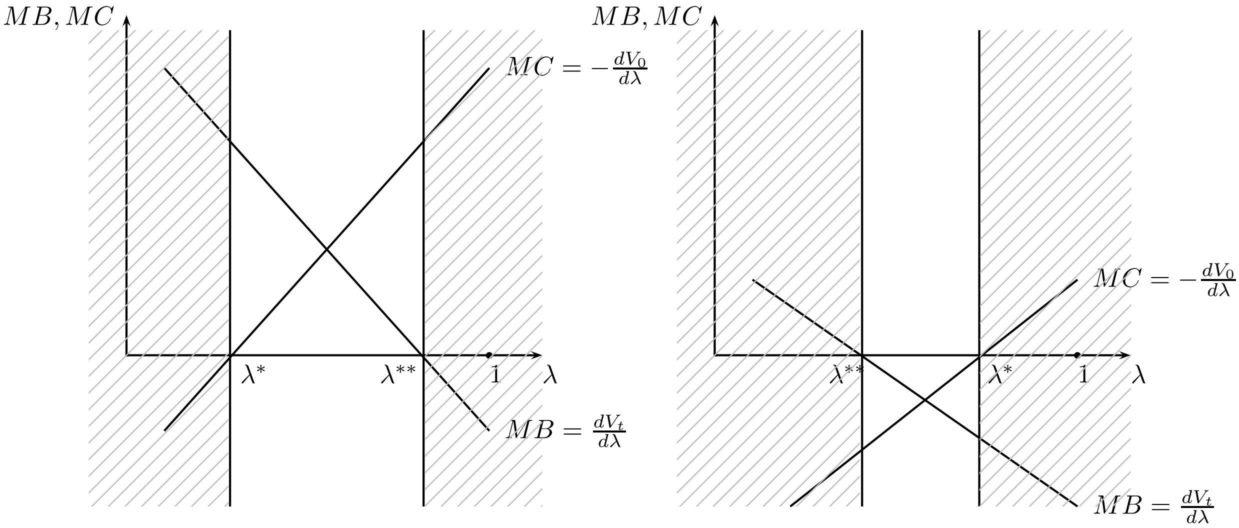
4.2.5 Efficiency

Inspection of our above analysis indicates that generation 0 will prefer a debt structure λ^*, that is a solution to (118), rather than λ^{**}, which solves (120).[48] If the government can control the composition of the public debt only, all debt structures located between λ^* and λ^{**} are Pareto-efficient. Raising λ beyond λ^* will increase expected utility of all unborn generations at the expense of generation 0. Starting with λ^{**}, the same applies when λ is lowered. Hence, we have the following proposition:

[48]For appropriate $(Ew - w_0, r - n, cov(w, R))$, λ^* may actually coincide with λ^{**}. In this case both generations prefer – though for different reasons – the same debt structure, and, except for choosing this structure, no additional government intervention is necessary. The same applies when corner solutions coincide.

Proposition 7. *If the government can only implement the debt structure that is used to roll over the initial debt, there exists a set* $[\lambda^*, \lambda^{**}] \subseteq [0, 1]$ *of efficient financing methods. This set is characterized by the conflicting interests of the current young and the yet unborn generations.*

Diagrams 10a and 10b illustrate this trade-off. We now trace out the set of Pareto-



10a 10b

Diagram 10: *Efficient debt structures.*
Diagrams 10a *and* 10b *illustrate the gains and losses of generation* 0 *and one representative member of the yet unborn generations. All debt structures located in the dashed area are inefficient. Diagram* 10b *depicts a situation that may occur if* $E[w] \gg w_0$ *and* $r \gg n$.

improving transitions from one debt scheme to another, which are available once the government can change both, the composition and the structure of the public debt. As we have stressed earlier, with these two instruments, it will be possible for the government to separate the risk sharing properties of the public debt from the crowding-out effect.

Efficiency with Government Intermediation Suppose now that the initial conditions are such that $\lambda = \lambda^* < \lambda^{**}$. In this case each member of the yet unborn generations is willing to accept a (slightly) higher level of public debt in exchange for a more favorable composition $\tilde{\lambda} > \lambda^*$ of the debt. At the same time members of the current young generation are willing to accept additional pension claims and safe bonds in exchange for the less favorable allocation of factor-price risks associated with $\tilde{\lambda}$. The government can now offer generation 0 to increase the per-capita (in terms of generation -1) size of the public debt by π. The new debt scheme has a per-capita (of generation 0) size of $\frac{g}{1+n} \equiv \frac{g_0+\pi}{1+n}$. The associated Lagrangian, which allows to trace-out the set of Pareto-improving pension reforms, is then given by:

$$\max_{\pi,\lambda,\mu} \mathcal{L} = V_0(\lambda, \pi) + \mu(V_t(\lambda, g) - \bar{V}); \quad V_t(\lambda, g) \geqq \bar{V} \equiv V_t(\lambda^*, g_0), \quad g \equiv g_0 + \pi. \tag{123}$$

Where the Lagrangian (123) consists of the indirect utility functions of the current young and the yet unborn generations which where discussed earlier in Section 4.2.4. The additional argument π in V_0 reflects that members of generation 0 receive additional safe consumption (after taxes) amounting to $(1-\lambda)\frac{1}{1+r}\pi$ and additional pension claims $\lambda\frac{\pi}{w_0}w_1$ once the debt scheme is increased in size. The partial derivative $\frac{\partial V_0}{\partial \pi}$ is therefore positive. Regarding future generations, we focus on the interesting case where resources are scarce and an increase per-capita debt alone is not Pareto-improving. That is, the partial derivative $\frac{\partial V_t}{\partial g}$, described in (122), is assumed to be negative. Finally, as per-capita debt does not grow over time it is sufficient to represent future generations using only one lagrangian multiplier μ. Regarding the first order conditions associated with (123) we have:

$$\frac{\partial \mathcal{L}}{\partial \pi} = \frac{\partial V_0}{\partial \pi} + \mu \frac{\partial V_t}{\partial g} = 0, \tag{124}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \frac{\partial V_0}{\partial \lambda} + \mu \frac{\partial V_t}{\partial \lambda} = 0. \tag{125}$$

Combining (124) and (125) we can drop the Lagrangian multiplier μ. The first order condition for the optimum structure for government debt λ^{***} is then:

$$\frac{\frac{\partial V_0}{\partial \lambda}}{\frac{\partial V_0}{\partial \pi}} = \frac{\frac{\partial V_t}{\partial \lambda}}{\frac{\partial V_t}{\partial g}}. \tag{126}$$

Condition (126) indicates that the optimum structure for government debt is associated with a point on the contract curve. It equalizes the marginal rates of

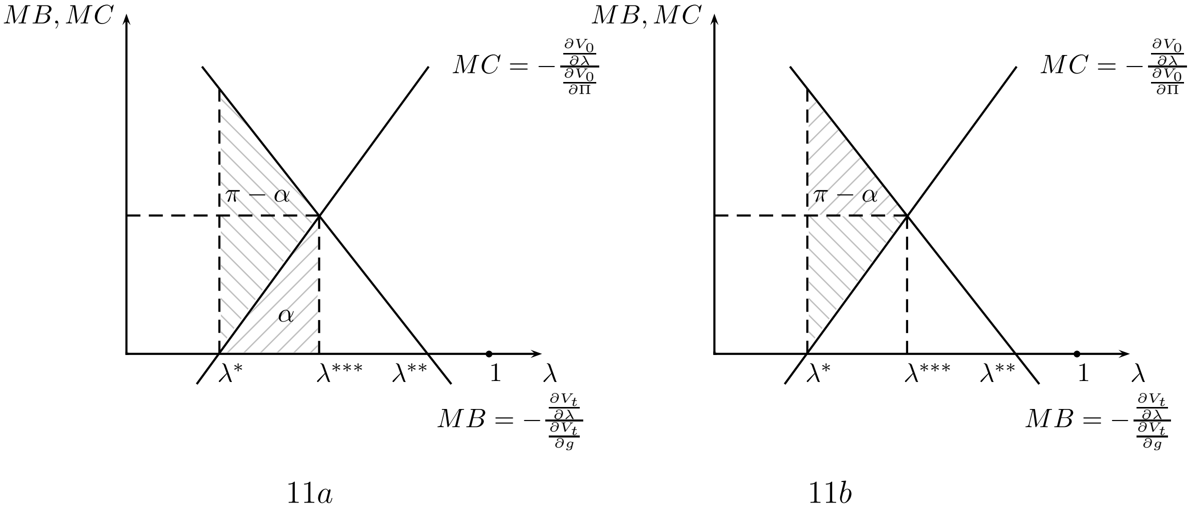


Diagram 11: *Efficiency gains from intertemporal compensation.*
Diagrams 11a *and* 11b *illustrate the compensation described in (126). In the case where* $\bar{U} = U_t(\lambda, g_0)$ *all efficiency gains* $\pi - \alpha$ *accrue to Generation* 0.

substitution between the burden of an additional unit of debt and risk sharing benefits between current and future generations. By varying the size and composition of the debt it is possible to recover the efficiency gains displayed in Diagram 11 in a Pareto-improving manner. We therefore have the following proposition:

Proposition 8. *If the government can vary both, the size of the public debt and its composition, it is possible to separate the crowding-out effect from the risk sharing properties of the public debt scheme. The efficiency gains associated with the optimum structure for government debt* λ^{***} *can be recovered in a Pareto-improving manner.*

Remark 1: *The optimum structure for debt* λ^{***} *may be at a corner solution.*

Remark 2: *Different reference levels* $\bar{V}_t$ *for the utility of future generations will change the distribution of the efficiency gains brought about by the implementation of* λ^{***}*. The associated income effects will slightly affect the location of* λ^{***}.

