Periodic solutions of the sinh-Gordon equation and integrable systems

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Dipl.-Math. Markus Knopf
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Dekan:	Professor Dr. Heinz Jürgen Müller, Universität Mannheim
Referent:	Professor Dr. Martin Schmidt, Universität Mannheim
Korreferent:	Professor Dr. Claus Hertling, Universität Mannheim

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Abstract

The elliptic sinh-Gordon equation arises in the context of particular surfaces of constant mean curvature. With the help of differential geometric considerations the space of periodic solutions is parametrized by means of spectral data consisting of a Riemann surface Y and a divisor D. It is investigated if the space $M_g^{\mathbf{p}}$ of real periodic finite type solutions with fixed period $\mathbf{p}$ can be considered as a completely integrable system $(M_g^{\mathbf{p}}, \Omega, H_2)$ with a symplectic form Ω and a series of commuting Hamiltonians $(H_n)_{n\in\mathbb{N}_0}$. In particular we relate the gradients of these Hamiltonians to the Jacobi fields $(\omega_n)_{n\in\mathbb{N}_0}$ from the Pinkall-Sterling iteration. Moreover, a connection between the symplectic form Ω and Serre duality is established.

Zusammenfassung

Die elliptische sinh-Gordon-Gleichung steht im Zusammenhang zu bestimmten Flächen konstanter mittlerer Krümmung. Mithilfe differentialgeometrischer Überlegungen lässt sich der Raum der periodischen Lösungen durch Spektraldaten, bestehend aus einer Riemannschen Fläche Y und einem Divisor D, parametrisieren. Es wird untersucht, ob der Raum $M_g^{\mathbf{p}}$ der reellen periodischen Lösungen von endlichem Typ mit festgehaltener Periode $\mathbf{p}$ als ein vollständig integrables System $(M_g^{\mathbf{p}}, \Omega, H_2)$ mit einer symplektischen Form Ω und einer Folge kommutierender Hamiltonfunktionen $(H_n)_{n\in\mathbb{N}_0}$ aufgefasst werden kann. Insbesondere werden die Gradienten dieser Hamiltonfunktionen mit den Jacobifeldern $(\omega_n)_{n\in\mathbb{N}_0}$ aus der Pinkall-Sterling-Iteration in Beziehung gebracht. Außerdem wird eine Verbindung zwischen der symplektischen Form Ω und der Serre-Dualität hergestellt.

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Contents

1 Introduction

1.1 The sinh-Gordon equation and spectral data

The elliptic *sinh-Gordon equation* is given by

$$\Delta u + 2\sinh(2u) = 0, \tag{1.1.1}$$

where Δ is the Laplacian of $\mathbb{R}^2$ with respect to the Euclidean metric and $u : \mathbb{R}^2 \to \mathbb{R}$ is a twice partially differentiable function which we assume to be real.

The sinh-Gordon equation arises in the context of particular surfaces of constant mean curvature (CMC) since the function u can be extracted from the conformal factor e^{2u} of a conformally parameterized CMC surface. The study of CMC tori in 3-dimensional space forms was strongly influenced by algebro-geometric methods (as described in [5]) that led to a complete classification by Pinkall and Sterling [45] for CMC-tori in $\mathbb{R}^3$. Moreover, Bobenko [8, 9] gave explicit formulas for CMC tori in $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$ in terms of theta-functions and introduced a description of such tori by means of *spectral data*. We also refer the interested reader to [10, 11]. Every CMC torus yields a doubly periodic solution $u : \mathbb{R}^2 \to \mathbb{R}$ of the sinh-Gordon equation. With the help of differential geometric considerations one can associate to every CMC torus a hyperelliptic Riemann surface Y, the so-called *spectral curve*, and a holomorphic line bundle E on Y (the so-called *eigenline bundle*) that is represented by a certain divisor D. Hitchin [30], and Pinkall and Sterling [45] independently proved that all doubly periodic solutions of the sinh-Gordon equation correspond to spectral curves of finite genus. We say that solutions of (1.1.1) that correspond to spectral curves of finite genus are of *finite type*.

In the present setting we will relax the condition on the periodicity and demand that u is only simply periodic with a fixed period. After rotating the domain of definition we can assume that this period is real. This enables us to introduce simply periodic *Cauchy data* with fixed period $\mathbf{p} \in \mathbb{R}$ consisting of a pair $(u, u_y) \in C^\infty(\mathbb{R}/\mathbf{p}) \times C^\infty(\mathbb{R}/\mathbf{p})$. Moreover, we demand that the corresponding solution u of the sinh-Gordon equation is of finite type.

In the following we will see that a finite type solution of the sinh-Gordon equation is uniquely determined by its *spectral data* (Y, D) and investigate how Y and D fit into the description of the sinh-Gordon equation as a completely integrable system. In order to understand the features that are provided by completely integrable system we introduce a simple example of such a system in the following section.

