Consumption, Savings, and Insurance with Incomplete Markets

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Chapter 1

General Introduction

Generally, people don't like to change their consumption habits abruptly or frequently. Once you are used to driving a luxury car, you typically would not want to have to trade it for a second-hand, low-quality alternative. In order to avoid such unplanned swings in consumption, individuals try to insure themselves against adverse events that would reduce their purchasing capacity. For example, a car-owner might want to buy an auto liability insurance, because accidents can happen suddenly and be very costly.

However, there are many risks for which individuals can't directly buy insurance. One intuitive example, which is at the heart of this dissertation, is an unforeseen reduction in your income. This might come about because your company has been hit by bad luck while its competitors are doing well. Or it might be that the whole economy is in a downturn and wages and returns fall for everybody. Individuals might want to purchase insurance against these risks, but we do not see insurance companies offering widely available contracts to that end. When such insurance contracts are not available, economists speak of incomplete markets. The question why markets are incomplete is interesting, but not the topic of this dissertation. Rather, I take this incompleteness as given, and ask how households or the government can react to it.

One way to react to missing insurance markets is to build up private savings. This way, households can use their financial wealth to partially compensate a drop in income, be it due to job loss, negative capital returns, or retirement. In chapter 2, I evaluate how efficient this type of self-insurance is in various settings. Chapter 3 looks at a specific government policy, namely an expansion of the social security system, and asks how much households value this form of mandatory insurance. Finally, in chapter 4, households have an additional insurance channel, in that they can default on their loans when they face hard times. All three chapters have in common that households face idiosyncratic, i.e. individual-specific, labor income risk, and that insurance markets against this risk are missing by assumption. Households can self-insure with a bond and have a finite lifetime. In addition, chapters 2 and 3 also feature aggregate , i.e. economy-wide, business-cycle risk. Markets for aggregate risk are incomplete, too, but now agents can self-insure by investing in stock in addition to bonds. Another difference to chapter 4 is that the model in these two chapters contains a production sector. In the remainder of this introduction, I discuss the three chapters and their findings in more detail.

In *Chapter 2* of this dissertation, I analyze how the combination of borrowing constraints and uninsurable, idiosyncratic income risk affects the equity premium, i.e. the expected difference in the returns of stock and bonds. These are two forms of market incompleteness that have received ample attention in the literature. In standard economies with infinitely-lived households, this kind of incompleteness doesn't affect the equity premium much, implying that consumers achieve a large degree of self-insurance (see, e.g., Lucas (1994), Krueger and Lustig (2010)). For life-cycle economies, on the other hand, the picture is less clear cut.

I build a large-scale, overlapping generations model that is calibrated to U.S. statistics and use it to conduct the following experiments. I first ask whether the large effect of the zero-borrowing constraint in Constantinides, Donaldson, and Mehra (2002) quantitatively survives in a richer environment with a large number of generations, production, and social security. These features should dampen their mechanism, but to what extent is an open, quantitative question. I then add idiosyncratic labor income risk that does not depend on aggregate risk. Given that the zero-borrowing constraint by itself can have a notable impact on the equity premium, the presence of idiosyncratic risk might or might not reinforce that. I find that introducing a zero-borrowing constraint in an economy without idiosyncratic risk increases the equity premium by 70 percent, which means that the mechanism described in Constantinides, Donaldson, and Mehra (2002) is dampened because of the large number of generations and production. With social security the effect of the zero-borrowing constraint is a lot weaker, because the retirement income allows retirement consumption to be less correlated with stock returns, thereby directly undoing the mechanism behind the borrowing constraint. In the next experiment, I introduce idiosyncratic labor income risk in an economy without a zero-borrowing constraint. I find that this increases the equity premium by 50 percent, even though the income shocks are independent of aggregate risk and are not permanent. The reason for this surprisingly strong increase is that idiosyncratic risk makes the implied natural borrowing limits much tighter, so that they have an effect similar to an exogenously imposed zero-borrowing constraint. This intuition is confirmed in the last experiment, where I add idiosyncratic risk in an economy with a zero-borrowing constraint. Here, neither the equity premium nor the Sharpe ratio change, because the zero-borrowing constraint is already tighter than the natural borrowing limits that result when idiosyncratic risk is added.

The finding that idiosyncratic risk with a homoscedastic variance increases the equity premium seems to oppose several irrelevance results, like Constantinides and Duffie (1996) or Krueger and Lustig (2010). However, the present setup differs in several respects. Probably the most important difference is the life-cycle, which gives rise to asset trading that results in a subgroup of the population holding most of the stock.

Chapter 3, which is joint work with Alexander Ludwig, studies the welfare effects of expanding a pay-as-you-go social security system. Like in the previous chapter, we consider an overlapping generations economy with aggregate and idiosyncratic risk, where markets against both risks are incomplete. In such a setting social security can increase economic efficiency by partially substituting for missing markets. The analysis is embedded in a general equilibrium framework to account for the costs of crowding out.

Prior research on social security has only considered either aggregate or idiosyncratic risk. Social security can partially substitute for incomplete markets against aggregate risk, because it enables the young to insure the old by pooling aggregate wage and return risks. It can partially substitute for incomplete markets against idiosyncratic risk, because in this model it pays the same benefits to everybody, which means that there is a lot of redistribution. By including both types of risk in one model, we can assess the contribution of each risk to the total insurance provided by the system. But more importantly, we can analyze the role played by interactions between the two risks.

We show analytically that aggregate and idiosyncratic risks interact due to the life-cycle structure of the economy. This interaction increases the welfare gains of a marginal introduction of an unfunded social security system. Adding a second interaction by making

the variance of idiosyncratic risk countercyclical further increases the welfare gains. In our quantitative experiment, raising the contribution rate from zero to two percent leads to long-run welfare gains of 3.5% of life-time consumption on average, even though the economy experiences substantial crowding out of capital. In contrast to the previous literature, this is a large, positive number. Approximately one third of these insurance gains can be attributed to the interactions between idiosyncratic and aggregate risk.
When compared to the cost of the business cycle as calculated by Lucas (1987), this welfare gain seems extraordinarily large. One reason is that we have idiosyncratic risk, which is substantially larger than the fluctuations considered by Lucas (see Krebs (2007)). Another reason is that the introduction of social security means that agents can achieve a higher life-time consumption on average, an effect that is not present in the Lucas analysis.

In *Chapter 4*, I look at an economy where households can default on their loans. This provides them with an additional channel to insure against idiosyncratic labor income risk that was not available in the previous two chapters. In contrast to the previous two chapters, I focus on an economy in partial equilibrium and derive results analytically rather than numerically.
Default is modeled as a full discharge of unsecured consumer debt like in Chapter 7 of the U.S. bankruptcy code. The market for loans is competitive and financial intermediaries adjust the price for loans so as to account for the probability of not being repaid. I show how the savings policy depends on the default punishment. Specifically, I look at a simple 3-period case with a quadratic utility function, where the agent can default only in the second period. If the punishment consists of a linear monetary cost, then the savings policy will be continuous, asymptotically approach a lower, endogenous bound, and it will have a flat section 'in the middle'. If, on the other hand, the punishment consists of exclusion from credit markets, then it will have a single discontinuity in addition.
More generally speaking, this paper addresses the question whether we can put enough structure on a model with a discrete default choice so that we can derive the properties of the policy functions analytically. The problem is that a discrete default decision will typically introduce points at which the value function is not differentiable. To overcome this, applied work frequently uses stochastic shocks to smooth out any kinks in the value function. However, I show that this approach may introduce new kinks in the value function, if the shock support is bounded.

Chapter 2

Asset Pricing in OLG Economies with Borrowing Constraints and Idiosyncratic Income Risk

2.1 Introduction

Asset prices are subject to large fluctuations that directly change a household's wealth. To what extent the fluctuations translate into consumption and welfare depends on the extent to which households can insure against them. The degree of insurance in turn is a main determinant for the average return that agents require to hold an asset. The study of asset prices, and the equity premium, is of continued interest, because it helps economists understand to what degree households are exposed to risks, to what degree they can insure against them, and what the mechanisms behind it are.

More specifically, much of the applied and theoretical literature studies under what circumstances agents can or can't efficiently self-insure when markets are incomplete. Two forms of incompleteness that have received much attention are missing insurance markets for labor income risk and borrowing constraints. In standard economies with infinitely-lived households, they typically do not affect asset prices much, implying that consumers achieve a large degree of self-insurance. For life-cycle economies, on the other hand, an exogenous borrowing constraint (Constantinides, Donaldson, and Mehra (2002), hereafter CDM) and idiosyncratic income risk (Storesletten, Telmer, and Yaron (2007)) can each individually increase the equity premium by a potentially large amount.

This paper analyzes how the combination of these two forms of market incompleteness affect asset prices in an overlapping generations (OLG) economy. I first ask whether the large effect of the zero-borrowing constraint in CDM quantitatively survives in a richer environment with a large number of generations, production, and social security. These features should dampen their mechanism, but to what extent is an open, quantitative question. I then add idiosyncratic labor income risk that does not depend on aggregate risk. Given that the zero-borrowing constraint by itself can have a notable impact on the equity premium, the presence of idiosyncratic risk might or might not reinforce that.

The intuition for why the CDM mechanism should be mitigated by a large number of generations, production, and social security is straight-forward. CDM analyze a model with only three generations: the young, the middle-aged, and the retired. This limits intertemporal consumption smoothing, as any capital income shock to the retired directly translates into a consumption shock for a third of the population. Similarly, the introduction of a zero-borrowing constraint immediately affects a third of the population, namely the young. A larger number of generations means that agents can smooth the capital income shocks during retirement, and that a potentially much smaller fraction is affected by the borrowing constraint. Essentially, increasing the number of generations means that we approach the irrelevance results of infinitely-lived agent economies. Introducing production allows for an endogenous response of aggregate capital to shocks. As a consequence, the supply of assets is not fixed to an exogenous amount like in CDM, which provides an additional margin along which the economy can respond to the introduction of a borrowing constraint. The third dampening force, social security, directly counteracts the high covariance of retirement consumption with stock returns, which is the crucial feature of the three-generations economy. The first question of the paper is by how much the three factors will mitigate the large increase in the equity premium that CDM find in their quantitative exercise.

The second question asks how the results change when households face uninsurable idiosyncratic labor income risk. Contrary to what might be expected, idiosyncratic risk in the present model increases the equity premium, even if there are no exogenous borrowing constraints. This is somewhat surprising given that the idiosyncratic risk is independent of aggregate risk. I show that this is due to tight natural borrowing limits which arise endogenously and act in a similar fashion as the exogenous borrowing limit. These natural borrowing limits arise because the household is not allowed to die in debt, which is similar to the No-Ponzi-scheme condition for infinitely-lived agents. Thus, when the exogenous borrowing constraint is introduced in an economy with idiosyncratic risk, the impact on the equity premium is not clear ex-ante. Given that both increase the equity premium individually, it could be that the combination of idiosyncratic risk and the exogenous borrowing constraint drive it even higher. On the other hand, one could offset the other, because they both prevent the young from holding stock.

To address the two questions, I build a large-scale OLG model with production and aggregate uncertainty. At every point in time, there are 65 generations, which differ due to a deterministic life-cycle profile for labor productivity. Households within a generation are identical ex-ante. In economies with idiosyncratic uncertainty there will be ex-post intragenerational heterogeneity caused by idiosyncratic shocks to labor income. Households choose how much to consume and how much to save in bonds and stock. The bond is one-period risk free, while the stock return depends on the realization of next period's aggregate shock. Trade is limited to these two assets by assumption and markets against aggregate risk are incomplete. Agents retire at the age of 65 and are not allowed to die in debt. In economies with a zero-borrowing constraint, agents cannot borrow in either asset. The social security system, if present, is a pure Pay-As-You-Go system with a fixed contribution rate. As an extension, I also look at an economy where the idiosyncratic labor income risk has a countercyclical variance (CCV).

There is a single consumption good produced by a representative firm with a Cobb-Douglas production function. The firm issues bonds and stock at an exogenously fixed debt-equity ratio to finance its capital requirements. The reason for modeling the firm's capital structure in this very simple way is that I want an exogenous supply of both assets so that there will be trade in both assets even with a zero-borrowing constraint.[1] Each period, the production function is hit by a TFP shock which directly affects the aggregate wage and the marginal product of capital. The latter is also affected by stochastic depreciation, which is a well-known mechanism to increase the variance of asset returns.

The model is parameterized in a similar way as the related literature and calibrated to

[1] An additional effect is that the stock return will be leveraged, which increases its mean and variance. The more standard case where the bond is in zero net supply and only the stock constitutes a claim on the firm's capital is nested for a debt-equity ratio of zero. Cf. e. g. Boldrin, Christiano, and Fisher (1995), or Croce (2010).

match key asset pricing statistics in the U. S., in particular the covariance of aggregate consumption growth with stock returns. The baseline economy, which has production and a large number of generations, but no idiosyncratic risk and no zero-borrowing constraint, features an equity premium of 1.6 percent. When the borrowing constraint is introduced, this increases to 2.7 percent. At the same time, the Sharpe ratio, which measures the market price per unit of risk, goes up from 0.14 to 0.23. While, as expected, this is is less than the increase that Constantinides, Donaldson, and Mehra (2002) report, it is more than the typical finding with infinitely-lived agents. However, when social security is added, the equity premium drops again to 2.0 percent. Thus our intuition that social security directly counteracts the asset pricing effect of borrowing constraints is confirmed.

When I introduce idiosyncratic labor income risk in the economy without a zero-borrowing constraint, I find that the equity premium increases to 2.4 percent, slightly less than when the zero-borrowing limit is introduced. In view of the fact that the shocks are neither permanent nor correlated with the aggregate shock, this seems surprisingly much.[2] I elaborate on this by showing that the natural borrowing limits, which are implied by the requirement that agents can't die in debt, are tight and act in a similar manner to the exogenous zero-borrowing limit. This claim is further substantiated when I look at an economy with both a zero-borrowing constraint and idiosyncratic risk: here, the equity premium and the Sharpe ratio are exactly the same as in the economy with only the exogenous borrowing constraint (and without idiosyncratic risk). The reason is that the zero-borrowing constraint is tighter than the natural borrowing limit, so the latter is ineffective. Consequently, we get an irrelevance result like in the case with infinitely-lived agents.

Finally, I perform the same experiment for labor income risk with a countercyclical variance (CCV). I find that the results are essentially the same as for idiosyncratic risk with a homoscedastic variance. This might seem like a stark difference to Storesletten, Telmer, and Yaron (2007). However, that paper and the companion papers do not explicitly analyze the difference between idiosyncratic risk with CCV and with a homoscedastic variance. One reason that CCV adds so little in the present setup is that the process does not contain a unit root, and the mapping of CCV to the aggregate state is different, as is detailed in the section 2.5.4.

[2] One table in Storesletten, Telmer, and Yaron (2008) seems to show something similar, but their numbers are inconclusive and they don't comment on it.

Related literature. The quantitative irrelevance of borrowing constraints and idiosyncratic risk in models with infinitely-lived agents has been documented by e. g. Lucas (1994), Telmer (1993), Heaton and Lucas (1996), and more recently Krusell, Mukoyama, and Smith (2011).[3] Krueger and Lustig (2010) obtain analytical results for the irrelevance of idiosyncratic risk for asset pricing with general borrowing constraints that cover those that I consider. In contrast to their setup, the present model has production, aggregate shocks that are not i.i.d., and a productivity life-cycle.

Constantinides and Duffie (1996) show that if idiosyncratic income follows a unit root process with a countercyclical variance, then it can have a large impact on asset prices. Krebs and Wilson (2004) extend their results to an endogenous growth model with production, and Storesletten, Telmer, and Yaron (2007) add the life-cycle. Krusell, Mukoyama, and Smith (2011) provide analytical and quantitative results confirming a large impact of CCV on the equity premium. Note that in the present paper, equilibria will not be autarkic.

Turning to OLG economies, Ríos-Rull (1994) and Ríos-Rull (1996) find that incompleteness of markets against aggregate risk do not matter much for asset prices and business cycles. Gomes and Michaelides (2008) have a very similar setup to the present one, but they focus on the impact of limited participation on the equity premium.

Finally, the present paper is related to the literature on endogenous borrowing limits. For the case without aggregate uncertainty, Aiyagari (1994) discusses the natural borrowing limit arising from a no-Ponzi-scheme condition. Magill and Quinzii (1994), Levine and Zame (1996), and Levine and Zame (2002) do this for economies with incomplete markets against aggregate risk. Their theoretical results are relevant for the present paper, but the approach here is a quantitative one similar to Aiyagari. Another strand of literature looks at endogenous borrowing limits arising from the possibility of default, or limited enforceability of debt contracts. Zhang (1997) and Alvarez and Jermann (2001) find that the asset pricing implications of such borrowing limits are large, which is very similar in spirit to the findings in the present paper. Alvarez and Jermann (2000) prove existence of competitive equilibrium in such economies, and Ábrahám and Cárceles-Poveda (2010) extend the setup to include production and an infinite number of agents, making it closer to the present paper, but they focus on taxation.

[3] In contrast, Krusell and Smith (1997) find a large effect of introducing borrowing constraints. This might be due to the zero net supply of the bond.

The next section presents the model. Section three gives details on the computation and the implementation of the natural borrowing limits. Section four presents the calibration. In section five the results are discussed, and in section six I conclude.

2.2 The Model

2.2.1 Demographics and Uncertainty

Time is discrete and runs from $t = 0, \dots, \infty$. At the beginning of each period t, an aggregate shock z_t hits the economy. For a given initial z_0, a date-event is uniquely identified by the history of shocks $z^t = (z_0, z_1, \dots, z_t)$. The shocks z_t follow a Markov chain with finite support $\mathcal{Z}$ and nonnegative transition matrix π_z. So $\pi_z(z_{t+1}|z_t)$ represents the probability of the shock next period given the current shock, and $\pi_z(z^t|z^{t_0})$ represents the probability of reaching date-event z^t from a given date-event z^{t_0}.

At every point in time t, the economy is populated by J overlapping generations indexed by $j = 1, \dots, J$. Each generation consists of a continuum of households of unit mass.[4] Agents within a cohort are ex-ante identical but receive an idiosyncratic shock s_j each period so that there is (ex-post) intragenerational heterogeneity with respect to the history of idiosyncratic shocks s^j. Like the aggregate shock, s_j follows a Markov chain with finite support $\mathcal{S}$ and strictly positive transition matrix π_s. The transition probabilities are $\pi_s(s_{j+1}|s_j)$ and the probability of a specific idiosyncratic shock history is $\pi_s(s^j)$. I assume that a Law of Large Numbers holds, so that $\pi_s(s^j)$ represents both the individual probability for s^j as well as the fraction of the population with that shock history; the same obtains for the transition probabilities $\pi_s(s_{j+1}|s_j)$. Finally, $\pi_s(s_j)$ denotes the unconditional probability of shock s_j.

2.2.2 Households

At any date-event z^t, a household is fully characterized by their age j and their history of idiosyncratic shocks s^j. Preferences over consumption c are represented by a recursive

[4] In contrast to the previous chapter, there is no population growth or survival risk.

utility $U_j(c, \cdot)$ of the Epstein-Zin form (Epstein and Zin (1989), Kreps and Porteus (1978)):

$$\begin{aligned} U_j(c, s^j, z^t) = & \Bigg(\left[c_j(s^j, z^t)\right]^{\frac{1-\theta}{\gamma}} \\ & + \beta \left(\sum_{z_{t+1}} \sum_{s_{j+1}} \pi_z(z_{t+1}|z_t) \pi_s(s_{j+1}|s_j) \left[U_{j+1}(c, s^{j+1}, z^{t+1})\right]^{1-\theta} \right)^{\frac{1}{\gamma}} \Bigg)^{\frac{\gamma}{1-\theta}} , \\ U_J(c, s^J, z^t) = & \; c_J(s^J, z^t) , \\ & c > 0 , \end{aligned} \tag{2.2.1}$$

where β is the discount factor and θ controls risk aversion. The parameter γ is defined as $\gamma = \frac{1-\theta}{1-\frac{1}{\phi}}$ with ϕ denoting the elasticity of intertemporal substitution. The CRRA utility specification is nested for $\theta = \frac{1}{\phi}$ which gives $\gamma = 1$.

Households inelastically supply one unit of labor until they retire at the fixed retirement age j_r. They are endowed with a deterministic life-cycle productivity profile e_j. Every period, each household receives an income shock η, which depends on his realization of s_j and may also depend on the current aggregate shock z_t. Labor income $y_j(s_j, z^t)$ is then given as

$$y_j(s_j, z^t) = w(z^t) e_j \eta(s_j, z_t) , \tag{2.2.2}$$

where $w(z^t)$ is the real, aggregate wage in terms of the consumption good at z^t. By construction, the unconditional expectation of idiosyncratic income shocks is equal to one, i.e. letting $\Pi(s_t)$ be the stationary distribution of s_t, we have that $\sum_{s_t} \Pi(s_t)\eta(s_t, z_t) = 1$ for all z_t. The idiosyncratic income shock $\eta(s_j, z_t)$ is the only channel through which s_t affects the household. Insurance markets against this risk are closed by assumption.

There are two assets that agents can trade to transfer wealth from one period to the next, called stocks and bonds. Since by assumption the cardinality of $\mathcal{Z}$ is greater than two, markets against aggregate risk are incomplete. Both the stock and the bond constitute a claim on the firm's capital in the following period. They only differ in their returns: the stock has a risky return $r_\sigma(z^{t+1})$ that depends on the realization of the aggregate uncertainty in the following period, whereas the bond pays an interest rate $r_b(z^t)$ that is one period risk-free. Households buy amounts $\sigma_j(s^j, z^t)$ of stock and $b_j(s^j, z^t)$ of bonds by selling the consumption good to the firm. The firm transforms the consumption good into next period capital. The sequential budget constraint is standard:

$$\begin{aligned} c_j(s^j, z^t) + \sigma_j(s^j, z^t) + b_j(s^j, z^t) &= (1 + r_\sigma(z^t))\sigma_{j-1}(s^{j-1}, z^{t-1}) \\ &+ (1 + r_b(z^{t-1}))b_{j-1}(s^{j-1}, z^{t-1}) \\ &+ (1 - \tau)y_j(s_j, z^t)I(j) + y_{ss}(z^t)(1 - I(j)) \ , \end{aligned} \tag{2.2.3}$$

where τ is a fixed social security contribution rate, $y_{ss}(z^t)$ is pension income from social security, and $I(j)$ is an indicator function which takes the value 1 if $j < j_r$ and 0 else (recall that j_r is the retirement age and that the process for labor income $y_j(s_j, z^t)$ is given in eq. 2.2.2). All households are born with zero assets, i.e. $\sigma_0(s^0, z^t) = b_0(s^0, z^t) = 0$.
In addition to the budget constraint, households face one of two borrowing constraints, which both are very common in the literature. The first constraint requires that households can not die with debt, or more precisely, with a negative net asset position:

$$\sigma_J(s^J, z^t) + b_J(s^J, z^t) \geq 0. \tag{NNB}$$

This is a standard constraint to rule out Ponzi-schemes in economies with finite lifetimes. Together with the requirement of positive consumption $c > 0 \forall z^t$, it implies a sequence of endogenous borrowing constraints on the net value the household can borrow at each date event.[5] Somewhat loosely, we can say that the household cannot borrow more than the present value of his worst future income stream. The second constraint an exogenously imposed zero-borrowing limit:

$$\begin{aligned} \sigma_j(s^j, z^t) &\geq 0 \\ b_j(s^j, z^t) &\geq 0 \end{aligned} \tag{ZB}$$

Households will face either the nonnegative bequest (NNB) constraint or the tighter zero-borrowing (ZB) constraint.

[5]Of course, we also need full enforceability of contracts, so that default is precluded.

2.2.3 Firms

There is a representative firm that uses capital $K(z^t)$ and labor $L(z^t)$ to produce the consumption good $Y(z^t)$. The production technology is Cobb-Douglas with capital share α and deterministic, labor-augmenting productivity growth λ. At each date-event, it is subject to a multiplicative shock to total factor productivity $\zeta(z_t)$ which depends only on the current aggregate shock:

$$Y(z^t) = \zeta(z_t)K(z^t)^{\alpha}((1+\lambda)^t L(z^t))^{1-\alpha}. \tag{2.2.4}$$

Households purchase the produced goods to satisfy their consumption needs. Alternatively, the firm can use the goods to invest in capital. Assuming zero capital adjustment costs and a stochastic depreciation rate $\delta(z_t)$, the capital stock evolves according to:

$$K(z^{t+1}) = I(z^t) + K(z^t)(1-\delta(z_t)) \tag{2.2.5}$$

The firm finances its capital requirements $K(z^{t+1})$ by issuing stock and bonds. Both one share of stock and a bond give the holder a claim on one unit of tomorrow's capital stock. The capital structure of the firm is exogenous and determined by a constant debt-equity ratio $\bar{d}$:

$$K(z^{t+1}) = \Sigma(z^{t+1}) + B(z^{t+1}) = \Sigma(z^{t+1})(1+\bar{d})\ , \tag{2.2.6}$$

where Σ and B are the amount of stock and bond issued by the firm.[6] The return on capital has to equal

$$r(z^{t+1})K(z^{t+1}) = r(z^{t+1})\Sigma(z^{t+1})(1+\bar{d}).$$

Out of this, bondholders receive

$$r_b(z^t)B(z^{t+1}) = r_b(z^t)\bar{d}\Sigma(z^{t+1})$$

[6]See, for example, Boldrin, Christiano, and Fisher (1995) or Croce (2010) for modeling capital structure this way.

and stock holders receive the rest, which is

$$r_\sigma(z^{t+1})\Sigma(z^{t+1}) = r(z^{t+1})\Sigma(z^{t+1})(1+\bar{d}) - r_b(z^t)\bar{d}\Sigma(z^{t+1}).$$

Consequently, the bond and stock returns can be calculated directly from the return on capital as

$$r_b(z^t) = \frac{1}{\bar{d}} E\left[r(z^{t+1})(1+\bar{d}) - r_\sigma(z^{t+1})|z^t\right] \tag{2.2.7}$$

$$r_\sigma(z^{t+1}) = r(z^{t+1})(1+\bar{d}) - \bar{d}r_b(z^t) \tag{2.2.8}$$

As one can see, the stock return is leveraged. This increases both its expected value as well as its variance. For $\bar{d} = 0$ we are back to the standard case where the return on capital equals the return on the risky asset.

2.2.4 Social Security

Social security is a pay-as-you-go system with a fixed contribution rate τ that is levied on labor income. Pension income $y_{ss}(z^t)$ adjusts to ensure that the social security budget is balanced in every date-event. By assumption $y_{ss}(z^t)$ does not depend on the idiosyncratic history, which means that every household receives the same pension income.[7] The case $\tau = 0.0$, i.e. an economy without a social security system, will be the baseline case.

2.2.5 Equilibrium

I will first define a competitive equilibrium, because it is economically intuitive and directly refers to the model as it has been set up. Also, we know that such equilibria exist. Then I will define the special case of a recursive competitive equilibrium, which is used in the quantitative experiments. There, I will restate all model elements in recursive form.

Definition 2.1. *For an initial aggregate state z_0, an initial distribution $\{\Pi_0(s^j)\}_j$ and associated initial stock and bond positions $\{\sigma_j(s^j, z_0), b_j(s^j, z_0)\}_j$, a competitive general equilibrium consists of sequences for household choices $\{c_j(s^j, z^t), \sigma_j(s^j, z^t), b_j(s^j, z^t)\}_j$, firm*

[7] As discussed in the previous chapter, one can argue that this is a reasonable approximation to the U.S. pension system.

choices $\{K(z^t), L(z^t)\}$*, social security settings* $\{\tau, y_{ss}(z^t)\}$*, factor prices* $\{w(z^t), r(z^t)\}$*, and asset returns* $\{r_b(z^t), r_\sigma(z^t)\}$ *such that for all* (s^j, z^t)*:*

a) *given prices and returns, household choices solve the households' problem of maximizing* (2.2.1) *subject to* (2.2.2), (2.2.3), *and either* (NNB) *or* (ZB)

b) *factor prices and firm choices are related by*

$$w(z^t) = (1-\alpha)(1+\lambda)^t \zeta(z_t) \left(\frac{K(z^t)}{L(z^t)}\right)^{\alpha} \tag{2.2.9}$$

$$r(z^t) = \alpha \zeta(z_t) \left(\frac{K(z^t)}{L(z^t)}\right)^{\alpha-1} - \delta(z_t) \tag{2.2.10}$$

c) *asset returns are given by* (2.2.7) *and* (2.2.8)

d) *the social security budget is balanced, i.e.*

$$\sum_{j=1}^{j_r-1} \sum_{s_j} (1-\tau) y_j(s_j, z^t) \pi_s(s_j) = (J - (j_r - 1)) y_{ss}(z^t) \tag{2.2.11}$$

e) *all markets clear:*

$$\begin{aligned} Y(z^t) + (1-\delta(z_t))K(z^t) & \\ &= \sum_{j=1}^{J} \sum_{s^j} c_j(s^j, z^t) \pi_s(s^j) + K(z^{t+1}) && (2.2.12) \\ K(z^t) &= \sum_{j=1}^{J} \sum_{s^j} \left(\sigma_j(s^{j-1}, z^{t-1}) + b_j(s^{j-1}, z^{t-1})\right) \pi_s(s^{j-1}) && (2.2.13) \\ \frac{1}{(1+\bar{d})} K(z^t) &= \sum_{j=1}^{J} \sum_{s^j} \sigma_j(s^{j-1}, z^{t-1}) \pi_s(s^{j-1}) && (2.2.14) \\ L(z^t) &= \sum_{j=1}^{j_r-1} e_j && (2.2.15) \end{aligned}$$

Recall that by the law of large numbers, $\pi_s(s^j)$ represents the fraction of of households with that specific idiosyncratic shock history, and that each generation has unit mass. The capital market clearing equation (2.2.13) shows that next period's capital is financed by

both stocks and bonds, and the following stock market clearing equation (2.2.14) states that total stock is always a constant fraction of aggregate capital, with bonds making up the remainder. This follows from our assumption of a constant debt-equity-ratio $\bar{d}$.
While we know that such competitive equilibria exist, we generally can't compute them.[8] To make the solution computationally feasible, the literature usually defines a recursive competitive equilibrium. I first de-trend the economy by dividing all aggregate and individual variables by the level of technology $(1+\lambda)^t$. Since in a recursive equilibrium all endogenous variables are functions of the current state, one needs to define a state space that is sufficient for solving the households' maximization problem. I follow the applied literature and use the current asset distribution, together with current idiosyncratic and aggregate shocks as the state.[9] Let the current probability distribution over current stock and bond holdings, current idiosyncratic shocks, and age be denoted by Φ.[10] The set of measures Φ is defined over $\mathcal{M}$, which is a family of subsets of $\{[\underline{\sigma}, \infty] \times [\underline{b}, \infty] \times \mathcal{S} \times \mathcal{J}\}$, where $\underline{\sigma}$ and $\underline{b}$ are implied by (NNB) or (ZB). In addition to Φ, each household needs to know their own current idiosyncratic state (σ, b, s) and the current aggregate shock z. Since a recursive equilibrium does not depend on the date-event, I drop the time index t, and use a prime for next period's variables.

Definition 2.2. *A recursive competitive equilibrium consists of a distribution Φ, measurable functions for household choices $\{c_j(\sigma, b, s; \Phi, z), \sigma'_j(\sigma, b, s; \Phi, z), b'_j(\sigma, b, s; \Phi, z)$ and an associated value function $U(\sigma, b, s; \Phi, z)$, firm choices $\{K(\Phi, z), L(\Phi, z)\}$, social security settings $\{\tau, y_{ss}(\Phi, z)\}$, factor prices $\{w(\Phi, z), r(\Phi, z)\}$, asset returns $\{r_b(\Phi), r_\sigma(\Phi, z)\}$, and a law of motion $H(\Phi, z)$ such that:*

a) given functions for prices and returns and the law of motion, the households' policy

[8] See Kubler and Polemarchakis (2004) for an existence proof in an OLG economy with stochastic production and a finite number of heterogeneous households. Miao (2006) considers the case of a continuum of infinitely-lived heterogeneous households subject to the zero-borrowing constraint zero borrowing (ZB), also with stochastic production. Alvarez and Jermann (2000) provide an existence proof for an economy with endogenous, state-dependent borrowing constraints that are similar in spirit to (NNB).

[9] See, e. g. Ríos-Rull (1996) or Krusell and Smith (1998). In general, only the existence of 'generalized Markov equilibria' can be proven, see e. g. Kubler and Polemarchakis (2004). However, Cao (2012) proves the existence of recursive equilibria with such a minimal state space consisting of the current distribution of wealth and shocks.

[10] We need a distribution which is continuous over (b, σ) because there is a continuum of agents in each generation. If there was a finite number of households, we could instead track each households' current asset holdings.

functions $\{c_j(\sigma, b, s; \Phi, z), \sigma'_j(\sigma, b, s; \Phi, z), b'_j(\sigma, b, s; \Phi, z)$ *solve*

$$\max_{c>0,\sigma',b'} U_j(\sigma, b, s; \Phi, z)$$
$$= \begin{cases} \left(c^{\frac{1-\theta}{\gamma}} + \tilde{\beta}\left(\sum_{z'}\sum_{s'}\pi_z(z'|z)\pi_s(s'|s)\, U_{j+1}^{1-\theta}\left(\sigma', b', s'; H(\Phi, z), z'\right)\right)^{\frac{1}{\gamma}}\right)^{\frac{\gamma}{1-\theta}} \\ c & \text{if } j = J \end{cases}$$

$$\begin{aligned} s.\ t. \qquad c + \sigma' + b' &= (1 + r_\sigma(\Phi, z))\sigma + (1 + r_b(\Phi))b \\ &\quad + (1-\tau)y_j(s, \Phi, z)I(j) + y_{ss}(\Phi, z)(1 - I(j)) \ , \\ y_j(s, \Phi, z) &= w(\Phi, z)e_j\eta(s, z) \ , \\ \sigma' + b' &\geq 0 \qquad\qquad \text{if } \ j = J. \end{aligned} \tag{NNB'}$$

b) *functions for prices and for firm choices are related by*

$$\begin{aligned} w(\Phi, z) &= (1-\alpha)\zeta(z)\left(\frac{K(\Phi)}{L(\Phi)}\right)^{\alpha} \\ r(\Phi, z) &= \alpha\zeta(z)\left(\frac{K(\Phi)}{L(\Phi)}\right)^{\alpha-1} - \delta(z) \end{aligned}$$

c) *functions for asset returns are given by*

$$\begin{aligned} r_b(\Phi) &= \frac{1}{\bar{d}}E\left[r(\Phi, z)(1+\bar{d}) - r_\sigma(\Phi, z)|z\right] \\ r_\sigma(\Phi, z) &= r(\Phi, z)(1+\bar{d}) - \bar{d}r_b(\Phi) \end{aligned}$$

d) *the function for social security settings ensures a balanced budget, i.e.*

$$\sum_{j=1}^{j_r-1}\sum_{s}(1-\tau)y_j(s, \Phi, z)\pi_s(s) = (J - (j_r - 1))y_{ss}(\Phi, z)$$

e) all markets clear:

$$\begin{aligned}
&\zeta(z)K(\Phi)^{\alpha}(L(\Phi))^{1-\alpha} + (1-\delta(z))K(\Phi) \\
&\quad = \sum_{j=1}^{J}\sum_{s}\int_{b}\int_{\sigma} c_j(\sigma, b, s; \Phi, z)\Phi(\sigma, b, s, j)\, db\, d\sigma + K(H(\Phi, z)) \\
&K(\Phi) = \sum_{j=1}^{J}\sum_{s}\int_{b}\int_{\sigma}(\sigma + b)\Phi(\sigma, b, s, j)\, db\, d\sigma \\
&\frac{K(\Phi)}{(1+\bar{d})} = \sum_{j=1}^{J}\sum_{s}\int_{b}\int_{\sigma}\sigma\, \Phi(\sigma, b, s, j)\, db\, d\sigma \\
&L(\Phi) = \sum_{j=1}^{j_r-1} e_j
\end{aligned}$$

f) the law of motion H is generated by the policy functions and the Markov transition matrix π_s so that

$$\Phi' = H(\Phi, z)$$

In the households' utility, $\tilde{\beta} = \beta(1+\lambda)^{\frac{1-\theta}{\gamma}}$ because of the normalization with the deterministic trend. As before, agents are born with zero assets, so that for $j = 1$, $\sigma = 0$ and $b = 0$. The condition of nonnegative bequests in the recursive equilibrium is (NNB'); recall that either this or the stricter zero-borrowing constraint (ZB') will be imposed:

$$\begin{aligned}
\sigma_j'(\sigma, b, s; \Phi, z) &\geq 0 \\
b_j'(\sigma, b, s; \Phi, z) &\geq 0
\end{aligned} \tag{ZB'}$$

By the law of large numbers, unconditional probability of receiving shock s, $\pi_s(s)$, is equal to the corresponding marginal distribution of Φ, i.e. $\pi_s(s) = \int_b\int_\sigma \Phi(\sigma, b, s, j)\, db\, d\sigma \quad \forall s, j$. This equilibrium is not stationary in the sense that Φ is not time-invariant.

2.3 Computation

2.3.1 Computational Solution

The computational procedure is the same as in the previous chapter. I restate the main elements with a focus on the application at hand, because the model notation differs. I compute the recursive equilibrium using global solution methods.[11] I follow the recent, applied literature and use the Krusell and Smith (1998) procedure to approximate the distribution Φ with a finite number of moments, and to approximate the law of motion $H(\cdot)$ by a specific functional form $\hat{H}$.[12] Intuitively, households need to know next period's prices w', r'_σ, r'_b in order to solve their maximization problem, and the approximate law of motion $\hat{H}(\cdot)$ should enable them to forecast these prices. Let the expected equity premium be $\mu(\Phi, z) = E\left[r_\sigma(H(\Phi, z), z') - r_b(H(\Phi, z))|z\right]$. The laws of motion households use are linear forecasts of next period's capital K' and next periods expected equity premium μ':

$$\hat{K}' = \psi_0^K(z) + \psi_1^K(z)\ln(K) + \psi_2^K(z)\ln(K)^2 \;, \tag{2.3.16}$$

$$\hat{\mu}'(z') = \psi_0^\mu(z') + \psi_1^\mu(z')\ln(\hat{K}') + \psi_2^\mu(z')\ln(\hat{K}')^2 \;, \tag{2.3.17}$$

where $\{\psi(z)\}_{0,1,2}^{K,\mu}$ are state-contingent coefficients. By forecasting K' households can calculate tomorrow's marginal productivity of capital and labor. Combining this with a forecast of μ' enables them to calculate the expected stock and bond returns.[13] The approximate law of motion (2.3.16-2.3.17) is close to the ones employed by Gomes and Michaelides (2008) Storesletten, Telmer, and Yaron (2007), and Krusell and Smith (1997). Note that one μ' is forecast for each z', and that the forecast depends on $\hat{K}'$. This mirrors the true equity premium one period ahead $\mu'(H(\Phi, z), z')$, which also depends on the the transition of Φ and on z'.

The coefficients $\{\psi(z)\}_{0,1,2}^{K,\mu}$ are estimated from simulations. Like Gomes and Michaelides (2008), I simulate the economy for $T = 5000$ periods and discard the first 500 to avoid the

[11]To be precise, all one can do is to approximate the recursive equilibrium numerically. So in general we compute ϵ-equilibria as defined by Kubler and Polemarchakis (2004). These are known to exist.

[12]Krueger and Kubler (2004) show that this method can yield a bad approximation if the number of generations J is not large. However, in a model with a large number of generations, like the present one, the method should perform better, since the model is more similar to an infinite horizon model.

[13]Using the equity premium instead of the bond return has two advantages: the equity premium fluctuates less, and we can prevent it from becoming negative.

impact of initial values. The initial distribution and the aggregate grids are initialized with the help of a degenerate equilibrium, which I call 'mean-shock' equilibrium and describe in the appendix 2.A. In each simulation period, I explicitly solve for the equity premium that clears bond and stock markets. While this is a time-consuming step, it improves the regressions. I use the quasi-Newton method described in Ludwig (2007) to find the fixed-point of $\left(\{\psi(z)\}_{0,1,2}^{K,\mu}\right)_{i+1} = \Psi\left(\left(\{\psi(z)\}_{0,1,2}^{K,\mu}\right)_i\right)$. The goodness of fit for the final approximation is $R^2 \geq 0.99$ for all experiments computed, which is in the usual range in the literature.[14]

For the solution of the household problem, I first transform the model so that the individual state space consists of cash-at-hand instead of stocks and bonds. This reduces the dimension of the state by one. The details on the transformations and the resulting equilibrium definitions can be found in chapter 2.A. I apply the endogenous grid method of Carroll (2006) when solving the household problem backwards. The well-known advantage is that Carroll's method avoids expensive root-finding steps in the consumption Euler equation. In the case of two assets, it has the additional advantage that instead of solving simultaneously for the optimal amount of two assets, which is a two-dimensional root-finding problem, I can keep the total savings amount fixed and solve for the optimal share invested in stock, which is only a one-dimensional problem. See appendix 2.A for details. The Fortran 2003 code and compiled binaries will be published on-line under the GNU General Public license, because its full object-orientation and parallelization contains some originality.

2.3.2 Implementation of Borrowing Constraints

Two types of borrowing constraints are central to this paper, and in general they are not trivial to implement computationally. The baseline economy requires the condition of nonnegative bequests (NNB') to hold. Together with $c > 0$, this condition implies a sequence of age- and state-dependent endogenous borrowing constraints, the natural borrowing limits $M_j(s, \Phi, z)$. They can be interpreted as the capitalized value of the worst future income stream and have to be calculated explicitly to guarantee that during the

[14] While this is the usual measure reported in the literature, it is not necessarily a good one to evaluate how close the solution is to a true equilibrium. Two complementary measures are the N-step-ahead forecast error Den Haan (2010), and the average and maximum Euler equation errors (see Judd (1992)).

simulations there are no negative bequests. The reason that agents might want to take more debt than $M_j(s,\Phi,z)$ is that the worst labor income may be very small, and agents expect high income at later ages due to the deterministic life-cycle component e_j. So if the $M_j(s,\Phi,z)$ are not explicitly calculated, then during the simulations it could happen that agents die in debt or have implicit negative consumption. However, the $M_j(s,\Phi,z)$ are endogenous objects, since they are a combined restriction on asset positions, asset returns, and labor income. To calculate them, I make the following assumption.