Remark 3: *If the initial debt structure is such that* $\lambda > \lambda^{***}$*, some of the efficiency gains associated with the implementation of* λ^{***} *can be passed forward to compensate the unborn generations. In this case, generation* 0 *gives up resources in exchange for lower labor income risk.*

Remark 4: *To keep in touch with the steady state as a reference point, Proposition 8 neglects the possibility of a repeated restructuring of the debt.*

Remark 5: *The golden rule of accumulation lends itself to the interpretation: maintaining a capital intensity that permanently exceeds the golden rule level is inefficient. In the present case we have a stronger result: maintaining any debt structure that permanently differs from* λ^{***} *is inefficient.*

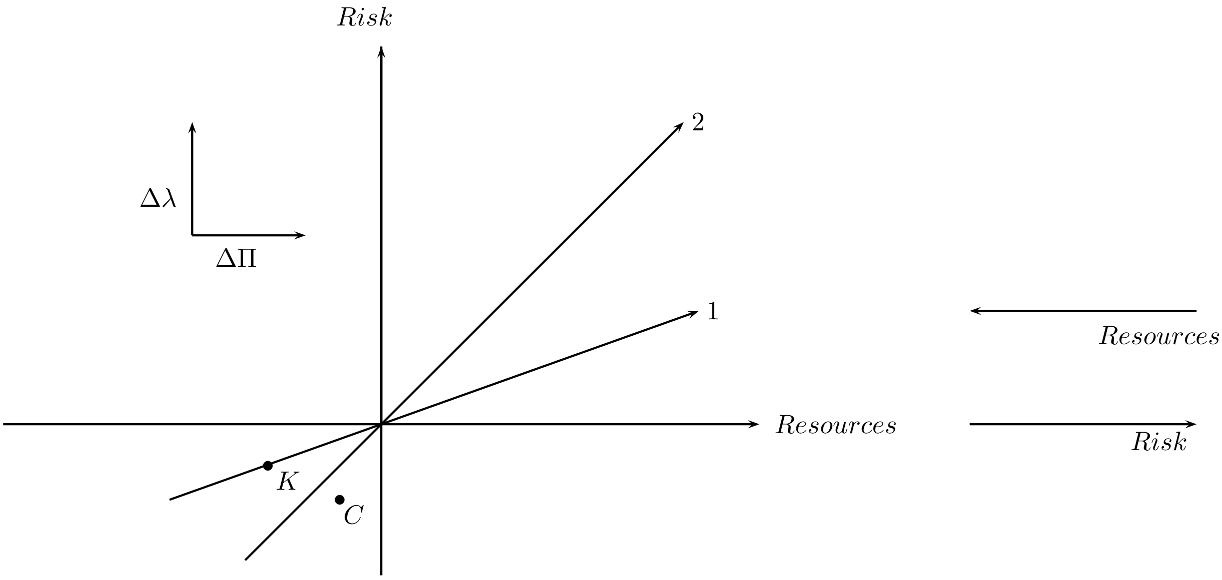


Diagram 12: *Separation of crowding-out and risk sharing*
A linear social security tax 1 *implies a combination of crowding-out and intergenerational reallocation of factor-price risks. Introducing a particular social security tax moves the economy from the origin to point K. Line* 2 *indicates the minimum reallocation of risks necessary to compensate future generations for the negative crowding-out effect. In the present case the government has two instruments available. It can therefore move freely in the risk-resource plane and implement the optimal allocation C.*

Interpretation At this point it is interesting to compare the present result on the possibility of Pareto-improving social security reforms with the earlier negative results by Green (1977), Krüger and Kubler (2006), Gottardi and Kubler (2008). In the case, where an initial debt is already present, a change in the composition of this debt reallocates factor-price risks but does not affect the size of the intergenerational transfer. By choosing λ^{***} as a debt structure it is now possible to tailor a particular exchange of risks and resources such that it is beneficial to both groups of agents. Namely, those living in the "long-run" and those who live today.

Diagram 12 illustrates this. Curve 1 represents the long-run consequences of a linear social security tax. As this tax increases, the economy moves from the origin to a certain point e.g. K. Curve 2, which is steeper than 1, shows the threshold where future generations are indifferent between the crowding-out of capital and the risk sharing benefits. Finally, point C is an allocation that can be reached in the manner described above: a change in the composition of the debt reallocates many risks via the linear social security tax. The change in the allocation of resources is mainly due to the change in the size of the debt π. Put differently, by introducing a linear social security tax alone the government can only move along arrow 1. If there is already an initial debt present it has two linearly independent instruments. In this case it can move in the entire plane, where point C is associated with an optimal pair λ^{***}, π^{***}.

Another Interpretation In a different interpretation (126) may be seen as an intertemporal version of the Samuelson (1954) condition for the efficient provision of a public good. Recalling equation (122) we can rewrite (126) such that:[49]

$$\begin{aligned} \frac{\frac{\partial V_0}{\partial \lambda}}{\frac{\partial V_0}{\partial \pi}} &= \frac{\frac{\partial V_t}{\partial \lambda}}{-E_{w_t}[U'(c^1)]\frac{r-n}{(1+r)(1+n)} + \frac{\lambda}{g}\frac{\partial V_t}{\partial \lambda}} \\ &= \sum_{t=1}^{\infty}\left(\frac{1+n}{1+r}\right)^t \frac{\frac{\partial V_t}{\partial \lambda}}{-E_{w_t}[U'(c^1)]\frac{1}{1+r} + \frac{\lambda}{g}\frac{\partial V_t}{\partial \lambda}\frac{1+n}{r-n}}. \end{aligned} \tag{127}$$

Condition (127) indicates that all future generations benefit from the public good “risk sharing” which is embodied in the debt scheme. The cost with the provision of this public good has to be incurred only once by generation 0, which bears additional wage-related risk. Depending on its position on the time axis, the present value of tax payments differs from cohort to cohort. The first element $-E[U'(c^1)]\frac{1}{1+r}$ in the numerator of the marginal rate of substitution of future generations indicates the negative crowding-out effect. The second element $\frac{\lambda}{g}\frac{\partial V_t}{\partial \lambda}$ is positive. As a share λ

[49]For $r > n$, we have $\sum_{t=1}^{\infty}\left(\frac{1+n}{1+r}\right)^t = \frac{1+n}{r-n}$. Note that the RHS of condition (126) is the marginal rate of substitution between an increase in λ and and increase of the debt level of one unit. The new formulation in (127) is the sum of the marginal rates of substitution between a marginal increase of λ and a marginal increase in the tax level. In Appendix 4.5.3 we develop the more intuitive case where only safe debt is used as a means of compensation. That is, the debt is not injected into the optimal debt scheme.

of the new debt π is injected into social security. This increases the willingness of future generations to accept a higher level of public debt.

The analogy to the problem of public good provision also extends to the aspect of income effects. Changing levels of $\bar{V}_t$ will require different compensation schemes. Hence, the exact location of λ^{***} depends on the particular compensation scheme as the associated income effects may slightly change preferences for λ, i.e. shift the marginal cost and benefit curves displayed in Diagram 11.

4.3 Extensions

So far attention was confined to an economy where the safe rate-of-return is constant over time. The prospects of a third debt instrument, namely a defined benefit social security system, have also been neglected. In a first step, we now show that a time-varying, safe rate-of-return does not alter the quality of the foregoing conclusions and that defined benefits are equivalent to safe bonds. Finally a second group of representative agents who do not invest in the stock market (risky technology) is introduced into our model. In this setting we show that both groups require different social security contribution rates, i.e. debt structures. If either is at a corner solution there is additional scope for an intragenerational reallocation of the public debt.

4.3.1 Time-Varying Safe Returns

To work out the pivotal elements, the safe rate of return was assumed to remain constant over time. However, the main results of our previous analysis carry over to an economy where r is now an i.i.d. random variable. Regarding generation 0, nothing is changed, i.e. the agents and the social planner start maximizing after r_1 is known. Except for the additional expectations regarding r the long-run planning problem (120) is also little changed:

$$\max_{\lambda} V_t = E_{w_t,r_{t+1}}\Big[U(w_t(1-\lambda\frac{g_0}{w_0(1+n)}) - \frac{r_{t+1}-n}{(1+r_{t+1})}\frac{(1-\lambda)g_0}{(1+n)} - a_t - h_t)\Big] + \beta E_{w_t w_{t+1} R_{t+1} r_{t+1}}\Big[U(a_t(1+r_{t+1}) + h_t(1+R) + \lambda\frac{g_0}{w_0}w_{t+1})\Big].$$

Employing the envelope conditions (114) and (115), yields:

$$\frac{dV_t}{d\lambda} = \frac{g_0}{(1+n)}\Big(E_{wr}\Big[\frac{r_{t+1}-n}{1+r_{t+1}}U'(c^1)\Big]\frac{w_0-E[w]}{w_0} - cov_{w_t r_{t+1}}(U'(c^1),\frac{w_t}{w_0}) + \beta(1+n)cov_{r_{t+1}w_t w_{t+1}R}(U'(c^2),\frac{w_{t+1}}{w_0})]\Big) = 0. \quad (128)$$

Due to the nature of the treasury's tax schedule (112), the initial interest rate r_1 does not, unlike the wage rate w_0, enter into the long-run first order condition. While there are now additional expectations regarding the safe rate-of-return, the principal structure of the first order condition is preserved. Regarding our Pareto-improving interventions that were discussed in Section 4.2.5, we note that the government can still reallocate gains and losses along its budget constraint. However, each compensation scheme will now require some sort of risk-taking.

4.3.2 Defined Benefits

We will now briefly show that a defined benefit system is equivalent to an explicit debt scheme. The budget constraint of a defined benefit system, which is used to roll over a fraction γ of the public debt, is given by:

$$\tau_t^{DB} w_t = \frac{\gamma g_0}{(1+n)}, \quad p_t^{DB} = \gamma g_0. \tag{129}$$

Once we recall that the young agent can consume c^1, invest an amount a into safe assets and an amount h into risky assets, the present value budget constraint is given by:

$$c_t^1 + a_t + h_t = w_t(1 - \tau_t^{DB} - \tau_t^t) + \frac{p_{t+1}^{DB}}{(1 + r_{t+1})}. \tag{130}$$

Utilizing (129) and (112) where $(1-\lambda)$ is replaced by $(1-\gamma)$, the right-hand side of (130) can now be rewritten such that:

$$c_t^1 + a_t + h_t = w_t - \frac{g_0(r_{t+1} - n)}{(1+n)(1+r_{t+1})}. \tag{131}$$

Hence the structure of debt γ is irrelevant, i.e. a defined benefit system is equivalent to a bond-financed debt scheme.

4.3.3 A Working Class

This final paragraph considers a society that is partitioned into a group of capitalists who are endowed with a large amount of efficient labor and a group of workers with a low labor endowment. While capitalists participate in the stock-market, workers invest in the safe technology only.[50,51] The working class is assumed to make up a fraction α of the population and each worker has only a fraction ϕ of the effective labor endowment of a capitalist. Hence, workers earn a fraction $\theta = \frac{\alpha\phi}{1+\alpha(\phi-1)}$ of aggregate wages. Consequently, with a linear social security tax, the debt rolled over on the shoulders of workers and capitalists is given by $g_0^w = \theta\frac{g_0}{1+n}$ and $g_0^c = (1-\theta)\frac{g_0}{1+n}$. Workers will now choose safe investment according to (114). The optimal shares of debt for the working class, $\lambda_w^*, \lambda_w^{**}$ are then characterized by (118) and (120), with the notable difference that $h = 0$.[52] For $Ew = w_0$, we therefore have $\frac{dV_0^w}{d\lambda_w}_{|\lambda_w=0} = 0$ and $\frac{dV_t^w}{d\lambda_w}_{|\lambda_w=0} > 0$ and $\frac{d^2V_t^w}{(d\lambda_w)^2} < 0$; i.e. a unique globally optimal debt structure λ_w^{***} exists if g_0 is large enough (see (140) in Appendix 4.5.2). If per capita debt $\frac{g_0}{1+n}\theta$ is not large enough to transport a sufficient amount of wage-related risk into the retirement period, we have $\lambda_w^{***} > 1$ and hence, $\frac{dV^w}{d\lambda_w}_{|\lambda_w=1} > 0$. Once $\lambda_c^{***} < 1$, bonds from the capitalists' debt scheme can be injected into the workers' pension scheme. If the capitalists, in turn, pay the implicit tax associated with this debt swap as a subsidy to the workers, the marginal increase in rent for workers is, recalling equations (122)-(126) with $\lambda_w = 1$, given by:

$$\frac{\partial \mathcal{L}^w}{\partial g^w} = \frac{1}{g^w}\frac{\partial V_0}{\partial g^w}\left(\frac{\frac{\partial V_0^w}{\partial \lambda^w}}{\frac{\partial V_0^w}{\partial g^w}} - \frac{\frac{\partial V_t^w}{\partial \lambda^w}}{\frac{\partial V_t^w}{\partial g^w}}\right) > 0. \tag{132}$$

Thus, while utility of the capitalists remains constant, the utility of workers has increased.

