Regularity of Market Impact Models

Time-Dependent Impact, Dark Pools and Multivariate Transient Impact

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Abstract

In this thesis, we consider three different market impact models and their regularity. Regularity of a market impact model is characterized by properties of optimal liquidation strategies. Specifically, we discuss absence of price manipulation and absence of transaction-triggered price manipulation. Moreover, we introduce a new regularity condition called positive expected liquidation costs.

The first market impact model under consideration allows for transient impact with a time-dependent liquidity parameter. This includes time-dependent permanent impact as a special case. In this model, we show an example for an arbitrage opportunity while the unaffected price process is a martingale. Furthermore, we show that regularity may depend strongly on the liquidation time horizon. We also find that deterministic strategies can be suboptimal for a risk-neutral investor even if the liquidity parameter is a martingale.

Second, we extend an Almgren-Chriss model with the possibility to trade in a dark pool. In particular, we model the cross impact of trading in the exchange onto prices in the dark pool and vice versa. We find that the model is regular if there is no temporary cross impact from the exchange to the dark pool, full permanent cross impact from the dark pool to the exchange, and an additional penalization of orders executed in the dark pool. In other cases, we show by a number of examples how the regularity depends on the interplay of all model parameters and on the liquidation time constraint.

Third, we consider a linear transient impact model in discrete time with the possibility to trade multiple assets including cross impact between the different assets. The model is regular if the matrix-valued decay kernel of market impact is a positive definite function. We characterize both symmetric and non-symmetric matrix-valued positive definite functions. We discuss nonnegative and nonincreasing decay kernels. If a decay kernel is additionally symmetric and convex, it is positive definite. Moreover, if it is also commuting, we show that the optimal discrete-time strategies converge to an optimal continuous-time strategy. For matrix-valued exponential functions, we provide explicit solutions.

Zusammenfassung

In dieser Arbeit untersuchen wir drei verschiedene Markteinflussmodelle und deren Regularität. Die Regularität eines Markteinflussmodells bestimmt sich durch die Eigenschaften von optimalen Liquidationsstrategien. Insbesondere untersuchen wir die Abwesenheit von Preismanipulationen und transaktionsinduzierten Preismanipulationen. Darüber hinaus führen wir eine neue Regularitätsbedingung ein, die positiven erwarteten Liquidationskosten.

Das erste Markteinflussmodell, das wir betrachten, beinhaltet transienten Preiseinfluss mit einem zeitabhängigen Liquiditätsparameter. Dies enthält zeitabhängigen permanenten Preiseinfluss als Spezialfall. Wir geben ein Beispiel für eine Arbitragegelegenheit an, während der unbeeinflusste Preisprozess ein Martingal ist. Außerdem zeigen wir, dass die Regularität stark vom Liquidationszeithorizont abhängen kann. Zudem können deterministische Strategien suboptimal für einen risikoneutralen Investor sein, auch wenn der Liquiditätsparameter ein Martingal ist.

Zweitens erweitern wir ein Almgren-Chriss Modell mit der Möglichkeit, in einem Dark Pool zu handeln. Insbesondere modellieren wir den wechselseitigen Preiseinfluss des Handels an der Börse auf Preise im Dark Pool und umgekehrt. Wir zeigen dass das Modell regulär ist, wenn es keinen temporären Preiseinfluss der Börse auf den Dark Pool, vollen permanenten Preiseinfluss vom Dark Pool auf die Börse und eine zusätzliche Penalisierung von im Dark Pool ausgeführten Orders gibt. Andernfalls zeigen wir durch Beispiele, wie die Regularität von dem Zusammenspiel aller Modellparameter und von der Liquidationszeitbedingung abhängt.

Drittens betrachten wir ein Modell linearen transienten Preiseinflusses mit der Möglichkeit verschiedene Wertpapiere zu handeln unter Berücksichtigung deren wechselseitigen Preiseinflusses. Das Modell ist regulär wenn die matrixwertige Abklingfunktion des Preiseinflusses eine positiv definite Funktion ist. Wir charakterisieren symmetrische und nichtsymmetrische positiv definite Funktionen und diskutieren nichtnegative und steigende Abklingfunktionen. Wenn eine Abklingfunktion zusätzlich symmetrisch und konvex ist, ist sie positiv definit. Wenn sie außerdem kommutierend ist, zeigen wir dass die optimalen zeitdiskreten Strategien gegen eine optimale zeitstetige Strategie konvergieren. Für matrixwertige Exponentialfunktionen geben wir explizite Lösungen an.

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Chapter 1

Introduction

1.1 The Optimal Liquidation Problem and Literature Overview

Many models in mathematical finance assume that arbitrary amounts of an asset can be traded at the current price either at no cost or with transaction costs that are linear in the quantity traded. However, when trading large amounts investors experience nonlinear adverse price effects of their own trading called *market impact*.

In former times, market makers stood ready to buy or sell large amounts of securities at huge spreads, even as large as 1/4th of a dollar on NASDAQ before 1994 (Christie and Schultz (1994)). Now markets offer much lower spreads, for liquid stocks often only one cent (Cont and de Larrard (2013)). This makes trading cheap for small investors. On the other hand, a large part of this liquidity is provided by high frequency traders who avoid to hold a large inventory, see Kirilenko *et al.* (2011) and Menkveld (2011). That is, they are not willing to buy or sell large amounts of securities at once. Therefore, large trades have to be split up over time.

With today's "fully automated stock exchanges" (Black (1971)) the task of trading is increasingly performed by algorithms without human intervention. This becomes most noticeable when these algorithms fail, e.g. in the Flash Crash on May 6, 2010 (Kirilenko *et al.* (2011)) or the recent loss of \$440 million for Knight Capital Group in just 45 minutes on August 1, 2012 due to a software bug.

To automate the order splitting problem one needs a model for the market impact incurred by a trade. Given this market impact model, the problem is then to compute an optimal splitting strategy. This is called the *optimal liquidation problem*, because this problem can be thought of liquidating a position (a buy trade is equivalent to liquidating a short position), or *optimal order execution problem.*

The first to set up such a model and solving for the optimal strategy that minimizes expected costs were Bertsimas and Lo (1998). However, the optimal liquidation problem is often seen as a trade-off between trading slow in order to reduce expected costs and trading fast in order to reduce volatility risk from the remaining position. This trade-off was discussed in Almgren and Chriss (2001) by introducing a mean-variance criterion for the cost of the trade. Moreover, they decomposed impact into temporary impact, which vanishes instantly, and permanent impact, which affects all future trades equally. We call models of this type Almgren-Chriss models although they were not the first to propose this decomposition. The continuous-time version of this model was introduced in Almgren (2003).

Closer to reality is a model where impact decays over time, i.e. where impact is transient. Such a model was introduced by Bouchaud *et al.* (2004). Obizhaeva and Wang (2013) were the first to use a transient impact model for the computation of optimal liquidation strategies.

For a market impact model, it is not only important to model the behavior of real markets as close as possible, but also that optimal strategies exist and are somehow reasonable, e.g. they should not switch between buying and selling large amounts very fast. This is the *regularity* of a market impact model. The first to analyze regularity of a market impact model were Huberman and Stanzl (2004). They found that permanent impact has to be linear in the trade size to exclude price manipulations, i.e. strategies that do neither buy nor sell shares in total but have a strictly positive expected profit. Gatheral (2010) analyzed regularity of transient impact models. In the case of linear transient impact, Alfonsi *et al.* (2012) found that another irregularity might occur, so-called transaction-triggered price manipulation. This occurs when an optimal strategy switches between buying in selling. The continuous-time version of this model was treated in Gatheral *et al.* (2012).

For the Almgren-Chriss model Schied and Schöneborn (2009) solved the utility maximization problem, which was already proposed by Bertsimas and Lo (1998): There an investor is considered who wants to maximize expected utility of the revenues generated by trading. The original mean-variance approach of Almgren and Chriss (2001) was confirmed by Schied *et al.* (2010) with showing that an investor with CARA utility function has a deterministic optimal strategy, which can be found using the mean-variance approach for Almgren-Chriss impact. Since mean-variance optimization is not time-consistent, Lorenz and Almgren (2011) consider adaptive mean-variance optimization.

The original Almgren-Chriss model assumes a Brownian motion as *unaffected price process*, i.e. when there is no trading by the large investor. This may be seen as a deficiency since it allows for negative prices in the model whereas the prices of many securities are always nonnegative. The situation with geometric Brownian motion was analyzed by Gatheral and Schied (2011). Furthermore, throughout this thesis and in many market impact models, the unaffected price process is assumed to be a martingale. This is necessary for studying the regularity of the model since models with drift will usually be irregular with respect to the regularity conditions we use. The martingale assumption is justified when the investor has no particular opinion on the assets traded. In an application it may be natural to allow for a drift in the unaffected price process. Such a model is considered in Lorenz and Schied (2012).

An important decision in the trading process is the decision whether to use limit orders or market orders. However, we adopt the view of Gatheral (2011), where the trading process is separated into three layers. The first layer is called macrotrader. This layer decides about the timing of trading and about the order sizes. Given a slice of the order (a child order) by the macrotrader, the second layer called microtrader decides whether to place market orders or limit orders and if so, at which price. The third layer, the smart order router, then decides which venue to send the orders to. The market impact models we consider are models for the first layer of the trading process, i.e. we do not take into account the decision between market and limit orders. Although it is desirable to have an integrated model for all layers of the trading process, such a model might be overly complex. Furthermore, improving the understanding of each single layer of the trading process seems to be still a promising area for further research.

Biais and Weill (2009) analyze limit orders in an equilibrium model, Guéant *et al.* (2012) are discussing the decision at which price to set limit orders, Guilbaud and Pham (2012) consider the problem of trading with limit orders in the futures market, i.e. in a pro-rata microstructure (that is, there is no priority for an earlier order as it is usual in equity markets). Cont and Kukanov (2012) and Huitema (2012) consider the decision on the usage of market orders versus limit orders.

Instead of assuming that impact is described by a function of the trading sizes, one can also model the limit-order book directly (at least one side, if trading is restricted to one side of the order book only). Obizhaeva and Wang (2013) assumed a constant order book height and exponential resilience. More general order book shapes were treated in Alfonsi *et al.* (2010), Alfonsi and Schied (2010) and Predoiu *et al.* (2011). Other approaches to model order books directly include Bayraktar and Ludkovski (2012), Cont *et al.* (2010), Osterrieder (2007) and Weiss (2010).

Since high-frequency traders are important liquidity providers in today's electronic markets, research in this area is also relevant for market impact modeling, as the implementors of market impact models consume liquidity provided by these market participants. Research on high-frequency trading and high-frequency data includes Avellaneda and Stoikov (2008), Brogaard (2010), Cartea and Jaimungal (2012), Fodra and Labadie (2012), Hasbrouck and Saar (2010), Hautsch and Podolskij (2013), Hendershott and Riordan (2012) and Hendershott *et al.* (2011).

In a market impact model we usually regard the problem from the point of view of a large trader. However, one can also consider the interaction of several traders. For example, if there is a large trader and other agents know about his trading intents, it is an interesting question whether it is more profitable for them to provide liquidity or to frontrun the large trader, i.e. first trade in the same direction as the large trader (and amplifying his impact), and unwind the position later when the price recovered. This problem is studied for the Almgren-Chriss model in Schöneborn and Schied (2009). Other publications concerned with multi-agent problems include Brunnermeier and Pedersen (2005), Carlin *et al.* (2007) and Moallemi *et al.* (2012).

Furthermore, survey papers and introductions include Bouchaud (2010), Bouchaud *et al.* (2008), Gökay *et al.* (2011) and Lehalle (2012). Empirical estimates of market impact are performed in Almgren *et al.* (2005) and Gerig (2007). Stochastic control problems in the context of market impact are considered by Bouchard *et al.* (2011), Bouchard and Dang (2013), Kato (2012), Kharroubi and Pham (2010) and Naujokat and Westray (2011). Risk measures for liquidity risk were studied by Acerbi and Scandolo (2008).

It can be argued there should be no price manipulation in market impact models since executing such a strategy might be an illegal market manipulation in many jurisdictions. Since it is not precisely defined what constitutes an illegal market manipulation, Kyle and Viswanathan (2008) make suggestions concerning the distinction of illegal market manipulations from legal strategies.

Some models for illiquidity assume that there are no liquidity costs for strategies of finite variation, e.g. Çetin *et al.* (2004) and Çetin *et al.* (2010). However, these models can not be applied to the optimal liquidation problem, since a buy- or sell-only strategy has finite variation, i.e. there is no cost associated to such a strategy.

In this thesis and almost all papers mentioned above it is assumed that the impact function is exogenously given. That is, we take the viewpoint of a single trader observing his impact on market prices. There is also a rich literature analyzing equilibrium models to understand why there is such impact and which economic factors are relevant for the magnitude of the impact, i.e. to understand impact as an endogenous factor. Although there are many possibilities to explain impact, one major insight is that some trading decisions carry more information (or a more careful analysis of the information available) than others. Since almost every trader has to assume that the trading counterparty may have more information, market participants demand a premium for providing liquidity. One of the first formal models for this line of thinking were Kyle (1985) and Glosten and Milgrom (1985). We refer to Vayanos and Wang (2012) for a recent survey over the available literature.

1.2 Summary of Results

In chapter 2 we introduce a market impact model with stochastic impact. Stochastic impact is motivated by the fact that liquidity in real markets is not constant but fluctuating. Furthermore we explain the impact of trading a listed derivative is naturally modeled non-deterministically. The focus of our analysis concerns the regularity of such a model. The classical regularity condition for market impact models is absence of price manipulation defined by Huberman and Stanzl (2004). Motivated by the fact that a market impact model can be free of price manipulation but obviously irregular, Alfonsi *et al.* (2012) introduced the notion of transaction-triggered price manipulation. We introduce a new regularity condition called positive expected liquidation costs that is between the two preceding conditions. The same condition was introduced independently by Roch and Soner (2011). It states that the expected revenues of a trading strategy should be at most the face value of the initial position, i.e. impact should cause a cost on average for the large trader. This condition has advantages compared to the condition of absence of price manipulations and absence of transaction-triggered price manipulations that will be shown throughout the thesis.

The first result in this chapter states that a model with stochastic permanent impact is regular if and only if the permanent impact coefficient is a submartingale. That is, such a model is clearly not suitable for market impact modeling without further modifications. Therefore, we analyze a model with stochastic transient impact. We give a necessary condition, and in special cases also sufficient conditions for regularity. Furthermore we present a numerical example that shows that transaction-triggered price manipulation might appear but the model and optimal strategies seem to be reasonable otherwise. This shows that requiring every market impact model to be free of transaction-triggered price manipulation could be a too strong requirement. Furthermore, we provide an example where the trader is risk-neutral and the liquidity parameter is a martingale, but contrary to the intuition one might have the optimal strategy is not deterministic.

In chapter 3 we consider the extension of an Almgren-Chriss model with a dark pool. We extend the model of Kratz and Schöneborn (2010) with the possibility of permanent impact. In contrast to Kratz and Schöneborn (2010), who focus on optimal liquidation strategies for risk-averse investors, we focus on the regularity of the model. In this context, we consider risk-neutral investors only. Especially with permanent impact it is not obvious how to model the interaction of exchange and dark pool. In Kratz and Schöneborn (2010) it was mostly assumed that there is no impact from the exchange onto the dark pool. When allowing for such an impact, it was observed that price manipulations may arise. Our model is flexible in this regard. Additionally, it allows for permanent impact from dark pool trades on exchange prices.

The main theorem, Theorem 3.4.1, states that a dark-pool extension is regular for all Almgren-Chriss models if and only if there is no temporary impact from the exchange to the dark pool, if there is permanent impact from the dark pool on the exchange and if there is an additional cost term for dark pool trades. For a fixed Almgren-Chriss model, we can only find weaker results. With different additional assumptions we present subsequently more concrete results. Furthermore, we show how to compute optimal liquidation strategies in the regular version of our model. Finally, we show that there is transaction-triggered price manipulation in the model which is an artifact from a too narrow set of admissible strategies.

In this regular version, dark pool prices and exchange prices are different and there are additional costs in the dark pool. That other model versions are irregular can have two interpretations: Either, dark pools allow for manipulations in reality, or our model is not reflecting all economic costs of trading in the dark pool. Independent of the reason, our results suggest that dark pools cannot be viewed simply as additional source of liquidity for larger trades at random times. A possible economic explanation could be adverse selection in the dark pool as explained in Kratz and Schöneborn (2010). However, modeling adverse selection explicitly would increase the complexity of the model considerably.

In the final chapter 4 we consider transient linear impact for multiple assets. That is, we extend the analysis of Alfonsi *et al.* (2012) to the multivariate case. In particular, we allow for cross impact between the assets traded. The model is regular if the matrix-valued decay kernel is positive definite. First, we characterize positive definite functions analogously to Bochner's theorem. In the multivariate case, it is important to differentiate between symmetric and nonsymmetric decay kernels. If the decay kernel is symmetric, optimal strategies can be computed as solution of a linear system of equations.

We explain that in the multivariate case the decay kernel should be nonincreasing and nonnegative, as in the one-dimensional setup. If it is additionally convex and symmetric, it is positive definite. This follows from the corresponding results in Alfonsi *et al.* (2012). Furthermore we consider transformations of the decay kernel. These allow us in the case of commuting decay kernels to compute optimal strategies via one-dimensional versions of the decay kernel. In contrast to the one-dimensional result, convex decay kernels may also admit transaction-triggered price manipulation in the multivariate case, but for commuting decay kernels we can prove a similar generalized result. In particular, this leads to convergence of the discrete-time optimal strategies to continuous time.

We illustrate our theoretical results with examples. First, we analyze componentwise linear and exponential decay of impact. We study the dependence of the properties discussed before on the model parameters. Then, we consider matrix functions, i.e. one-dimensional functions applied to matrices. They are always commuting and the other properties can be characterized via the generating one-dimensional function. Finally we analyze the exponential matrix function. We present a Hausdorff-Bernstein-Widder theorem for matrix-valued functions, i.e. every completely monotone matrix function can be characterized as Laplace transform of a nonnegative matrix-valued measure. We close by providing an explicit solution for optimal strategies both in discrete and continuous time for the exponential matrix function, i.e. for a generalized Obizhaeva and Wang (2013) model.

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Chapter 2

Time-Dependent Transient Impact

2.1 Introduction

Many market impact models assume that the impact of trades is constant and deterministic, i.e. every trade has the same impact and the impact of every future trade is known in advance. For several reasons it is interesting to consider impact which is stochastic or at least not constant:

First, there are intra-day seasonalities of liquidity, i.e. average spreads and depth of the order book depend on the time of day.

Second, liquidity also fluctuates randomly. Especially for less liquid assets this effect is significant, according to Almgren (2012). An adaptive algorithm which takes variable liquidity into account might perform better in practice.

Third, if we trade simultaneously in a derivative and its underlying, it is necessary to consider stochastic impact and cross impact, as we will explain in section 2.3.3. This might also be interesting for hedging derivatives under impact or for the study of the impact induced on the underlying through trading derivatives.

Fourth, when investigating the regularity of market impact models we find new effects with stochastic impact. So our model contributes to the question which regularity conditions are appropriate for market impact models.

Models for deterministic time-varying impact are discussed in Fruth *et al.* (2011) and Alfonsi and Infante Acevedo (2012). Stochastic impact is discussed in Almgren (2012), Fruth (2011) and Roch and Soner (2011). Both Fruth *et al.* (2011) and Fruth (2011) discuss exponential decay of market impact only, while Almgren (2012) concentrates on purely temporary impact. In this chapter, we want to allow also for non-exponential decay of market impact.

Transient impact was considered in Bouchaud *et al.* (2004), Gatheral (2010) and Gatheral *et al.* (2012). In this chapter, we present a stochastic extension of such a model.

We assume that impact is linear. Although Blais and Protter (2010) show empirical evidence for linear impact, using their proprietary data Tóth *et al.* (2011) argue that impact is nonlinear and present a corresponding model. Since the regularity of nonlinear transient impact models has not been treated systematically yet, we restrict ourselves to linear impact to keep the focus on stochastic liquidity.

As discussed in section 2.3.3, stochastic impact can also be motivated by derivatives. We think that stochastic impact models are better suited for impact modeling in the context of derivatives. Therefore, our model can probably serve as a basis for an impact model including derivatives of an underlying. Hedging with price impact and the impact of derivatives on the underlying are problems that attained significant interest in the literature, see for example Bank and Baum (2004), Çetin *et al.* (2010), Frey (1998), Gruber (2004), Horst and Naujokat (2011), Kraft and Kühn (2011), Li and Almgren (2011), Rogers and Singh (2010) and references therein.

We assume a market model without spread. In this setup, we find that the model is not regular for many parameters. Fruth *et al.* (2011) argue that in models without spread it might appear that there are price manipulations while in practice the spread precludes these price manipulations. However, Cont and de Larrard (2013) show that the spread is nearly always very small for liquid stocks and if it is not, then it recovers very fast. So the spread recovers much faster than the impact of large trades. In particular, in a limit order book the spread recovers from both sides of the order book and not only from the side where we are trading. Often, irregularities in the model remain if a spread is included that is small enough. Since the impact of trades cannot directly be estimated from anonymous trade and order book data, the estimation of the quantitative magnitude of these effects requires proprietary data. That is, although some irregularites in models without spread may be weaker when considering spread, it is not clear how to include spread into a market impact model exactly.

The chapter is structured as follows. In the next section, we present the market model and all notions of regularity. In section 2.3, we will discuss the special case of stochastic permanent impact, and we discuss briefly how to model the impact of derivatives in such a model. In section 2.4, we give conditions for the regularity of the market model, we discuss the case of a deterministic liquidity parameter, and we present a numerical example for a stochastic liquidity parameter. All proofs are presented in section 2.5.

2.2 Market model and regularity

2.2.1 Definitions

In this chapter, we consider a stochastic extension of Gatheral (2010) and Gatheral *et al.* (2012). We assume a large investor who wants to liquidate a given number $X_0 \in \mathbb{R}$ shares until a given time $T > 0$. For any time $t \in [0, T]$, we denote the number of shares held by the investor by X_t, the *trading strategy* chosen by the investor.

When the large investor is not trading, we assume that the unaffected price process is given by a continuous martingale $(P_t^0)_{t\geq 0}$ with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, where $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions.

We impose the following requirements on the trading strategy $X = (X_t)_{t\geq 0}$:

(a) The function $t \to X_t$ is leftcontinuous and adapted,

(b) the function $t \to X_t$ has finite and uniformly bounded variation and

(c) $X_t = 0$ for all $t > T$.

We call a strategy *admissible* if it satisfies the preceding conditions. The set of all admissible strategies is denoted by $\mathcal{X}(X_0, T)$.

For a comment on the preceding conditions we refer to Remark 2.1 in Gatheral *et al.* (2012). Note that from the uniform boundedness of the variation of X_t it follows that X_t itself is uniformly bounded.

If the large investor is using the strategy X, the price process is defined as

$$P_t = P_t^0 + \int_{s<t} g_s G(t-s)\, dX_s,$$

where $G : [0, T] \to [0, \infty)$ is a continuous nonincreasing function, the *decay kernel*. G describes the decay of the market impact of a trade. To allow for time-dependent impact, we include the process $(g_t)_{t\in[0,T]}$ which is assumed to be a continuous non-negative semimartingale.

The revenues of the investor from the trading strategy are given by

$$\mathcal{R}_T^X = X_0 P_0^0 + \int_0^T X_t\, dP_t^0 - \int_0^T \int_{s<t} g_s G(t-s)\, dX_s dX_t - \frac{1}{2} \sum_{t\in[0,T]} g_t G(0)(\Delta X_t)^2, \tag{2.1}$$

where ΔX_t denotes the jump size of X at time t. Starting from the definition $\mathcal{R}_T = -\int_0^T P_t\, dX_t$ for continuous strategies X, we show in section 2.5.2 that this is a natural definition for non-continuous strategies.

Usually we denote $\mathcal{R}_T^X$ as $\mathcal{R}_T$ if there is only one trading strategy in consideration. Throughout the chapter, we will consider maximization of expected revenues $\mathbb{E}[\mathcal{R}_T]$. That is, we assume a risk-neutral investor. Since the risk tolerance of investors varies, the regularity we analyze in the next section consider risk-neutral investors. Most regularity conditions are also satisfied for risk-averse investors, if they are satisfied for risk-neutral investors, as discussed in Remark 2.2.3.

Remark 2.2.1. To calibrate G, one could use statistical observations of the decay of market impact. g_t could be calibrated from the current state of the order book. Our model does not have a stochastic decay of market impact, since this is difficult to distinguish from random fluctuations of the price. That is, the fluctuations in the decay speed are difficult to distinguish from the unaffected price process P^0. This problem is discussed in more detail in chapter 4 of Fruth (2011).

2.2.2 Regularity of market impact models

We want to analyze our market model with respect to the following four regularity conditions:

Definition 2.2.2 (Regularity conditions)**.** (a) We say a market model has *bounded expected profits* for a certain X_0, if

$$\sup_{X\in\mathcal{X}(X_0,T)} \mathbb{E}[\mathcal{R}_T^X] < \infty.$$

Otherwise, we say the model admits *unbounded expected profits*.

(b) A market model does not admit *price manipulation*, if

$$\sup_{X \in \mathcal{X}(0,T)} \mathbb{E}[\mathcal{R}_T^X] = 0.$$

(c) A market model has *positive expected liquidation costs*, if for every $X_0 \in \mathbb{R}$

$$\sup_{X \in \mathcal{X}(X_0,T)} \mathbb{E}[\mathcal{R}_T^X] \leq X_0 P_0^0.$$

(d) A market model does not admit *transaction-triggered price manipulation*, if

$$\sup_{X \in \mathcal{X}(X_0,T)} \mathbb{E}[\mathcal{R}_T^X] = \sup_{X \in \mathcal{X}_{mon}(X_0,T)} \mathbb{E}[\mathcal{R}_T^X],$$

where $\mathcal{X}_{mon} \subset \mathcal{X}$ are the monotone admissible trading strategies, i.e. with X_t either nonincreasing or nondecreasing in $t \in [0,T]$.

Since an optimal liquidation strategy will aim at maximizing expected revenues, a model with unbounded expected profits is not suitable for finding optimal strategies.

The notion of *price manipulation* is due to Huberman and Stanzl (2004). They define a *round trip* as an admissible trading strategy with $X_0 = 0$. Furthermore, they define a *price manipulation* as a round trip X with $\mathbb{E}[\mathcal{R}_T^X] > 0$.

The condition of positive expected liquidation costs states that the expected revenues of a liquidation strategy should be at most the face value of the position at time 0. Since we assume that P_t^0 is a martingale, assuming that there is no impact would yield $\mathbb{E}[\mathcal{R}_T] = X_0 P_0^0$. So the condition says that impact should be a cost on average and not increase expected revenues. Roch and Soner (2011) also give in their Theorem 4.2 a result on this regularity condition.

Transaction-triggered price manipulation was defined in Alfonsi *et al.* (2012). If there is transaction-triggered price manipulation, it is possible to lower the costs of a buy (resp. sell) program by intermediate sell (resp. buy) orders. Note that our definition is equivalent with the original definition in Alfonsi *et al.* (2012), except if the supremum is attained on the left-hand side of the equation, but not on the right-hand side.

Remark 2.2.3. Consider a risk-averse investor, i.e. an investor who wishes to maximize $\mathbb{E}[u(\mathcal{R}_T)]$, where the utility function $u : \mathbb{R} \to \mathbb{R}$ is assumed to be concave and strictly increasing. The regularity conditions generalize as follows: The market model has bounded expected profits if

$$\sup_{X \in \mathcal{X}(X_0,T)} \mathbb{E}[u(\mathcal{R}_T^X)] < \lim_{r \to \infty} u(r),$$

no price manipulation if

$$\sup_{X \in \mathcal{X}(0,T)} \mathbb{E}[u(\mathcal{R}_T^X)] = u(0),$$

positive expected liquidation costs if for every $X_0 \in \mathbb{R}$ we have

$$\sup_{X \in \mathcal{X}(X_0,T)} \mathbb{E}[u(\mathcal{R}_T^X)] \leq u(X_0 P_0^0)$$

and no transaction-triggered price manipulation if

$$\sup_{X \in \mathcal{X}(X_0,T)} \mathbb{E}[u(\mathcal{R}_T^X)] = \sup_{X \in \mathcal{X}_{mon}(X_0,T)} \mathbb{E}[u(\mathcal{R}_T^X)].$$

Taking $u(r) = r$ yields again definition 2.2.2. Now, for the first three regularity criteria Jensen's inequality yields that the model is regular for the risk-averse investor, if it is regular in the risk-neutral case. However, there is no such result for transaction-triggered price manipulation.

Remark 2.2.4. Consider a lager set of admissible strategies $\tilde{\mathcal{X}} \supset \mathcal{X}$. The first three regularity conditions are stable against enlargement of the set of admissible strategies in the following sense: If a regularity criterion fails for $\mathcal{X}$, the regularity criterion will also fail for $\tilde{\mathcal{X}}$. However, there may be transaction-triggered price manipulation for $\mathcal{X}$, but not for $\tilde{\mathcal{X}}$, since also $\tilde{\mathcal{X}}_{mon} \supseteq \mathcal{X}_{mon}$. For an example, see section 3.4.4.

We also want to compare these regularity criteria to the classical regularity criterion in mathematical finance, no arbitrage. In our setup, we need first to define an arbitrage opportunity:

Definition 2.2.5. An *arbitrage opportunity* is an admissible trading strategy with

(a) $\mathbb{P}(\mathcal{R}_T \geq X_0 P_0^0) = 1$ and

(b) $\mathbb{P}(\mathcal{R}_T > X_0 P_0^0) > 0$.

The regularity conditions in Definition 2.2.2 form a hierarchy, i.e. *bounded expected profits* is the weakest condition, and *absence of transaction-triggered price manipulation* is the strongest condition, as stated in the following proposition. Note that Roch (2011) and Roch and Soner (2011) stated before that no arbitrage follows from positive expected liquidation costs.

Proposition 2.2.6 (Hierarchy of regularity conditions)**.** *(a) If there is no transaction-triggered price manipulation, then we have positive expected liquidation costs.*

(b) If we have positive expected liquidation costs, then there is no price manipulation, the market model has bounded expected profits for each X_0 and there are no arbitrage opportunities.

(c) If there is no price manipulation, then the market model has bounded expected profits for $X_0 = 0$.

For a moment, drop the condition that P^0 is a martingale. In the classical theory without impact, no-arbitrage is ensured by the existence of an equivalent martingale measure, i.e. the existence of a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that

P^0 is a $\mathbb{Q}$-martingale. However, as already noted by Roch and Soner (2011), this is not sufficient in our setup with impact, as will be shown in example 2.3.2. A sufficient condition for ensuring no-arbitrage is the condition of positive expected liquidation costs under the martingale measure, i.e. $\sup_{X\in\mathcal{X}(X_0,T)} \mathbb{E}^{\mathbb{Q}}[\mathcal{R}_T^X] \leq X_0 P_0^0$ for all $X_0 \in \mathbb{R}$.

2.3 Stochastic permanent impact

2.3.1 Regularity of permanent impact

Our market model contains permanent impact as a special case. Permanent impact corresponds to $G(t) \equiv \text{const}$. In this case, let $\gamma_t := G(0)\, g_t$. Then we can write

$$P_t = P_t^0 + \int_0^t \gamma_s \, dX_s.$$

For such a stochastic permanent impact, Roch (2011) has shown that there is no arbitrage if $(\gamma_t)_{t\geq 0}$ is a submartingale, see Theorem 2.6 in Roch (2011). Indeed, the model is regular with respect to any of our regularity conditions if and only if $(\gamma_t)_{t\geq 0}$ is a submartingale:

Proposition 2.3.1 (Regularity of permanent impact)**.** *If G is constant and $\gamma_t = G(0)\, g_t$, all the following conditions are equivalent:*

(a) There is no transaction-triggered price manipulation.

(b) We have positive expected liquidation costs.

(c) There is no price manipulation.

(d) For each X_0, the market model does not admit unbounded profits.

(e) The market model does not admit unbounded profits for any $X_0 \in \mathbb{R}$.

(f) $(\gamma_t)_{t\geq 0}$ is a submartingale.

Furthermore, if any of these conditions holds, an optimal strategy is given by $X_t = 0$ for $t \in (0, T]$, i.e. an immediate liquidation of the complete position at time 0.

So mean-reverting processes, or processes that model an intraday U-shape of liquidity should not be used in a model with permanent impact, since they violate the submartingale condition. If $(\gamma_t)_{t\geq 0}$ is not a submartingale the model is completely irregular and typically there do not exist optimal strategies. If one restricts strategies to be monotone, the model becomes more regular, but the optimal strategies in this class can have undesirable properties like a clustering of orders during a short time span as can be seen in Mönch (2009).

Example 2.3.2. We give an example for an arbitrage opportunity in the sense of Definition 2.2.5. Let $\gamma_t = 4 - t$. Clearly, this is not a submartingale, so we cannot apply Proposition 2.3.1. Let now $T = 3$, $X_0 = 0$, $P_0^0 = 1$ and assume $P_t^0 \geq 0$ for every $t \in [0, T]$ almost surely. Consider the strategy $X_t = +\mathbb{1}_{(0,3]}(t)$. Then $\mathcal{R}_T = P_3^0 - P_0^0 + \frac{3}{2} = P_3^0 + \frac{1}{2} \geq \frac{1}{2}$ almost surely, so this strategy is an arbitrage opportunity.

2.3.2 Additional temporary impact

The model in the previous section is regular only if $(\gamma_t)_{t\geq 0}$ is a submartingale. But in this case, an optimal strategy is the complete liquidation of the position at $t = 0$. To penalize such large trades, we can add temporary impact to the model as in Almgren and Chriss (2001). Temporary impact can also regularize models, so we want to discuss whether the regularity of this model improves with additional temporary impact. In the case of temporary impact, trading strategies have to be absolutely continuous so as to admit a derivative $\dot{X}_t$ almost everywhere.

So in this section, let the price process be given by

$$\tilde{P}_t = P_t^0 + \int_0^t \gamma_t \, dX_t + \eta \dot{X}_t,$$

with $\eta > 0$. If $(\gamma_t)_{t\geq 0}$ is a submartingale, then there is still no transaction-triggered price manipulation. On the other hand, if $(\gamma_t)_{t\geq 0}$ is not a submartingale, the regularity of the model may improve for small T, what we show by the following example:

Proposition 2.3.3. *Let $\gamma_t = C - \beta t$ with constants $\beta > 0$ and $C \geq \beta T$:*

(a) If T is small enough, for each X_0 there is no transaction-triggered price manipulation.

(b) If $T > \sqrt{24\frac{\eta}{\beta}}$, for each X_0 there are unbounded expected profits.

2.3.3 Permanent impact and derivatives

Assume we can trade in a derivative and in the underlying continuously at the same time. For example, plain vanilla options are listed on exchanges, so their liquidity can in principle be modeled the same way as for stocks. An interesting question in this context is how trading in the underlying impacts the derivative and vice versa.

In this section, we want to derive heuristically, why the impact of derivative trading should be stochastic. Assume an underlying with purely constant permanent impact, i.e. $P_t = P_t^0 + \gamma(X_t - X_0)$. In this section, we assume that all trading strategies are continuous to simplify the expressions. Assume that a derivative's price is given by

$$D_t = D_0 + \int_0^t \delta(s, P_s) \, dP_s - \int_0^t \theta(s, P_s) \, ds,$$

with the greeks $\delta, \theta : [0, T] \times [0, \infty) \to \mathbb{R}$, e.g. from a Black-Scholes model. We will use the short-hand notation $\delta_t = \delta(t, P_t)$ and $\theta_t = \theta(t, P_t)$.

Now we add the possibility for the investor to trade in the derivative. Let the number of derivatives held by the investor at time t be denoted as X_t^D. Let us assume $X_0^D = 0$ and that the seller of the derivatives hedges the derivative position in the underlying at time t with $\delta_t X_t^D$ shares. This hedge causes an impact on the underlying of $\gamma\delta_t X_t^D$. So the resulting price process for the underlying is

$$P_t = P_t^0 + \gamma(X_t - X_0) + \gamma\delta_t X_t^D. \tag{2.2}$$

Thus, we find

$$\begin{aligned} dD_t &= \delta_t\, dP_t - \theta_t\, dt \\ &= \delta_t\, dP_t^0 + \gamma\delta_t^2\, dX_t^D + \gamma X_t^D \delta_t\, d\delta_t + \delta_t\gamma\, dX_t - \theta_t\, dt. \end{aligned}$$

Here, the impact has a stochastic component via δ_t, although the impact in the underlying without trading in the derivative is not stochastic.

So there is a clear need for stochastic impact models. Permanent impact is not suited well for this purpose since the permanent impact parameter has to be a submartingale by Proposition 2.3.1. However, e.g. in (2.2) $\gamma\,\delta_t$ would be the permanent impact parameter which is typically not a submartingale. Even with temporary impact there may be problems as shown in Proposition 2.3.3. This motivates to consider stochastic transient impact as we will do in the next section.

2.4 Stochastic transient impact

2.4.1 A necessary condition for regularity

After analyzing permanent impact, we discuss the more general model presented in section 2.2.1. If g is constant and G is convex, then Theorem 1 in Alfonsi *et al.* (2012) guarantees the absence of transaction-triggered price manipulation. If g is not constant, in general convexity of G is not sufficient to guarantee regularity.

First, we present a necessary condition for the regularity of the model:

Proposition 2.4.1. *If there are bounded profits for any X_0, then for every $s, t \in [0, T]$ with $s \le t$*

$$\mathbb{E}[g_t|\mathcal{F}_s] \ge g_s \left(2\frac{G(t-s)}{G(0)} - 1\right). \tag{2.3}$$

The intuitive meaning of the inequality is that g may not decrease more than twice as fast as G. If G is constant, (2.3) simplifies to

$$\mathbb{E}[g_t|\mathcal{F}_s] \ge g_s,$$

i.e. we recover the submartingale condition from Proposition 2.3.1. Proposition 2.4.5 shows that (2.3) is a sufficient condition in a special case. But in general, even the stronger condition

$$\mathbb{E}[g_t|\mathcal{F}_s] \ge g_s \frac{G(t-s)}{G(0)}. \tag{2.4}$$

is not sufficient. This can be seen in section 2.4.4 where we discuss an example with a strict submartingale g and strictly convex G with unbounded expected profits if T is large enough.

2.4.2 A sufficient condition for exponential G

If G is exponential, we can also find a sufficient condition for regularity. In this case, our model is a special case of the model in Roch and Soner (2011). A direct application of Theorem 4.2 in Roch and Soner (2011) yields the following result.

Proposition 2.4.2. *Let $G(t) = \exp(-\beta t)$ with $\beta \geq 0$. If the process $\phi_t := \frac{\exp(-2\beta t)}{g_t}$ is a supermartingale, we have positive expected liquidation costs.*

Remark 2.4.3. If $\beta = 0$ and ϕ_t is a supermartingale, then g_t is a submartingale due to Jensen's inequality and we can conclude by Proposition 2.3.1 that there are positive expected liquidation costs. On the other hand, if g_t is a submartingale, $\frac{1}{g_t}$ is not necessarily a supermartingale. So the above condition is not a necessary condition on regularity since Proposition 2.3.1 states that g_t being a submartingale is a sufficient condition in the case $\beta = 0$.

Example 2.4.4. If g_t is geometric Brownian motion, then Proposition 2.4.2 is particularly easy to apply. Let

$$g_t = g_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t\right)$$

with a standard Brownian motion B_t and $\mu \in \mathbb{R}$, $\sigma > 0$. That is, g_t satisfies the stochastic differential equation

$$dg_t = \mu g_t \, dt + \sigma g_t \, dB_t.$$

In this case, we find that ϕ_t is a supermartingale if and only if

$$\beta \geq \frac{\sigma^2 - \mu}{2}, \tag{2.5}$$

compare also Remark 4.3 in Roch and Soner (2011). On the other hand, if $\beta = 0$, g_t is a submartingale if and only if $\mu \geq 0$, while (2.5) would require $\mu \geq \sigma^2$, so (2.5) is not a necessary condition.

However, Theorem 4.2 in Roch and Soner (2011) cannot be applied to our setting if G is not exponential, as we discuss in Remark 2.5.3.

2.4.3 Deterministic liquidity parameter g

In the following, we discuss the special case of deterministic g. First, we characterize regularity if both g and G are exponential functions.

Proposition 2.4.5. *Let $G(t) = e^{-\beta t}$ and $g_t = e^{-\alpha t}$ with $\alpha, \beta \geq 0$. The following conditions are equivalent:*

(a) There is an $X_0 \in \mathbb{R}$ such that there are bounded expected profits.

(b) There is no price manipulation.

(c) There are positive expected liquidation costs.

(d) Condition (2.3) is fulfilled.

(e) The process ϕ_t defined in Proposition 2.4.2 is a supermartingale, i.e. it is non-increasing.

(f) $\alpha \leq 2\beta$.