Assumption 2.1. $\forall\ (j,\sigma,b,s,\Phi,z)$:

$$\left(\sigma_j'(\sigma,b,s;\Phi,z)+b_j'(\sigma,b,s;\Phi,z)\right)\to -M_j(s,\Phi,z)\Rightarrow \sigma_j'(\sigma,b,s;\Phi,z)\to 0$$

This does not seem a strong assumption, since all it says is that as the agent approaches his maximum borrowing capacity, he will reduce his investment in the risky asset. This is plausible, because the agent only takes up so much debt to keep consumption positive. Also, if he was borrowing using the risky asset, he would reduce this short position, because in expectation borrowing in stock is much costlier than in bond. I check this assumption both in the household solution as well as in the simulations and never find it violated. Under assumption 2.1, the natural borrowing limits can be calculated recursively as

$$\begin{aligned}
&M_J(s,\Phi,z)=0\\
&M_j(s,\Phi,z)=\frac{1}{1+r_b(\Phi)}\Big[M_{j+1}(\underline{s},H(\Phi,z),\underline{z})\\
&\qquad -(1-\tau)y_{j+1}(\underline{s},H(\Phi,z),\underline{z})I(j+1)-y_{ss}(H(\Phi,z),\underline{z})(1-I(j+1))\Big]\ ,
\end{aligned}\tag{2.3.18}$$

where $\underline{s}$ is the smallest element of $\mathcal{S}$, which by construction yields the smallest value of the stochastic income component $\eta(s,z)$. Likewise, $\underline{z}$ is the smallest element of $\mathcal{Z}$ and by construction yields the smallest value for $\zeta(z)$. The equation formalizes the notion of the capitalized value of the worst future income stream: for every state today, I calculate the endogenous borrowing limit by subtracting the worst possible income realization tomorrow from the tightest possible borrowing constraint tomorrow and discounting that at the one-period risk-free rate $r_b(\Phi)$.[15] Assumption 2.1 essentially ensures that I can use $r_b(\Phi)$ for

[15] Agents are allowed to borrow against future pension income, if there is any. This does not correspond to the law in the U.S., which prohibits pension income to be pledged for debt. However, since in the model

discounting. Of course, in the computation I replace Φ and $H(\cdot)$ by their approximations given in (2.3.16-2.3.17). The natural borrowing limits are never binding, because a binding constraint would imply zero consumption at some date-event. Since they never bind, they do not affect the Euler equations in the solution, and during the simulations, I check that the fraction of agents at this lower bound of the distribution Φ is tiny.

When implementing the zero-borrowing constraint (ZB'), one usually faces the numerical difficulty of finding the 'kink point', i. e. the line in the state space where the constraint just binds. This is particularly problematic for high-dimensional state spaces like the present one. However, another advantage of Carroll's method of endogenous gridpoints is that it can deal well with exogenous constraints. We can simply set the lower bound of the grid for total savings $a' = \sigma' + b'$ to zero. More interestingly, note that we can deal with the lower bounds $\{M_j(s, \Phi, z)\}$ in a very similar manner, by setting the lowest gridpoint of a' slightly above the corresponding natural borrowing limit (see Hintermaier and Koeniger (2010) for a similar argument).

In the following, whenever I talk of an economy without an exogenous borrowing constraint, it means that (ZB') is not imposed, but the nonnegativity of bequests (NNB') is. On the other hand, note that (ZB') implies (NNB').

2.3.3 Computational Experiments

The experiments are designed to expose the effects of idiosyncratic risk and borrowing constraints on the equity premium and to explain the mechanisms behind it. Markets against aggregate risk are incomplete in all economies, so that we have an explicit market price of risk. All economies are recalibrated to have the same capital-output ratio, which implies that the exogenous supply of stocks and bonds remains constant. The details of the (re-)calibrations are described in the next section.

The conceptual sequence of experiments is the following. The baseline economy features complete insurance markets against idiosyncratic risk and no exogenous borrowing constraint. Then I first impose the exogenous zero-borrowing-constraint, and in the tables I call it the ZB economy. This is the thought experiment carried out by Constantinides, Donaldson, and Mehra (2002), so the results can be understood as a quantitative eval-

there is perfect enforcement of contracts, there is no reason to distinguish labor income from pension income. Kubler, Davis, and Willen (2006) have a similar specification.

uation of their mechanism in a large-scale model. Next, I look at an economy without an exogenous borrowing constraint and without insurance markets against idiosyncratic risk. I will say that idiosyncratic risk is present and call it the IR economy. Note that the nonnegativity of bequests (NNB') has to hold and that the implied endogenous borrowing constraints will differ from the baseline economy. The third economy features both an exogenous zero-borrowing constraint and idiosyncratic risk. So insurance markets against idiosyncratic risk are closed, and I call it the ZB,IR economy.

Then, all the exercises are repeated with a social security system. The corresponding economy names will have an SS attached. As discussed in the introduction, the reason for this specific extension is that social security directly counteracts the forces underlying the mechanism of Constantinides, Donaldson, and Mehra (2002). As can be seen from the equilibrium description, I limit attention to a defined contribution system with a flat pension scheme.

Finally, I will also analyze the effect of a counter-cyclical variance of the idiosyncratic income risk. The experiments will be analogous to the cases where I allow for idiosyncratic risk, and I call the economies the CCV economy and the ZB,CCV economy, respectively. Note that in the model description, the possibility CCV was included as the idiosyncratic income shock $\eta(s, z)$ was allowed to also depend on z.

2.4 Parametrization

2.4.1 Parametrization and Calibration Strategy

Most of the model parameters are directly set to the values in Gomes and Michaelides (2008) (GM) and Storesletten, Telmer, and Yaron (2007) (STY) to stay comparable to them, since both papers analyze similar asset pricing questions in a large-scale OLG economy with idiosyncratic and aggregate uncertainty. I then calibrate the model to match three statistics in the data that are crucial for asset pricing: the variance of aggregate consumption growth $var(\frac{C_{t+1}}{C_t})$, the covariance of aggregate consumption growth with the stock return $cov(\frac{C_{t+1}}{C_t}, r_{\sigma,t})$, and the capital-to-output ratio $E(\frac{K}{Y})$.[16] The variance and covariance of aggregate consumption growth are at the heart of the equity premium puzzle as

[16]STY also match $var\left(\frac{C_{t+1}}{C_t}\right)$ and $E\left(\frac{K}{Y}\right)$, but not $cov(\frac{C_{t+1}}{C_t}, r_{\sigma,t})$.

originally stated by Mehra and Prescott (1985). In section 2.4.3, I describe how I calibrate the model to match them.

The capital-to-output ratio has a strong impact on the level of returns: when it increases, the stock and the bond return decrease by roughly the same amount. The ratio also determines average aggregate output and the exogenous supply of stocks and bonds, as is clear from eq. (2.2.6). Therefore, I keep this ratio constant at the value of 3.3 across all economies. This is achieved by varying the discount factor β as shown in table 2.1.

Table 2.1: Values for discount factor β for all economies

	baseline	ZB	IR	ZB,IR	SS	ZB,SS	CCV	ZB,CCV
β	0.97	0.96	0.91	0.90	0.99	0.99	0.90	0.90

Notes: The names for the different economies are explained in section 2.3.3 and as well as in the results section.

2.4.2 Demographics, Technology, and Preferences

A model period corresponds to one year. Households enter the model at biological age 22, retire at the age of 65, and die at 85. The deterministic life-cycle productivity profile $\{e_j\}_1^J$ is estimated from PSID data and displayed in figure 2.1. The remaining parameters are standard and their value is shown in table 2.2 together with the source where they are taken from.

Table 2.2: Values for preference and technology parameters

Parameter	Value	Source
Discount factor, β	cf. table 2.1	–
Coefficient of relative risk aversion, θ	8.0	STY
Elasticity of intertemporal substitution, ϕ	0.5	GM
Capital share, α	0.36	STY
Debt-equity-ratio, $\bar{d}$	0.66	Croce (2010)
Technological growth, λ	0.00	GM

Notes: These parameters are directly set for all economies. If not stated otherwise, the values are taken from Gomes and Michaelides (2008) (GM) and Storesletten, Telmer, and Yaron (2007) (STY).

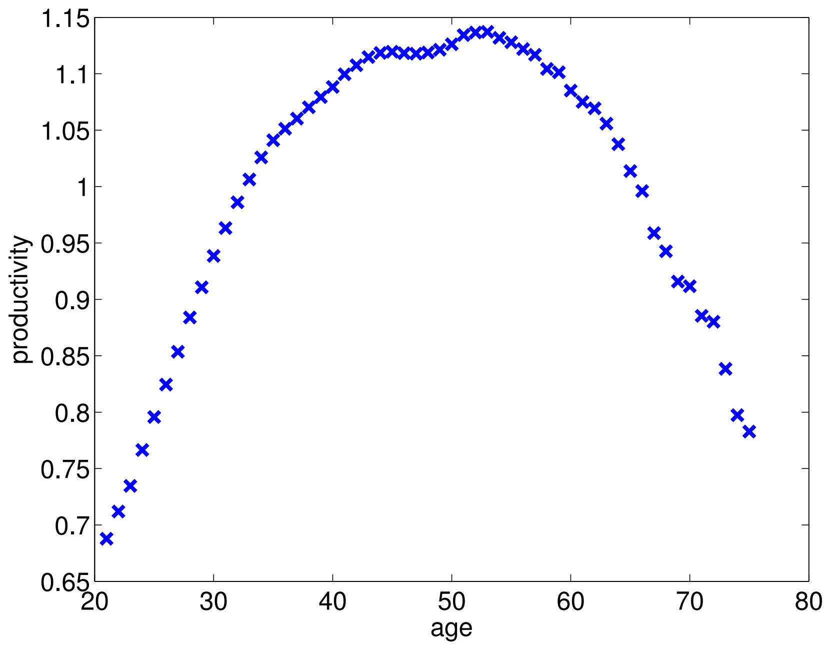

Figure 2.1: Deterministic life-cycle productivity profile $\{e_j\}_1^J$ estimated from PSID data.

2.4.3 Aggregate Shocks

There are two types of aggregate shocks: the TFP shock $\zeta(z)$ and the depreciation shock $\delta(z)$. Each can take on two values. I specify a symmetric 2x2 transition matrix for each, π_ζ and π_δ, and from this construct the 4x4 transition matrix for the aggregate shocks π_z.[17] This allows me to match the autocorrelation of TFP shocks together with the covariance of TFP and depreciation shocks. I jointly calibrate π_ζ, π_δ, and the variance of the depreciation shocks σ_δ to match the autocorrelation of TFP shocks, the variance of consumption growth $var(\frac{C_{t+1}}{C_t})$, and $cov(\frac{C_{t+1}}{C_t}, r_{\sigma,t})$. The target values for $var(\frac{C_{t+1}}{C_t}) = 0.00127$ (corresponding to a a standard deviation of 0.036), and $cov(\frac{C_{t+1}}{C_t}, r_{\sigma,t}) = 0.00219$ are those from Mehra and Prescott (1985).[18] The autocorrelation for TFP $cor(\zeta_t, \zeta_{t-1}) = 0.43$ is estimated from NIPA data after linearly detrending the the Solow residual. The values are shown in table 2.3.

2.4.4 Idiosyncratic Shocks

An idiosyncratic shock s affects the household only through the stochastic idiosyncratic component of income η. Consequently, use estimates of the empirical income process to

[17] Details on the construction of the transition matrices can be found in section 2.A.
[18] See also the values in Kocherlakota (1996).

Table 2.3: Parametrization of aggregate uncertainty

Parameter	Value	Target/ Source
Mean of TFP shocks, $\bar{\zeta}$	1.00	GM
Std. dev. of TFP shocks, σ_ζ	0.02	GM
Mean of depreciation shocks, $\bar{\delta}$	0.08	GM
Std. dev. of depreciation shocks, σ_δ	0.07	$var(\frac{C_{t+1}}{C_t}) = 0.00127$
Transition prob. TFP, $\pi^\zeta(1,1)$	0.66	$cor(\zeta_t, \zeta_{t-1}) = 0.43$
Cond. prob. depreciation, $\pi^\delta(1\|\zeta)$	0.50	$cov(\frac{C_{t+1}}{C_t}, r_{\sigma,t}) = 0.00219$

Notes: The first three parameters are taken from Gomes and Michaelides (2008) (GM); the last three parameters are jointly calibrated to match the three targets.

set the transition matrix for idiosyncratic shocks, π_s. Specifically, I take the estimates from Storesletten, Telmer, and Yaron (2004), because as an extension, I analyze the case of a countercyclical variance of the income risk CCV (see section 2.5.4). They estimate an income process of the following kind:

$$\ln(\eta)_{i,t} = \rho \ln(\eta)_{i,t} + \epsilon_{i,t} \quad , \ \epsilon_{i,t} \sim \mathcal{N}(0, \sigma^2_{\epsilon,t}) \tag{2.4.19}$$

The CCV enters through the time-dependence of the variance of the innovations, $\sigma^2_{\epsilon,t}$. However, note that in all experiments but the CCV extension, ϵ will be homoscedastic.[19] Their estimates are displayed in table 2.4. I then use the Rouwenhorst method to create the transition matrix π_s and the values for $\eta(s,z)$.[20] It is important to point how the CCV maps into the aggregate state of the economy: a high TFP shock is associated with the low CCV and v. v. Thus, the booms in Storesletten, Telmer, and Yaron (2004) are mapped to TFP shocks, not to depreciation shocks. However, I calibrate the correlation between TFP and depreciation shocks explicitly.

[19]The homoscedastic variance for ϵ is calculated as $\sigma_\epsilon = (\sigma_\epsilon(1) + \sigma_\epsilon(2))/2$.

[20]Kopecky and Suen (2010) show that the Rouwenhorst method usually yields a better approximation to the continuous process than traditional methods like Tauchen. More importantly, it fits the CCV case very well, because it will yield different the values for $\eta(s,z)$ for each z, but will leave π_s unchanged.

Table 2.4: Parametrization of idiosyncratic uncertainty

Parameter	Value	Source
Autocorrelation of $\ln(\eta)$, ρ	0.95	STY
Std. dev. of idios. income shock, σ_ϵ	0.17	STY
Std. dev. of idios. income shock with CCV, $\sigma_\epsilon(z)$	$\{0.21, 0.13\}$	STY
Mean of idios. income shock, $\bar{\eta}$	1.00	–

Notes: These parameters are directly set for all economies. The values are taken from Storesletten, Telmer, and Yaron (2007) (STY).

2.5 Results

2.5.1 Asset Prices

The asset pricing effects of zero-borrowing constraints and of idiosyncratic risk are displayed in table 2.5. The third column shows the data for the U. S. (1970-1998) as reported by Campbell (2003). There, one can see the well known stylized facts of a low risk-free rate - which here is the bond return $r_{b,t}$- and a high equity premium $E(r_{\sigma,t} - r_{b,t})$. The Sharpe ratio, defined as $\frac{E(r_{\sigma,t}-r_{b,t})}{\sqrt{Var(r_{\sigma,t})}}$, is a measure of the market price of risk and amounts to 0.31 in the U. S. over this period. Since consumption-based asset pricing models typically can't generate a high equity premium, the Sharpe ratio is helpful in understanding to what extent this is a failure of generating a high price of risk (as opposed to having a too small amount of risk in the economy). Finally, the standard deviation of the bond return much smaller than the standard deviation of the stock return (by a factor of about 10). The table does not report the values for the calibration targets, specifically $var(\frac{C_{t+1}}{C_t}) = 0.00127$, $cov(\frac{C_{t+1}}{C_t}, r_{\sigma,t}) = 0.00219$, and $E(\frac{K}{Y}) = 3.3$, because all model versions match them very closely. These and other aggregate statistics are relegated to appendix 2.B.

The model-generated moments of the baseline economy are shown in the second column. The mean and standard deviation of the bond return at 1.63 percent and 1.13 percent, respectively, are reasonably close to the data. The mean stock return, on the other hand, amounts to less than half the empirical value of 6.93 percent, and the standard deviation of stock returns is about two thirds that of the data. The classic 'equity premium puzzle' is apparent: the model-value of 1.60 percent is far below the empirical 5.44 percent. The Sharpe-ratio of the baseline economy of 0.14 is also far below its empirical counterpart, which implies that even if the standard deviation of stock returns did match the data, the

Table 2.5: Asset pricing moments

Variable	Moment	Data	Baseline	ZB	IR	ZB, IR
Bond return, $r_{b,t}$	Mean	1.49	1.63	1.31	1.51	1.17
	Std. Dev.	1.69	1.13	1.13	1.05	1.03
	AR(1)	0.57	0.92	0.91	0.90	0.90
Stock return, $r_{\sigma,t}$	Mean	6.93	3.23	4.05	3.86	3.91
	Std. Dev.	17.5	11.7	11.7	11.7	11.7
	AR(1)	0.05	0.03	0.03	0.03	0.03
Equity premium	Mean	5.44	1.60	2.74	2.38	2.74
Sharpe ratio	Mean	0.31	0.14	0.23	0.20	0.23

Notes: The empirical moments for the U. S. (1970-1998) are from Campbell (2003). The model equity premium is $E(r_{\sigma,t} - r_{b,t})$, and the model Sharpe ratio is $\frac{E(r_{\sigma,t}-r_{b,t})}{\sqrt{Var(r_{\sigma,t})}}$.

model wouldn't get close to the empirical equity premium.

The picture changes when a zero-borrowing constraint is introduced. The equity premium goes up by more than one percentage point, which represents an increase of 71 percent. This translates into a substantial increase in the Sharpe ratio: at a value of 0.23, the distance to the empirical value is half that of the baseline economy. Consequently, the mechanisms laid down by Constantinides, Donaldson, and Mehra (2002) in their stylized model survive to this much richer environment. Due to the large number of generations and production with physical capital accumulation, the impact of zero-borrowing constraints on the equity premium is somewhat more moderate than in their quantitative exercise. However, the impact on the Sharpe ratio is similar.

The sixth column shows the numbers for the economy with only idiosyncratic risk (IR), but no exogenous zero-borrowing constraint. Compared to the baseline, the equity premium increases by 0.8 percentage points, which represents an increase of 50 percent, somewhat less than the increase in the ZB case. The same goes for the Sharpe ratio. This result opposes the many studies with infinitely-lived agents that report basically zero changes in the equity premium (see, e. g. Lucas (1994), and Heaton and Lucas (1996)). Storesletten, Telmer, and Yaron (2008) find very similar numbers for the equity premium, but in their case, the Sharpe ratio actually decreases.[21] Before analyzing the reasons, let's look at the last column, which shows the economy with both a zero-borrowing-constraint and

[21]Note that this is a different paper from the better known Storesletten, Telmer, and Yaron (2007), in which they only look at the asset pricing implications of CCV. I elaborate on the reasons for my different finding in section 2.5.4.

idiosyncratic risk (ZB,IR). The equity premium and the Sharpe ratio are essentially the same as in the economy with only the zero-borrowing constraint (ZB). This is the most striking finding and a crucial stepping stone to understanding the mechanisms at work. To this aim, I will next show the effect that the exogenous zero-borrowing limit (ZB) and the endogenous natural borrowing limit have on portfolio choices.

2.5.2 Borrowing Limits and Portfolio Choices

Recall that in all economies households are subject to the nonnegative bequest constraint. In the baseline and IR case, it leads to state-dependent, endogenous natural borrowing limits, which are computed according to (2.3.18). Figure 2.2 shows these natural borrowing limits for the baseline and the IR case, together with the zero-borrowing limit and a natural borrowing limit for an infinitely-lived agent. Note that the limits are defined on total savings, i. e. $b'_j + \sigma'_j \geq M_j$, and that the graph shows the average over all states, i. e. the mean of state-dependent limits. The dotted line for the infinitely-lived agent is calculated like in Aiyagari (1994), using the idiosyncratic income shocks, the average wage, and the average bond return from the IR economy.[22] The three lines for the baseline, the IR, and the ZB case meet at zero when the agent retires, since he does not have retirement income. Before retirement, the natural borrowing limit in the IR economy is much tighter, because the worst possible income realization is much closer to zero, due to the presence of idiosyncratic shocks. Indeed it is closer to the ZB line, which could help explain why the equity premium in the IR case increases not quite as much as in the ZB case. The (ZB,IR) case is not shown in the figure, because its line corresponds to the ZB line. As was just discussed, the equity premium and the Sharpe ratio are also the same for ZB and (ZB,IR). Finally, the line for the infinitely-lived agent looks like a lower bound. It is, but only to the IR line, not necessarily to the ZB line. The reason that ZB lies above the line for the infinitely-lived agent is the numerical calibration. This sheds light on why idiosyncratic risk in infinitely-lived economies does not have an impact on the equity premium: even with idiosyncratic risk, the Aiyagari-style natural borrowing limit is very lax (see, e.g. Zhang (1997)).

The essence of figure 2.2 and the preceding paragraph is that it seems as if the endogenous,

[22] It is correct to take the average, aggregate wage and bond return, instead of the worst realizations, because in the graph we average over all states.

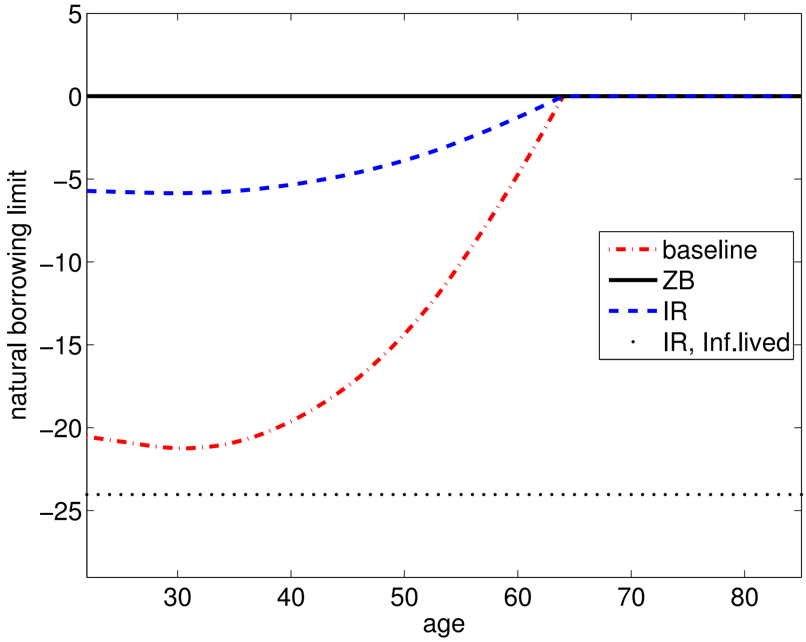

Figure 2.2: Natural borrowing limits M_j for the baseline economy, the economy with a zero-borrowing-constraint (ZB), the economy with idiosyncratic risk (IR), and an economy with infinitely-lived agents and idiosyncratic risk (IR, Inf.lived).

natural borrowing limit is responsible for the high equity premium and Sharpe ratio in the IR case. This is supported by the fact that the ZB case and the (ZB,IR) case have the same borrowing limit and the same equity premium and Sharpe ratio. To understand how the borrowing constraints affect the household, I now turn to portfolio choices.

Figure 2.3 shows the average total savings, $b'_j + \sigma'_j$, and the average share invested in stock, $\frac{\sigma'_j}{b'_j+\sigma'_j}$, over the life-cycle.[23] Negative total savings mean that the agent is borrowing to increase consumption. But he can also borrow to invest in stock, which is the case whenever the share invested in stock is larger than one. Generally speaking, both total savings and the share invested in stock correspond to the typical life-cycle profile. Households accumulate savings until retirement, then they dissave and reach zero in the final period. The young invest heavily in stock, because their life-time expected income is relatively safe and they want to benefit from the large returns on equity. As their capitalized value of labor income diminishes over age, they reduce their share in stock.

First, I want to highlight that in the baseline economy, the households do have negative

[23]Averages are taken over the distribution Φ.

total savings, i. e. net debt positions. Recall that only the average is plotted, so individual households might be taking up much more debt. They can do so because the natural borrowing limit in this economy is very lax. They want to do so in order to smooth consumption, both in response to aggregate income fluctuations, and in view of the larger future income due to the steeply increasing deterministic productivity profile.

Turning to the economy with a zero-borrowing constraint, we see in figure 2.3 that the young now are stuck at zero total savings. They would like to borrow by short-selling the bond and invest in stock, but are not allowed to. Consequently, the share invested in stock is flat at one until the age of 50, and only then starts decreasing. After the age of 45, households in the ZB economy hold a substantially larger fraction of their assets in stock than in the baseline economy. The reason is the following: since the young can't hold as much stock as in the baseline, but the stock supply remains constant, the equity premium has to rise to induce the middle aged and old to hold the stock. In fact, it has to rise substantially, because the middle-aged and old households do not like stock as an asset to save for retirement. Since they have no social security income, their consumption growth will covary strongly with returns, thus the large equity premium. This is the essence of Constantinides, Donaldson, and Mehra (2002).

Now we come to the most interesting part of figure 2.3, the economy with idiosyncratic risk (and no zero-borrowing constraint). The total savings graph shows only very few negative total savings. This is in line with the much tighter natural borrowing limit displayed in figure 2.2. At the same time, the young invest less in stock and the old more than in the baseline case. In other words, the share invested in stock in the IR case is right in between the baseline and the ZB case. This highlights the similarity of introducing a zero-borrowing constraint to introducing idiosyncratic risk. The tight endogenous borrowing limits not only prevent the young households from borrowing, but also shift the life-cycle of portfolios in a fashion similar to that of the zero-borrowing constraint. The young hold less stock to avoid being pushed to the endogenous borrowing limit by bad returns. This means that they accumulate assets less quickly, so that the older households have to hold a larger share of equity to realize enough consumption in retirement. However, this means that the covariance of consumption growth of the households facing or being in retirement with stock returns increases. This increases the equity premium. Obviously, the channel is basically the same as with the zero-borrowing-constraint.

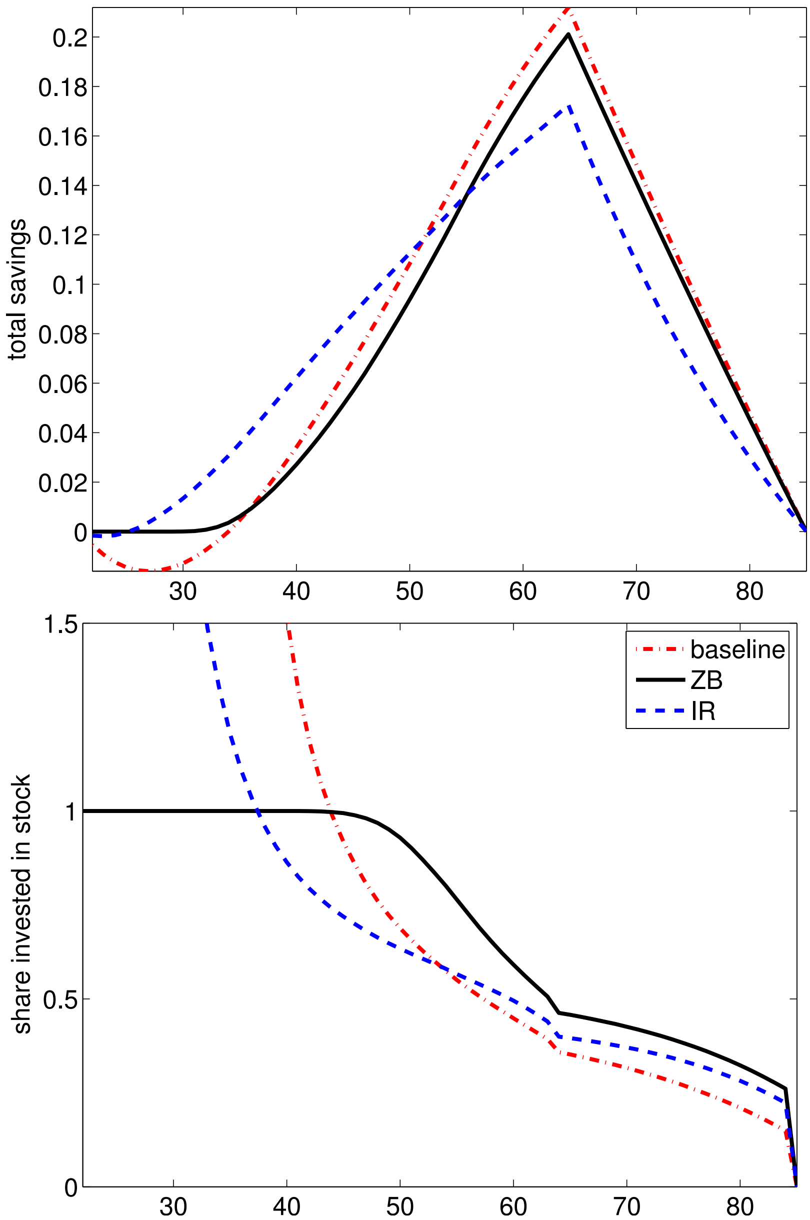

Figure 2.3: Life-cycle profiles for total savings, $b'_j + \sigma'_j$, and the share invested in stock, $\frac{\sigma'_j}{b'_j+\sigma'_j}$; baseline economy, economy with zero-borrowing constraint (ZB), and economy with idiosyncratic risk (IR).

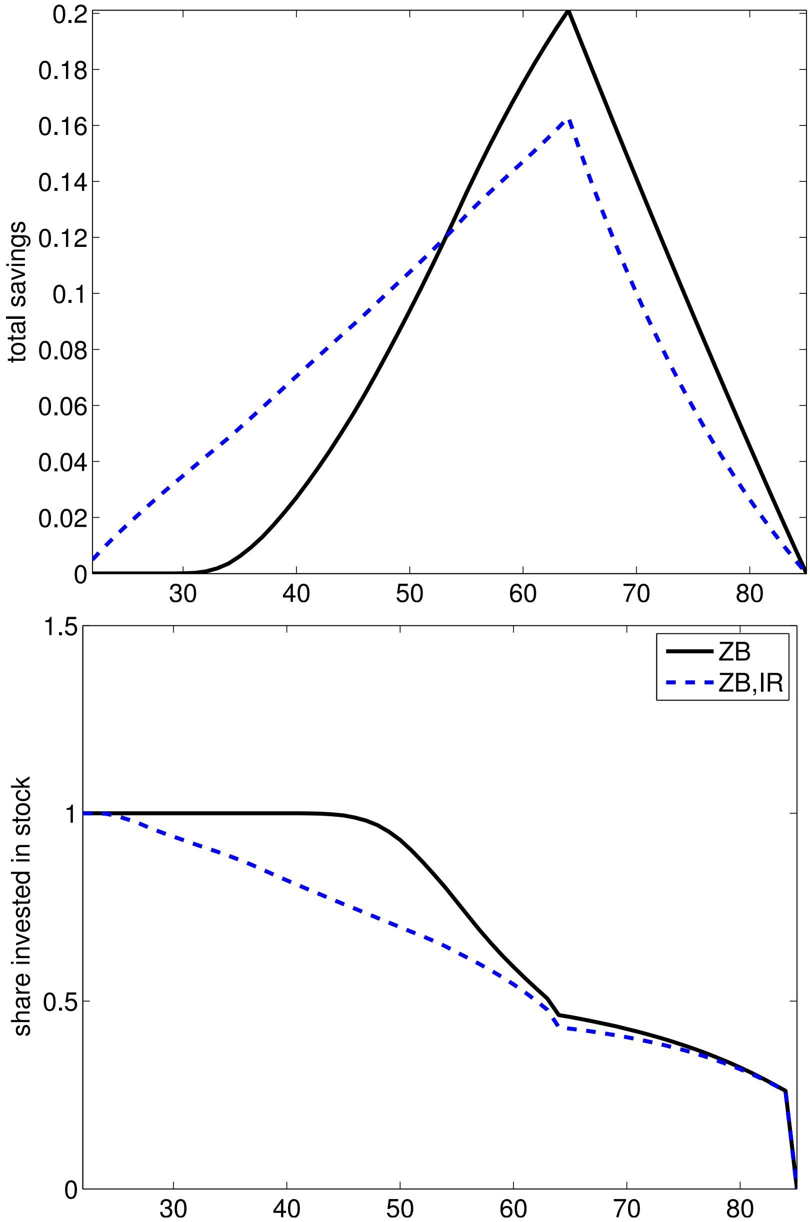

Figure 2.4: Life-cycle profiles for total savings, $b'_j + \sigma'_j$, and the share invested in stock, $\frac{\sigma'_j}{b'_j+\sigma'_j}$; zero-borrowing economy (ZB) and zero-borrowing economy with idiosyncratic risk (ZB,IR).

Figure 2.4 displays the average life-cycle profiles for the economy with both a zero-borrowing constraint and idiosyncratic risk. The lines for the ZB economy are plotted as a reference, because the two economies yield the same equity premium and Sharpe ratio. When households face both idiosyncratic risk and a zero-borrowing constraint, they start to accumulate positive savings immediately. The reason is that they want to avoid a binding borrowing constraint, because then a sequence of bad idiosyncratic shocks will directly translate into diminishing consumption. As a consequence, the (ZB,IR) line runs above the (ZB) line until the age of 55. On first glance, it might seem surprising that the life-cycle profiles of the share invested in stock differ so much, given that the two economies yield the same results for the equity premium and the Sharpe ratio. Specifically, households reduce their share invested in stock quickly when they also have idiosyncratic risk, which can be explained again by their desire to avoid the borrowing constraint. Thus the prefer the safer bond. However, from the age of 60 onward, the two curves differ only very slightly, which is not surprising, since there is no more idiosyncratic labor income risk after retirement at 65.[24] The drop in the relative demand for stock of the households aged 30 to 60 puts an upward pressure on the equity premium, even if the drop is smaller than those observed in 2.3. The fact that the premium doesn't move means that it is still the households aged 60 and over who price the stock. Therefore, the mechanisms laid down by Constantinides, Donaldson, and Mehra (2002) are robust to the introduction of idiosyncratic risk.

2.5.3 Social Security

The previous section has shown that the mechanism of Constantinides, Donaldson, and Mehra (2002), namely that the households facing retirement price the stock, survives to richer environments with production, long life-spans, and idiosyncratic risk. Their argument seems to crucially depend on the absence of social security, because only then does the consumption growth of the retired covary strongly with stock returns. If, on the other hand, the retired receive social security benefits, then they have a potentially very efficient way of smoothing their consumption, because the shocks to stock returns display a very low persistence in both the data and the model, as was shown in table 2.5.

[24]In the graph for the share invested in stock, one line seems to lie below the other everywhere. At the same time the total savings curve shifts, so that the aggregate demand for both assets doesn't change and is equal to the constant aggregate supply for each.

The results in table 2.6 confirm this intuition. The value for the social security contribution rate τ is set to the current value in the U. S. of 12 percent. Social security strongly decreases the equity premium if a zero-borrowing constraint is present, but it has basically no effect on the premium in the baseline economy. In the baseline economy, there are two

Table 2.6: Asset pricing moments with social security

Variable	Moment	Baseline	SS	ZB	ZB,SS
Bond return, $r_{b,t}$	Mean	1.63	1.93	1.31	1.56
Equity premium	Mean	1.60	1.58	2.74	2.07
Sharpe ratio	Mean	0.14	0.14	0.23	0.17

Notes: The numbers in columns 3 and 5 are the same as in table 2.5. Columns 4 and 6 show the baseline economy with social security (SS) and with a zero-borrowing constraint and social security (ZB,SS). The social security contribution rate is set to the U. S. value of 12 percent. See table 2.5 for further explanations.

opposing forces: the older households don't require such a high equity premium, because their retirement consumption covaries less with stock returns if they receive social security income. This decreases the equity premium. On the other side, the young invest a smaller share in stock, because they do not need to accumulate as much savings for retirement. This increases the equity premium. The net effect in the present calibration is zero. However, with zero borrowing constraints, only the first channel is at work, while the second is basically shut down. As can be seen in figure 2.5, the reduction of the share invested in stock is small, because both in (ZB) and (ZB,SS) the upper bound is binding. Figure 2.5 shows that households save less for retirement in the SS economy. However, they start saving much earlier because of the higher average returns, whereby they achieve a much flatter lifetime consumption profile. It is worth pointing out that figures 2.4 and 2.5 look qualitatively very similar, but lead to opposing effects on the equity premium. The reason for the different results is of course the differing effects on the consumption growth of the retired: idiosyncratic risk does not affect its covariance with stock returns, whereas social security does.

In order to evaluate whether the tight natural borrowing limits in an economy with idiosyncratic risk again play a similar role to the zero-borrowing constraints, I repeat the same comparison for the IR economy and the (IR,SS) economy. I find results similar, but less strong than the once just reported. The equity premium decreases by 0.4 percentage points (as opposed to 0.67 in table 2.6), and the life-cycle profiles shift in a similar way as

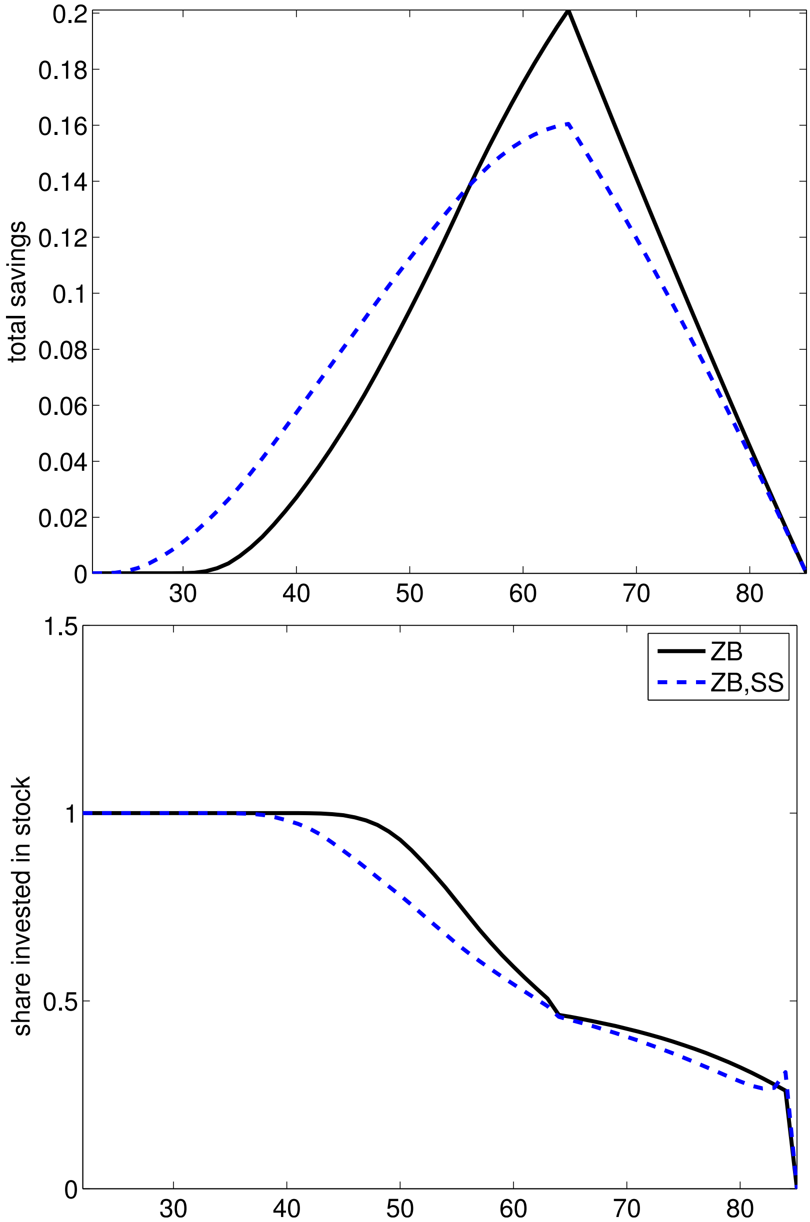

Figure 2.5: Life-cycle profiles for total savings, $b'_j + \sigma'_j$, and the share invested in stock, $\frac{\sigma'_j}{b'_j+\sigma'_j}$; zero-borrowing economy (ZB) and zero-borrowing economy with social security (ZB,SS).

in figure 2.5. This again lends support to the hypothesis that the natural borrowing limits play an important role.

2.5.4 Countercyclical Variance of Income Risk

As an extension, I analyze how the results change when the idiosyncratic income risk has a countercyclical variance (CCV). This is very similar to Storesletten, Telmer, and Yaron (2007) and Storesletten, Telmer, and Yaron (2008), but they do not look at zero-borrowing constraints. Table 2.7 compares the economy with homoscedastic idiosyncratic risk (IR) to the one with a countercyclical variance (CCV), and then compares these two cases with borrowing constraints present, i. e. (ZB,IR) to (ZB,CCV). I find that the equity premium and the Sharpe ratio don't change much in the first comparison, and even less in the second. The result that CCV does not have a strong impact in this model is also reflected in the portfolio choices of the household in figure 2.6, which barely change.

Table 2.7: Asset pricing moments with CCV

Variable	Moment	IR	CCV	ZB,IR	ZB,CCV
Bond return, $r_{b,t}$	Mean	1.51	1.74	1.17	1.06
Equity premium	Mean	2.38	2.49	2.74	2.78
Sharpe ratio	Mean	0.20	0.21	0.23	0.23

Notes: The numbers in columns 3 and 5 are the same as in columns 6 and 7 of table 2.5; economy with countercyclical variance of idiosyncratic income risk (CCV), and economy with a zero-borrowing constraint and CCV (ZB,CCV). See table 2.5 for further explanations.

The reason I find a smaller effect of CCV than Storesletten, Telmer, and Yaron (2008) lies in a slightly different calibration.[25] They have two aggregate states: the economic expansion features a 'good' TFP shock together with a small depreciation shock, and v. v. for the recession. The variance of idiosyncratic income shocks is larger in a recession. In contrast, I calibrate the co-movements of TFP and depreciation shocks to match the covariance of consumption growth with stock returns in the data. Like in STY, the variance

[25] Note that table 3 in Storesletten, Telmer, and Yaron (2008) shows results that are similar to the ones presented here for a risk aversion coefficient of 3. Specifically, in that case, they find that idiosyncratic risk *without* CCV accounts for most of the increase in the equity premium and the Sharpe ratio. Their results are qualitatively different for a risk aversion of 8, but in that case adding idiosyncratic risk actually decreases the equity premium and the Sharpe ratio. So overall their evidence is inconclusive. The better known paper Storesletten, Telmer, and Yaron (2007) does not report the case with homoscedastic variance.

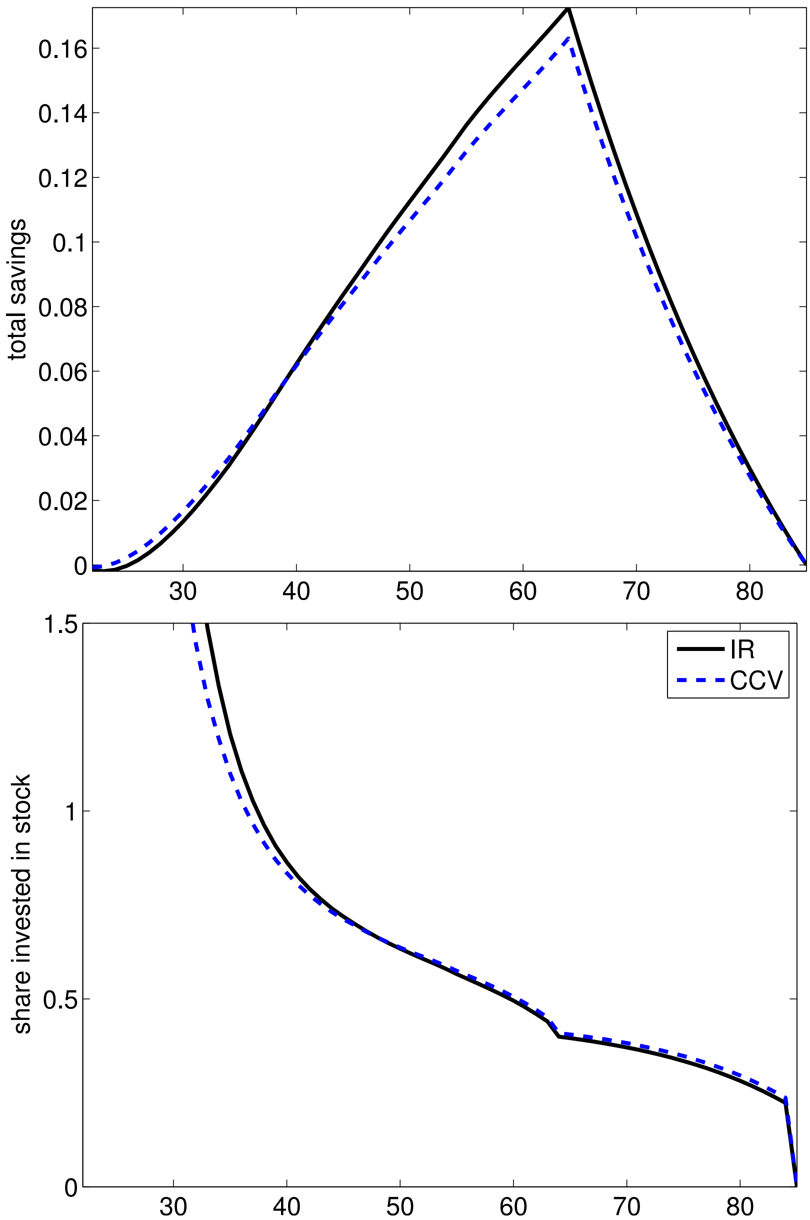

Figure 2.6: Life-cycle profiles for total savings, $b'_j + \sigma'_j$, and the share invested in stock, $\frac{\sigma'_j}{b'_j+\sigma'_j}$; economy with idiosyncratic risk (IR) and economy with idiosyncratic risk and a countercyclical variance of the idiosyncratic income shocks (CCV).

of the income risk is high when the TFP shock is 'bad', but this does not necessarily imply a large depreciation shock in my case. It is then clear that the effect of CCV must be stronger in STY, because households can receive a larger, adverse idiosyncratic income shock if and only if the aggregate wage and stock returns are low. To sum up this section, the main new finding regarding CCV is that it does not increase the equity premium if there is an exogenous zero borrowing constraint. This should not be too surprising given the previous results on homoscedastic, idiosyncratic risk.