To a certain extent this result illustrates the main point of our analysis. Given that we already have incurred the debt, the risk sharing capacities of the debt are a scarce resource. Transferring some of the debt from capitalists to workers improves risk sharing without any additional crowding-out of capital.

[50] At this point, we take the non-participation of workers in the stock-market as given; Abel (2001a) endogenizes the participation decision by introducing fixed costs that make it rational for agents with a small portfolio to abstain from the stock market. Regarding this non-participation decision, Diamond and Geanakoplos (2003) point out that roughly 50 percent of the working population in the US does not hold any stocks (this figure includes indirect holdings of stocks through pension plans).

[51] To focus on the intertemporal and intergenerational reallocation of risks, rather than intra-generational redistribution which can also be achieved without social security, we assume that the affiliation with the two groups of all agents is known in period $t = 0$, i.e. cannot be insured against.

[52] Given the different labor endowment and the different exposition to the covariance risk ($cov(R, w_{t+1})$), it is clear that it is not optimal to choose a "one-size-fits-all pension scheme". Hence we will right away allow for distinct debt structures λ_c, λ_w for capitalists and workers.

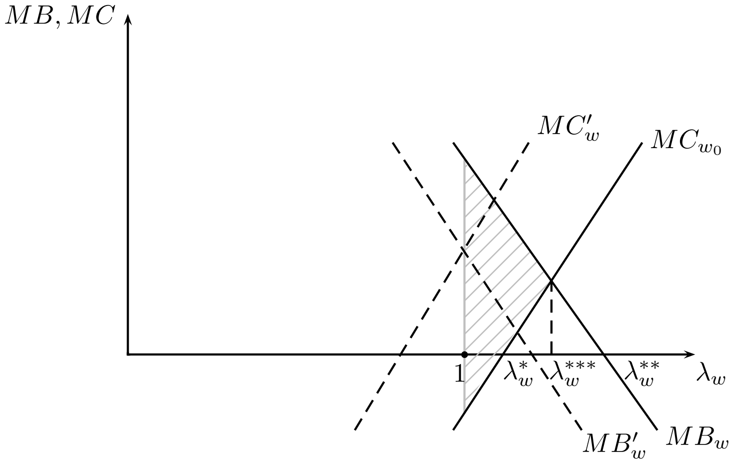


Diagram 13: *Intragenerational reallocation of the debt.*

The shaded area to the right of $\lambda = 1$ is the welfare gain associated with an intragenerational debt swap.

4.4 Conclusion

If a government can issue safe bonds and claims to an unfunded social security system to service a given obligation, there exists a set of Pareto-efficient financing policies. This set is characterized by the conflicting interests of agents who are currently alive and those who are yet unborn. The current young, who have already observed their wage income, will prefer safe debt, i.e. safe retirement benefits. The unborn generations on the contrary benefit from the ex-ante diversification of their wage risk if a large portion of the initial debt is injected into social security.

The government may now act as a representative of the unborn members of society. Through its budget constraint, it can offer generation 0 a compensation that reflects the willingness to pay of all unborn agents. Such an intermediation allows to collect the benefits, which are associated with the optimum structure for government debt λ^{***} in a Pareto-improving manner. If the initial conditions are such that $\lambda^{***} > 0$, an unfunded social security system is therefore always warranted.

Unlike the deterministic economy, where all debt policies are equally desirable, the current analysis shows that the structure of government debt has distinct implications for individual welfare. If we compare our analysis to the problem of optimal capital accumulation, the following analogy is notable: While the golden rule capital intensity maximizes long-run utility, it comes at the cost of lower consumption along the transition path. All capital intensities below the golden rule level are efficient and there is no compensation mechanism available.[53] All government interventions are either neutral or require a welfare criterion. Compared to the reallocation of aggregate risks, the situation without compensation is similar; there exists a whole set of efficient debt structures. In the present case, however, the government budget constraint *can* be used to reconcile the conflicting interests of the current young and those who live in the long run in a Pareto-improving manner. As a result, subject to our assumptions, the set of efficient debt structures can be narrowed.

4.5 Appendix

4.5.1 The Envelope Conditions

Derivation of condition (120): Equations (114) and (115) imply an investment behavior for each realization of the wage-income w_t, namely $a_t = a_t(w_t, \lambda)$, $h_t = h_t(w_t, \lambda)$. Hence, agents smooth consumption state by state with regard to first period wage income. At the same time, they smooth consumption in expectations when it comes to second period consumption. Taking expectations E_{w_t} of (114) and (115) yields:

$$E_{w_t}[U'(c^1)] = \beta(1+r_1)E_{w_t}\Big[E_{w_{t+1}R}[U'(c^2)]\Big], \tag{133}$$

$$E_{w_t}[U'(c^1)] = \beta E_{w_t}\Big[E_{w_{t+1}R}[(1+R)U'(c^2)]\Big]. \tag{134}$$

Writing out the first order condition for λ^{**}, we obtain:

$$\begin{aligned}\frac{dV_t}{d\lambda} = &\Big(E_{w_t}\Big[-U'(c^1)w_t + (1+n)\beta E_{w_{t+1}R}[w_{t+1}U'(c^2)]\Big]\frac{g_0}{w_0} \\ &+ \frac{r-n}{(1+r)}g_0 E_{w_t}\Big[U'(c^1)\Big]\Big)\frac{1}{1+n} \\ &- E_{w_t}[U'(c^1)(\frac{da}{d\lambda} + \frac{dh}{d\lambda}) - \beta E_{w_{t+1}R}[U'(c^2)((1+r)\frac{da}{d\lambda} + (1+R)\frac{dh}{d\lambda})]] = 0.\end{aligned} \tag{135}$$

To rearrange the first line in (135), equation (133) can be utilized as $\frac{E_{w_t}[U'(c^1)]}{1+r} = \beta E_{w_t}[U'(c^2)]$. Applying the covariance rule ($E[xy] = cov(x, y) + E[x]E[y]$) to the

[53] The lack of such a compensation mechanism led to the turnpike literature; see e.g. Samuelson (1968) or Blanchard and Fischer (1989). The absence of such an intertemporal compensation mechanism is of course also the reason for the intertemporal efficiency of pay-go schemes that we have been referring to in Footnote 32.

resulting expressions, we obtain (120). Noting that the derivatives $\frac{da}{d\lambda}$ and $\frac{dh}{d\lambda}$ are functions of w_t, the second line can be rearranged using the covariance rule such that:

$$\begin{aligned}
&-E_{w_t}[U'(c^1)]E_{w_t}[\frac{da}{d\lambda}] + (1+r)\beta E_{w_t w_{t+1} R}[U'(c^2)]E_{w_t}[\frac{da}{d\lambda}] \\
&-E_{w_t}[U'(c^1)]E_{w_t}[\frac{dh}{d\lambda}] + \beta E_{w_t w_{t+1} R}[(1+R)U'(c^2)]E_{w_t}[\frac{da}{d\lambda}] \\
&+cov_{w_t}(-U'(c^1) + (1+r)\beta E_{w_{t+1} R}[U'(c^2)], \frac{da}{d\lambda}) \\
&+cov_{w_t}(-U'(c^1) + \beta E_{w_{t+1} R}[(1+R)U'(c^2)], \frac{dh}{d\lambda}) = 0.
\end{aligned}$$

That is, recalling (114), (115), (133), and (134), the expressions related to changes in the investment behavior vanish by the envelope theorem.

4.5.2 Characteristics of the Long-run Optimum

This appendix examines the properties of condition (120). In a first step we note that (120) characterizes a "best" debt structure, which may or may not be interior. In a next step it is shown that interior solutions will exist for appropriate parameters. Finally the conditions, which ensure that $\frac{dV_t(\lambda)}{d\lambda}_{|\lambda=0} > 0$ and that $\frac{d^2V_t(\lambda)}{d\lambda^2} < 0$, are outlined.

Existence Since short sales of bonds or social security claims were ruled out, the set of feasible debt structures $[0,1]$ is a compact subset of $\mathbb{R}$. If $V_t(\lambda)$ is continuous and real-valued, it will therefore attain its bounds on this choice set according to the Weierstrass theorem.

Interior Solutions If $\frac{dcov(U'(c^1),w_t)}{d\tau^s}\frac{d\tau^s}{d\lambda}$ and $\frac{dcov(U'(c^2),w_{t+1})}{d\tau^s}\frac{d\tau^s}{d\lambda}$ are continuous and $\frac{dh}{d\tau^s} < 0$, it is obvious that for sufficiently large g_0, sufficiently small $cov(R, w_{t+1})$, and $E_w[w] = w_0$ or $r = n$, we have:

$$\frac{dV_t}{d\lambda}_{|\lambda=0} > 0, \quad \frac{dV_t}{d\lambda}_{|\lambda=1} < 0. \tag{136}$$

In this case, there exists one interior global optimum λ^{**} and there may exist several local optima.

Unique Optimum To interpret condition (120) in more detail, we will first show that $cov(U'(c^1), w_t) < 0$ and give a condition for $cov(U'(c^2), w_{t+1}) \gtreqless 0$:

$$\begin{aligned} cov(c^1, w_t) = &\quad cov((1-\tau^s)w_t - \frac{r-n}{1+r}\frac{g_0}{1+n}(1-\lambda) - s(w_t, \tau^s), w_t) \\ = &\quad cov((1-\tau^s)w_t - s(w_t, \tau^s), w_t) > 0, \end{aligned} \tag{137}$$

where the sign $cov(c^1, w_t) > 0$ is due to the normality of c^1; i.e. $\frac{\partial((1-\tau^s)w_t - s(w_t,\tau^s))}{\partial w_t} > 0$. Hence, since $U''() < 0$, $cov(U'(c^1), w_t) < 0$. For $cov(U'(c^2), w_{t+1})$ we have:

$$\begin{aligned} cov(c^2, w_{t+1}) = &\quad cov((1+r)a + (1+R)h + \tau^s(1+n)w_{t+1}, w_{t+1}) \\ = &\quad hcov(R, w_{t+1}) + \tau^s(1+n)\sigma_w^2 \gtreqless 0; \quad \tau^s = \lambda\frac{g_0}{(1+n)w_0}. \end{aligned} \tag{138}$$

Hence, depending on the amount of risky assets h, $cov(w, R) \gtreqless 0$ and the amount of debt that is injected in the pension system, we may have $cov(U'(c^2), w_{t+1}) \gtreqless 0$. Together with the ambiguous sign of $\frac{(n-r)(Ew-w_0)}{w_0(1+r)}$, we may or may not have $\frac{dV_t}{d\lambda}_{|\lambda=0} > 0$.