The condition for the absence of transaction-triggered price manipulation is given in Proposition 2.4.8.

For the remainder of this section, we want to discuss trading in discrete time, i.e. the investor may only trade on a finite set of deterministic times $\mathbb{T} = \{t_0, t_1, \dots t_N\}$ with $N \in \mathbb{N}$ and $0 \leq t_0 < t_1 < \dots t_N \leq T$. Furthermore, we denote by $\xi_{t_i} = \Delta X_{t_i}$ the jump size of the strategy X at time t_i. ξ is also called strategy, since the strategy X can be reconstructed from ξ by $X_t = X_0 + \sum_{t_i<t} \xi_{t_i}$. Thus, the liquidation condition is given by $X_0 + \sum_{i=0}^{N} \xi_{t_i} = 0$ and the expected revenues of a trading strategy are

$$\mathbb{E}[\mathcal{R}_T] = X_0 P_0^0 - \mathbb{E}\left[\frac{1}{2}\sum_{n=0}^{N}\sum_{m=0}^{N} \xi_{t_n}\xi_{t_m} g_{t_n \wedge t_m} G(|t_n - t_m|)\right]. \tag{2.6}$$

If g is deterministic and there is a unique optimal strategy, the optimal strategy is deterministic and consequently we can restrict ourselves to the class of deterministic strategies. For a deterministic strategy, the expected revenues are given by

$$\mathbb{E}[\mathcal{R}_T] = X_0 P_0^0 - \frac{1}{2}\xi^\top M \xi,$$

with $\xi = (\xi_{t_0}, \xi_{t_1}, \dots, \xi_{t_N}) \in \mathbb{R}^{N+1}$ and the symmetric matrix $M \in \mathbb{R}^{(N+1)\times(N+1)}$ given by $(M)_{i,j} = g_{t_i \wedge t_j} G(|t_i - t_j|)$. In this case, we have the following regularity result:

Proposition 2.4.6. *(a) We have positive expected liquidation costs if and only if M is a positive semidefinite matrix.*

(b) There is no price manipulation if and only if there exists an $X_0 \in \mathbb{R}$ such that there are bounded expected profits.

However, if M is not positive semidefinite, there may be or may not be price manipulation or unbounded expected profits. See also Example 2.4.7 below.

If M is even strictly positive definite, there is a unique optimal strategy. In this case, the optimal strategy ξ^* is given by

$$\xi^* = \frac{X_0}{\mathbb{1}^\top M^{-1} \mathbb{1}} M^{-1} \mathbb{1},$$

with $\mathbb{1} = (1, 1, \dots, 1)^\top \in \mathbb{R}^{N+1}$, as in Alfonsi *et al.* (2012).

The next example shows that it can be the case that there is no price manipulation even though M is not positive definite.

Example 2.4.7. Let $g_t = \frac{1}{(1+t)^2}$ and $G(t) = \frac{1}{1+t}$. Note that both functions are convex. Furthermore, condition (2.3) is fulfilled since for $t_j \geq t_i \geq 0$

$$\frac{g_{t_j}}{g_{t_i}} - 2\frac{G(t_j - t_i)}{G(0)} + 1 = \frac{(t_j - t_i)(t_j + t_j^2 + t_i + t_i^2)}{(1+t_j)^2(1+t_j-t_i)} \geq 0.$$

Let $\mathbb{T} = \{0, 1, 2\}$. Then $\det M = -\frac{1}{576}$ and thus, M is not positive semidefinite. However, we find that the expected revenues of the optimal strategy are given by $\mathbb{E}[\mathcal{R}_T] = X_0 P_0^0 + \frac{1}{15} X_0^2$, so there is no price manipulation and there are bounded expected profits, but the condition of positive expected liquidation costs is not satisfied. Consequently, there is also transaction-triggered price manipulation.

In discrete time, transaction-triggered price manipulation can be characterized using Lemma 1 from Alfonsi *et al.* (2012). In the following proposition, we use it to analyze exponential impact like in Proposition 2.4.5.

Proposition 2.4.8. *Let $G(t) = e^{-\beta t}$ and $g_t = e^{-\alpha t}$ with $\alpha, \beta \geq 0$. In the discrete-time setup, there is no transaction-triggered price manipulation if and only if $\alpha \leq \beta$.*

2.4.4 Numerical example: geometric Brownian motion

In the following, we discuss the example of g being geometric Brownian motion. As in Example 2.4.4, let

$$g_t = g_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right).$$

Furthermore, we assume that we are in the discrete-time setting as in the previous section. In the remainder of the section, we show numerical results for the optimal strategy.

For geometric Brownian motion, a straightforward application of dynamic programming yields the optimal strategy. Using the software *Mathematica* we obtained explicit expressions for the optimal strategy dependent on $g(\omega)$ evaluated on the time grid.

We have chosen geometric Brownian motion since the computation of optimal strategies by dynamic programming is easy in this case, while having a mean-reverting property might be more realistic, it is also more complicated.

In the following, we will use these concrete parameters, if not otherwise stated: Let $\sigma = 1, \mu = \frac{1}{4}, g_0 = 1, G(t) = \frac{1}{1+t}, X_0 = 1, T = 3$ be given. Furthermore, we will use equidistant trading points, i.e. $\mathbb{T} = \{\frac{i}{N}T | i \in \{0, 1, \ldots, N\}\}$ for $N \in \mathbb{N}$, where we use $N = 3$.

In Figure 2.1, we show the dependence of the optimal ξ_{t_1} on g_{t_1}. We see, that if liquidity increases strongly (that is, g_{t_1} being small), the optimal strategy will buy shares instead of selling. That is, there is transaction-triggered price manipulation. The rationale behind that is to profit of the (relatively) large impact generated at t_0, since with very high probability the position can be sold off again with (relatively) few costs. Since this occurs only with a small probability and the size of the buy

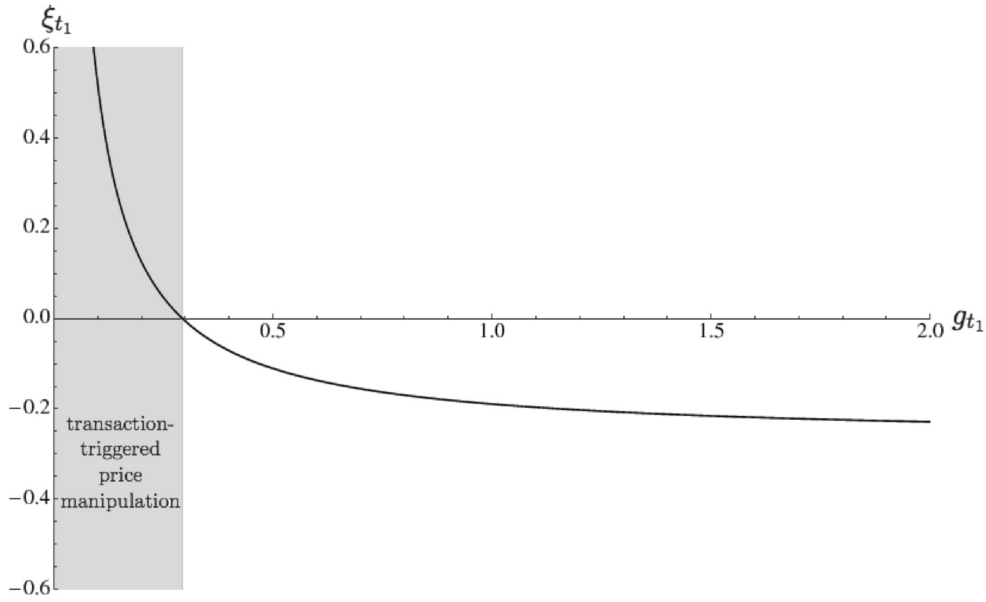

Figure 2.1: ξ_{t_1}, the optimal trade size at time t_1, in dependence of g_{t_1}

order is not large (if g_t is not very small), one can still consider the model regular, although it admits transaction-triggered price manipulation.

In Figure 2.2, we show how the expected revenues of the optimal strategy $\mathbb{E}[\mathcal{R}_T] - X_0 P_0^0$ depend on T. We see that for $T \leq 18.1$, there are positive expected liquidation costs, and for larger T this regularity condition is not fulfilled any longer. At $T^* \approx 23.8$, we observe a peak. That is, there are unbounded expected profits for $T \geq T^*$.

If we compute the optimal strategy for $X_0 = 0$, we see that there is no price manipulation for $T < T^*$ and unbounded expected profits for larger T. But we see, if $X_0 \neq 0$ and T approaches T^*, the expected revenues of the optimal strategy get arbitrary large. So the model cannot be considered regular for these values, although there is no price manipulation.

So for this simple stochastic impact model we find that the regularity depends strongly on the time horizon T. Also, we find that there is transaction-triggered price manipulation but here it is not necessary to reject the model for that reason.

We observe that the optimal strategy is not bounded uniformly, since $|X_t| \to \infty$ for $g_t \to 0$ and formally, it is not admissible. However, in practice g_t is bounded below since there is only a finite amount of shares or money available.

2.4.5 Adaptive strategies versus deterministic strategies

Now consider the model from the previous section with $\mu = 0$. In this case, $(g_t)_{t\geq 0}$ is a martingale. If the strategy ξ is deterministic, we could compute (because of the

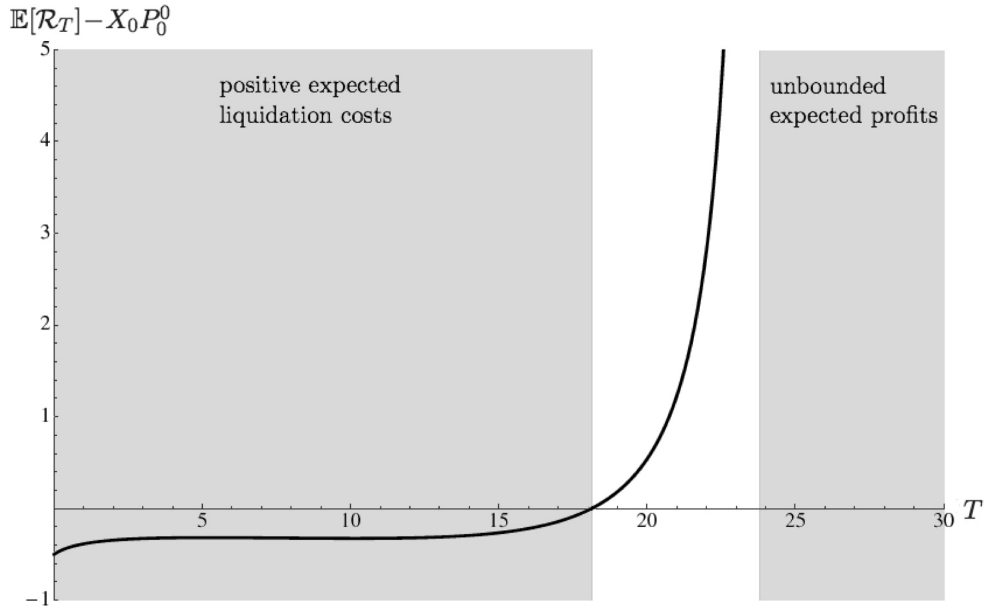

Figure 2.2: Expected revenues of the optimal strategy in dependence of the time horizon T

martingale property of g)

$$\begin{aligned}\mathbb{E}[\mathcal{R}_T] &= X_0P_0^0 - \mathbb{E}\left[\frac{1}{2}\sum_{n,m=0}^{N}\xi_{t_n}\xi_{t_m}g_{t_n\wedge t_m}G(|t_n-t_m|)\right]\\ &= X_0P_0^0 - \frac{1}{2}\sum_{n,m=0}^{N}\xi_{t_n}\xi_{t_m}G(|t_n-t_m|).\end{aligned}$$

With this expression, we can compute the optimal deterministic strategy like in section 2.4.3 and obtain $\mathbb{E}[\mathcal{R}_T]-X_0P_0^0 \approx -0.272$ for the optimal deterministic strategy, while the optimal adaptive strategy yields $\mathbb{E}[\mathcal{R}_T]-X_0P_0^0 \approx -0.265$. This shows that there can be an improvement by adaptive strategies, even if the liquidity parameter $(g_t)_{t\geq 0}$ is a martingale, contrary to the intuitive expectation one might have from investment strategies without impact: There the revenues are linear in ξ and do not depend on ξ at all if the price process is a martingale, i.e. there is also a deterministic optimal strategy.

2.5 Proofs and derivation of the revenues

2.5.1 Proofs

Proof of Proposition 2.2.6. (a): Let a monotone strategy $X \in \mathcal{X}_{mon}(X_0,T)$ be given. Since $g_sG(t-s) \geq 0$ for all $s,t \in [0,T]$, it is $\int_0^T\int_{s<t} g_sG(t-s)\,dX_sdX_t \geq 0$.

Furthermore we have for all strategies $\frac{1}{2}\sum_t g_t G(0)(\Delta X_t)^2 \geq 0$. Thus,

$$\begin{aligned}\mathcal{R}_T^X &= X_0 P_0^0 + \int_0^T X_t \, dP_t^0 - \int_0^T \int_{s<t} g_s G(t-s) \, dX_s dX_t - \frac{1}{2}\sum_t g_t G(0)(\Delta X_t)^2 \\ &\leq X_0 P_0^0 + \int_0^T X_t \, dP_t^0.\end{aligned}$$

So for any montone strategy we find $\mathbb{E}[\mathcal{R}_T^X] \leq X_0 P_0^0$ and therefore,

$$\sup_{X \in \mathcal{X}_{mon}(X_0,T)} \mathbb{E}[\mathcal{R}_T^X] \leq X_0 P_0^0.$$

By absence of transaction-triggered price manipulation, we obtain

$$\sup_{X \in \mathcal{X}(X_0,T)} \mathbb{E}[\mathcal{R}_T^X] \leq X_0 P_0^0.$$

(b): Inserting $X_0 = 0$, the condition of positive expected liquidation costs yields $\sup_{X\in\mathcal{X}(0,T)} \mathbb{E}[\mathcal{R}_T^X] \leq 0$. Since the strategy $X_t \equiv 0$ for all $t \in [0,T]$ yields $\mathbb{E}[\mathcal{R}_T] = 0$, we find $\sup_{X\in\mathcal{X}(0,T)} \mathbb{E}[\mathcal{R}_T^X] = 0$, which is absence of price manipulation.

Furthermore, bounded expected profits trivially follow from positive expected liquidation costs, since $X_0 P_0^0 < \infty$.

To see that there is no arbitrage, note that for any arbitrage opportunity X we have $\mathbb{E}[\mathcal{R}_T] > X_0 P_0^0$.

(c): This is trivial, since $0 < \infty$. □

Remark 2.5.1. Note that Proposition 2.2.6 holds in a quite general setting: Let the unaffected price process be given as $P_t = P_t^0 + I_t$, where the unaffected price process $(P_t^0)_{t\geq 0}$ is a martingale and I_t is the impact of the trading strategy, and let the revenues be given by $\mathcal{R}_T = -\int_0^T P_t \, dX_t$ for continuous strategies. If the impact is always positive for nondecreasing strategies and always negative for nonincreasing strategies, then (a) holds. (b) and (c) hold even without this condition.

For the proof of Proposition 2.3.1, we need the following Lemma:

Lemma 2.5.2. *In the case of permanent impact, i.e. $G(t) \equiv const$ with $\gamma_t = G(0)g_t$, the revenues of a trading strategy are given by*

$$\mathcal{R}_T = X_0 P_0^0 + \int_0^T X_t \, dP_t^0 - \frac{1}{2}\gamma_0 X_0^2 - \frac{1}{2}\int_0^T X_t^2 \, d\gamma_t.$$

Proof.

$$\begin{aligned}\mathcal{R}_T &= X_0 P_0^0 + \int_0^T X_t \, dP_t^0 - \int_0^T \int_{s<t} \gamma_s \, dX_s dX_t - \frac{1}{2}\sum_t \gamma_t (\Delta X_t)^2 \\ &= X_0 P_0^0 + \int_0^T X_t \, dP_t^0 + \int_0^T X_t \gamma_t \, dX_t + \frac{1}{2}\sum_t \gamma_t (\Delta X_t)^2 \\ &= X_0 P_0^0 + \int_0^T X_t \, dP_t^0 + \frac{1}{2}\int_0^T \gamma_t \, dX_t^2 \\ &= X_0 P_0^0 + \int_0^T X_t \, dP_t^0 - \frac{1}{2}\gamma_0 X_0^2 - \frac{1}{2}\int_0^T X_t^2 \, d\gamma_t,\end{aligned}$$

where we used integration by parts two times. □

Proof of Proposition 2.3.1. We will prove first, that if (γ_t) is not a submartingale, then there are unbounded profits for each X_0 (and by Proposition 2.2.6 there is price manipulation, transaction-triggered price manipulation and not positive expected liquidation costs). Second, we will prove, that if (γ_t) is a submartingale, then there is no transaction-triggered price manipulation (and by Proposition 2.2.6, there are positive expected liquidation costs, no price manipulation and bounded profits for each X_0).

Assume that (γ_t) is not a submartingale, i.e. there exist t_1, t_2 such that $0 < t_1 \leq t_2 \leq T$ and $A \in \mathcal{F}_{t_1}$ with $\mathbb{P}(A) > 0$ such that

$$\mathbb{E}[\gamma_{t_2}|\mathcal{F}_{t_1}] < \gamma_{t_1} \text{ on } A.$$

Consider the trading strategy

$$X_s = \begin{cases} x, & t_1 \leq s < t_2 \text{ and } \omega \in A \\ X_0, & s = 0 \\ 0, & \text{else.} \end{cases}$$

Note that the trading strategy is admissible. With Lemma 2.5.2 we find that the expected revenues of this strategy are

$$\mathbb{E}[\mathcal{R}_T] = X_0 P_0^0 - \frac{1}{2}\gamma_0 X_0^2 - \frac{1}{2}x^2 \mathbb{E}[\gamma_{t_2} - \gamma_{t_1}; A].$$

Since $E[\gamma_{t_2} - \gamma_{t_1}; A] < 0$, sending $x \to \infty$ shows, that there are unbounded profits for each X_0.

Conversely, assume that (γ_t) is a submartingale. Then for each strategy X

$$\mathbb{E}[\mathcal{R}_T] \leq X_0 P_0^0 - \frac{1}{2}\gamma_0 X_0^2,$$

by Lemma 2.5.2 since $\mathbb{E}[\int_0^T X_t^2 \, d\gamma_t] \leq 0$. Furthermore, for the monotone strategy $X_t = 0$ for all $t \in (0, T]$ we have $\mathbb{E}[\mathcal{R}_T] = X_0 P_0^0 - \frac{1}{2}\gamma_0 X_0^2$. This proves the absence of transaction-triggered price manipulation and the optimality of this trading strategy. □

Proof of Proposition 2.3.3. The revenues are given here by

$$\tilde{\mathcal{R}}_T = X_0 P_0^0 + \int_0^T X_t \, dP_t^0 - \frac{1}{2}C X_0^2 + \frac{1}{2}\beta \int_0^T X_t^2 \, dt - \eta \int_0^T \dot{X}_t^2 \, dt.$$

Proof of (a): We prove that for $T \leq \sqrt{2\frac{\eta}{\beta}}$, there is no transaction-triggered price manipulation. First, we proof that in this case, there are bounded expected profits. Let $X_0 \in \mathbb{R}$ and $(X_t)_{t \geq 0}$ an admissible trading strategy. Define $x := \sup_{t \in [0,T]} |X_t|$. Then

$$\frac{1}{2}\beta \int_0^T X_t^2 \, dt - \eta \int_0^T \dot{X}_t^2 \, dt \leq \frac{1}{2}\beta T x^2 - \eta \frac{x^2}{T} = x^2 (\frac{1}{2}\beta T - \frac{\eta}{T}). \tag{2.7}$$

Due to our assumption on T, the latter expression is nonpositive and thus, $\mathbb{E}[\tilde{\mathcal{R}}_T] \leq X_0 P_0^0$.

Observe that if there is a unique optimal deterministic strategy maximizing $\frac{1}{2}\beta \int_0^T X_t^2\, dt - \eta \int_0^T \dot{X}_t^2\, dt$, this strategy will be optimal in the class of all strategies. Since the optimal strategy is bounded due to (2.7), using calculus of variations (cf. Cesari (1983) Theorems 2.6.i-iii and 2.20.i) we find that there is a unique optimal deterministic strategy X, it is twice continuously differentiable and satisfies the following ordinary differential equation (Euler-Lagrange equation)

$$\beta X_t + 2\eta \ddot{X}_t = 0$$

with boundary conditions X_0 and $X_T = 0$. The unique solution of this equation is given by

$$X_t = X_0 \cos\left(t\sqrt{\frac{\beta}{2\eta}}\right) - X_0 \cot\left(T\sqrt{\frac{\beta}{2\eta}}\right)\sin\left(t\sqrt{\frac{\beta}{2\eta}}\right).$$

Due to our assumption on T, $\cot(\cdot)$ is positive and $\cos(\cdot)$ and $-\sin(\cdot)$ are decreasing for $t \in [0,T]$, and thus, the optimal strategy X_t is decreasing for $X_0 > 0$ and increasing for $X_0 < 0$, so there is no transaction-triggered price manipulation.

Proof of (b): Consider the trading strategy

$$X_t = \begin{cases} X_0 + \frac{2t}{T}x, & \text{for } t \leq \frac{T}{2}, \\ (X_0 + x)\frac{2(T-t)}{T}, & \text{for } \frac{T}{2} \leq t \leq T \end{cases}$$

with $x \in \mathbb{R}$ such that $x\, X_0 \geq 0$. The expected revenues of this strategies can be estimated by

$$\mathbb{E}[\tilde{\mathcal{R}}_T] \geq X_0 P_0^0 - \frac{1}{2}C X_0^2 + \beta x^2 \frac{T}{6} - 4\eta \frac{(X_0 + x)^2}{T},$$

and if $T > \sqrt{24\frac{\eta}{\beta}}$ this expression converges to $+\infty$ for $|x| \to \infty$. □

Proof of Proposition 2.4.1. Consider a trading strategy which is only trading at t_i and t_j. With (2.6) we find that the expected revenues of this strategy are

$$\mathbb{E}[\mathcal{R}_T] = X_0 P_0^0 - \mathbb{E}[\xi_{t_i}\xi_{t_j} g_{t_i} G(t_j - t_i)] - \frac{1}{2}G(0)\mathbb{E}[g_{t_i}\xi_{t_i}^2 + g_{t_j}\xi_{t_j}^2].$$

Using $\xi_{t_j} = -X_0 - \xi_{t_i}$ and the tower property of conditional expectation we find

$$\begin{aligned} \mathbb{E}[\mathcal{R}_T] \;=\; & X_0 P_0^0 + \mathbb{E}\left[\xi_{t_i}^2 \left(g_{t_i} G(t_j - t_i) - \frac{1}{2} g_{t_i} G(0) - \frac{1}{2}\mathbb{E}[g_{t_j}|\mathcal{F}_{t_i}]G(0)\right)\right] \\ & + \mathbb{E}\left[\xi_{t_i} X_0 \left(g_{t_i} G(t_j - t_i) - g_{t_j} G(0)\right) - \frac{1}{2} X_0^2 \mathbb{E}[g_{t_j}^2] G(0)\right]. \end{aligned}$$

Assume that (2.3) does not hold, i.e. there exists $A \in \mathcal{F}_{t_i}$ with $\mathbb{P}(A) > 0$ such that

$$\mathbb{E}[g_{t_j}|\mathcal{F}_{t_i}] < g_{t_i}\left(2\frac{G(t_j - t_i)}{G(0)} - 1\right) \text{ on } A,$$

i.e. $C := \mathbb{E}[g_{t_i}G(t_j - t_i) - \frac{1}{2}g_{t_i}G(0) - \frac{1}{2}\mathbb{E}[g_{t_j}|\mathcal{F}_{t_i}]G(0); A] > 0$. Consider the trading strategy

$$\xi_{t_i} = \begin{cases} x, & \text{if } \omega \in A \\ 0, & \text{otherwise.} \end{cases}$$

with $x \in \mathbb{R}$. The trading strategy is admissible. The expected revenues of this strategy are

$$\mathbb{E}[\mathcal{R}_T] = X_0 P_0^0 + x^2 C + x X_0 \mathbb{E}[g_{t_i}G(t_j - t_i) - g_{t_j}G(0); A] - \frac{1}{2}X_0^2 \mathbb{E}[g_{t_j}^2]G(0),$$

so we find $\lim_{x\to\infty} \mathbb{E}[\mathcal{R}_T] = \infty$, a contradiction to the condition of bounded expected profits. Thus, (2.3) holds. □

Proof of Proposition 2.4.2. We use Theorem 4.2 in Roch and Soner (2011). Let

$$l_t = P_t - P_t^0 = \int_{s<t} g_s G(t-s)\, dX_s$$

and

$$\kappa_t = -\frac{\int_{s<t} g_s G'(t-s)\, dX_s}{l_t} \quad \text{if } l_t \neq 0.$$

Since $-\frac{G'(t)}{G(t)}$ is the relative speed of the decay of market impact at time t by an order at time 0, κ_t can be seen as the cumulative relative speed of the decay of market impact of all past orders. Due to our assumption that $G(t) = e^{-\beta t}$ we find that $\kappa_t \equiv \beta$ for all t with $l_t \neq 0$. However, Roch and Soner (2011) only rely on the representation

$$dl_t = -\kappa_t l_t\, dt$$

in the absence of trading, and this holds also if we take $\kappa_t = \beta$ in case $l_t = 0$.

Furthermore, let

$$\begin{aligned} m_t &= \frac{1}{2}G(0)g_t, \\ z_t &= X_t - X_0, \\ \mathcal{R}_t &= X_0 P_0^0 + \int_0^t X_s\, dP_s^0 - \int_0^t \int_{r<s} g_s G(s-r)\, dX_r dX_s - \frac{1}{2}\sum_{s\in[0,t]} g_s G(0)(\Delta X_s)^2, \\ L_t &= \mathcal{R}_t - X_0 P_0^0 - \int_0^t X_s\, dP_s^0. \end{aligned}$$

Then we are in the setup of Roch and Soner (2011) and if ϕ_t is a supermartingale we can conclude by Theorem 4.2 that $\mathbb{E}[L_T] \geq 0$, which is equivalent to positive expected liquidation costs since $\mathbb{E}[\int_0^T X_t\, dP_t^0] = 0$. □

Remark 2.5.3. If G is not exponential, then κ_t is typically not bounded and can increase or decrease arbitrary fast. Thus, for any reasonable g the process ϕ_t will not be a supermartingale. To show that κ_t is not bounded it is sufficient to show that it can be that $\int_{s<t} g_s G'(t-s)\, dX_s \neq 0$ while $l_t = 0$. That is, if $P_t = P_t^0$, there

can be a nonzero difference between these two price process short after t even in the absence of trading.

To show this, assume G is not exponential. Furthermore assume g is strictly positive, assume $G(t) > 0$ for all t and G is C^1. In this case, we know that there is a $t \in [0, T)$ such that

$$\frac{G'(0)}{G(0)} \neq \frac{G'(t)}{G(t)}.$$

Then we choose a strategy that trades at 0 and t, but not in $(0, t)$. Take an arbitrary $\Delta X_0 \neq 0$. Furthermore, let $\Delta X_t = -\frac{g_0 G(0)}{g_t G(t)} \Delta X_0$. Then $l_{t+} = 0$, but $g_0 G'(0) \Delta X_0 + g_t G'(t) \Delta X_t = g_0 \Delta X_0 \left(G'(0) - G(0) \frac{G'(t)}{G(t)} \right) \neq 0$.

Proof of Proposition 2.4.5. The equivalence of (f) and (e) is obvious. From (e) follows (c) by Proposition 2.4.2, and from (c) follow (a) and (b) by Proposition 2.2.6. Furthermore from (b) follows (a) obviously. By Proposition 2.4.1, from (a) follows (d). If we show that from (d) follows (f), the equivalence of (a)-(f) is shown.

To this end, note that (d) is equivalent to

$$f_s(t) := e^{-\alpha(t-s)} - 2e^{-\beta(t-s)} + 1 \geq 0$$

for all $s, t \in [0, T]$ with $s \leq t$. Note that $f_s(s) = 0$ and the derivative of f is given by $f_s'(t) = -\alpha e^{-\alpha(t-s)} + 2\beta e^{-\beta(t-s)}$. Now assume that (f) does not hold, i.e. $a > 2\beta$. Then $f_s'(s) = -\alpha + 2\beta < 0$, so $f_s(t) < 0$ for $t - s$ small enough, so (d) does not hold. This finishes the proof. □

Proof of Proposition 2.4.6. (a): Let M be a positive semidefinite matrix. Then $\xi^\top M \xi \geq 0$ for all ξ, so $\mathbb{E}[\mathcal{R}_T] = X_0 P_0^0 - \frac{1}{2} \xi^\top M \xi \leq X_0 P_0^0$. That is, the condition of positive expected liquidation costs holds.

Assume conversely that M is not positive semidefinite, i.e. there exists a negative eigenvalue $\lambda < 0$ and an eigenvector $\xi \in \mathbb{R}^{N+1}$ with $M\xi = \lambda\xi$. Let $X_0 = \sum_{n=0}^N \xi_n$. Then for the strategy ξ we have $\mathbb{E}[\mathcal{R}_T] = X_0 P_0^0 - \frac{1}{2} \xi^\top M \xi = X_0 P_0^0 - \frac{1}{2} \lambda \xi^\top \xi > X_0 P_0^0$, i.e. the condition of positive expected liquidation costs is not satisfied.

(b): Assume that there is no price manipulation. Obviously, for $X_0 = 0$ there are bounded expected profits.

Conversely, assume that there exists $X_0 \in \mathbb{R}$ such that there are bounded expected profits. We have to show, that there is no price manipulation. Assume there is price manipulation, to arrive at a contradiction.

In this case, let ξ be a price manipulation. Let $\hat{\xi} := (\frac{X_0}{N+1}, \ldots, \frac{X_0}{N+1})^\top$. Define a sequence of trading strategies by $\xi^{(n)} := n \cdot \xi + \hat{\xi}$. The expected revenues of these trading strategies are given by

$$\begin{aligned} \mathbb{E}[\mathcal{R}_T^{\xi^{(n)}}] &= X_0 P_0^0 - \frac{1}{2} (\xi^{(n)})^\top M \xi^{(n)} \\ &= X_0 P_0^0 - \frac{1}{2} n^2 \xi^\top M \xi - n \, \xi^\top M \hat{\xi} - \hat{\xi}^\top M \hat{\xi}. \end{aligned}$$

Since ξ is a price manipulation it is $\xi^\top M \xi < 0$. Thus, $\mathbb{E}[\mathcal{R}_T^{\xi_n}]$ converges to $+\infty$ for $n \to \infty$, i.e. a X_0 such that there are bounded profits cannot exist. This a contradiction. □

For the proof of Proposition 2.4.8, we use Lemma 1 from Alfonsi *et al.* (2012):

Lemma 2.5.4. *Let M be an invertible symmetric matrix. We have $M^{-1}\mathbb{1} \geq 0$ or $M^{-1}\mathbb{1} \leq 0$ if and only if there is no vector z such that*

$$z^\top \mathbb{1} = 0, Mz > 0.$$

Proof of Proposition 2.4.8. In the following, we will assume that we have given $t_1 < t_2 < \ldots < t_N$ and the matrix M is given by $(M)_{i,j} = e^{-\alpha(t_i \wedge t_j)} e^{-\beta|t_i - t_j|} = e^{-(\alpha-\beta)(t_i \wedge t_j) - \beta(t_i \vee t_j)}$ for $i, j \in \{1, 2, \ldots, N\}$ in contrast to using t_0 for the first trading time before. We change the notation to avoid confusion about the 0th row or column of a matrix.

For $\alpha > 2\beta$, Proposition 2.4.5 provides the existence of transaction-triggered price manipulation. Then consider the special case $\alpha = 2\beta$. In this case, for the vector

$$\tilde{\xi} := -\frac{X_0}{1 - e^{-\beta(t_N - t_1)}} (-e^{-\beta(t_N - t_1)}, 0, 0, \ldots, 0, 1)$$

we have $M\tilde{\xi} = 0$. Therefore, for this strategy we have $\mathbb{E}[\mathcal{R}_T^{\tilde{\xi}}] = X_0 P_0$. On the other hand, if $X_0 \neq 0$, then for every admissible strategy ξ with $\xi \geq 0$ or $\xi \leq 0$, obviously $\xi^\top M \xi > \varepsilon$ for some $\varepsilon > 0$ since all entries of M are strictly positive, and therefore, $\mathbb{E}[\mathcal{R}_T^{\xi}] < X_0 P_0 - \varepsilon$. This proves the existence of transaction-triggered price manipulation.

We are left with the case $\alpha < 2\beta$. We will first show that M is invertible in this case. To this end, we show show that the determinant of M is given by

$$\det M = \exp\left(-2\beta \sum_{i=2}^{N} t_i - 2\alpha \sum_{i=1}^{N} t_i + \alpha t_N\right) \prod_{i=1}^{N-1} \left(e^{\alpha t_i + 2\beta t_{i+1}} - e^{2\beta t_i + \alpha t_{i+1}}\right) \tag{2.8}$$

We prove this by induction: Suppose (2.8) is shown for $N-1$. We define $M_{N,i}$ to be the $(N-1) \times (N-1)$ matrix that is obtained by removing the Nth row and ith column of M. The Laplace expansion along the Nth row yields

$$\det M = \sum_{i=1}^{N} (-1)^{N+i} m_{N,i} \det(M_{N,i}).$$

Next we observe that $\det(M_{N,i}) = 0$ for $i \leq N-2$ since for these i the vector

$$(0, 0, \ldots 0, -\exp(-\beta(t_N - t_{N-1})), 1)^\top$$

is in the null space of $M_{N,i}$. Furthermore, we have $\det(M_{N,N-1}) = \exp(-\beta(t_N - t_{N-1})) \det(M_{N,N})$, because $M_{N,N-1}$ can be obtained from $M_{N,N}$ by multiplying the last column with $\exp(-\beta(t_N - t_{N-1}))$. But $\det(M_{N,N})$ is known due to the induction hypothesis. Thus, we have

$$\begin{aligned}
\det M &= -e^{-\alpha t_{N-1}-\beta(t_N-t_{N-1})}\det(M_{N,N-1}) + e^{-\alpha t_N}\det(M_{N,N})\\
&= \left(-e^{-\alpha t_{N-1}-2\beta(t_N-t_{N-1})} + e^{-\alpha t_N}\right)\det(M_{N,N})\\
&= \left(-e^{-\alpha t_{N-1}-2\beta(t_N-t_{N-1})} + e^{-\alpha t_N}\right)\cdot\\
&\quad (-1)^N \exp\left(-2\beta\sum_{i=2}^{N-1} t_i - 2\alpha\sum_{i=1}^{N-1} t_i + \alpha t_{N-1}\right)\cdot\\
&\quad \prod_{i=1}^{N-2}\left(e^{2\beta t_i+\alpha t_{i+1}} - e^{\alpha t_i + 2\beta t_{i+1}}\right)\\
&= (-1)^{N+1}\exp\left(-2\beta\sum_{i=2}^{N} t_i - 2\alpha\sum_{i=1}^{N} t_i + \alpha t_N\right)\cdot\\
&\quad \prod_{i=1}^{N-1}\left(e^{2\beta t_i+\alpha t_{i+1}} - e^{\alpha t_i + 2\beta t_{i+1}}\right)\\
&= \exp\left(-2\beta\sum_{i=2}^{N} t_i - 2\alpha\sum_{i=1}^{N} t_i + \alpha t_N\right)\prod_{i=1}^{N-1}\left(e^{\alpha t_i+2\beta t_{i+1}} - e^{2\beta t_i + \alpha t_{i+1}}\right)
\end{aligned}$$

so (2.8) is proven. Now, if

$$\alpha t_i + 2\beta t_{i+1} > 2\beta t_i + \alpha t_{i+1},$$

then the determinant is strictly positive. Since this is equivalent to $\alpha(t_{i+1}-t_i) < 2\beta(t_{i+1}-t_i)$, in the case of $\alpha < 2\beta$ the determinant is strictly positive.

So we know that for $\alpha < 2\beta$ the matrix M is invertible. Thus, we can apply Lemma 2.5.4 and have to show, that there exists a vector z such that

$$z^\top \mathbb{1} = 0, Mz > 0 \tag{2.9}$$

if and only if $2\beta > \alpha > \beta$. Consider first the case $2\beta > \alpha > \beta$. In this case, we find that $(M)_{i,j}$ is strictly decreasing in j. Therefore, e.g. by using the vector $z = (1, -1, 0, \ldots, 0)$, Lemma 2.5.4 yields the existence of transaction-triggered price manipulation.

Let now $\alpha \le \beta$. The proof, that there exists no vector z such that (2.9) is satisfied, follows the lines of the proof of Theorem 1 in Alfonsi *et al.* (2012). The result is shown by induction over N. For $N = 1$ the result is obvious. Suppose now the assertion has already be proven for $N-1$ and let us assume that there is a vector $z \in \mathbb{R}^N$ that satisfies (2.9). Since $Mz > 0$, there exists a $k \in \{1, 2, \ldots, N\}$ such that $z_k > 0$. Let $\mathbb{1}_{N-1}$ denote the vector in $\mathbb{R}^{N-1}$ whose components are all equal to 1.

If $k = N$ we have that

$$m_{m,N} z_N \le m_{m,N-1} z_N \text{ for } m = 1, 2, \ldots, N-1.$$

Hence, the $N-1$-dimensional vector $\tilde{z} := (z_1, z_2, \ldots, z_{N-1} + z_N)^\top$ satisfies both $\tilde{z}^\top \mathbb{1}_{N-1} = 0$ and $\tilde{M}\tilde{z} > 0$, with $\tilde{M}$ being the matrix corresponding to the time grid $\{t_1, t_2, \ldots, t_{N-1}\}$. But this is a contradiction to the induction hypothesis.

If $k = 1$, the assumption that $\alpha \le \beta$ yields

$$m_{m,1} z_0 \le m_{m,2} z_0 \text{ for } m = 2, 3, \dots, N.$$

Hence, the vector $\hat{z} := (z_1 + z_2, z_3, \dots, z_N)^\top$ satisfies both $\hat{z}^\top \mathbb{1}_{N-1} = 0$ and $\hat{M}\hat{z} > 0$, with $\hat{M}$ corresponding to the time grid $\{t_2, t_3, \dots, t_N\}$. This is again a contradiction to the induction hypothesis.

Finally, consider $2 \le k \le N-1$. Let $\alpha \in [0,1]$ be such that $t_k = \alpha t_{k-1} + (1-\alpha)t_{k+1}$. We then have

$$m_{k,l} z_k \le m_{k-1,l} \alpha z_k + m_{k+1,l}(1-\alpha) z_k \text{ for } l \ne k,$$

since the function $t \mapsto e^{-\alpha(t \wedge t_l)} e^{-\beta|t-t_l|}$ is convex in t. Hence, the vector

$$\bar{z} := (z_1, z_2, \dots, z_{k-2}, z_{k-1} + \alpha z_k, z_{k+1} + (1-\alpha) z_k, z_{k+2}, \dots, z_N)^\top$$

satisfies both $\bar{z}^\top \mathbb{1}_{N-1} = 0$ and $\bar{M}\bar{z} > 0$, with $\bar{M}$ corresponding to the time grid $\{t_1, \dots, t_N\} \setminus \{t_k\}$. Since this is again impossible due to the induction hypothesis, the proof is finished. □

Remark 2.5.5. From (2.8) it follows, that $\det M > 0$ for $\alpha < 2\beta$. Since M is positive definite for $\alpha = 0$ (since in this case, G is a positive definite function) and the eigenvalues depend continuously on α, we know that M is strictly positive definite for $\alpha < 2\beta$. So there are positive expected liquidation costs for $\alpha \le 2\beta$, what was also shown in Proposition 2.4.5.

2.5.2 Derivation of the revenues

For continuous strategies $(X_t)_{t \ge 0}$, it is natural to define the revenues of the investor by $\mathcal{R}_T = -\int_0^T P_t \, dX_t$, since at time t the investor buys dX_t shares at price P_t.

There are three methods for deriving the revenues from non-continuous strategies, all leading to the same result. First, one may approximate a non-continuous strategy by suitable continuous strategies and take the limit of their revenues.

Second, linear impact corresponds to an order book with constant height. Calculating the revenues of a market order in such a limit order book yields the revenues of a jump in the trading strategy. This approach is presented in more detail in Alfonsi *et al.* (2012).

Third, observe that a trade of ΔX_t shares at time t moves the price from P_{t-} to $P_{t-} + g_t G(0) \Delta X_t$, where $P_{t-} = \lim_{s \uparrow t} P_s$. It is natural to assume that this trade takes place at the mid-price, i.e. $P_{t-} + \frac{1}{2} g_t G(0) \Delta X_t$. In the following, we show how the revenues (2.1) can be derived from this assumption.