2.6 Conclusion

This paper has shown that in an overlapping generations economy, idiosyncratic risk has a similar effect to an exogenous borrowing constraint, because it tightens the natural borrowing limit. As a consequence, idiosyncratic risk increases the equity premium by a similar amount like an exogenously imposed zero-borrowing-limit. This seems to oppose several irrelevance results, like Constantinides and Duffie (1996) or Krueger and Lustig (2010), which state that a countercyclical variance is necessary for idiosyncratic risk to have asset pricing effects. However, the present setup differs in several respects. Probably the most important difference the life-cycle, which gives rise to asset trading that results in a subgroup of the population holding most of the stock. Another interesting finding was that when idiosyncratic risk is added in an economy with a zero-borrowing constraint, then the equity premium does not increase. This is so because the zero-borrowing constraint is tighter than the natural borrowing limit, so that the latter is ineffective, and only the first affects the pricing of the assets.

I also looked at the case with an idiosyncratic variance of the income risk. In contrast to most of the quantitative literature, I find that it does not add much on top of the case of a homoscedastic variance. The reason is that I calibrate the depreciation and TFP shocks to match the covariance of aggregate consumption growth with stock returns. However, this implies that aggregate wages and returns are not perfectly negatively correlated like in Storesletten, Telmer, and Yaron (2007), but instead are closer to the empirical correlation. As a consequence, it can happen that in a recession, i. e. when an adverse TFP shock hits the economy, the stock return goes up. Then, agents do not mind the countercyclical variance as much, because they are somewhat hedged by the less-than-perfect correlation of wages with stock returns. In future work, it would be interesting to analyze the sensitivity of the countercyclical variance of income risk quantitatively in a structured way.

Finally, I have shown that the equity-premium effect of the borrowing constraint largely disappears when the retired receive social security income. However, I looked only at one specific system, namely a Pay-As-You-Go system with a constant contribution rate and full redistribution. While a constant benefits system should yield similar results, a system that does not fully redistribute and instead ties benefits to contributions might yield different results. If, for example, benefits are tied only to very few income realizations, then the system might even increase the equity premium. This is left for future research.

2.A Appendix: Computational Solution

Aggregate Problem

Mean Shock Equilibrium

As an initialization step, I solve for a degenerate path of the economy where the realizations of all aggregate shocks are at their respective means. I accordingly set $z = \bar{z} = \mathbb{E}z$ and $\delta = \bar{\delta} = \mathbb{E}\delta$. I assume that households accurately solve their forecasting problem for each realization of the aggregate state. This means that I approximate the above approximate law of motion as

$$(k', \mu') = \hat{\hat{H}}(k, \mu, \bar{z}, \bar{z}') \tag{2.A.20}$$

Observe that in the two stationary equilibria of the model, I have that fixed point relation

$$(k', \mu') = \hat{\hat{H}}(k, \mu, \bar{z}, \bar{z}') = (k, \mu) \tag{2.A.21}$$

With these assumptions, I can solve the mean shock path by standard Gauss-Seidel iterations as, e.g., described in Auerbach and Kotlikoff (1987). I adopt the modifications described in Ludwig (2007). While the numerical methods are the same as in the solution to a deterministic economy, the actual behavior of households fully takes into account the stochastic nature of the model. This also means that I solve the household problem using recursive methods and store the solutions to the household problem on grids of the idiosyncratic state x. The fixed-point computed in this auxiliary equilibrium gives k^{ms} and μ^{ms} as aggregate moments and cross-sectional distributions of agents as induced by the mean shock path. I denote these distributions by Φ^{ms}.

Recursive Equilibrium

In order to solve for the stochastic recursive equilibria of the model, I use simulation methods. To this end, I specify the approximate law of motion as:

$$\ln(k_{t+1}) = \psi_0^k(z) + \psi_1^k(z)\ln(k_t) + \psi_2^k(z)\ln(k_{t+1}) \tag{2.A.22a}$$

$$\ln(\mu_{t+1}) = \psi_0^\mu(z') + \psi_1^\mu(z')\ln(k_{t+1}) + \psi_2^\mu(z')\ln(k_{t+1}) \tag{2.A.22b}$$

Like in Krusell and Smith (1997), the forecast for k_{t+1} is used to forecast μ_{t+1}. Intuitively, k_{t+1} contains a lot of information on the savings choice of the agent and therefore on the returns next period. Note that, in each period, μ_t is an “endogenous state”, the realization of which has to be pinned down in that particular period (in contrast to k_t which is given in period t from decisions $t-1$). As in the standard application of the Krusell and Smith (1998) method, the coefficients also depend on the realization of the aggregate state, z.

To construct the grids for the the aggregate states k and μ, $\mathcal{G}^k$, $\mathcal{G}^\mu$, define scaling factors s^k and s^μ and the number of grid points, n. I set $s^k = 0.8$ $s^\mu = 0.6$, and $n = 7$. Using these factors, I construct symmetric grids around k^{ms}, μ^{ms}.

I assume that aggregate risk is driven by a four state Markov chain with support $Z = \{z_1, \ldots, z_4\}$ and transition matrix $\pi = (\pi_{ij})$. Each aggregate state maps into a combination of low or high technology shocks and low or high physical capital depreciation. To be concrete, I let

$$\zeta(z) = \begin{cases} 1 - \bar{\zeta} \text{ for } z \in z_1, z_2 \\ 1 + \bar{\zeta} \text{ for } z \in z_3, z_4 \end{cases} \quad \text{and } \delta(z) = \begin{cases} \delta_0 + \bar{\delta} \text{ for } z \in z_1, z_3 \\ \delta_0 - \bar{\delta} \text{ for } z \in z_2, z_4. \end{cases} \tag{2.A.23}$$

With this setup, z_1 corresponds to a low wage and a low return, while z_4 corresponds to a high wage and a high return.

To calibrate the entries of the transition matrix, denote by $\pi^\zeta = \pi(\zeta' = 1 - \bar{\zeta} \mid \zeta = 1 - \bar{\zeta})$ the transition probability of remaining in the low technology state. Assuming that the transition of technology shocks is symmetric, I then also that $\pi(\zeta' = 1 + \bar{\zeta} \mid \zeta = 1 + \bar{\zeta}) = \pi^\zeta$ and, accordingly $1 - \pi^\zeta = \pi(\zeta' = 1 - \bar{\zeta} \mid \zeta = 1 + \bar{\zeta}) = \pi(\zeta' = 1 + \bar{\zeta} \mid \zeta = 1 - \bar{\zeta})$.

To govern the correlation between technology and depreciation shocks, let the probability of being in the high (low) depreciation state conditional on being in the low (high) technology state, assuming symmetry, be $\pi^\delta = \pi(\delta' = \delta_0 + \bar{\delta} \mid \zeta' = 1 - \bar{\zeta}) = \pi(\delta' = \delta_0 - \bar{\delta} \mid \zeta' = 1 + \bar{\zeta})$. I then have that the transition matrix of aggregate states follows from the corresponding assignment of states in (2.A.23) as

$$\pi^z = \begin{bmatrix} \pi^\zeta \cdot \pi^\delta & \pi^\zeta \cdot (1-\pi^\delta) & (1-\pi^\zeta)\cdot(1-\pi^\delta) & (1-\pi^\zeta)\cdot \pi^\delta \\ \pi^\zeta \cdot \pi^\delta & \pi^\zeta \cdot (1-\pi^\delta) & (1-\pi^\zeta)\cdot(1-\pi^\delta) & (1-\pi^\zeta)\cdot \pi^\delta \\ (1-\pi^\zeta)\cdot \pi^\delta & (1-\pi^\zeta)\cdot(1-\pi^\delta) & \pi^\zeta \cdot (1-\pi^\delta) & \pi^\zeta \cdot \pi^\delta \\ (1-\pi^\zeta)\cdot \pi^\delta & (1-\pi^\zeta)\cdot(1-\pi^\delta) & \pi^\zeta \cdot (1-\pi^\delta) & \pi^\zeta \cdot \pi^\delta \end{bmatrix}$$

In sum, the Markov chain process of aggregate shocks is characterized by four parameters, $(\bar{\zeta}, \bar{\delta}, \pi^\zeta, \pi^\delta)$. All of these parameters are second stage parameters which I calibrate jointly to match the following targets: (i) an average variance of the cyclical component of TFP, again estimated from NIPA data, (ii) the average fluctuation of the risky return which features a standard deviation in the data of 0.16, (iii) the autocorrelation of the cyclical component of TFP in the data and (iv) the estimated correlation of the cyclical component of TFP with risky returns.

Household Problem

To define equilibrium I adopt a de-trended version of the household model. I therefore first describe transformations of the household problem and then proceed with the equilibrium definition.

Transformations

Following Deaton (1991), define cash-on-hand by $X_{i,j,t} = A_{i,j,t}(1+r_t^f+\kappa_{i,j-1,t-1}(r_t - r_t^f)) + \mathrm{Y}_{i,j,t}$. The dynamic budget constraint then rewrites as

$$X_{i,j+1,t+1} = (X_{i,j,t} - C_{i,j,t})(1 + r_{t+1}^f + \kappa_{i,j,t}(r_{t+1} - r_{t+1}^f)) + \mathrm{Y}_{i,j+1,t+1} \qquad (2.A.24)$$

I next transform the problem to de-trend the model and work with stationary variables throughout. That is, I de-trend with the deterministic trend component induced by technological progress. Along this line, define by $x_{i,j,t} = \frac{X_{i,j,t}}{\Upsilon_t}$ transformed cash-on-hand and all other variables accordingly. Using $\omega_t = \frac{w_t}{\Upsilon_t}$ to denote wages per efficiency unit I have

$$\mathrm{y}_{i,j,t} = \begin{cases} (1-\tau)\epsilon_j \omega_t \eta_{i,j,t} & \text{for } j < jr \\ b_t & \text{for } j \geq jr. \end{cases}$$

Now divide the dynamic budget constraint (2.A.24) by Υ_t and rewrite to get

$$x_{i,j+1,t+1} = (x_{i,j,t} - c_{i,j,t})\tilde{R}_{i,j+1,t+1} + \mathrm{y}_{i,j+1,t+1}. \tag{2.A.25}$$

where $\tilde{R}_{i,j+1,t+1} = \frac{(1+r^f_{t+1}+\kappa_{i,j,t}(r_{t+1}-r^f_{t+1}))}{1+g}$.

Transform the per period utility function accordingly and take an additional monotone transformation to get

$$u_{i,j,t} = \left[c_{i,j,t}^{\frac{1-\theta}{\gamma}} + \tilde{\beta}_{j+1} \left(\mathbb{E}_{i,j,t} \left[(u_{i,j+1,t+1})^{1-\theta} \right] \right)^{\frac{1}{\gamma}} \right]^{\frac{\gamma}{1-\theta}} \tag{2.A.26}$$

where $\tilde{\beta}_{j+1} = \beta \left(1+g\right)^{\frac{1-\theta}{\gamma}}$.

Recursive Solution

I iterate on the Euler equation, using ideas developed in Carroll (2006). The transformed dynamic programming problem of the household reads as

$$u(j,\cdot) = \max_{c,\kappa} \left\{ \left[c^{\frac{1-\theta}{\gamma}} + \tilde{\beta} \left(\mathbb{E} \left[u(j+1, (x-c)\tilde{R}' + \mathrm{y}', \cdot)^{1-\theta} \right] \right)^{\frac{1}{\gamma}} \right]^{\frac{\gamma}{1-\theta}} \right\} \tag{2.A.27}$$

where $x' = a'\tilde{R}' + \mathrm{y}'$, with $\tilde{R}' = \frac{(1+r^{f\prime}+\kappa(r'-r^{f\prime}))}{(1+g)}$, and $\tilde{\beta} = \beta\varsigma_{j+1}\left((1+g)\right)^{\frac{1-\theta}{\gamma}}$.

The first-order conditions can be rewritten as

$$\begin{aligned} c: \quad & c^{\frac{1-\theta-\gamma}{\gamma}} - \tilde{\beta}\left(\mathbb{E}\left[u(j+1,\cdot)^{1-\theta}\right]\right)^{\frac{1-\gamma}{\gamma}} \dots \\ & \cdot \mathbb{E}\left[u(j+1,\cdot)^{\frac{(1-\theta)(\gamma-1)}{\gamma}} (c')^{\frac{1-\theta-\gamma}{\gamma}} \tilde{R}'\right] = 0 \qquad \text{(2.A.28a)} \\ \kappa: \quad & \mathbb{E}\left[u(j+1,\cdot)^{\frac{(1-\theta)(\gamma-1)}{\gamma}} (c')^{\frac{1-\theta-\gamma}{\gamma}} \left(r' - r^{f\prime}\right)\right] = 0 \qquad \text{(2.A.28b)} \end{aligned}$$

With respect to the numerical solution, I interpolate the functions $u(j,\cdot)$ and $c(j,\cdot)$. Note that I can expect $u(j, \cdot)$ to be approximately linear, since in period J it is simply given by $u(J) = c_J = x_J$.

Next, notice that $u(j+1, \cdot)$ and c' are functions of $(x - c)$ so that c shows up on both sides of the equation in (2.A.28a). This would require calling a non-linear solver whenever I solve optimal consumption and portfolio shares. To alleviate this computational burden I employ the endogenous grid method of Carroll (2006). So instead of the usual exogenous grid for x (and the usual endogenous grid for savings, $s = x - c$), the exogenous grid is $s = x - c$ and the endogenous grid is x.

2.B Appendix: Additional Tables

Moments of Endogenous Variables

In the following tables, the equity premium is the average of the excess returns, $E(ER_t) = E(r_{\sigma,t} - r_{b,t})$.

Table 2.8: Aggregate statistics, baseline economy

Variable	AVG	STD	CV	AR(1)
Capital, K_t	6.94E+00	1.20E+00	1.73E-01	9.27E-01
Output, Y_t	2.00E+00	1.26E-01	6.29E-02	8.72E-01
Investment I_t	6.25E-01	5.94E-02	9.50E-02	1.91E-01
Excess return, $(r_{\sigma,t} - r_{b,t})$	1.60E-02	1.16E-01	7.25E+00	-5.36E-03
Stock return, $r_{\sigma,t}$	3.23E-02	1.17E-01	3.62E+00	3.15E-02
Bond return, $r_{b,t}$	1.63E-02	1.13E-02	6.96E-01	9.20E-01
Consumption, C_t	1.38E+00	1.07E-01	7.76E-02	9.27E-01
Cons. growth, $(\frac{C_{t+1}}{C_t} - 1)$	4.72E-04	3.15E-02	6.25E+01	-3.84E-02
Wage rate, w_t	1.28E+00	8.06E-02	6.29E-02	8.72E-01
TFP shocks, $zeta$	1.00E+00	2.00E-02	2.00E-02	4.38E-01
Capital-output-ratio, $\frac{K}{Y}$	3.44E+00	3.74E-01	1.09E-01	9.26E-01

Notes: AVG: average, STD: standard deviation, CV: coefficient of variation, AR(1): autocorrelation coefficient.

Table 2.9: Variance-covariance matrix, baseline economy

	$r_{\sigma,t}$	w_t	ζ_t	Y_t	I_t	ER_t
$r_{\sigma,t}$	1.37E-02					
w_t	-8.62E-04	6.50E-03				
ζ_t	6.87E-05	4.63E-04	4.00E-04			
Y_t	-1.35E-03	1.00E-02	7.23E-04	1.59E-02		
I_t	-4.78E-03	2.49E-03	7.23E-04	3.89E-03	3.52E-03	
C_t	3.43E-03	7.55E-03	5.57E-07	1.18E-02	4.91E-04	4.53E-03
$c_{g,t}$	2.53E-03	-3.64E-04	5.24E-05	-5.69E-04	-1.18E-03	-2.41E-04
δ_t	-8.15E-03	7.61E-05	2.84E-06	1.19E-04	2.82E-03	-8.14E-03

Notes: δ_t is the depreciation shock, $c_{g,t} = (\frac{C_{t+1}}{C_t} - 1)$, $ER_t = (r_{\sigma,t} - r_{b,t})$, all other variables explained in table 2.8.

Table 2.10: Aggregate statistics, zero-borrowing (ZB) economy

Variable	AVG	STD	CV	AR(1)
Capital, K_t	6.56E+00	1.06E+00	1.62E-01	9.16E-01
Output, Y_t	1.96E+00	1.17E-01	5.97E-02	8.57E-01
Investment I_t	5.91E-01	5.19E-02	8.79E-02	1.67E-01
Excess return, $(r_{\sigma,t} - r_{b,t})$	2.74E-02	1.16E-01	4.25E+00	-5.41E-03
Stock return, $r_{\sigma,t}$	4.05E-02	1.17E-01	2.89E+00	-3.37E-02
Bond return, $r_{b,t}$	1.32E-02	1.13E-02	8.62E-01	9.14E-01
Consumption, C_t	1.37E+00	1.10E-01	8.00E-02	9.39E-01
Cons. growth, $(\frac{C_{t+1}}{C_t} - 1)$	4.21E-04	3.05E-02	6.59E+01	1.02E-01
Wage rate, w_t	1.26E+00	7.50E-02	5.97E-02	8.57E-01
TFP shocks, $zeta$	1.00E+00	2.00E-02	2.00E-02	4.38E-01
Capital-output-ratio, $\frac{K}{Y}$	3.32E+00	3.39E-01	1.02E-01	9.15E-01

Notes: AVG: average, STD: standard deviation, CV: coefficient of variation, AR(1): autocorrelation coefficient.

Table 2.11: Variance-covariance matrix, zero-borrowing (ZB) economy

	$r_{\sigma,t}$	w_t	ζ_t	Y_t	I_t	ER_t
$r_{\sigma,t}$	1.37E-02					
w_t	-7.88E-04	5.62E-03				
ζ_t	6.88E-05	4.61E-04	4.00E-04			
Y_t	-1.23E-03	8.55E-03	7.21E-04	1.37E-02		
I_t	-4.22E-03	1.35E-03	6.30E-04	2.10E-03	2.69E-03	
C_t	2.98E-03	7.21E-03	9.11E-05	1.13E-02	-3.63E-04	4.11E-03
$c_{g,t}$	2.53E-03	-1.83E-04	9.34E-05	-2.86E-04	-8.42E-04	2.85E-04
δ_t	-8.15E-03	9.04E-05	3.45E-06	1.41E-04	2.55E-03	-8.14E-03

Notes: δ_t is the depreciation shock, $c_{g,t} = (\frac{C_{t+1}}{C_t} - 1)$, $ER_t = (r_{\sigma,t} - r_{b,t})$, all other variables explained in table 2.10.

Table 2.12: Aggregate statistics, idiosyncratic risk (IR) economy

Variable	AVG	STD	CV	AR(1)
Capital, K_t	6.58E+00	9.92E-01	1.51E-01	9.08E-01
Output, Y_t	1.97E+00	1.11E-01	5.63E-02	8.43E-01
Investment I_t	5.92E-01	5.91E-02	9.97E-02	1.39E-01
Excess return, $(r_{\sigma,t} - r_{b,t})$	2.36E-02	1.16E-01	4.93E+00	-4.75E-03
Stock return, $r_{\sigma,t}$	3.86E-02	1.17E-01	3.03E+00	-3.36E-02
Bond return, $r_{b,t}$	1.51E-02	1.05E-02	7.00E-01	9.02E-01
Consumption, C_t	1.37E+00	1.14E-01	8.32E-02	9.10E-01
Cons. growth, $(\frac{C_{t+1}}{C_t} - 1)$	6.54E-04	3.52E-02	5.37E+01	-4.66E-02
Wage rate, w_t	1.26E+00	7.08E-02	5.63E-02	8.43E-01
TFP shocks, $zeta$	1.00E+00	2.00E-02	2.00E-02	4.38E-01
Capital-output-ratio, $\frac{K}{Y}$	3.33E+00	3.18E-01	9.54E-02	9.07E-01

Notes: AVG: average, STD: standard deviation, CV: coefficient of variation, AR(1): autocorrelation coefficient.

Table 2.13: Variance-covariance matrix, idiosyncratic risk (IR) economy

	$r_{\sigma,t}$	w_t	ζ_t	Y_t	I_t	ER_t
$r_{\sigma,t}$	1.37E-02					
w_t	-6.85E-04	5.02E-03				
ζ_t	6.93E-05	4.70E-04	4.00E-04			
Y_t	-1.07E-03	7.84E-03	7.34E-04	1.23E-02		
I_t	-5.37E-03	8.57E-04	6.98E-04	1.34E-03	3.49E-03	
C_t	4.30E-03	6.99E-03	3.61E-05	1.09E-02	-2.15E-03	5.38E-03
$c_{g,t}$	3.07E-03	-4.16E-04	6.85E-05	-6.49E-04	-1.50E-03	-3.52E-04
δ_t	-8.15E-03	7.25E-05	2.52E-06	1.13E-04	3.30E-03	-8.14E-03

Notes: δ_t is the depreciation shock, $c_{g,t} = (\frac{C_{t+1}}{C_t} - 1)$, $ER_t = (r_{\sigma,t} - r_{b,t})$, all other variables explained in table 2.12.

Table 2.14: Aggregate statistics, zero-borrowing and idiosyncratic risk (ZB,IR)

Variable	AVG	STD	CV	AR(1)
Capital, K_t	6.67E+00	9.86E-01	1.48E-01	9.05E-01
Output, Y_t	1.98E+00	1.09E-01	5.54E-02	8.38E-01
Investment I_t	6.00E-01	6.02E-02	1.00E-01	1.49E-01
Excess return, $(r_{\sigma,t} - r_{b,t})$	2.74E-02	1.16E-01	4.24E+00	-5.07E-03
Stock return, $r_{\sigma,t}$	3.91E-02	1.17E-01	2.99E+00	-3.37E-02
Bond return, $r_{b,t}$	1.17E-02	1.03E-02	8.74E-01	9.00E-01
Consumption, C_t	1.38E+00	1.17E-01	8.48E-02	9.06E-01
Cons. growth, $(\frac{C_{t+1}}{C_t} - 1)$	7.07E-04	3.66E-02	5.18E+01	-4.69E-02
Wage rate, w_t	1.26E+00	7.00E-02	5.54E-02	8.38E-01
TFP shocks, $zeta$	1.00E+00	2.00E-02	2.00E-02	4.38E-01
Capital-output-ratio, $\frac{K}{Y}$	3.36E+00	3.14E-01	9.35E-02	9.04E-01

Notes: AVG: average, STD: standard deviation, CV: coefficient of variation, AR(1): autocorrelation coefficient.

Table 2.15: Variance-covariance matrix, zero-borrowing and idiosyncratic risk

	$r_{\sigma,t}$	w_t	ζ_t	Y_t	I_t	ER_t
$r_{\sigma,t}$	1.37E-02					
w_t	-6.58E-04	4.91E-03				
ζ_t	6.85E-05	4.73E-04	4.00E-04			
Y_t	-1.03E-03	7.66E-03	7.39E-04	1.20E-02		
I_t	-5.56E-03	6.44E-04	6.82E-04	1.01E-03	3.62E-03	
C_t	4.53E-03	7.02E-03	5.70E-05	1.10E-02	-2.62E-03	5.60E-03
$c_{g,t}$	3.23E-03	-4.27E-04	7.89E-05	-6.67E-04	-1.60E-03	-3.74E-04
δ_t	-8.15E-03	7.14E-05	2.52E-06	1.12E-04	3.43E-03	-8.14E-03

Notes: δ_t is the depreciation shock, $c_{g,t} = (\frac{C_{t+1}}{C_t} - 1)$, $ER_t = (r_{\sigma,t} - r_{b,t})$, all other variables explained in table 2.14.

Estimates of the Approximate Law of Motion

Table 2.16: Coefficients of law of motion, baseline economy

Dependent variable: $\log(K')$				
z	constant	$\log(K)$	$\log(K)^2$	R^2
1	1.86E-01	8.10E-01	2.97E-02	0.9999
2	2.37E-01	8.76E-01	1.59E-02	1.0000
3	1.84E-01	8.30E-01	2.31E-02	1.0000
4	2.47E-01	8.82E-01	1.31E-02	1.0000
Dependent variable: μ'				
z'	constant	$\log(K')$	$\log(K')^2$	R^2
1	1.28E-02	1.61E-04	6.60E-04	0.9965
2	1.34E-02	-2.87E-04	8.24E-04	0.9970
3	1.26E-02	3.05E-04	6.21E-04	0.9969
4	1.39E-02	-8.62E-04	9.74E-04	0.9963

Table 2.17: Coefficients of laws of motion, zero-borrowing (ZB) economy

Dependent variable: $\log(K')$				
z	constant	$\log(K)$	$\log(K)^2$	R^2
1	2.02E-01	7.95E-01	3.20E-02	0.9998
2	2.43E-01	8.75E-01	1.39E-02	0.9999
3	2.41E-01	7.68E-01	3.78E-02	0.9998
4	2.25E-01	9.11E-01	2.98E-03	0.9998
Dependent variable: μ'				
z'	constant	$\log(K')$	$\log(K')^2$	R^2
1	4.29E-02	-2.15E-02	6.90E-03	0.9127
2	4.44E-02	-2.45E-02	8.19E-03	0.9028
3	3.90E-02	-1.79E-02	6.06E-03	0.9117
4	4.46E-02	-2.53E-02	8.45E-03	0.8939

Table 2.18: Coefficients of law of motion, idiosyncratic risk (IR) economy

Dependent variable: $\log(K')$				
z	constant	$\log(K)$	$\log(K)^2$	R^2
1	1.95E-01	8.12E-01	2.49E-02	1.0000
2	2.82E-01	8.39E-01	2.19E-02	1.0000
3	2.24E-01	8.00E-01	2.68E-02	1.0000
4	3.06E-01	8.29E-01	2.32E-02	1.0000
Dependent variable: μ'				
z'	constant	$\log(K')$	$\log(K')^2$	R^2
1	1.73E-02	1.66E-03	7.69E-04	0.9981
2	1.82E-02	9.64E-04	1.03E-03	0.9983
3	1.71E-02	1.66E-03	7.81E-04	0.9980
4	1.88E-02	1.33E-04	1.25E-03	0.9981

Table 2.19: Coefficients of law of motion, zero-borrowing and idiosyncratic risk

Dependent variable: $\log(K')$				
z	constant	$\log(K)$	$\log(K)^2$	R^2
1	2.09E-01	8.04E-01	2.61E-02	1.0000
2	2.93E-01	8.31E-01	2.28E-02	1.0000
3	2.36E-01	7.92E-01	2.79E-02	1.0000
4	3.16E-01	8.22E-01	2.41E-02	1.0000
Dependent variable: μ'				
z'	constant	$\log(K')$	$\log(K')^2$	R^2
1	2.21E-02	4.90E-04	1.10E-03	0.9587
2	2.30E-02	-2.90E-04	1.38E-03	0.9708
3	2.07E-02	1.70E-03	8.08E-04	0.9576
4	2.33E-02	-7.76E-04	1.51E-03	0.9694

Chapter 3

The Welfare Effects of Social Security in a Model with Aggregate and Idiosyncratic Risk

3.1 Introduction

Many countries operate large social security systems. One reason is that social security can provide insurance against risks for which there are no private markets. However, these systems also impose costs by distorting prices and decisions. The question arises whether the benefits of social security outweigh the costs.

We address this question in a model which features both aggregate and idiosyncratic risk. We follow the literature and assume that insurance markets for both forms of risk are incomplete. In such a setting social security can increase economic efficiency by partially substituting for missing markets. The analysis is embedded in a general equilibrium framework to account for the costs of crowding out. The difference to the previous literature is that, so far, only models with one kind of risk were examined. One strand of the literature has looked at social security when only aggregate risk is present (e.g. Krueger and Kubler (2006)). There, social security can improve the intergenerational sharing of aggregate risks. The other strand included only idiosyncratic risk (e.g. Imrohoroğlu, Imrohoroğlu, and Joines (1995, 1998)). There, social security is valuable because of intragenerational insurance. However, households face both kinds of risk over their lifetime. To get a more complete picture of how much insurance social security can provide, the different risks need to be included in one model. By doing that, we can assess the contribution of each risk to total insurance. More importantly, we can analyze the role played by interactions between the two types of risk.

The first interaction is an interaction over the life-cycle and accordingly we call it the life-cycle interaction (LCI). To better understand this new effect, consider a standard model in which idiosyncratic wage risk is statistically independent of aggregate risk. Due to the nature of a life-cycle economy, aggregate and idiosyncratic risks directly interact despite their statistical independence. The reason is that when retired, consumption is mainly financed out of private savings. The level of private savings depends on the realizations of idiosyncratic wage risk and aggregate return risk during working life. As a consequence, the variance of private savings contains an interaction term between idiosyncratic and aggregate risk. Because households face these risks for many years before they go into retirement, this interaction term becomes large.

The second interaction operates via the so-called counter-cyclical cross-sectional variance of idiosyncratic productivity shocks (CCV). This means that the variance of idiosyncratic shocks is higher in a downturn than in a boom. The CCV has been documented in the data (Storesletten, Telmer, and Yaron (2004)), and has been analyzed with respect to asset pricing (Mankiw (1986), Constantinides and Duffie (1996), Storesletten, Telmer, and Yaron (2007)). We want to understand whether social security can provide insurance against this interaction.

In order to evaluate how much these interactions matter quantitatively, we build a large-scale overlapping generations (OLG) model in the tradition of Auerbach and Kotlikoff (1987), extended by various forms of risk. Aggregate wage risk is introduced through a standard shock to total factor productivity (TFP). Aggregate return risk is introduced through a depreciation shock. The two shocks enable us to calibrate the model in such a way that it produces realistic fluctuations of wages and returns, both of which are central to the welfare implications of social security.

The social security system is a pure pay-as-you-go (PAYG) system. Every period, all the contributions are paid out as a lump-sum to all the retirees. Households can also save privately by investing in a risk-free bond and risky stock. Having this portfolio choice in the quantitative model is important, because social security can be seen as an asset with a low risk and a low return. Therefore, the risk-return structure of the bond and the stock directly affect the value of social security. In order to match a high expected risky return and a low risk-free rate at the same time we need Epstein-Zin-preferences. Finally, households also face survival risk. Therefore, they value social security because it partially substitutes for missing annuity markets.

Our experiment consists of a marginal introduction of social security. We use a two-period model to expose the new life-cycle interaction LCI. We show analytically that social security provides insurance against both LCI and against the countercyclical variance CCV. We also show analytically that the benefit of the insurance against CCV becomes larger when aggregate risk in the economy increases.

When we calibrate the model to the U.S. economy, we find that the introduction of social security leads to a strong welfare gain. This stands in contrast to the previous literature, because social security in our model provides insurance against both idiosyncratic and aggregate risk, as well as their interactions. To be precise, increasing the contribution rate from zero to two percent leads to welfare gains of 3.5% in terms of consumption equivalent variation. This welfare improvement is obtained even though we observe substantial crowding out of capital. About one third of the welfare gains is attributed to the two interactions LCI and CCV.

The welfare gains are not caused by reducing an inefficient overaccumulation of capital in the sense of Samuelson (1958) or Diamond (1965). To control for that, we ensure in our calibration that the economy is dynamically efficient. The welfare numbers do not hinge on the specific experiment: when we increase the contribution rate from 12% to 14%, the welfare gains are still positive, though smaller. When we follow different calibration strategies, the welfare numbers also retain the same sign and relative magnitude.

Related Literature. The idea that social security can insure against aggregate risks goes back to Diamond (1977) and Merton (1983). They show how it can partially complete financial markets and thereby increase economic efficiency. Building on these insights, Shiller (1999) and Bohn (2001, 2009) show that social security can reduce consumption risk of all generations by pooling labor income and capital income risks across generations if labor income and capital returns are imperfectly correlated.

Gordon and Varian (1988), Ball and Mankiw (2007), Matsen and Thogersen (2004) and Krueger and Kubler (2006) use a two-period partial equilibrium model where households consume only in the second period of life, i.e. during retirement. For our analytical results, we extend the model by adding idiosyncratic risk. In our companion paper Harenberg and Ludwig (2011) we relax the assumption of zero first-period consumption and conclude that one of the results breaks down: a smaller or negative covariance between wages and risky returns does not necessarily improve intergenerational risk-sharing. In the present paper, we address this insight by analyzing two calibrations which differ with respect to this covariance.

Quantitative papers with aggregate uncertainty and social security are scarce. Krueger and Kubler (2006) is the closest to us.[1] They also look at a marginal introduction of a PAYG system and find that it does not constitute a Pareto-improvement. The concept of a Pareto-improvement requires that they take an ex-interim welfare perspective, whereas we calculate welfare from an ex-ante perspective. Our paper differs in that it adds idiosyncratic risks and analyzes the interactions.

Quantitative papers with idiosyncratic uncertainty and social security, on the other hand, are plenty (e.g. Conesa and Krueger (1999), Imrohoroğlu, Imrohoroğlu, and Joines (1995, 1998), Huggett and Ventura (1999) and Storesletten, Telmer, and Yaron (1999)). On a general level, a conclusion from this literature is that welfare in a stationary economy without social security is higher than in one with a PAYG system. That is, the losses from crowding out dominate the gains from completing insurance markets. The more recent work by Nishiyama and Smetters (2007) and Fehr and Habermann (2008) are examples of papers which focus on modeling institutional features of existing social security systems in detail. Our approach is less policy oriented than theirs and we abstract from such details. Our results show the benefits of a flat pension scheme without additionally optimizing over the exact design of the pension benefit formula.

Huggett and Parra (2010) argue that it is important to look at a simultaneous reform of both the social security system and of the general tax system. They report strong welfare gains from joint reforms of both systems. We instead follow the more standard approach and take the general income tax system as given. Consequently, we calibrate our model to income processes after taxation.

[1] Ludwig and Reiter (2010) ask how pension systems should optimally adjust to demographic shocks. Olovsson (2010) claims that pension payments should be very risky because this increases precautionary savings and thereby welfare improving capital formation.

The remainder of this paper is structured as follows. We derive our analytical results in section 3.2. Section 3.3 develops the quantitative model and section 3.4 presents the calibration. The main results of our quantitative analysis are presented in section 3.5, where we make much use of our analytical results. We conclude in section 3.6. Proofs, computational details, and robustness checks are relegated to separate appendices.

3.2 A Two-Generations Model

We first develop an analytical model that provides useful insights for our quantitative analysis. We develop our model in several steps. We start by adopting the partial equilibrium framework of Gordon and Varian (1988), Ball and Mankiw (2007), Matsen and Thogersen (2004), Krueger and Kubler (2006) and others who assume that members of each generation consume only in the second period of life. We show that the aforementioned literature—which focuses on aggregate risk only—misses important interaction mechanisms between idiosyncratic and aggregate risk. Furthermore, as shown in Harenberg and Ludwig (2011) a two period model misses an important aspect of the inter-temporal nature of the savings problem which biases results against social security if wages and returns are positively correlated. To avoid this discussion here—which would in any case lead us on a sidetrack—, we simply shut down the correlation between wages and returns.

We argue that this simple setup can only provide a partial characterization of the total welfare effects of social security. It misses the effects of taxation on reallocation of consumption and savings as well as the welfare losses induced by crowding out. To accommodate both channels at once we extend our model to a standard Diamond (1965) model with risk. Hence, consumption and savings decisions take place in the first period and wages and returns are determined in general equilibrium.[2]

3.2.1 Households

Each period t, a continuum of households is born. A household has preferences over consumption in two periods whereby second period consumption is discounted with the raw

[2]Without explicitly acknowledging how prices are determined in general equilibrium we cannot derive the effect of taxation on first period consumption. The reason is that a human capital wealth effect—caused by the discounted value of future pension income—inhibits closed form solutions in the partial equilibrium setup. Hence, moving to general equilibrium "kills two birds with one stone".

time discount factor β. In the first period of life, the household experiences an idiosyncratic productivity shock which we denote by η. This shock induces heterogeneity by household type which we denote by i. In addition, we index age by $j = 1, 2$. Consequently, all variables at the individual level carry indices i, j, t. The expected utility function of a household born in period t is given by

$$E_t U_t = E_t \left[v(c_{i,1,t}) + \beta u(c_{i,2,t+1}) \right],$$

where the per period Bernoulli utility functions are (weakly) increasing and concave, i.e., $v' \geq 0, u' > 0, v'' \leq 0, u'' < 0$.
Consumption in the two periods is given by

$$c_{i,1,t} + s_{i,1,t} = (1-\tau)\eta_{i,1,t} w_t \tag{3.2.1a}$$

$$c_{i,2,t+1} = s_{i,1,t}(1 + r_{t+1}) + b_{t+1} \tag{3.2.1b}$$

where $\eta_{i,1,t}$ is an idiosyncratic shock to wages in the first period of life. We assume that $E\eta_{i,1,t} = 1$ for all i, t. b_{t+1} are social security benefits to be specified next and τ is the contribution rate to social security.

3.2.2 Government

The government organizes a PAYG financed social security system. Pension benefits are lump-sum. Then the social security budget constraint writes as

$$b_t N_{2,t} = \tau w_{t+1} N_{1,t}$$

where $N_{j,t}$ is the population in period t of age j, i.e., $N_{j,t} = \int N_{i,j,t} di$. We ignore population growth, hence

$$b_t = \tau w_t.$$

We can therefore rewrite consumption in the second period as

$$c_{i,2,t+1} = s_{i,1,t}(1 + r_{t+1}) + w_{t+1}\tau.$$

3.2.3 Welfare

We take an ex-ante Rawlsian perspective and hence specify the social welfare function (SWF) of a cohort born in period t as the expected utility of a generation from the perspective of period $t-1$:

$$SWF_t \equiv E_{t-1}U_t = E_{t-1}\left[v(c_{i,1,t}) + \beta u(c_{i,2,t+1})\right].$$

3.2.4 Partial Equilibrium Analysis

We start by looking at a degenerate version of our model where first-period utility is zero. We assume that utility from consumption in the second period is CCRA with a coefficient of relative risk aversion of θ:

Assumption 3.1. *Let* $v(c_{i,1,t}) = 0$ *and* $u(c_{i,2,t+1}) = \frac{c_{i,2,t+1}^{1-\theta}-1}{1-\theta}$.

Stochastic Processes

Wages and interest rates are stochastic. We denote by ζ_t the shock on wages and by $\tilde{\varrho}_t$ the shock on returns. We further assume that wages grow deterministically at rate g. We therefore have:

$$\begin{aligned} w_t &= \bar{w}_t\zeta_t = \bar{w}_{t-1}(1+g)\zeta_t \\ R_t &= \bar{R}\tilde{\varrho}_t \end{aligned}$$

To simplify the analysis we assume that both ζ_t and $\tilde{\varrho}_t$ are not serially correlated. Despite the observed positive serial correlation of wages and asset returns in annual data, this assumption can be justified on the grounds of the long factual periodicity of each period in a two-period OLG model which is about 30 to 40 years. We also assume that ζ_t and $\tilde{\varrho}_t$ are statistically independent. We do so because, as we point out in Harenberg and Ludwig (2011), any conclusion from such a simple and inherently a-temporal model on the effects of the correlation structure between wages and returns is misleading. The idiosyncratic shock $\eta_{i,1,t}$ is not correlated with either of the two aggregate shocks. We relax this assumption once we introduce the CCV mechanism below. All shocks are assumed to have bounded support. We now summarize these assumptions:

Assumption 3.2. *a) Bounded support: $\zeta_t > 0$, $\tilde{\varrho}_t > 0$ for all t, $\eta_{i,1,t} > 0$ for all i,t.*

b) Means: $E\zeta_t = E\tilde{\varrho}_t = E\eta_{i,1,t} = 1$, for all i,t.

c) Statistical independence of (ζ_{t+1}, ζ_t) and $(\tilde{\varrho}_{t+1}, \tilde{\varrho}_t)$. Therefore: $E(\zeta_{t+1}\zeta_t) = E\zeta_{t+1}E\zeta_t$ for all t and, correspondingly, $E(\tilde{\varrho}_{t+1}\tilde{\varrho}_t) = E\tilde{\varrho}_{t+1}E\tilde{\varrho}_t$ for all t.

d) Statistical independence of $(\zeta_t, \tilde{\varrho}_t)$. Therefore: $E(\zeta_t\tilde{\varrho}_t) = E\zeta_t E\tilde{\varrho}_t$ for all t.

e) Statistical independence of $(\zeta_t, \eta_{i,1,t})$. Therefore: $E(\eta_{i,1,t}\zeta_t) = E\eta_{i,1,t}E\zeta_t$ for all i,t.

f) Statistical independence of $(\tilde{\varrho}_t, \eta_{i,1,t})$. Therefore: $E(\eta_{i,1,t}\varrho_t) = E\eta_{i,1,t}E\varrho_t$ for all i,t.

Life-Cycle Interaction

Under assumption 3.1, utility maximization implies that $c_{i,1,t} = 0$ and $s_{i,1,t} = (1-\tau)\eta_{i,1,t}\bar{w}\zeta_t$. Consumption in the second period can accordingly be rewritten as

$$c_{i,2,t+1} = \bar{w}\left(\eta_{i,1,t}\zeta_t\bar{R}\tilde{\varrho}_{t+1} + \tau\left((1+g)\zeta_{t+1} - \eta_{i,1,t}\zeta_t\bar{R}\tilde{\varrho}_{t+1}\right)\right). \tag{3.2.3}$$

We then have:

Proposition 3.1. *Under assumptions 3.1 and 3.2, a marginal introduction of social security increases ex-ante expected utility if*

$$(1+g)\frac{E_{t-1}\left[\frac{\zeta_{t+1}}{\tilde{\varrho}_{t+1}^{\theta}}\right]E_{t-1}\left[\frac{1}{\zeta_t^{\theta}}\right]E_{t-1}\left[\frac{1}{\eta_{i,1,t}^{\theta}}\right]}{E_{t-1}\left[\tilde{\varrho}_{t+1}^{1-\theta}\right]E_{t-1}\left[\zeta_t^{1-\theta}\right]E_{t-1}\left[\eta_{i,1,t}^{1-\theta}\right]} > \bar{R}. \tag{3.2.4}$$

The RHS of equation (3.2.4) reflects the costs of introducing social security represented here by the ex-risk return on savings. We speak of the LHS of equation (3.2.4) as the risk-adjusted implicit return of social security which reflects the value (or benefit) of introducing social security. Obviously, the implicit return increases if g increases. This is the standard Aaron condition.