Sufficient Condition To allow for a global optimum, it is a sufficient condition, that $\frac{dV_t}{d\lambda}$ is downward-sloping in λ:

$$\frac{d^2V_t}{(d\lambda)^2} = \frac{g_0}{1+n}\Big(\frac{n-r}{1+r}\frac{E[w]-w_0}{w_0}\frac{dE[U'(c^1)]}{d\lambda} \tag{139}$$

$$- \frac{dcov_{w_t}(U'(c^1), \frac{w_t}{w_0})}{d\lambda} + \beta(1+n)\frac{dcov_{w_{t+1}R}(U'(c^2), \frac{w_{t+1}}{w_0})}{d\lambda}\Big) < 0.$$

A first inspection of (139) indicates that for $Ew = w_0$ and/or $r = n$, we expect $\frac{dcov(U'(c^1),w_t)}{d\lambda} > 0$, $\frac{dcov(U'(c^2),w_{t+1})}{d\lambda} < 0$ and thus $\frac{d^2V_t}{d\lambda^2} < 0$.[54] With respect to $\frac{dcov(U'(c^1),w_t)}{d\lambda}$ we have:

$$\begin{aligned} \frac{dcov_{w_t}(U'(c^1), w_t)}{d\lambda} &= cov_{w_t}(U''(c^1)(-w\frac{d\tau^s}{d\lambda} - \frac{ds}{d\lambda}), w) \\ &\approx E_{w_t}[U''(c^1)]cov_{w_t}(-w\frac{d\tau^s}{d\lambda} - \frac{ds}{d\lambda}, w) \gtreqless 0, \end{aligned} \tag{140}$$

[54] Regarding the first element, which is inherently ambiguous, we note that for $U'''(c^1) > 0$, $\frac{dE[U(c^1)]}{d\lambda}$ is most likely negative, as the variance of first period consumption is decreasing in λ. However, at the same time an increase in λ may increase second period variance and if $U'''() > 0$, precautionary savings (see Green (1977) and Kimball (1990) for the coefficient of prudence) will increase $E[U'(c^1)]$.

where (140) holds with strict equality if $U'''(c^1) = 0$. Condition (140) indicates that $\frac{dcov_{w_t}(U'(c^1),w_t)}{d\lambda} > 0$ as long as $cov_{w_t}(\frac{ds}{d\lambda}, w_t)$ is not large and negative. Finally, by the same approximation as in (140) we have:

$$\frac{dcov(U'(c^2), w_{t+1})}{d\lambda} \approx E_{wR}[U''(c^2)]\Big(\frac{dh}{d\lambda}cov(R, w_{t+1}) + (1+n)\sigma_w^2\Big)\frac{d\tau^s}{d\lambda} \gtreqless 0, \quad (141)$$

where (141) is negative if $\frac{dh}{d\lambda}cov(R, w_{t+1}) + (1+n)\sigma_w^2 > 0$. If $cov(R, w_{t+1})$ is large and positive and the share of savings invested in the risky technology is also very large, the crowding-out effect (with regard to risky investment) of additional pension claims may in principle overcompensate the direct effect of the exposition to additional wage-related risks once λ is increased.

4.5.3 Lagrangian

In this appendix, we discuss the Lagrangian associated with the set of efficient debt structures and compensation schemes. However, in the current case, the premium π_t payed by members of generation t, is not injected into the general debt scheme. Instead π_t is issued in period 0 and redeemed (principal and interest) by generation t in period t. In this case, the analogy to the problem of the efficient provision of a public good is easier to conceive:

$$\max_{\{\pi_t\}_{t=1}^{t=\infty},\lambda} \quad \mathcal{L} = V_0(\lambda, \pi_0) + \sum_{t=1}^{\infty} \mu_t\Big(V_t(\lambda, \pi_t) - \bar{V}(\lambda^*, 0)\Big) \quad (142)$$

$$+\gamma\Big(\sum_{t=1}^{\infty}\Big(\frac{1+n}{1+r}\Big)^t \pi_t - \pi_0\Big)$$

Taking the first derivatives and eliminating γ yields:

$$\frac{\partial \mathcal{L}}{\partial \pi_t} = \Big(\frac{1+n}{1+r}\Big)^t \frac{\partial V_0}{\partial \pi_0} + \mu_t \frac{\partial V_t}{\partial \pi_t} = 0, \quad \forall t = 1, 2, ...\infty, \quad (143)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \frac{\partial V_0}{\partial \lambda} + \sum_{t=1}^{\infty} \mu_t \frac{\partial V_t}{\partial \lambda} = 0. \quad (144)$$

Substitution allows to drop μ, and we have

$$\frac{\frac{\partial V_0}{\partial \lambda}}{\frac{\partial V_0}{\partial \pi_0}} - \sum_{t=1}^{\infty}\Big(\frac{1+n}{1+r}\Big)^t \frac{\frac{\partial V_t}{\partial \lambda}}{\frac{\partial V_t}{\partial \pi_t}} = 0 \quad (145)$$

In the particular case where each agent pays the same premium, $\pi_t = \pi$, this simplifies to:

$$\frac{\frac{\partial V_0}{\partial \lambda}}{\frac{\partial V_0}{\partial \pi_0}} - \frac{1+n}{r-n}\frac{\frac{\partial V_t}{\partial \lambda}}{\frac{\partial V_t}{\partial \pi}} = 0. \tag{146}$$

If $\frac{\partial V_t}{\partial \lambda}, \frac{\partial V_0}{\partial \lambda} > 0$ and $\frac{\partial^2 V_0}{\partial \lambda^2}, \frac{\partial^2 V_t}{\partial \lambda^2} < 0$, as discussed in Appendix 4.5.2, the expression in (146) changes signs and an interior maximum exists.

4.5.4 The Covariance Risk

Technology and Covariance This appendix reflects on the sign of $cov(w_t, R_t)$. Taking the perspective of a small open economy, we examine the correlation of factor-prices received from the global economy. If global production uses capital and labor inputs, K and L, to produce aggregate net output $zF(K, L)$, where $zF(K, L)$ is concave and first-degree-homogenous, factor-prices are given by

$$\begin{aligned} w_t &= \frac{\partial z_t F(K_t, L_t)}{\partial L_t} = z_t F_{L_t}(K_t, L_t), \\ R_t &= \frac{\partial z_t F(K_t, L_t)}{\partial K_t} = z_t F_{K_t}(K_t, L_t). \end{aligned} \tag{147}$$

If the global supply L_t of (efficient) labor fluctuates over time, we have $cov(w_t, R_t) < 0$ since (by the Euler Theorem) $F_{K_t L_t} > 0$ and $F_{L_t L_t} < 0$. Examples for a stochastic global labor supply may be the entrance of the labor force from Eastern Europe into the EU labor market, or the rise of China. If total factor productivity z_t is stochastic, wages and profits are perfectly correlated and we have $cov(w_t, R_t) = \sigma^2_{z_t} F_{L_t} F_{K_t} > 0$.[55] If depreciation is stochastic, we have $cov(w_t, R_t) = 0$. Finally, unpredicted changes in global savings are associated with $cov(w_t, R_t) < 0$. Hence, depending on the relative magnitude of the respective effects, the sign the covariance between wages and profits is ambiguous.

[55] See Bohn (1998) and Smetters (2006) for the strong results that originate from perfect correlation.

5 Intertemporal Compensation with Incomplete Markets

This chapter briefly puts the results derived in chapters 2, 3, and 4 into a broader perspective. In previous chapters we have shown that the scope for government intervention increases with the number of missing markets in the economy. In this chapter we sketch the more general underlying structure of the problem. In particular, we show that the potential use of the government budget constraint as an intertemporal collusion device changes in nature once there are at least two goods and two government budget constraints available. That is, an increase in the number of goods and budget constraints from one to two changes the quality of the efficiency results obtained for OLG economies. Diagrams 14 and 15 illustrate this point. Further increases in the number of budget constraints beyond two, however, do not change the quality of the results.

For a simple exposition we discuss a stylized economy with the following properties:

Assumption 1: Households live for two periods. Each generation i has a smooth and (jointly) concave utility criterion $U_i = U_i(x_i^1, x_i^2) : \mathbb{R}_+^n \times \mathbb{R}_+^n \mapsto \mathbb{R}$. Where x_i^1, x_i^2 are vectors containing n different consumption goods. In the first period of life, each household is endowed with $\bar{x}_i \in \mathbb{R}_+^n$ units of consumption goods.

Assumption 2: Each consumption good can be stored/invested. The rate of return $r_l > 0$ for each good $l = 1, 2, ..., n$ is independent of the aggregate level of investment.

Assumption 3: Technological progress and population growth are zero.

Assumption 4: The government can transfer goods of type l from generation i to generation j. These transfers are denoted by $\tau_{i,j} \in \mathbb{R}^n$, $i, j = 0, 1, 2...\infty$. Where $\tau_{i,j,l}$, $l = 1, 2, ..., n$, are the components of $\tau_{i,j}$. These transfers are financed by appropriate borrowing and lending at the technologically determined interest rates.

With these assumptions in place it is useful to define the following indirect utility function for an agent born at node i:

$$V_i := arg \max_{s_i} \Big\{ U_i \Big(\bar{x}_i - s_i - \sum_{j=0}^{i-1} (1+r)^{i-j} \tau_{i,j} - \sum_{j=i+1}^{\infty} \mathbf{1} \tau_{i,j}, (1+r) s_i \Big) \Big\}, \quad (148)$$

where s_i is a $n \times 1$ vector and $(1+r)$ and $\mathbf{1}$ are diagonal matrixes of dimension $n \times n$ with diagonal entries $(1 + r_l)$ and 1 respectively.