The continuous part of the trading strategy is given by $X_t^C = X_t - \sum_{s<t} \Delta X_s$.

The revenues are then given by

$$\begin{aligned}
\mathcal{R}_T &= -\int_0^T P_t\,dX_t^C - \sum_t \Delta X_t \left(P_{t-} + \frac{1}{2} g_t G(0) \Delta X_t\right) \\
&= -\int_0^T P_t^0\,dX_t^C - \int_0^T \int_{s<t} g_s G(t-s)\,dX_s\,dX_t^C \\
&\quad - \sum_t \Delta X_t \left(P_t^0 + \int_{s<t} g_s G(t-s)\,dX_s\right) - \frac{1}{2}\sum_t g_t G(0)(\Delta X_t)^2 \\
&= -\int_0^T P_t^0\,dX_t - \int_0^T \int_{s<t} g_s G(t-s)\,dX_s\,dX_t - \frac{1}{2}\sum_t g_t G(0)(\Delta X_t)^2 \\
&= X_0 P_0^0 + \int_0^T X_t\,dP_t^0 - \int_0^T \int_{s<t} g_s G(t-s)\,dX_s\,dX_t - \frac{1}{2}\sum_t g_t G(0)(\Delta X_t)^2.
\end{aligned}$$

Chapter 3

A Market Impact Model with Dark Pool

3.1 Introduction

Recent years have seen a mushrooming of alternative trading platforms called *dark pools*. Orders placed in a dark pool are not visible to other market participants (hence the name) and thus do not influence the publicly quoted price of the asset. Thus, when dark-pool orders are executed against a matching order, no direct price impact is generated, although there may be certain indirect effects. Dark pools therefore promise a reduction of market impact and of the resulting liquidation costs. They are hence a popular platform for the execution of large orders.

Dark pools differ from standard limit order books in that they do not have an intrinsic price finding mechanism. Instead, the price at which orders are executed is derived from the publicly quoted prices at an exchange. Thus, by manipulating the price at the exchange through placing suitable buy or sell orders, the value of a possibly large amount of "dark liquidity" in the dark pool can be altered. For this reason, dark pools have drawn significant attention by regulators; see IOSCO (2011). We refer to Mittal (2008) for a practical overview on dark pools and some related issues of market manipulation. Dark pools were also considered by Altunata *et al.* (2010), Buti *et al.* (2010), Comerton-Forde and Putniņš (2012), Degryse *et al.* (2009), Ganchev *et al.* (2010), Preece (2012), Ray (2010), Ready (2012), Ye (2010) and Zhu (2012)

In this chapter, we consider a stochastic model for order execution at two possible venues: a dark pool and an exchange. This model is a continuous-time variant of the one proposed by Kratz and Schöneborn (2010). It is a natural model, because it extends the standard Almgren–Chriss market impact model for exchange prices by a dark pool, where incoming matching orders are described by a compound Poisson process. We refer to Almgren (2003) for details on the Almgren–Chriss model and also to Bertsimas and Lo (1998) for a discrete-time precursor. A different approach to modeling and analyzing dark pools was proposed by Laruelle *et al.* (2011).

Kratz and Schöneborn (2010) mainly investigate optimal order execution strategies for an investor who can trade at the exchange and in the dark pool. But they are also interested in price manipulation strategies in the sense of Huberman and Stanzl (2004). Their Propositions 7.1 and 7.2 provide some first results on the existence and the absence of such strategies, and they propose the further investigation of this problem. We refer to Huberman and Stanzl (2004), Gatheral (2010), Alfonsi *et al.* (2012), and our Section 3.3 for discussions on the importance of the absence of price manipulation strategies. In Section 3.3 we will argue in particular that the absence of price manipulation and related concepts can be regarded as a regularity condition that plays a similar role for a market impact model as the absence of arbitrage for a derivatives pricing model.

Our main goal in this chapter is to investigate in a systematic manner the existence and absence of price manipulation with dark pools and related topics. To this end, we modify the setup of Kratz and Schöneborn (2010) in several ways. On the one hand, we simplify their setup by using the concrete continuous-time, single-asset Almgren–Chriss model to describe market impact at the exchange and by restricting the possibilities for adjusting the sizes of orders in the dark pool[1]. On the other hand, we allow for additional possibilities of cross impact between the two venues and for additional "slippage" in dark-pool execution.

In Section 3.4.1, our first main result characterizes completely those models from our class that are sufficiently regular for all underlying Almgren–Chriss models, either in the sense of the absence of price manipulation or in terms of the new condition of "positive expected liquidation costs". The critical quantities will be the size of "slippage" and the degrees of permanent and temporary cross-venue impact. In Section 3.4.2, we then investigate the existence of model irregularities for special model characteristics. It will turn out that generation of such irregularities hinges in a subtle way on the interplay of all model parameters and on the liquidation time constraint. In Section 3.4.3 we illustrate in a simplified setting that our regularity condition guarantees the existence of optimal order execution strategies, and we show how such strategies can be computed.

The chapter is organized as follows. In the subsequent Section 3.2 we introduce the model and formulate our standing assumptions. In Section 3.3 we review and discuss several notions for the regularity of a market impact model, namely the absence of standard and transaction-triggered price manipulation and a new condition of positive expected liquidation costs. Our main results are stated in Section 3.4 and proved in Section 3.5.

[1] Kratz and Schöneborn (2010) allow for arbitrary adaptive adjustment of the sizes of orders in the dark pool. In our model, these orders can only be placed at the beginning of the trading period, and their remainder can be cancelled at a later time. The possibility of arbitrary adaptive adjustment of dark-pool orders influences the particular form of optimal order execution strategies, but it does not have a significant impact on the existence and absence of price manipulation in comparison to our setting, at least if we exclude so-called 'fishing' strategies (see Remark 3.3.5). Most dark pools operating in practice will probably have order placement and cancellation policies which lie in between these two possibilities.

3.2 Model setup

We will analyze a continuous-time variant of the market impact model with dark pool that was proposed in Kratz and Schöneborn (2010). This model is natural since it extends the continuous-time version of the standard Almgren–Chriss market impact model for an investor who can generate price impact by trading at an exchange; see Almgren (2003) for details on this model and also Bertsimas and Lo (1998) for a discrete-time precursor. The Almgren–Chriss model has been the basis of many academic studies pertaining to market impact and is also common in industry application.

In the Almgren–Chriss market impact model, it is assumed that the number of shares in the trader's portfolio is described by an absolutely continuous trajectory $t \mapsto X_t$, the *trading strategy*. Given this trading trajectory, the price at which transactions occur is

$$P_t = P_t^0 + \gamma(X_t - X_0) + h(\dot{X}_t). \tag{3.1}$$

Here, P_t^0 is the *unaffected stock price process*. The term $h(\dot{X}_t)$ describes the *temporary* or *instantaneous impact* of trading $\dot{X}_t\,dt$ shares at time t and only affects this current order. The term $\gamma(X_t - X_0)$ corresponds to the *permanent price impact* that has been accumulated by all transactions until time t. It is usually assumed to be linear in $X_t - X_0$ with γ denoting a positive constant, because linearity is also needed so as to exclude price manipulation; see Huberman and Stanzl (2004) or Gatheral (2010), see also Almgren *et al.* (2005) for empirical justification.

Assumption 3.2.1. We assume that the unaffected stock price process $(P_t^0)_{t\geq 0}$ is a càdlàg martingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ for which $\mathcal{F}_0$ is $\mathbb{P}$-trivial. The permanent-impact parameter γ is assumed to be strictly positive. The temporary-impact function $h : \mathbb{R} \to \mathbb{R}$ is assumed to have the following properties: h is continuous, strictly increasing, and satisfies $h(0) = 0$ and $|h(x)| \to \infty$ for $|x| \to \infty$. Moreover, the function $f(x) := xh(x)$ is assumed to be convex.

The condition that P^0 is a martingale is a standard assumption in the market impact literature. One reason is that drift effects can be ignored due to the usually short trading horizons. In addition, we are interested here in the qualitative effects of price impact on the stability of the model. A nonzero drift would lead to the existence of profitable "round trips" that would have to be distinguished from price manipulation strategies in the sense of Definition 3.3.1. Our assumptions on h are satisfied for the popular choices of linear temporary impact, $h(x) = \eta x$, or more generally for power-law impact,

$$h(x) = \eta \,\mathrm{sgn}(x)|x|^\nu \tag{3.2}$$

where η and ν are positive constants and, typically, $\nu \leq 1$; see Almgren *et al.* (2005) for a discussion. An Almgren–Chriss model is thus defined in terms of the parameters

$$(\gamma, h, P^0). \tag{3.3}$$

The Almgren–Chriss model is a market impact model for exchange-traded orders. We will now extend this model by allowing the additional execution of orders in a *dark pool.* A dark pool is an alternative trading venue in which unexecuted orders are invisible to all other market participants. In this dark pool, buy and sell orders are matched and executed at the current price at which the asset is traded at the exchange.

In addition to a trading strategy executed at the exchange, investors can place an order of $\hat{X}$ shares into the dark pool at time $t = 0$. This order will be matched with incoming orders of the opposite side. These orders arrive at random times $0 < \tau_1 < \tau_2 < \dots$ and we denote the size of incoming matching orders by $\tilde{Y}_1, \tilde{Y}_2, \dots > 0$. We consider only those orders that are a possible match. That is, the $\tilde{Y}_i$ will describe sell orders when $\hat{X} > 0$ is a buy order and buy orders when $\hat{X} < 0$ is a sell order. These incoming orders will then be matched piece by piece with the order $\hat{X}$ until it is cancelled or completely filled. That is,

$$Y_i := \begin{cases} \operatorname{sgn}(\hat{X})\tilde{Y}_i, & \text{if } \sum_{j=1}^{i} \tilde{Y}_j \le |\hat{X}|, \\ \hat{X} - \operatorname{sgn}(\hat{X}) \sum_{j=1}^{i-1} \tilde{Y}_j, & \text{if } \sum_{j=1}^{i-1} \tilde{Y}_j \le |\hat{X}| \text{ and } \sum_{j=1}^{i} \tilde{Y}_j > |\hat{X}|, \\ 0, & \text{if } \sum_{j=1}^{i-1} \tilde{Y}_j > |\hat{X}|, \end{cases}$$

is the part of the incoming order that is actually executed against the remainder of $\hat{X}$. By defining the counting process associated with the arrival times (τ_k),

$$N_t := \max\{k \in \mathbb{N} \mid \tau_k \le t\}, \tag{3.4}$$

the amount of shares that have been executed in the dark pool until time t can be conveniently denoted by

$$Z_t := \sum_{i=1}^{N_t} Y_i.$$

By $(\mathcal{G}_t)$ we denote the rightcontinuous filtration generated by $(\mathcal{F}_t)$ and Z.

In the first part of the chapter, we make some very mild assumptions on the laws and interdependence of the random variables (τ_i), $(\tilde{Y}_i)$, and P^0:

Assumption 3.2.2. We assume the following conditions:

$$0 < \tau_1 < \infty \ \mathbb{P}\text{-a.s. and } \lambda_0 := \inf_{0<\delta\le 1} \frac{1}{\delta} \mathbb{P}[\, \tau_1 \le \delta \,] > 0; \tag{3.5}$$

$$\text{there exists } x_0 > 0 \text{ such that } \lambda_1 := \inf_{\delta > 0} \mathbb{P}[\, \tilde{Y}_1 \ge x_0 \mid \tau_1 \le \delta \,] > 0. \tag{3.6}$$

We furthermore assume that P^0 does not jump in $\tau_1, \tau_2, \dots$ and that P^0 is a martingale also under the filtration $(\mathcal{G}_t)$ generated by $(\mathcal{F}_t)$ and Z.

Condition (3.5) means that the intensity for the arrival of the first matching order is bounded away from zero. Condition (3.6) states that there is a positive probability that the first incoming matching order has at least size x_0, conditional on the event that $\{\tau_1 \le \delta\}$. Clearly, these assumptions are very mild. The requirement that P^0 is a $(\mathcal{G}_t)$-martingale allows (τ_i) and $(\tilde{Y}_i)$ to depend on P^0 in an arbitrary manner but, conversely, limits the dependence of P^0 on these random variables. This limitation is entirely natural since we will explicitly model the possible dependence of exchange-quoted prices on dark-pool executions via (3.9). Note that Assumption 3.2.2 is satisfied in particular when τ_1 has an exponential distribution and (τ_i), $(\tilde{Y}_i)$, and P^0 are independent random variables, as we will assume in the second part of the chapter.

Now we consider an investor who must liquidate an initial asset position of $X_0 \in \mathbb{R}$ shares during the time interval $[0, T]$. The problem of how to do this in an optimal fashion is known as the optimal order execution problem; see, e.g., Gökay *et al.* (2011), Schöneborn (2008), Schied and Slynko (2011), and the references therein.

In the extended dark pool model, the investor will first place an order of $\hat{X} \in \mathbb{R}$ shares in the dark pool[2] and then choose a liquidation strategy of Almgren–Chriss-type for the execution of the remaining assets at the exchange. This latter strategy must be absolutely continuous in time. It will thus be described by a process (ξ_t) that parameterizes the speed by which shares are sold at the exchange. Moreover, until fully executed, the remaining part of the order $\hat{X}$ can be cancelled at a (possibly random) time $\rho < T$. Hence, the number of shares held by the investor at time t is

$$X_t := X_0 + \int_0^t \xi_s \, ds + Z^\rho_{t-}, \tag{3.7}$$

where Z^ρ_{t-} denotes the left-hand limit of $Z^\rho_t = Z_{\rho \wedge t}$.

Definition 3.2.3. Let an initial position $X_0 \in \mathbb{R}$ and a liquidation horizon $T > 0$ be given. An *admissible trading strategy* is a triple $\chi := (\hat{X}, \xi, \rho)$ where $\hat{X} \in \mathbb{R}$, ρ is a $(\mathcal{G}_t)$-stopping time such that $\rho < T$ $\mathbb{P}$-a.s., and ξ is a $(\mathcal{G}_t)$-predictable process that is $\mathbb{P}$-a.s. bounded uniformly in t and ω. In addition, the liquidation constraint

$$X_0 + \int_0^T \xi_t \, dt + Z_\rho = 0 \tag{3.8}$$

must be $\mathbb{P}$-a.s. satisfied. The set of all admissible strategies for given X_0 and T is denoted by $\mathcal{X}(X_0, T)$.

Due to (3.7) and (3.8), the terminal asset position of any admissible strategy is $X_T = 0$, since our requirement $\rho < T$ implies that $Z^\rho_{T-} = Z_\rho$.

Now we turn to the definition of the prices at which the orders at the exchange and in the dark pool are executed. In particular, we will specify the cross impacts of order execution in the dark pool on the exchange price and vice versa. Here, our approach is to introduce a model that is flexible enough to allow for a wide range of

[2] If at time $t = 0$ the dark pool contains an order $\tilde{Y}_0$ of the opposite side, then the investor could fill this order immediately and then start liquidating the remaining asset position $X_0 - \tilde{Y}_0$, maybe by resizing the dark-pool order. Therefore we can assume that the dark pool does not contain a matching order at $t = 0$. Moreover, restricting the placement of dark-pool orders to $t = 0$ lets us exclude so-called 'fishing' strategies; see Remark 3.3.5 below.

possible mutual influences of orders executed on both venues. Extending (3.1), the price at which assets can be traded at the exchange is defined as

$$P_t = P_t^0 + \gamma\left(\int_0^t \xi_s\,ds + \alpha Z_{t-}^{\rho}\right) + h(\xi_t). \tag{3.9}$$

Here $\alpha \in [0,1]$ describes the intensity of the possible permanent impact of an execution in the dark pool on the price quoted at the exchange. The existence of such a cross-venue impact can be made plausible by noting that without the dark pool the matching order would have been executed at the exchange and there would have generated permanent price impact in a favorable direction. Thus, the price impact generated by the execution of a dark-pool order can be understood in terms of a deficiency in opposite price impact.

The price at which the i^{th} incoming order is executed in the dark pool will be

$$\begin{aligned}\hat{P}_{\tau_i} &= P_{\tau_i}^0 + \gamma\left(\int_0^{\tau_i} \xi_s\,ds + \alpha Z_{\tau_i-} + \beta Y_i\right) + g(\xi_{\tau_i})\\ &= P_{\tau_i} + \beta\gamma Y_i + (g(\xi_{\tau_i}) - h(\xi_{\tau_i})).\end{aligned} \tag{3.10}$$

In this price, orders executed at the exchange have full permanent impact, but their possible temporary impact is described by a function $g : \mathbb{R} \to \mathbb{R}$. The parameter $\beta \geq 0$ in (3.10) describes additional "slippage" related to the dark-pool execution, which will result in transaction costs of the size $\beta\gamma Y_i^2$. It may also be used to account for hidden costs, which relate to dark pools but which are extremely difficult to model explicitly. For instance, one can think of costs arising from the phenomena of adverse selection or 'fishing'; see Mittal (2008) and Kratz and Schöneborn (2010). Moreover, due to the very nature of dark pools, data may be sparse so that there will be a high degree of model uncertainty. The parameter β can thus also serve as a penalization of dark-pool orders in view of adverse selection, model misspecification, fishing (see Remark 3.3.5), and other hidden costs that are difficult to model explicitly. In this case, $\hat{P}_{\tau_i}$ in (3.10) is not the actual price at which the dark-pool order is executed, but it is a virtual adjusted price that includes hidden costs and penalties.

Assumption 3.2.4. We assume that $\alpha \in [0,1]$, $\beta \geq 0$, and that g either is identically zero or satisfies the conditions on h in Assumption 3.2.1.

Definition 3.2.5. The *dark-pool extension* of a given Almgren–Chriss model is defined in terms of the new parameters

$$(\alpha, \beta, g, (\tau_i), (\tilde{Y}_i)) \tag{3.11}$$

satisfying Assumptions 3.2.2 and 3.2.4.

Our main goal in this chapter is to study the influence of these parameters on the stability and regularity of the model and, in particular, on the optimal execution problem. Our investigation will be based on an analysis of the revenues generated by a trading strategy. In such a strategy, $\xi_t\,dt$ shares are bought at price P_t at each time t. In addition, Y_i shares are bought at price $\hat{P}_{\tau_i}$ at each time τ_i. The revenues generated by the strategy until time T are thus given by

$$\mathcal{R}_T = -\int_0^T \xi_s P_s \, ds - \sum_{i=1}^{N_{T\wedge\rho}} Y_i \hat{P}_{\tau_i}. \tag{3.12}$$

To emphasize the dependence of $\mathcal{R}_T$ on the strategy $\chi = (\hat{X}, \xi, \rho)$ we will sometimes also write $\mathcal{R}_T^\chi$.

3.3 Price manipulation

Our main concern in this chapter is to investigate the stability and regularity of the dark-pool extension in dependence on the parameters (γ, h, P^0) and $(\alpha, \beta, g, (\tau_i), (\tilde{Y}_i))$. This question is analogous to establishing the absence of arbitrage in a derivatives pricing model, where absence of arbitrage is a necessary condition for the existence of replicating strategies of a given contingent claim.

But there must also be a difference in the notions of regularity of a derivatives pricing model and of a market impact model. In a derivatives pricing model, one is interested in constructing strategies that almost surely replicate a given contingent claim, and this is the reason why one must exclude the existence of arbitrage opportunities defined in the usual almost-sure sense. In a market impact model, one is interested in constructing optimal order execution strategies. These strategies are not defined in terms of an almost-sure criterion but as minimizers of a cost functional of a risk averse investor. Commonly used cost functionals involve expected value as in Bertsimas and Lo (1998) and Gatheral (2010), mean-variance criteria as in Almgren and Chriss (2001), expected utility as in Schied and Schöneborn (2009) and Schöneborn (2008), or alternative risk criteria as in Forsyth *et al.* (2012) and Gatheral and Schied (2011). Therefore, also the regularity conditions to be imposed on a market impact model need to be formulated in a similar manner. To make such regularity conditions independent of particular investors preferences, it is reasonable to formulate them in a risk-neutral manner:

Definition 3.3.1 (Huberman and Stanzl (2004))**.** A *round trip* is an admissible trading strategy with $X_0 = 0$. A *price manipulation strategy* is a round trip that has strictly positive expected revenues, $\mathbb{E}[\mathcal{R}_T] > 0$.

When the revenues are a concave functional of an order execution strategy, as it is often the case, the existence of price manipulation precludes the existence of optimal execution strategies for risk-neutral investors, because one can generate arbitrarily large expected revenues by adding a multiple of a price manipulation strategy. In most cases, the same argument also works for risk-averse investors provided that risk aversion is small enough. The problem of characterizing the absence of price manipulation in a dark-pool model was formulated in Kratz and Schöneborn (2010), along with some first results in that direction. Analyses of the absence of price manipulation in various other market impact models were given, e.g., by Huberman and Stanzl (2004), Gatheral (2010), Alfonsi and Schied (2010), Alfonsi *et al.* (2012), and Gatheral *et al.* (2012).

It was observed in Alfonsi *et al.* (2012) that the absence of price manipulation may not be sufficient to guarantee the stability of the model, because optimal order execution strategies can still oscillate strongly between alternating buy and sell trades, a property one should exclude for various reasons: They seem not to be optimal in real markets, or when a spread between bid and ask prices are included in the model (which we omit for simplicity of the model). Furthermore, such trading strategies may be considered illegal market manipulation in many jurisdictions. This was the reason for introducing the following notion, where execution costs are defined as minus expected revenues.

Definition 3.3.2 (Alfonsi *et al.* (2012))**.** A market impact model admits *transaction-triggered price manipulation* if the expected execution costs of a sell (buy) program can be decreased by intermediate buy (sell) trades.

Equivalently, there is transaction-triggered price manipulation if there is a strategy where X_t is not monotone that has strictly higher expected revenues than all strategies where X_t is monotone a.s.

In our situation there would be transaction-triggered price manipulation if there exists $X_0 \in \mathbb{R}$ and a strategy $(\hat{X}, \xi, \rho)$ for which either $\hat{X}$ or some ξ_t have the same sign as X_0 and that has strictly higher expected revenues than all strategies $(\hat{X}', \xi', \rho')$ for which both $\hat{X}'$ and ξ_t' have always the opposite sign of X_0. We will also consider the following notion:

Definition 3.3.3. The model has *positive expected liquidation costs* if for all $X_0 \in \mathbb{R}$, $T > 0$, and every corresponding order execution strategy

$$\mathbb{E}[\mathcal{R}_T] \leq X_0 P_0^0. \tag{3.13}$$

Condition (3.13) states that on average it is not possible to make a profit beyond the face value of a position out of the market impact generated by one's own trades. We have the following hierarchy of regularity conditions in our model.

Proposition 3.3.4. (a) *If there is no transaction-triggered price manipulation, we have positive expected liquidation costs.*

(b) *If we have positive expected liquidation costs, then there is no price manipulation.*

Implication (a) holds for every market impact model in which buy orders increase the price and sell orders decrease the price. This implication is particularly useful in models where the condition of positive expected liquidation costs is violated, since it immediately yields the existence of transaction-triggered price manipulation. Implication (b) clearly holds for every market impact model.

Remark 3.3.5. A common price manipulation strategy is the so-called 'fishing' strategy in dark pools; see Mittal (2008). In a fishing strategy, agents first send small orders to dark pools so as to detect dark liquidity. Once a dark-pool order is detected, the visible price at the exchange is manipulated for a short period in a direction that is unfavorable for that order. Finally, a large order is sent to the dark pool so as to be executed against the dark liquidity at the manipulated price.

Here, we are not interested in the profitability of such predatory fishing strategies but primarily in the stability and regularity of optimal order execution algorithms in dark pool and exchange. We therefore exclude fishing strategies by allowing the placement of orders in the dark pool only at time $t = 0$. Allowing for the placement of dark-pool orders at times $t > 0$ will increase the class of admissible strategies, i.e. we would consider a class of admissible strategies $\tilde{\mathcal{X}} \supset \mathcal{X}$. Since $\sup_{\chi\in\tilde{\mathcal{X}}} \mathbb{E}[\mathcal{R}_T^{\chi}] \geq \sup_{\chi\in\mathcal{X}} \mathbb{E}[\mathcal{R}_T^{\chi}]$, in such an extended setting, the conditions of no-price manipulation or of positive expected liquidation costs will be violated as soon as they are violated in our present setting.

3.4 Results

An Almgren–Chriss model is specified by the parameters (γ, h, P^0) satisfying Assumption 3.2.1. Its extension incorporating a dark pool is based on the additional parameter set $(\alpha, \beta, g, (\tau_i), (\tilde{Y}_i))$, which will always be assumed to satisfy Assumptions 3.2.2 and 3.2.4. We are interested in the conditions we need to impose on these parameters such that the extended market model is regular. Here, regularity refers to the absence of price manipulation and related notions as explained in the preceding section.

3.4.1 General regularity results

Our first result characterizes completely those parameters $(\alpha, \beta, g, (\tau_i), (\tilde{Y}_i))$ for which the dark-pool extension of *every* Almgren–Chriss model is sufficiently regular for all time horizons.

Theorem 3.4.1. *For given $(\alpha, \beta, g, (\tau_i), (\tilde{Y}_i))$, the following conditions are equivalent.*

(a) *For any Almgren–Chriss model, the dark-pool extension has positive expected liquidation costs.*

(b) *For any Almgren–Chriss model, the dark-pool extension does not admit price manipulation for every time horizon $T > 0$.*

(c) *We have $\alpha = 1, \beta \geq \frac{1}{2}$ and $g = 0$.*

Remark 3.4.2. Let us comment on the three conditions in part 3 of the preceding theorem.

(i) The requirement $\alpha = 1$ means that an execution of a dark-pool order must generate the same permanent impact on the exchange-quoted price as a similar order that is executed at the exchange. At first sight, this requirement might seem artificial. At second thought, however, one realizes that price impact generated by the execution of a dark-pool order can be understood in terms of a deficiency in opposite-price impact; see the discussion following (3.9).

(ii) The condition $\beta \geq \frac{1}{2}$ means that the execution of a dark-pool order of size Y_i needs to generate "slippage" of at least $\frac{\gamma}{2}Y_i^2$. This latter amount is just equal to the costs from permanent impact one would have incurred by executing the order at the exchange. With this amount of slippage, the savings by executing an order not at the exchange but at a dark pool would thus be equal to the costs generated by permanent impact. It seems that dark pools that are currently operative do not charge transaction costs or taxes of this magnitude. Nevertheless, our theorem states that a penalization with a factor $\beta \geq \frac{1}{2}$ is needed for a robust stabilization of the model against irregularities.

(iii) The requirement $g = 0$ means that temporary impact from trades executed at the exchange must not affect the price at which dark-pool orders are executed. This requirement may not be surprising, although the $(\mathcal{G}_t)$-predictability of the exchange-traded part (ξ_t) of an admissible strategy excludes short-term manipulation in immediate response of the arrival of a matching order in the dark pool.

In Theorem 3.4.1, it is crucial that we may vary at least the parameter h of the underlying Almgren-Chriss model. If all parameters are fixed, we can only obtain the following implication instead of an equivalent characterization of regular models.

Theorem 3.4.3. *Suppose an Almgren–Chriss model with parameters (γ, h, P^0) has been fixed. When a dark-pool extension $(\alpha, \beta, g, (\tau_i), (\tilde{Y}_i))$ of this model does not admit price manipulation for all $T > 0$, then*

$$\beta \geq \alpha - \frac{1}{2}. \tag{3.14}$$

If, in addition, there is equality in (3.14) and $g(x) = \kappa h(x)$ for some constant $\kappa \geq 0$, then $\kappa = 0$ and $\alpha = 1$.

In the next section, we will analyze several concrete situations in which some of the model parameters are chosen in a particular way. Our corresponding results will first illustrate that (3.14) cannot be improved to a result as strong as in part (c) in the previous Theorem 3.4.1. For instance, it will follow from Corollary 3.4.11 that even in the case $\alpha = \beta = 0$ it may happen that there is no price manipulation for all $T > 0$, but this situation is then characterized in terms of relations between γ, h, and the law of τ_1.

3.4.2 Regularity and irregularity for special model characteristics

In this section, we will investigate in more detail the regularity and irregularity of a dark-pool extension of a *fixed* Almgren–Chriss model. To this end, we will assume throughout this section that slippage is zero, $\beta = 0$, which is the natural (naive) first guess in setting up a dark-pool extension of an Almgren–Chriss model. We know from Theorem 3.4.1, though, that there must be some Almgren–Chriss model such that there is price manipulation for sufficiently large time horizon T.

First, we will look into the role played by T in the existence of price manipulation. We show there exists a critical threshold $T^* \geq 0$ such that there is no price manipulation for $T < T^*$ but price manipulation does exist for $T > T^*$. We will show that all three cases $T^* = \infty$, $0 < T^* < \infty$, and $T^* = 0$ can occur. Second, we will analyze the stronger requirements of absence of transaction-triggered price manipulation and of positive expected liquidation costs. We will find situations in which there is no price manipulation for all $T > 0$ but where the condition of positive expected liquidation costs fails and where there is transaction-triggered price manipulation for sufficiently large T.

We will make the following simplifying but natural assumption on the dark-pool extension defined through $(\alpha, \beta, g, (\tau_i), (\tilde{Y}_i))$.

Assumption 3.4.4. We assume the following conditions throughout Section 3.4.2.

(a) Slippage is zero: $\beta = 0$.

(b) The process (N_t), as defined in (3.4), is a standard Poisson process with parameter $\theta > 0$ and $(\tilde{Y}_i)$ are i.i.d. random variables with common distribution μ on $(0, \infty]$. We also assume that the stochastic processes (P_t^0), (N_t), and $(\tilde{Y}_i)$ are independent.

Note that Assumption 3.4.4 (b) implies that

$$\lim_{t \uparrow \infty} \sum_{i=1}^{N_t} Y_i = \hat{X} \qquad \mathbb{P}\text{-a.s.} \tag{3.15}$$

Note also that we do not exclude the possibility that $\tilde{Y}_i$ takes the value $+\infty$ with positive probability. The particular case $\tilde{Y}_i = +\infty$ $\mathbb{P}$-a.s., corresponding to $\mu = \delta_\infty$, can be regarded as the limiting case of infinite liquidity in the dark pool. It results in $Y_1 = \hat{X}$ and hence in an immediate execution of the entire order $\hat{X}$. In fact, many dark pools allow the specification of lower limits on the size of matching orders, for instance to avoid the effects of 'fishing'. So, in principle, it should be possible to set this lower limit equal to $\hat{X}$. Unless $\mu = \delta_\infty$, setting such a limit will however lower the arrival rate θ of matching orders.

In Propositions 3.4.5 and 3.4.7, we will consider the situation in which the execution of a dark-pool order has full permanent impact on the price at the exchange, i.e., $\alpha = 1$. In view of our assumption $\beta = 0$, Theorem 3.4.3 implies that there will be price manipulation for sufficiently large T. The following proposition shows that one then can also generate arbitrarily large expected revenues. In contrast to the situation in many other market impact models, this conclusion is not obvious in our case, because the expected revenues are typically not a concave functional of admissible strategies.

Proposition 3.4.5. *Suppose that an Almgren–Chriss model has been fixed and that* $\alpha = 1$. *Then, for any* $X_0 \in \mathbb{R}$,

$$\lim_{T\uparrow\infty} \sup_{\chi\in\mathcal{X}(X_0,T)} \mathbb{E}[\mathcal{R}_T^{\chi}] = +\infty.$$

In particular, the condition of positive expected liquidation costs is violated.

Now we examine in more detail the role played by T in the existence of price manipulation. First, we show that for a certain class of models there is no price manipulation for small T.

Proposition 3.4.6. *Let* $g = 0$ *and* $h(x) = \eta x$. *If* $T \leq \frac{2\eta}{\gamma}$, *then there is no price manipulation. If moreover* $\alpha = 1$, *then there is no price manipulation if and only if* $T \leq \frac{2\eta}{\gamma}$.

Since the class $\mathcal{X}(X_0, T)$ of admissible strategies increases with T, the existence of price manipulation for one T implies the existence of price manipulation for any $T' \geq T$. Hence there exists a critical value T^* such that there is no price manipulation for $T < T^*$ but price manipulation does exist for $T > T^*$. For $\alpha = 1$ we have that $T^* = \frac{2\eta}{\gamma}$. In the next proposition, we show that $T^* = 0$ for sublinear impact and $\mu = \delta_\infty$ (which can be interpreted as setting the minimum execution size in the dark pool to $\hat{X}$).

Proposition 3.4.7. *Suppose that an Almgren–Chriss model has been fixed and that* $\alpha = 1$. *If* $\mu = \delta_\infty$ *and* h *has sublinear growth, i.e.,*

$$\lim_{|x|\to\infty} \frac{h(x)}{x} = 0,$$

then there is price manipulation for every $T > 0$.

After considering the case $\alpha = 1$, we will assume in the following that

$$\alpha = 0, \qquad g = 0 \qquad \text{and} \qquad h(x) = \eta x, \tag{3.16}$$

in addition to Assumption 3.4.4. Note that we assume $g = 0$ since this is required by Theorems 3.4.1 and 3.4.3 for regularity. Other choices for g will usually make the model less regular. By Theorem 3.4.1, we know that there exists an Almgren–Chriss model for which there is price manipulation and for which the condition of positive expected liquidation costs is violated for sufficiently large T. Our aim is to give a refined analysis for the class of Almgren–Chriss models with linear temporary price impact. We first take a look at the condition of positive expected liquidation costs.

Proposition 3.4.8. *Consider a fixed Almgren–Chriss model and suppose that condition* (3.16) *holds.*

(a) *If* $\frac{\gamma}{\eta} < 2\theta$, *we have for any* $X_0 \in \mathbb{R} \setminus \{0\}$,

$$\lim_{T\uparrow\infty} \sup_{\chi\in\mathcal{X}(X_0,T)} \mathbb{E}[\mathcal{R}_T^{\chi}] \geq X_0 P_0^0 + \frac{1}{2}\gamma^2 X_0^2 \frac{1}{2\eta\theta - \gamma} > X_0 P_0^0. \tag{3.17}$$

(b) *If either* $\frac{\gamma}{\eta} = 2\theta$ *and* $X_0 \neq 0$ *or* $\frac{\gamma}{\eta} > 2\theta$*, then*

$$\lim_{T\uparrow\infty} \sup_{\chi\in\mathcal{X}(X_0,T)} \mathbb{E}[\mathcal{R}_T^{\chi}] = +\infty.$$

In particular, the condition of positive expected liquidation costs is violated in both cases.

Proposition 3.3.4 immediately yields that there is transaction-triggered price manipulation in the situations considered in Proposition 3.4.8. We are interested in the form of these manipulations.

Proposition 3.4.9. *Consider a fixed Almgren–Chriss model and suppose that condition* (3.16) *holds. If* $X_0 \geq 0$ *and* $\xi_t \leq 0$ *for all* t *(or* $X_0 \leq 0$ *and* $\xi_t \geq 0$ *for all* t*), then* $\mathbb{E}[\mathcal{R}_T] \leq X_0 P_0^0$*, i.e. the violation of positive expected liquidation costs in Proposition 3.4.8 can only be obtained by intermediate buy (sell) trades at the exchange during an overall sell (buy) program.*

Therefore, if T is large enough, only a strategy that manipulates the exchange-quoted price can be more profitable than other strategies.

Some of the preceding results can be strengthened in the infinite-liquidity limit $\mu = \delta_\infty$. We refer to the paragraph after Assumption 3.4.4 for a discussion of this condition. We first show that (3.17) actually becomes an equality.

Proposition 3.4.10. *Consider a fixed Almgren–Chriss model. Suppose moreover that condition* (3.16) *holds and that* $\mu = \delta_\infty$*. Then, for* $X_0 \in \mathbb{R}$ *and* $\frac{\gamma}{\eta} < 2\theta$*,*

$$\lim_{T\uparrow\infty} \sup_{\chi\in\mathcal{X}(X_0,T)} \mathbb{E}[\mathcal{R}_T^{\chi}] = X_0 P_0^0 + \frac{1}{2}\gamma^2 X_0^2 \frac{1}{2\eta\theta - \gamma}. \tag{3.18}$$

Equation (3.18) is remarkable, because it implies on the one hand that the condition of positive expected liquidation costs is violated. By taking $X_0 = 0$ we see, on the other hand, that there is no price manipulation and $T^* = \infty$. In fact, we have the following result.

Corollary 3.4.11. *Consider a fixed Almgren–Chriss model. Suppose moreover that condition* (3.16) *holds and that* $\mu = \delta_\infty$*. Then there is no price manipulation for every* $T > 0$ *if and only if* $\frac{\gamma}{\eta} \leq 2\theta$*.*

By comparing the preceding result with Propositions 3.3.4 and 3.4.10, we arrive at the following statement.

Corollary 3.4.12. *Under the assumptions of Corollary 3.4.11 there is always transaction-triggered price manipulation for sufficiently large* T*. Standard price manipulation, however, exists only for* $\frac{\gamma}{\eta} > 2\theta$*.*

3.4.3 Optimal order execution strategies

In this section, we illustrate some of our results by determining an optimal strategy for selling $X_0 > 0$ shares. To this end, we will make a number of simplifying assumptions, because our main goal is to analyze the regularity of the model. In particular, for us, optimality of a strategies refers to the maximization of the expected revenues. For a detailed analysis of optimal order execution strategies in a discrete-time model with dark pool we refer to Kratz and Schöneborn (2010).

We fix an Almgren–Chriss model and assume that Assumption 3.4.4 (b) holds and that

$$\alpha = 1,\ \beta = \frac{1}{2},\ g = 0. \tag{3.19}$$

Then Theorem 3.4.1 guarantees that there is no price manipulation. For simplicity, we will also assume that there is infinite liquidity in the dark pool in the sense that

$$\mu = \delta_\infty. \tag{3.20}$$

Then the entirety of the dark-pool order $\hat{X}$ will either be filled when $\tau_1 \leq \rho$ or it will be cancelled when $\tau_1 > \rho$. In this setting, an admissible strategy $(\hat{X}, \xi, \rho)$ will be called a *single-update strategy* if ρ is a deterministic time in $[0, T)$ and ξ is predictable with respect to the filtration generated by the stochastic process $\mathbb{1}_{\{\tau_1 \leq t\}}$, $t \geq 0$.

Note that the process ξ of a single-update strategy evolves deterministically until there is an execution in the dark pool, i.e., until time τ_1. At that time, ξ can be updated. But the update will only depend on the time τ_1 and not on any other random quantities. In particular, ξ can be written as

$$\xi_t = \begin{cases} \xi_t^0, & \text{if } t \leq \tau_1 \text{ or } \tau_1 > \rho, \\ \xi_t^1, & \text{if } t > \tau_1 \text{ and } \tau_1 \leq \rho, \end{cases} \tag{3.21}$$

where ξ^0 is deterministic and ξ^1 depends on τ_1.

Proposition 3.4.13. *Suppose that Assumption 3.4.4* (b), (3.19), *and* (3.20) *hold. For any $X_0 \in \mathbb{R}$ and $T > 0$ there exists a single-update strategy that maximizes the expected revenues $\mathbb{E}[\mathcal{R}_T]$ in the class of all admissible strategies.*

Now we show how an optimal single-update strategy can be computed. To this end, we make the additional simplifying assumption that temporary impact is linear, $h(x) = \eta x$. It will follow from Equation (3.39) in the proof of Proposition 3.4.13 that the expected revenues of a single-update strategy are given by

$$\begin{aligned} \mathbb{E}[\mathcal{R}_T] = X_0 P_0^0 - \frac{1}{2}\gamma X_0^2 - \int_0^\rho \eta (\xi_s^0)^2 e^{-\theta s}\, ds - \eta e^{-\theta\rho} \frac{(X_0 + \int_0^\rho \xi_s^0\, ds)^2}{T - \rho} \\ - \int_0^\rho \eta\theta e^{-\theta t} \frac{(X_0 + \int_0^t \xi_s^0\, ds + \hat{X})^2}{T - t}\, dt. \end{aligned} \tag{3.22}$$

A standard calculation shows that the strategy $X_t^0 := X_0 - \int_0^t \xi_s^0\, ds$, $0 \leq t \leq \rho$, minimizing this expression is the solution of the Euler–Lagrange equation

$$-\ddot{X}_t^0 + \theta \dot{X}_t^0 + \theta \frac{X_t^0 + \hat{X}}{T - t} = 0 \tag{3.23}$$

with initial condition $X_0^0 = X_0$ and a terminal condition X_ρ^0 that will be determined later. By using the computer algebra software Mathematica, we found the analytic solution

$$\begin{aligned} X_t^0 = &-\hat{X}+ \\ &\Big(\theta T e^{\theta T}(T-\rho)(\mathrm{Ei}(-T\theta) - \mathrm{Ei}(\theta(\rho - T))) - \rho + T(1-e^{\theta\rho})\Big)^{-1}\Big\{ - e^{\theta t}\rho X_0 \\ &+(t-T)\big(X_0 e^{\theta\rho} - X_\rho^0 - \hat{X}\big) + \theta(T-t)e^{\theta T}\Big[\mathrm{Ei}((t-T)\theta)(T(X_0 - X_\rho^0 - \hat{X}) \\ &-\rho X_0) + X_0(\rho - T)\mathrm{Ei}(\theta(\rho - T)) + T(X_\rho^0 + \hat{X})\mathrm{Ei}(-T\theta)\Big] \\ &+e^{\theta t}T(X_0 - X_\rho^0 - \hat{X})\Big\}, \end{aligned}$$

where $\mathrm{Ei}(t) = \int_{-\infty}^t s^{-1}e^s\,ds$ is the exponential integral function. The constants ρ, $\hat{X}$ and X_ρ^0 can then be determined by optimizing the expression (3.22) numerically. Finally, the part ξ^1 of the strategy, which describes the trades to be executed at the exchange after time $\rho \wedge \tau_1$ is given by

$$\xi_t^1 = \begin{cases} \dfrac{-X_{\tau_1} - \hat{X}}{T - \tau_1} & \text{on } \{\tau_1 \le \rho\}, \\[2ex] \dfrac{-X_\rho}{T-\rho} & \text{on } \{\rho < \tau_1\}; \end{cases}$$

see (3.38) in the proof of Proposition 3.4.13.