To interpret the risk adjustment, we next assume that all stochastic variables are jointly distributed as log-normal.

Assumption 3.3. *Joint log-normality:* $\eta_{i,1,t}, \zeta_t, \zeta_{t+1}, \tilde{\varrho}_{t+1}$ *are jointly distributed as log-normal with parameters* $\mu_{\ln\eta}$, $\mu_{\ln\zeta}$, $\mu_{\ln\tilde{\varrho}}$, $\sigma^2_{\ln(\eta)}$, $\sigma^2_{\ln(\zeta)}$, $\sigma^2_{\ln(\tilde{\varrho})}$ *for means and variances, respectively.*

We then have:

Proposition 3.2. *Under assumptions 3.1 through 3.3, a marginal introduction of social security increases ex-ante expected utility if*

$$(1+g)\cdot(1+TR)^{\theta} > \bar{R}, \tag{3.2.5}$$

where

$$TR \equiv var(\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}) = \underbrace{\sigma^2_\eta}_{IR} + \underbrace{\sigma^2_\zeta + \sigma^2_{\tilde{\varrho}} + \sigma^2_\zeta\sigma^2_{\tilde{\varrho}}}_{AR} + \underbrace{\sigma^2_\eta\left(\sigma^2_\zeta + \sigma^2_{\tilde{\varrho}} + \sigma^2_\zeta\sigma^2_{\tilde{\varrho}}\right)}_{LCI=IR\cdot AR}. \tag{3.2.6}$$

To interpret this condition, observe that, according to equation (3.2.6), term TR – standing in for "total risk" – consists of three components, reflecting the effect of idiosyncratic risk in term IR, total aggregate risk in term AR and a mechanical interaction between idiosyncratic and aggregate risk in term LCI. To understand the nature of these terms notice that, in absence of social security, savings cum interest in our simple model is given by $s_{i,1,t}R_{t+1} = \bar{w}_t\bar{R}\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}$. Hence, from the ex-ante perspective, the product of three sources of risk are relevant, idiosyncratic wage risk, $\eta_{i,1,t}$, aggregate wage risk, ζ_t, and aggregate return risk, $\tilde{\varrho}_{t+1}$. Term TR is the variance of the product of these stochastic elements. It can be derived by applying the exact product formula of variances presented in Goodman (1960).

For standard random variables, an interaction term involving products of variances—such as LCI in our context—would be small. However, we here deal with long horizons so that the single variance terms might be quite large. Illustration 1 in the appendix gives a simple numerical example which uses parameters of our calibrated income processes. Based on this example we conclude that LCI adds about 40 percent times AR. Whatever the exact size of AR is, this interaction is clearly a non-negligible increase in overall income risk.

We next address how the utility consequences of a marginal introduction of social security—by a percentage point increase of $d\tau$—translate into utility. To measure this we compute the consumption equivalent variation (CEV). That is, we express utility gains from introducing social security at rate $d\tau > 0$ as the compensation in a policy regime without social security ($\tau = 0$) in units of a percent increase of consumption g_c. We denote by $g_c(AR, IR)$ the CEV required if both risks, idiosyncratic and aggregate, are present. We decompose this total CEV into various components. We accordingly denote the CEV in a deterministic setting by $g_c(0,0)$, with only aggregate risk by $g_c(AR, 0)$ and with only idiosyncratic risk by $g_c(0, IR)$, respectively. Observe from these definitions that $g_c(AR, 0) = g_c(0,0) + dg_c(AR)$, where $dg_c(AR)$ denotes the additional CEV due to aggregate risk. Correspondingly, we have $g_c(0, IR) = g_c(0,0) + dg_c(IR)$. With these definitions, it is also straightforward to define the additional effects, in terms of CEV, of the interaction between idiosyncratic and aggregate risks. It is given as the residual, namely, $dg_c(LCI) = g_c(AR, IR) - (g(0,0) + dg_c(AR) + dg_c(IR))$. In the appendix, we show that $g_c(AR, IR)$ can be expressed—in a logarithmic approximation—as

$$g_c = \left(\frac{1+g}{\bar{R}} (1+V)^{\theta} - 1 \right) d\tau \tag{3.2.7}$$

Taking a first-order Taylor series expansion of the above around $V = 0$ gives

$$g_c(AR, IR) \approx \left(\underbrace{\frac{1+g}{\bar{R}} - 1}_{g_c(0,0)/d\tau} + \underbrace{\theta \frac{1+g}{\bar{R}} AR}_{dg_c(AR)/d\tau} + \underbrace{\theta \frac{1+g}{\bar{R}} IR}_{dg_c(IR)/d\tau} + \underbrace{\theta \frac{1+g}{\bar{R}} LCI}_{dg_c(LCI)/d\tau} \right) d\tau \tag{3.2.8}$$

These equations have a straightforward interpretation. First, utility losses in a dynamically efficient economy—where $\bar{R} > 1 + g$—are approximately linear in the size of the social security system, $d\tau$. Additional gains due to insurance against risk—the risk components being AR, IR and LCI, respectively—increase in the size of risk whereby this increase is exponentially in risk aversion, cf. equation (3.2.7). Finally, the proportional increase of risk via the interaction translates—in a first-order approximation given by equation (3.2.8)—into corresponding utility consequences as measured by CEV because $dg_c(LCI) = IR \cdot dg_c(AR)$.

Modification: Counter-Cyclical Conditional Variance

We now return to condition (3.2.4) and modify assumption 3.3 slightly in order to reflect the CCV mechanism. Observe that CCV, by definition, does away with assumption 3.2e.

Assumption 3.4. *$\zeta_t \in [\zeta_l, \zeta_h]$ for all t where $\zeta_h > \zeta_l > 0$. We let $\zeta_h = 1 + \Delta_\zeta$ and $\zeta_l = 1 - \Delta_\zeta$ where $\Delta_\zeta < 1$. Notice that $\frac{1}{2}(\zeta_l + \zeta_h) = 1$. $\eta_{i,1,t}$ is distributed as log-normal whereby*

$$\eta_{i,1,t} = \begin{cases} \eta_{i,1,l} & \text{for } \zeta_t = \zeta_l \\ \eta_{i,1,h} & \text{for } \zeta_t = \zeta_h. \end{cases}$$

and $E \ln \eta_{i,1,l} = E \ln \eta_{i,j,h} = E \ln \eta_{i,j,t} = E \ln \eta$ and

$$\sigma^2_{\ln \eta} = \begin{cases} \sigma^2_{\ln \eta h} = \sigma^2_{\ln \eta} + \Delta & \text{for } \zeta_t = \zeta_l \\ \sigma^2_{\ln \eta l} = \sigma^2_{\ln \eta} - \Delta & \text{for } \zeta_t = \zeta_h. \end{cases}$$

For simplicity, we focus only at the log-uility case, hence $\theta = 1$. The RHS of equation (3.2.4) then rewrites as

$$(1+g)E_{t-1}\left[\frac{\zeta_{t+1}}{\tilde{\varrho}_{t+1}}\right] E_{t-1}\left[\frac{1}{\zeta_t}\right] E_{t-1}\left[\frac{1}{\eta_{i,1,t}}\right]$$

Under assumption 3.4, the expression rewrites as

$$(1+g)E_{t-1}\left[\frac{\zeta_{t+1}}{\tilde{\varrho}_{t+1}}\right] \frac{1}{2}\left(\frac{1}{\zeta_l}E_{t-1}\left[\frac{1}{\eta_{i,1,l}}\right] + \frac{1}{\zeta_h}E_{t-1}\left[\frac{1}{\eta_{i,1,h}}\right]\right) \tag{3.2.9}$$

and, without CCV, the corresponding expression is

$$(1+g)E_{t-1}\left[\frac{\zeta_{t+1}}{\tilde{\varrho}_{t+1}}\right] \frac{1}{2}\left(\frac{1}{\zeta_l} + \frac{1}{\zeta_h}\right) E_{t-1}\left[\frac{1}{\eta_{i,j,t}}\right]. \tag{3.2.10}$$

We can then show the following:

Proposition 3.3. *a) the LHS of eq. (3.2.9) is larger than the LHS of eq. (3.2.10).*

b) the difference between the LHS of eq. (3.2.9) and the LHS of eq. (3.2.10) increases in the variance of aggregate shocks.

We can therefore conclude that, on top of the previously illustrated mechanical interaction between idiosyncratic and aggregate risk, the direct interaction via the CCV mechanism will further increase the beneficial effects of social security. Importantly, finding 3.3b establishes that the effect of CCV is larger when the variance of aggregate risk is higher.

3.2.5 General Equilibrium Analysis

In general equilibrium, we relax assumption 3.1, thereby modeling utility from consumption in the first period. To derive analytical solutions we have to restrict attention to log-utility in both periods. Furthermore, again for analytical reasons, we assume absence of idiosyncratic shocks:[3]

Assumption 3.5. *a)* $v(\cdot) = u(\cdot) = \ln(\cdot)$.

b) $\eta_{i,1,t} = E\eta_{i,1,t} = 1$ *for all* i, t.

As a consequence of discarding idiosyncratic risk, our analysis in this subsection does not contribute anything particularly new to the literature on social security so that the quick reader may wish to proceed with section 3.3. We nevertheless regard the general equilibrium extension as very useful to provide guidance for interpretation of our quantitative results in section 3.5 where we will occasionally refer back to our analytical expressions.

Firms

To close the model in general equilibrium, we add a firm sector. We take a static optimization problem. Firms maximize profits operating a neo-classical production function. Let profits of the firm be

$$\Pi = \zeta_t F(K_t, \Upsilon_t L_t) - (\bar{\delta} + r_t)\varrho_t^{-1} K_t - w_t L_t$$

where ζ_t is a technology shock with mean $E\zeta_t = 1$. Υ_t is the technology level growing at the exogenous rate g, hence $\Upsilon_{t+1} = (1+g)\Upsilon_t$. ϱ_t is an exogenous shock to the unit user costs of capital with mean $E\varrho_t = 1$. We add this non-standard element in order to model additional shocks to the rate of return to capital. These shocks are multiplicative in the user costs to capital for analytical reasons. In our full-blown quantitative model, these

[3] Our proof of equilibrium dynamics requires that all households are ex-ante identical which is not the case if idiosyncratic is present in the first period of life. We reintroduce idiosyncratic risk by slightly altering our model in a separate appendix. In this appendix we also shed more light on the analytical reasons for discarding idiosyncratic risk in the first place. Our extension features a subperiod structure where households also work a constant fraction in the second period. This allows us to reintroduce idiosyncratic risk in the second period while otherwise preserving the structure of the model. We briefly summarize our findings from this extension once we complete the discussion of the simpler version.

shocks will be replaced by shocks to the depreciation rate.[4]
Throughout we assume full depreciation and Cobb-Douglas production, hence $\bar{\delta} = 1$ and

$$F(K_t, \Upsilon_t L_t) = K_t^{\alpha} \left(\Upsilon_t L_t\right)^{1-\alpha},$$

where α is the capital elasticity of production. The firm first-order conditions then give:

$$1 + r_t = \alpha k_t^{\alpha-1} \zeta_t \varrho_t = \bar{R}_t \zeta_t \varrho_t \tag{3.2.11a}$$

$$w_t = (1-\alpha)\Upsilon_t k_t^{\alpha} \zeta_t = \bar{w}_t \zeta_t. \tag{3.2.11b}$$

Hence, ϱ_t is a shock to the gross return on savings. We also define by $\bar{R}_t \equiv \alpha k_t^{\alpha-1}$ the "ex-shock" component of the gross return and, correspondingly, by $\bar{w}_t = (1-\alpha)\Upsilon_t k_t^{\alpha}$ the "ex-shock" component of per capita wages.

Stochastic Processes in General Equilibrium

Observe that, relative to the partial equilibrium model of subsection 3.2.4, instead of shocks to wages and returns, shocks to productivity and the user costs of capital take over as stochastic primitives of the model. This also implies an explicit linkage the deterministic and stochastic components of wages and asset returns.[5] Consequently, assumption 3.2d is dropped. Also notice that assumptions 3.2e–f are irrelevant because of the absence of idiosyncratic risk. Replacing $\tilde{\varrho}_t$ by ϱ_t, assumptions 3.2a–c remain unaltered.

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[4] Let shocks to the depreciation rate be δ_t. Our formulation with shocks to the user costs to capital can be translated into shocks to the depreciation rate to capital. In our quantitative model, we have that $R_t = \zeta_t \bar{R}_t - \delta_t$. Let $\zeta_t = 1$. We then have $\delta_t = \bar{R}_t(1 - \varrho_t)$.

[5] Notice that, in the notation of the partial equilibrium version of the model, $\tilde{\varrho}_t = \zeta_t \varrho_t$.

General Equilibrium Dynamics

Proposition 3.4. *Equilibrium dynamics in the economy are given by*

$$k_{t+1} = \frac{1}{1+g}\chi(1-\tau)(1-\alpha)\zeta_t k_t^\alpha \tag{3.2.12a}$$

where the savings rate χ is given by

$$\chi \equiv \frac{1}{1+\left(\beta\alpha\bar{E}\right)^{-1}} \tag{3.2.12b}$$

and

$$\bar{E} \equiv E_t\left[\frac{1}{\alpha+(1-\alpha)\varrho_{t+1}^{-1}\tau}\right]. \tag{3.2.12c}$$

Analyzing the system of equations in (3.2.12) we find that, for $\tau > 0$, increasing the variance of return shocks, ϱ_t, reduces $\bar{E}$ and therefore decreases the saving rate, χ if $\tau > 0$. This is the human capital wealth effect of discounted pension income.[6] Notice that the discounted value of pension income is given by $\frac{b_{t+1}}{R_{t+1}}$. An increase in the variance of R_{t+1} hence increases discounted pension income in expectation. This increase the (expected) human capital wealth which, as in a standard deterministic model, increases first-period consumption and thereby decreases the saving rate. Increasing τ decreases $\bar{E}$ and therefore decreases the saving rate, χ. This is the crowding-out of private capital formation. Increasing α or β increases the savings rate.

A useful concept are mean shock equilibria which we will refer to below to evaluate welfare. Mean shock equilibria will also play an important role below in our computational analysis of our more elaborate quantitative model.

Definition 3.1. *In a mean shock equilibrium the realizations of all aggregate shocks are at their respective unconditional means, hence $\zeta_t = E\zeta_t = 1$, $\varrho_t = E\varrho_t = 1$ for all t. The corresponding equilibrium dynamics follow from (3.2.12a) as*

$$k_{t+1,ms} = \frac{1}{1+g}\chi(1-\tau)(1-\alpha)k_{t,ms}^\alpha \tag{3.2.13}$$

where χ is defined in (3.2.12b).

A stationary mean shock equilibrium is equivalent to a stochastic steady state:

[6] Since we assume log utility, income and substitution effects of stochastic interest rates cancel out.

Definition 3.2. *In a stationary mean shock equilibrium (=stochastic steady state) all variables grow at constant rates. In particular, we have that $k_{t,ms} = k_{ms}$ for all t which is given by*

$$k_{ms} = \left(\frac{1}{1+g}\chi(1-\tau)(1-\alpha)\right)^{\frac{1}{1-\alpha}}. \tag{3.2.14}$$

We then have:

Proposition 3.5. *A no social security ($\tau = 0$) stationary mean-shock equilibrium is dynamically efficient, i.e., $\bar{R}^{ms} > 1+g$, iff $\frac{\alpha}{1-\alpha} > \frac{\beta}{1+\beta}$.*

The proof is trivial and therefore omitted in the appendix. It immediately follows from the definition of $\bar{R}^{ms}$ which is given by

$$\bar{R}_{ms} = \alpha k_{ms}^{\alpha-1} = (1+g)\frac{\alpha}{(1-\alpha)}\frac{(1+\beta)}{\beta}. \tag{3.2.15}$$

and the fact that the golden rule capital stock is $k^\star = \left(\frac{\alpha}{1+g}\right)^{\frac{1}{1-\alpha}}$.
In what follows, we assume a dynamically efficient economy. This implies that households loose in terms of welfare from decreasing the capital stock. Discounting is sufficiently strong such that induced gains from increasing returns (caused by decreasing the capital stock) are offset by decreasing wages.

Welfare Analysis in General Equilibrium

To simplify the analysis, we compare two long-run mean shock equilibria and thereby (again) ignore endogenous fluctuations of k_t and all transitional dynamics:

Proposition 3.6. *A marginal introduction of social security increases ex-ante expected utility in the long-run mean shock stationary equilibrium if*

$$\underbrace{\frac{\beta(1-\alpha)}{\alpha}E_{t-1}\left[\frac{1}{\zeta_t}\right]E_{t-1}\left[\frac{1}{\varrho_{t+1}}\right] - (1+\beta)}_{=A} + \underbrace{B}_{=0} + \underbrace{-\left(\alpha(1+\beta) - \beta(1-\alpha)\right)\frac{1}{1-\alpha}\left(1 + \frac{1}{1+\beta}\frac{1-\alpha}{\alpha}E_t\left[\frac{1}{\varrho_{t+1}}\right]\right)}_{=C} > 0 \tag{3.2.16}$$

In equation (3.2.16), term A reflects the implicit rate of return condition. Combining equation (3.6) with equation (3.2.15) we get that $A > 0$ iff

$$(1+g)\left(E_{t-1}\left[\frac{1}{\zeta_t}\right]E_{t-1}\left[\frac{1}{\varrho_{t+1}}\right]\right) > \bar{R}_{ms}.$$

Observe that this condition immediately follows from condition (3.2.4) under assumption 3.5, equation (3.2.15) and acknowledging that $\tilde{\varrho}_t = \zeta_t \varrho_t$. It is therefore the general equilibrium equivalent to the partial characterization we gave above. Term B, which is equal to zero, encompasses two effects that exactly offset each other in the mean shock equilibrium of our model with log utility. Introducing social security crowds out savings and thereby increases period 1 consumption which is beneficial to the household. The offsetting effect is that, by lowering capital income, it reduces period 2 consumption. Finally, by the assumption of dynamic efficiency, term C is negative. The term reflects the welfare losses incurred by crowding out of capital.

3.3 The Quantitative Model

Our quantitative model extends our simple model along several dimensions. First, rather than considering a stylized setup with two generations we take a periodicity of one calendar year and consider J overlapping generations. Second, we introduce one period ahead risk-free bonds. The primary reason for this extension is to impose discipline on calibration. Having a bond in the model means that our model entails predictions about general equilibrium asset prices. Any model on the welfare effects of social security should have realistic asset pricing implications. By providing a bond, we give households an additional asset to self-insure against idiosyncratic and aggregate risk. Ceteris paribus, this reduces the beneficial effects of social security. However, the presence of the bond also reduces the effect of decreasing savings on the crowding out of productive capital because part of the reduced savings is absorbed by the bond market.

3.3.1 Risk and Time

Time is discrete and runs from $t = 0, \ldots, \infty$. Risk is represented by an event tree. The economy starts with some fixed event z_0, and each node of the tree is a history of exogenous shocks $z^t = (z_0, z_1, ..., z_t)$. The shocks are assumed to follow a Markov chain with finite support Z and strictly positive transition matrix $\pi^z(z' \mid z)$. Let Π^z denote the invariant distribution associated with π^z. In our notation, we will make all aggregate and idiosyncratic shocks contingent on z_t. For notational convenience, we will suppress the dependency of all other variables on z^t but history dependence of all choice variables is understood.

3.3.2 Demographics

In each period t, the economy is populated by J overlapping generations of agents indexed by $j = 1, \ldots, J$, with a continuum of agents in each generation. Population grows at the exogenous rate of n. Households face an idiosyncratic (conditional) probability to survive from age j to age $j+1$ which we denote by ς_{j+1}, hence $\varsigma_1 = 1$ and $\varsigma_{J+1} = 0$. Consequently, given an initial population distribution $\{N_{0,j}\}_{j=1}^J$ which is consistent with constant population growth for all periods $t = 0, 1, \ldots$ and normalized such that $N_0 = \sum_{j=1}^J N_{t,j} = 1$, the exogenous law of motion of population in our model is given by

$$\begin{aligned} N_{t+1,1} &= (1+n)N_{t,1} \\ N_{t+1,j+1} &= \varsigma_{j+1} \cdot N_{t,j} \quad \text{for } j = 1, \ldots, J. \end{aligned}$$

Households retire at the fixed age j_r. Labor supply is exogenous in our model and during the working period $j = 1, \ldots, j_r - 1$ each household supplies one unit of labor. Observe that constant population growth implies that population shares, e.g., the working age to population ratio, are constant.

3.3.3 Firms

Production of the final good takes place with a standard Cobb-Douglas production function with total output at time t given by

$$Y_t = F(\zeta(z_t), K_t, L_t) = \zeta(z_t) K_t^{\alpha} (\Upsilon_t L_t)^{1-\alpha} \tag{3.3.17}$$

where K_t is the aggregate stock of physical capital, L_t is labor, $\zeta(z_t)$ is a stochastic shock to productivity and Υ_t is a deterministic technology level growing at the exogenous rate g.

The economy is closed. The consumption good can either be consumed in the period when it is produced or can be used as an input into a production technology producing capital. We ignore capital adjustment costs. Accordingly, the production technology for capital is

$$\begin{aligned} K_{t+1} &= I_t + K_t(1-\delta(z_t)) \\ &= Y_t - C_t + K_t(1-\delta(z_t)) \end{aligned} \tag{3.3.18}$$

where $\delta(z_t)$ is the stochastic depreciation rate of physical capital.
Firms maximize profits and operate in perfectly competitive markets. Accordingly, the rate of return to capital and the wage rate are given by

$$w_t = (1-\alpha)\Upsilon_t \zeta(z_t) k_t^{\alpha} \tag{3.3.19a}$$

$$r_t = \alpha \zeta(z_t) k_t^{\alpha-1} - \delta(z_t) \tag{3.3.19b}$$

where $k_t = \frac{K_t}{\Upsilon_t L_t}$ is the capital stock per unit of efficient labor which we refer to as "capital intensity".

3.3.4 Endowments

Agents are endowed with one unit of labor which is supplied inelastically for ages $j = 1, \ldots, j_r - 1$. After retirement, labor supply is zero. Households have access to two savings storage technologies. Either they save in the risky technology at rate of return r_t or in a one-period risk-free bond at return r_t^f which is in zero net supply. Households are subject to idiosyncratic shocks to their labor productivity. This shock induces heterogeneity by household type which we denote by i. We denote total assets by $A_{i,j,t}$, and the share invested in the risky asset by $\kappa_{i,j,t}$.
Additional elements of the dynamic budget constraint are income, $\Upsilon_{i,j,t}$, to be specified below and consumption, $C_{i,j,t}$. The dynamic budget constraint of a household at age j then reads as

$$A_{i,j+1,t+1} = A_{i,j,t}(1 + r_t^f + \kappa_{i,j-1,t-1}(r_t - r_t^f)) + \Upsilon_{i,j,t} - C_{i,j,t} \tag{3.3.20}$$

where $\kappa_{i,j-1,t-1} \in [-\underline{\kappa}, \overline{\kappa}]$, for all i, j, t. This restricts the leverage in stocks in our model.[7] Income is given by

$$Y_{i,j,t} = \begin{cases} (1-\tau)\epsilon_j w_t \eta_{i,j,t} & \text{for } j < jr \\ B_{i,j,t} & \text{for } j \geq jr \end{cases} \tag{3.3.21}$$

where ϵ_j is age-specific productivity and $\eta_{i,j,t}$ is an idiosyncratic stochastic component. We assume that $\eta_{i,j,t}$ follows a time and age-independent Markov chain whereby the states of the Markov chain are contingent on aggregate states z. Accordingly, let the states be denoted by $\mathcal{E}_z = \{\eta_{z1}, \ldots, \eta_{zM}\}$ and the transition matrices be $\pi^\eta(\eta' \mid \eta) > 0$. Let Π^η denote the invariant distribution associated with π^η.

As for pension income, we assume that pension payments are lump-sum, hence

$$B_{i,j,t} = b_t \Upsilon_t \tag{3.3.22}$$

where b_t is some normalized pension benefit level which only depends on t. Accordingly, the pension system fully redistributes across household types. This is an approximation to the U.S. pension system.[8]

3.3.5 Preferences

We take Epstein-Zin preferences. Let θ be the coefficient of relative risk-aversion and φ denote the inter-temporal elasticity of substitution. Then

$$U_{i,j,t} = \left[C_{i,j,t}^{\frac{1-\theta}{\gamma}} + \beta \varsigma_{j+1} \left(\mathbb{E}_{i,j,t} \left[U_{i,j+1,t+1}^{1-\theta} \right] \right)^{\frac{1}{\gamma}} \right]^{\frac{\gamma}{1-\theta}} \tag{3.3.23}$$

where $\gamma = \frac{1-\theta}{1-\frac{1}{\varphi}}$, and $\beta > 0$ is the standard discount factor. For $\theta = \frac{1}{\varphi}$ we have $\gamma = 1$ and are back to CRRA preferences. $\mathbb{E}_{i,j,t}$ is the expectations operator and expectations,

[7] In a model without a constraint of the form $\kappa_{i,j-1,t-1} \in [-\underline{\kappa}, \overline{\kappa}]$ we have a singularity at $X_{i,j,t} - C_{i,j,t} = 0$ so that, for $X_{i,j,t} - C_{i,j,t} \to +0$, $\kappa_{i,j,t} \to +\infty$ and for $X_{i,j,t} - C_{i,j,t} \to -0$, $\kappa_{i,j,t} \to -\infty$. The presence of the singularity has consequences for aggregation because the set for κ will not be compact. We set the constraint in order to rule out this technicality, but we set the bounds so high that the constraint will rarely be binding in equilibrium.

[8] The U.S. pension system links contributions to AIME, the average indexed monthly earnings and has an additional distributional component by the so-called bend point formula. From an ex-ante perspective, given this distributional component and provided that income shocks are non-permanent, an approximation with lump-sum pension benefits is a good first-order approximation.

conditional on information for household i, j, t, are taken with respect to idiosyncratic wage shocks and aggregate productivity and depreciation shocks. As $\varsigma_{J+1} = 0$, equation (3.3.23) implies that $U_J = C_J$.

3.3.6 The Government

The government organizes a PAYG financed social security system. We take the position that social security payments are not subject to political risk. We assume that the budget of the social security system is balanced in all periods. We describe various social security scenarios below. We further assume that the government collects all accidental bequests and uses them up for government consumption which is otherwise neutral.

3.3.7 Equilibrium

To define equilibrium we adopt a de-trended version of the household model. We therefore first describe transformations of the household problem and then proceed with the equilibrium definition.

Transformations

Following Deaton (1991), define cash-on-hand by $X_{i,j,t} = A_{i,j,t}(1 + r_t^f + \kappa_{i,j-1,t-1}(r_t - r_t^f)) + \mathrm{Y}_{i,j,t}$. The dynamic budget constraint (3.3.20) then rewrites as

$$X_{i,j+1,t+1} = (X_{i,j,t} - C_{i,j,t})(1 + r_{t+1}^f + \kappa_{i,j,t}(r_{t+1} - r_{t+1}^f)) + \mathrm{Y}_{i,j+1,t+1} \tag{3.3.24}$$

We next transform the problem to de-trend the model and work with stationary variables throughout. That is, we de-trend with the deterministic trend component induced by technological progress. Along this line, define by $x_{i,j,t} = \frac{X_{i,j,t}}{\Upsilon_t}$ transformed cash-on-hand and all other variables accordingly. Using $\omega_t = \frac{w_t}{\Upsilon_t}$ to denote wages per efficiency unit we have

$$\mathrm{y}_{i,j,t} = \begin{cases} (1-\tau)\epsilon_j \omega_t \eta_{i,j,t} & \text{for } j < jr \\ b_t & \text{for } j \geq jr. \end{cases}$$

Now divide the dynamic budget constraint (3.3.24) by Υ_t and rewrite to get

$$x_{i,j+1,t+1} = (x_{i,j,t} - c_{i,j,t})\tilde{R}_{i,j+1,t+1} + \mathrm{y}_{i,j+1,t+1}. \tag{3.3.25}$$

where $\tilde{R}_{i,j+1,t+1} = \frac{(1+r^f_{t+1}+\kappa_{i,j,t}(r_{t+1}-r^f_{t+1}))}{1+g}$.
Transform the per period utility function accordingly and take an additional monotone transformation to get

$$u_{i,j,t} = \left[c_{i,j,t}^{\frac{1-\theta}{\gamma}} + \tilde{\beta}_{j+1}\left(\mathbb{E}_{i,j,t}\left[(u_{i,j+1,t+1})^{1-\theta}\right]\right)^{\frac{1}{\gamma}}\right]^{\frac{\gamma}{1-\theta}} \tag{3.3.26}$$

where $\tilde{\beta}_{j+1} = \beta\varsigma_{j+1}(1+g)^{\frac{1-\theta}{\gamma}}$.

Definition of Equilibrium

Individual households, at the beginning of period t are indexed by their age j, their idiosyncratic productivity state η, their cash on hand holdings x, and a measure $\Phi(j,x,\eta)$ which describes the beginning of period wealth distribution in the economy, i.e., the share of agents at time t with characteristics (j,x,η). We normalize such that $\int d\Phi = 1$. Existence of aggregate shocks implies that Φ evolves stochastically over time. We use H to denote the law of motion of Φ which is given by

$$\Phi' = H(\Phi, z, z') \tag{3.3.27}$$

Notice that z' is a determinant of Φ' because it determines $\tilde{R}_{i,j+1,t+1}$ and therefore the distribution over $x\prime$.
The de-trended version of the household problem writes as

$$u(j,x,\eta;z,\Phi) = \max_{c,\kappa,x'}\left\{\left[c^{\frac{1-\theta}{\gamma}} + \tilde{\beta}\left(\mathbb{E}\left[(u(j+1,x',\eta';z',\Phi'))^{1-\theta}\right]\right)^{\frac{1}{\gamma}}\right]^{\frac{\gamma}{1-\theta}}\right\} \tag{3.3.28a}$$

$$\text{s.t. } x' = (x-c)\tilde{R}' + \mathrm{y}' \tag{3.3.28b}$$

$$\tilde{R}' = \frac{(1+r^{f\prime}+\kappa(r'-r^{f\prime}))}{1+g} \tag{3.3.28c}$$

$$\Phi' = H(\Phi, z, z'). \tag{3.3.28d}$$

We therefore have the following definition of the recursive equilibrium of our economy:[9]

Definition 3.3. *A recursive competitive equilibrium is a value function u, policy functions for the household, $x'(\cdot)$, $a'(\cdot)$, $c(\cdot)$, $\kappa(\cdot)$, policy functions for the firm, $K(\cdot)$, $L(\cdot)$, pricing functions $r(\cdot)$, $q(\cdot)$, $w(\cdot)$, policies, τ, b, aggregate measures $\Phi(\cdot)$ and an aggregate law of motion, H_t such that*

1. *$u(\cdot), x'(\cdot), a'(\cdot), c(\cdot), \kappa(\cdot)$ are measureable, $u(\cdot)$ satisfies the household's recursive problem and $x'(\cdot), a'(\cdot), c(\cdot), \kappa(\cdot)$ are the associated policy functions, given r, q, ω, τ and b.*

2. *K, L satisfy, given $r(\Phi, z)$ and $w(\Phi, z)$,*

$$\omega(\Phi, z) = (1-\alpha)\zeta(z)k(\Phi, z)^{\alpha} \tag{3.3.29a}$$

$$r(\Phi, z) = \alpha\zeta(z)k(\Phi, z)^{\alpha-1} - \delta(z). \tag{3.3.29b}$$

where $k(\Phi, z) = \frac{K(\Phi,z)}{\Upsilon L}$ is the capital stock per efficiency unit (or "capital intensity") and $\Upsilon = (1+g)\Upsilon_{-1}$ is the technology level in period t.

3. *neutral government consumption financed by bequests is given by*

$$gc' = \frac{\int (1-\varsigma_{j+1})a'(j, x, \eta; z, \Phi)\, R'(\kappa(\cdot))d\Phi}{\ell(1+n)(1+g)} \tag{3.3.30}$$

where

$$R'(\kappa(\cdot)) = (1 + r^{f'} + \kappa(j, x, \eta; z, \Phi)(r' - r^{f'})).$$

4. *the pension system budget constraint holds, i.e.*

$$\tau(\Phi, z)\omega(\Phi, z) = b(\Phi, z)p \tag{3.3.31}$$

where p is the economic dependency ratio which is stationary in our model.[10]

[9] We use the integration operator $\int$ as a short-cut notation for all sums and integrals involved but discreteness of the characteristics (j, z) is understood. When integrating out with respect to all characteristics of the distribution, we simply write $d\Phi$, hence $\int \cdot d\Phi = \int \cdot\Phi(dj \times dx \times d\eta)$. When we integrate only with respect to a subset of characteristics, we make this explicit by, e.g., writing $\int \cdot\Phi(j, dx \times d\eta)$.

[10] It is given by $p = \frac{\sum_{j=j_r}^{J}(1+n)^{J-j}\prod_{i=1}^{j-1}\varsigma_i}{\sum_{j=1}^{jr-1}(1+n)^{J-j}\epsilon_j\prod_{i=1}^{j-1}\varsigma_i}$.

5. *For all Φ and all z*

$$k(H(\Phi, z, z'), z')(1+g)(1+n) = \frac{1}{\ell}\int \kappa(j, x, \eta; z, \Phi)a'(j, x, \eta; z, \Phi)d\Phi \tag{3.3.32a}$$

$$0 = \int (1 - \kappa(j, x, \eta; z, \Phi))a'(j, x, \eta; z, \Phi)d\Phi \tag{3.3.32b}$$

$$i(\Phi, z) = f(k(\Phi, z)) - gc - \frac{1}{\ell}\int c(j, x, \eta; z, \Phi)d\Phi \tag{3.3.32c}$$

$$k(H(\Phi, z, z'), z')(1+g)(1+n) = k(\Phi, z)(1 - \delta(z)) + i(\Phi, z) \tag{3.3.32d}$$

where ℓ is the working age to population ratio[11]*, equation (3.3.32b) is the bond market clearing condition and the bond price q is determined such that it clears the bond market in each period t and $i(\cdot) = \frac{I(\cdot)}{\Upsilon L}$ is investment per efficiency unit.*

6. *The aggregate law of motion H is generated by the exogenous population dynamics, the exogenous stochastic processes and the endogenous asset accumulation decisions as captured by the policy functions x'.*

Definition 3.4. *A stationary recursive competitive equilibrium is as described above but with time constant individual policy functions $x'(\cdot)$, $a'(\cdot)$, $c(\cdot)$, $\kappa(\cdot)$ and a time constant aggregate law of motion H.*

3.3.8 Welfare Criteria

At this stage, we only compare two stationary equilibria and do not take into account transitional dynamics. Our welfare concept is the consumption-equivalent variation for a newborn before any shocks are realized. It is an ex-ante perspective where the agent does not know the aggregate state nor the level of capital that he will be born into. A positive number then states the amount an agent would be willing to give up in order to be born into the second long-run equilibrium (i.e. into an economy with some social security).
Note that this comparison between the long-run equilibria provides a lower bound on the expected welfare gains for the newborns along the transition, because they are spared some of the negative effects of crowding out, and because they get to save less and consume more as the level of capital moves toward its new, lower level.

[11]It is given by $\ell = \frac{\sum_{j=1}^{j_r-1}(1+n)^{J-j}\epsilon_j\prod_{i=1}^{j-1}\varsigma_i}{\sum_{j=1}^{J}(1+n)^{J-j}\prod_{i=1}^{j-1}\varsigma_i}$.

3.3.9 Thought Experiment

At this stage, we only compare two stationary equilibria. In our initial equilibrium a social security system does not exist. In the second equilibrium, the economy features a social security system with a contribution rate of 2 percent. One can think of this as the introduction of a 'marginal' social security system as described in Krueger and Kubler (2006). We use their proposition 1 to ensure that the initial economy is dynamically efficient so as to rule out any welfare gains that would come from curing dynamic inefficiency.

We then use the exact same economy to conduct partial equilibrium (PE) experiments that enable us to disentangle the welfare gains due to insurance from the welfare losses due to crowding out and its associated price changes. In this partial equilibrium, we feed in the sequence of shocks and prices $\{z_t, r_t, r_t^f, w_t\}_{t=1}^T$ obtained from the associated general equilibrium (GE). It is like a small open economy, where aggregate prices are determined by the world and fluctuate over time and are not influenced by domestic policy changes. If we do not change any other parameter, then the results are naturally exactly the same as in the associated GE. To isolate the total insurance effects, we let agents optimize under the new policy, i.e. $\tau = 0.2$, but with the 'old' approximate laws of motion that still hold for the evolution of aggregate prices. Then we simulate by feeding in the old sequence of shocks and prices, but with the new policy functions and the new social security system. In a very similar fashion, we isolate the insurance against aggregate risk, idiosyncratic risk, CCV, and survival risk.

In order to also isolate the interaction effect LCI, we proceed by relating back to equation (3.2.8) of our 2-generations model. Recall that $g_c(AR, IR)$—the total welfare gains with full aggregate and idiosyncratic risk at work, ignoring CCV and survival risk—can be decomposed as $g_c(AR, IR) = g_c(0,0) + dg_c(AR) + dg_c(IR) + dg_c(LCI)$ where $g_c(0,0)$ is the welfare gain—expressed in terms of CEV—in an economy with zero aggregate risk and zero idiosyncratic risk and $dg(X)$ is the additional gain attributed to component X, our objects of interest. Also recall that welfare gains in an economy with zero aggregate risk and full idiosyncratic risk, $g_c(0, IR)$, can be written as $g_c(0, IR) = g_c(0,0) + dg_c(IR)$. With the objects $g_c(0,0)$ and $g_c(0, IR)$ at hand we can, with these definitions, determine $dg(X)$ for $X = AR, IR, LCI$.

3.3.10 Computational Details

Following Gomes and Michaelides (2008) and Storesletten, Telmer, and Yaron (2007) we compute an approximate equilibrium of our model by applying the Krusell and Smith (1998) method. We approximate the solution by considering forecast functions of the average capital stock in the economy and the ex-ante equity premium. In the general equilibrium version of our model, we loop on the postulated laws of motion until convergence. We do so by simulating the economy for $T = 5000$ periods and discard the first 500 initialization periods. In each period, we compute the market clearing bond price. The goodness of fit of the approximate laws of motion is $R^2 = 0.99$.
We compute solution to the household model by adopting Carroll's endogenous grid method, which reduces computational time strongly. Written in Fortran 2003, the model takes about one hour to converge to a solution, given a decent initial guess for the laws of motion. A more detailed description of our computational methods can be found in appendix 3.B.

3.4 Calibration

3.4.1 Overview

Part of our parameters are exogenously calibrated either by reference to other studies or directly from the data. We refer to these parameters as first stage parameters. A second set of parameters is calibrated by informally matching simulated moments to respective moments in the data. Accordingly, we refer to those parameters as second stage parameters. The theoretical discussion in section 3.2.4 emphasized that the correlation between TFP and returns will play a crucial role when evaluating social security benefits. To address this, we take two views with regard to the data generating process of observed TFP (or wage) fluctuations. First, we detrend the data with a linear trend, thereby following the approach of Krueger and Kubler (2006), and -like them- find that the correlation is negative. Second, we assume a unit root process for (the log of) TFP and detrend by first differences, which yields a highly significant positive correlation. We will argue that this is a more appropriate approach. In our discussion of robustness in section 3.5.3 we show that when a Hodrick-Prescott filter is used, the correlation is again positive and significant.
Table 3.1 summarizes the calibration. Table 3.2 contains the information on the stochas-

tic processes of log TFP and the corresponding approximation according to our two approaches: NC stands for the negative correlation between TFP and returns which results from the linear trend specification, while PC stands for the positive correlation which results from the de-trending with first differences. Since all other targets in the calibration remain the same, we need to slightly adjust some endogenous parameters, which are displayed in table 3.3 for both specifications. The next subsections contain a detailed description of our methodology.

3.4.2 Production Sector

We set the value of the capital share parameter, a first stage parameter, to $\alpha = 0.32$. This is directly estimated from NIPA data (1960-2005) on total compensation as a fraction of (adjusted) GDP. Our estimated value is in the range of values considered as reasonable in the literature. It is close to the preferred value of 0.3 as used by Krueger and Kubler (2006). To estimate α, we take data on total compensation of employees (NIPA Table 1.12) and deflate it with the GDP inflator (NIPA Table 1.1.4). In the numerator, we adjust GDP (NIPA Table 1.1.5), again deflated by the GDP deflator, by nonfarm proprietors' income and other factors that should not be directly related to wage. Without these adjustments, our estimate of α would be considerably higher, i.e., at $\alpha = 0.43$.

To determine the mean depreciation rate of capital, a first stage parameter in our model, we proceed as follows. We first estimate the capital output ratio in the economy. To measure capital, we take the stock of fixed assets (NIPA Table 1.1), appropriately deflated. We relate this to total GDP. This gives an estimate of the capital output ratio of $K/Y = 2.65$, in line with the estimates by, e.g., Fernández-Villaverde and Krueger (2011), or of the ratio of output to capital of 0.38. This implies an average marginal product of capital $E[mpk] = \alpha E[Y/K] = 0.12$. Given this estimate for the marginal product of capital and our estimate for the average risky return on capital of 0.079 based on data since 1950 provided by Rob Shiller, we set $E[\delta] = E[mpk] - E[r] = 0.042$.[12]

Our estimate of the deterministic trend growth rate, also a first stage parameter, is $g = 0.018$ which is in line with other studies. We determine it by estimating the Solow residual from the production function, given our estimate of α, our measure for capital, and a

[12] The data was downloaded from Rob Shillers webpage and is available under the address http://www.econ.yale.edu/∼shiller/data.htm.

Table 3.1: Calibration: Summary

Parameter	Value	Target (source)	Stage
Preferences			
Discount factor, β	cf. table 3.3	Capital output ratio, 2.65 (NIPA)	2
Coefficient of relative risk aversion, θ	cf. table 3.3	Average equity premium, 0.056 (Shiller)	2
Elasticity of intertemporal substitution, φ	1.5	Consumption Profile	2
Technology			
Capital share, α	0.32	Wage share (NIPA)	1
Leverage, b	0.66	Croce (2010)	1
Technology growth, g	0.018	TFP growth (NIPA)	1
Mean depreciation rate of capital, δ_0	0.0418	Risky return, 0.079 (Shiller)	1
Std. of depreciation $\bar{\delta}$	cf. table 3.3	Std. of risky return, 0.168 (Shiller)	2
Aggregate productivity states, $1 \pm \bar{\zeta}$	$\{1.029, 0.971\}$	Std. of TFP, 0.029 (NIPA)	1
Transition probabilities of productivity, π^{ζ}	0.941	Autocorrelation of TFP, 0.88 (NIPA)	1
Conditional prob. of depreciation shocks, π^{δ}	cf. table 3.3	Corr.(TFP, returns), 0.36 (NIPA, Shiller)	2
Idiosyncratic Productivity			
Age productivity, $\{\epsilon_j\}$	-	Earnings profiles (PSID)	1
CCV $\sigma_{\nu(z)}$	$\{0.21, 0.13\}$	Storesletten, et al. (2007)	1
Autocorrelation ρ	0.952	Storesletten, et al. (2007)	1
Demographics: Exogenous parameters			
Biological age at $j = 1$	21		1
Model age at retirement, j_r	45		1
Model age maximum, J	70		1
Survival rates, $\{s_j\}$	-	Population data (HMD)	1
Population growth, n	0.011	U.S. Social Sec. Admin. (SSA)	1

Table 3.2: Calibration: Estimates of aggregate risk

	NC	PC
Corr. (TFP, returns), $cor(\zeta_t, r_t)$	-0.08 (0.57)	0.50 (0.00)
Corr. (wages, returns), $cor(w_t, r_t)$	-0.33 (0.016)	0.306 (0.025)

Notes: NC: Negative correlation between TFP shocks and returns (linear trend estimation), PC: Positive correlation between TFP shocks and returns (first differences estimation). p-values are reported in brackets.