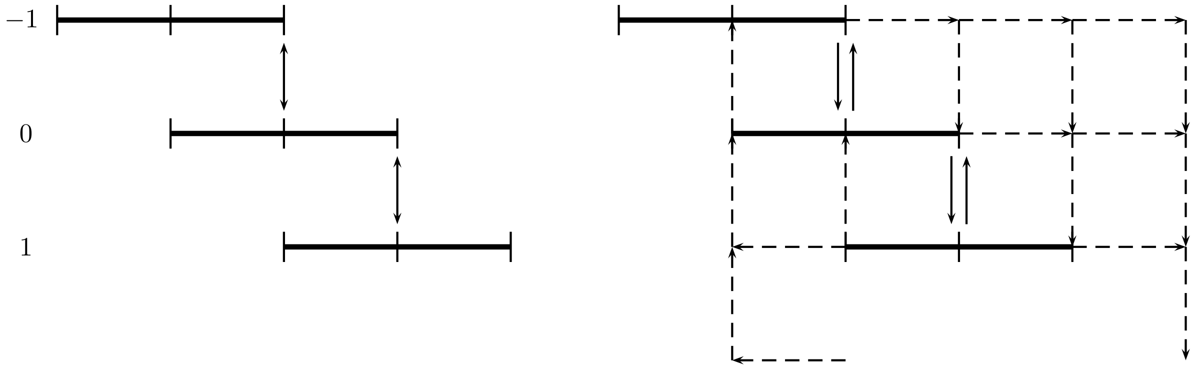


Diagram 14a Diagram 14b

Diagram 14: *Unfolding the missing markets and intertemporal compensation*

In Diagram 14a *there is only one government budget constraint along which goods can be transferred. Every increase in consumption for generation* i *lowers consumption for members of some generation* j. *If* $r > 0$ *and marginal utility is positive, each allocation (transfer scheme) is Pareto-efficient. In Diagram* 14b *on the contrary, there is a vector of taxes available in each period and intertemporal compensation is possible. Depending on the respective preferences, there exists a large set of inefficient allocations (transfer schemes). Using these taxes, the government can make up for missing markets. However, private savings must be large enough to support the government borrowing along the diagonal of the diagram.*

We now proceed as in the previous chapters. To find the set of Pareto-improving intertemporal reallocations we write the associated Lagrangian:

$$\begin{aligned}\max_{\{\tau_{i,j},\mu_{i,j},\lambda_i,\gamma_i\}_{i,j=0}^{\infty}} \mathcal{L} \quad &= U_0(\bar{x}_0 - \sum_{i=1}^{\infty} \tau_{0,i}) \\ &+ \sum_{i=1}^{\infty} \lambda_i \Big(V_i(\bar{x}_i - \sum_{j=0}^{i-1} (1+r)^{i-j} \tau_{i,j} - \sum_{j=i+1}^{\infty} \mathbf{1}\tau_{i,j}) - V_i(\bar{x}_i) \Big) \\ &+ \sum_{j=0}^{\infty} \sum_{i=j+1}^{\infty} \mu_{i,j} (\tau_{i,j} + \tau_{j,i}) \\ &+ \sum_{i=0}^{\infty} \gamma_i \Big(\sum_{k=0}^{i} \sum_{j=i+1}^{\infty} \tau_{k,j} (1+r)^{i-k} + s_i(\bar{x}_i - \sum_{j=0}^{i-1} (1+r)^{i-j} \tau_{i,j} - \sum_{j=i+1}^{\infty} \mathbf{1}\tau_{i,j}) \Big).\end{aligned} \tag{149}$$

Where the Lagrangian multipliers $\mu_{i,j}$ ensure that a transfer from agent i to agent j increases consumption for agent j and lowers it for agent i and viceversa. The second restriction associated with γ_i is of the Kuhn-Tucker type. It requires that the government debt in the respective goods may not exceed households savings. That is, if the government borrows in good $\tilde{l}$ to implement trades between the generations i and j, the generations in between must have sufficient savings to absorb the debt which is rolled over from period i to period j. Put differently, if this condition binds as in Diagram 15, households savings are a bottleneck to intergenerational transfers.[56] The savings decision would be associated with an externality. If the reallocation scheme is sufficiently small the savings condition γ_i is not binding, i.e. $\gamma_i = 0$, and the first order conditions for $\tau_{i,j}$ are:

$$-\nabla V_0 + \mu_{i,0} = 0 \quad \forall i = 1, 2, ...\infty, \tag{150}$$

$$-\lambda_i(1+r)^{i-j}\nabla V_i + \mu_{j,i} = 0 \quad \forall i > j, i = 1, 2, ...\infty, j = 0, 1, 2, ...\infty, \tag{151}$$

$$-\lambda_i \nabla V_i + \mu_{i,j} = 0 \quad \forall i < j, i = 1, 2, ...\infty, j = 0, 1, 2, ...\infty, \tag{152}$$

$$\tau_{i,j} + \tau_{j,i} = 0 \quad \forall i, j = 0, 1, 2, ...\infty, \tag{153}$$

$$V_i \geqq \bar{V}_i \quad \forall i = 1, 2, ...\infty, \tag{154}$$

$$\sum_{k=0}^{i} \sum_{j=i+1}^{\infty} (1+r)^{i-k} \tau_{k,j} \tag{155}$$

$$+s_i(\bar{x}_i - \sum_{j=0}^{i-1}(1+r)^{i-j}\tau_{i,j} - \sum_{j=i+1}^{\infty} \mathbf{1}\tau_{i,j}) \geqq 0, \quad \forall i = 1, 2, ...\infty.$$

Where λ_i is a scalar and $\gamma_i, \mu_{i,j}, s_i, \tau_{i,j}$ are appropriate $n \times 1$ vectors. Finally $(1+r)^{i-j}$ is a diagonal matrix of dimension $n \times n$ with diagonal entries $(1+r_l)^{i-j}$. Rewriting these conditions yields the first order condition for the respective tax rates $\tau_{i,j,l}$:

$$\frac{\frac{\frac{\partial V_j}{\partial x_l}}{\frac{\partial V_j}{\partial x_{l+1}}}}{\frac{\frac{\partial V_i}{\partial x_l}}{\frac{\partial V_i}{\partial x_{l+1}}}} = \left(\frac{1+r_l}{1+r_{l+1}}\right)^{i-j}, \quad \forall \quad i, j = 0, 1, 2, ...\infty, \quad l = 1, 2, ...n-1, \tag{156}$$

[56] If one cohort i has a very low endowment (when compared to the surrounding generations) it will save little and therefore, it can absorb only little amounts of public debt. In this case this savings constraint is likely to bind, i.e. $\gamma_i \neq 0$. In a different interpretation the condition associated with γ_i requires that aggregate assets of the different types cannot be negative at any point i.

where (156) indicates that taxes should be chosen such that a point on the contract curve is reached.[57] Moreover, if we have only $m < n$ tax rates available per agent, $n - m$ optimality conditions are lost. Regarding the prospects of intertemporal compensation we therefore have the following proposition:

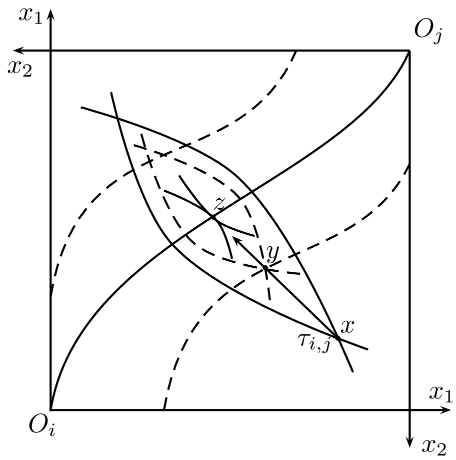


Diagram 15: *The contract curve*

The two origins represent members of generation i and j, respectively. Starting from the market allocation at x the government can raise a tax such that a point z on the contract curve is obtained. If γ_i is binding at some point, we can only move towards y where savings are no longer sufficient to support further transfers. The dashed lines surrounding the contract curve represent the "savings constraint".

Proposition 9. *If there are $n > 1$ goods and the government has $m > 1$ intertemporal budget constraints it can reallocate resources such that a point on the contract curve is reached in a Pareto-improving manner. Moreover, only those tax and transfer schemes that steer the economy towards a point on the contract curve are efficient. If the government can only reallocate resources of one type (one budget constraint) a Pareto-improving change in the tax scheme is not possible. Each initial allocation is constrained efficient.*

Proof. The first part is obvious. For the second part we note that an increase in period 0 consumption, i.e. $\tau_{0,i} < 0$, requires higher taxes $\tau_{i,0} > 0$ at some point i. If

[57]This first order condition is of particular interest with regard to the problem of natural resources. If the first generation is the only one in possession of a particular good/resource, condition (156) defines how these resources can be exhausted optimally. That is, future households would be willing to live with a higher debt in a different consumption good in exchange for more (natural) resources. Similar to the change in the validity of the golden rule there will be corresponding changes to results like, e.g., the "green golden rule".

preferences are locally non satiated this violates the condition $V_i(\bar{x}_i - \tau_{i,0}(1+r)^i) \leqq V_i(\bar{x}_i)$. $\square$

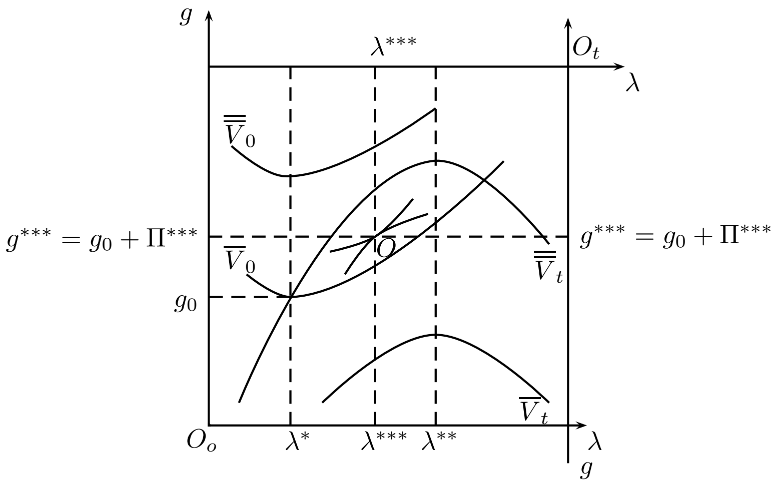


Diagram 16: *The optimum structure for government debt*

Compared to our problem of the optimum structure for government debt two differences are notable: (i) we have changed the debt structure once for all to keep in touch with the literature and the steady state as a reference point. (ii) The current setting is linear regarding the resources transferred. In the setting with the optimum structure for government debt we have the aspect of non-linearity. This is reflected in the fact that there are points of satiation, namely λ^* and λ^{**}. If we would introduce non-linearity into the current model, there would also be the possibility of satiation as with the optimal capital intensity in the Diamond (1965) model. Moreover, there would be more externalities of a similar type as the current savings externality (associated with the γ_i constraint). Diagram 16 gives an illustration of the effects of nonlinearity of the derivatives of the indirect utility functions associated with the FOC (126) in Chapter 4, for the optimum structure for government debt.

The interesting point with Diagram 16 is the fact that the set of efficient allocations now depends on both, the curvature of the utility function and the technology. In the case with only one good, this is different: positive marginal utility and the interest rate condition are the only data required to know that all transfer schemes are efficient as long as $r_l > 0$ for all goods $l = 1, 2, ...n$.

5.1 Conclusion

The concept of dynamic efficiency in the basic Diamond (1965) model has a remarkable property: the set of efficient capital intensities does not depend on preferences. It is defined by the curvature of $f(\cdot)$ and the growth rate of the aggregate economy. The mere comparison of the marginal product of capital and the aggregate growth rate suffices to decide whether or not the economy grows on an efficient or inefficient path. The results derived in Chapter 3 showed that this result was sensitive once preferences are heterogeneous. In Chapter 4, where two goods were available, this conclusion was also altered for a representative agent economy for the case with two budget constraints and two missing markets. The present chapter shows that the efficiency properties of the economy in general change once there are more than two markets missing and there are at least two intertemporal budget constraints available. In the current model, the government can steer the economy to a certain point on the contract curve. In this case the set of inefficient policies is quite large. The notable difference compared to the Arrow-Debreu setting is the fact that the particular location on the contract curve is not an outcome of a market process. It has to be chosen by the planner rather than the households. Moreover, if the economy is closed, the life-cycle savings condition still limits intergenerational trades.