3.4.4 Existence of transaction-triggered price manipulation and adaptive dark pool strategies

In this section, we show that there is transaction-triggered price manipulation in the model considered in the previous section. Furthermore, we show that transaction-triggered price manipulation disappears if we allow for an adaptive adjustment of the order in the dark pool, i.e. transaction-triggered price manipulation is an artifact from choosing the class of admissible strategies too restrictive.

Proposition 3.4.14. *Suppose that Assumption 3.4.4* (b), (3.19), *and* (3.20) *hold. If* $X_0 > 0$, *for the optimal strategy* $(\hat{X}^*, \xi^*, \rho^*)$ *we have that*

(i) $\hat{X}^* < 0$ *and*

(ii) if $\tau_1 \ge \rho^*$: $X_{\rho^*}^* + \hat{X}^* < 0$.

If we are in a sell program, (i) says that there the optimal order in the dark pool is always a sell order, while (ii) leads to the existence of transaction-triggered price manipulation:

Corollary 3.4.15. *Under the assumptions of Proposition 3.4.14, there is transaction-triggered price manipulation.*

The intuition is the following: If we may not adjust the position in the dark pool, we would have to cancel the dark pool order as soon as we sold $X_0 + X^*$ shares on the exchange to avoid a short position in the shares. However, it is better to wait with the cancellation at least a little bit longer: In this case, we risk a very small short position that has to be liquidated subsequently, but we save a larger amount on temporary impact in the exchange by the dark pool execution. But this means there is transaction-triggered price manipulation, although it might exist only with low probability (if the execution in the dark pool happens short before cancellation) and with a low number of shares bought during a sell program, depending on the parameters used.

Next, we show that there is no transaction-triggered price manipulation if we allow for adaptive strategies in the dark pool. To this end, let $\hat{X} = (\hat{X}_t)_{t\geq 0}$ be an adapted process. We assume again $\mu = \delta_\infty$. Then the number of shares executed in the dark pool up to time t is given by $Z_t = \mathbb{1}_{\{\{\tau_1 \leq t\}\}}\hat{X}_{\tau_1}$. Based on the observation that the optimal strategy in this setup is to place all remaining shares in the dark pool (i.e. $\hat{X}_t = -X_t$) as in Kratz and Schöneborn (2010), we have the following result.

Proposition 3.4.16. *Suppose that Assumption 3.4.4* (b), (3.19), (3.20) *hold and assume that f is C^2. Furthermore allow for adaptive strategies in the dark pool, i.e. let $\hat{X} = (\hat{X})_t$ be an adapted process. Then there is no transaction-triggered price manipulation.*

Given the assumptions of the preceding proposition, we find in the proof that the strategy X^0 (before the dark-pool execution) satisfies $f'(\dot{X}^0_t) = C\exp(\theta t)$ with a constant $C \in \mathbb{R}$ which can be determined from the constraint $\int_0^T \dot{X}^0_t\,dt = -X_0$. Thus, for linear impact corresponding to $f(x) = \eta x^2$ with $\eta > 0$, we find that $\dot{X}^0_t = -X_0\frac{\theta}{e^{\theta T}-1}\exp(\theta t)$ and $X^0_t = X_0\frac{e^{\theta T}-e^{\theta t}}{e^{\theta T}-1}$. Note that the optimal strategy does not depend on η.

3.5 Proofs

Recall from Assumption 3.2.2 that the martingale property of (P^0_t) is retained by passing to the enlarged filtration $(\mathcal{G}_t)$. Next, for an admissible strategy, the asset position process X defined in (3.7) is an admissible integrand for P^0 since it is leftcontinuous, $(\mathcal{G}_t)$-adapted, and hence $(\mathcal{G}_t)$-predictable. Recall also that $f(x) = x\,h(x)$.

Lemma 3.5.1. *The terminal revenues of an admissible strategy for given X_0 and T are given by*

$$\begin{aligned}\mathcal{R}_T &= X_0P^0_0 + \int_0^T X_t\,dP^0_t - \frac{1}{2}\gamma\Big(X_0 + \sum_{i=1}^{N_\rho} Y_i\Big)^2 - \int_0^T f(\xi_t)\,dt \\ &\quad - \sum_{i=1}^{N_\rho} Y_i\left(\gamma\int_0^{\tau_i}\xi_s\,ds - \alpha\gamma X_{\tau_i+} + \beta\gamma Y_i + g(\xi_{\tau_i})\right).\end{aligned}$$

Proof. First we prove that $-\int_0^T \xi_t P_t^0\,dt - \sum_{i=1}^{N_\rho} Y_i P_{\tau_i}^0 = X_0 P_0^0 + \int_0^T X_t\,dP_t^0$.

To this end, we first define $\tilde{X}_t := \int_0^t \xi_s\,ds$ and note that $\mathbb{P}$-a.s. $\tilde{X}_T = -X_0 - Z_{T-}^\rho = -X_0 - Z_T^\rho$. Since P^0 does not jump in τ_i, we have that $\mathbb{P}$-a.s. $P_{\tau_i-}^0 = P_{\tau_i}^0$. In particular, the quadratic co-variations $[P^0, N]$ and $[P^0, Z]$ vanish $\mathbb{P}$-a.s. It follows that $\mathbb{P}$-a.s.

$$\begin{aligned}
X_0 P_0^0 + \int_0^T X_t\,dP_t^0 &= X_0 P_0^0 + \int_0^T \left(X_0 + \int_0^t \xi_s\,ds + Z_{t-}^\rho\right) dP_t^0 \\
&= X_0 P_0^0 + X_0(P_T^0 - P_0^0) + \int_0^T \tilde{X}_t\,dP_t^0 + \int_0^T Z_{t-}^\rho\,dP_t^0 \\
&= -P_T^0 Z_{T-}^\rho - \int_0^T \xi_t P_t^0\,dt + Z_T^\rho P_T^0 - \int_0^T P_{t-}^0\,dZ_t^\rho \\
&= -\int_0^T \xi_t P_t^0\,dt - \sum_{i=1}^{N_\rho} Y_i P_{\tau_i-}^0 \\
&= -\int_0^T \xi_t P_t^0\,dt - \sum_{i=1}^{N_\rho} Y_i P_{\tau_i}^0.
\end{aligned}$$

Thus, $\mathbb{P}$-a.s.,

$$\begin{aligned}
\mathcal{R}_T &= -\int_0^T \xi_t P_t\,dt - \sum_{i=1}^{N_\rho} Y_i \hat{P}_{\tau_i} \\
&= -\int_0^T \xi_t \left(P_t^0 + \gamma\left(\int_0^t \xi_s\,ds + \alpha Z_{t-}^\rho\right) + h(\xi_t)\right) dt \\
&\quad - \sum_{i=1}^{N_\rho} Y_i \left(P_{\tau_i}^0 + \gamma\left(\int_0^{\tau_i} \xi_s\,ds + \alpha \sum_{j=1}^{i-1} Y_j\right) + \beta\gamma Y_i + g(\xi_{\tau_i})\right) \\
&= X_0 P_0^0 + \int_0^T X_t\,dP_t^0 - \gamma \int_0^T \int_0^t \xi_s\,ds\,\xi_t\,dt - \int_0^T \xi_t \gamma\alpha Z_{t-}^\rho\,dt - \int_0^T f(\xi_t)\,dt \\
&\quad - \sum_{i=1}^{N_\rho} Y_i \left(\gamma \int_0^{\tau_i} \xi_s\,ds + \alpha\gamma \sum_{j=i+1}^{N_\rho} Y_j + \beta\gamma Y_i + g(\xi_{\tau_i})\right) \\
&= X_0 P_0^0 + \int_0^T X_t\,dP_t^0 - \frac{1}{2}\gamma\left(\int_0^T \xi_t\,dt\right)^2 - \gamma\alpha \sum_{i=1}^{N_\rho} Y_i \int_{\tau_i}^T \xi_t\,dt - \int_0^T f(\xi_t)\,dt \\
&\quad - \sum_{i=1}^{N_\rho} Y_i \left(\gamma \int_0^{\tau_i} \xi_s\,ds + \alpha\gamma \sum_{j=i+1}^{N_\rho} Y_j + \beta\gamma Y_i + g(\xi_{\tau_i})\right) \\
&= X_0 P_0^0 + \int_0^T X_t\,dP_t^0 - \frac{1}{2}\gamma\left(X_0 + \sum_{i=1}^{N_\rho} Y_i\right)^2 - \int_0^T f(\xi_t)\,dt \\
&\quad - \sum_{i=1}^{N_\rho} Y_i \left(\gamma \int_0^{\tau_i} \xi_s\,ds - \alpha\gamma X_{\tau_i+} + \beta\gamma Y_i + g(\xi_{\tau_i})\right).
\end{aligned}$$

In the last step, we have again used the fact that $X_T = X_{T+} = 0$ $\mathbb{P}$-a.s. □

Proof of Proposition 3.3.4. (a): Assume $X_0 \geq 0$, and let the trading strategy be selling only, i.e. $\xi_t \leq 0$ for all t and $Y_i \leq 0$ for all i. Then $P_t \leq P_t^0$ for all t and $\hat{P}_{\tau_i} \leq P_{\tau_i}^0$ for all i. Using integration by parts, we find that

$$\mathcal{R}_T \leq -\int_0^T \xi_s P_s^0 \, ds - \sum_{i=1}^{N_{T\wedge\rho}} Y_i P_{\tau_i}^0 = X_0 P_0^0 + \int_0^T X_t \, dP_t^0.$$

Since (P^0) is a martingale, $\mathbb{E}[\mathcal{R}_T] \leq X_0 P_0^0$ for such a trading strategy. Absence of transaction-triggered price manipulation implies that the expected revenues cannot be increased by intermediate sell trades and therefore, we have $\mathbb{E}[\mathcal{R}_T] \leq X_0 P_0^0$ for all trading strategies. The case $X_0 \leq 0$ works analogously.

(b): By setting $X_0 = 0$ in (3.13) we find that $\mathbb{E}[\mathcal{R}_T] \leq 0$ for round trips. □

In the following, we will consider round trips which cancel the order in the dark pool after the first execution, i.e. $X_0 = 0$ and $\rho = \tau_1 \wedge r$ with some $r < T$. With Lemma 3.5.1 we find that the revenues of such a round trip are

$$\begin{aligned}\mathcal{R}_T \;\; = \;\; & \int_0^T X_t \, dP_t^0 - \frac{\gamma}{2} \mathbb{1}_{\{\tau_1 \leq r\}} Y_1^2 - \int_0^T f(\xi_t) \, dt \\ & -\mathbb{1}_{\{\tau_1 \leq r\}} Y_1 \left(\gamma X_{\tau_1 -} - \alpha\gamma (X_{\tau_1 -} + Y_1) + \beta\gamma Y_1 + g(\xi_{\tau_i}) \right).\end{aligned}$$

Furthermore, we will consider strategies that do not depend on P^0, i.e. they only depend on τ_1 and Y_1. In particular, these strategies can be written as

$$\xi_t = \begin{cases} \xi_t^0, & \text{if } t \leq \tau_1 \text{ or } \tau_1 > r, \\ \xi_t^1, & \text{if } t > \tau_1 \text{ and } \tau_1 \leq r, \end{cases} \tag{3.24}$$

where ξ^0 is deterministic and ξ^1 depends on τ_1 and Y_1. As in Section 3.4.3, we will call these strategies *single-update round trips.* We define further $X_t^0 = \int_0^t \xi_s^0 \, ds$.

The expected revenues of a single-update round trip are

$$\begin{aligned}\mathbb{E}[\mathcal{R}_T] \;\; = \;\; & -\int_0^r f(\xi_t^0) \mathbb{P}[\, t \leq \tau_1 \,] \, dt - \mathbb{P}[\, \tau_1 > r \,] \int_r^T f(\xi_t^0) \, dt \\ & -\mathbb{E}\left[\int_{\tau_1}^T f(\xi_t^1) \, dt; \tau_1 \leq r \right] \\ & -\mathbb{E}\Big[\gamma\phi Y_1^2 + \gamma(1-\alpha) X_{\tau_1}^0 Y_1 + g(\xi_{\tau_1}^0) Y_1 \, ; \tau_1 \leq r \Big]\end{aligned} \tag{3.25}$$

where

$$\phi := -\alpha + \frac{1}{2} + \beta. \tag{3.26}$$

We will next prove Theorem 3.4.3. The proof of Theorem 3.4.1 will be based on Theorem 3.4.3.

Proof of Theorem 3.4.3. We first show that we must have that $\phi \geq 0$ when there is no price manipulation. To this end, we assume by way of contradiction that $\phi < 0$ but that there is no price manipulation for all T. Consider the single-update round trip with $r = T/2$, $\hat{X} > 0$, and

$$\xi_t = \begin{cases} \frac{-2Y_1}{T} & \text{if } t > r \text{ and } \tau_1 \leq r \\ 0 & \text{otherwise.} \end{cases}$$

The expected revenues of this strategy satisfy

$$\begin{aligned} \mathbb{E}[\mathcal{R}_T] &= -\mathbb{E}\left[\frac{T}{2} f\left(-\frac{2Y_1}{T}\right) + \gamma\phi Y_1^2\,;\, \tau_1 \leq \frac{T}{2}\right] \\ &= \mathbb{E}\left[Y_1 h\left(-\frac{2Y_1}{T}\right) - \gamma\phi Y_1^2\,;\, \tau_1 \leq \frac{T}{2}\right]. \end{aligned}$$

The continuity of $h(x)$ at $x = 0$ yields that $h(-2Y_1/T) \nearrow 0$ for $T \uparrow \infty$. Dominated convergence and our assumption $\phi < 0$ hence imply that

$$\lim_{T\uparrow\infty} \mathbb{E}[\mathcal{R}_T] = -\gamma\phi\mathbb{E}\left[\, Y_1^2 \,\right] > 0.$$

It follows that for sufficiently large T the expected revenues are strictly positive, and so there is price manipulation. But this contradicts our assumption.

We now consider the special case in which $\phi = 0$ and $g(x) = \kappa h(x)$ for some $\kappa \geq 0$ and deduce $\kappa = 0$. By way of contradiction, we will show that there is price manipulation for sufficiently large T when $\kappa > 0$. Consider any single-update round trip. Assume that r is fixed and both ξ^0 and ξ^1 are fixed on $[0, r)$. When taking T arbitrarily large, we can liquidate the asset position $X_{\tau_1 \wedge r}$ arbitrarily slowly during $[r, T]$ and thus achieve that both $\xi_t^0 \searrow 0$ and $\xi_t^1 \searrow 0$ for $t \geq r$ as $T \uparrow \infty$. By sending T to infinity in (3.25), it follows that we can achieve via monotone convergence that

$$\begin{aligned} \lim_{T\uparrow\infty} \mathbb{E}[\mathcal{R}_T] = -\mathbb{E}\bigg[&\int_0^r f(\xi_t^0)\mathbb{1}_{\{t\leq\tau_1\}}\,dt \\ &+ \Big(\gamma\phi Y_1^2 + \gamma(1-\alpha)X_{\tau_1}^0 Y_1 + g(\xi_{\tau_1}^0)Y_1\Big)\mathbb{1}_{\{\tau_1\leq r\}}\bigg], \end{aligned} \tag{3.27}$$

where we keep the term with ϕ for the moment, although $\phi = 0$ here, because this and the subsequent formulas will also be used later on. Now we take some $\delta \in (0, 1)$, which will be specified later, and let $r = \delta$ and $\xi_t^0 = -\delta$ for $0 \leq t \leq \delta$. We also suppose that $\hat{X} > 0$. Then

$$\begin{aligned} \lim_{T\uparrow\infty} \mathbb{E}[\mathcal{R}_T] &= -\mathbb{E}\bigg[f(-\delta)\int_0^\delta \mathbb{1}_{\{t\leq\tau_1\}}\,dt \\ &\qquad + \Big(\gamma\phi Y_1^2 + Y_1\big(\gamma(1-\alpha)X_{\tau_1}^0 + g(-\delta)\big)\Big)\mathbb{1}_{\{\tau_1\leq\delta\}}\bigg] &(3.28) \\ &\geq -\delta f(-\delta) - \mathbb{E}\bigg[\Big(\gamma\phi Y_1^2 + Y_1 g(-\delta)\Big)\mathbb{1}_{\{\tau_1\leq\delta\}}\bigg] &(3.29) \\ &= -\kappa h(-\delta)\Big(\frac{\delta f(-\delta)}{\kappa h(-\delta)} + \mathbb{E}[\,Y_1\,|\,\tau_1 \leq \delta\,]\mathbb{P}[\,\tau_1 \leq \delta\,]\Big) \\ &\geq -\kappa h(-\delta)\delta\Big(-\frac{\delta}{\kappa} + \lambda_0\lambda_1(x_0 \wedge \hat{X})\Big), \end{aligned}$$

where we have used (3.5) and (3.6) in the last step. Due to the assumption $\lambda_0\lambda_1 x_0 > 0$, this expression is strictly positive as soon as $\delta > 0$ is small enough. This implies the desired existence of price manipulation for sufficiently large T.

We now show that we must have $\alpha = 1$ when $\phi = 0$ and $g = 0$. To this end, we assume by way of contradiction that $\alpha < 1$. As before, we may assume that (3.27) holds. When taking $r = 1$ and $\xi^0_t := -\delta\mathbb{1}_{[0,1]}$ for $\delta \in (0,1)$, we have $X^0_{\tau_1} = -\delta\tau_1$ on $\{\tau_1 \le r\}$, and (3.27) yields that

$$\begin{aligned}\lim_{T\uparrow\infty}\mathbb{E}[\mathcal{R}_T] &= -\mathbb{E}\Big[\int_0^1 f(-\delta)\mathbb{1}_{\{t\le\tau_1\}}\,dt + \gamma(1-\alpha)X^0_{\tau_1}Y_1\mathbb{1}_{\{\tau_1\le r\}}\Big]\\ &\ge \delta\Big[h(-\delta) + \gamma(1-\alpha)\mathbb{E}\big[\,\tau_1 Y_1\mathbb{1}_{\{\tau_1\le r\}}\,\big]\Big].\end{aligned}$$

But the latter expression is strictly positive as soon as δ is small enough, because $\mathbb{E}[\,\tau_1 Y_1\mathbb{1}_{\{\tau_1\le r\}}\,]$ is strictly positive by (3.5) and (3.6). Hence, $\alpha < 1$ implies the existence of price manipulation for sufficiently large T. $\square$

Proof of Theorem 3.4.1. The implication (a)$\Rightarrow$(b) follows immediately by taking $X_0 = 0$.

(b)$\Rightarrow$(c): We already know from Theorem 3.4.3 that we must have $\phi \ge 0$, where ϕ is as in (3.26). Thus, it remains to show that $g = 0$ and $\alpha = 1$.

We start by showing that $g = 0$. To this end, we assume by way of contradiction that there is no price manipulation but $g \neq 0$. Then g must satisfy the conditions on a temporary-impact function in Assumption 3.2.1. When $\phi = 0$, we can take $h := g$ and get the desired contradiction from the second part of Theorem 3.4.3. So let us now consider the case $\phi > 0$. To this end, we consider again the situation in the proof of Theorem 3.4.3 in which (3.27) holds and where $r = \delta$, $\delta \in (0,1)$, and $\xi^0_t = -\delta$ for $0 \le t \le \delta$. Then we have from (3.29) that

$$\lim_{T\uparrow\infty}\mathbb{E}[\mathcal{R}_T] \ge -\delta f(-\delta) - \gamma\phi\mathbb{E}\big[\,Y_1^2\mathbb{1}_{\{\tau_1\le\delta\}}\,\big] - g(-\delta)\mathbb{E}\big[\,Y_1\mathbb{1}_{\{\tau_1\le\delta\}}\,\big].$$

On the one hand, we have $0 < Y_1 = \tilde{Y}_1 \wedge \hat{X} \le \hat{X}$ and hence

$$\mathbb{E}\big[\,Y_1^2\mathbb{1}_{\{\tau_1\le\delta\}}\,\big] \le \hat{X}^2\mathbb{P}[\,\tau_1 \le \delta\,].$$

On the other hand, our assumption (3.6) implies that for all $\hat{X}$ such that $0 < \hat{X} \le x_0$ we have $\mathbb{E}[\,Y_1\,|\,\tau_1 \le \delta\,] \ge \lambda_1\hat{X}$. Thus, for $0 < \hat{X} \le x_0$ we have

$$\lim_{T\uparrow\infty}\mathbb{E}[\mathcal{R}_T] \ge -\delta f(-\delta) - \big(\gamma\phi\hat{X}^2 + g(-\delta)\lambda_1\hat{X}\big)\mathbb{P}[\,\tau_1 \le \delta\,].$$

Choosing $\hat{X} = -g(-\delta)\lambda_1/(2\gamma\phi)$ yields

$$\lim_{T\uparrow\infty}\mathbb{E}[\mathcal{R}_T] \ge -\delta f(-\delta) + \frac{g(-\delta)^2\lambda_1^2}{4\phi\gamma}\mathbb{P}[\,\tau_1 \le \delta\,] \ge \delta g(-\delta)^2\Big(\frac{\delta h(-\delta)}{g(-\delta)^2} + \frac{\lambda_1^2}{4\phi\gamma}\lambda_0\Big), \quad (3.30)$$

where we have used (3.5) in the last step. One can check that the function $h(x) := xg(x)^2$ satisfies Assumption 3.2.1. But with this choice, the right-hand side of (3.30) becomes strictly positive for sufficiently small $\delta > 0$, and we obtain price manipulation for sufficiently large T. This completes the proof of $g = 0$.

Now we show that we must have $\alpha = 1$. To this end, we assume by way of contradiction that $\alpha < 1$ and start from the identity (3.28), which holds for $r = \delta$, $\hat{X} > 0$, $\xi_t^0 = -\delta$ for $0 \le t \le \delta$, and a suitable choice for ξ^1 and ξ_t^0 $(t > \delta)$, depending on T. We take $\delta = 1$ and hence have $X_{\tau_1}^0 = -\tau_1$ on $\{\tau_1 \le 1\}$. Since $g = 0$, (3.28) implies that

$$\begin{aligned}\lim_{T\uparrow\infty}\mathbb{E}[\mathcal{R}_T] &= -\mathbb{E}\Big[f(-1)\int_0^1 \mathbb{1}_{\{t\le\tau_1\}}\,dt + \Big(\gamma\phi Y_1^2 - \gamma(1-\alpha)Y_1\tau_1\Big)\mathbb{1}_{\{\tau_1\le 1\}}\Big]\,dt\\ &\ge -f(-1) + \mathbb{E}\Big[\Big(-\gamma\phi Y_1^2 + \gamma(1-\alpha)Y_1\tau_1\Big)\mathbb{1}_{\{\tau_1\le 1\}}\Big]. \qquad (3.31)\end{aligned}$$

Next we consider Almgren–Chriss models with fixed permanent-impact parameter $\gamma > 0$ and with temporary impact function εh, where h is fixed and $\varepsilon > 0$. Suppose first that $\phi = 0$. Then we get in the limit $\varepsilon \downarrow 0$,

$$\lim_{\varepsilon\downarrow 0}\lim_{T\uparrow\infty}\mathbb{E}[\mathcal{R}_T] \ge \gamma(1-\alpha)\mathbb{E}\big[Y_1\tau_1\mathbb{1}_{\{\tau_1\le 1\}}\big] > 0,$$

which implies that there is price manipulation for small enough ε and large enough T.

For $\phi > 0$, we get

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\lim_{T\uparrow\infty}\mathbb{E}[\mathcal{R}_T] &\ge \mathbb{E}\Big[\Big(-\gamma\phi Y_1^2 + \gamma(1-\alpha)Y_1\tau_1\Big)\mathbb{1}_{\{\tau_1\le 1\}}\Big]\\ &= \gamma\phi\hat{X}\mathbb{E}\Big[\Big(-\frac{(\tilde{Y}_1\wedge\hat{X})^2}{\hat{X}} + \frac{1-\alpha}{\phi}\cdot\frac{Y_1\wedge\hat{X}}{\hat{X}}\tau_1\Big)\mathbb{1}_{\{\tau_1\le 1\}}\Big].\end{aligned}$$

But it is easy to see that the expectation on the right will be strictly positive as soon as $\hat{X}$ is sufficiently small, since

$$\frac{(Y_1\wedge\hat{X})^2}{\hat{X}} \longrightarrow 0 \qquad \text{and} \qquad \frac{Y_1\wedge\hat{X}}{\hat{X}} \longrightarrow 1 \qquad \text{as } \hat{X}\downarrow 0.$$

This shows that there is price manipulation for small enough ε and large enough T when $\alpha < 1$.

(c)⇒(a): Assume that $\alpha = 1$, $\beta \ge \frac{1}{2}$, $g \equiv 0$. Note that

$$\int_0^{\tau_i}\xi_s\,ds - X_{\tau_i+} = -\sum_{j=1}^{i} Y_i - X_0. \qquad (3.32)$$

With Lemma 3.5.1 we get for the revenues of an admissible strategy $(\hat{X}, \rho, \xi)$

$$\begin{aligned}\mathcal{R}_T &= X_0P_0^0 + \int_0^T X_t\,dP_t^0 - \frac{1}{2}\gamma\left(X_0 + \sum_{i=1}^{N_\rho} Y_i\right)^2 - \int_0^T f(\xi_t)\,dt\\ &\quad + \sum_{i=1}^{N_\rho} Y_i\left(\gamma\sum_{j=1}^{i} Y_i + \gamma X_0 - \beta\gamma Y_i\right)\\ &= X_0P_0^0 + \int_0^T X_t\,dP_t^0 - \frac{1}{2}\gamma X_0^2 - \int_0^T f(\xi_t)\,dt - \big(\beta - \frac{1}{2}\big)\gamma\sum_{i=1}^{N_\rho} Y_i^2.\end{aligned}$$

Therefore,

$$\mathbb{E}[\mathcal{R}_T] = X_0 P_0^0 - \frac{1}{2}\gamma X_0^2 - \mathbb{E}\left[\int_0^T f(\xi_t)\,dt\right] - \big(\beta - \frac{1}{2}\big)\gamma \mathbb{E}\left[\sum_{i=1}^{N_\rho} Y_i^2\right] \leq X_0 P_0^0.$$

This establishes (a). $\square$

Proof of Proposition 3.4.5. Let $X_0 \in \mathbb{R}$ and $\alpha = 1, \beta = 0$. The revenues for a strategy are given by

$$\begin{aligned}\mathcal{R}_T &= X_0 P_0^0 + \int_0^T X_t\,dP_t^0 - \frac{1}{2}\gamma X_0^2 - \int_0^T f(\xi_t)\,dt \\ &\quad + \frac{1}{2}\gamma \sum_{i=1}^{N_\rho} Y_i^2 - \sum_{i=1}^{N_\rho} Y_i g(\xi_{\tau_i}).\end{aligned}$$

Consider the following trading strategy with $\rho = \frac{T}{2}$ and given $\hat{X} \neq 0$,

$$\xi_t = \begin{cases} 0, & \text{if } 0 \leq t \leq \rho, \\ -2\frac{X_0 + Z_\rho}{T}, & \text{if } \rho < t \leq T. \end{cases}$$

The expected revenues of this strategy are

$$\begin{aligned}\mathbb{E}[\mathcal{R}_T] &= X_0 P_0^0 - \frac{1}{2}\gamma X_0^2 - \mathbb{E}\left[\frac{T}{2} f\left(-2\frac{X_0 + Z_{T/2}}{T}\right)\right] + \frac{1}{2}\gamma \mathbb{E}\left[\sum_{i=1}^{N_{T/2}} Y_i^2\right] \\ &= X_0 P_0^0 - \frac{1}{2}\gamma X_0^2 + \mathbb{E}\left[\big(X_0 + Z_{T/2}\big)\, h\left(-2\frac{X_0 + Z_{T/2}}{T}\right)\right] + \frac{1}{2}\gamma \mathbb{E}\left[\sum_{i=1}^{N_{T/2}} Y_i^2\right].\end{aligned}$$

Recall that $|Z_{T/2}|$ is bounded by $|\hat{X}|$ for all T and that Y_i is nonzero only as long as $|\sum_{j=1}^{i-1} \tilde{Y}_j| < |\hat{X}|$. Hence, with probability one, only finitely many Y_i are nonzero. Therefore, and by dominated convergence,

$$\lim_{T\uparrow\infty} \mathbb{E}[\mathcal{R}_T] = X_0 P_0^0 - \frac{1}{2}\gamma X_0^2 + \frac{1}{2}\gamma \mathbb{E}\left[\sum_{i=1}^{\infty} Y_i^2\right].$$

When sending $|\hat{X}|$ to infinity, $\sum_{i=1}^{\infty} Y_i^2$ tends to infinity with probability one. Hence, we can make $\lim_{T\uparrow\infty} \mathbb{E}[\mathcal{R}_T]$ arbitrarily large. So we can find a sequence of strategies $\chi_n \in \mathcal{X}(X_0, T_n)$ such that $\mathbb{E}[\mathcal{R}_{T_n}^{\chi_n}] \geq n$ which implies the assertion. $\square$

Proof of Proposition 3.4.6. (a): Lemma 3.5.1 and (3.32) yield

$$\begin{aligned}
\mathcal{R}_T &= \int_0^T X_t\,dP_t^0 - \frac{1}{2}\gamma\left(\sum_{i=1}^{N_\rho} Y_i\right)^2 - \eta\int_0^T \xi_t^2\,dt \\
&\quad -\gamma\sum_{i=1}^{N_\rho} Y_i\left((1-\alpha)\int_0^{\tau_i}\xi_s\,ds - \alpha\sum_{j=1}^{i} Y_j\right) \\
&= \int_0^T X_t\,dP_t^0 + \alpha\left(\frac{1}{2}\gamma\sum_{i=1}^{N_\rho} Y_i^2 - \eta\int_0^T \xi_t^2\,dt\right) \\
&\quad +(1-\alpha)\left(-\eta\int_0^T \xi_t^2\,dt - \gamma\sum_{i=1}^{N_\rho} Y_i\int_0^{\tau_i}\xi_s\,ds - \frac{1}{2}\gamma\left(\sum_{i=1}^{N_\rho} Y_i\right)^2\right).
\end{aligned}$$

Note that

$$\eta\int_0^T \xi_t^2\,dt \geq \frac{\eta}{T}\Big(\int_0^T \xi_t\,dt\Big)^2 = \frac{\eta}{T}\Big(\sum_{i=1}^{N_\rho} Y_i\Big)^2 \geq \frac{\eta}{T}\sum_{i=1}^{N_\rho} Y_i^2,$$

where we have used Jensen's inequality in the first step. Thus,

$$\frac{1}{2}\gamma\sum_{i=1}^{N_\rho} Y_i^2 - \eta\int_0^T \xi_t^2\,dt \leq \Big(\frac{\gamma}{2} - \frac{\eta}{T}\Big)\sum_{i=1}^{N_\rho} Y_i^2. \tag{3.33}$$

Furthermore, let $\Xi := \sup_{t\in[0,T]}|\int_0^t \xi_s\,ds|$. Then, by Jensen's inequality,

$$\int_0^T \xi_t^2\,dt \geq T\Big(\frac{1}{T}\int_0^T |\xi_t|\,dt\Big)^2 \geq \frac{\Xi^2}{T}.$$

We can estimate

$$-\eta\int_0^T \xi_t^2\,dt - \gamma\sum_{i=1}^{N_\rho} Y_i\int_0^{\tau_i}\xi_s\,ds \leq -\eta\frac{\Xi^2}{T} + \sum_{i=1}^{N_\rho}|Y_i|\gamma\Xi.$$

The right-hand side is maximized by

$$\Xi = \frac{\gamma T}{2\eta}\sum_{i=1}^{N_\rho}|Y_i|.$$

Therefore,

$$\begin{aligned}
&-\eta\int_0^T \xi_t^2\,dt - \gamma\sum_{i=1}^{N_\rho} Y_i\int_0^{\tau_i}\xi_s\,ds - \frac{1}{2}\gamma\left(\sum_{i=1}^{N_\rho} Y_i\right)^2 \\
&\quad \leq \frac{\gamma}{2}\left(\sum_{i=1}^{N_\rho}|Y_i|\right)^2\left(\frac{\gamma T}{2\eta} - 1\right).
\end{aligned} \tag{3.34}$$

Combining (3.33) and (3.34) yields

$$\begin{aligned}\mathcal{R}_T &\leq \int_0^T X_t\,dP_t^0 + \alpha\left(\frac{\gamma}{2} - \frac{\eta}{T}\right)\sum_{i=1}^{N_\rho} Y_i^2 + (1-\alpha)\frac{\gamma}{2}\left(\sum_{i=1}^{N_\rho}|Y_i|\right)^2\left(\frac{\gamma T}{2\eta} - 1\right) \\ &\leq \int_0^T X_t\,dP_t^0\end{aligned}$$

for $T \leq 2\eta/\gamma$ and we find $\mathbb{E}[\mathcal{R}_T] \leq 0$.

(b): Let now $\alpha = 1$. Necessity follows from part (a). For the proof of sufficiency, let us assume that $T > 2\eta/\gamma$. Then there exists $\varepsilon \in (0, T)$ such that

$$\frac{1}{2}\gamma - \frac{\eta}{T-\varepsilon} > 0.$$

Consider the round trip with $\rho = \tau_1 \wedge \varepsilon$, arbitrary $\hat{X} \neq 0$, and

$$\xi_t = \begin{cases} -\frac{Y_1}{T-\varepsilon}, & \text{if } t > \varepsilon \text{ and } \tau_1 \leq \varepsilon, \\ 0, & \text{otherwise.}\end{cases}$$

The expected revenues of this strategy are

$$\mathbb{E}[\mathcal{R}_T] = \left(\frac{1}{2}\gamma - \frac{\eta}{T-\varepsilon}\right)\mathbb{E}[Y_1^2; \tau_1 \leq \varepsilon] > 0.$$

Hence, there is price manipulation for $T > 2\eta/\gamma$. □

Proof of Proposition 3.4.7. Let $T > 0$ and fix $\hat{X}$ such that

$$\frac{\gamma}{2}\hat{X}^2 - \frac{T}{2}f\Big(2\frac{\hat{X}}{T}\Big) = \frac{\hat{X}^2}{2}\Big(\gamma - \frac{h(2\hat{X}/T)}{\hat{X}}\Big) > 0,$$

which is possible due to the sublinearity of h. Now we take $\rho = T/2$ and

$$\xi_t := \begin{cases} 0, & t \leq \rho, \\ 0, & t > \rho \text{ and } \tau > \rho, \\ -2\hat{X}/T, & t > \rho \text{ and } \tau \leq \rho.\end{cases}$$

The expected revenues of this strategy are

$$\begin{aligned}\mathbb{E}[\mathcal{R}_T] &= -\mathbb{E}\left[\int_0^T f(\xi_t)\,dt\right] + \int_0^\rho \theta e^{-\theta t}\left(\frac{1}{2}\gamma\hat{X}^2 + \hat{X}g(0)\right)dt \\ &= (1 - e^{-\theta\rho})\left(\frac{1}{2}\gamma\hat{X}^2 - \frac{T}{2}f\left(2\frac{\hat{X}}{T}\right)\right) \\ &> 0.\end{aligned}$$

So there is price manipulation. □

Now we prove the results pertaining to the assumptions that $\alpha = \beta = 0$, $g \equiv 0$, and $h(\xi) = \eta\xi$. Under this conditions, Lemma 3.5.1 yields

$$\mathcal{R}_T = X_0 P_0^0 + \int_0^T X_t \, dP_t^0 - \frac{1}{2}\gamma \left(X_0 + \sum_{i=1}^{N_\rho} Y_i \right)^2 - \eta \int_0^T \xi_t^2 \, dt - \sum_{i=1}^{N_\rho} Y_i \left(\gamma \int_0^{\tau_i} \xi_s \, ds \right). \tag{3.35}$$

Proof of Proposition 3.4.8. Proof of (a): Take $\rho = \frac{T}{2}$ and

$$\xi_t = \begin{cases} -\frac{\gamma}{2\eta}\hat{X}, & \text{if } t \le \tau_1, t \le \rho, \\ 0, & \text{if } t > \tau_1, t \le \rho \\ -\frac{X_{\rho+}}{\rho}, & \text{if } t > \rho, \end{cases}$$

where $\hat{X}$ will be specified later. By (3.35), we find that

$$\mathcal{R}_T = X_0 P_0^0 + \int_0^T X_t \, dP_t^0 - \frac{1}{2}\gamma \left(X_0 + \sum_{i=1}^{N_\rho} Y_i \right)^2 - \eta(\rho \wedge \tau_1) \frac{\gamma^2}{4\eta^2}\hat{X}^2 - \eta \frac{X_{\rho+}^2}{\rho} + \sum_{i=1}^{N_\rho} Y_i \tau_1 \frac{\gamma^2}{2\eta}\hat{X}.$$

In the limit $T \uparrow \infty$ we will have

$$\sum_{i=1}^{N_\rho} Y_i = \sum_{i=1}^{N_{T/2}} Y_i \longrightarrow \hat{X}.$$

Hence, using the fact that $\mathbb{E}[\tau_1] = \frac{1}{\theta}$,

$$\lim_{T\uparrow\infty} \mathbb{E}[\mathcal{R}_T] = X_0 P_0^0 - \frac{1}{2}\gamma(X_0 + \hat{X})^2 + \frac{1}{\theta}\frac{\gamma^2}{4\eta}\hat{X}^2. \tag{3.36}$$

Choosing

$$\hat{X} = -\frac{2X_0\eta\theta}{\gamma - 2\eta\theta}$$

yields

$$\lim_{T\uparrow\infty} \mathbb{E}[\mathcal{R}_T] = X_0 P_0^0 + \frac{1}{2}\gamma^2 X_0^2 \frac{1}{2\eta\theta - \gamma} > X_0 P_0^0.$$

This concludes the proof of part (a).

Proof of (b): We first consider the case in which $\frac{\gamma}{\eta\theta} = 2$ and $X_0 \neq 0$. With the same strategy as in part (a) we find with (3.36) that

$$\lim_{T\uparrow\infty} \mathbb{E}[\mathcal{R}_T] = X_0 P_0^0 - \frac{1}{2}\gamma X_0^2 - \gamma X_0 \hat{X}.$$

For $X_0 \neq 0$, the right-hand side can be made arbitrarily large by taking $\hat{X}$ with the opposite sign of X_0 and making $|\hat{X}|$ large.

Now we consider the case in which $\frac{\gamma}{\eta\theta} > 2$. With (3.36) we find

$$\lim_{T\uparrow\infty} \mathbb{E}[\mathcal{R}_T] = X_0 P_0^0 - \frac{1}{2}\gamma X_0^2 - \gamma X_0 \hat{X} + \varepsilon \hat{X}^2,$$

where $\varepsilon > 0$. Again, the right-hand side can be made arbitrarily large by sending $\hat{X}$ to infinity. $\square$

Proof of Proposition 3.4.9. In view of Proposition 3.4.8, the assertion will be implied by the following claim: If, for $0 \le t < \rho$, we have $\xi_t \le 0$ when $X_0 > 0$ or $\xi_t \ge 0$ when $X_0 < 0$, then

$$\mathbb{E}[\mathcal{R}_T] \le X_0 P_0^0.$$

In proving this claim, we will consider the case $X_0 > 0$. The case $X_0 < 0$ is analogous. With Lemma 3.5.1 we find that

$$\mathcal{R}_T = X_0 P_0^0 + \int_0^T X_t \, dP_t^0 - \frac{1}{2}\gamma \left(X_0 + \sum_{i=1}^{N_\rho} Y_i \right)^2 - \int_0^T f(\xi_t)\, dt - \sum_{i=1}^{N_\rho} Y_i \gamma \int_0^{\tau_i} \xi_s \, ds.$$

Consider first the case $\hat{X} \le 0$. Then

$$-\sum_{i=1}^{N_\rho} Y_i \gamma \int_0^{\tau_i} \xi_s \, ds \le 0$$

and $\mathbb{E}[\mathcal{R}_T] \le X_0 P_0^0$ follows.