Table 3.3: Calibration: Endogenous parameters

	NC	PC
Preferences		
Discount factor, β	0.96	0.97
Relative risk aversion, θ	12	8
Technology		
Std. of depreciation $\bar{\delta}$	0.10	0.11
Cond. prob. depr. shocks, π^δ	0.435	0.86

Notes: NC: Negative correlation between TFP shocks and returns (linear trend estimation), PC: Positive correlation between TFP shocks and returns (first differences estimation).

measure of labor supply determined by multiplying all full- and part-time employees in domestic employment (NIPA Table 6.4A) with an index for aggregate hours (NIPA Table 6.4A).[13] We then fit a linear trend specification to the Solow residual. Acknowledging the labor augmenting technological progress specification chosen, this gives the aforementioned point estimate.

3.4.3 Aggregate States and Shocks

We assume that aggregate risk is driven by a four state Markov chain with support $Z = \{z_1, \ldots, z_4\}$ and transition matrix $\pi = (\pi_{ij})$. Each aggregate state maps into a combination of low or high technology shocks and low or high physical capital depreciation. To be concrete, we let

$$\zeta(z) = \begin{cases} 1 - \bar{\zeta} \text{ for } z \in z_1, z_2 \\ 1 + \bar{\zeta} \text{ for } z \in z_3, z_4 \end{cases} \quad \text{and } \delta(z) = \begin{cases} \delta_0 + \bar{\delta} \text{ for } z \in z_1, z_3 \\ \delta_0 - \bar{\delta} \text{ for } z \in z_2, z_4. \end{cases} \tag{3.4.33}$$

With this setup, z_1 corresponds to a low wage and a low return, while z_4 corresponds to a high wage and a high return.

To calibrate the entries of the transition matrix, denote by $\pi^\zeta = \pi(\zeta' = 1 - \bar{\zeta} \mid \zeta = 1 - \bar{\zeta})$ the transition probability of remaining in the low technology state. Assuming that the transition of technology shocks is symmetric, we then also that $\pi(\zeta' = 1+\bar{\zeta} \mid \zeta = 1+\bar{\zeta}) = \pi^\zeta$ and, accordingly $1 - \pi^\zeta = \pi(\zeta' = 1 - \bar{\zeta} \mid \zeta = 1 + \bar{\zeta}) = \pi(\zeta' = 1 + \bar{\zeta} \mid \zeta = 1 - \bar{\zeta})$.

To govern the correlation between technology and depreciation shocks, let the probability of being in the high (low) depreciation state conditional on being in the low (high) technology state, assuming symmetry, be $\pi^\delta = \pi(\delta' = \delta_0 + \bar{\delta} \mid \zeta' = 1 - \bar{\zeta}) = \pi(\delta' = \delta_0 - \bar{\delta} \mid \zeta' = 1 + \bar{\zeta})$.

We then have that the transition matrix of aggregate states follows from the corresponding

[13] Notice that we thereby ignore age-specific productivity which should augment our measure of employment.

assignment of states in (3.4.33) as

$$\pi^z = \begin{bmatrix} \pi^\zeta \cdot \pi^\delta & \pi^\zeta \cdot (1-\pi^\delta) & (1-\pi^\zeta)\cdot(1-\pi^\delta) & (1-\pi^\zeta)\cdot\pi^\delta \\ \pi^\zeta \cdot \pi^\delta & \pi^\zeta \cdot (1-\pi^\delta) & (1-\pi^\zeta)\cdot(1-\pi^\delta) & (1-\pi^\zeta)\cdot\pi^\delta \\ (1-\pi^\zeta)\cdot\pi^\delta & (1-\pi^\zeta)\cdot(1-\pi^\delta) & \pi^\zeta \cdot (1-\pi^\delta) & \pi^\zeta \cdot \pi^\delta \\ (1-\pi^\zeta)\cdot\pi^\delta & (1-\pi^\zeta)\cdot(1-\pi^\delta) & \pi^\zeta \cdot (1-\pi^\delta) & \pi^\zeta \cdot \pi^\delta \end{bmatrix}$$

In sum, the Markov chain process of aggregate shocks is characterized by four parameters, $(\bar{\zeta}, \bar{\delta}, \pi^\zeta, \pi^\delta)$. All of these parameters are second stage parameters which we calibrate jointly to match the following targets: (i) an average variance of the cyclical component of TFP, again estimated from NIPA data, (ii) the average fluctuation of the risky return which features a standard deviation in the data of 0.16, (iii) the autocorrelation of the cyclical component of TFP in the data and (iv) the estimated correlation of the cyclical component of TFP with risky returns.

As to the latter targets we calibrate two versions which reflect different views on the nature of the data generating process of observed TFP fluctuations.

First, we adopt the Krueger and Kubler (2006) approach to the data by assuming a linear trend as a filter to get the stochastic components in the data. Such a linear trend specification can be justified on the grounds that the model features such a trend, and that the underlying covariance structures should remain unaffected.[14] The results are in line with Krueger and Kubler (2006) and are shown in the first column (labeled NC) of table 3.2.[15] The correlation between wages and returns is estimated to be negative, while the correlation between TFP and returns is negative but statistically not different from zero.

In our second approach, we assume a unit root process for the log of TFP. Applying a first difference filter to the data, we find that the correlation between TFP and returns as well as wages and returns is positive, the former being larger in magnitude than the latter. This finding coincides with our economic intuition as we would expect these variables to

[14] Rather than to TFP fluctuations, Krueger and Kubler (2006) refer to stochastic processes of aggregate wages. By the static first order conditions of the firm problem, these two are highly related, cf. equation (3.3.19).

[15] The differences between their and our estimates arise because they have to aggregate the data to 6-year intervals to match the period length of their model, whereas we use the yearly data since we have 1-year intervals.

co-move over the cycle.

For sake of consistency, we then transform the numbers to an equivalent deterministic trend specifications in the following way. We stick to the Krueger and Kubler (2006) calibration and only adopt the new correlation structure between TFP innovations and returns. This means that we implicitly compute the average horizon h in the unit root model such that the unconditional variance over h periods coincides with the KK calibration. This gives an average horizon of $h = 19.2751$ years.[16]

In order to check the robustness of our findings, we also adopt a standard RBC view of the data and de-trend with the Hodrick-Prescott filter. This yields a highly significant, positive correlation which is comparable in magnitude to our preferred PC (finite difference) calibration. Details can be found in the robustness section 3.5.3.

3.4.4 Population Data

We assume that agents start working at the biological age of 21, which therefore corresponds to $j = 1$. We set $J = 70$, implying that agents die with certainty at biological age 90, and $jr = 45$, corresponding to a statutory retirement age of 65. Population grows at the rate of 1.1% which reflects the current trend growth of the US population. The conditional survival rates ς_j are imputed from mortality data retrieved from the Human Mortality Database (HMD).

3.4.5 Household Sector

The value of household's raw time discount factor, β, and the coefficient of relative risk aversion θ are calibrated endogenously (second stage parameters) such that our model produces a capital output ratio of 2.65 and an average equity premium of 0.056.

We determine the intertemporal substitution elasticity as a second-stage parameter such that our model generates a hump-shaped consumption profile. This is achieved via a relatively high value of $\varphi = 1.5$. It is consistent with the range discussed in Bansal and

[16] Observe that the unit root estimates in fact imply even stronger aggregate fluctuations. Adjusting the variance in the linear trend specification such that the average horizon equals the average horizon of households in our model, appropriately adjusted to account for the correlation of TFP innovations, gives an average horizon of 34.88 years. This implies a standard deviation of 0.039. Relative to the PC calibration this means that the standard deviation of innovations increases by roughly 76 percent. However, the overall effects of this additional increase in risk are small. Results are available upon request.

Yaron (2004) and lower than their benchmark value of 2. We document the sensitivity of our results with respect to this parameter in section 3.5.3.

The age-specific productivity profile ϵ_j is calibrated to match PSID data applying the method of Huggett (2011).

Our calibration of states $\mathcal{E}_z$ and transition probabilities π^η of the idiosyncratic Markov chain income processes is based on estimates of Storesletten, Telmer, and Yaron (2004), henceforth STY, for individual wage income processes. STY postulate that the permanent shocks obey an $AR(1)$ process given as

$$\ln(\eta)_{i,j,t} = \rho \ln(\eta)_{i,j-1,t-1} + \epsilon_{i,j,t} \tag{3.4.34}$$

where

$$\epsilon_{i,j,t} \sim \mathcal{N}(0, \sigma_t^2) \tag{3.4.35}$$

Building on Constantinides and Duffie (1996), STY assume a countercyclical, cross-sectional variance of the innovations (CCV). Their estimates are $\rho = 0.952$ and

$$\sigma_t^2 = \begin{cases} \sigma_c^2 = 0.0445 \text{ for } z \in z_1, z_2 \\ \sigma_e^2 = 0.0156 \text{ for } z \in z_3, z_4 \end{cases} \tag{3.4.36}$$

where e stands for expansion and c for contraction.

We approximate the above process by discrete two-state Markov process. Denoting state contingency of the innovations by $\sigma^2(z)$, observe that $\sigma(z)^2_{\ln \eta} = \frac{\sigma^2(z)}{1-\rho^2}$. We then approximate the underlying η by the following symmetric Markov process:

$$\mathcal{E}_z = [\eta_1(z), \eta_2(z)] = [\eta_-(z), \eta_+(z)] \tag{3.4.37}$$

$$\pi^\eta = \begin{bmatrix} \bar{\pi}^\eta & 1 - \bar{\pi}^\eta \\ 1 - \bar{\pi}^\eta & \bar{\pi}^\eta \end{bmatrix} \tag{3.4.38}$$

$$\Pi = [0.5, 0.5]$$

so that the unconditional mean of the state vector is equal to 1.

Our approximation is different from standard approximations of log income processes in two respects. First, standard approximations do not condition on aggregate states. Second, standard approximations ignore a bias term which gets large when the variance of the estimates increases. We describe the details of our procedure in appendix 3.B. Resulting estimates are

$$\eta_1 = \eta_- = \begin{cases} 0.4225 & \text{for } z = z_1, z_2 \\ 0.6196 & \text{for } z = z_3, z_4 \end{cases} \qquad \eta_2 = \eta_+ = \begin{cases} 1.5775 & \text{for } z = z_1, z_2 \\ 1.3804 & \text{for } z = z_3, z_4 \end{cases}$$

and $\bar{\pi}^{\eta} = 0.9741$.

3.5 Results

In the discussion of the main results of our quantitative analysis we will refer to the insights derived from the simple model of section 3.2. In particular, we will highlight the insurance effects against idiosyncratic risk (IR), aggregate risk (AR), and their interaction (LCI) as defined in equation (3.2.8), and oppose them with the costs of crowding out denoted by term C in equation (3.2.16).

In order to expose the commonalities and differences to Krueger and Kubler (2006), we will first analyze the results of the calibration with a negative correlation between TFP and returns. Then we compare that to the calibration with a positive correlation. As argued by KK, a negative correlation should increase the intergenerational insurance provided by social security. However, our companion paper Harenberg and Ludwig (2011) exposes this to be a fallacy resulting from the essentially atemporal structure of the simple toy model, and shows that a positive correlation should increase the insurance effects of social security. The next two subsections will give some quantitative answers on how much these effects matter.

In the third subsection, we discuss the robustness of our results when we either conduct a different thought experiment, or change the elasticity of intertemporal substitution φ, or use a calibration with a nonlinear trend.

3.5.1 Calibration with Negative Correlation between TFP and Returns

This calibration matches the aggregate statistics in the column labeled NC in table 3.2. The corresponding endogenous parameters are displayed in the first column of table 3.3. As expected, the negative correlation between TFP and returns, $cor(\zeta, r)$, leads to a negative correlation between wages and returns, $cor(w, r)$, which is documented in table 3.4. While we target $cor(\zeta, r)$, KK target a negative $cor(w, r)$ directly. The table also shows that the standard deviation of aggregate consumption growth, $std(\Delta C/C)$, is counterfactually high, which is a direct consequence of the depreciation shocks that we employ to match the variance of risky returns.[17]

Table 3.4: Model-generated moments (NC)

$cor(w, r)$	$std(\Delta C/C)$
-0.079	0.073

The effects of our social security experiment on welfare, capital, and prices are documented in table 3.5. In the first column (labeled 'GE' for general equilibrium), we compare the two long-run equilibria without any transition. We see that the increase of the contribution rate from $\tau = 0.0$ to $\tau = 0.02$ leads to welfare gains of +0.51%. This number represents the percent of lifetime consumption the agent would be willing to give up to be born into the economy with some social security. There is substantial crowding out of capital of -5.91%, which leads to the displayed price changes, but this adverse effect is not strong enough to overturn the benefits from insurance.

In order to isolate those insurance benefits, we conduct the partial equilibrium (PE) experiment described in section 3.3.9. One can think of it as a small open economy, where aggregate prices are determined in the world, and social security is introduced in the small home country. As the second column in table 3.5 shows, the net welfare gains attributable to the total insurance provided by social security amount to +5.10%. Aggregate prices in this world do not change by construction, and that is why we can isolate the insurance effects. Therefore, the difference between the two welfare numbers $0.51\% - 5.10\% = -4.61\%$

[17] We can reduce $std(\Delta C/C)$ significantly by introducing a capital structure of the firm as described in section 2.2.3.

can be attributed to the crowding out of capital, which corresponds to term C in equation (3.2.16). Finally, the $\Delta K/K = -24.90\%$ in PE should be interpreted as 'less capital being invested abroad': of course agents save much less for old-age retirement, and this effect is much smaller in GE because of the mitigating price adjustments.

Table 3.5: The social security experiment (NC)

	GE	PE
ΔWelf/Welf	+0.51%	+5.10%
$\Delta K/K$	-5.91%	-24.90%
$\Delta E(r)$	+0.29%	*0.00%*
Δr_f	+0.59%	*0.00%*
$\Delta w/w$	-3.88%	*0.00%*

But where do these +5.10% of total net insurance come from, how much can be attributed to insurance against aggregate risk, how much to idiosyncratic wage risk, how much to CCV, and to survival risk? That is answered in table 3.6, where we start with an economy with only aggregate risk, then add idiosynratic wage risk on top, then add CVV, and finally also include survival risk. For each economy, we look at the welfare gains from the experiment in PE, so that in the last column, we end up with the same +5.10% that we just saw. The first column looks at an economy with only aggregate risk, which therefore is comparable to the partial equilibrium of KK. The welfare gains of +0.18% represent the intergenerational insurance against aggregate risk. This number is close to zero, because the additional insurance only just outweighs the cost of the higher contributions, which are painful in particular for those agents that are close to the natural borrowing limit. So the actual insurance itself is larger, and we will quantify it below.[18]

The second column of table 3.6 looks at an economy with both aggregate and idiosyncratic risk. Introducing social security in this economy leads to substantially larger welfare gains of +1.99%, which -since this is still PE- are attributable to the intergenerational insurance against aggregate risk plus the intergenerational insurance against idiosyncratic risk. When we add CCV risk, insurance gains go up by another 0.47% (calculated as 2.46% – 1.99%), and looking at the last column we see that adding survival risk adds another 2.64%. Summing up, insurance against idiosyncratic wage risk and survival risk is very large, that

[18]Optimally, we would compute the welfare numbers in a model without aggregate risk, but this would require such drastic recalibration that any comparison would very hard to interpret.

against aggregate risk and CCV only moderate.

Table 3.6: Insurance against sources of risk (NC)

	aggr. risk	+ idios. wage risk	+ CCV	+ surv. risk
ΔWelf/Welf	+0.18%	+1.99%	+ 2.46%	+5.10%

Table 3.7: Identification of the direct interaction term LCI (NC)

$\sigma\zeta, \sigma_\delta$	aggr. risk (AR)	+ idios. wage risk	$IR + LCI$	ΔLCI
benchmark	+0.18%	+1.99%	+1.81	-
-10%	+0.09%	+1.83%	+1.74	-0.072

We next turn to the interaction effect IAR. To isolate it, we conduct the difference-in-difference calculation described in section 3.3.9. The logic of this calculation becomes transparent in table 3.7. The first row looks at our benchmark economy, and the second row at the same economy with aggregate risk reduced by -10%.[19] The first two columns simply show the insurance against aggregate risk (AR) and against aggregate risk + idiosynratic wage risk in the same way we just discussed. Indeed the two numbers in the first row are simply copied from table 3.6.

By taking the difference between column two and column one we are left with $IR + LCI$, since AR drops out, and this number is displayed in the third column. We now take the difference of the numbers in the third column and thereby get ΔLCI, because IR remains constant by construction. So the term ΔLCI represents by how much LCI changes if we reduce AR by -10%. Relating the change in insurance against LCI to the change in insurance against AR, we get $\frac{-0.072}{0.09-0.18} = 0.8$. In other words, LCI increases insurance by 80% of AR.

With these numbers at hand, we can now easily recoup the levels of insurance against IR and LCI for the benchmark economy. We already know from the first number in table 3.7 that insurance against AR is 0.18%. The other two numbers are calculated as $LCI \approx 0.8 \cdot 0.18\% = 0.144\%$ and $IR = 1.81\% - 0.144\% = 1.036\%$. The total interaction

[19]Reducing aggregate risk by -10% is achieved by reducing the standard deviations of the TFP and depreciation shocks by -10% each.

between aggregate and idiosyncratic risk is the sum of CCV and LCI which is $0.47\% + 0.144\% = 0.61\%$.

3.5.2 Calibration with Positive Correlation between TFP and Returns

The discussion of this calibration will be much more concise than the previous one, because all components were already explained there, and the exposition is structured in exactly the same way. The value of the targeted correlation $cor(\zeta, r)$ is shown in column two (labeled PC) of table 3.2, and in contrast to before it is now positive. In order to match it, we now need the conditional probability of depreciation shocks to be $\pi^\delta = 0.86$. In comparison to the NC calibration, we need to adjust the discount factor β and the standard deviation of depreciation $\bar{\delta}$ slightly, and considerably reduce the coefficient of relative risk aversion θ so as to match all statistics, see table 3.3 column two. The large, positive $cor(\zeta, r)$ induces a large, positive $cor(w, r) = 0.236$, and also drives up a bit the standard deviation of consumption growth, cf. table 3.8.

Table 3.8: Model-generated moments (PC)

$cor(w, r)$	$std(\Delta C/C)$
0.236	0.076

Introducing social security into this economy leads to welfare gains of +3.52% when comparing the long-run general equilibria (table 3.9). Since we are talking about consumption-equivalent variations, this is a very large number. Note that it is much larger than in the NC calibration although we reduced θ substantially. The crowding out of capital and its associated price changes take virtually the same values as in the NC calibration. This surprising similarity is probably due to the fact that we still have the same elasticity of inter-temporal substitution, see section 3.5.3 for a sensitivity analysis with respect to this parameter.

When we repeat the PE experiment by again keeping prices fixed, we see that the net benefits attributable to the total insurance amount to +9.37%, so that the welfare costs of crowding out can be calculated as $3.52\% - 9.37\% = -5.85\%$. As before, the $\Delta K/K = -29.39\%$ in this PE should be interpreted as 'less capital being invested abroad'.

Table 3.9: The social security experiment (PC)

	GE	PE
ΔWelf/Welf	+3.52%	+9.37%
$\Delta K/K$	-5.90%	-29.39%
$\Delta E(r)$	+0.30%	*0.00%*
Δr_f	+0.67%	*0.00%*
$\Delta w/w$	-3.87%	*0.00%*

Table 3.10: Insurance against sources of risk (PC)

	aggr. risk	+ idios. wage risk	+ CCV	+ surv. risk
ΔWelf/Welf	+1.26%	+3.92%	+5.69%	+9.37%

Table 3.10 decomposes the total insurance into its four sources. Insurance against aggregate risk is a lot higher than before. This suggests that the mechanism described in our companion paper Harenberg and Ludwig (2011) obtains in our model. To put it in a nutshell, a positive $cor(\zeta, r)$ (and correspondingly positive $cor(w, r)$) increases the value of social security, because it increases the variance of lifetime income. This effect quantitatively dominates the effect from a negative $cor(\zeta, r)$, which would increase the value of social security as a hedge against volatile savings income at old age.

The additional insurance when idiosyncratic wage risk is included amounts to $+3.92\% - 1.26\% = 2.66\%$, which is larger than in the NC calibration and which already indicates that LCI will be larger than before, because IR remained the same. Similarly, including CCV has a much larger impact at $5.69\% - 3.92\% = 1.77\%$ as opposed to 0.47%.[20] Finally, the impact of survival risk is also larger.

Table 3.11: Identification of the direct interaction term LCI (PC)

$\sigma\zeta, \sigma_\delta$	aggr. risk (AR)	+ idios. wage risk	$IR + LCI$	ΔLCI
benchmark	+1.26%	+3.92%	+2.66	-
-10%	+0.83%	+3.10%	+2.27	-0.392

In order to isolate the interaction term LCI and the pure idiosyncratic risk term IR, we again proceed with the difference-in-difference calculation that is explained both in the

[20] The impact of CCV on both the equity premium and insurance is larger, the larger $cor(\zeta, r)$.

previous subsection and in section 3.3.9. The results in table 3.11 show that ΔLCI is much larger than before. Also in relation to the change in AR it is larger than before: $\frac{\Delta LCI}{\Delta AR} = \frac{-0.392\%}{0.83\%-1.26\%} \approx 0.91$, i.e. LCI adds 90% on top of AR in terms of insurance. From this, we can again compute the levels of insurance against IR and LCI in our benchmark economy. Since insurance against AR is +1.26% in the table, we get that the welfare gains from the other two sources amount to $LCI \approx 0.91 \times 1.26\% = 1.15\%$ and $IR = 2.66\% - 1.15\% = 1.51\%$. The total interaction between aggregate and idiosyncratic risk as the sum of CCV and LCI is hence $1.77\% + 1.15\% = 2.92\%$.

3.5.3 Robustness

This section discusses the sensitivity of our results with respect to the three most crucial model elements. First, we show that when the thought experiment is not a marginal introduction of a contribution rate, but instead a marginal increase from the current level in the U.S., all results remain qualitatively unchanged. Next, we document the sensitivity of the welfare and crowding out numbers when a much smaller elasticity of intertemporal substitution is used. Finally, we show that using a standard Hodrick-Prescott filter yields both empirical estimates as well as computational results that support the findings and conclusions from our preferred PC calibration.

Robustness of the Thought Experiment

We chose to perform the thought experiment of a marginal introduction of social security mainly because it conforms to both the approach taken in our theory section and the approach taken by KK. However, it is crucial to understand that the positive welfare results do not hinge on this specific example. Since it is beyond this paper to check all possible experiments, we chose one that is sufficiently different and that still seems relevant from an empirical perspective. For that, we take the current U.S. value of social security contributions $\tau = 0.12$ and increase it by 2% to keep it comparable to the previous results. We perform this new experiment for both the NC and PC calibrations, and document all findings in the same way as before. Here, we will summarize the key insights, and relegate the tables to appendix 3.C.

Since now we start from a situation with substantial old age income, we need to adjust the coefficient of relative risk aversion θ and the discount factor β considerably in order to match the same aggregate statistics. Specifically, θ has to be increased by five in both calibrations, because higher social security income puts downward pressure on the equity premium, as agents have more safe income and thus demand more of the risky stock. At the same time, β needs to be increased by approximately 0.02 in both calibrations in order to match the same capital-output-ratio, because agents' savings are of course reduced by the higher social security income. The model-generated, not-targeted moments $cor(w, r)$ and $std(\Delta C/C)$ remain basically unchanged.

Despite the fact that the changes in θ and β should, ceteris paribus, both increase the welfare gains of social security, we find these to be substantially smaller throughout. Still, welfare gains are positive in GE, with +0.17% for NC and +1.94% for PC, while at the same time, the crowding out of capital is half as large as it was before. The relative ordering of the various sources of insurance remains the same with the exception of survival risk, which now is much less important than before. The most relevant difference is that LCI now only adds approximately $0.3 * AR$ in terms of insurance in both cases, which -while being a notable reduction to before - still is a substantial amount. Finally, for the NC calibration, we find that insurance against aggregate risk is negative at -0.31%, meaning that the intergenerational insurance in this case does not outweigh the utility costs of having to pay the contributions, which is painful in particular for agents close to the natural borrowing limit.

Sensitivity with Respect to the Elasticity of Intertemporal Substitution

This section discusses the effects of a reduction in the elasticity of intertemporal substitution φ from 1.5 to 0.5. The first value resulted from our calibration strategy as described in section 3.4, where this parameter was set so as to get a more hump-shaped life-cycle consumption profile.[21] The value of 0.5 is substantially smaller and does not deliver a clear hump in the life-cycle profile, but we use this value for the sensitivity analysis because it is the value chosen by Krueger and Kubler (2006). Both values can be defended on empirical grounds, as the recent estimates in the literature range from close to zero to a value of two.

[21] A higher IES implies that agents are more willing to accept higher growth rates of consumption in return for higher mean consumption. In our model this generates an empirically plausible peak at the age of 55.

One word of caution before we present the results. With a high EIS, agents react to the changes in interest rates to a stronger degree and thereby mitigate the crowding out of capital, whereas with a low EIS, the crowding out will be larger.[22] However, we only asses the negative welfare impact of crowding out since we ignore the transition to the new long-run equilibrium. If crowding out of capital between the long-run equilibria is larger, then agents will save less and consume more along the transition, which would enhance their welfare. So the welfare numbers we discuss now should be seen as a lower bound, and we expect that once we include the transition, welfare will react less sensitively to changes in this parameter.

The tables are shown in appendix 3.C. Note that due to the nature of the experiment, they need to be compared the tables in appendix 3.C. As shown in the last section, that experiment is much more unfavorable for social security, and welfare gains for our original experiment should be larger. As before, we document the numbers for both the calibration with the positive and the negative correlation between TFP and returns (PC and NC, respectively).

The reduction in the EIS necessitates a recalibration, which can be effectuated by adjusting only two parameters. The coefficient of relative risk aversion has to be reduced by two (three for the PC calibration) so that the model generates the same equity premium. The discount factor β needs to be reduced only very slightly. The endogenously generated $cor(w, r)$ is smaller than before, and more importantly, the standard deviation of aggregate consumption growth diminishes substantially to $std(\Delta C/C) = 0.043$ ($std(\Delta C/C) = 0.049$ for PC), which is much closer to the data. The reason for this decrease is that agents now prefer a smooth consumption path to high consumption growth.[23]

We find that the welfare numbers in general equilibrium are reduced by approximately 3.5%. This is a lot, and for the experiment conducted means that overall welfare gains are clearly negative, as opposed to the positive numbers for $\varphi = 1.5$. When looking at the GE, it seems that crowding out causes much of these losses, as the percentage of capital lost more than doubles. However, the PE experiment reveals that also the insurance

[22] Higher social security contributions crowd out capital, which increases interest rates, which incentivizes agents to save more, and agents will respond to this incentive more strongly when their EIS is higher.

[23] Note that such small values for $std(\Delta C/C)$ are achieved only with a capital structure of the firm, as described in section 2.2.3.

gains are reduced by approx. 2.5%, which means that the capital loss accounts only for approximately 1% of the welfare losses. The fact that agents seem to value social security less with a smaller EIS even when prices are kept fixed seems an interesting finding. Our hypothesis is that agents still value the reduction in the variance of old-age consumption, but they dislike that their consumption profile becomes somewhat steeper, even though that means higher average consumption. On the contrary, when agents have a high EIS, they like this second effect. To verify this, we will look at mean consumption and the variance of consumption over the life-cycle for the GE and PE experiments.
Turning to the sources of risk, insurance effects drop for all of them. Insurance against aggregate risk drops the most, insurance against survival risk the least. The interaction term, on the other hand, is of about the same order of magnitude as before: it adds about 30% on top of aggregate risk, the same number as in the previous section.

Nonlinear Trend Calibration

We now take take a standard RBC or Hodrick and Prescott (1997) perspective according to which, while the model is stationary, the data are rather non stationary and driven by some deterministic trend component of unknown functional form. This view implies that we merge a large proportion of observed fluctuations into the deterministic component of the data. The autocorrelation of the cyclical component of log TFP is 0.43 with an unconditional standard deviation of 0.0125, cf. table 3.12. We also find that the correlation between stock returns and TFP is strongly positive and highly significant, whereas the correlation between returns and wages is not significantly different from zero. This gives strong support to our view that the PC calibration is the more relevant. While the value of $corr(\zeta_t, R_t)$ lies between the NC and PC calibrations, it is clearly closer to the latter.
The results from the experiments confirm our findings. All welfare numbers are between those found in the NC and PC calibrations, and slightly closer to the latter. Of course, insurance against aggregate risk decreases substantially when compared to the PC calibration, but the interaction effect LCI still amounts to about $0.4 * \cdot AR$, our prediction from the toy model. The tables are relegated to appendix 3.C.

Table 3.12: Aggregate Risk, HP-filter

Point Estimates	
Autocorrelation of TFP	0.43 (0.00)
Standard deviation of TFP	0.012
Corr. TFP, R, $corr(\zeta_t, R_t)$	0.35 (0.01)
Corr. w, R, $corr(w_t, R_t)$	-0.03 (0.82)
Markov Approximation	
Aggregate states, $1 \pm \bar{\zeta}$	$[1.012, 0.988]$
Transition probabilities, π^{ζ}	0.7152

Notes: p-values are reported in brackets.

Table 3.13: Endogenous Parameters, HP-filter

θ	φ	β	$\bar{\delta}$	π^{δ}
14	1.5	0.98	0.1	0.7

3.6 Conclusion

In a life-cycle model, idiosyncratic and aggregate risk interact despite the fact that they are statistically independent. This interaction increases the value of social security. In our general equilibrium analysis, the introduction of a PAYG system leads to strong welfare gains. This stands in contrast to the related literature. The reason for this difference is that in our model, social security provides partial insurance against both idiosyncratic and aggregate risk, as well as their interactions. In fact, the interactions account for one third of the total welfare gains.

In our analysis, we abstracted from endogenous labor supply. This biases the results in favor of social security, because a higher contribution rate would distort the households' labor supply decision. In addition to the crowding out of capital, we would also see less labor being supplied. While it would be interesting to see this extension, one would probably have to restrict the model in some other way in order to clearly expose the mechanisms.

While our results do not depend on the calibration, we have seen that the covariance between wages and risky returns plays a crucial role. Interestingly, a positive correlation leads to substantially larger welfare gains. Previous analyses suggested that a negative correlation should increase the welfare gains, because then social security income is a better hedge against volatile asset income at old age. It became apparent that this mechanism is opposed by other forces. We elaborate on this in our companion paper (Harenberg and Ludwig (2011)).

In our economy, the intergenerational sharing of aggregate risks is limited to those generations alive at the same point in time. From a social planner's point of view, it would be desirable to share the risk also with future, unborn generations. This could be achieved by allowing the government to take up debt to smooth shocks over time. That would open up an additional insurance channel, which would increase the welfare gains of introducing social security.

Finally, we document in our robustness section that increasing the contribution rate from the current level in the U.S. of 12 percent to 14 percent also leads to welfare gains. While the welfare gains are still large, they are smaller than when the contribution rate is increased from zero to two percent. It seems that the higher the current level of contributions, the smaller the welfare gains are for a fixed percentage point increase in contributions. From these results it seems that there is an optimal level of social security, and that it lies somewhere above the current level observed in the U.S. today. We leave this and the other extensions to future research.

3.A Appendix: Proofs

Proof of proposition 3.1. Maximize

$$E_{t-1}u(c_{i,2,t+1})=E_{t-1}\left(\bar{w}_t\left(\bar{R}\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}+\tau\left((1+g)\zeta_{t+1}-\bar{R}\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}\right)\right)\right)^{1-\theta},$$

where we already removed the constant $\frac{1}{1-\theta}$. This is equivalent to maximizing

$$\max E_{t-1}R_{p,t,t+1}^{1-\theta}$$

where $R_{p,t,t+1}\equiv\eta_{i,1,t}\zeta_t\bar{R}\tilde{\varrho}_{t+1}+\tau\left((1+g)\zeta_{t+1}-\bar{R}\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}\right)$ is a consumption (or portfolio) return. Increasing ex-ante utility for a marginal introduction of social security requires the first-order condition w.r.t. τ to exeed zero, hence:

$$E_{t-1}\left[R_{p,t,t+1}^{-\theta}\frac{\partial R_{p,t,t+1}}{\partial\tau}\right]|_{\tau=0}>0 \tag{3.A.39}$$

Evaluated at $\tau=0$ we have

$$\begin{aligned}R_{p,t,t+1}^{-\theta}\ |_{\tau=0} &= \left(\eta_{i,1,t}\zeta_t\bar{R}\varrho_{t+1}\right)^{-\theta}\\ \frac{\partial R_{p,t,t+1}}{\partial\tau}\ |_{\tau=0} &= (1+g)\zeta_{t+1}-\eta_{i,1,t}\zeta_t\bar{R}\varrho_{t+1}\end{aligned}$$

Equation (3.A.39) therefore rewrites as

$$(1+g)E_{t-1}\left[\left(\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}\right)^{-\theta}\zeta_{t+1}\right]>\bar{R}E_{t-1}\left[\left(\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}\right)^{1-\theta}\right]. \tag{3.A.40}$$

Rewriting the above and imposing assumption 3.2 we get equation (3.2.4). □

Proof of proposition 3.2. Define

$$\begin{aligned}Z_1&\equiv\left(\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}\right)^{-\theta}\zeta_{t+1}\\ Z_2&\equiv\left(\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}\right)^{1-\theta}.\end{aligned}$$

By log-normality we have that $EZ_i = \exp(E\ln Z_i + \frac{1}{2}\sigma^2_{\ln Z_i})$, $i = 1, 2$. Observe that

$$\begin{aligned} E\ln Z_1 &= -\theta\left(E\ln\eta_{i,1,t} + E\ln\tilde{\varrho}\right) + (1-\theta)E\ln\zeta \\ \sigma^2_{\ln Z_1} &= \theta^2\left(\sigma^2_{\ln\eta} + \sigma^2_{\ln\tilde{\varrho}}\right) + (1+\theta^2)\sigma^2_{\ln\zeta} \end{aligned}$$

Therefore

$$\begin{aligned} E_{t-1}[Z_1] = &\exp\left(-\theta\left(E\ln\eta_{i,1,t} + \frac{\sigma^2_{\ln\eta}}{2}\right)\right)\cdot\exp\left(-\theta\left(E\ln\tilde{\varrho} + \frac{\sigma^2_{\ln\tilde{\varrho}}}{2}\right)\right)\cdot \\ &\cdot\exp\left((1-\theta)\left(E\ln\zeta + \frac{\sigma^2_{\ln\zeta}}{2}\right)\right)\exp\left(\frac{1}{2}\theta(1+\theta)\left(\sigma^2_{\ln\eta} + \sigma^2_{\ln\tilde{\varrho}} + \sigma^2_{\ln\zeta}\right)\right) \\ = &\left(E[\eta_{i,1,t}]\right)^{-\theta}\left(E[\tilde{\varrho}]\right)^{-\theta}\left(E[\zeta]\right)^{1-\theta}\cdot\exp\left(\frac{1}{2}\theta(1+\theta)\left(\sigma^2_{\ln\eta} + \sigma^2_{\ln\tilde{\varrho}} + \sigma^2_{\ln\zeta}\right)\right) \\ = &\exp\left(\frac{1}{2}\theta(1+\theta)\left(\sigma^2_{\ln\eta} + \sigma^2_{\ln\tilde{\varrho}} + \sigma^2_{\ln\zeta}\right)\right) \end{aligned}$$

whereby the last line follows from assumption 3.2b.
Next, observe that log-normality implies that

$$\begin{aligned} \sigma^2_\eta = var_{t-1}(\eta_{i,1,t}) &= \exp\left(2E\ln\eta_{i,1,t} + \sigma^2_{\ln\eta}\right)\left(\exp\left(\sigma^2_{\ln\eta}\right) - 1\right) \\ &= \left(E\eta_{i,1,t}\right)^2\left(\exp\left(\sigma^2_{\ln\eta}\right) - 1\right) \\ &= \left(\exp\left(\sigma^2_{\ln\eta}\right) - 1\right) \end{aligned}$$

whereby the last line again follows from assumption 3.2b. Hence:

$$\sigma^2_{\ln\eta} = \ln\left(1 + \sigma^2_\eta\right)$$

with corresponding expressions for $\sigma^2_{\ln\zeta}$ and $\sigma^2_{\ln\tilde{\varrho}}$. Therefore:

$$\exp\left(\frac{1}{2}\theta(1+\theta)\left(\sigma^2_{\ln\eta} + \sigma^2_{\ln\tilde{\varrho}} + \sigma^2_{\ln\zeta}\right)\right) = \left((1+\sigma^2_\eta)(1+\sigma^2_\zeta)(1+\sigma^2_{\tilde{\varrho}})\right)^{\frac{1}{2}\theta(1+\theta)}$$

We consequently have

$$E_{t-1}[Z_1] = \left((1+\sigma^2_\eta)(1+\sigma^2_\zeta)(1+\sigma^2_{\tilde{\varrho}})\right)^{\frac{1}{2}\theta(1+\theta)}$$

As to $E_{t-1}[Z_2]$ observe that

$$
\begin{aligned}
E\ln Z_2 &= (1-\theta)\left(E\ln\eta_{i,1,t} + E\ln\zeta + E\ln\tilde{\varrho}\right) \\
\sigma^2_{\ln Z_2} &= (1-\theta)^2\left(\sigma^2_{\ln\eta} + \sigma^2_{\ln\zeta} + \sigma^2_{\ln\tilde{\varrho}}\right)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
E_{t-1}[Z_2] =& \exp\left((1-\theta)\left(E\ln\eta_{i,1,t} + \frac{\sigma^2_{\ln\eta}}{2}\right)\left(E\ln\tilde{\varrho} + \frac{\sigma^2_{\ln\tilde{\varrho}}}{2}\right)\left(E\ln\zeta + \frac{\sigma^2_{\ln\zeta}}{2}\right)\right) \\
&\cdot\exp\left(\frac{1}{2}\theta(\theta-1)\left(\sigma^2_{\ln\eta} + \sigma^2_{\ln\tilde{\varrho}} + \sigma^2_{\ln\zeta}\right)\right) \\
=& \left((1+\sigma^2_{\eta})(1+\sigma^2_{\zeta})(1+\sigma^2_{\tilde{\varrho}})\right)^{\frac{1}{2}\theta(\theta-1)}
\end{aligned}
$$

Hence

$$
\frac{E_{t-1}[Z_1]}{E_{t-1}[Z_2]} = \left((1+\sigma^2_{\eta})(1+\sigma^2_{\zeta})(1+\sigma^2_{\tilde{\varrho}})\right)^{\theta}
$$

$\square$

Illustration 1. *Let us provide a simplified numerical illustration. Below, we calibrate our model with an annual income processes given by* $\ln(\eta_{i,j,t}) = \rho\ln(\eta_{i,j-1,t-1}) + \epsilon_{i,j,t}$ *where* j *is actual age of a working household,* t *is time,* $\epsilon_{i,j,t} \sim \mathcal{N}(0,\sigma^2_t)$*, hence* $\eta_{i,j,t}$ *is distributed as log-normal for all* i, j, t *and* ρ *is the autocorrelation coefficient. While we consider time variation in variances below, let us assume constant variances for now. Our calibration has an average variance of* $\sigma^2 \approx 0.03$*. We also calibrate* $\rho = 0.952$*. Consider the overall variance of income risk at retirement, that is, after a period in the work force of about* 45 *years. For AR(1) processes with such a long horizon, the approximate infinite horizon*[24] *formula to compute the variance of* $\ln\eta_{i,1,t}$ *at retirement is given by* $\frac{1}{1-\rho^2}\sigma^2_\epsilon$*. Using our numbers we accordingly have that the variance of* $\ln\eta_{i,1,t}$ *at retirement is given by* $\frac{1}{1-0.952^2}\cdot 0.03 = 10.67\cdot 0.03$*. By the formula for log-normal random variables, the variance of* $\eta_{i,1,t}$ *at retirement is therefore* $var(\eta_{i,1,t}) = (E[\eta_{i,1,t}])^2\left(\exp\left(\sigma^2_{\ln\eta}\right)-1\right) = \exp(10.67\cdot 0.03) - 1 = 0.37$.[25]

[24] The exact formula is $\frac{1-\rho^{2(j_r-1)}}{1-\rho^2}$ where j_r is the retirement age but the term $\rho^{2(j_r-1)}$ is negligible.

[25] As we describe in our main text, our estimates are based on Storesletten, Telmer, and Yaron (2004) who use after tax earnings data and control for aggregate fluctuations. Observe that these numbers are

Derivation of equation 3.2.8. We want to evaluate CEV between two scenarios, i.e., comparing $E_{t-1}u(c_{i,2,t+1_{\tau>0}})$ with $E_{t-1}u(c_{i,2,t+1_{\tau=0}})$. To simplify, let us use that

$$E_{t-1}u(c_{i,2,t+1_{\tau>0}}) = E_{t-1}u(c_{i,2,t+1_{\tau=0}}) + \frac{\partial E_{t-1}u(c_{i,2,t+1_{\tau=0}})}{\partial \tau} d\tau.$$

and evaluate this expression at $\tau = 0$.
We have that, evaluated at $\tau = 0$,

$$\begin{aligned}
\frac{\partial E_{t-1}u(c_{i,2,t+1_{\tau=0}})}{\partial \tau} &= \bar{w}_t^{1-\theta} E_{t-1}\left[\left(\bar{R}\eta\zeta_t\tilde{\varrho}_{t+1}\right)^{-\theta} \cdot \left((1+g)\zeta_{t+1} - \bar{R}\eta\zeta_t\tilde{\varrho}_{t+1}\right)\right] \\
&= \bar{w}_t^{1-\theta}\left(\bar{R}^{-\theta}(1+g)E_{t-1}\left[\left(\eta\zeta_t\tilde{\varrho}_{t+1}\right)^{-\theta}\zeta_{t+1}\right] - \bar{R}^{1-\theta}E_{t-1}\left[\left(\eta\zeta_t\tilde{\varrho}_{t+1}\right)^{1-\theta}\right]\right) \\
&= \bar{w}_t^{1-\theta}\bar{R}^{1-\theta}\left(\frac{1+g}{\bar{R}}E_{t-1}Z_1 - E_{t-1}Z_2\right)
\end{aligned}$$

where Z_1, Z_2 are defined in our proof to proposition 3.2.
We also have that

$$\begin{aligned}
E_{t-1}u(c_{i,2,t+1_{\tau=0}}) &= \frac{1}{1-\theta}\bar{w}_t^{1-\theta}\bar{R}^{1-\theta}E_{t-1}\left(\eta_{i,1,t}\zeta_t\tilde{\varrho}_{t+1}\right)^{1-\theta} \\
&= \frac{1}{1-\theta}\bar{w}_t^{1-\theta}\bar{R}^{1-\theta}E_{t-1}Z_2.
\end{aligned}$$

Therefore:

$$\begin{aligned}
E_{t-1}u(c_{i,2,t+1_{\tau>0}}) = &\frac{1}{1-\theta}\bar{w}_t^{1-\theta}\bar{R}^{1-\theta}E_{t-1}Z_2 \\
&+ \bar{w}_t^{1-\theta}\bar{R}^{1-\theta}\left(\frac{1+g}{\bar{R}}E_{t-1}Z_1 - E_{t-1}Z_2\right)d\tau.
\end{aligned}$$

a conservative estimate of the overall dispersion of earnings inequalities at retirement because we ignore the dispersion of skills and learning abilities at the beginning of the life-cycle. The more recent work by Huggett (2011) attributes about 60 of the overall variation in life-time income to variations in initial conditions. However, it is rather education policies than pension policies and social insurance that should target such differences. Huggett (2011)'s specification for income shocks is a unit root process. Their estimate of the standard deviation of the innovation of this process is 0.111. This would roughly double the relevance of the interaction term at retirement to 0.74. However, the estimates of Huggett (2011) are based on pre-tax earnings data and the authors do not control for the business cycle. This may explain these substantial differences.