Taken together, chapters 3, 4 and the current one suggest that the role of preferences tends to increase when the economy becomes more complex. Along-side the scope for Pareto-improving government intervention also increases. On the contrary, the importance of technology diminishes. The set of efficient capital intensities can no-longer be determined by the shape of $f(\cdot)$ and the aggregate growth rate n alone.

6 Demographic Change and the Rates of Return to Risky Capital and Safe Debt

This chapter studies how the upcoming demographic transition will affect the returns to risky capital and safe government debt. In a neoclassical two-generations-overlapping model we show that the entrance of smaller cohorts into the labor market will lower both interest rates. The risky rate, however, will react more sensitive than the risk free rate. Consequently, the risk premium deteriorates during the transition.

6.1 Introduction

Neoclassical models of the Solow (1956), Swan (1956) type predict a positive relation between the growth rate for population and the rate of return to capital. In more refined models, involving uncertainty and more than one asset, several authors have recently examined the consequences of the demographic transition with regard to the rates of return to risky capital and safe debt.[58] However, while these studies tend to agree that the overall level of interest will fall when the baby boom generation retires, it remains an open question whether the demographic transition will affect both rates of return in the same manner. In particular, Brooks (2002) and Geanakoplos et al. (2004) project that the risky rate will deteriorate more severely than the return to safe government bonds. That is, the equity premium would fall during the demographic transition. At the same time, Brooks (2004) and Börsch-Supan et al. (2007) project an increase in the equity premium for that period.

The purpose of the current note is to complement these previous studies, which were based exclusively on simulations, by developing a modified version of the Diamond (1965) model which allows to analyze the relation between the growth rate for population and the two key interest rates.

This chapter is organized as follows: In Section 6.2 we discuss a baseline setting where households work in the first period only, aggregate shocks are log-normal, and utility is of the Epstein and Zin (1989) type. In this setting we show that (i) both rates of return increase with the growth rate for population, (ii) the risky rate to capital reacts more sensitive than the risk free rate, i.e. the equity premium increases with the growth rate of population. In Section 6.3 we show that the results from Section 6.2 carry over to a setting with a general concave utility function and shocks

[58] See Poterba (2001) and Poterba et al. (2005) for surveys.

which are no longer log-normal. Finally, we discuss a model where households work in both periods of life. In this setting it turns out that there is a "human capital effect" to our portfolio choice problem which thwarts the positive relation between the growth rate for population and the equity premium. While both rates fall, the safe return will be affected more severely if the human capital effect is sufficiently large.

6.2 The Model

6.2.1 Technology and factor-prices

The economy is inhabited by overlapping generations who live for two periods; one period of work is followed by one period of retirement. During the first period of life, each individual supplies one unit of labor inelastically and population evolves according to:

$$N_{t+1} = (1 + n_{t+1})N_t, \tag{157}$$

where N_t is the size of the cohort born at time t and $1 + n_{t+1}$ is the number of children raised by each member of cohort t.

Production is characterized by a continuous, concave, constant returns to scale, aggregate production function $F(K_t, N_t) \equiv \tilde{F}(K_t, L_t) + (1 - \delta)K_t$. This production process is subject to an aggregate technology shock z_t, which follows a log-normal distribution. For simplicity we assume that this shock is on average neutral. Per capita output y_t is therefore given by:

$$y_t = z_t f(k_t); \quad f'() > 0, \quad f''() < 0, \quad E[z_t] = 1, \quad \forall t. \tag{158}$$

Once the respective realization of the shock z_t is known, each firm will rent capital and hire labor up to the point where the respective marginal products are equal to the market prices:

$$R_t = z_t \frac{\partial F(K_t, N_t)}{\partial K_t} = z_t f'(k_t), \tag{159}$$

$$w_t = z_t \frac{\partial F(K_t, N_t)}{\partial N_t} = z_t (f(k_t) - f'(k_t)k_t). \tag{160}$$

6.2.2 Government Debt

Contrary to the approach, where safe debt/consumption loans are issued by the households (zero net supply), we will take note of the fact that the government is the only entity that can supply safe debt. The budget constraint of the government is given by:

$$B_t + N_t \tau_t = r_t B_{t-1}, \tag{161}$$

where B_{t-1} is the amount of outstanding and B_t the amount of newly issued debt in period t. Lump-sum taxes are denoted by τ_t. The rate of (gross) interest on government debt which was issued at time $t-1$ is denoted by r_t. This rate of interest earned on government debt is deterministic, i.e. at time t the government issues debt with a guaranteed rate of return r_{t+1}. Risk averse individuals will therefore be willing to hold safe debt even if its rate of return is below the expected risky rate. As in Bohn (1998) and Smetters (2006) we assume that the government holds the debt to GDP ratio constant over time. This assumption is indeed consistent with the Maastricht criterion on government debt for countries in the Euro-zone. If policy is characterized by such a constant debt output ratio ρ, we have:[59]

$$\frac{B_t}{Y_t} = \rho \quad \forall t. \tag{162}$$

Solving (161) for per capita taxes τ, using (162) and (157), yields:

$$\tau_t = \Big(\frac{1}{(1+n_t)} r_t y_{t-1} - y_t\Big)\rho. \tag{163}$$

6.2.3 Households

The representative household lives for two periods and supplies labor inelastically in the first period only. Towards the end of the first period the household faces a consumption/saving and a portfolio allocation decision. Preferences over current and future consumption, $c_{t,1}$ and $c_{t+1,2}$, are described by a simplified Epstein and Zin (1989) utility function:

$$U_t = \ln(c_{t,1}) + \frac{\beta}{1-\phi}\ln(E_t[(c_{t+1,2})^{1-\phi}]); \quad 0 < \phi, \quad 0 < \beta < 1. \tag{164}$$

[59] There are obviously many different debt policies perceivable. However, the following results will be valid for all perceivable debt policies provided that taxes τ_t and the amount of debt B_t which is issued at time t do not depend on variables that are not yet realized in period t, e.g., the future capital intensity k_{t+1}.

The utility function in (164) exhibits an elasticity of intertemporal substitution of unity. Hence, the individual savings/consumption decision is independent of the interest rate, since income and substitution effects cancel and precautionary savings neither dampen nor amplify private thrift. This assumption is reasonable as long as the (ambiguous) influence of changes in the rate of interest on savings is not too large. The advantage of this specification can be seen in the coefficient of relative risk aversion ϕ with respect to second period consumption, which allows to study the entire scope of the portfolio choice problem.

Recalling the taxes levied by the government τ_t, the value of wealth owned by the consumer when young can be written as:[60]

$$\Omega_t \equiv w_t - \tau_t. \tag{165}$$

For given values of lifetime wealth Ω, the individual chooses to hold assets amounting to:

$$a_t \equiv b_t + h_t, \tag{166}$$

where b_t and h_t are the amounts of riskless bonds and risky capital respectively. Denoting the portfolio share of risky assets by $\gamma_t \equiv \frac{h_t}{a_t}$ and the share of riskfree assets by $(1-\gamma_t) \equiv \frac{b_t}{a_t}$ yields, according to (164) and (165), the following household problem:

$$\max_{a,\gamma} U_t = \ln(\Omega_t - a_t) + \beta \ln(a_t) + \frac{\beta}{1-\phi} \ln\Big(E_t\Big[(\gamma_t R_{t+1} + (1-\gamma_t) r_{t+1})^{1-\phi}\Big]\Big), \tag{167}$$

with the corresponding first order condition for the optimal portfolio size:

$$a_t = \frac{\beta}{1+\beta}\Omega_t, \tag{168}$$

where the propensity to save out of wealth is $\frac{\beta}{1+\beta}$. The portfolio choice is characterized by the familiar implicit condition for γ_t:

$$E_t\Big([\gamma_t R_{t+1} + (1-\gamma_t) r_{t+1}]^{-\phi}(R_{t+1} - r_{t+1})\Big) = 0. \tag{169}$$

Using a second order Taylor series approximation, Campbell and Viceira (2002) show that the corresponding optimal portfolio share can be approximated as:

$$\gamma_t(R_{t+1}, r_{t+1}; \phi) = \frac{E_t[\ln(R_{t+1})] - \ln(r_{t+1}) + \frac{1}{2}\sigma_t^2}{\phi\sigma_t^2} = \ln\Big(\frac{E_t[R_{t+1}]}{r_{t+1}}\Big)\frac{1}{\phi\sigma_t^2}, \tag{170}$$

[60]The individual receives his wage w_t after the realization of z_t is known. Note also that taxes are known once z_t is known.

where $\sigma_t^2 = Var[\ln(R_{t+1})] = Var[\ln(z_{t+1})]$ and $\ln(E_t[R_{t+1}]) = E_t[\ln(R_{t+1})] + \frac{1}{2}\sigma_t^2$.[61] Once the investment opportunities are changing, the individual will adjust his portfolio according to:

$$\gamma_r \equiv \frac{\partial \gamma_t}{\partial r_{t+1}} = -\frac{1}{r_{t+1}}\frac{1}{\phi\sigma_t^2} < 0, \quad \gamma_{f'} \equiv \frac{\partial \gamma_t}{\partial(f'(k_{t+1}))} = \frac{1}{E_t[R_{t+1}]}\frac{1}{\phi\sigma_t^2} > 0. \quad (171)$$

The decisive property of the portfolio adjustment behavior in (171) is:

$$\frac{d(f'(k_{t+1}))}{dr_{t+1}}_{|d\gamma_t=0} = -\frac{\gamma_r}{\gamma_{f'}} = \frac{f'(k_{t+1})}{r_{t+1}} > 1, \quad (172)$$

where (172) indicates that, for positive expected equity premia, the share devoted to the risky asset reacts more sensitive with respect to the riskfree rate than the risky rate. That is, an increase in both rates of return, which leaves the equity premium unchanged, will result in a lower portfolio share in the risky asset.

6.2.4 Equilibrium

Having completed the partial analysis of the firm, the government and the household, we can now turn towards the equilibrium conditions for the bond and equity markets. Capital market clearing requires:

$$(1+n_{t+1})k_{t+1} = \gamma_t \frac{\beta}{1+\beta}\Omega_t. \quad (173)$$

The bond market equilibrium condition reads:

$$y_t\rho = (1-\gamma_t)\frac{\beta}{1+\beta}\Omega_t. \quad (174)$$

Taken together, equations (173), (174), (170) and (159) define the time path of the capital intensity k, the safe rate of return r, the optimal portfolio share γ and the risky rate R. Finally, the resulting expected equity premium $E_t[\Pi_{t+1}]$ is given by:

$$E_t[\Pi_{t+1}] = E_t[R_{t+1}] - r_{t+1} = f'(k_{t+1}) - r_{t+1}. \quad (175)$$

[61] The rate of return $R_{t+1} = z_{t+1}f'(k_{t+1})$ inherits its log-normal distribution from the technology shock z_{t+1}. Thus, $\ln(R_{t+1})$ follows a normal distribution.