Consider next the case $\hat{X} > 0$. Since $\xi_t \le 0$ this implies $X_t \ge 0$ for all t. Especially, $X_{\tau_i -} \ge 0$, or equivalently

$$\int_0^{\tau_i} \xi_s \, ds \ge -X_0 - \sum_{j=1}^{i-1} Y_i.$$

Therefore, we find that

$$\mathcal{R}_T \le X_0 P_0^0 + \int_0^T X_t \, dP_t^0 - \frac{1}{2}\gamma X_0^2 - \frac{1}{2}\gamma \sum_{i=1}^{N_\rho} Y_i^2 - \int_0^T f(\xi_t)\, dt$$

and $\mathbb{E}[\mathcal{R}_T] \le X_0 P_0^0$ follows. $\square$

Proof of Proposition 3.4.10. The revenues in this case are

$$\begin{aligned} \mathcal{R}_T &= X_0 P_0^0 + \int_0^T X_t \, dP_t^0 - \frac{1}{2}\gamma (X_0 + \mathbb{1}_{\{\tau_1 \le \rho\}} \hat{X})^2 \\ &\quad - \eta \int_0^T \xi_t^2 \, dt - \mathbb{1}_{\{\tau_1 \le \rho\}} \gamma \hat{X} (X_{\tau_1 -} - X_0) \\ &\le X_0 P_0^0 + \int_0^T X_t \, dP_t^0 \\ &\quad + \mathbb{1}_{\{\tau_1 \le \rho\}} \left(-\frac{1}{2}\gamma (X_0 + \hat{X})^2 - \eta \frac{(X_{\tau_1 -} - X_0)^2}{\tau_1} - \gamma \hat{X} (X_{\tau_1 -} - X_0) \right). \end{aligned}$$

The rightmost expression is maximized by

$$X_{\tau_1 -} = X_0 - \frac{\gamma}{2\eta} \tau_1 \hat{X}$$

and we find

$$\mathcal{R}_T \le X_0 P_0^0 + \int_0^T X_t \, dP_t^0 + \mathbb{1}_{\{\tau_1<\rho\}} \left(-\frac{1}{2}\gamma (X_0 + \hat{X})^2 + \frac{\gamma^2}{4\eta} \tau_1 \hat{X}^2 \right) \tag{3.37}$$

and thus

$$\mathbb{E}[\mathcal{R}_T] \le X_0 P_0^0 + \mathbb{E}[f(\rho, \hat{X})],$$

where

$$f(\rho, \hat{X}) := \int_0^\rho \theta e^{-\theta t} \left(-\frac{1}{2}\gamma (X_0 + \hat{X})^2 + \frac{\gamma^2}{4\eta} t \hat{X}^2 \right) dt.$$

We see that $f(0, \hat{X}) = 0$ and the term in parenthesis is increasing in t. Therefore, if $\hat{X}$ is such that $f(\infty, \hat{X}) > 0$, then we have $f(\rho, \hat{X}) \le f(\infty, \hat{X})$ for all $\rho < \infty$. For $\hat{X}$ with $f(\infty, \hat{X}) \le 0$ we have $f(\rho, \hat{X}) \le 0$ for all $\rho < \infty$. Thus,

$$\mathbb{E}[\mathcal{R}_T] \le X_0 P_0^0 + 0 \vee f(\infty, \hat{X}) = X_0 P_0^0 + \Big(-\frac{1}{2}\gamma (X_0 + \hat{X})^2 + \frac{\gamma^2}{4\eta\theta} \hat{X}^2 \Big)^+.$$

The right-hand side is maximized by taking

$$\hat{X} = 2X_0 \frac{\eta\theta}{\gamma - 2\eta\theta},$$

and so

$$\mathbb{E}[\mathcal{R}_T] \le X_0 P_0^0 + \frac{1}{2}\gamma^2 X_0^2 \frac{1}{2\eta\theta - \gamma}.$$

The statement now follows with Proposition 3.4.8. □

Proof of Corollary 3.4.11. We already know from Proposition 3.4.8 (b) that there is price manipulation for $\frac{\gamma}{\eta} > 2\theta$. On the other hand, Proposition 3.4.10 implies that is no price manipulation for $\frac{\gamma}{\eta} < 2\theta$. Hence, it remains to analyze the case $\frac{\gamma}{\eta} = 2\theta$. For a round trip with $X_0 = 0$, our estimate (3.37) yields that in this case

$$\mathcal{R}_T \ \le \ \int_0^T X_t \, dP_t^0 + \mathbb{1}_{\{\tau_1<\rho\}} \gamma \left(\frac{\gamma}{4\eta} \tau_1 - \frac{1}{2} \right) \hat{X}^2.$$

Hence,

$$\mathbb{E}[\mathcal{R}_T] \le \gamma \hat{X}^2 \mathbb{E}[g(\rho)]$$

where

$$\begin{aligned} g(\rho) \ &:= \ \int_0^\rho \theta e^{-\theta t} \left(\frac{\gamma}{4\eta} t - \frac{1}{2} \right) dt \\ &= \ \frac{\gamma}{4\eta\theta} \Big(1 - e^{-\theta\rho}(1 + \theta\rho) \Big) - \frac{1}{2}(1 - e^{-\theta\rho}) \\ &= \ -\frac{1}{2}\theta\rho e^{-\theta\rho} \\ &\le \ 0. \end{aligned}$$

This gives $\mathbb{E}[\mathcal{R}_T] \le 0$. □

Proof of Proposition 3.4.13. Under the assumptions $\alpha = 1$, $\beta = \frac{1}{2}$, and $g = 0$, the revenues of an admissible strategy are given by

$$\begin{aligned}\mathcal{R}_T &= X_0 P_0^0 + \int_0^T X_t \, dP_t^0 - \frac{1}{2}\gamma \left(X_0 + \mathbb{1}_{\{\tau_1 \le \rho\}} \hat{X} \right)^2 - \int_0^T f(\xi_t) \, dt \\ &\quad + \gamma \hat{X} \left(\frac{1}{2}\hat{X} + X_0 \right) \mathbb{1}_{\{\tau_1 \le \rho\}} \\ &= X_0 P_0^0 + \int_0^T X_t \, dP_t^0 - \frac{1}{2}\gamma X_0^2 - \int_0^T f(\xi_t) \, dt.\end{aligned}$$

Taking the conditional expectation with respect to $\mathcal{F}_{\tau_1 \wedge \rho}$ and using optional sampling yields

$$\begin{aligned}\mathbb{E}[\, \mathcal{R}_T \,|\, \mathcal{F}_{\tau_1 \wedge \rho} \,] &= X_0 P_0^0 + \int_0^{\tau_1 \wedge \rho} X_t \, dP_t^0 - \frac{\gamma}{2} X_0^2 - \int_0^{\tau_1 \wedge \rho} f(\xi_t) \, dt \\ &\quad - \mathbb{E}\Big[\int_{\tau_1 \wedge \rho}^T f(\xi_t) \, dt \,\Big|\, \mathcal{F}_{\tau_1 \wedge \rho} \Big].\end{aligned}$$

Due to the liquidation constraint, we must have $\int_{\tau_1 \wedge \rho}^T \xi_t \, dt = -X_{\tau_1 \wedge \rho} - \mathbb{1}_{\{\tau_1 < \rho\}} \hat{X}$, and so the convexity of f and Jensen's inequality yield that

$$\int_{\tau_1 \wedge \rho}^T f(\xi_t) \, dt \ge (T - \tau_1 \wedge \rho) f\Big(\frac{-X_{\tau_1 \wedge \rho} - \mathbb{1}_{\{\tau_1 < \rho\}} \hat{X}}{T - \tau_1 \wedge \rho} \Big)$$

with equality if, for $\tau_1 \wedge \rho \le t \le T$,

$$\xi_t = \begin{cases} \dfrac{-X_{\tau_1} - \hat{X}}{T - \tau_1} & \text{on } \{\tau_1 \le \rho\} \\[2ex] \dfrac{-X_\rho}{T - \rho} & \text{on } \{\rho < \tau_1\}. \end{cases} \tag{3.38}$$

These two possibilities will correspond to the single update of the optimal strategy at τ_1.

Note next that, due to the $(\mathcal{G}_t)$-predictability of the processes (ξ_t) and $(\rho \wedge t)_{t \ge 0}$, $(\xi_s)_{s \le t}$ and $\rho \wedge t$ are independent of τ_1, conditional on $\{t \le \tau_1\}$. It follows that

$$\begin{aligned}&\mathbb{E}[\, \mathcal{R}_T \,] = \mathbb{E}[\, \mathbb{E}[\, \mathcal{R}_T \,|\, \mathcal{F}_{\tau_1 \wedge \rho} \,] \,] \\ &\quad \le X_0 P_0^0 - \frac{\gamma}{2} X_0^2 - \mathbb{E}\Big[\int_0^{\tau_1 \wedge \rho} f(\xi_t) \, dt + (T - \tau_1 \wedge \rho) f\Big(\frac{-X_{\tau_1 \wedge \rho} - \mathbb{1}_{\{\tau_1 \le \rho\}} \hat{X}}{T - \tau_1 \wedge \rho} \Big) \Big] \\ &\quad = X_0 P_0^0 - \frac{\gamma}{2} X_0^2 - \mathbb{E}\big[\, F(\hat{X}, \xi, \rho) \,\big],\end{aligned} \tag{3.39}$$

where the functional F maps $\hat{X} \in \mathbb{R}$, $\xi \in L^1[0,T]$, and $r \in [0,T]$ to

$$\begin{aligned}&F(\hat{X}, \xi, r) \\ &\quad = \int_0^\infty du \, \theta e^{-\theta u} \Big\{ \int_0^{u \wedge r} f(\xi_t) \, dt + (T - u \wedge r) f\Big(\frac{-X_0 - \int_0^{u \wedge r} \xi_t \, dt - \mathbb{1}_{\{u \le r\}} \hat{X}}{T - u \wedge r} \Big) \Big\}.\end{aligned}$$

When F admits a minimizer $(\hat{X}^*, \xi^*, r^*)$, then concatenating ξ^* with (3.38) in $r^* \wedge \tau_1$ yields an optimal strategy that is a single-update strategy.

To show the existence of a minimizer of F, take any triple $(\tilde{X}, \tilde{\xi}, \tilde{r})$ for which $C := F(\tilde{X}, \tilde{\xi}, \tilde{r}) < \infty$. We then only need to look into those triples $(\hat{X}, \xi, r)$ for which $F(\hat{X}, \xi, r) \le C$. Without loss of generality, we can pick the component ξ from the set

$$K_C := \Big\{ \xi \in L^1[0,T] \,\Big|\, \int_0^T f(\xi_t)\,dt \le Ce^{\theta T} \Big\},$$

because we clearly have

$$F(\hat{X}, \xi, r) \ge \int_T^\infty du\,\theta e^{-\theta u} \int_0^{u \wedge r} f(\xi_t)\,dt = e^{-\theta T} \int_0^r f(\xi_t)\,dt$$

and we can set $\xi_t := 0$ for $t > r$.

The set K_C is a closed convex subset of $L^1[0,T]$. Hence it is also weakly closed in $L^1[0,T]$. It is also uniformly integrable according to the criterion of de la Vallée Poussin and our assumption that f has superlinear growth. Hence, the Dunford–Pettis theorem (Dunford and Schwartz, 1958, Corollary IV.8.11) implies that K_C is weakly sequentially compact in $L^1[0,T]$. From now on we will endow K_C with the weak topology.

It follows in particular that

$$\sup_{\xi \in K_C} \int_0^T |\xi_t|\,dt < \infty. \tag{3.40}$$

Since

$$F(\hat{X}, \xi, r) \ge \int_0^r du\,\theta e^{-\theta u}(T-u) f\Big(\frac{-X_0 - \int_0^u \xi_t\,dt - \hat{X}}{T-u}\Big),$$

the superlinear growth of f and (3.40) imply that there is a constant $C_1 \ge 0$ such that $|\hat{X}| \le C_1$ when $F(\hat{X}, \xi, r) \le C$. Hence we can restrict our search of a minimizer to the sequentially compact set

$$\mathcal{K} := [-C_1, C_1] \times K_C \times [0,T].$$

Next,

$$[0,T] \times K_C \ni (r, \xi) \longrightarrow \int_0^r \xi_t\,dt = \int_0^T \xi_t \mathbb{1}_{[0,r]}(t)\,dt$$

is a continuous map. Moreover, denoting by f^* the Fenchel-Legendre transform of the convex function f, we have $f^{**} = f$ due to the biduality theorem, and so

$$[0,T] \times K_C \ni (r, \xi) \longmapsto \int_0^r f(\xi_t)\,dt = \sup_{\varphi \in L^\infty} \Big[\int_0^T \mathbb{1}_{[0,r]}(t)\xi_t\varphi_t\,dt - \int_0^r f^*(\varphi_t)\,dt \Big];$$

see, e.g., Theorem 2 in Rockafellar (1968). It follows that this map is lower semicontinuous as the supremum of continuous maps.

Altogether, it follows that F is lower semicontinuous on the sequentially compact set $\mathcal{K}$ and so admits a minimizer. $\square$

Proof of Proposition 3.4.14. We denote $\hat{X} = \hat{X}^*, \xi = \xi^*, \rho = \rho^*$. Recall that $\mathcal{R}_T = X_0 P_0^0 + \int_0^T X_t \, dP_t^0 - \frac{1}{2}\gamma X_0^2 - \int_0^T f(\xi_t)\, dt$, so for fixed X_0 and γ we have $\mathbb{E}[\mathcal{R}_T] = C - \mathbb{E}[\int_0^T f(\xi_t)\, dt]$ with a constant C.

(i): Assume $\hat{X} \geq 0$. Since $0 = X_T = X_0 + \hat{X}\mathbb{1}_{\{\tau_1 \leq \rho\}} + \int_0^T \xi_t \, dt \geq X_0 + \int_0^T \xi_t \, dt$, it follows that $\int_0^T \xi_t \, dt \leq -X_0$. By Jensen's inequality,

$$\int_0^T f(\xi_t)\, dt \geq T f\left(-\frac{X_0}{T}\right).$$

On the other hand, consider the single-update strategy $(\tilde{X}, \tilde{\xi}, \tilde{\rho})$ with $\tilde{\rho} \in (0,T), \tilde{X} \in (X_0(\frac{\tilde{\rho}}{T} - 1), 0)$ and

$$\tilde{\xi}_t = \begin{cases} -\frac{X_0}{T} & \text{if } \tau_1 > \tilde{\rho} \wedge t, \\ -\frac{X_0 + \tilde{X} - \tau_1 \frac{X_0}{T}}{T - \tau_1} & \text{if } \tau_1 \leq \tilde{\rho} \wedge t. \end{cases}$$

We have that

$$\int_0^T f(\tilde{\xi}_t)\, dt = \int_0^T \mathbb{1}_{\{\tau_1 \geq \tilde{\rho} \wedge t\}} f\left(-\frac{X_0}{T}\right) dt + \int_0^T \mathbb{1}_{\{\tau_1 < \tilde{\rho} \wedge t\}} f\left(-\frac{X_0 + \tilde{X} - \tau_1 \frac{X_0}{T}}{T - \tau_1}\right) dt.$$

Since f is strictly decreasing on $(-\infty, 0]$ we have that $f\left(-\frac{X_0 + \tilde{X} - \tau_1 \frac{X_0}{T}}{T - \tau_1}\right) < f\left(-\frac{X_0}{T}\right)$. Subsequently $\int_0^T f(\tilde{\xi}_t)\, dt \leq T f\left(-\frac{X_0}{T}\right)$ and $\int_0^T f(\tilde{\xi}_t)\, dt < T f\left(-\frac{X_0}{T}\right)$ for $\tau_1 \leq \tilde{\rho}$. Since $\mathbb{P}(\tau_1 \leq \tilde{\rho}) > 0$, we have that $\mathbb{E}[\tilde{\mathcal{R}}_T] > \mathbb{E}[\mathcal{R}_T]$, so $\hat{X} \geq 0$ cannot be optimal. Hence $\hat{X} < 0$.

(ii): Assume $X_\rho \geq -\hat{X}$. Consider a strategy $(\tilde{X}, \tilde{\xi}, \tilde{\rho})$ with $\tilde{X} = \hat{X}$. For $\varepsilon > 0$, let $\tilde{\rho} = \rho + \varepsilon$. Let furthermore $\tilde{\xi} = \xi$ for $\tau_1 \notin (\rho, \tilde{\rho}]$. For $\tau_1 \in (\rho, \tilde{\rho}]$, let $\tilde{\xi}_t = \xi_t$ on $[0, \tilde{\rho}]$ and $\tilde{\xi}_t = -\frac{X_{\tilde{\rho}} + \hat{X}}{T - \tilde{\rho}} = -\frac{X_\rho(1 - \frac{\varepsilon}{T-\rho}) + \hat{X}}{T - \tilde{\rho}}$ on $(\tilde{\rho}, T]$. Then

$$\begin{aligned}
\mathbb{E}[\mathcal{R}_T] - \mathbb{E}[\tilde{\mathcal{R}}_T] &= \mathbb{E}\left[\int_{\tilde{\rho}}^T f(\tilde{\xi}_t) - f(\xi_t)\, dt; \tau_1 \in (\rho, \tilde{\rho}]\right] \\
&= E\left[\int_{\tilde{\rho}}^T f\left(-\frac{X_\rho(1 - \frac{\varepsilon}{T-\rho}) + \hat{X}}{T - \tilde{\rho}}\right) - f\left(-\frac{X_\rho}{T - \rho}\right) dt; \tau_1 \in (\rho, \tilde{\rho}]\right]
\end{aligned}$$

We have that $-\frac{X_\rho(1 - \frac{\varepsilon}{T-\rho}) + \hat{X}}{T - \tilde{\rho}} \to -\frac{X_\rho + \hat{X}}{T - \rho}$ as $\varepsilon \to 0$. By (i) and by assumption, we have furthermore that $-\frac{X_\rho}{T-\rho} < -\frac{X_\rho + \hat{X}}{T-\rho} \leq 0$. Since f is continuous everywhere and decreasing on $(-\infty, 0]$, we conclude that $f\left(-\frac{X_\rho(1 - \frac{\varepsilon}{T-\rho}) + \hat{X}}{T - \tilde{\rho}}\right) < f\left(-\frac{X_\rho}{T-\rho}\right)$ if $\varepsilon > 0$ is small enough. Thus, $\mathbb{E}[\tilde{\mathcal{R}}_T] > \mathbb{E}[\mathcal{R}_T]$, contradicting the optimality of ξ_t. Thus, $X_\rho < -\hat{X}$. □

Proof of Corollary 3.4.15. By Proposition 3.4.14, (ii) we have that $X^*_{\rho^*} + \hat{X}^* < 0$ given $\tau_1 \geq \rho^*$. Since X^*_t is continuous for $t \neq \tau_1$, there is an $\varepsilon > 0$ such that $X^*_{\tau_1 +} = X^*_{\tau_1 -} + \hat{X}^* < 0$ for $\tau_1 \in [\rho^* - \varepsilon, \rho^*]$. Since $X_0 > 0$ and $X_T = 0$, there is transaction-triggered price manipulation in this case. □

Proof of Prop. 3.4.16. $\hat{X}_t = -X_t$ and $\rho = T$ is the unique strategy[3] such that $\mathbb{E}[\mathcal{R}_T] = X_0 P_0^0 - \frac{1}{2}\gamma X_0^2 - \mathbb{E}[\int_0^{\tau_1 \wedge T} f(\xi_t)\,dt]$. Since $f(x) > 0$ for all $x \neq 0$, for all other dark-pool strategies $\hat{X}_t$ we have that $\mathbb{E}[\mathcal{R}_T] = X_0 P_0^0 - \frac{1}{2}\gamma X_0^2 - \mathbb{E}[\int_0^T f(\xi_t)\,dt] < X_0 P_0^0 - \frac{1}{2}\gamma X_0^2 - \mathbb{E}[\int_0^{\tau_1 \wedge T} f(\xi_t)\,dt]$, so the choice $\hat{X}_t = -X_t, \rho = T$ is optimal.

As in Theorem 3.4.13 one can show that the optimal strategy is a single-update strategy. Thus,

$$\begin{aligned}
\mathbb{E}\left[\int_0^{\tau_1 \wedge T} f(\xi_t)\,dt\right] &= \mathbb{E}\left[\int_0^T \mathbb{1}_{\{\tau_1 > t\}} f(\xi_t^0)\,dt\right] \\
&= \int_0^T \mathbb{P}(\tau_1 > t) f(\xi_t^0)\,dt \\
&= \int_0^T e^{-\theta t} f(\xi_t^0)\,dt.
\end{aligned}$$

By calculus of variations (cf. Cesari (1983) Theorems 2.6.i-iii and 2.20.i) an optimal strategy $X_t^0 := X_0 + \int_0^t \xi_s^0\,ds$ exists and satisfies the Euler-Lagrange equation

$$\theta f'(\dot{X}_t^0) - f''(\dot{X}_t^0)\ddot{X}_t^0 = 0,$$

i.e. $\frac{d}{dt} f'(\dot{X}_t^0) = \theta f'(\dot{X}_t^0)$. The unique solution of this equation is $f'(\dot{X}_t^0) = C\exp(\theta t)$ with a constant $C \in \mathbb{R}$. It follows that $\operatorname{sgn}(\dot{X}_t^0) = \operatorname{sgn}(f'(\dot{X}_t^0)) = \operatorname{sgn}(C\exp(\theta t)) = \operatorname{sgn}(C)$, i.e. it is constant. That is, there is no transaction-triggered price manipulation. □

[3] Note that we required $\rho < T$ a.s. before. However, this is to ensure that $X_{T+} = 0$, which implies for a non-adaptive $\hat{X} \neq 0$ that $\rho < T$ a.s. However, when $\hat{X}_t \to 0$ for $t \to T$, we can also allow for $\rho = T$.

Chapter 4

Transient Impact for Multiple Assets

4.1 Introduction

In a market impact model for trading with multiple assets it is important to take into account the cross impact of the assets, i.e. the impact of trading in one asset to the other assets. Risk-averse investors will also consider correlation of the assets, but we restrict ourselves to risk-neutral investors throughout the chapter.

We consider a transient impact model, i.e. a model where a decay kernel describes the decay of market impact over time. Transient impact was first discussed within the market impact literature in Bouchaud *et al.* (2004) and Obizhaeva and Wang (2013). Gatheral (2010) analyzed different decay kernels and their regularity, in particular the absence of price manipulation in the sense of Huberman and Stanzl (2004). In the case of linear transient impact, Alfonsi *et al.* (2012) analyzed the regularity of the market impact model in discrete time, whereas Gatheral *et al.* (2012) discussed the same setup in continuous time, including the analysis whether optimal strategies exist in continuous time.

The optimal liquidation problem for multiple assets was analyzed be Konishiy and Makimoto (2001) in the Almgren-Chriss model (see Almgren and Chriss (2001)). Furthermore, Schöneborn (2011) solved the problem of maximizing expected utility of the revenues generated by an investor with an Almgren-Chriss impact model for infinite time horizons using adaptive, i.e. in general non-deterministic strategies. In case the utility function of the investor has constant absolute risk aversion, Schied *et al.* (2010) showed that the optimal strategy is deterministic.

The chapter is structured as follows. In section 4.2 the market impact model and the cost function is presented. Furthermore, we define optimal strategies and positive definiteness of the decay kernel. In section 4.3 we present our results. At first, we characterize matrix-valued positive definite functions as Fourier transforms of nonnegative matrix-valued measures. Then we show that optimal strategies for symmetric decay kernels are solutions of a linear system of equations. We explain that the decay kernel should be nonincreasing and nonnegative. In case it is additionally symmetric, convex and all one-dimensional versions are nonconstant, it follows from the corresponding one-dimensional result that it is strictly positive definite. If every entry of the decay kernel, i.e. every impact and cross impact, decays at the same speed, we show that the optimal strategy is independent of the impact structure. That is, in this case the optimal strategy can be computed separately for each asset. Finally, if the decay kernel is in particular convex and commuting, we discuss the convergence to continuous time. In section 4.4, we present examples. First, we treat general linear and exponential decay kernels for two assets. We study for which parameters they are suitable decay kernels. Then we discuss matrix functions, i.e. one-dimensional functions applied to matrices. We give an explicit solution for exponential functions, i.e. a generalized Obizhaeva and Wang (2013) model for multiple assets. All proofs are presented in the final section 4.5.

4.2 Cost function and positive definiteness

4.2.1 Market impact model

Assume an investor is trading in K assets. At time 0, the investor holds $X_0 \in \mathbb{R}^K$ shares in these assets. In absence of trading of the large investor, the *unaffected price process* of these assets is given by a continuous martingale $(P_t^0)_{t\in[0,T]}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ satisfying the usual conditions.

Until a time horizon $T > 0$, the shares have to be liquidated. To accomplish this, the investor may trade on a finite set of given deterministic trading times $0 = t_1 < t_2 < \ldots < t_N \leq T$ with $N \in \mathbb{N}$. The size of the trade at time t_k is denoted by $\xi_{t_k} : \Omega \to \mathbb{R}^K$, where positive values denote buys and negative values denote sells. We abbreviate ξ_{t_k} by ξ_k. We assume ξ_k to be $\mathcal{F}_{t_k}$-measurable and uniformly bounded. For $t \in [0, T]$, the number of shares held by the investor is given by

$$X_t = X_0 + \sum_{t_k<t} \xi_{t_k},$$

where we require $X_{T+} = 0$, or equivalently, $\sum_{k=1}^N \xi_k = -X_0$. We call $\xi = (\xi_1, \ldots, \xi_N)$ the trading strategy of the investor; if ξ is deterministic, we can identify ξ with a $K \times N$-real matrix putting ξ_i in the i-th column of the matrix.

Including the impact of the investor, let the price process be given by

$$P_t = P_t^0 + \sum_{t_k<t} G(t - t_k)\, \xi_{t_k}, \tag{4.1}$$

where the matrix-valued function $G : [0, \infty) \to \mathbb{R}^{K\times K}$, the *decay kernel*, describes the decay of the impact. The diagonal of G quantifies the impact of trading an asset on the price of the same asset, the off-diagonal elements represent the cross impact on other assets.

4.2.2 Cost function and optimal strategies

Now, we define the cost function of a trading strategy. The definition will be motivated in the sequel by approximation with continuous-time strategies.

Definition 4.2.1. The *cost function* $C : \mathbb{R}^{K\times N} \to \mathbb{R}$ is given by

$$C(\xi) = \frac{1}{2}\sum_{k=1}^{N} \xi_k^\top G(0)\xi_k + \sum_{k=1}^{N}\sum_{l=1}^{k-1} \xi_k^\top G(t_k - t_l)\xi_l.$$

Let a discrete-time strategy $\xi_1, \dots \xi_N$ be given. For $\varepsilon > 0$, define its continuous-time approximation by $X_t^\varepsilon = X_0 + \sum_{n=0}^{N} \xi_{t_n} f_{t_n}^\varepsilon(t)$ where $f_s^\varepsilon(t) = \frac{1}{\varepsilon}((t-s)^+ \wedge \varepsilon)$. Since $f_s^\varepsilon(t) \to \mathbb{1}_{\{(s,\infty)\}}(t)$ for $\varepsilon \downarrow 0$, we have $\lim_{\varepsilon\downarrow 0} X_t^\varepsilon = X_t$ a.s., so the continuous-time approximation converges to the discrete-time strategy as $\varepsilon \downarrow 0$.

For such a trading strategy $(X_t^\varepsilon)_{t\in[0,T+\varepsilon]}$ that is absolutely continuous on $[0, T+\varepsilon]$ it is natural to define the revenues of this trading strategy by $\mathcal{R}^\varepsilon = -\int_0^{T+\varepsilon} (P_t^\varepsilon)^\top \, dX_t^\varepsilon$, since during an interval $[t, t+dt]$ we buy dX_t^ε shares at a price of P_t^ε. There the price process P^ε is given analogously to (4.1) by $P_t^\varepsilon = P_t^0 + \int_0^t G(t-s)\, dX_s^\varepsilon$, i.e. we have $\lim_{\varepsilon\downarrow 0} P_t^\varepsilon = P_t$ a.s. The costs $\mathcal{C}^\varepsilon$ are given by $\mathcal{R}^\varepsilon = X_0^\top P_0^0 - \mathcal{C}^\varepsilon$, i.e. revenues of trading are face value $X_0^\top P_0^0$ minus costs (note that $\mathbb{E}[\mathcal{R}^\varepsilon] = X_0^\top P_0^0$ if there is no impact, i.e. if $G \equiv 0$). Thus we have $\mathcal{C}^\varepsilon = X_0^\top P_0^0 - \mathcal{R}^\varepsilon$.

We have the following result, which justifies our definition of the cost function.

Proposition 4.2.2. *Let $\xi = (\xi_1, \dots, \xi_N)$ be a trading strategy. Then*

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}[\mathcal{C}^\varepsilon] = \mathbb{E}[C(\xi)].$$

Alternatively, the cost function can be derived by assuming that trading at time t_k takes place at the midprice, i.e. at $\frac{1}{2}(P_{t_k} + P_{t_k+})$. That is, the revenues would by given by $\mathcal{R} = -\sum_{k=1}^{N} \xi_k^\top \frac{1}{2}(P_{t_k} + P_{t_k+})$. With this definition, a direct calculation yields $\mathbb{E}[\mathcal{R}] = X_0^\top P_0^0 - \mathbb{E}[C(\xi)]$.

Finally, we need to define optimal strategies for the investor. If there is a (probably non-deterministic) strategy $\xi = (\xi_1, \dots, \xi_N)$ that minimizes $\mathbb{E}[C(\xi)]$ in the class of all strategies such that $\sum_{k=1}^{N} \xi_k = X_0$ for some $X_0 \in \mathbb{R}^K$, then there is also a minimizer in the class of deterministic strategies. We will assume the investor to be risk-neutral in the sequel, i.e. the investor aims at minimizing $\mathbb{E}[C(\xi)]$. So we can concentrate on deterministic strategies only and define optimal strategies as follows.

Definition 4.2.3. An *optimal strategy* is a deterministic trading strategy $\xi \in \mathbb{R}^{K\times N}$ that minimizes the cost function $C(\xi)$, given the constraint $\sum_{k=1}^{N} \xi_k = -X_0$.

4.2.3 Positive definiteness

One objective of a market impact model is the computation of optimal strategies. Furthermore, we expect market impact to be a cost on average and not to be something profitable. These considerations lead to the following definition.

Definition 4.2.4. We call G *positive definite*, if $C(\xi) \geq 0$ for all $N \in \mathbb{N}$, all $\xi_1, \xi_2, \dots, \xi_N \in \mathbb{R}^K$ and all $0 \leq t_1 < t_2 < \dots < t_N$. G is called *strictly positive definite*, if additionally equality holds only for $\xi_1 = \dots = \xi_N = 0$.

For $K = 1$, this is equivalent with the usual definition found in the literature. In the multidimensional case, i.e. $K > 1$, we need for equivalence of our definition to the usual one that $G(0)$ is symmetric or G is continuous in 0. For details see Definition 4.5.3 and subsequent results.

For positive definite G the costs of any trading strategy are nonnegative, i.e. there are positive expected liquidation costs as defined in section 2.2.2. If G is even strictly positive definite, every nontrivial trading strategy has strictly positive costs. Positive definiteness of G guarantees the existence of optimal strategies:

Proposition 4.2.5. *If G is positive definite, there is an optimal strategy. If G is strictly positive definite, there is a unique optimal strategy.*

4.3 Results

4.3.1 Characterization of positive definiteness

In this section, we discuss how positive definiteness can be characterized. In particular, we discuss criteria to decide whether a given function G is positive definite.

For $K = 1$, continuous positive definite decay kernels are characterized by Bochner's Theorem (Bochner (1932)) as the Fourier transform of a nonnegative Borel measure. In the multivariate case, $K > 1$, we find an analogous characterization as Fourier transform of a complex nonnegative matrix-valued measure as defined below. Before, we need to define nonnegative matrices.

Definition 4.3.1. We call a real matrix $M \in \mathbb{R}^{n\times n}$ *nonnegative* if $x^\top Mx \geq 0$ for every $x \in \mathbb{R}^n$ (often this is called positive semidefinite). Furthermore, we call a complex matrix $N \in \mathbb{C}^{n\times n}$ *nonnegative* if $y^*Ny \geq 0$ for every $y \in \mathbb{C}^n$.

If furthermore $x^\top Mx > 0$ for every $x \neq 0$, M is called *strictly positive*. Analogously, if $y^*Ny > 0$ for every $y \neq 0$, N is called *strictly positive*.

By $S(K)$ we denote the set of all real symmetric $K \times K$-matrices and by $S_+(K)$ the cone of all real nonnegative matrices in $S(K)$.

Note that a real matrix $M \in \mathbb{R}^{n\times n}$ is nonnegative if and only if its symmetrization $\frac{1}{2}(M + M^\top)$ is nonnegative. A nonnegative complex matrix $N \in \mathbb{C}^{n\times n}$ is necessarily Hermitian, i.e. $N = N^* = \bar{N}^\top$. A real matrix $M \in \mathbb{R}^{n\times n}$ is nonnegative as a complex matrix, i.e. $y^*My \geq 0$ for all $y \in \mathbb{C}^n$, if and only if it is in $S_+(n)$.

Definition 4.3.2. Let $\mathcal{B}(\mathbb{R})$ be the Borel sigma-algebra on $\mathbb{R}$. We call a mapping $M : \mathcal{B}(\mathbb{R}) \to \mathbb{C}^{K\times K}$ *complex nonnegative matrix-valued measure*, if M is a measure (that is, M is σ-additive and $M(\emptyset) = 0$) and $M(E)$ is nonnegative for every $E \in \mathcal{B}(\mathbb{R})$.

Theorem 4.3.3. *Let G be continuous. Then the following are equivalent:*

(a) *G is positive definite.*

(b) *G is the Fourier transform of a complex nonnegative matrix-valued measure M, i.e.*

$$G(t) = \int_{\mathbb{R}} e^{i\gamma t} M(d\gamma) \qquad \text{for } t \geq 0. \tag{4.2}$$

The proof uses the fact that G is positive definite if and only if there is a vector-valued random field such that G is its correlation function. But due Cramér (1940), these correlation functions are precisely the Fourier transforms as in part (b). For Hilbert space operator-valued function the theorem was shown by Naimark (1943).

Now, assume for a moment that G is continuous and positive definite. Thus, we can characterize G via (4.2). The right-hand side of (4.2) is also well-defined for $t < 0$. So we can define $\tilde{G} : \mathbb{R} \to \mathbb{R}^{K\times K}$ by setting

$$\tilde{G}(t) = \int_{\mathbb{R}} e^{i\gamma t} M(d\gamma)$$

for all $t \in \mathbb{R}$. For any $t \geq 0$, we then find that

$$\tilde{G}(-t) = \int_{\mathbb{R}} e^{-i\gamma t} M(d\gamma) = \int_{\mathbb{R}} \overline{e^{i\gamma t}} M(d\gamma) = \int_{\mathbb{R}} e^{i\gamma t} M^{\top}(d\gamma) = G(t)^{\top},$$

since G is real and $\overline{M(E)}^{\top} = M(E)$ for every $E \in \mathcal{B}(\mathbb{R})$, since $M(E)$ is nonnegative. Furthermore, for any $t \geq 0$ we have $\tilde{G}(t) = G(t)$. This observation motivates the following definition.

Definition 4.3.4. Let $G : [0, \infty) \to \mathbb{R}^{K\times K}$. We define the *extension of G to $\mathbb{R}$* as $\tilde{G} : \mathbb{R} \to \mathbb{R}^{K\times K}$ given by

$$\tilde{G}(t) = \begin{cases} G(-t)^{\top} & \text{for } t < 0 \\ G(t) & \text{for } t \geq 0. \end{cases}$$

Note that in one dimension (i.e. $K = 1$) the extension of G is simply $\tilde{G}(t) = G(|t|)$.

If the inverse Fourier transform of $\tilde{G}$ exists, then M in (4.2) is the inverse Fourier transformation of $\tilde{G}$ and necessarily Hermitian. So in view of Theorem 4.3.3, if a continuous decay kernel G is given with inverse Fourier transformation M of $\tilde{G}$, G is positive definite if and only if $M(E)$ is nonnegative for all $E \in \mathcal{B}(\mathbb{R})$.

With this notation, we can rewrite the cost function C in a particular simple form.

Proposition 4.3.5. *For any $\xi \in \mathbb{R}^{K\times N}$*

$$C(\xi) = \frac{1}{2} \sum_{k,l=1}^{N} \xi_k^{\top} \tilde{G}(t_k - t_l) \xi_l. \tag{4.3}$$

Next, we will consider the special case of symmetric decay kernels G.

Definition 4.3.6. G is called *symmetric* if $G(t) = G(t)^{\top}$ for all $t \geq 0$.

Note that for a symmetric decay kernel G we have $\tilde{G}(-t) = \tilde{G}(t)$ for all $t \in \mathbb{R}$. Since the (inverse) Fourier transform of a symmetric function is real, M in (4.2) will become a real measure, as defined in the following.

Definition 4.3.7. Let $\mathcal{B}(T)$ be the Borel sigma-algebra on $T \subset \mathbb{R}$. We call $M : \mathcal{B}(T) \to \mathbb{R}^{K\times K}$ a *symmetric nonnegative matrix-valued measure on* T, if M is a measure (that is, M is σ-additive and $M(\emptyset) = 0$) and $M(E) \in S_+(K)$ for every $E \in \mathcal{B}(T)$.

In the symmetric case, we can characterize positive definiteness via one-dimensional functions g^ζ defined in the following.

Definition 4.3.8. For any $\zeta \in \mathbb{R}^K$, we define the one-dimensional version of G by $g^\zeta : [0,\infty) \to \mathbb{R}, t \mapsto \zeta^\top G(t)\zeta$. Furthermore, g^ζ is called positive definite if it satisfies Definition 4.2.4 with $K = 1$ and $G = g^\zeta$.

The definition of positive definiteness for g^ζ is equivalent with the common definition in the literature, i.e. for all $N \in \mathbb{N}$, all $0 \le t_1 \le t_2 \le \ldots \le t_N$ and all $x_1, x_2, \ldots, x_N \in \mathbb{R}$ we have $\sum_{k,l=1}^N x_k x_l g^\zeta(|t_k - t_l|) \ge 0$.

The following theorem is a variant of Theorem 4.3.3 for symmetric decay kernels G. In this case, the measure M is real symmetric instead of complex Hermitian. Furthermore, there is a characterization via the one-dimensional functions g^ζ.

Theorem 4.3.9. *Let G be continuous and symmetric. Then the following are equivalent:*

(a) *G is positive definite.*

(b) *For all $\zeta \in \mathbb{R}^K$, the function g^ζ is positive definite.*

(c) *G is the Fourier transform of a symmetric nonnegative matrix-valued measure M on $\mathbb{R}$, i.e.*

$$G(t) = \int_{\mathbb{R}} e^{i\gamma t}\, M(d\gamma).$$

The characterization via the one-dimensional functions g^ζ is due to Falb (1969), from where we also took parts of the proof.

4.3.2 Optimal strategies for symmetric decay kernels

For symmetric decay kernels, an optimal strategy is characterized by a linear system of equations.

Proposition 4.3.10. *Let G be symmetric and $(\xi_1, \ldots \xi_N)$ be an optimal strategy. Then there exists $\lambda \in \mathbb{R}^K$ such that for all $p \in \{1, \ldots, N\}$ we have*

$$\sum_{k=1}^N \xi_k^\top \tilde{G}(t_k - t_p) = \lambda^\top. \tag{4.4}$$

Futhermore, the cost of the strategy is given by $C(\xi) = -\frac{1}{2}\lambda^\top X_0$. Conversely, if G is symmetric and positive definite and $(\xi_1, \ldots \xi_N)$ satisfies (4.4), then $(\xi_1, \ldots \xi_N)$ is an optimal strategy.

The proof is similar to that of Theorem 2.11 in Gatheral *et al.* (2012). The method of Lagrange multipliers yields the same result.

If G is strictly positive definite, the unique optimal strategy can also be given explicitely. Furthermore, we find that the costs of the optimal strategy is a quadratic form of X_0. This is natural since we assume that impact is linear.

Proposition 4.3.11. *Let G be symmetric and strictly positive definite. Furthermore, let*

$$\begin{aligned} &L : \mathbb{R}^{K\times N} \to \mathbb{R}^{K\times N}, \\ &\qquad (\xi_1, \dots \xi_N) \mapsto \left(\sum_{l=1}^{N} \tilde{G}(t_1 - t_l)\xi_l, \sum_{l=1}^{N} \tilde{G}(t_2 - t_l)\xi_l, \dots, \sum_{l=1}^{N} \tilde{G}(t_N - t_l)\xi_l \right). \end{aligned}$$

L is invertible, so we can define $M : \mathbb{R}^K \to \mathbb{R}^K, \lambda \mapsto L^{-1}(\lambda \mathbb{1}_N^\top)\mathbb{1}_N$. M is also invertible, the unique optimal strategy ξ^* is given by $\xi^* = -L^{-1}(\mathbb{1}_N(M^{-1}(X_0))^\top)$ and its cost is given by $C(\xi^*) = \frac{1}{2}(M^{-1}(X_0))^\top X_0$. I.e., the cost is a quadratic form in X_0.