The CEV, denoted by g_c, is defined by the relationship:

$$E_{t-1}u(c_{i,2,t+1_{\tau=0}}(1+g_c)) = E_{t-1}u(c_{i,2,t+1_{\tau>0}}),$$

from which, using the above formulae, we get

$$\begin{aligned}(1+g_c)^{1-\theta}\frac{1}{1-\theta}\bar{w}_t^{1-\theta}\bar{R}^{1-\theta}E_{t-1}Z_2 = \frac{1}{1-\theta}\bar{w}_t^{1-\theta}\bar{R}^{1-\theta}E_{t-1}Z_2 \\ + \bar{w}_t^{1-\theta}\bar{R}^{1-\theta}\left(\frac{1+g}{\bar{R}}E_{t-1}Z_1 - E_{t-1}Z_2\right)d\tau.\end{aligned}$$

Hence:

$$\begin{aligned}(1+g_c)^{1-\theta} &= 1 + \frac{\bar{w}_t^{1-\theta}\bar{R}^{1-\theta}\left(\frac{1+g}{\bar{R}}E_{t-1}Z_1 - E_{t-1}Z_2\right)}{\frac{1}{1-\theta}\bar{w}_t^{1-\theta}\bar{R}^{1-\theta}E_{t-1}Z_2}d\tau \\ &= 1 + (1-\theta)\left(\frac{1+g}{\bar{R}}\frac{E_{t-1}Z_1}{E_{t-1}Z_2} - 1\right)d\tau \\ &= 1 + (1-\theta)\left(\frac{1+g}{\bar{R}}(1+V)^{\theta} - 1\right)d\tau\end{aligned}$$

where the last line again follows from the proof to proposition 3.2.
Hence,

$$g_c = \left(1 + (1-\theta)\left(\frac{1+g}{\bar{R}}(1+V)^{\theta} - 1\right)d\tau\right)^{\frac{1}{1-\theta}} - 1.$$

or, expressed in logs, i.e., $g_c \approx \ln(1+g_c)$, we get

$$\begin{aligned}g_c &\approx \frac{1}{1-\theta}\cdot\ln\left(1 + (1-\theta)\left(\frac{1+g}{\bar{R}}(1+V)^{\theta} - 1\right)d\tau\right) \\ &\approx \left(\frac{1+g}{\bar{R}}(1+V)^{\theta} - 1\right)d\tau\end{aligned}$$

Taking a first-order Taylor series expansion of the above round $V = 0$ we get

$$g_c \approx \left(\frac{1+g}{\bar{R}} - 1 + \theta\frac{1+g}{\bar{R}}V\right)\cdot d\tau$$

The first term in brackets is the deterministic part. The second term is the additional gain due to risk which is linear in V. $\square$

Proof of proposition 3.3. To establish proposition 3.3a we have to show that

$$\begin{aligned} & \frac{1}{\zeta_l} E_{t-1} \frac{1}{\eta_{i,1,l}} + \frac{1}{\zeta_h} E_{t-1} \frac{1}{\eta_{i,1,h}} > \left(\frac{1}{\zeta_l} + \frac{1}{\zeta_h} \right) E_{t-1} \frac{1}{\eta_{i,1,t}} \\ \Leftrightarrow \quad & \frac{1}{\zeta_l} \left(E_{t-1} \frac{1}{\eta_{i,1,l}} - E_{t-1} \frac{1}{\eta_{i,1,t}} \right) + \frac{1}{\zeta_h} \left(E_{t-1} \frac{1}{\eta_{i,1,h}} - E_{t-1} \frac{1}{\eta_{i,1,t}} \right) > 0. \end{aligned} \tag{3.A.41}$$

Under assumption 3.4 we have that

$$\begin{aligned} E_{t-1} \frac{1}{\eta_{i,1,t}} &= \exp \left(- \left(E \ln \eta + \frac{1}{2} \sigma^2_{\ln \eta} \right) \right) \\ E_{t-1} \frac{1}{\eta_{i,1,l}} &= \exp \left(- \left(E \ln \eta + \frac{1}{2} \left(\sigma^2_{\ln \eta} - \Delta \right) \right) \right) \\ E_{t-1} \frac{1}{\eta_{i,1,h}} &= \exp \left(- \left(E \ln \eta + \frac{1}{2} \left(\sigma^2_{\ln \eta} + \Delta \right) \right) \right). \end{aligned}$$

Therefore

$$\begin{aligned} & E_{t-1} \frac{1}{\eta_{i,1,l}} - E_{t-1} \frac{1}{\eta_{i,1,t}} \\ & = \exp \left(- \left(E \ln \eta + \frac{1}{2} \left(\sigma^2_{\ln \eta} - \Delta \right) \right) \right) - \exp \left(- \left(E \ln \eta + \frac{1}{2} \sigma^2_{\ln \eta} \right) \right) \\ & = \exp \left(- \left(E \ln \eta + \frac{1}{2} \sigma^2_{\ln \eta} \right) \right) \exp \left(\frac{1}{2} \Delta \right) - \exp \left(- \left(E \ln \eta + \frac{1}{2} \sigma^2_{\ln \eta} \right) \right) \\ & = \exp \left(- \left(E \ln \eta + \frac{1}{2} \sigma^2_{\ln \eta} \right) \right) \left(\exp \left(\frac{1}{2} \Delta \right) - 1 \right) \\ & E_{t-1} \frac{1}{\eta_{i,1,h}} - E_{t-1} \frac{1}{\eta_{i,1,t}} \\ & = \exp \left(- \left(E \ln \eta + \frac{1}{2} \left(\sigma^2_{\ln \eta} + \Delta \right) \right) \right) - \exp \left(- \left(E \ln \eta + \frac{1}{2} \sigma^2_{\ln \eta} \right) \right) \\ & = \exp \left(- \left(E \ln \eta + \frac{1}{2} \sigma^2_{\ln \eta} \right) \right) \exp \left(- \frac{1}{2} \Delta \right) - \exp \left(- \left(E \ln \eta + \frac{1}{2} \sigma^2_{\ln \eta} \right) \right) \\ & = \exp \left(- \left(E \ln \eta + \frac{1}{2} \sigma^2_{\ln \eta} \right) \right) \left(\exp \left(- \frac{1}{2} \Delta \right) - 1 \right) \end{aligned}$$

Equation (3.A.41) therefore rewrites as

$$\frac{1}{\zeta_l}\left(\exp\left(\frac{1}{2}\Delta\right)-1\right)+\frac{1}{\zeta_h}\left(\exp\left(-\frac{1}{2}\Delta\right)-1\right)>0. \tag{3.A.42}$$

Observe that $\exp\left(-\frac{1}{2}\Delta\right)-1<0,\ \ \exp\left(\frac{1}{2}\Delta\right)-1>0$ and convexity of the exponential function implies that

$$|\exp\left(-\frac{1}{2}\Delta\right)-1|<|\exp\left(\frac{1}{2}\Delta\right)-1|.$$

Therefore

$$\begin{aligned}
&\frac{1}{\zeta_l}\left(\exp\left(\frac{1}{2}\Delta\right)-1\right)+\frac{1}{\zeta_h}\left(\exp\left(-\frac{1}{2}\Delta\right)-1\right)\\
&>\frac{1}{\zeta_l}\left(\exp\left(\frac{1}{2}\Delta\right)-1\right)-\frac{1}{\zeta_h}\left(\exp\left(\frac{1}{2}\Delta\right)-1\right)\\
&=\left(\exp\left(\frac{1}{2}\Delta\right)-1\right)\left(\frac{1}{\zeta_l}-\frac{1}{\zeta_h}\right)\\
&=\left(\exp\left(\frac{1}{2}\Delta\right)-1\right)\left(\frac{1}{\zeta_l}-\frac{1}{2-\zeta_l}\right)\\
&=\left(\exp\left(\frac{1}{2}\Delta\right)-1\right)\frac{2(1-\zeta_l)}{\zeta_l(2-\zeta_l)}>0.
\end{aligned}$$

To establish proposition 3.3b use assumption 3.4 to rewrite equation (3.A.42) as

$$f(\Delta_\zeta)\equiv\frac{1}{1-\Delta_\zeta}\left(\exp\left(\frac{1}{2}\Delta\right)-1\right)+\frac{1}{1+\Delta_\zeta}\left(\exp\left(-\frac{1}{2}\Delta\right)-1\right)>0.$$

Observe that

$$\begin{aligned}
\frac{\partial f(\Delta_\zeta)}{\partial\Delta_\zeta}&=\left(\frac{1}{1-\Delta_\zeta}\right)^2\left(\exp\left(\frac{1}{2}\Delta\right)-1\right)-\left(\frac{1}{1+\Delta_\zeta}\right)^2\left(\exp\left(-\frac{1}{2}\Delta\right)-1\right)\\
&=\left(\frac{1}{1-\Delta_\zeta}\right)^2\underbrace{\left(\exp\left(\frac{1}{2}\Delta\right)-1\right)}_{>0}+\left(\frac{1}{1+\Delta_\zeta}\right)^2\underbrace{\left(1-\exp\left(-\frac{1}{2}\Delta\right)\right)}_{>0}\\
&>0
\end{aligned}$$

Hence, a mean preserving spread of ζ increases the effect of CCV. □

Proof of Proposition 3.4. The proof is by guessing and verifying. We guess that

$$\begin{aligned} s_{1,t} &= \chi(1-\tau)w_t \\ &= \chi(1-\tau)(1-\alpha)\Upsilon_t\zeta_t k_t^\alpha. \end{aligned}$$

If this is correct, then the equilibrium dynamics are given by

$$\begin{aligned} K_{t+1} &= N_{1,t}s_{1,t} \\ &= N_{1,t}\chi(1-\tau)(1-\alpha)\Upsilon_t\zeta_t k_t^\alpha \\ \Leftrightarrow \qquad k_{t+1} &= \frac{N_{1,t}\chi(1-\tau)(1-\alpha)\Upsilon_t\zeta_t k_t^\alpha}{\Upsilon_{t+1}N_{1,t+1}} \\ &= \frac{1}{1+g}\chi(1-\tau)(1-\alpha)\zeta_t k_t^\alpha \end{aligned}$$

To verify this, notice that our assumptions on savings implies that

$$c_{1,t} = (1-\chi)(1-\tau)(1-\alpha)\Upsilon_t\zeta_t k_t^\alpha \tag{3.A.43}$$

and, by the budget constraint, we have

$$c_{2,t+1} = \chi(1-\tau)(1-\alpha)\Upsilon_t\zeta_t k_t^\alpha \alpha\zeta_{t+1}\varrho_{t+1}k_{t+1}^{\alpha-1} + (1-\alpha)\Upsilon_{t+1}\zeta_{t+1}k_{t+1}^\alpha\tau. \tag{3.A.44}$$

Using (3.2.12a) in (3.A.44) we get

$$\begin{aligned} c_{2,t+1} &= k_{t+1}(1+g)\Upsilon_t\alpha\zeta_{t+1}\varrho_{t+1}k_{t+1}^{\alpha-1} + (1-\alpha)\Upsilon_{t+1}\zeta_{t+1}k_{t+1}^\alpha\tau \\ &= \Upsilon_{t+1}\alpha\zeta_{t+1}\varrho_{t+1}k_{t+1}^\alpha + (1-\alpha)\Upsilon_{t+1}\zeta_{t+1}k_{t+1}^\alpha\tau \\ &= (\alpha\varrho_{t+1} + (1-\alpha)\tau)\,\Upsilon_{t+1}\zeta_{t+1}k_{t+1}^\alpha \end{aligned}$$

Next, notice that the first-order-condition of household maximization gives:

$$1 = \beta E_t\left[\frac{c_{1,t}(1+r_{t+1})}{c_{2,t+1}}\right]. \tag{3.A.45}$$

Using the above equations for consumption in the two periods, we can rewrite (3.A.45) as:

$$\begin{aligned}
1 &= \beta E_t \left[\frac{c_{1,t}\alpha\zeta_{t+1}\varrho_{t+1}k_{t+1}^{\alpha-1}}{\left(\alpha\varrho_{t+1}+(1-\alpha)\tau\right)\Upsilon_{t+1}\zeta_{t+1}k_{t+1}^{\alpha}} \right] \\
&= \beta E_t \left[\frac{c_{1,t}\alpha\varrho_{t+1}}{\left(\alpha\varrho_{t+1}+(1-\alpha)\tau\right)\Upsilon_{t+1}k_{t+1}} \right] \\
&= \beta E_t \left[\frac{(1-\chi)(1-\tau)(1-\alpha)\Upsilon_t\zeta_t k_t^{\alpha}\alpha\varrho_{t+1}}{\left(\alpha\varrho_{t+1}+(1-\alpha)\tau\right)\Upsilon_{t+1}\frac{1}{1+g}\chi(1-\tau)(1-\alpha)\zeta_t k_t^{\alpha}} \right] \\
&= \beta E_t \left[\frac{(1-\chi)\alpha\varrho_{t+1}}{\left(\alpha\varrho_{t+1}+(1-\alpha)\tau\right)\chi} \right] \\
&= \frac{\beta(1-\chi)\alpha}{\chi} E_t \left[\frac{\varrho_{t+1}}{\alpha\varrho_{t+1}+(1-\alpha)\tau} \right] \\
&= \frac{\alpha\beta(1-\chi)}{\chi}\bar{E}
\end{aligned}$$

where

$$\bar{E} \equiv E_t \left[\frac{1}{\alpha+(1-\alpha)\varrho_{t+1}^{-1}\tau} \right].$$

It follows that

$$\chi = \frac{1}{1+\left(\alpha\beta\bar{E}\right)^{-1}}$$

Uniqueness is established by convexity of the problem. Given that the solution is unique and given that we have characterized one solution, this is the solution to the problem. □

Proof of proposition 3.6. Rewrite the above consumption equations to get

$$c_{1,t,ms} = (1-\chi)(1-\tau)\Upsilon_t\zeta_t(1-\alpha)k_{ms}^{\alpha}$$

and

$$c_{2,t+1,ms} = \left(\chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1} + \tau\left((1+g)-\chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1}\right)\right)\Upsilon_t\zeta_{t+1}(1-\alpha)k_{ms}^{\alpha}$$

We now look at ex-ante utility:

$$\begin{aligned} E_{t-1}u_t =& E_{t-1}\mathbf{c} + E_{t-1}\left[\ln(1-\chi) + \ln(1-\tau)\right] + \alpha \ln k_{ms} \\ &+ \beta E_{t-1}\left[\ln\left(\chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1} + \tau\left((1+g) - \chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1}\right)\right)\right] \\ &+ \beta\alpha \ln k_{ms} \\ =& E_{t-1}\mathbf{c} + E_{t-1}\left[\ln(1-\chi) + \ln(1-\tau)\right] + \alpha(1+\beta)\ln k_{ms} \\ &+ \beta E_{t-1}\left[\ln\left(\chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1} + \tau\left((1+g) - \chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1}\right)\right)\right] \end{aligned} \tag{3.A.46}$$

where $\mathbf{c}$ encompasses all elements that are not affected by τ.

We evaluate the derivative of the above at $\tau = 0$. To this end, we look separately at the different terms. Let us define

$$B_1 \equiv E_{t-1}\frac{\partial \ln(1-\chi)}{\partial \tau} = -E_{t-1}\frac{1}{1-\chi}\frac{\partial \chi}{\partial \tau} > 0 \tag{3.A.47}$$

The sign of this term is due to the fact that an increase of the contribution rate crowds out savings, hence $\frac{\partial \chi}{\partial \tau} < 0$. We return to this equation in more detail below.

The second term—which we define as A_1—captures the effect of taxation on income (evaluated again at $\tau = 0$):

$$A_1 \equiv E_{t-1}\frac{\partial \ln(1-\tau)}{\partial \tau} = -\frac{1}{1-\tau} = -1. \tag{3.A.48}$$

We next investigate the implicit return equation for social security. We get, evaluated at $\tau = 0$, that:

$$\begin{aligned} &\frac{\partial \ln\left(\chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1} + \tau\left((1+g) - \chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1}\right)\right)}{\partial \tau} \\ &= \frac{1}{\chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1}} \\ &\quad\times\left(\underbrace{(1+g) - \chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1}}_{\equiv\tilde{A}_2} + \alpha\zeta_t\varrho_{t+1}\left(\underbrace{\frac{\partial \chi}{\partial \tau}k_{ms}^{\alpha-1}}_{\equiv\tilde{B}_2} + \underbrace{\chi(\alpha-1)k_{ms}^{\alpha-2}\frac{\partial k_{ms}}{\partial \tau}}_{\equiv\tilde{C}}\right)\right). \end{aligned}$$

First, look at

$$\begin{aligned}\frac{1}{\chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1}}\tilde{A}_2 &= \frac{1+g}{\chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1}} - 1 \\ &= \frac{1+g}{\chi\zeta_t\varrho_{t+1}\alpha\frac{1+g}{\chi(1-\tau)(1-\alpha)}} - 1 \\ &= \frac{1-\alpha}{\alpha\zeta_t\varrho_{t+1}} - 1.\end{aligned}$$

Multiplying the above term by β (cf. equation (3.A.46)), taking expectations and subtracting -1 in order to acknowledge the effects of taxation on income (from equation (3.A.48)), term A accordingly writes as

$$A \equiv \frac{\beta(1-\alpha)}{\alpha}E_{t-1}\left[\frac{1}{\zeta_t}\right]E_{t-1}\left[\frac{1}{\varrho_{t+1}}\right] - (1+\beta)$$

Next, look at

$$\frac{1}{\chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1}}\alpha\zeta_t\varrho_{t+1}\tilde{B}_2 = \frac{1}{\chi k_{ms}^{\alpha-1}}\frac{\partial\chi}{\partial\tau}k_{ms}^{\alpha-1} = \frac{\partial\chi}{\partial\tau}\frac{1}{\chi}$$

hence, this gives the percentage change in the saving rate. Multiplying the above by β (cf. equation (3.A.46)) and combining the resulting term with equation (3.A.47), we get

$$B \equiv \frac{\partial\chi}{\partial\tau}\left(\beta\frac{1}{\chi} - \frac{1}{1-\chi}\right).$$

To evaluate this expression observe that

$$\chi = \frac{\alpha\beta\bar{E}}{1+\alpha\beta\bar{E}}$$

and $\bar{E}$ evaluated at $\tau = 0$ gives

$$\bar{E} = \frac{1}{\alpha}.$$

Hence

$$\chi = \frac{\beta}{1+\beta}$$

and

$$1 - \chi = \frac{1}{1+\beta}$$

Therefore:

$$B \equiv \frac{\partial \chi}{\partial \tau}\left(\beta\frac{1}{\chi} - \frac{1}{1-\chi}\right) = \frac{\partial \chi}{\partial \tau}\left(\beta\frac{1+\beta}{\beta} - (1+\beta)\right) = 0.$$

In our model with log utility (where income and substitution effects just cancel) and no second period income (no human capital wealth effect), the beneficial effects of decreasing savings in the first period and the offsetting effects of lower asset income and the second just cancel each other out.

Finally, look at

$$\begin{aligned}\frac{1}{\chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1}}\alpha\zeta_t\varrho_{t+1}\tilde{C} &= \frac{1}{\chi\zeta_t\varrho_{t+1}\alpha k_{ms}^{\alpha-1}}\alpha\zeta_t\varrho_{t+1}\chi(\alpha-1)k_{ms}^{\alpha-2}\frac{\partial k_{ms}}{\partial \tau}\\ &= -(1-\alpha)k_{ms}^{-1}\frac{\partial k_{ms}}{\partial \tau}\\ &= -(1-\alpha)\frac{\partial \ln k_{ms}}{\partial \tau}.\end{aligned}$$

Multiplying the above by β, taking expectations and combining it with equation (3.A.46), all terms incorporating $\frac{\partial \ln k_{ms}}{\partial \tau}$ are given by

$$C \equiv (\alpha(1+\beta) - \beta(1-\alpha))\,\frac{\partial \ln k_{ms}}{\partial \tau} \tag{3.A.49}$$

Turning to $\frac{\partial \ln k_{ms}}{\partial \tau}$ we find that, at $\tau = 0$, we have

$$\begin{aligned}\frac{\partial \ln k_{ms}}{\partial \tau} &= \frac{1}{1-\alpha}\left(\frac{\partial \ln \chi}{\partial \tau} + \frac{\partial \ln(1-\tau)}{\partial \tau}\right)\\ &= \frac{1}{1-\alpha}\left(\frac{1}{\chi}\frac{\partial \chi}{\partial \bar{E}}\frac{\partial \bar{E}}{\partial \tau} - 1\right).\end{aligned} \tag{3.A.50}$$

We have

$$\frac{\partial \chi}{\partial \bar{E}} = \alpha\beta\left(\frac{1}{1+(\alpha\beta\bar{E})^{-1}}\right)^2\left(\frac{1}{\alpha\beta\bar{E}}\right)^2.$$

and, since

$$\chi = \frac{1}{1 + (\alpha\beta\bar{E})^{-1}}$$
$$\Leftrightarrow \qquad \alpha\beta\bar{E} = \frac{\chi}{1-\chi}$$

we get

$$\begin{aligned}
\frac{\partial \chi}{\partial \bar{E}} &= \alpha\beta \left(\frac{1}{1 + \frac{1-\chi}{\chi}} \right)^2 \left(\frac{1-\chi}{\chi} \right)^2 \\
&= \alpha\beta\chi^2 \left(\frac{1-\chi}{\chi} \right)^2 \\
&= \alpha\beta(1-\chi)^2 \\
&= \alpha\beta \frac{1}{(1+\beta)^2}
\end{aligned}$$

Evaluated at $\tau = 0$ we further get

$$\frac{\partial \bar{E}}{\partial \tau} = -\frac{1-\alpha}{\alpha^2} E_t \left[\varrho_{t+1}^{-1} \right]$$

Therefore:

$$\begin{aligned}
\frac{\partial \chi}{\partial \tau} &= \frac{\partial \chi}{\partial \bar{E}} \frac{\partial \bar{E}}{\partial \tau} \\
&= -\alpha\beta \frac{1}{(1+\beta)^2} \frac{1-\alpha}{\alpha^2} E_t \left[\varrho_{t+1}^{-1} \right] \\
&= -\frac{\beta(1-\alpha)}{\alpha(1+\beta)^2} E_t \left[\varrho_{t+1}^{-1} \right] < 0.
\end{aligned}$$

and, consequently,

$$\begin{aligned}
\frac{1}{\chi} \frac{\partial \chi}{\partial \tau} &= -\frac{1+\beta}{\beta} \frac{\beta(1-\alpha)}{\alpha(1+\beta)^2} E_t \left[\varrho_{t+1}^{-1} \right] \\
&= -\frac{1-\alpha}{\alpha(1+\beta)} E_t \left[\varrho_{t+1}^{-1} \right].
\end{aligned}$$

Therefore, equation (3.A.50) rewrites as

$$\begin{aligned}\frac{\partial \ln k_{ms}}{\partial \tau} &= \frac{1}{1-\alpha}\left(\frac{1}{\chi}\frac{\partial \chi}{\partial \bar{E}}\frac{\partial \bar{E}}{\partial \tau} - 1\right)\\ &= \frac{1}{1-\alpha}\left(-\frac{1-\alpha}{\alpha(1+\beta)}E_t\left[\varrho_{t+1}^{-1}\right] - 1\right)\\ &= -\frac{1}{1-\alpha}\left(1+\frac{1-\alpha}{\alpha(1+\beta)}E_t\left[\frac{1}{\varrho_{t+1}}\right]\right) < 0.\end{aligned}$$

Combining the above with equation (3.A.49) gives term C as

$$C \equiv -\frac{\alpha(1+\beta)-\beta(1-\alpha)}{1-\alpha}\left(1+\frac{1-\alpha}{\alpha(1+\beta)}E_t\left[\frac{1}{\varrho_{t+1}}\right]\right). \tag{3.A.51}$$

□

3.B Appendix: Computational Solution

Aggregate Problem

In order to compute the stationary competitive equilibrium of our model, we apply the Krusell and Smith (1997) method. Specifically, we follow Storesletten, Telmer, and Yaron (2007) (STY) and approximate the aggregate law of motion as

$$(k', \mu') = \hat{H}(t; k, \mu, z, z') \tag{3.B.52}$$

where k is the capital stock per efficiency unit and $\mu = \mathbb{E}r' - r^{f'}$ is the equity premium. That is, we approximate the distribution Φ by two "moments" where the equity premium captures information about equity and bond holding moments. Our approach differs from STY in three ways: (i) we plan to explicitly compute transitional dynamics between two stationary competitive equilibria (which fluctuate in two ergodic sets), (ii) we do not use simulation techniques to aggregate on the idiosyncratic states of the distribution and (iii) we compute an approximate equilibrium, referred to as a "mean shock equilibrium", which serves three purposes: first, it enables us to initialize the cross-sectional distribution of agents second, we use it in order to calibrate our model in the initial competitive equilibrium in all periods $t \leq 0$ (for $\tau_t = \tau_0$) and third, it determines the means of the aggregate grids which we employ in the stochastic solution of our model. Computation of the "mean shock equilibrium" is by standard methods to solve OLG models without any aggregate risk. But in contrast to fully deterministic models, the mean shock equilibrium gives rise to an equity premium.

Mean Shock Equilibrium

As an initialization step, we solve for a degenerate path of the economy where the realizations of all aggregate shocks are at their respective means. We accordingly set $z = \bar{z} = \mathbb{E}z$ and $\delta = \bar{\delta} = \mathbb{E}\delta$. We assume that households accurately solve their forecasting problem for each realization of the aggregate state. This means that we approximate the above approximate law of motion as

$$(k', \mu') = \hat{\hat{H}}(k, \mu, \bar{z}, \bar{z}') \tag{3.B.53}$$

Observe that in the two stationary equilibria of our model, we have that fixed point relation

$$(k', \mu') = \hat{\hat{H}}(k, \mu, \bar{z}, \bar{z}') = (k, \mu) \tag{3.B.54}$$

With these assumptions, we can solve the mean shock path by standard Gauss-Seidel iterations as, e.g., described in Auerbach and Kotlikoff (1987). We adopt the modifications described in Ludwig (2007). While the numerical methods are the same as in the solution to a deterministic economy, the actual behavior of households fully takes into account the stochastic nature of the model. This also means that we solve the household problem using recursive methods and store the solutions to the household problem on grids of the idiosyncratic state x. The fixed-point computed in this auxiliary equilibrium gives k^{ms} and μ^{ms} as aggregate moments and cross-sectional distributions of agents as induced by the mean shock path. We denote these distributions by Φ^{ms}.

Grids

To construct the grids for the the aggregate states k and μ, $\mathcal{G}^k$, $\mathcal{G}^\mu$, define scaling factors s^k and s^μ and the number of grid points, n. We set $s^k = 0.8$ $s^\mu = 0.6$, and $n = 7$. Using these factors, we construct symmetric grids around k^{ms}, μ^{ms}.

Stochastic Solution

In order to solve for the stochastic recursive equilibria of our model, we use simulation methods. To this end, we specify the approximate law of motion in (3.B.52) as:

$$\ln(k_{t+1}) = \psi_0^k(z) + \psi_1^k(z)\ln(k_t) + \psi_2^k(z)\ln(\mu_t) \tag{3.B.55a}$$

$$\ln(\mu_{t+1}) = \psi_0^\mu(z) + \psi_1^\mu(z)\ln(k_{t+1}) + \psi_2^\mu(z)\ln(\mu_t) \tag{3.B.55b}$$

Like in Krusell and Smith (1997), the forecast for k_{t+1} is used to forecast μ_{t+1}. Intuitively, k_{t+1} contains a lot of information on the savings choice of the agent and therefore on the returns next period. Note that, in each period, μ_t is an "endogenous state", the realization of which has to be pinned down in that particular period (in contrast to k_t which is given in period t from decisions $t-1$). As in the standard application of the Krusell and Smith (1998) method, the coefficients also depend on the realization of the aggregate state, z.

Stationary Equilibria

Define a number M of stochastic simulations and a number of $s < M$ of simulations to be discarded. We follow GM and set $M = 5500$ and $s = 500$. Also define a tolerance ζ. Further, draw a sequence for z for periods $t = -M, \ldots, 0$ and denote these realizations by $z_{-M}, \ldots, z_0$. Notice that we thereby use the same sequence of aggregate shocks (as given by a random number generator) in each iteration. Collecting coefficients as $\Psi = [\psi_0^k, \psi_1^k, \psi_2^k, \psi_0^\mu, \psi_1^\mu, \psi_2^\mu]'$, the iteration is as follows:

1. Initialization: Guess Ψ.
2. In each iteration i do the following:
   (a) Solution of household problem. We store the solutions of the household problem on the $\mathcal{G}^{hh} = \mathcal{G}^J \times \mathcal{G}^x \times \mathcal{G}^z \times \mathcal{G}^k \times \mathcal{G}^\mu$. This gives us policy functions for all households, e.g., $c(j, x; z, k, \mu)$, $\kappa(j, x; z, k, \mu)$, $a'(j, x; z, k, \mu)$.
   (b) Simulation and aggregation. We simulate the model economy for the M realizations of aggregate shocks, z. To aggregate on the idiosyncratic states, we start

in period $t = -M$ with the initial distribution generated by the mean shock path, Φ^{ms}. We then loop forward using the transition functions Q to update distributions. Notice that, conditional on the realization of z, this aggregation is by standard methods that are used in OLG models with idiosyncratic risk. Simulation and aggregation then gives us M realizations of k_t and μ_t for $t = -M, \dots, 0$. Observe that, in order to compute the realizations for μ_t, we have to solve for the bond market clearing equilibrium in each t. We do so by using a univariate function solver (Brent's method). We are thereby more accurate than Gomes and Michaelides (2008) who simply interpolate on $\mathcal{G}^{\mu}$.

(c) From the stochastic simulations, discard the first s observations and, for the remaining periods $t = s, \dots, 0$ run regressions on:

$$\ln(k_{t+1}) = \tilde{\psi}_0^k(z) + \tilde{\psi}_1^k(z)\ln(k_t) + \tilde{\psi}_2^k(z)\ln(\mu_t) + \vartheta_{t+1}^k \tag{3.B.56a}$$

$$\ln(\mu_{t+1}) = \tilde{\psi}_0^{\mu}(z) + \tilde{\psi}_1^{\mu}(z)\ln(k_t) + \tilde{\psi}_2^{\mu}(z)\ln(\mu_t) + \vartheta_{t+1}^{\mu} \tag{3.B.56b}$$

and collect the resulting coefficient estimates in the vector $\tilde{\Psi}$.

(d) `IF` $\|\Psi_i - \tilde{\Psi}_i\| < \zeta$ then `STOP`, `ELSE` define

$$g(\Psi) = \Psi - \tilde{\Psi}(\Psi) \tag{3.B.57}$$

as the distance function (=root finding problem) and update Ψ_{i+1} as

$$\Psi_{i+1} = \Psi_i - sJ(\Psi)^{-1}g(\Psi) \tag{3.B.58}$$

where $J(\Psi)$ is the Jacobi matrix of the system of equations in (3.B.57) and s is a scaling factor. Continue with step 2a. We solve the root finding problem using Broyden's method, see Ludwig (2007).

Household Problem

We iterate on the Euler equation, using ideas developed in Carroll (2006). As derived in section 3.3.7, the transformed dynamic programming problem of the household reads as

$$
\begin{aligned}
&u(t, j, x, \eta; z, k, \mu) \\
&\qquad = \max_{c,\kappa,a'} \left\{ \left[c^{\frac{1-\theta}{\gamma}} + \tilde{\beta} \left(\mathbb{E} \left[u\left(t+1, j+1, x', \eta'; z', k', \mu'\right)^{1-\theta} \right] \right)^{\frac{1}{\gamma}} \right]^{\frac{\gamma}{1-\theta}} \right\} \\
&\text{s.t.} \qquad x = a' + c,
\end{aligned}
$$

where $x' = a'\tilde{R}' + \mathrm{y}'$, with $\tilde{R}' = \frac{(1+r^{f\prime}+\kappa(r'-r^{f\prime}))}{(1+g)}$, and $\tilde{\beta} = \beta\varsigma_{j+1}\left((1+g)\right)^{\frac{1-\theta}{\gamma}}$. Dropping the time index to simplify notation and using the dynamic budget constraint in the continuation value we get

$$
u(j, \cdot) = \max_{c,\kappa} \left\{ \left[c^{\frac{1-\theta}{\gamma}} + \tilde{\beta} \left(\mathbb{E} \left[u(j+1, (x-c)\tilde{R}' + \mathrm{y}', \cdot)^{1-\theta} \right] \right)^{\frac{1}{\gamma}} \right]^{\frac{\gamma}{1-\theta}} \right\} \tag{3.B.59}
$$

The first-order conditions are given by:

$$
\begin{aligned}
c:\quad & c^{\frac{1-\theta-\gamma}{\gamma}} - \tilde{\beta} \left(\mathbb{E}\left[u(j+1, \cdot)^{1-\theta} \right] \right)^{\frac{1-\gamma}{\gamma}} \\
& \cdot \mathbb{E}\left[u(j+1, \cdot)^{-\theta} u_{x'}(j+1, \cdot) \tilde{R}' \right] = 0 && \text{(3.B.60a)} \\
\kappa:\quad & \mathbb{E}\left[u(j+1, \cdot)^{-\theta} u_{x'}(j+1, \cdot) \left(r' - r^{f\prime} \right) \right] = 0 && \text{(3.B.60b)}
\end{aligned}
$$

and the envelope condition reads as:

$$
\begin{aligned}
u_x = & \left(c^{\frac{1-\theta}{\gamma}} + \tilde{\beta} \left(\mathbb{E}\left[u(j+1, \cdot)^{1-\theta} \right] \right)^{\frac{1}{\gamma}} \right)^{\frac{\gamma-1+\theta}{1-\theta}} \cdot \ldots \\
& \cdot \tilde{\beta} \left(\mathbb{E}\left[u(j+1, \cdot)^{1-\theta} \right] \right)^{\frac{1-\gamma}{\gamma}} \mathbb{E}\left[\tilde{R}' u(j+1, \cdot)^{-\theta} u_{x'}(j+1, \cdot) \right] \\
= & u(j, \cdot)^{\frac{\gamma-1+\theta}{\gamma}} \tilde{\beta} \left(\mathbb{E}\left[u(j+1, \cdot)^{1-\theta} \right] \right)^{\frac{1-\gamma}{\gamma}} \cdot \ldots \\
& \cdot \mathbb{E}\left[\tilde{R}' u(j+1, \cdot)^{-\theta} u_{x'}(j+1, \cdot) \right] \\
= & \left(\frac{c}{u(j, \cdot)} \right)^{\frac{1-\theta-\gamma}{\gamma}}, && \text{(3.B.61)}
\end{aligned}
$$

where the last line follows from equation (3.B.60a) and is exactly the result one would get from direct application of the Benveniste/Scheinkman theorem to recursive preferences, namely $v_x = u_1(c, Ev)$ (see Weil (1989)). Plugging this into the FOCs we get

$$\begin{aligned} c: \quad & c^{\frac{1-\theta-\gamma}{\gamma}} - \tilde{\beta}\left(\mathbb{E}\left[u(j+1,\cdot)^{1-\theta}\right]\right)^{\frac{1-\gamma}{\gamma}} \dots \\ & \cdot \mathbb{E}\left[u(j+1,\cdot)^{\frac{(1-\theta)(\gamma-1)}{\gamma}} (c')^{\frac{1-\theta-\gamma}{\gamma}} \tilde{R}'\right] = 0 \qquad (3.B.62a) \\ \kappa: \quad & \mathbb{E}\left[u(j+1,\cdot)^{\frac{(1-\theta)(\gamma-1)}{\gamma}} (c')^{\frac{1-\theta-\gamma}{\gamma}} \left(r' - r^{f'}\right)\right] = 0 \qquad (3.B.62b) \end{aligned}$$

With respect to our numerical solution, we will interpolate the functions $u(j,\cdot)$ and $c(j,\cdot)$. Note that we can expect $u(j,\cdot)$ to be approximately linear, since in period J it is simply given by $u(J) = c_J = x_J$.

Next, notice that $u(j+1,\cdot)$ and c' are functions of $(x-c)$ so that c shows up on both sides of the equation in (3.B.62a). This would require calling a non-linear solver whenever we solve optimal consumption and portfolio shares. To alleviate this computational burden we employ the endogenous grid method of Carroll (2006). Accordingly, instead of working with an *exogenous* grid for x (and thereby an *endogenous* grid for savings, $s = x - c$) we revert the order and work with an *exogenous* grid for $s = x - c$ and an *endogenous* grid for x.

So, roughly speaking, for each age j and each grid point in the savings grid $\mathcal{G}^s$, our procedure is the following:

1. Solve equation (3.B.62b) for κ using a univariate equation solver (Brent's method).

2. Given the solution to (3.B.62b) invert (3.B.62a) to compute

$$c = \left(\tilde{\beta}\left(\mathbb{E}\left[u(j+1,\cdot)^{1-\theta}\right]\right)^{\frac{1-\gamma}{\gamma}} \mathbb{E}\left[v(j+1,\cdot)^{-\theta}\tilde{R}'\right]\right)^{\frac{\gamma}{1-\theta-\gamma}}. \qquad (3.B.63)$$

3. Update u, u_x and v.

More precisely, our procedure is as follows:

1. Loop on the grids of the aggregate states, $\mathcal{G}^z$, $\mathcal{G}^k$, $\mathcal{G}^\mu$.

2. For each (z, k, μ) use (3.B.55) to compute the associated k', μ'.

3. Initialize the loop on age for $j = J$ by setting $c_J = x_J$ and compute $u(x_J) = c_J$, $u_x(x_J) = 1$ and $v(x_J) = u(x_J) = c_J$.

4. Loop backwards in age from $j = J-1, \dots, 0$ as follows:

   (a) As $k' \notin \mathcal{G}^k$, $\mu' \notin \mathcal{G}^\mu$, interpolate on the aggregate states and store the interpolated objects $u(k', \mu', x', z', j+1)$, $v(k', \mu', x', z', j+1)$ as a projection on $\mathcal{G}^x$. Do so for each $z' \in \mathcal{G}^z$. Denote the interpolated objects as $\bar{u}(x', z', j+1)$, $\bar{v}(x', z', j+1)$.

   (b) For each s in $\mathcal{G}^s$ first solve (3.B.62b) for κ. To so, we have to loop on $z' \in \mathcal{G}^z$ (as well as the idiosyncratic shock) to evaluate the expectation taking into account the Markov transition matrix $\pi(z'|z)$. In this step, we also use the law of motion of the idiosyncratic state x:

   $$x' = s\tilde{R}' + \mathrm{y}' \tag{3.B.64}$$

   As, generally, $x' \notin \mathcal{G}^x$ we have to interpolate on $\bar{u}(x', z', j+1)$ and $\bar{v}(x', z', j+1)$ before evaluating the expectation.

   (c) Taking the optimal κ as given, next compute c from (3.B.63). Again we interpolate on $\bar{u}(x', z', j+1)$ and $\bar{v}(x', z', j+1)$ before evaluating the expectation.

Calibration of Income Process

We determine $\eta\mp(z)$ and $\bar{\pi}^\eta$ such that we match the unconditional variance of the STY estimates, i.e.,

$$E[(\ln \eta')^2 \mid z'] = \sigma(z')^2_{\ln \eta} \tag{3.B.65}$$

and the unconditional autocorrelation, i.e.,

$$\frac{E[\ln \eta' \ln \eta]}{E[(\ln \eta')^2]} = \rho. \tag{3.B.66}$$

To match the variance we specify the states of the Markov process as

$$\eta_\mp(z) = \frac{2\exp(1 \mp \tilde{\sigma}(z))}{\exp(1-\tilde{\sigma}(z)) + \exp(1+\tilde{\sigma}(z))}$$

so that the unconditional mean equals one.

We pick $\tilde{\sigma}(z)$ such that the unconditional variance—which of course preserves its contingency on z—satisfies (3.B.65). To achieve this, observe that

$$\ln \eta_{\mp} = \ln\left(\frac{2}{\exp(1-\tilde{\sigma}(z)) + \exp(1+\tilde{\sigma}(z))}\right) + 1 \mp \tilde{\sigma}(z) \equiv \phi(\tilde{\sigma}(z)) \mp \tilde{\sigma}(z).$$

Hence

$$\begin{aligned} E[(\ln \eta')^2 \mid \eta = \eta_-, z'] &= \bar{\pi}^\eta(\phi(\tilde{\sigma}(z')) - \tilde{\sigma}(z'))^2 + \\ &\qquad (1-\bar{\pi}^\eta)(\phi(\tilde{\sigma}(z')) + \tilde{\sigma}(z'))^2 \\ E[(\ln \eta')^2 \mid \eta = \eta_+, z'] &= \bar{\pi}^\eta(\phi(\tilde{\sigma}(z')) + \tilde{\sigma}(z'))^2 + \\ &\qquad (1-\bar{\pi}^\eta)(\phi(\tilde{\sigma}(z')) - \tilde{\sigma}(z'))^2. \end{aligned}$$

The unconditional mean of the above—conditional on z'—is

$$E[(\ln \eta')^2 \mid z'] = \phi(\tilde{\sigma}(z'))^2 + \tilde{\sigma}(z')^2.$$

To determine $\tilde{\sigma}(z)$, we then numerically solve the distance function

$$f(\tilde{\sigma}(z)) = \phi(\tilde{\sigma}(z))^2 + \tilde{\sigma}(z)^2 - \sigma(z)^2_{\ln \eta} = 0$$

for all z. Standard procedures ignore the bias term $\phi(\tilde{\sigma}(z))^2$.

To determine $\bar{\pi}^\eta$ observe that, in the stationary invariant distribution, we have

$$\begin{aligned} E[\ln \eta' \ln \eta \mid \eta = \eta_-] &= \sum_z \Pi^z(z) \left\{\bar{\pi}^\eta(\phi(\tilde{\sigma}(z)) - \tilde{\sigma}(z))^2 + \right. \\ &\qquad \left. (1-\bar{\pi}^\eta)(\phi(\tilde{\sigma}(z)) - \tilde{\sigma}(z))(\phi(\tilde{\sigma}(z)) + \tilde{\sigma}(z))\right\} \\ E[\ln \eta' \ln \eta \mid \eta = \eta_+] &= \sum_z \Pi^z(z) \left\{\bar{\pi}^\eta(\phi(\tilde{\sigma}(z)) + \tilde{\sigma}(z))^2 + \right. \\ &\qquad \left. (1-\bar{\pi}^\eta)(\phi(\tilde{\sigma}(z)) + \tilde{\sigma}(z))(\phi(\tilde{\sigma}(z)) - \tilde{\sigma}(z))\right\} \end{aligned}$$

and

$$E[\ln \eta' \ln \eta] = \sum_\eta \Pi^\eta(\eta) E[\ln \eta' \ln \eta \mid \eta]$$

as well as

$$E[(\ln \eta')^2] = \sum_{z'} \Pi^{z'} E[(\ln \eta')^2 \mid z'].$$

Noticing that

$$\frac{E[\ln \eta' \ln \eta]}{E[(\ln \eta')^2]} = \rho$$

we use the above relationships to determine $\bar{\pi}^\eta$.