6.2.5 Baby-Boom and Equity-Premium

We can now consider the consequences of the entrance of a large/small cohort into the labor market. Taking the current state of the economy $(k_{t-1}, k_t, z_{t-1}, z_t)$ as given, we differentiate (173) and (174) with respect to dn_{t+1}, $d\gamma_t$, dk_{t+1}. Hence, after recalling (159) and (170), we have:

$$\frac{dk_{t+1}}{dn_{t+1}} = -\frac{k_{t+1}}{(1+n_{t+1})} < 0, \quad \therefore \quad \frac{d(f'(k_{t+1}))}{dn_{t+1}} = f''(k_{t+1})\frac{dk_{t+1}}{dn_{t+1}} > 0 \tag{176}$$

and

$$\frac{d\gamma_t}{dn_{t+1}} = 0, \quad \therefore \quad \gamma_{f'}\frac{d(f'(k_{t+1}))}{dn_{t+1}} + \gamma_r\frac{dr_{t+1}}{dn_{t+1}} = 0. \tag{177}$$

The expressions in (176) indicate that a change in the growth rate of population does not change government taxes (163). Hence the value of life-cycle income Ω_t out of which individuals save a constant fraction remains unchanged. Thus, an increase in the relative size of the next cohort lowers the capital intensity and increases the expected future return on risky investments. The expressions in (177) follow from the bond market equilibrium condition. They indicate that, for $\frac{d(f'(k_{t+1}))}{dn_{t+1}} > 0$, the government has to offer a higher riskless rate $\frac{dr_{t+1}}{dn_{t+1}} > 0$ to sell a given amount of debt. With respect to the expected equity premium (175), we can now use the individual portfolio adjustment behavior described in (172) to show that:

$$\frac{d(E_t[\Pi_{t+1}])}{dn_{t+1}} = \Big(\frac{d(f'(k_{t+1}))}{dr_{t+1}} - 1\Big)\frac{dr_{t+1}}{dn_{t+1}} = \Big(\frac{E_t[\Pi_{t+1}]}{r_{t+1}}\Big)\frac{dr_{t+1}}{dn_{t+1}} > 0. \tag{178}$$

Equation (178) indicates that, due to the higher sensitivity of the portfolio shares with respect to the riskfree rate, the government does not need to raise its riskfree rate one for one with the expected risky rate to sell its debt. Diagram 17 illustrates this link. The demographic transition affects the returns to physical capital through the capital widening effect. In turn, households demand a higher safe rate. Due to the asymmetric portfolio adjustment effect, the equity premium rises.

6.3 Extensions

In this section, we will briefly discuss the robustness of our foregoing results. In particular we analyze the role of a potential human capital effect. Moreover, we consider a more general portfolio choice setting, which is no longer based on CRRA preferences and log-normal shocks.

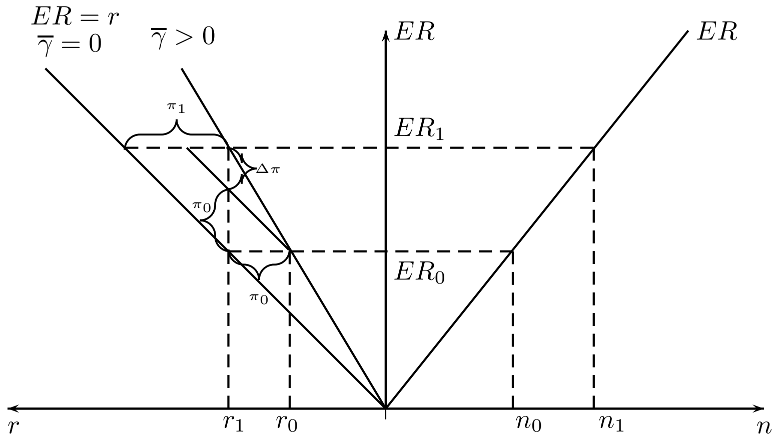


Diagram 17: *Demographic change and portfolio adjustment.*

6.3.1 The Effect of Human Capital

If households second period labor endowment is given by θ, the demographic transition will not only affect the rates of return. It also changes the present value of the labor endowment through the induced factor-price changes. Consequently, the household problem now reads:

$$\max_{a,\gamma} U_t = \ln(\Omega_t - a_t) + \beta \ln(a_t) + \frac{\beta}{1-\phi} \ln \left(E_t \left[(\gamma_t R_{t+1} + (1-\gamma_t) r_{t+1})^{1-\phi} \right] \right), \quad (179)$$

where life-cycle wealth is given by $\Omega_t \equiv w_t - \tau_t + \theta \frac{w_{t+1}}{R_{t+1}}$.[62] Solving the household problem in the same manner as before, we obtain our two market clearing conditions:

$$(1 + n_{t+1})k_{t+1} = \gamma_t \frac{\beta}{1+\beta}\Omega_t - \theta \frac{w_{t+1}}{R_{t+1}} \quad (180)$$

$$y_t \rho = (1-\gamma_t)\frac{\beta}{1+\beta}\Omega_t. \quad (181)$$

While the bond market clearing condition remains unaltered, the capital market clearing condition has to reflect that households only buy capital in excess to what they already hold as human capital. Moreover, Ω_t is now a function of the future capital intensity. To trace out the comparative statics of our system, it is useful to note that (180), (181) and (170) are separable. Beginning with (180) and (181), we

[62]Given our specification of the production sector, second period wage income and capital are perfectly correlated (perfect substitutes), i.e. $corr(w_{t+1}, R_{t+1}) = \frac{cov(w_{t+1},R_{t+1})}{\sigma_w \sigma_R} = 1$.

determine how changes in the growth rate for population affect the risky rate and the portfolio share. Subsequently we differentiate (170) to determine the change in the safe rate. Taken together, we obtain:

$$\frac{dk_{t+1}}{dn_{t+1}} = \frac{k_{t+1}}{\frac{1}{1+\beta}\theta\frac{f(k_{t+1})f''(k_{t+1})}{f'^2(k_{t+1})} - (1+n_{t+1})} < 0, \tag{182}$$
$$\therefore \quad \frac{df'(k_{t+1})}{dn_{t+1}} = f''(k_{t+1})\frac{dk_{t+1}}{dn_{t+1}} > 0,$$

$$\frac{d\gamma_t}{dn_{t+1}} = \frac{(1-\gamma_t)\theta\frac{-f(k_{t+1})f''(k_{t+1})}{f'^2(k_{t+1})}}{\Omega_t}\frac{dk_{t+1}}{dn_{t+1}} < 0, \tag{183}$$
$$\therefore \quad \gamma_{f'}\frac{d(f'(k_{t+1}))}{dn_{t+1}} + \gamma_r\frac{dr_{t+1}}{dn_{t+1}} < 0.$$

The relations in (182) and (183) reveal that an increase in the population growth rate lowers the capital intensity and increases both rates of return.[63] The decrease in the share of the risky asset in the portfolio reflects, that the present value of life-cycle wealth decreases, as the human capital deteriorates with higher birth rates. Hence, as the supply of government debt is unchanged, the share of the safe asset must be larger in the new equilibrium. Consequently, as the demand for safe bonds shrinks due to the decrease in Ω_t, the government has to offer a higher safe interest rate to sell its debt. Differentiation of (175) and (170) now yields the induced changes in the equity premium:

$$\frac{dE_t[\Pi_{t+1}]}{dn_{t+1}} = \left(\frac{d(f'(k_{t+1}))}{dr_{t+1}} - 1\right)\frac{dr_{t+1}}{dn_{t+1}} = \left(\frac{\frac{d\gamma_t}{dn_{t+1}}\frac{dn_{t+1}}{dr_{t+1}}}{\gamma_{f'}} - \frac{\gamma_r}{\gamma_{f'}} - 1\right) \gtreqless 0, \tag{184}$$

The second expression in (184) reflects the pure portfolio adjustment effect, which, as we have observed earlier, will once again increase the equity premium, i.e. $-\frac{\gamma_r}{\gamma_{f'}} - 1 > 0$. However, due to the human capital effect, the portfolio share of the risky asset decreases. Hence, as $\frac{\frac{d\gamma_t}{dn_{t+1}}\frac{dn_{t+1}}{dr_{t+1}}}{\gamma_{f'}} < 0$, the resulting change in the risk premium is ambiguous. In particular, (183) shows that the human capital effect rises with the labor endowment θ. This may explain why Börsch-Supan et al. (2007) and Brooks (2004) find that falling birth rates increase the equity premium in their large-scale OLG models where households hold lots of unrealized human capital.

[63] From (182), we have $\frac{df'(k_{t+1})}{dn_{t+1}} > 0$ in turn we find that $\frac{d\gamma}{dn_{t+1}} < 0$, together with $\gamma_r < 0$ and $\gamma_{f'} > 0$ implies that $\frac{dr_{t+1}}{dn_{t+1}} > 0$.

Accordingly, the simulations of Brooks (2002) and Geanakoplos et al. (2004) yield the opposite result for smaller three- and four-generation-overlapping models. Diagram 18 illustrates the human capital effect to the portfolio adjustment problem.

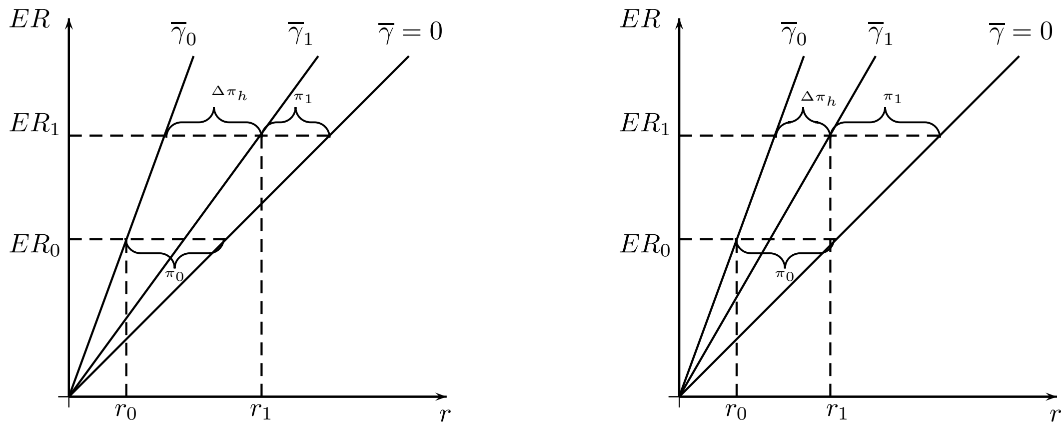


Diagram 18: *The human capital effect and portfolio adjustment.*
If the human capital effect is large (small), the equity premium is destined to fall (rise) if a large cohort enters the labor market.