4.3.3 Nonincreasing, nonnegative and convex decay kernels

Not every positive definite function G is a "reasonable" model of the decay of market impact. In one dimension ($K = 1$), one assumes that G is nonincreasing and nonnegative. We assume that it is nonincreasing, since the impact of a trade should be smaller if there is more time gone since the trade was executed. It should be nonnegative since a buy trade should move prices up over all time horizons and a sell trade should move prices down over all time horizons. In fact, one can show that positive definite and nonincreasing implies nonnegative, cf. Proposition 4.3.15.

Now we want to transfer these concepts to the case of more assets ($K > 1$). Let two trades ξ_1 and ξ_2 at times $t_1 < t_2$ be given and assume that they have the same size, i.e. $\xi_1 = \xi_2 = \xi$. Now by inspection of the cost function we observe that the quantity $\xi^\top G(t_2 - t_1)\xi$ is the cost of the impact incurred by the first trade on the second trade. By the same reasoning as in the one-dimensional case, this quantity should be nonnegative and nonincreasing in $t_2 - t_1$. Recalling the notation in Definition 4.3.8 this motivates the following definition.

Definition 4.3.12. A matrix-valued function $G : [0, \infty) \to \mathbb{R}^{K\times K}$ is called

(a) *nonincreasing*, if for every $\zeta \in \mathbb{R}^K$ the function g^ζ is nonincreasing,

(b) *nonnegative*, if for every $\zeta \in \mathbb{R}^K$ and every $t \in [0, \infty)$ we have $g^\zeta(t) \geq 0$, i.e. if $G(t)$ is a nonnegative matrix for all $t \in [0, \infty)$,

(c) (strictly) *convex*, if for all $\zeta \in \mathbb{R}^K$ the function g^ζ is (strictly) convex,

Note that the properties introduced in the preceding definition depend only on the symmetric part of G.

In Propositions 4.5.11 and 4.5.13 we give characterizations of nonincreasing and convex functions for general functions G. If G is smooth, these results simplify to the following statement.

Proposition 4.3.13. (a) *Let G be absolutely continuous with (component-wise) derivative G'. G is nonincreasing if and only if $-G'(t)$ is nonnegative for almost all $t \geq 0$.*

(b) *Let G be twice continuously differentiable. G is convex if and only if $G''(t)$ is nonnegative for all $t \geq 0$, where G'' is the second derivative of G.*

If G is nonincreasing, nonnegative, and convex, then so is the function g^ζ for each $\zeta \in \mathbb{R}^K$. Hence, g^ζ is a positive definite function due to a criterion sometimes called "Pólya criterion" after Pólya (1949), although this fact is also an easy consequence of the much older work of Young (1913). It therefore follows from Theorem 4.3.9 that the matrix-valued function G is also positive definite as soon as it is symmetric. In fact, a stronger result is possible: G is even *strictly* positive definite as soon as g^ζ is nonconstant for each nonzero $\zeta \in \mathbb{R}^K$.

Theorem 4.3.14. *Suppose that G is nonnegative, nonincreasing, convex and symmetric. When g^ζ is nonconstant for each nonzero $\zeta \in \mathbb{R}^K$, then G is strictly positive definite.*

Finally, we show that G being nonnegative follows from being positive definite and nonincreasing. In fact, this is a result for one-dimensional functions, but since nonincreasing and nonnegative are characterized fully by one-dimensional versions of G, the result carries over to the multi-dimensional case.

Proposition 4.3.15. *If G is positive definite and nonincreasing, then G is nonnegative.*

4.3.4 Transformations of the decay kernel

In this section we discuss certain transformations of the decay kernel G. We are interested whether the resulting transformed decay kernel is positive definite and how optimal strategies change. We begin with a result on decay kernels with equal decay speed in every component, i.e. that consist of a one-dimensional decay kernel and a "co-impact" matrix.

Proposition 4.3.16. *Let $L \in \mathbb{R}^{K\times K}$ be a symmetric and strictly positive matrix and $g : [0,\infty) \to \mathbb{R}$ a strictly positive definite function. Then $G_1(t) = Lg(t)$ is strictly positive definite, and the optimal strategy is also an optimal strategy for $G_2(t) = \tilde{L}g(t)$ for any strictly positive definite symmetric $\tilde{L} \in \mathbb{R}^{K\times K}$, i.e. the optimal strategy does not depend on L.*

So the optimal strategy only depends on the one-dimensional decay function g and not on the "co-impact" matrix L, i.e. the the optimal strategy can be computed independently for every asset by observing that we can take the identity matrix for $\tilde{L}$. The optimal strategy in one asset will be a multiple of the optimal strategies for the other assets depending solely on the initial position in this asset. This implies that cross impact does not have to be considered in this case. Cross impact is only relevant when decay speeds are different. Though, correlation will be relevant for risk-averse investors.

For linear transformations of the decay kernel, there is a similar result concerning the optimal strategies.

Proposition 4.3.17. *Let $0 \leq t_1 < t_2 < \ldots < t_N$ be given, let G_1 be symmetric and strictly positive definite and $\xi^{(1)} = (\xi_1^{(1)}, \ldots, \xi_N^{(1)})$ be the associated optimal strategy. Let $L \in \mathbb{R}^{K \times K}$ such that $G_2(t) = G_1(t)L$ is symmetric and strictly positive definite. Then the optimal strategy for G_2 is $\xi^{(1)}$.*

Note that the assumption that G_2 has to be symmetric and strictly positive definite is very strong. For example, symmetry of G_2 implies that $G_1(t)L = L^\top G_1(t)$ for all $t \geq 0$.

Next, a congruence transform of a positive definite decay kernel remains positive definite, but the optimal strategy changes.

Proposition 4.3.18. *Let G_1 be a (strictly) positive definite decay kernel, L be an invertible $K \times K$ matrix, $0 \leq t_1 < t_2 < \ldots < t_N$ be given, and $\xi^{(1)} = (\xi_1^{(1)}, \ldots, \xi_N^{(1)})$ be a corresponding optimal strategy for G_1 such that $\sum_{k=1}^N \xi_k^{(1)} = X_0$. The decay kernel $G_2(t) := L^\top G_1(t) L$ is also (strictly) positive definite. Moreover, $\xi^{(2)} := (L^{-1}\xi_1^{(1)}, \ldots, L^{-1}\xi_N^{(1)})$ is an optimal strategy for G_2 such that $\sum_{k=1}^N \xi_k^{(2)} = L^{-1}X_0$.*

Finally, the transpose of a positive definite decay kernel is positive definite and any convex combination. In particular, the symmetrization of any positive definite decay kernel is positive definite.

Proposition 4.3.19. *If G is (strictly) positive definite, its transpose $t \mapsto G(t)^\top$ and every convex combination $t \mapsto \alpha G(t) + (1-\alpha)G(t)^\top$ (with $\alpha \in [0,1]$) is (strictly) positive definite. In particular, the symmetrization $t \mapsto \frac{1}{2}(G(t) + G(t)^\top)$ is (strictly) positive definite.*

4.3.5 Commuting decay kernels

We will now discuss the class of decay kernels that are commuting. With this property we discuss convergence of optimal strategies to continuous time.

Definition 4.3.20. A decay kernel $G : [0, \infty) \to \mathbb{R}^{K \times K}$ is called *commuting* if $G(t)G(s) = G(s)G(t)$ holds for all $s, t \geq 0$.

If the decay kernel is commuting, it may be simultaneously diagonalized as set out in the following lemma.

Lemma 4.3.21. *A symmetric decay kernel G is commuting if and only if there exists an orthogonal matrix O and functions $g_1, \ldots, g_K : [0, \infty) \to \mathbb{R}$ such that*

$$G(t) = O^\top \mathrm{diag}(g_1(t), \ldots, g_K(t))\, O. \tag{4.5}$$

The properties of G are characterized via the one-dimensional functions g_i.

Proposition 4.3.22. *Let G be a symmetric and commuting decay kernel, and let O and $g_1, \dots, g_K$ be as in Lemma 4.3.21.*

(a) *G is (strictly) positive definite if and only if the functions $t \mapsto g_i(t)$ are all (strictly) positive definite for all i.*

(b) *G is nonnegative if and only if $g_i(t) \geq 0$ for all i and t.*

(c) *G is nonincreasing if and only if g_i is nonincreasing for all i.*

(d) *G is convex if and only if g_i is convex for all i.*

Remark 4.3.23. The preceding proposition characterizes when a symmetric and commuting decay kernel admits optimal strategies. These strategies can be computed by means of Proposition 4.3.18. More precisely, write X_0 as $X_0 = \sum_{i=1}^K \mu_i v_i$, where $\mu_i \in \mathbb{R}$ and $v_1, \dots, v_K \in \mathbb{R}^K$ are the (transposed) rows of O. That is, we have $(\mu_1, \dots, \mu_k)^\top = OX_0$. Let $\widetilde{\xi}^i = (\widetilde{\xi}_1^i, \dots, \widetilde{\xi}_N^i)$ be a one-dimensional optimal strategy for the initial position μ_i and the one-dimensional decay kernel $g_i(t) = g^{v_i}(t)$, and set

$$\xi_j := O^\top (\widetilde{\xi}_j^1, \dots, \widetilde{\xi}_j^K)^\top.$$

Then $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N)$ is an optimal strategy for X_0 and G.

The next result is an extension of Theorem 1 in Alfonsi *et al.* (2012).

Proposition 4.3.24. *Let G be a symmetric, nonnegative, nonincreasing, convex and commuting decay kernel. Then there exists an orthonormal basis $v_1, \dots, v_K$ of $\mathbb{R}^K$ such that, for given $0 = t_1 < t_2 < \cdots < t_N$, each initial portfolio of the form $X_0 = v_i$ admits an optimal strategy $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N)$ whose components are such that $\xi_n^k \xi_m^k \geq 0$ for all k, n, and m.*

For $K = 1$, the statement simplifies to Theorem 1 in Alfonsi *et al.* (2012), i.e. for any nonnegative, nonincreasing and convex decay kernel we have $\xi_n \xi_m \geq 0$ for an optimal strategy ξ and every n, m. Note that in case $K = 1$ every decay kernel is symmetric and commuting.

It follows from the preceding proposition that the variation of optimal strategies is uniformly bounded independent of the number of trades N. Note that it is open whether commuting is a necessary condition in Proposition 4.3.25.

Proposition 4.3.25 (Bounded variation for optimal strategies)**.** *If G is symmetric, nonnegative, nonincreasing, convex and commuting, then there is a constant $C > 0$ independent of $N \in \mathbb{N}$ and $0 \leq t_1 \leq \ldots \leq t_N \leq T$ such that for an optimal strategy ξ we have that $\sum_{n=1}^N \|\xi_n\|_1 < C$.*

4.3.6 Continuous-time strategies

Commuting decay kernels as in the last section are of particular interest since we can show existence of optimal strategies in continuous time for them. Following Gatheral *et al.* (2012), in continuous time we require a strategy $X = (X_t)_{t \geq 0}$ to be leftcontinuous and adapted, with finite and uniformly bounded variation and we require $X_{T+} = 0$. The price process is then given by

$$P_t = P_t^0 + \int_{[0,t)} G(t-s)\, dX_s$$

and the cost function is given by

$$\begin{aligned} C(X) &= \int_{[0,T]} \left(\int_{[0,t)} G(t-s)\, dX_s \right)^\top dX_t + \frac{1}{2} \sum_{t\in[0,T]} \Delta X_t^\top G(0) \Delta X_t \\ &= \frac{1}{2} \int_{[0,T]} \left(\int_{[0,T]} \tilde{G}(t-s)\, dX_s \right)^\top dX_t. \end{aligned}$$

For strategies that consist only of finitely many jumps, i.e. discrete-time strategies, it is easily seen that these definitions are equivalent to those used throughout this chapter.

To show existence of optimal strategies in continuous time, let G satisfy the requirements of Proposition 4.3.24 and let $X_0 = v_i$ with a v_i from Proposition 4.3.24. As in the proof in Theorem 2.20 in Gatheral *et al.* (2012) it follows that a continuous-time optimal strategy exists. For more general X_0 note that the optimal strategy is linear in X_0, so continuous-time optimal strategies exist for them as well.

4.4 Examples

4.4.1 Linear and exponential decay for two assets

In this section, we study to examples for $K = 2$ assets. We will discuss linear decay of impact and exponential decay of impact. These are the two most tractable and simple forms of impact. We allow for general forms of these impacts and check under which conditions the decay kernel G is nonnegative, nonincreasing, convex or positive definite. Finally we study an example with exponential decay and show an optimal strategy.

First we study linear decay of impact. Note that the positive definiteness of similar decay kernels is analyzed in a more general setup in Bevilacqua *et al.* (2012).

Proposition 4.4.1. *Let*

$$G(t) = \begin{pmatrix} (a_{11} - b_{11}t)^+ & (a_{12} - b_{12}t)^+ \\ (a_{21} - b_{21}t)^+ & (a_{22} - b_{22}t)^+ \end{pmatrix}$$

with $a_{11}, a_{12}, a_{21}, a_{22}, b_{11}, b_{12}, b_{21}, b_{22} > 0$.

(a) *G is nonnegative if and only if* $\max\{\frac{a_{12}}{b_{12}}, \frac{a_{21}}{b_{21}}\} \le \min\{\frac{a_{11}}{b_{11}}, \frac{a_{22}}{b_{22}}\}$ *and* $\frac{1}{4}(a_{12} + a_{21})^2 \le a_{11}a_{22}$.

(b) *G is nonincreasing if and only if* $\max\{\frac{a_{12}}{b_{12}}, \frac{a_{21}}{b_{21}}\} \le \min\{\frac{a_{11}}{b_{11}}, \frac{a_{22}}{b_{22}}\}$ *and* $\frac{1}{4}(b_{12} + b_{21})^2 \le b_{11}b_{22}$.

(c) *Let* $\max\{\frac{a_{12}}{b_{12}}, \frac{a_{21}}{b_{21}}\} \leq \min\{\frac{a_{11}}{b_{11}}, \frac{a_{22}}{b_{22}}\}$. *Then G is positive definite if and only if G is symmetric (i.e. $a_{12} = a_{21}$ and $b_{12} = b_{21}$), $\frac{a_{11}}{b_{11}} = \frac{a_{12}}{b_{12}} = \frac{a_{22}}{b_{22}}$ and $b_{12}^2 \leq b_{11}b_{22}$.*

(d) *G is nonnegative and positive definite if and only if it is symmetric, $\frac{a_{11}}{b_{11}} = \frac{a_{12}}{b_{12}} = \frac{a_{22}}{b_{22}}$, $a_{12}^2 \leq a_{11}a_{22}$ and $b_{12}^2 \leq b_{11}b_{22}$. In this case, G is also nonincreasing and convex.*

(e) *If G is symmetric and $\frac{a_{11}}{b_{11}} = \frac{a_{12}}{b_{12}} = \frac{a_{22}}{b_{22}}$, G is commuting if and only if $a_{12}(b_{11} - b_{22}) = b_{12}(a_{11} - a_{22})$.*

Next, we study component-wise exponential decay of impact.

Proposition 4.4.2. *Let*

$$G(t) = \begin{pmatrix} a_{11}\exp(-b_{11}t) & a_{12}\exp(-b_{12}t) \\ a_{21}\exp(-b_{21}t) & a_{22}\exp(-b_{22}t) \end{pmatrix}$$

with $a_{11}, a_{12}, a_{21}, a_{22}, b_{11}, b_{12}, b_{21}, b_{22} > 0$.

(a) *G is nonnegative if and only if $\min\{b_{12}, b_{21}\} \geq \frac{1}{2}(b_{11} + b_{22})$ and $\frac{1}{4}(a_{12} + a_{21})^2 \leq a_{11}a_{22}$.*

(b) *G is nonincreasing if and only if $\min\{b_{12}, b_{21}\} \geq \frac{1}{2}(b_{11} + b_{22})$ and $\frac{1}{4}(a_{12}b_{12} + a_{21}b_{21})^2 \leq a_{11}b_{11}a_{22}b_{22}$.*

(c) *G is convex if and only if $\min\{b_{12}, b_{21}\} \geq \frac{1}{2}(b_{11} + b_{22})$ and $\frac{1}{4}(a_{12}b_{12}^2 + a_{21}b_{21}^2)^2 \leq a_{11}b_{11}^2a_{22}b_{22}^2$.*

(d) *Let G be nonincreasing, $a_{12} = a_{21}$ and $b_{11}^2 + b_{22}^2 \leq b_{12}^2 + b_{21}^2$. Then G is positive definite.*

(e) *G is commuting if and only if either $b_{11} = b_{12} = b_{21} = b_{22}$, or $b_{11} = b_{22}$ and $b_{12} = b_{21}$ and $a_{11} = a_{22}$.*

For the following simpler decay kernel, the results follow immediately from the preceding proposition.

Corollary 4.4.3. *Let $\rho, \kappa, \tilde{\kappa} > 0$ and*

$$G(t) = \begin{pmatrix} \exp(-\kappa t) & \rho\exp(-\tilde{\kappa}t) \\ \rho\exp(-\tilde{\kappa}t) & \exp(-\kappa t) \end{pmatrix}.$$

(a) *G is nonnegative if and only if $\frac{\kappa}{\tilde{\kappa}} \leq 1$ and $\rho \leq 1$.*

(b) *G is nonincreasing if and only if $\rho \leq \frac{\kappa}{\tilde{\kappa}} \leq 1$.*

(c) *G is convex if and only if $\rho \leq \frac{\kappa^2}{\tilde{\kappa}^2} \leq \frac{\kappa}{\tilde{\kappa}} \leq 1$.*

(d) *If G is nonincreasing, G is positive definite. In this case, it is also nonnegative.*

(e) *G is commuting.*

Example 4.4.4. Let G as in Corollary 4.4.3 with $\kappa = 1, \tilde{\kappa} = 1.8, \rho = 0.3$. Since $\rho \leq \frac{\kappa^2}{\tilde{\kappa}^2} \leq 1$, G is convex. In Figure 4.1 an example for an optimal strategy with this decay kernel G is plotted. It can be seen that the optimal strategy switches from selling to buying in between. In the one-dimensional case, in Theorem 1 in Alfonsi *et al.* (2012) it is proven that for convex decay kernels there is no transaction-triggered price manipulation, i.e. an optimal strategy does not switch between buying and selling but is monotone. The example shows that such a result does not hold in the multivariate case with multiple assets. However, in Proposition 4.3.25 we find a bound on the variation of the optimal strategy which excludes a behavior as in Example 4 in Alfonsi *et al.* (2012).

Proposition 4.4.5. *Let*

$$G(t) = \begin{pmatrix} \exp(-t \wedge 1) & \frac{1}{8}\exp(-2(t \wedge 1)) \\ \frac{1}{8}\exp(-3(t \wedge 1)) & \exp(-t \wedge 1) \end{pmatrix}.$$

G is continuous, convex, nonincreasing and nonnegative, but not positive definite.

This shows that we need symmetry in Theorem 4.3.14. Furthermore, the characterization via one-dimensional functions g as in part (b) of Theorem 4.3.9 does not hold in the nonsymmetric case.

4.4.2 Matrix functions

We are now going to discuss a special situation in which commuting decay kernels arise in a natural manner. To this end, let $g : [0, \infty) \to \mathbb{R}$ a function and $B \in S_+(K)$ be a nonnegative symmetric matrix. Then there exists an orthogonal matrix O such that $B = O^\top \mathrm{diag}(\rho_1, \ldots, \rho_K)O$, where $\rho_1, \ldots, \rho_K \geq 0$ are the eigenvalues of B. As usual, the matrix $g(B) \in S(K)$ is defined as $g(B) := O^\top \mathrm{diag}(g(\rho_1), \ldots, g(\rho_K))O$; see, e.g., Donoghue (1974). We can thus define a matrix-valued function $G : [0, \infty) \to S(K)$ by

$$G(t) = g(tB) = O^\top \mathrm{diag}(g(t\rho_1), \ldots, g(t\rho_K))O, \qquad t \geq 0. \tag{4.6}$$

Clearly, G is commuting. Moreover, it is of the form (4.5) with $g_i(t) = g(t\rho_i)$, and so Proposition 4.3.22 characterizes the properties of G. If $g : [0, \infty) \to [0, \infty)$ is nonincreasing, convex and not constant, it follows immediately from Theorem 4.3.14 and Proposition 4.3.22 that G is strictly positive definite.

Example 4.4.6 (Matrix exponentials)**.** Here we extend the discrete-time model of Obizhaeva and Wang (2013) and the results of Alfonsi *et al.* (2008) to a multivariate setting. For $B = O^\top \mathrm{diag}(\rho_1, \ldots, \rho_K)O \in S_+(K)$, the matrix exponential,

$$G(t) = \exp(-tB),$$

is of the form (4.6) with $g(t) = e^{-t}$. It follows that G is nonnegative, nonincreasing, and convex. In particular, G is positive definite. When B is strictly positive, as we

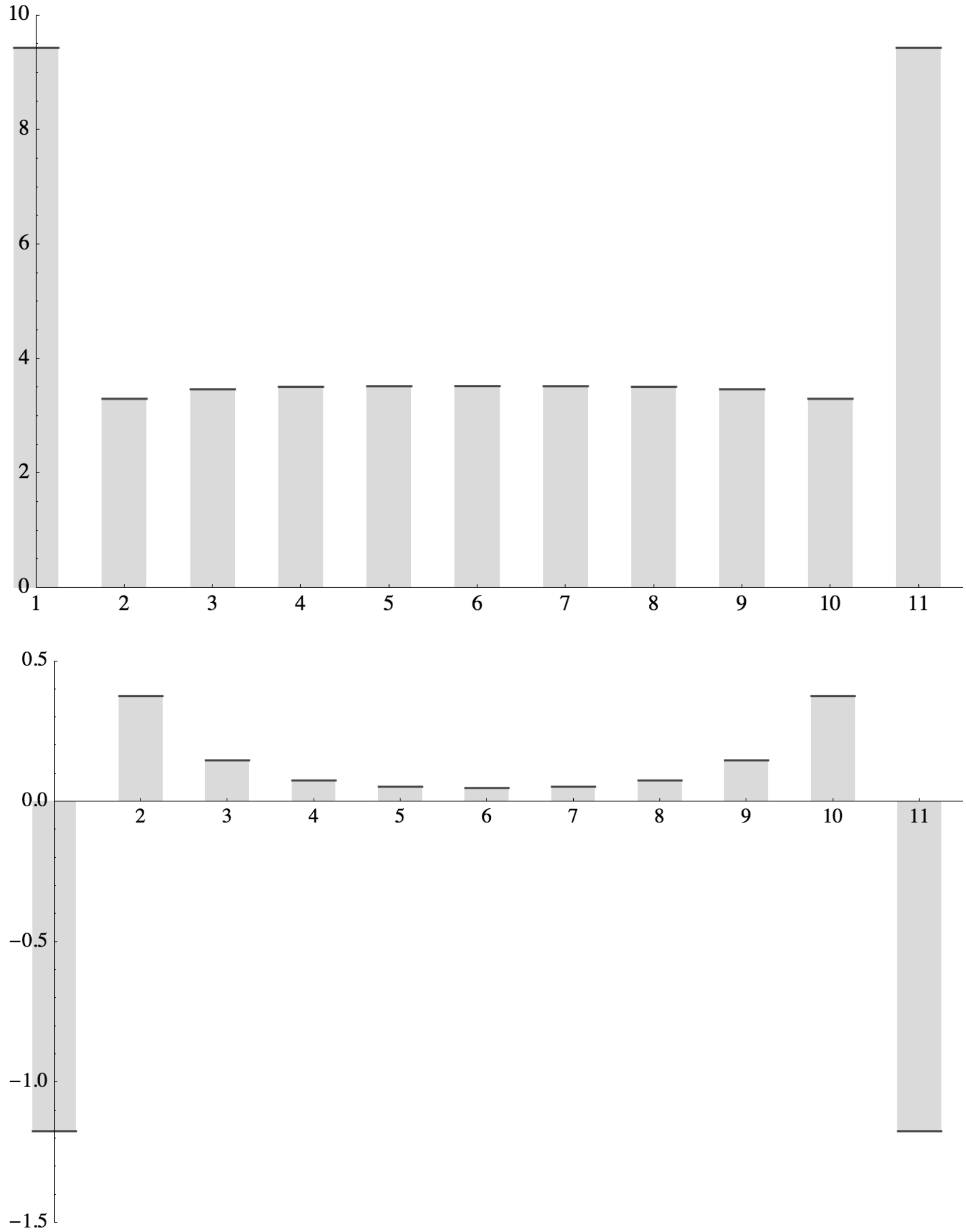

Figure 4.1: Optimal strategy ξ for Example 4.4.4 with $X_0 = (-50, 1)^\top, T = 5, N = 11$. Top: $\xi_1^1, \dots \xi_{11}^1$, bottom: $\xi_1^2, \dots, \xi_{11}^2$.

will assume from now on, G is even strictly positive definite. We now compute the optimal strategy $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_N)$ for an initial portfolio $X_0 \in \mathbb{R}^K$ and time points $0 = t_1 < t_2 < \cdots < t_N$. To this end, we proceed as in Remark 4.3.23 and the proof of Proposition 4.3.24 and first compute the optimal strategy $(x_1^i, \ldots, x_N^i)$ for the initial position y^i and for the one-dimensional decay kernel $g_i(t) = e^{-t\rho_i}$. Theorem 3.1 in Alfonsi *et al.* (2008) implies that, with the notations

$$a_n^i := e^{-(t_n - t_{n-1})\rho_i} \qquad \text{and} \qquad \lambda_i := \frac{-y_i}{\frac{2}{1+a_2^i} + \sum_{n=3}^N \frac{1-a_n^i}{1+a_n^i}},$$

the optimal strategy $(x_1^i, \ldots, x_N^i)$ is given by the initial market order is

$$x_1^i = \frac{\lambda_i}{1+a_2^i},\; x_n^i = \lambda_i\Big(\frac{1}{1+a_n^i} - \frac{a_{n+1}^i}{1+a_{n+1}^i}\Big) \text{ for } n = 2, \ldots, N-1, \text{ and } x_N^i = \frac{\lambda_i}{1+a_N^i}.$$

Using the recipe of Remark 4.3.23 we can now compute the optimal strategy $\boldsymbol{\xi}$. Consider the optimal strategy $\tilde{\xi}$ for the decay kernel $\operatorname{diag}(\exp(-\rho_1 t), \ldots, \exp(-\rho_K t))$ and initial position OX_0. When defining

$$Q_n = \operatorname{diag}(\exp(-\rho_1(t_n - t_{n-1})), \ldots, \exp(-\rho_K(t_n - t_{n-1})))$$

and $\tilde{\lambda} = -\Big(2(\mathrm{Id} + Q_2)^{-1} + \sum_{n=3}^N (\mathrm{Id} - Q_n)(\mathrm{Id} + Q_n)^{-1}\Big)^{-1} OX_0$, the optimal strategy is given by $\tilde{\xi}_1 = (1 + Q_2)^{-1}\tilde{\lambda}$, $\tilde{\xi}_n = (\mathrm{Id} + Q_n)^{-1}\lambda - Q_{n+1}(\mathrm{Id} + Q_{n+1})^{-1}\lambda$ for $n = 2, \ldots, N-1$ and $\tilde{\xi}_n = (\mathrm{Id} + Q_N)^{-1}\lambda$. By Remark 4.3.23, the optimal strategy for G and X_0 is now given by $\boldsymbol{\xi} = O^T\tilde{\xi}$.

To remove O from these expressions, define $A_n = e^{-(t_n - t_{n-1})B} = O^\top Q_n O$ and

$$\lambda := -\Big[2\big(\mathrm{Id} + A_2\big)^{-1} + \sum_{i=3}^N \big(\mathrm{Id} - A_i\big)\big(\mathrm{Id} + A_i\big)^{-1}\Big]^{-1} X_0.$$

By observing that $(\mathrm{Id} + A_n)^{-1} = O^\top(\mathrm{Id} + Q_n)^{-1}O$ and subsequently $\lambda = O^\top\tilde{\lambda}$, we find that the components of the optimal stratey $\boldsymbol{\xi}$ are

$$\begin{aligned} \xi_1 &= \big(\mathrm{Id} + A_2\big)^{-1}\lambda, \\ \xi_n &= \big(\mathrm{Id} + A_n\big)^{-1}\lambda - A_{n+1}\big(\mathrm{Id} + A_{n+1}\big)^{-1}\lambda \quad \text{for } n = 2, \ldots, N-1, \\ \xi_N &= \big(\mathrm{Id} + A_N\big)^{-1}\lambda. \end{aligned}$$

Let us finally consider the situation of an equidistant time grid, $t_i = \frac{i-1}{N-1}$. In this case, all matrices A_i are equal to a single matrix A. Our formula for λ then becomes

$$\lambda = -(\mathrm{Id} + A)\Big(N\mathrm{Id} - (N-2)A\Big)^{-1} X_0$$

The formula for the optimal strategy thus simplifies to

$$\begin{aligned} \xi_1 &= -\Big(N\mathrm{Id} - (N-2)A\Big)^{-1} X_0, \\ \xi_i &= (\mathrm{Id} - A)\xi_1, \qquad \text{for } i = 2, \ldots, N-1, \\ \xi_N &= \xi_1. \end{aligned}$$

Example 4.4.7 (Completely monotone decay)**.** Let μ be a Borel measure on the set of symmetric nonnegative definite $K \times K$ matrices. Then the function

$$G(t) := \int e^{-tB}\, \mu(dB) \tag{4.7}$$

can be regarded as a matrix-valued completely monotone function. It is also a mixture of the matrix exponential functions from Example 4.4.6 and thus inherits the properties of being symmetric, positive definite, nonincreasing, and convex. It therefore is positive definite.

These completely monotone functions can be characterized by a generalized Hausdorff-Bernstein-Widder Theorem, i.e. they can be represented as the Laplace transform of a nonnegative matrix-valued measure.

Theorem 4.4.8 (Hausdorff-Bernstein-Widder Theorem for matrix-valued functions)**.** *Let $G : [0, \infty) \to \mathbb{R}^{K\times K}$ be symmetric. It is equivalent:*

(a) *G is completely monotone, i.e. continuous on $[0, \infty)$, infinitely differentiable on $(0, \infty)$ and $(-1)^n G^{(n)}(t)$ be nonnegative for all $n = 0, 1, \ldots$ and $t > 0$.*

(b) *There is a symmetric nonnegative matrix-valued measure M such that*

$$G(t) = \int_0^\infty e^{-tx} M(dx). \tag{4.8}$$

(c) *There is a Borel measure μ on the set of symmetric nonnegative definite $K \times K$ matrices such that (4.7) holds.*

Remark 4.4.9. Note the difference in the representations (4.7) and (4.8). Given a representation (4.7), we give the corresponding representation (4.8) in the following. Note that the (simpler) other way round is presented in the proof. Since $\mu = \int \delta_B \mu(dB)$, we will restrict ourselves for simplicity to simple measures $\mu(dB) = \delta_A(dB)$ with $A \in S_+(K)$, i.e. $\mu(E) = \mathbb{1}_{\{A\in E\}}$A. So $G(t) = e^{-tA} = O^\top e^{-tD} O$ with a diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_K)$. We can also write $D = \mathrm{diag}(d_1, 0, \ldots, 0) + \mathrm{diag}(0, d_2, 0, \ldots, 0) + \ldots + \mathrm{diag}(0, \ldots, 0, d_K)$ and we thus easily see that the measure M in (4.8) is $M(dx) = O^\top \mathrm{diag}(1, 0, \ldots, 0) O\, \delta_{d_1}(dx) + O^\top \mathrm{diag}(0, 1, 0, \ldots, 0) O\, \delta_{d_2}(dx) + \ldots + O^\top \mathrm{diag}(0, \ldots, 0, 1) O\, \delta_{d_K}(dx)$.

We now provide a short discussion of decay kernels that arise by applying a matrix function to a *nonsymmetric* matrix B. For the sake of simplicity, we concentrate on the case in which B is a 2×2 Jordan block of the form

$$B = \begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix},$$

where $b > 0$. Assume that $f : [0, \infty) \to [0, \infty)$ is analytic, i.e. we have $f(x) = \sum_{n=0}^\infty f^{(n)}(0)x^n$. In this case we can define $f(M) = \sum_{n=0}^\infty f^{(n)}(0)M^n$ for any quadratic matrix M, if the series converges. For $t \geq 0$, we now set

$$G(t) := f(tB) = \begin{pmatrix} f(tb) & tf'(tb) \\ 0 & f(tb) \end{pmatrix}.$$

Let us suppose next that f satisfies $f(t) \to 0$ and $f'(t) \to 0$ for $t \uparrow \infty$. Furthermore assume that f'' is integrable on $[0, \infty)$. Then

$$f(t) = \int_0^\infty (x-t)^+ f''(x)\,dx \qquad \text{and} \qquad f'(t) = -\int_t^\infty f''(x)\,dx. \tag{4.9}$$

It follows that

$$\tilde{G}(t) = \int_0^\infty \begin{pmatrix} (x-|t|b)^+ & -t\mathbb{1}_{\{0<t<x/b\}} \\ t\mathbb{1}_{\{0>t>-x/b\}} & (x-|t|b)^+ \end{pmatrix} f''(x)\,dx. \tag{4.10}$$

As a function of t, $(x-|t|b)^+$ is the Fourier transform of $z \mapsto \frac{b}{\pi z^2}(1-\cos(xz/b))$; see, e.g., Lemma 4.2 in Gatheral *et al.* (2012). Using the software Mathematica we found that $-t\mathbb{1}_{\{\{0<t<x/b\}\}}$ is the Fourier transform of $z \mapsto \frac{1}{\pi z^2}(1-e^{-ixz/b}(1+ixz/b))$. It follows that it can be represented as the Fourier transform of the following function $M(z)$, which takes values in the set of Hermitian matrices:

$$M(z) = \frac{1}{\pi z^2}\int_0^\infty \begin{pmatrix} b(1-\cos(xz/b)) & \frac{1}{2}(1-e^{-ixz/b}(1+ixz/b)) \\ \frac{1}{2}(1-e^{ixz/b}(1-ixz/b)) & b(1-\cos(xz/b)) \end{pmatrix} f''(x)\,dx. \tag{4.11}$$

This representation is quite similar to the one in Proposition 4.5.14, but there is also a significant difference: the integrand matrix is in general not nonnegative. To see this, it is sufficient to compute its determinant:

$$\frac{6b^4 + 2b\left(b^3\cos\left(\frac{2xz}{b}\right) + (b-4b^3)\cos\left(\frac{xz}{b}\right) + xz\sin\left(\frac{xz}{b}\right)\right) - 2b^2 - x^2z^2}{4b^2},$$

which is negative for sufficiently large xz.

Example 4.4.10 (Nonsymmetric matrix exponential decay)**.** Let $f(t) = e^{-t}$ so that

$$G(t) = e^{-tB} = \begin{pmatrix} \exp(-tb) & -t\exp(-tb) \\ 0 & \exp(-tb). \end{pmatrix}$$

G is not symmetric, not nonnegative, not nonincreasing and not convex. But G is positive definite if and only if $b \geq 1/2$. To see this, we integrate (4.11) with $f''(x) = e^{-x}$ and get

$$M(z) = \frac{1}{\pi}\begin{pmatrix} \frac{b}{b^2+z^2} & \frac{1}{2(z-ib)^2} \\ \frac{1}{2(ib+z)^2} & \frac{b}{b^2+z^2} \end{pmatrix}.$$

We have

$$\det M(z) = \frac{4b^2-1}{4(b^2+z^2)^2},$$

so $M(z)$ is nonnegative if and only if $b \geq 1/2$.

Example 4.4.11 (Nonsymmetric power law decay)**.** Let $f(t) = (c+t)^{-\alpha}$ for $c, \alpha > 0$. Then

$$G(t) = f(tB) = \begin{pmatrix} (c+tb)^{-\alpha} & -\alpha t(c+tb)^{-\alpha-1} \\ 0 & (c+tb)^{-\alpha} \end{pmatrix}.$$

For $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^2$ we have

$$g^{\zeta}(t) = (c+tb)^{-\alpha}\Big(\zeta_1^2 + \zeta_2^2 - \frac{\alpha t}{c+tb}\zeta_1\zeta_2\Big).$$

Hence, G is nonnegative if and only if $\alpha \leq 2b$, nonincreasing if and only if $\alpha \leq 2b$ and $b \geq \frac{1}{2}$ and convex if and only if $\alpha \leq 2b$ and $b \geq 1$.

4.4.3 Exponential decay in continuous time

The conditions of Proposition 4.3.24 are clearly satisfied when $G(t) = e^{-tB}$ as in Example 4.4.6, and so optimal strategies exist also in continuous time for this decay kernel. In fact, for every $X_0 \in \mathbb{R}^K$ the corresponding optimal strategy over the interval $[0, T]$ can be given in closed form. To this end, recall first that B is assumed to be symmetric and nonnegative, so that the matrix $C := 2\mathrm{Id} + TB$ is symmetric and strictly positive definite, hence invertible. We will show below that the unique optimal strategy is then given by

$$X_t^* = (\mathrm{Id} + (T-t)B)C^{-1}X_0, \qquad 0 < t < T,$$

which extends the main result from Obizhaeva and Wang (2013). Since X^* satisfies the side conditions $X_0^* = X_0$ and $X_T^* = 0$, the strategy X^* must have the following initial and terminal jumps:

$$\Delta X_0^* = (\mathrm{Id} + TB)C^{-1}X_0 - X_0 = (\mathrm{Id} + TB - C)C^{-1}X_0 = -C^{-1}X_0$$

and

$$\Delta X_T^* = -C^{-1}X_0.$$

We therefore have

$$dX_t^* = -(\mathrm{Id}\,\delta_0(dt) + B\,dt + \mathrm{Id}\,\delta_T(dt))C^{-1}X_0. \tag{4.12}$$

To check that this is indeed the unique optimal strategy, we use the fact that $\frac{d}{ds}e^{-sB} = -e^{-sB}B$ and compute from (4.12) that

$$\begin{aligned}
&\int_{[0,T]} e^{-|t-s|B}\,dX_s^* \\
&= -\Big(e^{-tB} + \int_0^t e^{-(t-s)B}B\,ds + \int_t^T e^{-(s-t)B}B\,ds + e^{-(T-t)B}\Big)C^{-1}X_0 \\
&= -2C^{-1}X_0,
\end{aligned}$$

independently of $t \in [0, T]$. An analogon to Proposition 4.3.10 holds in continuous time, see Theorem 2.11 in Gatheral *et al.* (2012). Thus, X^* is indeed optimal. This argument extends the one from Example 2.12 in Gatheral *et al.* (2012).

4.5 Proofs

Proof of Prop. 4.2.2. For $\varepsilon \leq \min_{k\in\{1,\dots,N-1\}}(t_{k+1}-t_k)$

$$
\begin{aligned}
\mathcal{R}^\varepsilon &= -\int_0^{T+\varepsilon} (P_t^\varepsilon)^\top \, dX_t^\varepsilon \\
&= -\int_0^{T+\varepsilon} (P_t^0)^\top \, dX_t^\varepsilon - \sum_{k=1}^N \int_{t_k}^{t_k+\varepsilon} \left(\int_0^t G(t-s)\, dX_s^\varepsilon \right)^\top \xi_k \frac{1}{\varepsilon}\, dt \\
&= -\int_0^{T+\varepsilon} (P_t^0)^\top \, dX_t^\varepsilon \\
&\quad - \sum_{k=1}^N \int_{t_k}^{t_k+\varepsilon} \left(\sum_{l=1}^{k-1} \int_{t_l}^{t_l+\varepsilon} G(t-s)\xi_l \frac{1}{\varepsilon}\, ds + \int_{t_k}^t G(t-s)\xi_k \frac{1}{\varepsilon}\, ds \right)^\top \xi_k \frac{1}{\varepsilon}\, dt \\
&= -\int_0^{T+\varepsilon} (P_t^0)^\top \, dX_t^\varepsilon - \sum_{k=1}^N \sum_{l=1}^{k-1} \frac{1}{\varepsilon^2} \xi_k^\top \left(\int_{t_k}^{t_k+\varepsilon} \int_{t_l}^{t_l+\varepsilon} G(t-s)\, ds\, dt \right) \xi_l \\
&\quad - \sum_{k=1}^N \frac{1}{\varepsilon^2} \xi_k^\top \left(\int_{t_k}^{t_k+\varepsilon} \int_{t_k}^t G(t-s)\, ds\, dt \right) \xi_k.
\end{aligned}
$$

Integration by parts yields $-\int_0^{T+\varepsilon}(P_t^0)^\top\, dX_t^\varepsilon = X_0^\top P_0^0 - X_{T+\varepsilon}^\top P_{T+\varepsilon}^0 + \int_0^{T+\varepsilon}(X_t^\varepsilon)^\top\, dP_t^0$. Since $X_{T+\varepsilon} = 0$ and P^0 is a martingale, we find $\mathbb{E}[-\int_0^{T+\varepsilon}(P_t^0)^\top\, dX_t^\varepsilon] = X_0^\top P_0^0$. Furthermore,

$$
\lim_{\varepsilon\downarrow 0} \sum_{k=1}^N \sum_{l=1}^{k-1} \frac{1}{\varepsilon^2} \xi_k^\top \left(\int_{t_k}^{t_k+\varepsilon} \int_{t_l}^{t_l+\varepsilon} G(t-s)\, ds\, dt \right) \xi_l = \sum_{k=1}^N \sum_{l=1}^{k-1} \xi_k^\top G(t_k - t_l)\xi_l
$$

and $\lim_{\varepsilon\downarrow 0} \sum_{k=1}^N \frac{1}{\varepsilon^2}\xi_k^\top \left(\int_{t_k}^{t_k+\varepsilon} \int_{t_k}^t G(t-s)\, ds\, dt \right) \xi_k = \frac{1}{2}\xi_k^\top G(0)\xi_k$, so by dominated convergence the result follows. □

Now we prepare for the proof of Proposition 4.2.5. We need to prove Proposition 4.3.5 first. Then we show a relation between positive definiteness of G and convexity of C. Furthermore, we show that C can be represented as a quadratic form with respect to a scalar product.