3.C Appendix: Robustness

The tables shown below are all discussed in section 3.5.3 of the main text.

Robustness of the Thought Experiment

Here we report the results of the experiment $\tau = 0.12 \to \tau = 0.14$ ('HL experiment'). The value $\tau = 0.12$ is the average contribution rate in the U.S. today. As in the main text, we first report the NC calibration (negative correlation between TFP and returns), then the PC calibration (positive correlation between TFP and returns).

Calibration with Negative Correlation between TFP and Returns (NC)

Parameters					Moments	
θ	φ	β	$\bar{\delta}$	π^δ	$cor(r,w)$	$std(\Delta C/C)$
17	1.5	0.975	0.1	0.435	-8.27 E-2	6.84 E-2

	GE	PE
ΔWelf/Welf	+0.17%	+2.62%
$\Delta K/K$	-2.88%	-21.27%
$\Delta E(r)$	+0.16%	*0.00%*
Δr_f	+0.28%	*0.00%*
$\Delta w/w$	-3.18%	*0.00%*

	aggr. risk	+ wage risk	+ CCV	+ surv. risk
ΔWelf/Welf	-0.31%	+1.01%	+ 1.26%	+2.62%

Welfare gains					
	aggr. risk	+ wage risk	Δ	IAR	$\widehat{IAR}$
$orig$	-0.31%	+1.01%	+1.32	-	-
-10%	-0.39%	+0.90%	+1.28	-0.028	-0.031

Calibration with Positive Correlation between TFP and Returns (PC)

Parameters					Moments	
θ	φ	β	$\bar{\delta}$	π^{δ}	$cor(r, w)$	$std(\Delta C/C)$
13	1.5	0.99	0.11	0.86	25.1 E-2	7.40 E-2

	GE	PE
ΔWelf/Welf	+1.94%	+4.18%
$\Delta K/K$	-2.21%	-17.41%
$\Delta E(r)$	+0.12%	*0.00%*
Δr_f	+0.26%	*0.00%*
$\Delta w/w$	-2.98%	*0.00%*

	aggr. risk	+ wage risk	+ CCV	+ surv. risk
ΔWelf/Welf	+0.87%	+2.65%	+3.33%	+4.18%

Welfare gains					
	aggr. risk	+ wage risk	Δ	IAR	$\widehat{IAR}$
$orig$	+0.87%	+2.65%	+1.78	-	-
-10%	+0.35%	+1.98%	+1.63	-0.146	-0.207

Sensitivity to the Elasticity of Intertemporal Substitution

We now set the value of the elasticity of intertemporal substitution to $\varphi = 0.5$, the value used by Krueger and Kubler (2006). Of course we have to recalibrate the model, but the changes are restricted to a few parameters, which we document. First we report the results for the NC calibration, then for the PC calibration.

Calibration with Negative Correlation between TFP and Returns (NC)

Parameters					Moments	
θ	φ	β	$\bar{\delta}$	π^{δ}	$cor(r,w)$	$std(\Delta C/C)$
15	0.5	0.975	0.1	0.44	-10.27 E-2	4.32 E-2

	GE	PE
ΔWelf/Welf	-3.10%	–
$\Delta K/K$	-5.91%	–
$\Delta E(r)$	+0.44%	*0.00%*
Δr_f	+0.47%	*0.00%*
$\Delta w/w$	-4.14%	*0.00%*

	aggr. risk	+ wage risk	+ CCV	+ surv. risk
ΔWelf/Welf	-0.89%	+0.04%	–	–

Welfare gains					
	aggr. risk	+ wage risk	Δ	IAR	$\widehat{IAR}$
orig	-0.89%	+0.04%	+0.92	-	-
+10%	-0.80%	+0.15%	+0.96	+0.032	+0.034

Calibration with Positive Correlation between TFP and Returns (PC)

Parameters					Moments	
θ	φ	β	$\bar{\delta}$	π^{δ}	$cor(r,w)$	$std(\Delta C/C)$
10	0.5	0.98	0.11	0.86	19.1 E-2	4.85 E-2

	GE	PE
ΔWelf/Welf	-2.51%	+1.74%
$\Delta K/K$	-5.87%	-20.55%
$\Delta E(r)$	+0.47%	*0.00%*
Δr_f	+0.48%	*0.00%*
$\Delta w/w$	-4.14%	*0.00%*

	aggr. risk	+ wage risk	+ CCV	+ surv. risk
ΔWelf/Welf	-0.41%	+0.60%	+ 1.07%	+1.74%

Welfare gains					
	aggr. risk	+ wage risk	Δ	IAR	$\widehat{IAR}$
orig	-0.41%	+0.60%	+1.01	-	-
-10%	-0.35%	+1.98%	+0.94	-0.075	-0.142

Nonlinear Trend Calibration

Parameters					Moments	
θ	φ	β	$\bar{\delta}$	π^{δ}	$cor(r, w)$	$std(\Delta C/C)$
14	1.5	0.98	0.1	0.7	6.45 E-2	6.53 E-2

	GE	PE
ΔWelf/Welf	+0.8%	+3.0%
$\Delta K/K$	-2.7%	-22.1%
$\Delta E(r)$	+0.2%	*0.0%*
Δr_f	+0.3%	*0.0%*
$\Delta w/w$	-3.2%	*0.0%*

	aggr. risk	+ wage risk	+ CCV	+ surv. risk
ΔWelf/Welf	-0.2%	+1.4%	+ 1.7%	+3.0%

Welfare gains					
	aggr. risk	+ wage risk	Δ	IAR	$\widehat{IAR}$
orig	-0.21%	+1.35%	+1.56	-	-
-10%	-0.33%	+1.18%	+1.51	-0.054	-0.045

Chapter 4

Saving Policies When Consumers Can Default on Unsecured Loans

4.1 Introduction

The rise in debt and default over the last two decades and the peak in the recent crisis have sparked a renewed research activity in various economic areas. One important distinction in the different strands of literature is the kind of debt and the default rules that are analyzed. The present paper focuses on unsecured consumer loans that will be fully discharged when the agent decides to default. Thus, the bankruptcy option is close in spirit to Chapter 7 of the US bankruptcy code.

The present paper follows the recent work in the dynamic macroeconomics literature on unsecured consumer bankruptcy, which features equilibrium default. In that literature, neither the applied nor the theoretical work makes use of the agent's first-order conditions to solve the model or derive results. This is of course due to the obvious difficulties: the first-order conditions might not exist because the discrete choice introduces non-differentiabilities, or, when they exist, they might not be sufficient or even necessary due to the non-convexity of the choice set and the non-concavity of the objective function. However, as I will show, the specific form of the repayment scheme under a Chapter 7-style bankruptcy does impose enough structure so that we can make use of first-order conditions. For this, we need to split up the agent's problem into differentiable parts. What we get in return is a precise characterization of the agent's default and savings policies, which might be of interest for both theoretical and quantitative work.

Specifically, I look at a simple 3-period case with a quadratic utility function (although all result obtain for log-utility with $\beta = 1$ and CARA), where the agent can default only in the second period. As in most of the mentioned quantitative literature, filing for bankruptcy entails a discharge of all debt, but also exclusion from asset markets in the current period. This is usually not an overly strong assumption because consumers are not allowed to accumulate assets while declaring bankruptcy, and they typically also find it harder to take up more loans in the following periods. In the present three-period context this punishment might seem strict because it implies that the agent lives in autarky for the rest of his life. I show that there is a policy that we can find using the first-order conditions with an additional condition. Furthermore, this policy typically has a unique discontinuity despite the fact that both the asset choice and the income shock are continuous. This jump does not occur at the point where for the first time the probability of default is greater zero; instead it occurs at a point where the agent cannot be 'too poor to default' anymore - something that will become clear in the respective section.

In order to introduce the main ideas and to show how the shape of the policy depends on the punishment regime, I start with a two-period model where default only entails a pecuniary cost instead of exclusion. This might be more similar to a Chapter 13 filing, where some of the agent's income can be garnished, or to the situation in Germany prior to 1999, where the consumer remained liable for the rest of his life. By comparing the two regimes it will become clear why and when the discontinuity will occur.

Most of the directly related literature is quantitative in nature and tries to assess the welfare effects of the option to default in different economic environments, like the recent change in the US law (Bankruptcy Abuse Prevention and Consumer Protection Act of 2005), which is discussed for example in Chatterjee, Corbae, Nakajima, and Ríos-Rull (2007), Athreya (2002), or Li and Sarte (2006). Athreya (2008) and Sánchez (2007) analyze the consequences of different information structures, while Livshits, MacGee, and Tertilt (2007) show that it matters how persistent we model the income process. Much less has been done on the theory side due to the fact that the non-convexity introduced by a discrete choice creates various difficulties. In the theoretical part of the paper, Chatterjee, Corbae, Nakajima, and Ríos-Rull (2007) prove existence of the household solution and existence of a steady state, whereas Chatterjee, Corbae, and Ríos-Rull (2008) is a purely theoretical

analysis of signaling and endogenous punishment in a three-period economy.
This differs from the default option typically found in general equilibrium settings with incomplete markets, as e. g. in Zame (1993) or Dubey, Geanakoplos, and Shubik (2005), where default can be in any percentage of the promise of delivery. In the models by Zhang (1997), Alvarez and Jermann (2000), or Ábrahám and Cárceles-Poveda (2010), among others, agents face a participation constraint so that, in contrast to the present paper, there will be no default in equilibrium.
The next section describes the asset markets and the financial intermediaries. Section three presents the model and the subsections discuss the results for the different punishment regimes. In section four I conclude.

4.2 Credit Markets

There is a single, risk-free asset in the economy. Agents can borrow and save by selling or buying one-period discount bonds in this asset. Denote by $a_{t+1} \in \mathcal{A} \subset \mathbf{R}$ the amount the agent wants to borrow ($a_{t+1} < 0$) or save ($a_{t+1} \geq 0$). In contrast to most of the quantitative macro literature, we assume that $\mathcal{A}$ is compact and convex. There is no obvious reason why one should limit the set of loans to be discrete, as for example Chatterjee, Corbae, Nakajima, and Ríos-Rull (2007).[1] If $a_{t+1} < 0$, then the agent can decide in the following period to default on his debt, i.e. to not pay back the principal.
There are many risk-neutral financial intermediaries (banks) each offering all possible a_{t+1}, and entry and exit into this sector are completely free. As a consequence, banks charge a price $q(a_{t+1})$ which takes into account the probability default $P(d|a_{t+1})$ in such a way that in expectation, they make zero profits on each loan. They can refinance at the exogenous interest rate $1 + r$, so that the price scheme has to satisfy

$$q(a_{t+1}) = \begin{cases} \frac{1}{1+r} & \text{if } a_{t+1} \geq \tilde{a}_{t+1} \\ \frac{1-P(d|a_{t+1})}{1+r} & \text{else} \end{cases} \tag{4.2.1}$$

where $\tilde{a}_t \leq 0$ denotes the lowest asset position at which the agent will never default.

[1] There could be the issue that otherwise the market for a specific loan would become thin, since there are infinitely many loan markets in the continuous case. In that case, one cannot invoke a law of large numbers to ensure zero profits. However, it should be sufficient to have zero expected profits.

In principle, banks could condition on more than just the loan size. However, the current asset position a_t does not provide any additional information on the probability of defaulting tomorrow, because we are looking at one-period bonds. If, say, $q(a_{t+1})a_{t+1} < a_t < 0$, then the agent is effectively rolling over his debt and increasing his consumption only by $a_t - q(a_{t+1})a_{t+1}$. Another source of information on future default would be the current income state, in particular because the stochastic income realization in $t+1$ is the driver for the default decision in $t+1$. In the simple model described next, income is constant in all periods but one, so that no information can be retrieved. For more general setups, having an i.i.d. income process would again yield the same situation that conditioning on current income doesn't yield anything. But one could easily allow for a Markovian income process, and let banks condition on it, without qualitatively changing any of the results below. However, it is crucial that banks cannot write contracts contingent on the realization of tomorrow's income state, because then there would not be any default in equilibrium. Thus I assume that insurance markets for idiosyncratic risk are closed, so markets are incomplete.

4.3 A Model with Analytic Solutions

In this section, we will analyze a model that is simple enough to allow for analytic solutions, and that nonetheless displays all the characteristics that are at the center of this paper. I will first describe the details of the model without specifying the punishment for default. In the two subsections that follow, we will look at two different, admittedly very special forms of punishment, and see how they influence the shape of the savings policy.

The strongest assumption that we make is that utility is of a very simple quadratic form:

$$u(c) = -(c-\gamma)^2 \ , \quad 0 < c < \gamma. \tag{4.3.2}$$

The reason for choosing a quadratic utility is just that no other common functional form yields an analytic solution in the present setup. One disadvantage is that we need to heed the additional restrictions guaranteeing that consumption will be positive and in the increasing part of the function.

The agent is born into the economy at $t = 1$ with assets (a bequest) $a_1 \in \mathcal{A}_1 \subset \mathbf{R}$, which can be positive or negative. He then decides how much to borrow or save $a_2 \in \mathcal{A}_2 \subset \mathbf{R}$ and how much to consume. Both $\mathcal{A}_1$ and $\mathcal{A}_2$ are assumed to be convex and compact. As mentioned in the last section, most of the literature (e.g. Chatterjee, Corbae, Nakajima, and Ríos-Rull (2007)) limits the choice set to be discrete. At the start of $t = 2$, he receives a stochastic income y_2, which, for the present section, we assume to be distributed uniformly with $\mathcal{U}(\underline{y}_2, \bar{y}_2)$ and $\underline{y}_2 > 0$. A continuous income distribution is important to introduce smoothness into value and policy functions, and it also avoids the multiplicity of equilibria that can arise with discrete income shocks. Given this realization and the assets he brought forward from the first period, the agent decides whether he wants to repay his debt or default. In the case of default, the agent's debt is discharged (similar to Chapter 7 of the US bankruptcy code), but he faces a punishment. Denote by $V^{nd}(a_2, y_2)$ the continuation value of not defaulting and by $V^d(y_2)$ the continuation value of default. The latter is independent of a_2 because debt is discharged. Then we can write the agent's maximization problem as

$$\max_{a_2} -(a_1 - a_2 q(a_2) - \gamma)^2 + \beta E\left[\max\left\{V^{nd}(a_2, y_2), V^d(y_2)\right\}\right], \tag{4.3.3}$$

where the expectation is taken over the income shock y_2. In the following two subsections, we will look at different punishment rules, which of course entail different $V^{nd}(a_2, y_2)$ and $V^d(y_2)$. Of particular interest will be the cut-off income levels $\tilde{y}_2$ for which the agent is indifferent between defaulting and not defaulting:

$$V^{nd}(a_2, \tilde{y}_2) = V^d(\tilde{y}_2). \tag{4.3.4}$$

4.3.1 Linear Monetary Punishment

We start with a simple punishment rule, because it allows to derive analytic results that transfer to the more complicated cases. If the agent defaults, he has to pay a constant fraction $\tau \in (0, 1)$ of his income, plus a fixed cost $\kappa \in [0, \bar{y}_2)$. After his default decision, he consumes all he has and dies. Thus we have

$$V^{nd}(a_2, y_2) = -(a_2 + y_2 - \gamma)^2 \tag{4.3.5}$$

$$V^d(y_2) = -((1-\tau)y_2 - \kappa - \gamma)^2 \tag{4.3.6}$$

Economically, one could see this as an approximation to a bankruptcy filing under Chapter 13, where some of the income will be garnished to repay the loans. Even more closely, the situation in Germany before 1999 was such that the agent had to repay the debt for the rest of his life. Alternatively, one could argue that it is a simple way of subsuming the utility cost of stigma, or worsened credit conditions in the future, and the pecuniary costs of filing for bankruptcy. The advantages of this punishment are of technical nature, and consist of the combination of three aspects: first, it makes the default choice contingent on the income realization, which is non-contractable due to the assumption of incomplete markets. If, for example, we had only a fixed pecuniary cost of default (shoeleather cost), i.e. $\tau = 0$ and $\kappa > 0$, then the agent would default for any $a_2 < \kappa$, independent of y_2. This would result in an equilibrium without default, as banks would charge a price of $q(a_2) = 0$ for such loans. Second, it affects directly the budget constraint, which is what a bankruptcy filing under Chapter 7 does, and in this regard is similar to exclusion from asset (or credit) markets. This is an important element which is not present if one models only a utility shock. And third, it allows us to limit the lifetime of the agent to $T = 2$, which is not the case for many other punishments (like credit market exclusion, discussed in the next subsection). This doesn't seem so important at first glance, but it turns out that for longer time-horizons, the model in its present form doesn't have a closed-form solution anymore, as discussed in section 4.3.3.[2]

Now that we have completed the description of the agent's problem, we can solve backwards. First, let me mention that the requirement $0 < c < \gamma$ now translates into

$$\frac{\kappa}{1-\tau} < \underline{y}_2 < \bar{y}_2 < \gamma \tag{4.3.7}$$

$$-\underline{y}_2 < a_2 < \gamma - \bar{y}_2. \tag{4.3.8}$$

They ensure that consumption remains positive and in the increasing part of utility in the default and the no-default case. The upper is a requirement on the primitives of the economy, and is not very strict. Indeed, setting $\kappa = 0$ doesn't change any of the results qualitatively. The lower defines the bounds on the maximization problem, as we will see below. Its left-hand part $-\underline{y}_2 < a_2$ is just the natural debt limit.

[2]If one gives up the additivity of assets and stochastic income in the bugdet constraint or works with stigma as a penalty, the model might have closed-form solutions.

In $t = 2$, the agent has to decide whether to default or not. Given his income realization y_2 and $a_2 < \tilde{a}_2$, he will default iff

$$V^d(y_2) < V^{nd}(a_2, y_2) \;\Leftrightarrow\; -((1-\tau)y_2 - \kappa - \gamma)^2 < -(a_2 + y_2 - \gamma)^2 \tag{4.3.9}$$

$$\Leftrightarrow y_2 < \frac{-a_2}{\tau} - \frac{\kappa}{\tau} \;\equiv\; \tilde{y}_2(a_2) \tag{4.3.10}$$

The cut-off income level $\tilde{y}_2(a_2)$ will play a crucial role in this and the following section. It directly gives us the probability of default,

$$P(d|a_2) = F\left(\tilde{y}_2(a_2)\right) \tag{4.3.11}$$

where $F(\cdot)$ denotes the cdf of y_2. Also, the agent will never default on a_2 iff

$$a_2 \geq -\tau \underline{y}_2 - \kappa \;\equiv\; \tilde{a}_2 \tag{4.3.12}$$

That means that there are debt levels $a_2 < 0$ for which the agent will always prefer to pay back, even if $\kappa = 0$, because $\underline{y}_2 > 0$ is implied by condition (4.3.7). Similarly, the agent will always default on a_2 iff

$$a_2 \leq -\tau \overline{y}_2 - \kappa. \tag{4.3.13}$$

Putting this together, it is instructive to plot $\tilde{y}_2(a_2)$ as shown in figure 4.1. Note that it is linear, strictly bounded away from the origin, and defined on an open interval. Also, we can rewrite the pricing function (4.2.1) as

$$q(a_2) = \begin{cases} \frac{1}{1+r} & \text{if } a_2 \geq -\tau \underline{y}_2 - \kappa \\ \frac{1-F(\tilde{y}_2(a_2))}{1+r} & \text{if } -\tau \overline{y}_2 - \kappa < a_2 < -\tau \underline{y}_2 - \kappa \\ 0 & \text{if } a_2 \leq -\tau \overline{y}_2 - \kappa \end{cases} \tag{4.3.14}$$

Obviously, the agent would never choose $a_2 \leq -\tau \overline{y}_2 - \kappa$, because he gets effectively nothing and has to repay something. In order to find the optimal a_2, we will distinguish the two cases 'no default' ($a_2 \geq -\tau \underline{y}_2 - \kappa$) and 'some default' ($-\tau \overline{y}_2 - \kappa < a_2 < -\tau \underline{y}_2 - \kappa$) and look at the maximization problems separately.

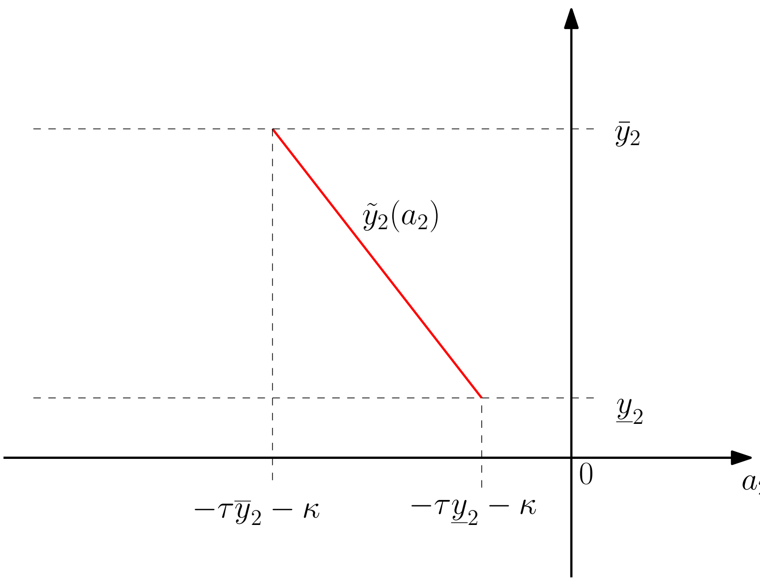

Figure 4.1: The cut-off income level $\tilde{y}_2(a_2)$

When the agent will never default, the problem is standard.

$$\max_{\gamma-\bar{y}_2 \geq a_2 \geq -\tau\underline{y}_2-\kappa} -\left(a_1 - \frac{1}{1+r}a_2 - \gamma\right)^2 + \beta E\left[-(a_2 + y_2 - \gamma)^2\right] \tag{4.3.15}$$

The lower bound we have just discussed (see eq. 4.3.12), the upper bound arises from the quadratic utility, see eq. 4.3.8. An interior solution has to satisfy the following FOC, which we write down because we will need it later:

$$\left(a_1 - \frac{1}{1+r}a_2 - \gamma\right)\frac{1}{1+r} = \beta(a_2 - \gamma) + \beta\frac{\bar{y}_2 + \underline{y}_2}{2} \tag{4.3.16}$$

Since everything is concave, the FOC is also sufficient for an interior solution, which is then linear and given by:

$$a_2^{ND}(a_1) = \frac{a_1}{\frac{1}{1+r} + (1+r)\beta} + \frac{((1+r)\beta - 1)\gamma - (1+r)\beta E y_2}{\frac{1}{1+r} + (1+r)\beta} \tag{4.3.17}$$

and the full policy is

$$a_1^{*,ND}(a_1) = \begin{cases} -\tau \underline{y}_2 - \kappa & \text{for } a_2^{ND}(a_1) < -\tau \underline{y}_2 - \kappa \\ a_2^{ND}(a_1) & \text{for } \gamma - \bar{y}_2 \geq a_2^{ND}(a_1) \geq -\tau \underline{y}_2 - \kappa \\ \gamma - \bar{y}_2 & \text{for } a_2^{ND}(a_1) > \gamma - \bar{y}_2 \end{cases} \tag{4.3.18}$$

Now let's turn to the case with some default, that is, now the agent chooses asset levels on which he will default in some states tomorrow. As discussed, the price $q(a_2)$ is now adjusted by the default premium, and we will always have $\tilde{y}_2(a_2) \in (\underline{y}_2, \bar{y}_2)$ due to the bounds of the problem. The maximization problem can thus be written as

$$\begin{aligned} &\max -(a_1 - q(a_2)a_2 - \gamma)^2 + \beta E\left[\max\left[u(a_2 + y_2), u((1-\tau)y_2 - \kappa)\right]\right] && (4.3.19) \\ &\quad \text{s. t.} - \tau \overline{y}_2 - \kappa < a_2 < -\tau \underline{y}_2 - \kappa \\ &\quad \Leftrightarrow \\ &\max -(a_1 - q(a_2)a_2 - \gamma)^2 + \\ &\quad \beta \int_{\tilde{y}_2(a_2)}^{\overline{y}_2} -(a_2 + y_2 - \gamma)^2 f(y_2) dy_2 + \beta \int_{\underline{y}_2}^{\tilde{y}_2(a_2)} -((1-\tau)y_2 - \gamma)^2 f(y_2) dy_2 && (4.3.20) \\ &\quad \text{s. t.} - \tau \overline{y}_2 - \kappa < a_2 < -\tau \underline{y}_2 - \kappa \end{aligned}$$

Transforming a problem with two maximizations into one with a cut-off rule is not new, but it is still worth remembering that the original maximization involves a discrete choice, which doesn't directly appear in eq. (4.3.20). On the flipside, problem (4.3.20) is difficult because the choice a_2 enters both the bounds of the integral and the pricing function $q(a_2)$. As a consequence, it is not clear that the objective function is concave in the choice. Indeed, at least the expectations part is not, it is convex.

Let's define *effective debt* as $g(a_2) = a_2 q(a_2)$. This object will play a central role in many of the findings below. Marginal effective debt, which will enter the foc, is given by

$$\begin{aligned}\frac{\partial g(a_2)}{\partial a_2} &= q(a_2) + \frac{\partial q(a_2)}{\partial a_2} a_2 \\ &= \left(\frac{1}{1+r} \left(1 - \frac{-\frac{a_2}{\tau} - \frac{\kappa}{\tau} - \underline{y}_2}{\bar{y}_2 - \underline{y}_2} \right) \right) + \frac{a_2}{(\bar{y}_2 - \underline{y}_2)(1+r)\tau} \\ &= \frac{\tau \bar{y}_2 + 2a_2 + \kappa}{\tau(\bar{y}_2 - \underline{y}_2)(1+r)}\end{aligned} \tag{4.3.21}$$

It is easily shown that $g(a_2)$ is strictly concave on the interval defined by the bounds of the current maximization problem, and that it has a unique minimum at

$$a_2^{*min} = -\frac{1}{2}\kappa - \frac{\tau}{2}\bar{y}_2 \,. \tag{4.3.22}$$

This endogenous value is greater than the lower bound of the maximization problem $-\tau\bar{y}_2 - \kappa$. Nobody will want to borrow below a_2^{*min} because it becomes prohibitively expensive in the sense that there is another debt level which dominates it, i. e. another a_2 for which the agent gets same effective debt, but pays back less. Thus only debt levels $a_2 > a_2^{*min}$ can be a solution, we will call this the non-dominated region. In that region, $g(a)$ is strictly increasing, i.e. $\frac{\partial g(a_2)}{\partial a_2} > 0$.
Equipped with this, we can now turn to the FOC, which is given by

$$\begin{aligned}(a_1 - q(a_2)a_2 - \gamma)\left[q(a_2) + a_2 \frac{\partial q(a_2)}{\partial a_2} \right] &= \\ \beta(a_2 - \gamma)(1 - F(\tilde{y}_2(a_2))) + \beta \frac{\bar{y}_2^2 - \tilde{y}_2^2}{2(\bar{y}_2 - \underline{y}_2)}\end{aligned} \tag{4.3.23}$$

On the one hand, this is simpler than one might have expected from inspection of the maximization problem (4.3.20). The reasons are that the value of defaulting is independent of a_2, which is why the second integral drops out, and that the changes in the integral bounds exactly offset each other. The detailed derivation can be found in appendix 4.A. On the other hand, this FOC is still significantly more complicated than in the standard quadratic utility case. In fact, it is a cubic function instead of a linear one as is typical for quadratic utility. This happens because here, effective debt is a quadratic and is multiplied with its own derivative. Nonetheless, we get the following result.

Proposition 4.1. *Assume that* $\tau > \frac{\kappa-1}{2}$*. Then, in the non-dominated region* $(a_2 > a_2^{*min})$*, there is a unique solution to the FOC given in eq. (4.3.23).*

Proof. See appendix 4.A. □

This result means the FOC exists and together with the requirement $a_2 > a_2^{*min}$ is necessary, which is not immediate since we have a discrete choice in the original model. The condition $\tau > \frac{\kappa-1}{2}$ doesn't seem very strict, as it is always fulfilled for $\kappa < 1$. To show that this solution is the unique policy function that solves the maximization problem (4.3.19), we would need to prove sufficiency, so as to rule out minima. Appendix 4.A discusses the difficulties of proving this.

Denote the solution to problem (4.3.19) by $a_1^D(a_1)$. It is possible to calculate a closed-form solution, but it is surprisingly convoluted - written down it down takes more than four pages of space. I find that worth mentioning because it indicates how quickly the complexity of an analytic solution in this model grows, and it helps to understand why extensions or generalizations will usually not have a closed form solution, as discussed in section 4.3.3. From the above proof we also get the following two corollaries.

Corollary 4.1. *If* $a_1^D(a_1)$ *exists, then it is nonlinear and monotonically increasing.*

Corollary 4.2. *The endogenous lower bound* a_2^{*min} *is greater than the lower bound in the maximization problem and is never binding.*

Now that we have found and characterized the solutions to the two separate maximization problems, we want to combine them to get the policy for the original, unconstrained maximization problem (4.3.3), which we denote $a_2^*(a_1)$. For this, we need to know what happens as we approach the common bound $\tilde{a}_2 = -\tau \underline{y}_2 - \kappa$ from the right and from the left.

Lemma 4.1. *There exists a nonempty interval* $[\tilde{a}_1^1\ ,\ \tilde{a}_1^2]$*, where neither of the two FOCs has a solution. In that reqion the lower bound of the no-default maximization problem (4.3.15) is binding.*

Proof. Recall that the common bound of the two problems is given by $\tilde{a}_2 = -\tau \underline{y}_2 - \kappa$. Let $\tilde{a}_1^2$ be defined as the a_1 that solves the no-default foc (4.3.16), given that the optimal choice is $\tilde{a}_2$:

$$\left(\tilde{a}_1^2 - \frac{1}{1+r}\tilde{a}_2 - \gamma\right)\frac{1}{1+r} = \beta(\tilde{a}_2 - \gamma) + \beta\frac{\bar{y}_2 + \underline{y}_2}{2}$$

Similarly, $\tilde{a}_1^1$ is defined as the a_1 that solves the default foc (4.3.23), given that the optimal choice is $\tilde{a}_2$. However, since the interval for which (4.3.23) is valid does not include its bounds, we need to take the limit from the left first:

$$\begin{aligned}\lim_{a_2^- \to \tilde{a}_2} &\Bigg((a_1 - q(a_2)a_2 - \gamma)\left[q(a_2) + a_2\frac{\partial q(a_2)}{\partial a_2}\right] \\ &\quad -\beta(a_2-\gamma)(1 - F(\tilde{y}_2(a_2))) - \beta\frac{\bar{y}_2^2 - \tilde{y}_2^2}{2(\bar{y}_2 - \underline{y}_2)}\Bigg) \\ &= \left(\tilde{a}_1^1 - \frac{1}{1+r}\tilde{a}_2 - \gamma\right)\left[\frac{1}{1+r} + \tilde{a}_2\frac{\partial q(a_2)}{\partial a_2}|_{\tilde{a}_2}\right] - \beta(\tilde{a}_2 - \gamma) - \beta\frac{\bar{y}_2 - \underline{y}_2}{2}\end{aligned}$$

This is so because $\lim_{a_2^- \to \tilde{a}_2} \tilde{y}_2(a_2) = \underline{y}_2$, and $\lim_{a_2^- \to \tilde{a}_2} q(a_2) = \frac{1}{1+r}$. Thus we see that the two focs differ only by the term $\tilde{a}_2\frac{\partial q(a_2)}{\partial a_2}|_{\tilde{a}_2} = \frac{\tilde{a}_2}{(\bar{y}_2 - \underline{y}_2)(1+r)\tau} < 0$. Given that everything else is equal, it must be that $\tilde{a}_1^1 < \tilde{a}_1^2$ (recall that $(a_1 - q(a_2)a_2 - \gamma) < 0$ because of quadratic utility, see condition (4.3.8)). By construction, no $a_1 \in [\tilde{a}_1^1\ ,\ \tilde{a}_1^2]$ can satisfy either foc. The lower bound to the no-default problem must bind. □

I have not relegated this proof to the appendix, because we will use a very similar logic for a crucial result in the next section. The following proposition basically just sums up our findings so far.

Proposition 4.2. *Consider the original maximization problem given by (4.3.3) and (4.3.5-4.3.6). The savings policy $a_2^*(a_1)$ that solves this problem is continuous and consists of three parts: a nonlinearly increasing section, a flat section, and a linearly increasing section.*

Proof. Follows from corollaries 4.1 and 4.2 and lemma 4.1. □

Figure 4.2 graphically reproduces the statements in proposition 4.2, and as such is at the core of the present section. It shows the policy $a_1^*(a_1)$, which is the object we have been looking for. Let me restate the interesting points. Although we have a discrete choice in the model, the savings policy does not have one or more discontinuities. However, it has a flat section, which hovers at the highest debt level the agent can take up without

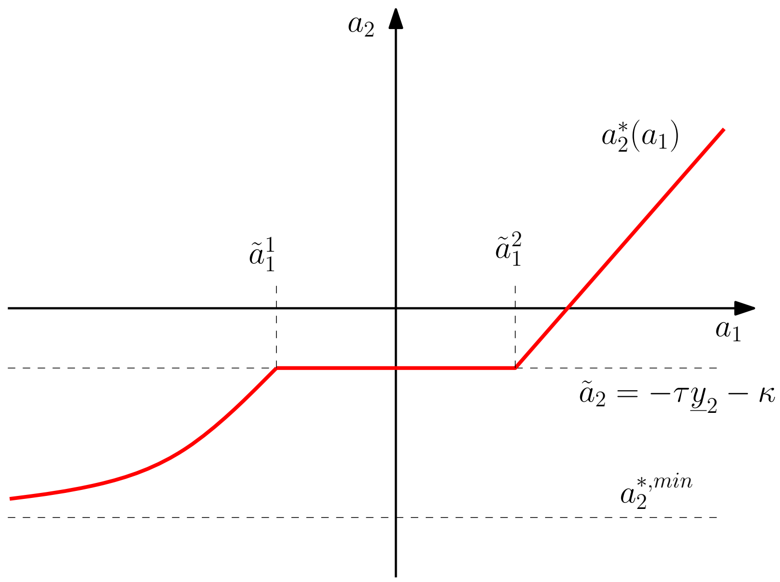

Figure 4.2: Full savings policy $a_1^*(a_1)$

paying a default-premium. If the agent is born with $a_1 \in (\tilde{a}_1^1\ ,\ \tilde{a}_1^2)$, he would like to take up more debt at the risk-free rate, but given that he can only get it at a higher price, he prefers to stay at $\tilde{a}_2$. It is kind of a 'borrowing constraint in the middle', given that there is a second, lower borrowing constraint that is never binding. Once the agent is 'poor enough' ($a_1 < \tilde{a}_1^1$), he will be willing to pay the default premium. In this part, the policy is convex, because taking up more and more debt becomes more and more expensive and thus unattractive. This property is better understood if we remember that we have quadratic utility which usually leads to a linear policy, as in the part $a_1 > \tilde{a}_1^2$, and that in more standard models, we usually observe a weakly concave savings policy, independent of the utility specification. Finally, the policy asymptotically approaches $a_2^{*,min}$, which is not the level of debt at which the agent will always default and at which $q(a_2) = 0$. Instead, $q(a_2^{*,min}) > 0$ and $P(d|a_2^{*,min}) < 1$. To my knowledge, there is no discussion of such features in the related literature. Taking a more applied perspective, it is clear that simply interpolating this policy without further thought can easily lead to very misleading results, whereas it becomes simple and accurate when one takes into account the results presented here.

4.3.2 Exclusion from Asset Markets

As argued above, the previous case was primarily meant to introduce the main ideas and to have a simple benchmark. As we will see, much of the logic carries over to this section, and the benchmark will help us to understand why some important differences arise. While the pecuniary punishment rather resembled bankruptcy under Chapter 13, the present section is meant to capture bankruptcy under Chapter 7, where pecuniary costs are negligible and income cannot be garnished (see White (1998) for a concise description). While not mentioned in the law, the literature often models the punishment as an exclusion from asset markets, because the agent is not allowed to accumulate assets or incur more debt during the filing period. In the present paper it means that if the agent defaults, all his debt will again be discharged, but he will not be able to borrow or save in the current period. For this punishment to have any bite, we need a third period, because otherwise, the agent would not care and always default. With $T = 3$, default in $t = 2$ implies that the agent has to live in autarky for the rest of his life, since it consists of only one more period. To keep the model as simple as possible and comparable to the last one, we assume that the agent receives a fixed income $y_3 > 0$ in the third period, and can default only in the second. Thus we have

$$V^{nd}(a_2, y_2) = -(a_2 + y_2 - \frac{1}{1+r} a_3^* - \gamma)^2 + \beta \left[-(a_3^* + y_3 - \gamma)^2 \right] \tag{4.3.24}$$

$$V^{d}(y_2) = -(y_2 - \gamma)^2 + \beta \left[-(y_3 - \gamma)^2 \right] \tag{4.3.25}$$

where a_3^* is the solution to

$$\max_{a_3 > -\bar{y}_3} - \left(a_2 + y_2 - a_3 \frac{1}{1+r} - \gamma \right)^2 - \beta (a_3 + \bar{y}_3 - \gamma)^2.$$

The maximization problem is as simple as it gets, and so is the policy a_3^*. Note that in equilibrium the agent will never hit the lower bound: he would rather default than choose $a_3 = -\bar{y}_3$, so we will ignore this situation. Turning to the default decision, the agent will again default on $a_2 < \tilde{a}_2$ iff

$$V^{d}(y_2) > V^{nd}(a_2, y_2) \tag{4.3.26}$$

In contrast to before, we now get two cut-off levels for income instead of one. This happens because now, some agents prefer to keep their access to the credit market when they get a bad income realization. This way, they can take up more debt and thereby shift consumption from the future to the present, whereas if they defaulted, they could consume only their small income. While this result has already been mentioned in Chatterjee, Corbae, Nakajima, and Ríos-Rull (2007), they do not discuss it in much detail. Denote the two cut-off levels by $\tilde{y}^l(a_2)$ and $\tilde{y}^u(a_2)$ (for lower and upper, respectively). They are plotted in figure 4.3, and their closed form solution is

$$\begin{aligned}
\tilde{y}^u(a_2) &= (a_2(1+r)+\bar{y}_3-\gamma)\,(1+r)\beta + B + \gamma \\
\tilde{y}^l(a_2) &= (a_2(1+r)+\bar{y}_3-\gamma)\,(1+r)\beta - B + \gamma \\
\text{where } B &= \sqrt{a_2(1+r)\beta\left((1+r)^2\beta+1\right)(a_2(1+r)+2\bar{y}_3-2\gamma)}
\end{aligned}$$

The range of income realizations for which the agent will choose to default on a given a_2

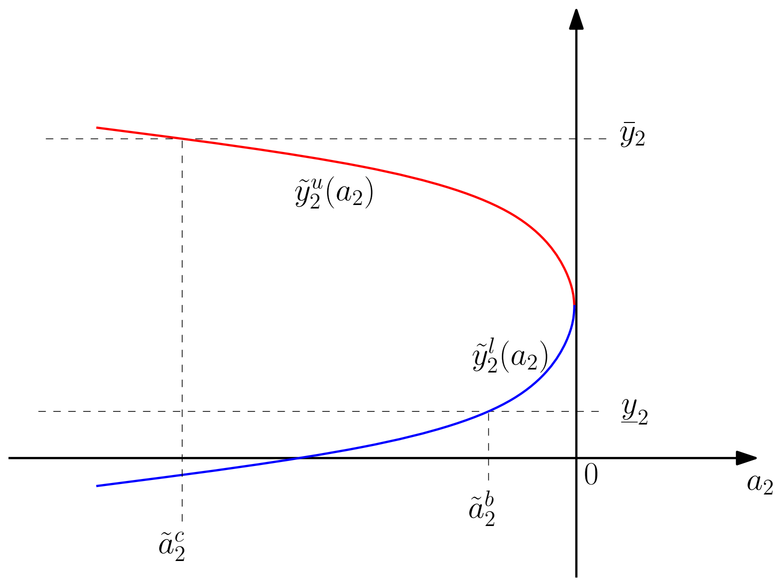

Figure 4.3: The two cut-off levels of income $\tilde{y}^l(a_2)$ (blue) and $\tilde{y}^u(a_2)$ (red)

is the line between the two curves, or $\tilde{y}^u(a_2)-\tilde{y}^l(a_2)$. There are several interesting features to note. First of all, $\tilde{a}=0$ whereas it was strictly smaller than zero before. That is, for any level of debt, there is some probability of default. The next thing that catches the eye is that, in contrast to before, the cut-off levels are nonlinear in a_2: as the debt level falls, both the range of incomes which are so high that the agent doesn't want to default as well as those that are too small to default increase at an increasing rate. That is eqivalent to saying that $\tilde{y}^u(a_2)$ is falling and concave in a_2 and $\tilde{y}^l(a_2)$ is increasing and convex in a_2. Another striking feature is that $\lim_{a_2\to 0}\tilde{y}^u(a_2) = \lim_{a_2\to 0}\tilde{y}^l(a_2)$, and that they both have an infinite slope at this limit. This is why they look like a parabola turned by 90 degrees. The symmetry of the two functions, though not exact, will simplify most of the expressions below. A more subtle point that will turn out to have significant impact is the following. Analogously to the previous section, $\tilde{y}_2^l(a_2)$ is only defined for $\tilde{y}_2^l(a_2) > \underline{y}_2$ and $\tilde{y}_2^u(a_2)$ is only defined for $\tilde{y}_2^u(a_2) < \bar{y}_2$. But typically, $\tilde{y}_2^l(a_2)$ will hit its bound first, so that for some $a_2 < \tilde{a}_2^b$ there exists only $\tilde{y}_2^u(a_2)$. While it could be the other way around, we will concentrate on this case for the sake of exposition.[3] Thus, for the pricing, we now need to distinguish four cases:

$$q(a_2) = \begin{cases} \frac{1}{1+r} & \text{if } a_2 \geq 0 \\ \frac{1-\left(F\left(\tilde{y}_2^u(a_2)\right)-F\left(\tilde{y}_2^l(a_2)\right)\right)}{1+r} & \text{if } \tilde{a}_2^b < a_2 < 0 \\ \frac{1-F\left(\tilde{y}_2^u(a_2)\right)}{1+r} & \text{if } \tilde{a}_2^c < a_2 \leq \tilde{a}_2^b \\ 0 & \text{if } a_2 \leq \tilde{a}_2^c \end{cases} \tag{4.3.27}$$

That is, $\tilde{y}_2^l(\tilde{a}_2^b) = \underline{y}_2$ and $\tilde{y}_2^u(\tilde{a}_2^c) = \bar{y}_2$, as shown in the figure. Note that $q(a_2)$ is continuous on the whole domain, strictly increasing on $\tilde{a}_2^c < a_2 < 0$, and convex on the two intervals $\tilde{a}_2^c < a_2 \leq \tilde{a}_2^b$ and $\tilde{a}_2^b < a_2 < 0$. So now we have to distinguish one maximization problem more than before. We can again drop the last case, because nobody would buy a one-period bond at price zero. We will now look at the three problems in turn. The first is the one with no default:

$$\max_{\gamma - y_3 > a_2 \geq 0} -(a_1 - \frac{1}{1+r}a_2 - \gamma)^2 + \beta E V_2^{nd}(a_2, y_2) \tag{4.3.28}$$

where the expectations is taken over y_2 and $V_2^{nd}(a_2, y_2)$ is given in eq. (4.3.24). Since there is no default, we are in the standard quadratic case and there is a unique, linear solution

[3] The reason why usually $\tilde{y}_2^l(a_2)$ will hit its bound first is twofold: for one, $\underline{y}_2 > 0$ so that we cannot decrease the lower bound arbitrarily, whereas we could increase $\bar{y}_2$ as much as we want. And second, $\frac{\partial \tilde{y}_2^l(a_2)}{\partial a_2} > -\frac{\partial \tilde{y}_2^u(a_2)}{\partial a_2}$ due to the small asymmetry in the two formulae. This comes from the fact that the agents at $\tilde{y}_2^l(a_2)$ have a higher curvature of utility because they are consuming little in any case.

given by $a_2^{nd}(a_1)$. As before, we will need the FOC

$$(a_1 - a_2 q(a_2) - \gamma)\left(q(a_2) + a_2 \frac{\partial q(a_2)}{\partial a_2}\right) \\ = \beta \int_{\underline{y}_2}^{\bar{y}^l} \left(a_2 + y_2 - a_3^*(a_2, y_2)\frac{1}{1+r} - \gamma\right) f(y_2) dy_2 \tag{4.3.29}$$

Now we turn to the problem where both $\tilde{y}^l(a_2)$ and $\tilde{y}^l(a_2)$ exist:

$$\max_{\tilde{a}_2^b < a_2 < 0} - (a_1 - a_2 q(a_2) - \gamma)^2 + \beta E\left[\max\left\{V^{nd}(a_2, y_2)\ ,\ V^d(y_2)\right\}\right] \tag{4.3.30}$$

which can be rewritten as

$$\begin{aligned}
&\max_{\tilde{a}_2^b < a_2 < 0} -(a_1 - a_2 q(a_2) - \gamma)^2 \\
&+ \beta \int_{\underline{y}_2}^{\tilde{y}^l(a_2)} - \left(a_2 + y_2 - \frac{a_3^*(a_2, y_2)}{1+r} - \gamma\right)^2 - \beta(a_3^*(a_2, y_2) + \bar{y}_3 - \gamma)^2 f(y_2) dy_2 \\
&+ \beta \int_{\tilde{y}^l(a_2)}^{\tilde{y}^u(a_2)} - \left(y_2 - \gamma\right)^2 - \beta\left(\bar{y}_3 - \gamma\right)^2 f(y_2) dy_2 \\
&+ \beta \int_{\tilde{y}^u(a_2)}^{\bar{y}_2} - \left(a_2 + y_2 - \frac{a_3^*(a_2, y_2)}{1+r} - \gamma\right)^2 - \beta(a_3^*(a_2, y_2) + \bar{y}_3 - \gamma)^2 f(y_2) dy_2
\end{aligned}$$

Using the same tricks as in the previous section, plus the envelope theorem, we again get a FOC that looks simple:

$$\begin{aligned}
(a_1 - a_2 q(a_2) - \gamma)\left(q(a_2) + a_2 \frac{\partial q(a_2)}{\partial a_2}\right) \\
= \beta \int_{\underline{y}_2}^{\bar{y}_2} \left(a_2 + y_2 - a_3^*(a_2, y_2)\frac{1}{1+r} - \gamma\right) f(y_2) dy_2 \\
- \beta \int_{\tilde{y}_2^l(a_2)}^{\tilde{y}_2^u(a_2)} \left(a_2 + y_2 - a_3^*(a_2, y_2)\frac{1}{1+r} - \gamma\right) f(y_2) dy_2
\end{aligned} \tag{4.3.31}$$

Let's first highlight the differences, or rather the similarity, to the no-default FOC given in eq. (4.3.29). The left-hand side only differs by marginal effective debt, something we already observed in the last section. On the right-hand side, the first term is actually identical, and the second term only does not appear in (4.3.29). Basically, the marginal expected value of the no-default situation is adjusted by subtracting the value of not defaulting in the range where the agent does default. Even after integrating and plugging in the formulae for $a_3^*(a_2, y_2)$, $\tilde{y}_2^u(a_2)$, and $\tilde{y}_2^l(a_2)$, the FOC doesn't look too bad, because a lot cancels out due to the linearity of $a_3^*(a_2, y_2)$ and the symmetry of the two cut-off rules. Yet, it turns out that the closed form is a polynomial of seventh degree, so that no general analytic solution exists. For the second time, we see how quickly the complexity of this model grows. Nevertheless, we can still analyze the critical points of the policy in the same fashion as before. We start with the following proposition:

Proposition 4.3. *In the non-dominated region ($a_2 > a_2^{*min}$), there is a unique solution to the FOC given in eq. (4.3.31).*

Proof. The proof goes along the lines of the proof to proposition 4.1. We again show that the left-hand-side of the FOC is strictly decreasing and the right-hand-side is strictly increasing. The details are in appendix 4.A. □

Corollary 4.3. *If there exists a policy function that solves the maximization problem in eq. (4.3.30), then it is nonlinear and monotonically increasing.*

In the remainder, I assume that such a policy, $a_2^{d1}(a_1)$, exists. We now look at its behavior at the upper bound $\tilde{a}_2 = 0$.