6.3.2 The Portfolio Decision

In the previous section attention was confined to an economy where the TFP shock was log-normal and utility of the CRRA variety. In this section we show, that the asymmetry in the portfolio adjustment, which was driving our earlier results, carries over to a more general setting where utility is only assumed to be concave. Moreover, TFP shocks are no longer assumed to be log-normal. In this case, the two-period two-asset problem is given by:[64]

$$\max_{\gamma_t} U(\Omega_t - s_t) + \beta E_t[U((r_{t+1} + \gamma_t(R_{t+1} - r_{t+1}))s_t)], \tag{186}$$

performing a second order Taylor-series expansion of the objective function at the point where $\gamma = 0$ yields the portfolio problem:

$$\begin{aligned}\max_{\gamma_t, s_t} & U(s_t r_{t+1}) + U'(s_t r_{t+1}) E_t[\gamma_t s_t (R_{t+1} - r_{t+1})] \\ & + \frac{1}{2} U''(s_t r_{t+1}) E_t[(\gamma_t s_t (R_{t+1} - r_{t+1}))^2],\end{aligned} \tag{187}$$

where we have dropped the constant β in (187). The corresponding optimal portfolio share is thus given by:

$$\begin{aligned}\gamma_t^* &= -\frac{U'(s_t r_{t+1})}{U''(s_t r_{t+1})} \frac{E[R_{t+1} - r_{t+1}]}{s_t(\sigma_R^2 + E_t[(R_{t+1} - r_{t+1})]^2)} \\ &=_{|CRRA, r\approx 1} \frac{1}{\phi} \frac{E[R_{t+1} - r_{t+1}]}{\sigma_R^2 + E_t[(R_{t+1} - r_{t+1})]^2}.\end{aligned} \tag{188}$$

[64] Note that there are two ways to think about the savings decision:

$$\max_{\gamma_t, s_t} U(\Omega_t - s_t) + \beta E_t[U((r_{t+1} + \gamma_t(R_{t+1} - r_{t+1}))s_t)].$$

expanding (186) at the point $\gamma = 0$ and $s = \bar{s}$, we have:

$$\begin{aligned}& U(\Omega_t - \bar{s}_t) + \beta U(\bar{s}_t r_{t+1}) \\ & + \beta U'(\bar{s}_t r_{t+1}) E[R_{t+1} - r_{t+1}] \bar{s}_t \gamma_t + \frac{1}{2} \beta U''(\bar{s}_t r_{t+1}) \Big(E[R_{t+1} - r_{t+1}] \bar{s}_t \gamma_t \Big)^2 \\ & + \Big(- U'(\Omega_t - \bar{s}_t) + \beta U'(\bar{s}_t r_{t+1}) r_{t+1} \Big) (s_t - \bar{s}_t) \\ & + \frac{1}{2} \Big(U''(\Omega - \bar{s}_t) + \beta U''(\bar{s}_t r_{t+1}) r_{t+1}^2 \Big) (s_t - \bar{s}_t)^2 \\ & + \gamma_t E[R_{t+1} - r_{t+1}] \beta U'(\bar{s}_t r_{t+1}) \Big(\frac{U''(\bar{s}_t r_{t+1})}{U'(\bar{s}_t r_{t+1})} r_{t+1} \bar{s}_t + 1 \Big) (s_t - \bar{s}_t).\end{aligned} \tag{185}$$

The last term in (185) indicates the interaction between the size and the composition of the portfolio (cross derivative). There are now two ways to think of our approximation in the main text: (i) The household chooses savings according to the usual Euler equation and chooses the portfolio shares according to the Taylor approximation (187). Put differently, the household chooses savings according to a precise rule. The portfolio shares, however, rely on an approximation. (ii) The household chooses both s and γ according to (185). In this (less appealing) case there would be an additional component in the FOC for γ of ambiguous sign.

If we assume $\gamma_{f'} > 0$, we find that $\gamma_r < 0$.[65] Moreover, the portfolio adjustment is once again asymmetric:[66]

$$\gamma_r + \gamma_{f'} = -\frac{2E_t[\Pi_{t+1}]f'(k_{t+1})\sigma_z^2}{\phi(\sigma_R^2 + E_t[(R_{t+1} - r_{t+1})]^2)^2} < 0 \quad \therefore -\frac{\gamma_r}{\gamma_{f'}} > 1. \tag{189}$$

For small equity premia we may follow Campbell and Viceira (2001, 2002) and regard $E_t[(R_{t+1} - r_{t+1})]^2$ as very small. This simplifies the approximate portfolio share in (188) such that we have:

$$\gamma_t^* = -\frac{U'(s_t r_{t+1})}{U''(s_t r_{t+1})} \frac{E[R_{t+1} - r_{t+1}]}{s_t \sigma_R^2}. \tag{190}$$

Compared to the special case studied by Campbell and Viceira (2001, 2002) the formula in (190) has the advantage that we neither require log-normal shocks nor CRRA utility. Moreover, it is interesting to note that the numerator now reads $E[R_{t+1} - r_{t+1}]$ rather than $log(E[R_{t+1}]) - log(r_{t+1})$.[67] For the portfolio adjustment, with CRRA utility, we now obtain:

$$\gamma_{f'} \leqq |\gamma_r|. \tag{191}$$

Where the inequality holds if shocks are multiplicative as specified in (158). Otherwise, if shocks are additive, the adjustment is symmetric. In reality one can expect some sort of multiplicative (TFP) component and thus our comparative statics with $\gamma_{f'} < |\gamma_r|$ shall point in the right direction.

6.3.3 Discussion

We have considered the consequences of the demographic transition with regard to the two key interest rates. In our log-linear example at the out-set, the asymmetric

[65]The respective partial derivatives are $\gamma_{f'} = \frac{\sigma_R^2 - E[\Pi]^2 - E[\Pi]2f'\sigma_z^2}{C}$ and $\gamma_r = \frac{-\sigma_R^2 + E[\Pi]^2}{C}$ where C is a positive constant. Hence, if $\gamma_{f'} > 0$, then $\gamma_r < 0$.

[66]For large equity premia, the term $E_t[(R_{t+1} - r_{t+1})]^2$ in the denominator grows very large compared to $E_t[(R_{t+1} - r_{t+1})]$ in the numerator. Increases in $f'(k)$ may now in principle decrease the portfolio share in the risky asset.

[67]Apparently, for small rates of return, there is not much difference between the two formulas. For small x, we have the first order Taylor-series $log(1 + x) = 0 + 1 \cdot x = x$. Thus, $log(E[R_{t+1}]) - log(r_{t+1}) \approx E[R_{t+1} - r_{t+1}]$. Moreover, in the denominator, $var(zf') \approx var(log(z))$ if the variance of z is small and the (net) rate of return close to zero, i.e. $f' \approx 1$. With regard to our analysis of the equity premium, however, the expression in (190) has the disadvantage that the relative risk aversion is now a function of savings and changes during the transition. In this case it is not possible to derive appealing conditions for the relation between the risk premium and the growth rate for population.

portfolio adjustment behavior was relating the equity-premium positively to the growth rate of population. A higher level of interest was associated with a higher risk-premium and viceversa.

In a more general setting with a human-capital effect this result was dampened. An increase in the growth rate for population lowers the present value of second period human-capital and increases savings in the risky asset. Moreover, the present value of life cycle wealth falls. This effect lowers the demand for safe second-period consumption and forces the government to increase the safe rate to sell its debt. If this effect is sufficiently large, the initial conclusion may be reversed. Finally, we relaxed our assumptions on the stochastic processes and the utility function to conclude that the asymmetry in the portfolio adjustment is robust.

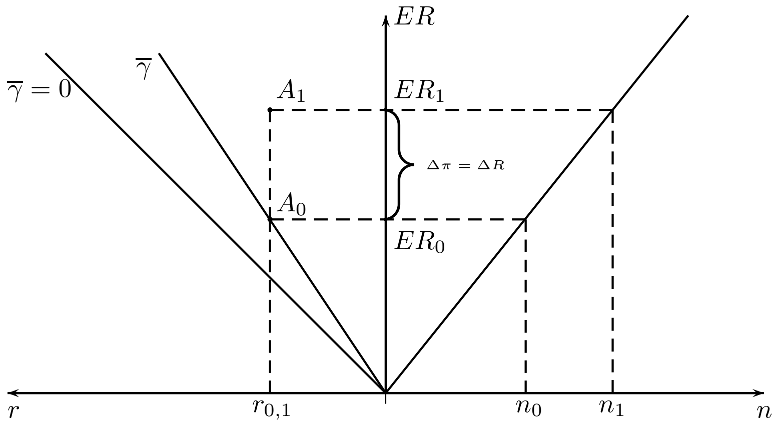


Diagram 19: *Myopic adjustment.*
If the demographic transition is unanticipated or the link between the rate of return to capital not understood, the demographic transition will lower the expected risk premium one for one.

Against this background it may be interesting to note that the relation between the demographic structure and the risk-free rate is a fragile one. If households do not understand that faster population growth dilutes the capital stock and raises the risky rate, they do not demand a higher safe return to by safe bonds. In this case, the equity premium increases one for one with the risky rate as Diagram 19 indicates. Apparently, the same effect is at work when fertility is stochastic, i.e., when each change in the growth rate for population is unanticipated as in the models of Abel (2001b, 2003).

6.4 Conclusion

Motivated by the conflicting results of previous simulation-based studies, we have discussed the link between the entrance of smaller cohorts into the labor market and the equity premium in a simple two-generations-overlapping economy. In this framework it was shown that the entrance of a large (small) cohort into the labor market will lead to a higher (lower) expected equity premium. While both rates of return rise (fall), the risky rate will rise (fall) by more than the riskless rate.

In this setting, the positive link between the growth rate for population and the equity premium is indirect. It operates through the capital widening effect which increases the expected risky rate and forces the government to offer a higher riskless rate to sell its debt. The increase in the equity premium is solely due to the asymmetric portfolio adjustment behavior, i.e. $\gamma_{f'} < |\gamma_r|$.

Myopia on the side of the households will amplify these effects. In the special case of fully myopic households, there is a one for one relation between the risky rate and the equity premium. However, in the case where households also anticipate changes in their implicit human capital holdings, our conclusions are no-longer unambiguous. Increasing population growth lowers the present value of human capital. This reduces the demand for safe assets. Consequently, the government needs to raise its riskfree rate more than in the case without human capital. That is, the current model predicts that the rates of return to safe and risky assets will fall during the demographic transition. The consequences for the equity premium, however, are ambiguous.

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Eidesstattliche Erklärung

Hiermit erkläre ich, dass ich diese Dissertation selbständig angefertigt und mich anderer als der in ihr angegebenen Hilfsmittel nicht bedient habe, insbesondere, dass aus anderen Schriften Entlehnungen, soweit sie in dieser Dissertation nicht ausdrücklich als solche gekennzeichnet und mit Quellenangaben versehen sind, nicht stattgefunden haben.

Mannheim, 25.4.2010 Wolfgang Kuhle

Lebenslauf

Wolfgang Kuhle
Geburtsort: Göttingen
Staatsangehörigkeit: deutsch
Familienstand: ledig

Ausbildung:

- (2001) Theodor-Heuss-Gymnasium Göttingen: Abitur in den Fächern Physik, Geschichte, Chemie und Deutsch
- (2006) Freie Universität Berlin: Diplom in Volkswirtschaftslehre
- (2007)-(2010) Promotionsstudium in den Fächern VWL und Mathematik an der Universität Mannheim

Berufserfahrung:

- (2001-2002) Wehrdienst: Jägertruppe in Hammelburg
- (2005-2007) Studentischer Mitarbeiter am Lehrstuhl für Wirtschaftstheorie Freie Universität Berlin
- (2007-2010) Wissenschaftlicher Mitarbeiter am MEA Universität Mannheim