Proof of Proposition 4.3.5. We can write

$$
C(\xi) = \frac{1}{2}\sum_{k=1}^N \sum_{l=k}^{k} \xi_k^\top \tilde{G}(t_k - t_l)\xi_l + \frac{1}{2}\sum_{k=1}^N \sum_{l=1}^{k-1} \xi_k^\top \tilde{G}(t_k - t_l)\xi_l + \frac{1}{2}\sum_{k=1}^N \sum_{l=k+1}^{N} \xi_l^\top \tilde{G}(t_l - t_k)\xi_k.
$$

Since $\xi_l^\top \tilde{G}(t_l - t_k)\xi_k = \xi_k^\top \tilde{G}(t_l - t_k)^\top \xi_l = \xi_k^\top \tilde{G}(t_k - t_l)\xi_l$, the result follows. Note that in the last equality, we used $t_k - t_l \neq 0$ for $l > k$. If this would not be required, it is sufficient to have $G(0) = G(0)^\top$, since in this case $G(t_l - t_k)^\top = G(t_k - t_l)$ is also true for $t_k = t_l$. □

Proposition 4.5.1. *G is (strictly) positive definite if and only if C is (strictly) convex.*

Proof. We define the bilinear form $\mathcal{C}(\tilde{\xi}, \hat{\xi}) := \frac{1}{2}\sum_{i,j=1}^N \tilde{\xi}_i^\top \tilde{G}(t_i - t_j)\hat{\xi}_j$, so by Proposition 4.3.5 it is $C(\xi) = \mathcal{C}(\xi, \xi)$. G is positive definite if and only if we have $C(\tilde{\xi} - \hat{\xi}) = C(\tilde{\xi}) - \mathcal{C}(\tilde{\xi}, \hat{\xi}) - \mathcal{C}(\hat{\xi}, \tilde{\xi}) + C(\hat{\xi}) \geq 0$ for all $\tilde{\xi}, \hat{\xi}$. If $t \in (0, 1)$, this is equivalent to

$$\begin{aligned} C(t\tilde{\xi} + (1-t)\hat{\xi}) &= t^2 C(\tilde{\xi}) + t(1-t)\mathcal{C}(\tilde{\xi}, \hat{\xi}) + t(1-t)\mathcal{C}(\hat{\xi}, \tilde{\xi}) + (1-t)^2 C(\hat{\xi}) \\ &\leq tC(\tilde{\xi}) + (1-t)C(\hat{\xi}), \end{aligned}$$

which is the definition of convexity. Equivalence of strict convexity and strictly positive definiteness of C follows analogously. □

Remark 4.5.2. For $A, B \in \mathbb{R}^{K\times N}$, let $\langle A, B\rangle = \operatorname{tr}(A^\top B) = \sum_{k=1}^K \sum_{l=1}^N a_{k,l} b_{k,l}$ be the Hilbert-Schmidt inner product on $R^{K\times N}$. We can arrange the vectors $\xi_1, \ldots \xi_N \in \mathbb{R}^K$ as a matrix $\xi \in \mathbb{R}^{K\times N}$, where we denote by ξ_i the ith column of ξ. If we define the linear operator L on $R^{K\times N}$ by

$$L(\xi) = \left(\sum_{l=1}^N \tilde{G}(t_1 - t_l)\xi_l, \sum_{l=1}^N \tilde{G}(t_2 - t_l)\xi_l, \ldots, \sum_{l=1}^N \tilde{G}(t_N - t_l)\xi_l \right),$$

we have $C(\xi) = \frac{1}{2}\langle \xi, L(\xi)\rangle$. Note that L is self-adjoint.

Proof of Prop. 4.2.5. By Remark 4.5.2, we can write $C(\xi) = \langle \xi, L(\xi)\rangle$. Since L is self-adjoint, there exists an orthonormal basis of eigenvectors $v_1, \ldots v_{K\cdot N}$ with eigenvalues $\lambda_1, \ldots, \lambda_{K\cdot N}$ of L. If G is positive definite, then all eigenvalues are nonnegative. Let the eigenvalues be ordered such that $\lambda_1 = \ldots = \lambda_{i-1} = 0$ and $\lambda_i \leq \ldots \leq \lambda_{K\cdot N}$. The affine space of admissible strategies is denoted by $A = \{\xi \in \mathbb{R}^{K\times N} | \sum_{k=1}^N \xi_k = -X_0\}$. With the representation $\xi = \sum_{n=1}^{K\cdot N} c_n v_n$, we have that $\|\xi\|_L = \langle \xi, L(\xi)\rangle = \sum_{n=i}^{K\cdot N} \lambda_n c_n^2$ is a seminorm, and a norm on the quotient space $\mathbb{R}^{K\times N}/\operatorname{span}(v_1, \ldots, v_{i-1})$. Obviously a minimizer of $\min_{\xi\in A} \|\xi\|_L$ exists in $\mathbb{R}^{K\times N}/\operatorname{span}(v_1, \ldots, v_{i-1})$, which leads also to a minimizer of the original problem.

When G is even strictly positive definite, C is strictly convex by Proposition 4.5.1. This yields immediately the uniqueness of optimal strategies. □

We prepare now for the proof of Theorem 4.3.3. First, we have to check under which conditions our definition of positive definiteness is equivalent to the common definition in the literature, which we call "complex positive definiteness". It will turn out that these definitions are equivalent if and only if $G(0)$ is symmetric. But if G is continuous and positive definite, $G(0)$ is necessarily symmetric. So both definitions are equivalent if G is continuous.

Definition 4.5.3. A function $H : \mathbb{R} \to \mathbb{C}^{K\times K}$ is called *complex positive definite*, if

$$\sum_{k,l=1}^N \xi^* H(t_k - t_l)\xi_l \geq 0$$

for all $N \in \mathbb{N}$, all $\xi_1, \ldots \xi_N \in \mathbb{C}^K$ and all $t_1, \ldots, t_N \in \mathbb{R}$. We call H *strictly* complex positive definite, if additionally equality holds only for $\xi_1 = \ldots = \xi_N = 0$.

The next result is well known.

Proposition 4.5.4. *Let $H : \mathbb{R} \to \mathbb{C}^{K\times K}$ be complex positive definite. Then*

(a) *$H(0)$ is positive definite.*

(b) *$H(t) = H(-t)^*$ for all $t \in \mathbb{R}$.*

Proof. (a) follows from $N = 1$.

To show (b), choose $N = 2$ and $t_2 = t_1 + t$. It follows immediately that for every $\xi_1, \xi_2 \in \mathbb{C}^K$ we need to have $\xi_1^* H(-t)\xi_2 + \xi_2^* H(t)\xi_1 \in \mathbb{R}$. Let $\xi_1 = c_1 e_i$ and $\xi_2 = c_2 e_j$ with $c_1, c_2 \in \mathbb{C}$ and $i, j \in \mathbb{N}$. It follows that $\overline{c_1} c_2 H_{ij}(t) + c_1 \overline{c_2} H_{ji}(t) \in \mathbb{R}$. Choosing $c_1 = c_2 = 1$ yields $\mathrm{Im}(H_{ij}(t)) = -\mathrm{Im}(H_{ji}(t))$ and $c_1 = 1, c_2 = i$ yields $\mathrm{Re}(H_{ij}(t)) = \mathrm{Re}(H_{ji}(t))$. So (b) is shown. □

Proposition 4.5.5. *Let $G : [0, \infty) \to \mathbb{R}^{K\times K}$. The following are equivalent:*

(a) *G is (strictly) positive definite and $G(0)$ is symmetric.*

(b) *$\tilde{G}$ is (strictly) complex positive definite.*

Proof. (b)⇒(a) follows with Proposition 4.5.4(a). So it remains to show (a)⇒(b).

Let G be positive definite and $G(0)$ be symmetric. Let furthermore $N \in \mathbb{N}$, $\xi_1, \dots, \xi_N \in \mathbb{C}^K$ and $t_1, \dots, t_N \in \mathbb{R}$. We have

$$\begin{aligned}
&\sum_{k,l=1}^{N} \xi_k^* \tilde{G}(t_k - t_l)\xi_l = \\
&= \sum_{k,l=1}^{N} (\mathrm{Re}(\xi_k) - i\,\mathrm{Im}(\xi_k))^\top \tilde{G}(t_k - t_l)(\mathrm{Re}(\xi_l) + i\,\mathrm{Im}(\xi_l)) \\
&= \sum_{k,l=1}^{N} \mathrm{Re}(\xi_k)^\top \tilde{G}(t_k - t_l)\,\mathrm{Re}(\xi_l) + \sum_{k,l=1}^{N} \mathrm{Im}(\xi_k)^\top \tilde{G}(t_k - t_l)\,\mathrm{Im}(\xi_l) \\
&\quad + i\left(-\sum_{k,l=1}^{N} \mathrm{Im}(\xi_k)^\top \tilde{G}(t_k - t_l)\,\mathrm{Re}(\xi_l) + \sum_{k,l=1}^{N} \mathrm{Re}(\xi_k)^\top \tilde{G}(t_k - t_l)\,\mathrm{Im}(\xi_l) \right).
\end{aligned}$$

Because of the symmetry of $G(0)$ we know that for all $t_k, t_l \in \mathbb{R}$ we have $\tilde{G}(t_k - t_l)^\top = \tilde{G}(t_l - t_k)$, and thus, $\mathrm{Re}(\xi_k)^\top \tilde{G}(t_k - t_l)\,\mathrm{Im}(\xi_l) = (\mathrm{Re}(\xi_k)^\top \tilde{G}(t_k - t_l)\,\mathrm{Im}(\xi_l))^\top = \mathrm{Im}(\xi_l)^\top \tilde{G}(t_k - t_l)^\top\,\mathrm{Re}(\xi_k) = \mathrm{Im}(\xi_l)^\top \tilde{G}(t_l - t_k)\,\mathrm{Re}(\xi_k)$. So

$$\begin{aligned}
&-\sum_{k,l=1}^{N} \mathrm{Im}(\xi_k)^\top \tilde{G}(t_k - t_l)\,\mathrm{Re}(\xi_l) + \sum_{k,l=1}^{N} \mathrm{Re}(\xi_k)^\top \tilde{G}(t_k - t_l)\,\mathrm{Im}(\xi_l) \\
&= -\sum_{k,l=1}^{N} \mathrm{Im}(\xi_k)^\top \tilde{G}(t_k - t_l)\,\mathrm{Re}(\xi_l) + \sum_{k,l=1}^{N} \mathrm{Im}(\xi_l)^\top \tilde{G}(t_l - t_k)\,\mathrm{Re}(\xi_k) \\
&= -\sum_{k,l=1}^{N} \mathrm{Im}(\xi_k)^\top \tilde{G}(t_k - t_l)\,\mathrm{Re}(\xi_l) + \sum_{k,l=1}^{N} \mathrm{Im}(\xi_k)^\top \tilde{G}(t_k - t_l)\,\mathrm{Re}(\xi_l) \\
&= 0.
\end{aligned}$$

So we find that

$$\sum_{k,l=1}^{N} \xi_k^* \tilde{G}(t_k - t_l)\xi_l = \sum_{k,l=1}^{N} \operatorname{Re}(\xi_k)^\top \tilde{G}(t_k - t_l) \operatorname{Re}(\xi_l) + \sum_{k,l=1}^{N} \operatorname{Im}(\xi_k)^\top \tilde{G}(t_k - t_l) \operatorname{Im}(\xi_l).$$

With Proposition 4.3.5 it follows that

$$\sum_{k,l=1}^{N} \xi_k^* \tilde{G}(t_k - t_l)\xi_l = 2\, C(\operatorname{Re}(\xi)) + 2\, C(\operatorname{Im}(\xi)).$$

Note that we can apply this result also if $t_k = t_l$ for $k \neq l$ due to our assumption $G(0) = G(0)^\top$, see the proof of Proposition 4.3.5. If we can show that $C(\xi) \geq 0$ for all $\xi_1, \dots \xi_N \in \mathbb{R}^K$, this will finish the proof (analogously $C(\xi) > 0$ for $\xi \neq 0$ if G is strictly positive definite).

To show this, let $0 = \tilde{t}_1 < \tilde{t}_2 < \ldots < \tilde{t}_{\tilde{N}}$ with $\tilde{N} \leq N$ such that $\{\tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_{\tilde{N}}\} = \{t_1, t_2, \dots, t_N\} - \min_{i=1,2,\dots,N} t_i$. Define $f : \{1, \dots, N\} \to \{1, \dots, \tilde{N}\}$ such that $\tilde{t}_{f(i)} = t_i - \min_{i=1,2,\dots,N} t_i$. Furthermore, let $\tilde{\xi}_i = \sum_{j \in f^{-1}(\{i\})} \xi_j$. Intuitively, we add up trades at the same point in time to one trade. We find that

$$\sum_{k,l=1}^{N} \xi_k^\top \tilde{G}(t_k - t_l)\xi_l = \sum_{k,l=1}^{\tilde{N}} \tilde{\xi}_k \tilde{G}(\tilde{t}_k - \tilde{t}_l)\tilde{\xi}_l.$$

Since the right-hand side is nonnegative due to the positive definiteness of G, also the left-hand side is nonnegative (resp. strictly positive for $\xi \neq 0$ if G is strictly positive definite) and everything is shown. □

Proposition 4.5.6. *If G is positive definite and continuous in 0, then $G(0)$ is symmetric.*

Proof. Assume that $G(0)$ is not symmetric, i.e. there exists $k, l \in \{1, \dots, K\}$ such that $G(0)_{k,l} > G(0)_{l,k}$. For $\varepsilon > 0$, set $t_1 = 0$, $t_2 = \varepsilon$, $t_3 = 2\varepsilon$ and $t_4 = 3\varepsilon$. Furthermore, set $N = 4$, $(\xi_1)_l = +1$, $(\xi_2)_k = -1$, $(\xi_3)_l = -1$, $(\xi_4)_k = +1$, and all other entries of $\xi_1, \dots \xi_4$ to zero. Then $C(\xi) = (G(0)_{l,l} - G(2\varepsilon)_{l,l}) + (G(0)_{k,k} - G(2\varepsilon)_{k,k}) + (G(3\varepsilon)_{k,l} - 2G(\varepsilon)_{k,l} + G(\varepsilon)_{l,k}) \to G(0)_{l,k} - G(0)_{k,l} < 0$ for $\varepsilon \to 0$, since G is continuous in 0. So G is not positive definite in this case. Therefore, $G(0)$ has to be symmetric. □

Corollary 4.5.7. *If G is continuous in 0, then G is (strictly) positive definite if and only if $\tilde{G}$ is (strictly) complex positive definite.*

Proof. This follows from Proposition 4.5.5 and Proposition 4.5.6. □

Proof of Theorem 4.3.3. By Corollary 4.5.7, $\tilde{G}$ is complex positive definite if and only if G is positive definite. So by the remark after chapter IV, §1, Theorem 1 in Gihman and Skorohod (1974) (see Chapter III, §1, pp 146f in Gihman and Skorohod (1974) for a proof), $\tilde{G}$ is the matrix correlation function of a homogeneous vector-valued random field if and only if G is positive definite. But since G is continuous, this is equivalent with (b) due to chapter IV, §2, Theorem 5 in Gihman and Skorohod (1974). This finishes the proof. □

We prepare now for the proof of Theorem 4.3.9.

Lemma 4.5.8 (Polarization)**.** *Let $S, T \subset \mathbb{R}$, $G : S \to \mathbb{R}^{K\times K}$ be symmetric and $f : S\times T \to \mathbb{C}$ be measurable. Assume for every $\xi \in \mathbb{R}^K$ there is a unique nonnegative finite Borel measure μ_ξ on T such that*

$$\xi^\top G(t)\xi = \int_T f(t,x)\,\mu_\xi(dx) \qquad \text{for all } t \in S.$$

Then there is a symmetric nonnegative matrix-valued measure M such that

$$G(t) = \int_T f(t,x)\,M(dx) \qquad \text{for all } t \in S.$$

Proof. Let $\mathcal{B}(T)$ be the Borel sets on T and $t \in S$. For $E \in \mathcal{B}(T)$, let $m_E : \mathbb{R}^K \to \mathbb{R}$ with $m_E(\xi) = \mu_\xi(E)$ and $B_E(\xi,\zeta) = \frac{1}{4}(m_E(\xi+\zeta) - m_E(\xi-\zeta))$. Furthermore we define $\mu_{\xi,\zeta} : \mathcal{B}(T) \to \mathbb{R}$ with $\mu_{\xi,\zeta}(E) = B_E(\xi,\zeta)$. Now we show that B_E is a symmetric bilinear functional on $\mathbb{R}^K$, i.e. it can be represented by a symmetric matrix. Since

$$\begin{aligned}
&(\xi_1+\xi_2+\zeta)^\top G(t)(\xi_1+\xi_2+\zeta) - (\xi_1+\xi_2-\zeta)^\top G(t)(\xi_1+\xi_2-\zeta)\\
&= (\xi_1+\zeta)^\top G(t)(\xi_1+\zeta) - (\xi_1-\zeta)^\top G(t)(\xi_1-\zeta)\\
&\quad +(\xi_2+\zeta)^\top G(t)(\xi_2+\zeta) - (\xi_2-\zeta)^\top G(t)(\xi_2-\zeta),
\end{aligned}$$

by the uniqueness of μ_ξ for every $\xi \in \mathbb{R}^K$ we know that $\mu_{\xi_1+\xi_2+\zeta} - \mu_{\xi_1+\xi_2-\zeta} = \mu_{\xi_1+\zeta} - \mu_{\xi_1-\zeta} + \mu_{\xi_2+\zeta} - \mu_{\xi_2-\zeta}$, which leads to $B_E(\xi_1+\xi_2,\zeta) = B_E(\xi_1,\zeta) + B_E(\xi_2,\zeta)$. Similarly, since for every $\lambda \in \mathbb{R}$

$$\begin{aligned}
&(\lambda\xi+\zeta)^\top G(t)(\lambda\xi+\zeta) - (\lambda\xi-\zeta)^\top G(t)(\lambda\xi-\zeta)\\
&= \lambda\left((\xi+\zeta)^\top G(t)(\xi+\zeta) - (\xi-\zeta)^\top G(t)(\xi-\zeta)\right),
\end{aligned}$$

we know that $\mu_{\lambda\xi+\zeta} - \mu_{\lambda\xi-\zeta} = \lambda(\mu_{\xi+\zeta} - \mu_{\xi-\zeta})$, which leads to $B_E(\lambda\xi,\zeta) = \lambda B_E(\xi,\zeta)$. Furthermore, we have $(\xi-\zeta)^\top\tilde{G}(t)(\xi-\zeta) = (\zeta-\xi)^\top\tilde{G}(t)(\zeta-\xi)$ and thus, $\mu_{\xi-\zeta} = \mu_{\zeta-\xi}$. It follows that $B_E(\xi,\zeta) = B_E(\zeta,\xi)$. So for each $E \in \mathcal{B}(T)$ we find a symmetric matrix $M(E)$ such that $B_E(\xi,\zeta) = \xi^\top M(E)\zeta$ for all $\xi,\zeta \in \mathbb{R}^K$. Since

$$\xi^\top M(E)\xi = B_E(\xi,\xi) = \mu_{\xi,\xi}(E) = \mu_\xi(E) \geq 0 \tag{4.13}$$

for every $\xi \in \mathbb{R}^K$, we know that $M(E)$ is nonnegative for every $E \in \mathcal{B}(T)$.

Furthermore, $M(\emptyset) = 0$ and let E_n be a sequence of disjoint sets in $\mathcal{B}(T)$. Then for $\xi,\zeta \in \mathbb{R}^K$ we have

$$\begin{aligned}
\xi^\top M\left(\bigcup_{n=1}^\infty E_n\right)\zeta &= \frac{1}{4}\left(\mu_{\xi+\zeta}\left(\bigcup_{n=1}^\infty E_n\right) - \mu_{\xi-\zeta}\left(\bigcup_{n=1}^\infty E_n\right)\right)\\
&= \frac{1}{4}\left(\sum_{n=1}^\infty \mu_{\xi+\zeta}(E_n) - \sum_{n=1}^\infty \mu_{\xi-\zeta}(E_n)\right)\\
&= \sum_{n=1}^\infty \frac{1}{4}\left(\mu_{\xi+\zeta}(E_n) - \mu_{\xi-\zeta}(E_n)\right)\\
&= \sum_{n=1}^\infty \xi^\top M(E_n)\zeta.
\end{aligned}$$

Since this holds for every $\xi, \zeta \in \mathbb{R}^K$, we have $\sum_{n=1}^{\infty} \xi^\top M(E_n)\zeta = \xi^\top(\sum_{n=1}^{\infty} M(E_n))\zeta$ and thus, $M(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} M(E_n)$, i.e. M is a measure.

For all $\xi, \zeta \in \mathbb{R}^K$ we have

$$\begin{aligned}
\xi^\top G(t)\zeta &= \frac{1}{4}((\xi+\zeta)^\top G(t)(\xi+\zeta) - (\xi-\zeta)^\top G(t)(\xi-\zeta)) \\
&= \frac{1}{4}\left(\int_T f(t,x)\,\mu_{\xi+\zeta}(dx) - \int_T f(t,x)\,\mu_{\xi-\zeta}(dx)\right) \\
&= \int_T f(t,x)\,\mu_{\xi,\zeta}(dx) \\
&= \int_T f(t,x)\,(\xi^\top M(dx)\zeta) \\
&= \xi^\top\left(\int_T f(t,x)\,M(dx)\right)\zeta,
\end{aligned}$$

where we used the symmetry of G in the first equality. So we have

$$G(t) = \int_T f(t,x)\,M(dx).$$

□

The following lemma shows that matrix-valued measures have a particularly simple structure.

Lemma 4.5.9. *Let $M : \mathcal{B}(\mathbb{R}) \to \mathbb{R}^{K\times K}$ be a symmetric nonnegative matrix-valued measure. Then there exists a finite (real-valued) measure $\mu : \mathcal{B}(\mathbb{R}) \to [0,\infty)$ and a bounded measurable function $\Sigma : \mathbb{R} \to S_+(K)$ such that*

$$M(E) = \int_E \Sigma(t)\,\mu(dt) \qquad \text{for every } E \in \mathcal{B}(\mathbb{R}). \tag{4.14}$$

Proof. Define a real-valued signed measure $M_{ij} : \mathcal{B}(\mathbb{R}) \to [0,\infty)$ by letting $M_{ij}(E)$ be the (i,j)-component of the matrix $M(E)$. In a first step, we show that the signed measure M_{ij} is of finite total variation. To this end, note first that $M_\zeta(E) := \zeta^\top M(E)\zeta$ is a real-valued and nonnegative Borel measure for each $\zeta \in \mathbb{R}$. In particular, it has the finite total variation $M_\zeta(\mathbb{R}) = \zeta^\top M(\mathbb{R})\zeta$. Now polarization implies that

$$M_{ij}(E) = \frac{1}{4}\big(M_{e_i+e_j}(E) - M_{e_i-e_j}(E)\big),$$

so M_{ij} is of finite total variation as the difference of two finite nonnegative measures.

We thus can define μ_{ij} as the total variation measure of the signed measure M_{ij}. When defining $\mu := \sum_{i,j=1}^{K} \mu_{ij}$, then each M_{ij} is absolutely continuous with respect to μ and admits a Radon-Nikodym derivative $\widetilde{\Sigma}_{ij} = dM_{ij}/d\mu$, which takes values in $[-1,1]$. Clearly, $\widetilde{\Sigma}_{ij} = \widetilde{\Sigma}_{ji}$ μ-a.e., and so we can assume without loss of generality that the matrix-valued function $\widetilde{\Sigma}(t) = (\widetilde{\Sigma}_{ij}(t))_{ij}$ is symmetric for all t. Moreover, (4.14) clearly holds with $\widetilde{\Sigma}$ in place of Σ. In particular,

$$\zeta^\top M(E)\zeta = \int_E \zeta^\top \widetilde{\Sigma}(t)\zeta\,\mu(dt)$$

holds for all $E \in \mathcal{B}(\mathbb{R})$ and all $\zeta \in \mathbb{R}^K$. Thus, for every $\zeta \in \mathbb{R}^K$ there is a μ-nullset N_ζ such that $\zeta^\top \widetilde{\Sigma}(t)\zeta \geq 0$ for all $t \notin N_\zeta$. Hence, $\widetilde{\Sigma}(t) \in S_+(K)$ for all t not belonging to the μ-nullset $N := \bigcap_{\zeta \in \mathbb{Q}^K} N_\zeta$. Thus, $\Sigma(t)$ defined as $\widetilde{\Sigma}(t)$ for $t \notin N$ and as 0 otherwise is as desired. □

Proof of Theorem 4.3.9. (a) ⇒ (b). This follows immediately by choosing $\xi_k = x_k \zeta$.

(b) ⇒ (c). By Bochner's Theorem, for each $\xi \in \mathbb{R}^K$ there exists a unique nonnegative finite Borel measure μ_ξ on $\mathbb{R}$ such that for all $t \geq 0$

$$\xi^\top G(t)\xi = \int_{\mathbb{R}} e^{it\gamma} \mu_\xi(d\gamma). \tag{4.15}$$

With Lemma 4.5.8, the result follows.

(c) ⇒ (a). Since G is symmetric, we have that $\tilde{G}(-t) = \tilde{G}(t)$ for all $t \in \mathbb{R}$, and thus, we have $\tilde{G}(t) = \int_{\mathbb{R}} e^{it\gamma} M(d\gamma)$ for all $t \in \mathbb{R}$. Take time points $0 \leq t_1 \leq t_2 \leq \cdots \leq t_N$, $\xi_1, \ldots, \xi_N \in \mathbb{R}^K$, and let Σ and μ be as in Lemma 4.5.9. Then

$$\begin{aligned} C(\xi) &= \frac{1}{2} \sum_{k,l=1}^{N} \xi_k^\top \tilde{G}(t_k - t_l)\xi_l = \frac{1}{2} \sum_{k,l=1}^{N} \xi_k^\top \left(\int_{\mathbb{R}} e^{i(t_k - t_l)\gamma} M(d\gamma) \right) \xi_l \\ &= \frac{1}{2} \sum_{k,l=1}^{N} \xi_k^\top \left(\int_{\mathbb{R}} e^{i(t_k - t_l)\gamma} \Sigma(\gamma)\, \mu(d\gamma) \right) \xi_l \\ &= \frac{1}{2} \int \Big(\sum_{k=1}^{N} e^{it_k\gamma} \xi_k \Big)^\top \Sigma(\gamma) \Big(\sum_{k=1}^{N} e^{-it_k\gamma} \xi_k \Big)\, \mu(d\gamma) \qquad (4.16) \\ &= \frac{1}{2} \int v(\gamma)^\top \Sigma(\gamma) \overline{v(\gamma)}\, \mu(d\gamma), \end{aligned}$$

where the vector field $v : \mathbb{R} \to \mathbb{C}^K$ is defined as $v(\gamma) := \sum_{k=1}^{N} e^{it_k\gamma} \xi_k$. Since $v(\gamma)^\top \Sigma(\gamma) \overline{v(\gamma)} \geq 0$ for all γ, condition (a) follows. □

For the proof of Proposition 4.3.10, we first need a lemma.

Lemma 4.5.10 (First order condition)**.** *Let G be symmetric, $(\xi_1, \ldots \xi_N)$ be an optimal strategy and $(\tilde{\xi}_1, \ldots, \tilde{\xi}_N)$ with $\sum_{k=0}^{N} \tilde{\xi}_k = 0$. Then*

$$\sum_{k,l=1}^{N} \tilde{\xi}_k^\top \tilde{G}(t_k - t_l)\xi_l = \sum_{k,l=1}^{N} \xi_k^\top \tilde{G}(t_k - t_l)\tilde{\xi}_l = 0$$

Proof. Since $(\xi_1, \ldots x_N)$ is an optimal strategy, we have that

$$\left.\frac{d}{dh}\right|_{h=0} \sum_{k,l=1}^{N} (\xi_k + h\tilde{\xi}_k)^\top \tilde{G}(t_k - t_l)(\xi_l + h\tilde{\xi}_l) = 0.$$

It follows that

$$\sum_{k,l=1}^{N} \tilde{\xi}_k^\top \tilde{G}(t_k - t_l)\xi_l + \sum_{k,l=1}^{N} \xi_k^\top \tilde{G}(t_k - t_l)\tilde{\xi}_l = 0.$$

Since G is symmetric, the result follows. □

Proof of Prop. 4.3.10. Let $m \in \{1, \dots K\}$, $q \in \{1, \dots, N\}$ and

$$(\tilde{\xi}_i)_j = \begin{cases} 1, & \text{if } i = p, j = m, \\ -1, & \text{if } i = q, j = m, \\ 0 & \text{else.} \end{cases}$$

By $\tilde{G}_m(t_k - t_p)$ we denote the mth row of $\tilde{G}(t_k - t_p)$. By Lemma 4.5.10, we know that

$$0 = \sum_{k=1}^{N} \xi_k^\top G_m(t_k - t_p) - \sum_{k=1}^{N} \xi_k^\top G_m(t_k - t_q).$$

Varying q shows, that there is $\lambda_m \in \mathbb{R}$ such that $\sum_{k=1}^{N} \xi_k^\top G_m(t_k - t_p) = \lambda_m$. This shows the existence of the asserted λ.

Using Proposition 4.3.5, we find that

$$C(\xi) = \frac{1}{2} \sum_{l=1}^{N} \left(\sum_{k=1}^{N} \xi_k^\top \tilde{G}(t_k - t_l) \right) \xi_l = \frac{1}{2} \lambda^\top \sum_{l=1}^{N} \xi_l = -\frac{1}{2} \lambda^\top X_0.$$

Now assume that $(\xi_1, \dots \xi_N)$ satisfies (4.4) and let $(\tilde{\xi}_1, \dots \tilde{\xi}_N)$ be any other admissible strategy. Let $(\zeta_1, \dots, \zeta_N) = (\tilde{\xi}_1 - \xi_1, \dots, \tilde{\xi}_N - \xi_N)$. Using the notation from the proof of Proposition 4.5.1, we find

$$\mathcal{C}(\xi, \zeta) = \frac{1}{2} \sum_{j=1}^{N} \left(\sum_{i=1}^{N} \xi_i^\top \tilde{G}(t_i - t_j) \right) \zeta_j = \frac{1}{2} \lambda^\top \sum_{j=1}^{N} \zeta_j = 0$$

and thus

$$C(\tilde{\xi}) = C(\xi + \zeta) = C(\xi) + C(\zeta) + 2\mathcal{C}(\xi, \zeta) = C(\xi) + C(\zeta) \geq C(\xi),$$

so ξ is optimal. □

Proof of Prop. 4.3.11. By Remark 4.5.2, L is a linear map and $L(\xi) = 0$ implies $C(\xi) = 0$. Since G is strictly positive definite, it follows that $L(\xi) = 0$ implies $\xi = 0$, so L is invertible. Since L^{-1} is also linear, M is linear.

For each $X_0 \in \mathbb{R}^K$, there is a unique optimal strategy ξ^* by Proposition 4.2.5. Proposition 4.3.10 implies that $L(\xi^*) = \lambda \mathbb{1}_N^\top$, i.e. $\xi^* = L^{-1}(\lambda \mathbb{1}_N^\top)$. Since $\xi^* \mathbb{1}_N = -X_0$, we have that $M(\lambda) = -X_0$. Since we can find such a λ for every $X_0 \in \mathbb{R}^K$, M is onto and therefore invertible. So $\lambda = -M^{-1}(X_0)$, and $\xi^* = -L^{-1}(M^{-1}(X_0)\mathbb{1}_N^\top)$ follows. By Proposition 4.3.10, we have that $C(\xi^*) = -\frac{1}{2}\lambda^\top X_0 = \frac{1}{2}(M^{-1}(X_0))^\top X_0$. □

We now prepare for the proof of Theorem 4.3.14.

Proposition 4.5.11. *For a right-continuous symmetric matrix-valued function G, the following conditions are equivalent.*

(a) *G is nonincreasing.*

(b) *There exists a nonnegative Radon measure μ on $[0,\infty)$ and a measurable function $\Gamma : [0,\infty) \to S_+(K)$ such that $G(t) = G(0) - \int_{[0,t]} \Gamma(s)\,\mu(ds)$.*

If, moreover, G is nonincreasing and nonnegative, then $G_{ij}(\infty) := \lim_{t\uparrow\infty} G_{ij}(t)$ exists for all i, j, and the corresponding matrix $G(\infty)$ is symmetric and nonnegative. Moreover, $\int_{[0,\infty)} \Gamma(s)\,\mu(ds)$ converges, and $G(\infty) = G(0) - \int_{[0,\infty)} \Gamma(s)\,\mu(ds)$. It follows in particular that

$$G(t) = G(\infty) + \int_{(t,\infty)} \Gamma(s)\,\mu(ds). \tag{4.17}$$

Remark 4.5.12. When all components $G_{ij}(t)$ of G are absolutely continuous functions, we can take for μ the Lebesgue measure.

Proof. The implication (b)⇒(a) is obvious. To prove (a)⇒(b), we use polarization to see that $G(t)$ is determined by the numbers $g^\zeta(t)$, where ζ runs through the finite set $Z := \{e_i + e_j \,|\, i, j = 1, \dots, K\}$. Here, e_i denotes as usual the i^{th} unit vector. Since the function $t \mapsto g^\zeta(t)$ is right-continuous and nonincreasing for each ζ, there exists a nonnegative Radon measure μ_ζ on $[0,\infty)$ such that $g^\zeta(t) = g^\zeta(0) - \mu_\zeta[0,t]$. We set $\mu := \sum_{\zeta\in Z} \mu_\zeta$. Then each μ_ζ with $\zeta \in Z$ is absolutely continuous with respect to μ and has the Radon-Nikodym derivative $\gamma_\zeta = d\mu_\zeta/d\mu$. We set

$$\widetilde{\Gamma}_{ij}(t) := \frac{1}{2}\gamma_{e_i+e_j}(t) - \frac{1}{8}\gamma_{2e_i}(t) - \frac{1}{8}\gamma_{2e_j}(t). \tag{4.18}$$

Clearly, $\widetilde{\Gamma}_{ij}(t) = \widetilde{\Gamma}_{ji}(t)$ and $G_{ij}(t) = G_{ij}(0) - \int_{[0,t]} \widetilde{\Gamma}_{ij}(s)\,\mu(ds)$. It remains to show that there exists a μ-nullset N such that the matrix $(\widetilde{\Gamma}_{ij}(t))$ is nonnegative for $t \notin N$. Once this has been established, we can set $\Gamma_{ij}(t) = \widetilde{\Gamma}_{ij}(t)$ for $t \notin N$ and $\Gamma_{ij}(t) = 0$ otherwise. To this end, we note that for every $\zeta = (\zeta_1, \dots, \zeta_K) \in \mathbb{Q}^K$ we have $g^\zeta(t) = g^\zeta(0) - \int_{[0,t]} \gamma_\zeta(s)\mu(ds)$, where $\gamma_\zeta(s) := \sum_{i,j=1}^K \zeta_i\zeta_j\widetilde{\Gamma}_{ij}(s)$. Since g^ζ is nonincreasing, we must have $\gamma_\zeta(s) \geq 0$ for all s outside some μ-nullset N_ζ. Thus, $N := \bigcap_{\zeta\in\mathbb{Q}^K} N_\zeta$ is as desired.

Now suppose that G is nonincreasing and nonnegative. Then, for each $\zeta \in \mathbb{R}^K$, the limit $g^\zeta(\infty) := \lim_{t\uparrow\infty} g^\zeta(t)$ exists as a nonnegative real number. Polarization thus implies that the limits $G_{ij}(\infty) := \lim_{t\uparrow\infty} G_{ij}(t)$ exists for all i, j and that the corresponding matrix $G(\infty)$ is symmetric and nonnegative. The remaing assertions are easy to prove. □

Proposition 4.5.13. *For a right-continuous symmetric matrix-valued function G, the following conditions are equivalent.*

(a) *G is convex.*

(b) *There exists a right-continuous and nonincreasing function $\Gamma : [0,\infty) \to S(K)$ such that $G(t) = G(0) - \int_0^t \Gamma(s)\,ds$.*

If, moreover, G is convex, nonincreasing, and nonnegative, then there exists a nonnegative Radon measure μ on $(0,\infty)$ and a measurable function $\Lambda : (0,\infty) \to S_+(K)$ such that

$$G(t) = G(\infty) + \int_{(0,\infty)} (r-t)^+ \Lambda(r)\,\mu(dr). \tag{4.19}$$

Proof. The implication (b)⇒(a) is obvious. We now prove (a)⇒(b). Since each function g^ζ is convex, it is of the form $g^\zeta(t) = g^\zeta(0) - \int_0^t \gamma_\zeta(s)\,ds$ for a right-continuous and nonincreasing function γ_ζ. When defining $\Gamma_{ij}(t)$ in the same way as $\widetilde{\Gamma}_{ij}(t)$ is defined in (4.18), then we have $\gamma_\zeta(t) := \sum_{i,j=1}^K \zeta_i\zeta_j\Gamma_{ij}(t)$ for all t and ζ, due to right-continuity. It follows that Γ is as desired.

Now assume that G is convex, nonincreasing, and nonnegative. It follows that (4.17) holds, and from (b) that $\mu(ds) = ds$ and that the corresponding function Γ is right-continuous, nonnegative, and nonincreasing. Since $\int_t^\infty \Gamma(s)\,ds$ is finite, we must have $\Gamma(\infty) = 0$. Applying (4.17) to Γ thus yields the existence of a nonnegative Radon measure μ and a right-continuous, nonnegative function Λ such that $\Gamma(s) = \int_{(s,\infty)} \Lambda(r)\,\mu(dr)$. Combining the representations for G and Γ yields

$$G(t) = G(\infty) + \int_t^\infty \int_{(s,\infty)} \Lambda(r)\,\mu(dr)\,ds.$$

Using Fubini's theorem finally implies (4.19). □

Proposition 4.5.14. *When G is convex, nonincreasing, nonnegative, symmetric and continuous, it is the Fourier transform of the symmetric nonnegative matrix-valued measure*

$$M(d\gamma) = G(\infty)\,\delta_0(d\gamma) + \Phi(\gamma)\,d\gamma,$$

where $\Phi : \mathbb{R} \to S_+(K)$ is the continuous function given by

$$\Phi(x) = \frac{1}{\pi}\int_{(0,\infty)} \frac{1-\cos xy}{x^2}\Lambda(y)\,\mu(dy)$$

for Λ and μ as in (4.19).

Proof. By Lemma 4.2 in Gatheral *et al.* (2012), we find that for every $\xi \in \mathbb{R}^K$ the function $g^\xi(t) = \xi^\top G(t)\xi$ is the Fourier transform of the nonnegative Radon measure

$$\mu_\xi(d\gamma) = \xi^\top(G(\infty)\delta_0(d\gamma) + \Phi(\gamma)d\gamma)\xi$$

on $\mathbb{R}$. By polarization as in Lemma 4.5.8, the result follows. □

Proof of Theorem 4.3.14. First we argue that we can assume without loss of generality that G is continuous. To this end, let $G^{cont}(t) = \lim_{s\downarrow t} G(s)$ for all $t \geq 0$. G^{cont} is continuous, since a convex function on $[0,\infty)$ is continuous on $(0,\infty)$. So $G(t) = G^{cont}(t) + \delta_0(t)\Delta G(0)$, where $\Delta G(0)$ denotes the jump of G in 0. So for any $\xi \in \mathbb{R}^K$ we have $g^\xi(t) = \xi^\top G^{cont}(t)\xi + \delta_0(t)\xi^\top(\Delta G(0))\xi$. Since g^ξ is convex for any ξ, we have that $\xi^\top(\Delta G(0))\xi \geq 0$ for any ξ, i.e. $\Delta G(0)$ is nonnegative. Let C^{cont} the cost function corresponding to G^{cont}. Then $C(\xi) = C^{cont}(\xi) + \frac{1}{2}\sum_{k=1}^{N} \xi_k^\top G(0)\xi_k \geq C^{cont}(\xi)$. So we can restrict ourselves to the case of continuous G.