Lemma 4.2. *There exists $\tilde{a}_1^1$ where the two policy functions $a_2^{nd}(a_1)$ and $a_2^{d1}(a_1)$ join, i.e. $a_2^{nd}(\tilde{a}_1^1) = \lim_{a_1^- \to \tilde{a}_1^1} a_2^{d1}(a_1) = 0$. The joint function has a kink at this point.*

Proof. The proof is very similar to the proof of lemma 4.1, which I discussed in detail for that reason. The difference is that we now have $\lim_{a_2 \to 0} a_2 \frac{\partial q(a_2)}{\partial a_2} = 0$, which means that the two first-order-conditions corresponding to problems 4.3.28 and 4.3.30 are identical at $\tilde{a}_1^1$. This difference arises because $\tilde{a}_2 = 0$, whereas before $\tilde{a}_2 < 0$, and $\frac{\partial q(a_2)}{\partial a_2}$ approaches infinity at a less than linear rate. The proof requires multiple application of L'Hopital's rule. □

In stark contrast to the previous punishment, the policy does not display a flat section once a default-premium is charged. Instead, it remains strictly increasing. Before looking at the lower bound, we need to solve the third maximization problem, which is very similar to the last, just that the bounds change. Since in this region $\tilde{y}^l(a_2)$ doesn't exist, the pricing is different, as shown in eq. (4.3.27).

$$\max_{\tilde{a}_2^c < a_2 \leq \tilde{a}_2^b} -(a_1 - a_2 q(a_2) - \gamma)^2 + \beta E\left[\max\left\{V^{nd}(a_2, y_2)\ ,\ V^d(y_2)\right\}\right] \tag{4.3.32}$$

$$\begin{aligned}
&\max_{\tilde{a}_2^c < a_2 \leq \tilde{a}_2^b} -(a_1 - a_2 q(a_2) - \gamma)^2 \\
&+ \beta \int_{\tilde{y}^u(a_2)}^{\bar{y}_2} - \left(a_2 + y_2 - \frac{a_3^*(a_2, y_2)}{1+r} - \gamma\right)^2 - \beta(a_3^*(a_2, y_2) + \bar{y}_3 - \gamma)^2 f(y_2) dy_2 \\
&+ \beta \int_{\underline{y}_2}^{\tilde{y}^u(a_2)} - \left(y_2 - \gamma\right)^2 - \beta\left(\bar{y}_3 - \gamma\right)^2 f(y_2) dy_2
\end{aligned}$$

Note in particular how there are only two integral signs left as opposed to eq. (4.3.30). Also, we limit attention to $a_2 > \tilde{a}_2^c$ because at debt levels below that, the agent will always default. The debt that is dominated is given again by $a_2 < a_2^{*min}$ with

$$a_2^{*min} = -\frac{4\tau}{9}(\bar{y}_2 - \kappa)^2. \tag{4.3.33}$$

We have that $a_2^{*min} > \tilde{a}_2^c$, and from the following FOC we can show that again a_2^{*min} will never bind. The FOC is

$$\begin{aligned}
(a_1 - a_2 q(a_2) - \gamma)\left(q(a_2) + a_2 \frac{\partial q(a_2)}{\partial a_2}\right) \\
= \beta \int_{\tilde{y}_2^u(a_2)}^{\bar{y}_2} \left(a_2 + y_2 - a_3^*(a_2, y_2)\frac{1}{1+r} - \gamma\right) f(y_2) dy_2
\end{aligned} \tag{4.3.34}$$

Conjecture 4.1. *In the non-dominated region ($a_2 > a_2^{*min}$), there is a unique solution to the FOC given in eq. (4.3.34). This solution is the unique policy function that solves the maximization problem (4.3.32).*

Surprisingly, this is more difficult to show than for the previous maximization problem where we had two cut-off income levels. I did manage to prove this conjecture for log-utility with $\beta = 1$. We now turn to the common bound of the two problems with default. The next result is probably the most important so far.

Lemma 4.3. *There exists a nonempty interval $[\tilde{a}_1^4\ ,\ \tilde{a}_1^3]$ where the two FOCs corresponding to problems 4.3.30 and 4.3.32 each admit a different solution.*

Proof. Proof similar to proof of lemma 4.1. See appendix 4.A. □

In other words, there is a range of bequests a_1 for which there are two debt levels that satisfy one of the two FOCs each. This happens because of the new effect that appears when agents can be 'too poor to default', i.e. when they have an income so low that they want to keep their access to credit markets in order to take up more debt. In that range, both the first problem with default (4.3.30) and the second problem with default (4.3.32) have a valid solution given by the respective policy functions $a_2^{d1}(a_1)$ and $a_2^{d2}(a_1)$. The next proposition shows how these two policies yield the policy that solves the original problem (4.3.3) and (4.3.24-4.3.25)).

Proposition 4.4. *Consider the original problem given by (4.3.3) and (4.3.24-4.3.25), and call the policy function that solves it $a^*(a_1)$. $a^*(a_1)$ has the following characteristics: it is linearly increasing for $a_1 > \tilde{a}_1^1$, it has a kink at $\tilde{a}_1^1$, it is nonlinearly increasing $\tilde{a}_1^4 > a_1 > \tilde{a}_1^1$ and has a discontinuity at $\tilde{a}_1^j \in (\tilde{a}_1^4\ ,\ \tilde{a}_1^3)$.*

Proof. The kink follows from lemma 4.2. Non-linearity and monotonicity follow from the application of the implicit function theorem to the FOC. The discontinuity follows from lemma 4.3, for details see appendix 4.A. □

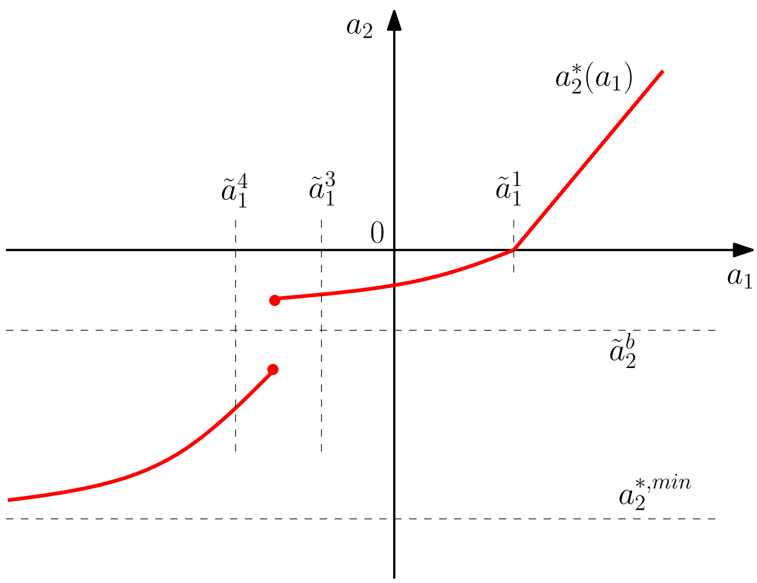

Figure 4.4: The full policy $a_1^*(a_1)$ with exclusion

Figure 4.4 visualizes the findings that are described in proposition 4.4. Comparing it with figure 4.2, which displayed the same policy under the monetary punishment, there are two striking differences: first, there is a discontinuity, and second, there is no flat section. Let's focus on the discontinuity. Contrary to what one might at first expect, the jump does not occur at a_1^1 where default starts and the pricing starts to include a default premium. At that point, there is only a kink and the policy is still continuous. Instead, the discontinuity occurs at a higher level of debt, namely close to the point where the effect of being 'too poor to default' disappears. If the agent chose $\tilde{a}_2^b$, then he would be exactly at that point where this additional effect vanishes; however, this can't be a solution because if it was, both FOCs should yield the same solution, which they don't (see proof). So the jump is such that $\tilde{a}_2^b$ is never chosen, and it must be at a point where both FOCs obtain. This gives us the interval $[\tilde{a}_1^4\ ,\ \tilde{a}_1^3]$; at the jump itself, the agent is indifferent between two courses of action. So the crucial feature that causes this jump is the fact that we have the two different cut-off income levels $\tilde{y}_2^u$ and $\tilde{y}_2^l$. Indeed we can copy this situation with a monetary punishment that is quadratic and no exclusion, whereas we do not observe this jump in the linear monetary punishment discussed earlier.

From an applied computational perspective, it is easy to see how knowledge of the jump can be very helpful. Even the robust discrete value function iteration could lead to large Euler equation errors in consumption close to the discontinuity (see e. g. Judd (1992)). Refining a grid around this point is easy and will yield better solutions. But trying to use interpolation schemes to interpolate the policy function without respecting the jump (and the kink) might yield misleading results. Indeed, if we follow the method of this paper of splitting up the original problem into subproblems, we can even use the FOCs to find a solution, which is much more efficient that using a grid search and discrete value function iteration.

From a theoretical perspective the question arises whether the discontinuity could affect proofs of existence in general equilibrium. Since the agent is indifferent between two courses of action, the aggregate demand function could be discontinuous as well. In the present model this seems unlikely, because the indifference exists only at a point, not over an interval, so that it would wash out in the aggregate.[4] However, many papers only allow

[4]This would of course only happen if the distribution of agents over initial assets does not have point mass at exactly this point.

discrete asset choices, so it could be that the agent is indifferent between two debt levels over an interval.

4.3.3 Discussion

It would be very interesting to show the same results for more periods and more general utility specifications. First, let me point out which results will still go through: for log-utility with $\beta = 1$ and for CARA utility, all results go through. Both for more periods and for more general utility, all but the uniqueness of a solution to the FOC go through, i.e. we could have optimal policy correspondences instead of a policy function. Note that the discontinuity still obtains in the correspondence sense: the correspondence will be upper hemi-continuous but not convex-valued at exactly the same spot. Similarly, the kink will show up, and the flat line from the monetary punishment regime will still be a flat line.
The big obstacle in showing that the optimal policy is single-valued is that it is not clear whether the value function resulting from the original maximization problem (4.3.3) is still concave. This is so because the expectations part might be nonconcave, given that the agent might default over an interval of income. As a matter of fact, in order to apply to a maximum theorem it would suffice if the expected value function was concave in the non-dominated region of the maximization problem one period prior. An even weaker sufficient statement is that the objective functions of the various subproblems is strictly quasiconcave in the non-dominated region. If any of this obtained, all results from above would go through for both more periods and more general utility, which would make the results much more interesting for applied work.
As a last point, note that other punishments (like utility costs of stigma) or other income distributions can be analyzed in the same framework. The reason is that the above proofs only require to check whether the lower cut-off level $\tilde{y}_2^l(a_2)$ exists, and what happens at $\lim_{a_2^- \to \tilde{a}_2} a_2 \frac{\partial q(a_2)}{\partial a_2}$. These two questions can be answered either analytically or numerically and are sufficient to establish whether there will be the discontinuity and the kink that we discussed, as well as the flat section from the first figure.

4.4 Conclusion

This paper addresses the question whether we can put enough structure on a model with a discrete default choice so that we can derive the properties of the policy functions analytically. The problem is that a discrete default decision will in general introduce points at which the value function is not differentiable. To overcome this, applied work frequently uses stochastic shocks to smooth out any kinks. However, if the shock support is bounded then additional kinks may appear at the bounds. These kinks translate into jumps of the policy function. While the jumps were characterized in a very simple setting, it seems hard to generalize this to richer settings.

It was shown that the punishment rules that are assumed determine whether the savings policy will display jumps. There might be default punishments that preserve the differentiability of the value function and at the same time are a good representation of the U. S. laws. This would be an interesting way to solve large-scale computational default models with continuous methods.

Since the non-differentiable points of the policy function are hard to characterize, applied work has to resort to numerical methods to try to approximate them. A promising approach seems to be to use endogenous grid points which are set close to the jumps of the policy function, which Fella (2011) tries. However, since it is not known where the jumps occur, the computational steps needed to find them might take more time than standard value function iteration.

Another avenue is to apply suitable envelope theorems - like the one recently proposed by Clausen and Strub (2012) - to a default model. This is left for future research.

4.A Appendix: Proofs

Derivation of the FOC with Default

Here we derive in detail the FOC with default given in eq. (4.3.23).

$$\max -(a_1 - q(a_2)a_2 - \gamma)^2 + \beta E\left[\max\left[u(a_2 + y_2), u((1-\tau)y_2 - \kappa)\right]\right]$$
$$\text{s. t.} - \tau\overline{y}_2 - \kappa < a_2 < -\tau\underline{y}_2 - \kappa$$

Rewrite the expectations

$$\beta\int_{\tilde{y}_2(a_2)}^{\overline{y}_2} -(a_2 + y_2 - \gamma)^2 f(y_2)dy_2 + \beta\int_{\underline{y}_2}^{\tilde{y}_2(a_2)} -((1-\tau)y_2 - \gamma)^2 f(y_2)dy_2 \tag{4.A.35}$$

Now differentiating the last equation w.r.t a_2 using Leibnitz' rule for differentiation

$$\begin{aligned}
&\beta\int_{\tilde{y}_2(a_2)}^{\overline{y}_2} -2(a_2 + y_2 - \gamma)f(y_2)dy_2 - \beta\left(-(a_2 + \tilde{y}_2(a_2) - \gamma)^2 f(\tilde{y}_2(a_2))\right)\frac{\partial\tilde{y}_2(a_2)}{\partial a_2}\\
&\quad + \beta\int_{\underline{y}_2}^{\tilde{y}_2(a_2)} 0\cdot f(y_2)dy_2 + \beta\left(-((1-\tau)\tilde{y}_2(a_2) - \gamma)^2 f(\tilde{y}_2(a_2))\right)\frac{\partial\tilde{y}_2(a_2)}{\partial a_2}\\
&\Leftrightarrow\\
&-2\beta(a_2 - \gamma)(1 - F(\tilde{y}_2(a_2))) - 2\beta\int_{\tilde{y}_2(a_2)}^{\overline{y}_2} y_2 f(y_2)dy_2\\
&\quad + \left(-\left((1-\tau)\tilde{y}_2(a_2) - \gamma\right)^2 - \left(-(a_2 + \tilde{y}_2(a_2) - \gamma)^2\right)\beta f(\tilde{y}_2(a_2))\right)\frac{\partial\tilde{y}_2(a_2)}{\partial a_2}
\end{aligned}$$

The term in the brackets in the last line is equal to zero, by definition of $\tilde{y}_2(a_2)$: the agent is indifferent between defaulting and not, the utilities are equal. Then:

$$-2\beta(a_2 - \gamma)(1 - F(\tilde{y}_2(a_2))) - 2\beta\frac{\bar{y}_2^2 - \tilde{y}_2^2}{2(\bar{y}_2 - \underline{y}_2)} \tag{4.A.36}$$

Thus, the FOC is

$$(a_1 - q(a_2)a_2 - \gamma)\left[q(a_2) + a_2\frac{\partial q(a_2)}{\partial a_2}\right] = \beta(a_2 - \gamma)(1 - F(\tilde{y}_2(a_2))) + \beta\frac{\bar{y}_2^2 - \tilde{y}_2^2}{2(\bar{y}_2 - \underline{y}_2)}$$

For convenience, I restate here Leibniz' rule for differentiation under integrals, which is used for the derivation of the foc.

$$\frac{d}{dz}\int\limits_{a(z)}^{b(z)} f(x,z)\;dx = \int\limits_{a(z)}^{b(z)} \frac{\partial f(x,z)}{\partial z}\;dx + f(b(z),z)\frac{\partial b(z)}{\partial z} - f(a(z),z)\frac{\partial a(z)}{\partial z}$$

Proof of Proposition 4.1

We begin by restating the FOC given in eq. 4.3.23 in the main text and making some transformations.

$$(a_1 - q(a_2)a_2 - \gamma)\left[q(a_2) + a_2\frac{\partial q(a_2)}{\partial a_2}\right] = \beta(a_2 - \gamma)(1 - F(\tilde{y}_2(a_2))) + \beta\frac{\bar{y}_2^2 - \tilde{y}_2^2}{2(\bar{y}_2 - \underline{y}_2)} \tag{4.A.37}$$

which can be rewritten as

$$\beta\left(\frac{\overline{y}_2 - \tilde{y}_2(a_2)}{\bar{y}_2 - \underline{y}_2}\right)(a_2 - \gamma) + \beta\frac{\bar{y}_2^2 - \tilde{y}_2^2}{2(\bar{y}_2 - \underline{y}_2)} = \left(a_1 - \frac{\overline{y}_2 - \tilde{y}_2(a_2)}{(\bar{y}_2 - \underline{y}_2)(1+r)}a_2 - \gamma\right)\left(\frac{1}{\bar{y}_2 - \underline{y}_2}\frac{a_2}{\tau(1+r)} + \frac{\overline{y}_2 - \tilde{y}_2(a_2)}{(\bar{y}_2 - \underline{y}_2)(1+r)}\right)$$

which in turn can be written as

$$\beta(1+r)\left((\overline{y}_2 - \tilde{y}_2(a_2))(a_2 - \gamma) + \frac{\overline{y}_2^2 - \tilde{y}_2^2}{2}\right) = \left(a_1 - \frac{\overline{y}_2 - \tilde{y}_2(a_2)}{(\bar{y}_2 - \underline{y}_2)(1+r)}a_2 - \gamma\right)\left(\frac{a_2}{\tau} + \overline{y}_2 - \tilde{y}_2(a_2)\right)$$

The final step then yields

$$
\begin{aligned}
\frac{\beta(1+r)}{\frac{a_2}{\tau}+\overline{y}_2-\tilde{y}_2(a_2)} &\left((\overline{y}_2-\tilde{y}_2(a_2))(a_2-\gamma)+\frac{\overline{y}_2^2-\tilde{y}_2^2}{2}\right) \\
&= a_1-\frac{\overline{y}_2-\tilde{y}_2(a_2)}{(\bar{y}_2-\underline{y}_2)(1+r)}a_2-\gamma \qquad (4.A.38)
\end{aligned}
$$

The last row is eq. 4.3.23 of the main text. We will now show that the LHS is strictly increasing and the RHS strictly decreasing, which means that they have a unique intersection. The trick is to limit attention to the relevant regions, where positive consumption holds and where effective debt is not dominated (as explained in the last paragraph). Of course, this intersection can only constitute part of the solution if it lies within the bounds of the maximization problem. Differentiate the LHS of (4.A.38) with respect to a_2 and examine under which conditions it is positive:

$$
\begin{aligned}
-&\frac{\left(\frac{1}{\tau}-\frac{\partial\tilde{y}(a_2)}{\partial a_2}\right)\left((\bar{y}_2-\tilde{y}_2(a_2))(a_2-\gamma)+\frac{\bar{y}_2^2-\tilde{y}(a_2)^2}{2}\right)}{\left(\frac{a_2}{\tau}+\bar{y}_2-\tilde{y}_2(a_2)\right)^2} \\
&+\frac{\left(-\frac{\partial\tilde{y}(a_2)}{\partial a_2}(a_2-\gamma)+(\bar{y}_2-\tilde{y}(a_2))-\tilde{y}(a_2)\frac{\partial\tilde{y}(a_2)}{\partial a_2}\right)}{\frac{a_2}{\tau}+\bar{y}-\tilde{y}(a_2)} > 0
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
-&\frac{2}{\tau}\left((\bar{y}_2-\tilde{y}(a_2))(a_2-\gamma)+\frac{\bar{y}_2^2-\tilde{y}(a_2)^2}{2}\right) \\
&+\left(\frac{a_2}{\tau}+\bar{y}_2-\tilde{(}a_2)\right)\left(\frac{1}{\tau}(a_2-\gamma)+(\bar{y}_2-\tilde{y}(a_2))+\frac{1}{\tau}\tilde{y}(a_2)\right) > 0
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
-&\frac{2}{\tau}\left(\bar{y}_2-\tilde{y}(a_2)\right)\left(a_2-\gamma+\frac{\bar{y}_2+\tilde{y}(a_2)}{2}+\frac{a_2-\gamma}{\tau}+(\bar{y}_2-\tilde{y}(a_2))+\frac{\tilde{y}(a_2)}{\tau}\right) \\
&+\frac{a_2}{\tau}\left(\frac{1}{\tau}(a_2-\gamma)+(\bar{y}_2-\tilde{y}(a_2))+\frac{1}{\tau}\tilde{y}(a_2)\right) > 0
\end{aligned}
$$

Lets look at the two terms in turn, beginning with the latter one:

$$\frac{a_2}{\tau}\left(\frac{1}{\tau}(a_2-\gamma)+(\bar{y}_2-\tilde{y}(a_2))+\frac{1}{\tau}\tilde{y}(a_2)\right)$$
$$\Leftrightarrow\frac{a_2}{\tau}\left((\bar{y}_2-\tilde{y}(a_2))-\frac{1}{\tau}(\gamma-a_2-\tilde{y}(a_2))\right),$$

which is greater zero, because $0<\tau<1$ by assumption, $\gamma>\bar{y}_2$ by construction, $a_2<0$ and $\bar{y}_2>\tilde{y}_2(a_2)$ by the bounds of the maximization problem. Now the first term:

$$-\frac{2}{\tau}\left(\bar{y}_2-\tilde{y}(a_2)\right)\left(a_2-\gamma+\frac{\bar{y}_2+\tilde{y}(a_2)}{2}+\frac{a_2-\gamma}{\tau}+(\bar{y}_2-\tilde{y}(a_2))+\frac{\tilde{y}(a_2)}{\tau}\right)$$

Of course, $-\frac{2}{\tau}\left(\bar{y}_2-\tilde{y}(a_2)\right)<0$. Now note that

$$-\left(\frac{1}{\tau}\gamma-\frac{1}{\tau}\tilde{y}(a_2)+\tilde{y}(a_2)-\bar{y}_2\right)<0\ ,$$

because $\gamma>\bar{y}_2$ and $(\bar{y}-\tilde{y}(a_2))(\frac{1}{\tau}-1)>0$. This leaves us with $-(\gamma-\frac{1}{2}\bar{y}_2)+a_2(1+\frac{1}{\tau}-\frac{1}{2\tau}-\frac{\kappa}{2\tau})<0$ if $\tau>\frac{\kappa-1}{2}$. That means that the whole term is positive.
Thus, for $\tau>\frac{\kappa-1}{2}$, the LHS of the foc is strictly increasing. This condition is assumed in the proposition. It doesn't seem very strict, as it is always fulfilled for $\kappa<1$, and doesn't directly collide with the conditions above. Note that it is sufficient but not necessary.
We now turn to the RHS of eq. (4.A.38), which is just marginal utility. Differentiating we get

$$-\frac{\bar{y}_2+\frac{1}{\tau}(2a_2+\kappa)}{(\bar{y}_2-\underline{y}_2)(1+r)},\tag{4.A.39}$$

which is negative for $a_2>\frac{\tau}{2}\bar{y}_2-\frac{\kappa}{2}=a_2^{*\,\min}$. Again note how we need that effective debt is not dominated.
Finally, to see that a solution always exists note that

$$\lim_{a_2^+\to a_2^{*min}}\frac{\beta(1+r)}{\frac{a_2}{\tau}+\overline{y}_2-\tilde{y}_2(a_2)}\left((\overline{y}_2-\tilde{y}_2(a_2))(a_2-\gamma)+\frac{\bar{y}_2^2-\tilde{y}_2^2}{2}\right)=-\infty\tag{4.A.40}$$

because the denominator goes to zero and the numerator is negative:

$$
\begin{aligned}
&(\overline{y}_2 - \tilde{y}_2(a_2))(a_2 - \gamma) + \frac{\overline{y}_2^2 - \tilde{y}_2^2}{2} < 0 \\
\Leftrightarrow &(\overline{y}_2 - \tilde{y}_2(a_2))(a_2 - \gamma) + \frac{1}{2}(\bar{y}_2 + \tilde{y}_2(a_2))(\bar{y}_2 - \tilde{y}_2(a_2)) < 0 \\
\Leftrightarrow &a_2 - \gamma + \frac{1}{2}(\bar{y}_2 + \tilde{y}_2(a_2)) < 0 \\
\Leftrightarrow &a_2 - \gamma + \frac{1}{2}(\bar{y}_2 + \tilde{y}_2(a_2)) < 0
\end{aligned}
$$

which is true because $\gamma > \bar{y}_2 > \tilde{y}_2(a_2)$.

Second-Order Condition of the Simple Model

Here, I show that second-order-condition to problem 4.3.20 does not help us in any way. In particular, we can't show that the maximization problem is strictly concave for the non-dominated region.

Recall from the main section that $a_2^{*min} = -\frac{1}{2}\kappa - \frac{\tau}{2}\overline{y}_2$, which is greater than the lower bound of the maximization problem $-\tau\overline{y}_2 - \kappa$. Nobody will want to borrow below a_2^{*min}, because it becomes prohibitively expensive in the sense that there is another debt level which dominates it (i.e. get same effective debt, but pay back less). Thus only debt levels $a_2 > a_2^{*min}$ can be a solution.

I first restate the SOC from the main text:

$$
\begin{aligned}
&-[(q_a(a_2)a_2 + q(a_2))^2 - 2q_a(a_2)(a_1 - q(a_2)a_2 - \gamma) \\
&+ \beta(1 - F(\tilde{y}_2)(a_2)) - \beta\frac{\partial\tilde{y}_2(a_2)}{\partial a_2}\left((a_2 - \gamma)\frac{f(\tilde{y}_2(a_2))}{1+r} - \frac{\tilde{y}_2(a_2)}{\bar{y}_2 - \underline{y}_2}\right)] < 0
\end{aligned}
$$

$$
\begin{aligned}
&(q_a(a_2)a_2 + q(a_2))^2 - 2q_a(a_2)(a_1 - q(a_2)a_2 - \gamma) \\
&+ \beta(1 - F(\tilde{y}_2)(a_2)) - \beta\frac{\partial\tilde{y}_2(a_2)}{\partial a_2}\left((a_2 - \gamma)\frac{f(\tilde{y}_2(a_2))}{1+r} - \frac{\tilde{y}_2(a_2)}{\bar{y}_2 - \underline{y}_2}\right) > 0
\end{aligned}
$$

The first term is obviously positive. By construction $a_1 - q(a_2)a_2 - \gamma < 0$, but we haven't established conditions on the primitives for this to hold. In particular, we should show under which conditions it holds that $a_1 - q(a_2^{*\,\min})a_2^{*\,\min} < \gamma$. Since $q_a(a_2) > 0$, the second term also evaluates to a positive summand. The third term is obvious (recall that in the present maximization, we always have $\bar{y}_2 < \tilde{y}_2(a_2) < \underline{y}_2$). Since $\frac{\partial \tilde{y}_2(a_2)}{\partial a_2} < 0$, we only need to show that

$$
\begin{aligned}
&(a_2 - \gamma)\frac{f(\tilde{y}_2(a_2))}{1+r} - \frac{\tilde{y}_2(a_2)}{\bar{y}_2 - \underline{y}_2} > 0 \\
\Leftrightarrow &(a_2 - \gamma)\frac{1}{\bar{y}_2 - \underline{y}_2}\frac{1}{1+r} + \left(\frac{a_2}{\tau} + \frac{\kappa}{\tau}\right)\frac{1}{\bar{y}_2 - \underline{y}_2} > 0 \\
\\
\Leftrightarrow &a_2\left(\frac{1}{1+r} + \frac{1}{\tau}\right) > \frac{\gamma}{1+r} - \frac{\kappa}{\tau} \\
\\
\Leftrightarrow &a_2 > \frac{\left(\frac{1}{1+r} + \frac{1}{\tau}\right)}{\frac{\gamma}{1+r} - \frac{\kappa}{\tau}}
\end{aligned}
$$

If the last inequality holds for $a_2^{*\,\min}$, then it holds for all valid a_2. Thus we only need to show

$$
\begin{aligned}
&-\frac{1}{2}\kappa - \frac{\tau}{2}\bar{y}_2 > \frac{\left(\frac{1}{1+r} + \frac{1}{\tau}\right)}{\gamma\frac{1}{1+r} - \frac{\kappa}{\tau}} \\
\\
\Leftrightarrow &\left(-\frac{1}{2}\kappa - \frac{\tau}{2}\bar{y}_2\right)\left(\gamma\frac{1}{1+r} - \frac{\kappa}{\tau}\right) > \frac{1}{1+r} + \frac{1}{\tau}
\end{aligned}
$$

Maybe one could assume something about $\frac{1}{1+r}(\tau\bar{y}_2 + \kappa)$, since it is weaker than condition (4.3.8). However, it seems there is no way to say something about negativity of the second-order condition.

Convexity of Effective Debt

Here I show that effective debt is strictly convex in its subintervals for quadratic utility with exclusion penalty. It suffices to focus on the interval $\tilde{a}_2 < a_2 < 0 \;\Leftrightarrow\; \exists\, \tilde{y}_2^l(a_2) \wedge \tilde{y}_2^u(a_2)$.

Marginal effective debt is given by

$$\begin{aligned}\frac{\partial g(a_2)}{\partial a_2} &= q(a_2) + \frac{\partial q(a_2)}{\partial a_2}a_2 \\ &= \frac{a_2\sqrt{a_2(r+1)\beta\left((r+1)^2\beta+1\right)(a_2r+a_2+2y_3-2\gamma)}}{(\underline{y}_2-\bar{y}_2)(a_2r+a_2+2y_3-2\gamma)} \\ &+ \frac{3\sqrt{a_2(r+1)\beta\left((r+1)^2\beta+1\right)(a_2r+a_2+2y_3-2\gamma)}}{(r+1)(\underline{y}_2-\bar{y}_2)} + \frac{1}{r+1}\end{aligned}$$

And the second derivative can be simplified to

$$\begin{aligned}\frac{\partial^2 g(a_2)}{\partial a_2^2} =& 2\frac{\partial q(a_2)}{\partial a_2} + \frac{\partial^2 q(a_2)}{\partial a_2^2}a1 \\ =& -\frac{2a_2(r+1)\beta^2\left((r+1)^2\beta+1\right)^2}{(\bar{y}_2-\underline{y}_2)} \\ &\times \frac{(2a_2^2(r+1)^2+6a_2(r+1)(y_3-\gamma)+3(y_3-\gamma)^2)}{\left(a_2(r+1)\beta\left((r+1)^2\beta+1\right)(a_2r+a_2+2y_3-2\gamma)\right)^{3/2}}\end{aligned}$$

which is strictly positive (numerator negative, denominator positive).

Proof of Proposition 4.3

The proof goes along the lines of the proof to proposition 4.1. We again show that the lhs of the FOC is decreasing and the rhs is increasing. We start by restating the foc in eq. 4.3.31:

$$\begin{aligned}(a_1 - a_2q(a_2) - \gamma)\left(q(a_2) + a_2\frac{\partial q(a_2)}{\partial a_2}\right) = \\ \beta\int_{\underline{y}_2}^{\bar{y}_2}\left(a_2 + y_2 - a_3^*(a_2,y_2)\frac{1}{1+r} - \gamma\right)f(y_2)dy_2 \\ -\beta\int_{\tilde{y}_2^l(a_2)}^{\tilde{y}_2^u(a_2)}\left(a_2 + y_2 - a_3^*(a_2,y_2)\frac{1}{1+r} - \gamma\right)f(y_2)dy_2\end{aligned}$$

Differentiate the lhs with respect to a_2:

$$-\left(q(a_2) + a_2\frac{\partial q(a_2)}{\partial a_2}\right) + (a_1 - a_2q(a_2) - \gamma)\left(2\frac{\partial q(a_2)}{\partial a_2} + \frac{\partial^2 q(a_2)}{\partial a_2^2}a1\right)$$

$2\frac{\partial q(a_2)}{\partial a_2} + \frac{\partial^2 q(a_2)}{\partial a_2^2}a1 > 0$ because effective debt is strictly increasing and strictly convex in this interval, as shown in appendix 4.A. Thus the whole expression is negative, and the lhs is decreasing in a_2.

As for the rhs, we plug in the analytical expressions for $a_3^*(a_2, y_2)$, $\tilde{y}_2^u(a_2)$, $\tilde{y}_2^l(a_2)$, evaluate the integrals, and simplify to get

$$
\begin{aligned}
&(1+r)\beta^2 \frac{(1+r)(a_2 + \frac{\underline{y}_2+\bar{y}_2}{2} - \gamma) + y_3 - \gamma}{(1+r)^2\beta + 1} \\
&\quad + (1+r)\beta^2 \frac{2(a_2(1+r) + y_3 - \gamma)}{(\bar{y}_2 - \underline{y}_2)} \\
&\quad \times \sqrt{a_2(1+r)\beta\left((1+r)^2\beta + 1\right)\left(a_2(1+r) + 2y_3 - 2\gamma\right)}
\end{aligned}
$$

Differentiating this with restpect to a_2 and simplifying yields

$$
\begin{aligned}
&\frac{2(1+r)\sqrt{a_2(1+r)\beta\left((1+r)^2\beta + 1\right)\left(a_2(1+r) + 2y_3 - 2\gamma\right)}}{(\bar{y}_2 - \underline{y}_2)} \\
&+ \frac{a_2(1+r) + y_3 - \gamma}{\bar{y}_2 - \underline{y}_2} \\
&\times \left(\frac{(1+r)\beta\left((1+r)^2\beta + 1\right)\left(a_2(1+r) + 2y_3 - 2\gamma\right)}{\sqrt{a_2(1+r)\beta\left((1+r)^2\beta + 1\right)\left(a_2(1+r) + 2y_3 - 2\gamma\right)}} \right. \\
&\quad \left. + \frac{a_2(1+r)^2\beta\left((1+r)^2\beta + 1\right)}{\sqrt{a_2(1+r)\beta\left((1+r)^2\beta + 1\right)\left(a_2(1+r) + 2y_3 - 2\gamma\right)}} \right) \\
&+ \frac{(1+r)}{(1+r)^2\beta + 1}
\end{aligned}
$$

This is positive, thus the rhs is strictly increasing.

Proof of Lemma 4.3

The proof is similar to the proof of lemma 4.1 in the main text, in that we look at what the solutions to the two FOCs are at the common boundary of the two respective problems. For convenience, we first rewrite the FOCs. The one from the first default problem (4.3.30) is

$$(a_1 - a_2 q(a_2) - \gamma)\left(q(a_2) + a_2 \frac{\partial q(a_2)}{\partial a_2}\right) \tag{4.A.41}$$
$$= \beta \int_{\tilde{y}_2^u(a_2)}^{\bar{y}_2} \left(a_2 + y_2 - a_3^*(a_2, y_2)\frac{1}{1+r} - \gamma\right) f(y_2) dy_2$$
$$+ \beta \int_{\underline{y}_2}^{\tilde{y}_2^l(a_2)} \left(a_2 + y_2 - a_3^*(a_2, y_2)\frac{1}{1+r} - \gamma\right) f(y_2) dy_2$$

and the one from the second default problem (4.3.32) is

$$(a_1 - a_2 q(a_2) - \gamma)\left(q(a_2) + a_2 \frac{\partial q(a_2)}{\partial a_2}\right) \tag{4.A.42}$$
$$= \beta \int_{\tilde{y}_2^u(a_2)}^{\bar{y}_2} \left(a_2 + y_2 - a_3^*(a_2, y_2)\frac{1}{1+r} - \gamma\right) f(y_2) dy_2$$

Taking the limit of (4.A.41) as $a_2 \to \tilde{a}_2^b$ we see that the RHS becomes equal to that of (4.A.42). The LHS differs only in the term $\frac{\partial q(a_2)}{\partial a_2}$ which in the first case equals

$$-\frac{1}{1+r}\left(f(\tilde{y}_2^u(a_2))\frac{\partial \tilde{y}_2^u(a_2)}{\partial a_2} - f(\tilde{y}_2^l(a_2))\frac{\partial \tilde{y}_2^l(a_2)}{\partial a_2}\right), \tag{4.A.43}$$

whereas in the second case it is $-\frac{1}{1+r} f(\tilde{y}_2^u(a_2))\frac{\partial \tilde{y}_2^u(a_2)}{\partial a_2}$. Since $f(\tilde{y}_2^l(a_2))\frac{\partial \tilde{y}_2^l(a_2)}{\partial a_2} > 0$, the $\tilde{a}_1^3$ that solves (4.A.42) at the bound is greater than the $\tilde{a}_1^4$ that solves (4.A.41). Since both focs always have a solution on their respective interval, we have that for $[\tilde{a}_1^4,\ \tilde{a}_1^3]$ there are two solutions, one corresponding to each foc.

Proof of Proposition 4.4

Here we show that while the two focs admit a solution each on the compact interval $[\tilde{a}_1^4,\ \tilde{a}_1^3]$, the solution to the original maximization problem will be single-valued everywhere on this interval but at one single point, which must lie on the interior.

First, there must be at least one point on the interval where the solution is multivalued. If not, then the policy $a_2^*(a_1)$ would not be upper hemi-continuous, which it must be by the maximum theorem. Next, there can be at most one point on the interval where the solution is multivalued. If not, then the two resulting value functions would have the same value and the same slope over an interval. But the slope must differ: we know that for a given a_1' we have $a_2^{*,d1}(a_1') > a_2^{*,d2}(a_1')$. The first derivative of the two value functions are (by the envelope theorem)

$$\begin{aligned}\frac{\partial V^{d1}(a_1)}{\partial a_1} =& u_c\left(a_1 - a_2^{*,d1}(a_1)q\left(a_2^{*,d1}(a_1)\right)\right)\left(-q(a_2^{*,d1}(a_1)) - \frac{\partial q(a_2)}{\partial a_2}|_{a_2^{*,d1}(a_1)}\right)\\ \frac{\partial V^{d2}(a_1)}{\partial a_1} =& u_c\left(a_1 - a_2^{*,d2}(a_1)q\left(a_2^{*,d2}(a_1)\right)\right)\left(-q(a_2^{*,d2}(a_1)) - \frac{\partial q(a_2)}{\partial a_2}|_{a_2^{*,d2}(a_1)}\right)\end{aligned}$$

We see that the two slopes will differ for any a_1', because $a_2 q(a_2)$ and $q(a_2)$ are monotonically increasing in the non-dominated part, and because

$$\frac{\partial q(a_2)}{\partial a_2}|_{a_2^{*,d1}(a_1)} > \frac{\partial q(a_2)}{\partial a_2}|_{a_2^{*,d2}(a_1)}$$

as was discussed in appendix 4.A. Thus, there will be only one element in $[\tilde{a}_1^4,\ \tilde{a}_1^3]$ with a multivalued solution to the original problem. It remains to show that it must be strictly in the interior. Assume the multivaluedness occured at the lower bound of the interval $\tilde{a}_1^4$. Then by construction $a_2^{*,d1}(a_1') = \tilde{a}_2^b$. But at $\tilde{a}_2^b$ the two value functions would attain exactly the same value, so if it was the solution to one at a given a_1', it would have to be the solution of the other as well. But the FOCs yield different values, so the multivaluedness can't occur at this point. The same argument goes through for the other bound.

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