Let now M, Φ, Λ, and μ be as in Proposition 4.5.14. It follows from that proposition and (4.16) that for $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbb{R}^K \otimes \mathbb{R}^N$ and all $0 \leq t_1 < t_2 < \ldots < t_N$

$$C(\boldsymbol{\xi}) = \frac{1}{2} v(0)^\top G(\infty) v(0) + \frac{1}{2} \int v(\gamma)^\top \Phi(\gamma) \overline{v(\gamma)}\, d\gamma,$$

where $v(\gamma) := \sum_{k=1}^{N} e^{it_k\gamma}\xi_k$. We are now going to show that the integral on the right is strictly positive unless $\boldsymbol{\xi} = 0$. To this end, we note first that the components of the vector field $v(\cdot)$ are holomorphic functions of $\gamma \in \mathbb{C}$. When $\boldsymbol{\xi} \neq 0$, at least one of these components is nonconstant and hence vanishes only on a set which has at most countably many elements. It follows that $v(\gamma) \neq 0$ for all but countably many $\gamma \in \mathbb{R}$. Moreover, we are going to argue next that the matrix $\Phi(\gamma)$ is strictly positive for all but countably many $\gamma \in \mathbb{R}$. Thus, $v(\gamma)^\top \Phi(\gamma)\overline{v(\gamma)} > 0$ for Lebesgue-almost every $\gamma \in \mathbb{R}$, and it follows that $C(\boldsymbol{\xi}) > 0$.

So let us show now that $\Phi(\gamma)$ is strictly positive for all but countably many $\gamma \in \mathbb{R}$. To this end, we first note that from (4.19),

$$g^\zeta(t) = \zeta^\top G(\infty)\zeta + \int_{(0,\infty)} (r-t)^+ \zeta^\top \Lambda(r)\zeta\, \mu(dr).$$

Since $\Lambda(r)$ is nonnegative, the fact that g^ζ is nonconstant for $\zeta \neq 0$ implies that

$$\int_{(0,\infty)} \zeta^\top \Lambda(r)\zeta\, \mu(dr) > 0 \qquad \text{for } \zeta \neq 0. \tag{4.20}$$

Now let A be the set of all $y > 0$ such that $\mu(\{y\}) > 0$, and let

$$\mu_d(E) := \mu(A \cap E) \quad \text{and} \quad \mu_c(E) := \mu(A^c \cap E)$$

be the discrete and continuous parts of μ, respectively. Clearly,

$$N := \{x \in \mathbb{R} \,|\, \cos xy = 1 \text{ for some } y \in A\} = \bigcup_{y \in A} \{x \in \mathbb{R} \,|\, \cos xy = 1\}$$

is at most countable. Moreover, the set $\{y > 0 \,|\, \cos xy = 1\}$ is a μ_c-nullset for all $x \neq 0$. It thus follows that the measure $\frac{1-\cos xy}{x^2}\,\mu(dy)$ is equivalent to μ for all $x \notin N \cup \{0\}$. Therefore (4.20) implies that

$$\zeta^\top \Phi(x)\zeta = \frac{1}{\pi}\int_{(0,\infty)} \frac{1-\cos xy}{x^2} \zeta^\top \Lambda(r)\zeta\, \mu(dy) > 0$$

for all $\zeta \neq 0$ as long as $x \notin N \cup \{0\}$. This concludes the proof. □

Proof of Proposition 4.3.15. Assume that G is not nonnegative, i.e. there exists $\zeta \in \mathbb{R}^K$, $t^* > 0$ and $\varepsilon > 0$ such that $g^\zeta(t^*) = -\varepsilon$. Now we want to show that g^ζ cannot be positive definite in this case. Set $t_k = k \cdot t^*$ and $x_k = 1$ for $k \in \mathbb{N}$. Since $|t_k - t_l| \geq t^*$ for $k \neq l$ and g^ζ is nonincreasing, we have $g^\zeta(|t_k - t_l|) \leq -\varepsilon$ for $k \neq l$. Thus, $\sum_{k,l=1}^{n} x_k x_l g^\zeta(|t_k - t_l|) \leq n g^\zeta(0) - (n^2 - n)\,\varepsilon$. If n is large enough, the latter expression is negative. Thus, g^ζ is not positive definite, and so G can not be positive definite. □

Proof of Proposition 4.3.16. First we prove that G_1 is strictly positive definite. Let $L = AA^\top$ be the Cholesky decomposition of L, so $A \in \mathbb{R}^{K\times K}$ is invertible. Let $\xi_1, \dots, \xi_N \in \mathbb{R}^K$ such that at least one of these vectors is nonzero. Let $\tilde{\xi}_k = A^\top \xi_k$ for $k = 1, \dots, N$. Since A is invertible, at least one of the vectors $\tilde{\xi}_1, \dots, \tilde{\xi}_N$ is nonzero. Since g is strictly positive definite, it follows that

$$\begin{aligned} C(\xi) &= \frac{1}{2}\sum_{k,l=1}^N \xi_k^\top \tilde{G}_1(t_k - t_l)\xi_l \\ &= \frac{1}{2}\sum_{k,l=1}^N \xi_k^\top A g(|t_k - t_l|)A^\top \xi_l \\ &= \frac{1}{2}\sum_{k,l=1}^N \tilde{\xi}_k^\top g(|t_k - t_l|)\tilde{\xi}_l \\ &> 0. \end{aligned}$$

So G_1 is strictly positive definite. By Proposition 4.2.5, there exists a unique optimal strategy $\xi^{(1)}$ for G_1. By Proposition 4.3.10, it satisfies $\sum_{k=1}^N (\xi_k^{(1)})^\top \tilde{G}_1(t_k - t_p) = \lambda_1^\top$ for some $\lambda_1 \in \mathbb{R}^K$. Multiplying this equation with $L^{-1}\tilde{L}$ from the right yields $\sum_{k=1}^N (\xi_k^{(1)})^\top \tilde{G}_2(t_k - t_p) = \lambda_1^\top L^{-1}\tilde{L}$, i.e. $\xi^{(1)}$ is also the unique optimal strategy for G_2 by Proposition 4.3.10. □

Proof of Prop. 4.3.17. Since G_3 is symmetric and strictly positive definite, its unique optimal strategy is characterized by $\sum_{k=1}^N (\xi_k^{(3)})^\top \tilde{G}_3(t_k - t_p) = \lambda_3^\top$ with some $\lambda_3 \in \mathbb{R}^K$. It follows that $\sum_{k=1}^N (\xi_k^{(3)})^\top \tilde{G}_1(t_k - t_p) = \lambda_3^\top L^{-1}$. But $\xi^{(1)}$ is the unique solution to the latter equation, so it follows that $\xi^{(3)} = \xi^{(1)}$. Furthermore, $\lambda_3 = L^\top \lambda_1$. □

Proof of Proposition 4.3.18. Let $C_i(\xi)$ denote the cost of a strategy ξ with respect to the decay kernel G_i. From (4.3) it follows that $C_1(L\xi) = C_2(\xi)$ for all strategies $\xi = (\xi_1, \dots, \xi_N)$, where $L\xi$ is defined componentwise, i.e., $L\xi = (L\xi_1, \dots, L\xi_N)$. In particular, G_2 is positive definite. Since L is invertible, G_2 is even strictly positive definite when G_1 is. It is also clear that $C_2(\xi)$ is minimized by $\xi = \xi^{(2)}$ if $C_1(\xi)$ is minimized by $\xi = L\xi^{(2)}$, which proves our formula for the optimal strategy. □

Proof of Proposition 4.3.19. Let G be positive definite. First, we show that the transpose is positive definite. That is, we have to show that for all $N \in \mathbb{N}$, $0 \le t_1 < t_2 < \dots t_N$ and $\xi_1, \dots, \xi_N \in \mathbb{R}^K$ we have

$$\frac{1}{2}\sum_{k=1}^N \xi_k^\top G(0)^\top \xi_k + \sum_{k=1}^N \sum_{l=1}^{k-1} \xi_k^\top G(t_k - t_l)^\top \xi_l \ge 0.$$

For $i \in \{1, \dots, N\}$, we define $\tilde{\xi}_i = \xi_{N+1-i}$ (so $\xi_i = \tilde{\xi}_{N+1-i}$) and $\tilde{t}_i = t_N - t_{N+1-i}$.

Then we have

$$\begin{aligned}
\sum_{k=1}^{N}\sum_{l=1}^{k-1}\xi_k^\top G(t_k-t_l)^\top\xi_l &= \sum_{k=1}^{N}\sum_{l=1}^{k-1}\tilde{\xi}_{N+1-k}^\top G(t_k-t_l)^\top\tilde{\xi}_{N+1-l}\\
&= \sum_{k=1}^{N}\sum_{l=1}^{k-1}\tilde{\xi}_l^\top G(t_{N+1-l}-t_{N+1-k})^\top\xi_k\\
&= \sum_{k=1}^{N}\sum_{l=1}^{k-1}\tilde{\xi}_k^\top G(t_{N+1-l}-t_{N+1-k})\xi_l\\
&= \sum_{k=1}^{N}\sum_{l=1}^{k-1}\tilde{\xi}_k^\top G(\tilde{t}_k-\tilde{t}_l)\xi_l
\end{aligned}$$

The latter expression is nonnegative, since G is positive definite. Since

$$\frac{1}{2}\sum_{k=1}^{N}\xi_k^\top G(0)^\top\xi_k=\frac{1}{2}\sum_{k=1}^{N}\xi_k^\top G(0)\xi_k=\frac{1}{2}\sum_{k=1}^{N}\tilde{\xi}_k^\top G(0)\tilde{\xi}_k,$$

we conclude that $t\mapsto G(t)^\top$ is positive definite.

The proof, that every convex combination $t\mapsto\alpha G(t)+(1-\alpha)G(t)^\top$ with $\alpha\in[0,1]$ is positive definite, is now straightforward, since the cost functional is linear in G. For every $N\in\mathbb{N}$, $0\le t_1<t_2<\dots t_N$ and $\xi_1,\dots,\xi_N\in\mathbb{R}^K$ we have

$$\begin{aligned}
&\frac{1}{2}\sum_{k=1}^{N}\xi_k^\top(\alpha G(0)+(1-\alpha)G(0)^\top)\xi_k+\sum_{k=1}^{N}\sum_{l=1}^{k-1}\xi_k^\top(\alpha G(t_k-t_l)+(1-\alpha G(t_k-t_l)^\top)\xi_l\\
&= \alpha\left(\frac{1}{2}\sum_{k=1}^{N}\xi_k^\top G(0)\xi_k+\sum_{k=1}^{N}\sum_{l=1}^{k-1}\xi_k^\top G(t_k-t_l)\xi_l\right)\\
&\quad +(1-\alpha)\left(\frac{1}{2}\sum_{k=1}^{N}\xi_k^\top G(0)^\top\xi_k+\sum_{k=1}^{N}\sum_{l=1}^{k-1}\xi_k^\top G(t_k-t_l)^\top\xi_l\right)\\
&\ge 0,
\end{aligned}$$

since G and $t\mapsto G(t)^\top$ are positive definite.

Choosing $\alpha=\frac{1}{2}$ yields that the symmetrization is positive definite.

Finally, in case G is strictly positive definite the computations above show immediately that the transpose and every convex combination are also strictly positive definite. □

Proof of Lemma 4.3.21. It is well known that two symmetric matrices commute if and only if they can be simultaneously diagonalized. Thus, for any pair $t,s\ge 0$ there exists an orthogonal matrix O and numbers $g_i(s)$, $g_i(t)$ ($i=1,\dots,K$) such that (4.5) holds for t and s. But this means that the matrices $G(t)$ and $G(s)$ have the same eigenvectors. If $r\ge 0$ is given, then $G(r)$ also must have the same eigenvectors as $G(t)$ and, hence, as $G(s)$. Therefore, the matrix O is in fact independent of the choice of the pair s,t, and the result follows. □

Proof of Proposition 4.3.22. Let v_i be the i^{th} column of O. Then v_i is the eigenvector of $G_i(t)$ for the eigenvalue $g_i(t)$. A given $\zeta \in \mathbb{R}^K$ can be written as $\zeta = \sum_{i=1}^K \alpha_i v_i$. Then $O\zeta = \sum_{i=1}^K \alpha_i e_i$, where e_i is the i^{th} unit vector. It follows from (4.5) that $g^\zeta(t) = \sum_{i=1}^K \alpha_i^2 g_i(t)$. From here, the assertions are obvious. □

Proof of Proposition 4.3.24. Take O and g_i as in (4.5) and let $v_1, \dots, v_K$ be the column vectors of O. Then $g^{v_i}(t) = g_i(t)$. By Theorem 1 in Alfonsi *et al.* (2012), there is a (one-dimensional) optimal strategy $(x_1, \dots, x_N)$ corresponding to the initial position $-\sum_{n=1}^N x_n = -1$ and to the nonnegative, convex, and nonincreasing decay kernel $g_i(t)$ that has only nonnegative components. But by Proposition 4.3.18, the optimal strategy for $X_0 = v_i$ is given by $(\xi_1, \dots, \xi_N) = (x_1, \dots, x_N)v_i$. This proves the result. □

Proof of Proposition 4.3.25. Let $v_1, \dots v_K$ as in Proposition 4.3.24. Furthermore, let $\alpha, \dots \alpha_K \in \mathbb{R}$ such that $X_0 = \sum_{k=1}^K \alpha_k v_k$ and $\xi^{(k)}$ the optimal strategy to the initial portfolio v_k. The optimal strategy is linear in the initial portfolio X_0, i.e. the optimal strategy is $\xi^* = \sum_{k=1}^K \alpha_k \xi^{(k)}$. Now $\sum_{n=1}^N \|\xi_n^*\|_1 \le \sum_{n=1}^N \sum_{k=1}^K |\alpha_k| \|\xi_n^{(k)}\|_1$. With the proof of Proposition 4.3.24, we find that $\sum_{n=1}^N \|\xi_n^{(k)}\|_1 = \|v_k\|_1$. So $\sum_{n=1}^N \|\xi_n^*\|_1 \le \sum_{k=1}^K |\alpha_k| \|v_k\|_1$, which shows the result. □

To study the examples for $K = 2$ assets, we will frequently use the following simple lemma. While it is well-known that for Hermitian matrices to be positive definite it is necessary and sufficient that all their eigenvalues are nonnegative, note that for real nonsymmetric matrices it is *not* sufficient.

Lemma 4.5.15. (a) *Let $a, d \ge 0$ and $b \in \mathbb{C}$. The Hermitian matrix $\begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix}$ is positive definite if and only if $|b|^2 \le ad$.*

(b) *Let $a, b, c, d \ge 0$. The real matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is positive definite if and only if $\frac{1}{4}(b+c)^2 \le ad$.*

Proof. (a): The second eigenvalue of the matrix is $\frac{1}{2}(a + d - \sqrt{(a-d)^2 + 4|b|^2})$. This quantity is nonnegative if and only if $|b|^2 \le ad$.

(b): The matrix $M := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is positive definite if and only if its symmetrization $\frac{1}{2}(M + M^\top)$ is positive definite. According to (a), this is the case if and only if $(\frac{1}{2}(b+c))^2 \le ad$. □

Proof of Proposition 4.4.1. (a): By Lemma 4.5.15, G is nonnegative if and only if for every $t \ge 0$

$$\frac{1}{4}((a_{12} - b_{12}t)^+ + (a_{21} - b_{21}t)^+)^2 \le (a_{11} - b_{11}t)^+(a_{22} - b_{22}t)^+.$$

Assume that G is nonnegative. Choosing $t = 0$ yields $\frac{1}{4}(a_{12} + a_{21})^2 \le a_{11}a_{22}$. Choosing $t = \min\{\frac{a_{11}}{b_{11}}, \frac{a_{22}}{b_{22}}\}$ yields that the right-hand side of the preceding equation is zero. So the left-hand side has to be zero which implies that $\max\{\frac{a_{12}}{b_{12}}, \frac{a_{21}}{b_{21}}\} \le t$.

Conversely, assume that $\frac{1}{4}(a_{12}+a_{21})^2 \le a_{11}a_{22}$ and $\max\{\frac{a_{12}}{b_{12}},\frac{a_{21}}{b_{21}}\} \le \min\{\frac{a_{11}}{b_{11}},\frac{a_{22}}{b_{22}}\}$. So for any $t \ge 0$, we have that $\max\{(1-\frac{b_{12}}{a_{12}}t)^+,(1-\frac{b_{21}}{a_{21}}t)^+\} \le \min\{(1-\frac{b_{11}}{a_{11}}t)^+,(1-\frac{b_{22}}{a_{22}}t)^+\}$. Thus,

$$\begin{aligned}
&\frac{1}{4}((a_{12}-b_{12}t)^+ + (a_{21}-b_{21}t)^+)^2\\
&= \frac{1}{4}\left(a_{12}\left(1-\frac{b_{12}}{a_{12}}t\right)^+ + a_{21}\left(1-\frac{b_{21}}{a_{21}}t\right)^+\right)^2\\
&\le \frac{1}{4}\left((a_{12}+a_{21})\max\left\{\left(1-\frac{b_{12}}{a_{12}}t\right)^+,\left(1-\frac{b_{21}}{a_{21}}t\right)^+\right\}\right)^2\\
&\le \frac{1}{4}\left((a_{12}+a_{21})\min\left\{\left(1-\frac{b_{11}}{a_{11}}t\right)^+,\left(1-\frac{b_{22}}{a_{22}}t\right)^+\right\}\right)^2\\
&\le a_{11}a_{22}\left(\min\left\{\left(1-\frac{b_{11}}{a_{11}}t\right)^+,\left(1-\frac{b_{22}}{a_{22}}t\right)^+\right\}\right)^2\\
&\le a_{11}a_{22}\left(1-\frac{b_{11}}{a_{11}}t\right)^+\left(1-\frac{b_{22}}{a_{22}}t\right)^+\\
&= (a_{11}-b_{11}t)^+(a_{22}-b_{22}t)^+.
\end{aligned}$$

So G is nonnegative.

(b): G is absolutely continuous with derivative

$$G'(t) = \begin{pmatrix} -b_{11}\mathbb{1}_{\{t<\frac{a_{11}}{b_{11}}\}} & -b_{12}\mathbb{1}_{\{t<\frac{a_{12}}{b_{12}}\}} \\ -b_{21}\mathbb{1}_{\{t<\frac{a_{21}}{b_{21}}\}} & -b_{22}\mathbb{1}_{\{t<\frac{a_{22}}{b_{22}}\}} \end{pmatrix}$$

By Proposition 4.3.13 and Lemma 4.5.15, G is nonincreasing if and only if for almost all $t > 0$

$$\frac{1}{4}(b_{12}\mathbb{1}_{\{t<\frac{a_{12}}{b_{12}}\}} + b_{21}\mathbb{1}_{\{t<\frac{a_{21}}{b_{21}}\}})^2 \le b_{11}\mathbb{1}_{\{t<\frac{a_{11}}{b_{11}}\}}b_{22}\mathbb{1}_{\{t<\frac{a_{22}}{b_{22}}\}}.$$

Assume G is nonincreasing. Then choosing t small enough shows $\frac{1}{4}(b_{12}+b_{21})^2 \le b_{11}b_{22}$. Choosing any $t \ge \min\{\frac{a_{11}}{b_{11}},\frac{a_{22}}{b_{22}}\}$ yields that the right-hand side of the preceding equation is zero. So the left-hand-side has to be zero which implies $\max\{\frac{a_{12}}{b_{12}},\frac{a_{21}}{b_{21}}\} \le \min\{\frac{a_{11}}{b_{11}},\frac{a_{22}}{b_{22}}\}$.

Conversely, if $\frac{1}{4}(b_{12}+b_{21})^2 \le b_{11}b_{22}$ and $\max\{\frac{a_{12}}{b_{12}},\frac{a_{21}}{b_{21}}\} \le \min\{\frac{a_{11}}{b_{11}},\frac{a_{22}}{b_{22}}\}$, it is obvious that G is nonincreasing.

(c): Assume that $a_{12} = a_{21}$. Then we have that $G(t) = \int_{\mathbb{R}} e^{i\gamma t}M(d\gamma)$ with $M(d\gamma) = \frac{1}{2\pi\gamma^2}N(\gamma)d\gamma$, with the Hermitian matrix

$$N(\gamma) = \begin{pmatrix} 2b_{11}(1-\cos(\frac{a_{11}}{b_{11}}\gamma)) & b_{12}(1-e^{\frac{-ia_{12}\gamma}{b_{12}}}) + b_{21}(1-e^{\frac{ia_{12}\gamma}{b_{21}}}) \\ b_{21}(1-e^{\frac{-ia_{12}\gamma}{b_{21}}}) + b_{12}(1-e^{\frac{ia_{12}\gamma}{b_{12}}}) & 2b_{22}(1-\cos(\frac{a_{22}}{b_{22}}\gamma)) \end{pmatrix}.$$

G is positive definite if and only if $N(\gamma)$ is positive definite for every $\gamma \in \mathbb{R}$. Using Lemma 4.5.15, $N(\gamma)$ is positive definite if and only if

$$|b_{12}(1-e^{\frac{-ia_{12}\gamma}{b_{12}}}) + b_{21}(1-e^{\frac{ia_{12}\gamma}{b_{21}}})|^2 \le b_{11}(1-\cos(\frac{a_{11}}{b_{11}}\gamma))b_{22}(1-\cos(\frac{a_{22}}{b_{22}}\gamma)),$$

i.e. if and only if

$$\left(b_{12}(1-\cos(\frac{a_{12}}{b_{12}}\gamma))+b_{21}(1-\cos(\frac{a_{12}}{b_{21}}\gamma))\right)^2+\left(b_{12}\sin(\frac{a_{12}}{b_{12}}\gamma)-b_{21}\sin(\frac{a_{12}}{b_{21}}\gamma)\right)^2 \\ \leq 4b_{11}(1-\cos(\frac{a_{11}}{b_{11}}\gamma))b_{22}(1-\cos(\frac{a_{22}}{b_{22}}\gamma)).$$

Assume that $a_{12}=a_{21}$, $b_{12}=b_{21}$, $\frac{a_{11}}{b_{11}}=\frac{a_{12}}{b_{12}}=\frac{a_{22}}{b_{22}}$ and $b_{12}^2\leq b_{11}b_{22}$. Since $\frac{a_{12}}{b_{12}}=\frac{a_{12}}{b_{21}}$, the sin term vanishes in the preceding equation and it simplifies to $b_{12}b_{21}\leq b_{11}b_{22}$. So G is positive definite.

Conversely, assume that G is positive definite. We have that $a_{12}=a_{21}$ since $G(0)$ is symmetric due to Proposition 4.5.6. Assume that $\min\{\frac{a_{12}}{b_{12}},\frac{a_{21}}{b_{21}}\}<\max\{\frac{a_{11}}{b_{11}},\frac{a_{22}}{b_{22}}\}$. Choose $\gamma=2\pi(\max\{\frac{a_{11}}{b_{11}},\frac{a_{22}}{b_{22}}\})^{-1}$. So the right-hand side of the preceding display is zero, but the left-hand side is strictly positive since $0<\min\{\frac{a_{12}}{b_{12}},\frac{a_{21}}{b_{21}}\}\gamma<2\pi$. This yields that $\min\{\frac{a_{12}}{b_{12}},\frac{a_{21}}{b_{21}}\}\geq\max\{\frac{a_{11}}{b_{11}},\frac{a_{22}}{b_{22}}\}$ and together with the assumption $\max\{\frac{a_{12}}{b_{12}},\frac{a_{21}}{b_{21}}\}\leq\min\{\frac{a_{11}}{b_{11}},\frac{a_{22}}{b_{22}}\}$ it follows that $\frac{a_{11}}{b_{11}}=\frac{a_{12}}{b_{12}}=\frac{a_{21}}{b_{21}}=\frac{a_{22}}{b_{22}}$. So $b_{12}=b_{21}$, i.e. G is symmetric. As before, the preceding display simplifies to $b_{12}b_{21}\leq b_{11}b_{22}$.

(d): The first part follows from (a) and (c). From (b) it follows that G is nonincreasing in this case. To show convexity, note that $G'(t)=\begin{pmatrix}-b_{11} & -b_{12}\\ -b_{12} & -b_{22}\end{pmatrix}=:B$ for $t<\frac{a_{11}}{b_{11}}$ and $G'(t)=0$ for $t>\frac{a_{11}}{b_{11}}$. By Proposition 4.5.13, we have to show that G' is nondecreasing, i.e. that B is negative definite, i.e. $-B$ is positive definite. By Lemma 4.5.15, this follows from $b_{12}^2\leq b_{11}b_{22}$.

(e): Note that if either $s\geq\frac{a_{11}}{b_{11}}$ or $t\geq\frac{a_{11}}{b_{11}}$, then $G(t)$ or $G(s)$ is the zero matrix, so we have $G(t)G(s)=G(s)G(t)$. If $0\leq s,t<\frac{a_{11}}{b_{11}}$, then

$$G(t)G(s)-G(s)G(t)= \\ (s-t)\begin{pmatrix}0 & a_{12}(b_{11}-b_{22})-b_{12}(a_{11}-a_{22})\\ -a_{12}(b_{11}-b_{22})+b_{12}(a_{11}-a_{22}) & 0\end{pmatrix}$$

so G is commuting if and only if $a_{12}(b_{11}-b_{22})=b_{12}(a_{11}-a_{22})$. □

Proof of Proposition 4.4.2. (a): By Lemma 4.5.15, G is nonnegative if and only if for every $t\geq 0$

$$\frac{1}{4}(a_{12}\exp(-b_{12}t)+a_{21}\exp(-b_{21}t))^2\leq a_{11}\exp(-b_{11}t)a_{22}\exp(-b_{22}t),$$

i.e. if and only if

$$\frac{1}{4}(a_{12}^2\exp(-2b_{12}t)+a_{12}a_{21}\exp(-(b_{12}+b_{21})t)+a_{21}^2\exp(-2b_{21}t)) \\ \leq\ a_{11}a_{22}\exp(-(b_{11}+b_{22})t).$$

If G is nonnegative, taking $t=0$ shows $\frac{1}{4}(a_{12}+a_{21})^2\leq a_{11}a_{22}$, while $t\to\infty$ shows $\min\{b_{12},b_{21}\}\geq\frac{1}{2}(b_{11}+b_{22})$. Conversely, if these inequalities hold, G is nonnegative.

(b): G is C^1. By Proposition 4.3.13 G is nonincreasing if and only if for every $t\geq 0$

$$-G'(t)=\begin{pmatrix}a_{11}b_{11}\exp(-b_{11}t) & a_{12}b_{12}\exp(-b_{12}t)\\ a_{21}b_{21}\exp(-b_{21}t) & a_{22}b_{22}\exp(-b_{22}t)\end{pmatrix}$$

is positive definite. Analogously to (a), the result follows.

(c): Analogously to (b), by Proposition 4.3.13 G is convex if and only if for every ≥ 0 its second derivative

$$G''(t) = \begin{pmatrix} a_{11}b_{11}^2 \exp(-b_{11}t) & a_{12}b_{12}^2 \exp(-b_{12}t) \\ a_{21}b_{21}^2 \exp(-b_{21}t) & a_{22}b_{22}^2 \exp(-b_{22}t) \end{pmatrix}$$

is positive definite. The result follows analogously to (a).

(d): Assume that $a_{12} = a_{21}$. We have that $G(t) = \int_{\mathbb{R}} e^{i\gamma t} M(d\gamma)$, where $M(d\gamma) = \frac{1}{2\pi} N(\gamma)\, d\gamma$ with the Hermitian matrix

$$N(\gamma) = \begin{pmatrix} 2\frac{a_{11}b_{11}}{b_{11}^2+\gamma^2} & \frac{a_{12}}{b_{21}-i\gamma} + \frac{a_{12}}{b_{12}+i\gamma} \\ \frac{a_{12}}{b_{12}-i\gamma} + \frac{a_{12}}{b_{21}+i\gamma} & 2\frac{a_{22}b_{22}}{b_{22}^2+\gamma^2} \end{pmatrix}.$$

G is positive definite if and only if $N(\gamma)$ is positive definite for all $\gamma \in \mathbb{R}$. According to Lemma 4.5.15, this is equivalent to

$$\left|\frac{a_{12}}{b_{21} - i\gamma} + \frac{a_{12}}{b_{12} + i\gamma}\right| \leq 4\frac{a_{11}b_{11}}{b_{11}^2 + \gamma^2} \frac{a_{22}b_{22}}{b_{22}^2 + \gamma^2},$$

i.e.

$$\frac{a_{12}^2(b_{12} + b_{21})^2}{(b_{12}^2 + \gamma^2)(b_{21}^2 + \gamma^2)} \leq 4\frac{a_{11}b_{11}}{b_{11}^2 + \gamma^2} \frac{a_{22}b_{22}}{b_{22}^2 + \gamma^2},$$

i.e.

$$a_{12}^2(b_{12} + b_{21})^2(b_{11}^2 + \gamma^2)(b_{22}^2 + \gamma^2) \leq 4a_{11}b_{11}a_{22}b_{22}(b_{12}^2 + \gamma^2)(b_{21}^2 + \gamma^2).$$

Comparing the coefficients for γ^0, γ^2 and γ^4, we see that it is sufficient to have

$$\begin{aligned} a_{12}^2(b_{12} + b_{21})^2 b_{11}b_{22} &\leq 4a_{11}a_{22}b_{12}^2b_{21}^2 && (4.21)\\ a_{12}^2(b_{12} + b_{21})^2(b_{11}^2 + b_{22}^2) &\leq 4a_{11}b_{11}a_{22}b_{22}(b_{12}^2 + b_{21}^2) && (4.22)\\ a_{12}^2(b_{12} + b_{21})^2 &\leq 4a_{11}b_{11}a_{22}b_{22}. && (4.23) \end{aligned}$$

Note that (4.23) follows immediately from (b), since G is nonincreasing and $a_{12} = a_{21}$. To show (4.21), note that $\sqrt{b_{11}b_{22}} \leq \frac{1}{2}(b_{11} + b_{22}) \leq \min\{b_{12}, b_{21}\}$, so $b_{11}^2b_{22}^2 \leq (\min\{b_{12}, b_{21}\})^4 \leq b_{12}^2b_{21}^2$. Together with (4.23) the result follows. Finally, (4.22) follows from (4.23) and the assumption $b_{11}^2 + b_{22}^2 \leq b_{12}^2 + b_{21}^2$. This finishes the proof.

Note that (4.21), (4.23) and $a_{12} = a_{21}$ are necessary for G being positive definite. However, (4.22) is not necessary, but the necessary condition is rather lengthy.

(e): We find that the left upper entry of $G(0)G(t) - G(t)G(0)$ is $a_{12}a_{21}(-e^{-b_{12}t} + e^{-b_{21}t})$, so $G(0)G(t) = G(t)G(0)$ implies $b_{12} = b_{21}$. Given that, a direct calculation shows that $G(0)G(t) = G(t)G(0)$ is equivalent to $a_{11}(e^{-b_{11}t} - e^{-b_{12}t}) + a_{22}(e^{-b_{12}t} - e^{-b_{22}t}) = 0$. If $a_{11} = a_{22}$, this implies $b_{11} = b_{22}$. If $a_{11} \neq a_{22}$, by the equivalent equation $a_{22} - a_{11} = a_{22}e^{-(b_{22}-b_{12})t} - a_{11}e^{-(b_{11}-b_{12})t}$ we see that $b_{11} = b_{22}$ and finally $b_{11} = b_{12} = b_{21} = b_{22}$.

Conversely, if either $a_{11} = a_{22}$ and $b_{12} = b_{21}$ and $b_{11} = b_{22}$, or $b_{11} = b_{12} = b_{21} = b_{22}$, a direct calculation shows that $G(s)G(t) = G(t)G(s)$ for all $s, t \geq 0$. □

Proof of Proposition 4.4.5. G is obviously continuous and Proposition 4.4.2 yields that G is convex, nonincreasing and nonnegative.

To show that G is not positive definite, using Mathematica we find that $G(t) = \int_{\mathbb{R}} e^{i\gamma t} M(d\gamma)$, where $M(d\gamma) = CN(\gamma)\, d\gamma + D\delta_0(d\gamma)$ with a constant $C > 0$, a matrix $D \in \mathbb{R}^{2\times 2}$, the Dirac measure δ_0 at 0 and the Hermitian matrix $N(\gamma)$ given by

$$\begin{pmatrix} \frac{2e^2(-\cos(\gamma)\gamma + e\gamma - \sin(\gamma))}{\gamma^3+\gamma} & f(\gamma) \\ \overline{f(\gamma)} & \frac{2e^2(-\cos(\gamma)\gamma + e\gamma - \sin(\gamma))}{\gamma^3+\gamma} \end{pmatrix}$$

with

$$f(\gamma) = \frac{5e^3\gamma - ((3+2e)\gamma + 6i(-1+e))\cos(\gamma) + (i(-3+2e)\gamma - 6(1+e))\sin(\gamma)}{8\gamma(\gamma(\gamma+i)+6)}.$$

If G was positive definite, then all eigenvalues of $N(\gamma)$ would be positive for $\gamma \neq 0$. But using Mathematica we find that one eigenvalue of $N(\gamma)$ is

$$\begin{aligned} &\frac{1}{8\left(\gamma^2+4\right)\left(\gamma^2+9\right)\left(\gamma^3+\gamma\right)^2}\Big(16e^3\left(\gamma^2+1\right)\left(\gamma^2+4\right)\left(\gamma^2+9\right)\gamma^2 \\ &-16e^2\left(\gamma^2+1\right)\left(\gamma^2+4\right)\left(\gamma^2+9\right)\gamma(\sin(\gamma)+\gamma\cos(\gamma)) \\ &-\Big(\gamma^2\left(\gamma^2+1\right)^4\left(\gamma^2+4\right)\left(\gamma^2+9\right)\Big(\left(9+4e^2+25e^6\right)\gamma^2 \\ &-10e^3(3+2e)\gamma^2\cos(\gamma)+12e\left(\gamma^2-6\right)\cos(2\gamma) \\ &-60e\gamma\sin(\gamma)\left(-2\cos(\gamma)+e^3+e^2\right)+36\left(1+e^2\right)\Big)\Big)^{\frac{1}{2}}\Big), \end{aligned}$$

which is negative for all γ with $0 < |\gamma| < 0.02$. So G is not positive definite. □

Proof of Theorem 4.4.8. (a)⇒(b): By the standard Hausdorff-Bernstein-Widder Theorem we know that for each $\xi \in \mathbb{R}^K$ there is a nonnegative finite Borel measure μ_ξ such that

$$\xi^\top G(t)\xi = \int_0^\infty e^{-tx}\mu_\xi(dx).$$

Since $\xi^\top G(t)\xi$ is the Laplace transform of μ_ξ, the measure μ_ξ is unique. So by Lemma 4.5.8, (b) follows.

(b)⇒(c): Since $M = \int \delta_x\, M(dx)$, it is sufficient to consider a simple measure of the form $M(dx) = \delta_y(dx)N$ for some $y \in [0,\infty)$ and $N \in S_+$, so $G(t) = e^{-ty}N = e^{-ty}\,\mathrm{Id}\,N = e^{-t(y\mathrm{Id})}N$. Choosing $\mu(dB) = \delta_{y\mathrm{Id}}(dB)N$ yields that there is a representation (4.7) for G.

(c)⇒(a): For $n \geq 0$ we have that

$$G^{(n)}(t) = (-1)^n \int B^n e^{-tB}\,\mu(dB).$$

Since B^n is nonnegative for all $n \geq 0$ for nonnegative B, (a) follows. □

Chapter 5

Conclusion

Our focus was on the regularity of the models, in particular the regularity conditions of no price manipulation of Huberman and Stanzl (2004) and no transaction-triggered price manipulation of Alfonsi *et al.* (2012). Additionally we have proposed the condition of positive expected liquidation costs, which is between these two conditions. It is important to investigate the regularity of a market impact model, since in irregular models optimal strategies may not exist or show unexpected behavior. Irregularity can also be a hint for misspecification of the model. As we have shown, regularity often depends strongly on the parameters. Since liquidity parameters vary significantly over time (compare Westray (2010)), it is important to examine the regularity of a model for different parameters.

In chapter 2 we have presented a model for stochastic transient impact. Stochastic impact is motivated by seasonalities and by the impact generated by trading derivatives. In the case of stochastic permanent impact the liquidity parameter has to be a submartingale to ensure regularity. Adding temporary impact to the model regularizes it only for small time horizons. Since the submartingale condition for the liquidity parameter is very restrictive, we have considered stochastic transient impact to allow for more flexible choices of the liquidity parameter.

For exponential decay of price impact with time-dependent liquidity parameter we have given a sufficient condition for regularity. In the non-exponential case, we have taken a specific decay kernel and discussed the behavior of optimal strategies when modeling liquidity with a geometric Brownian motion in a numerical example. In particular, transaction-triggered price manipulation exists in this case. If the liquidation horizon T is large enough, one obtains unbounded expected profits. Furthermore, even if the liquidity parameter is a martingale we have shown that it may be beneficial to adapt the strategy to the liquidity parameter.

Stochastic impact is interesting both from a practical and a theoretical point of view. Thus, it may be promising to explore stochastic transient impact models and their regularity further, in particular for non-exponential decay of market impact. Since the complexity of the computation of optimal strategies increases strongly in the number of trading times N, it might also be interesting to investigate efficient algorithms for the computation of optimal strategies.

In chapter 3 we have analyzed the regularity of a class of dark-pool extensions of an Almgren–Chriss model and found that such models admit price manipulation strategies unless the model parameters satisfy certain restrictions. These restrictions are satisfied for every Almgren–Chriss model when the penalty parameter β is at least $\frac{1}{2}$, the cross-venue impact parameter α is 1, and there is no temporary price impact from the exchange on dark-pool prices. With these choices, the dark-pool extension of any Almgren–Chriss model is free of price manipulation, has positive expected liquidation costs, and hence admits reasonable optimal order execution strategies. In this sense, the model is then regular.

For other parameter choices we have illustrated how regularity might fail. We found in many cases that serious problems arise. Note that the strategies in the dark pool have been very restrictive in allowing only one order at time 0. However, if more strategies in the dark pool were allowed, regularity would be even worse.

It should be noted, however, that the parameter values $\alpha = 1$ and $\beta \geq \frac{1}{2}$ will typically not correspond to values found in empirical analysis or calibration of real-world dark pools. Our results can therefore provide some indication that dark pools may create market inefficiencies and disturb the price finding mechanism of markets, although further empirical analysis will be needed to support this conjecture. On the other hand, it may be that not all economic costs are included the model, for example adverse selection (i.e. dependence of P^0 and N). However, it is challenging to extend the model while keeping tractability and simplicity of the model.

In chapter 4 we have considered transient impact for multiple assets. Such a market impact model is regular if the matrix-valued decay kernel of market impact is a positive definite function. We have given characterizations of such positive definite functions as Fourier transforms of nonnegative matrix-valued measures. For symmetric decay kernels, we have shown that an optimal strategy is a solution of a linear system of equations.

We have discussed nonincreasing, nonnegative and convex decay kernels. If a decay kernel is additionally symmetric, it is positive definite. Furthermore, decay kernels remain positive definite under congruence transformations and the transformation of the associated optimal strategies can be given explicitly. Using these results, we find the optimal strategy of a nonincreasing, nonnegative, convex, symmetric and commuting decay kernel has bounded variation, uniformly in the number of trading times N. This allows for convergence to continuous-time strategies.

To illustrate these theoretical results we have analyzed linear and exponential decay for two assets in detail. Moreover, we have discussed matrix functions. For the exponential function, i.e. a generalized Obizhaeva and Wang (2013) model, we have given explicit solutions in discrete and continuous time.

In this chapter there are two major open questions remaining. While symmetry is often a necessary condition for many results, as we show with counterexamples, there are also nonsymmetric positive definite functions. For these it would be desirable to have results which allow the computation of optimal strategies. Another open question is whether convergence to continuous time can also be shown for decay kernels that are not commuting. This is especially interesting since commuting is a very strong property and seems to have no financial interpretation.

For the transient impact models in this thesis (chapter 2 and 4) we have considered linear impact only. Since practitioners prefer nonlinear impact functions, it would be promising to investigate nonlinear models with transient impact. However, this will complicate the mathematical analysis significantly. For example, when modeling nonlinearity as in Gatheral (2010) decay kernels that are continuous in 0 allow for price manipulation, see Proposition 23 in Slynko (2010).

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