1.2 An example of a completely integrable system

We want to treat the sinh-Gordon equation (1.1.1) as a completely integrable system (compare with [23]) and illustrate its features by introducing the simplest example, i.e. the symplectic manifold $(\mathbb{R}^{2n}, \Omega)$ with coordinates $(q,p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ (see [3]). The coordinates q and p are often called *positions* and *moments*. The corresponding symplectic form Ω is

$$\Omega = \sum_{i=1}^{n} dq_i \wedge dp_i,$$

i.e. for $v, w \in \mathbb{R}^{2n} \simeq T_p\mathbb{R}^{2n}$ (with $p \in \mathbb{R}^{2n}$) one has

$$\Omega(v,w) = \langle v, Jw \rangle_{\mathbb{R}^{2n}} \quad \text{with} \quad J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}.$$

For a smooth map $H : \mathbb{R}^{2n} \to \mathbb{R}$, the so-called *Hamiltonian*, one can consider its gradient ∇H and define the *Hamiltonian vector field* as

$$X_H := J\nabla H.$$

Given $H : \mathbb{R}^{2n} \to \mathbb{R}$ we study the *equations of motion* given by

$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = X_H(q,p) = J\nabla H(q,p)$$

or written out in coordinates

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

We immediately note that H is constant along the integral curves $(q(t), p(t))$ for the Hamiltonian vector field X_H since there holds

$$\frac{d}{dt}H(q(t),p(t)) = \langle \nabla H(q,p), \tfrac{d}{dt}(q,p) \rangle_{\mathbb{R}^{2n}} = \langle \nabla H(q,p), J\nabla H(q,p) \rangle_{\mathbb{R}^{2n}} = 0$$

due to the skew-symmetry of J. A function $f : \mathbb{R}^{2n} \to \mathbb{R}$ is called *integral of motion* for the *Hamiltonian system* $(\mathbb{R}^{2n}, \Omega, H)$ if f is preserved under the flow Φ_{X_H} of the Hamiltonian vector field X_H. Expanding this condition leads to

$$\dot{f} = \langle \nabla f, \tfrac{d}{dt}(q,p) \rangle_{\mathbb{R}^{2n}} = \langle \nabla f, J\nabla H \rangle_{\mathbb{R}^{2n}} \overset{!}{=} 0.$$

In particular the Hamiltonian H is an integral of motion. We define $\{f, g\} := \langle \nabla f, J\nabla g \rangle_{\mathbb{R}^{2n}}$ as the *Poisson bracket* of two smooth functions $f, g : U \subset \mathbb{R}^{2n} \to \mathbb{R}$ and say that f and g are *in involution* if $\{f, g\} = 0$ holds. Thus a function f is an integral of motion if and only if f and H are in involution, i.e. $\{f, H\} = 0$.

The Hamiltonian system $(\mathbb{R}^{2n}, \Omega, H)$ is called *completely integrable* in the sense of Liouville if there exist functions $f_1 = H, f_2, \ldots, f_n$ such that

(i) the functions $f_1, \dots, f_n$ are pairwise in involution, i.e. $\{f_i, f_j\} = 0$ for $1 \leq i, j \leq n$,

(ii) their gradients $\nabla f_1, \dots, \nabla f_n$ are linearly independent and

(iii) their Hamiltonian vector fields $X_{f_1}, \dots, X_{f_n}$ are complete.

Considering the map $f := (f_1, \dots, f_n) : U \to \mathbb{R}^n$ for an open neighborhood $U \subset \mathbb{R}^{2n}$ one can show that f is a submersion. Moreover, every value is a regular value and thus every non-empty leaf

$$M^c := f^{-1}[c] = \{(q,p) \in U \mid f(q,p) = c\}$$

is a smooth manifold of dimension n. Therefore $U \subset \mathbb{R}^{2n}$ is foliated into these leaves. Now we arrive at the following

Theorem 1.1 (Liouville). *Let $U \subset \mathbb{R}^{2n}$ be an open subset, $x_0 \in U$ a point and $f := (f_1, \dots, f_n) : U \to \mathbb{R}^n$ a smooth map such that*

(i) the functions $f_1, \dots, f_n$ are pairwise in involution, i.e. $\{f_i, f_j\} = 0$ for $1 \leq i, j \leq n$,

(ii) their gradients $\nabla f_1, \dots, \nabla f_n$ are linearly independent on $N := f^{-1}[f(x_0)]$ and

(iii) their Hamiltonian vector fields $X_{f_1}, \dots, X_{f_n}$ restricted to N are complete.

Then the connected components of N are homeomorphic to $\mathbb{R}^n/\Gamma$, where Γ is a discrete subgroup of $\mathbb{R}^n$.

If the rank of Γ equals n we see that the connected components of N are homeomorphic to

$$\mathbb{R}^n/\Gamma \simeq \mathbb{R}^n/\mathbb{Z}^n \simeq (\mathbb{R}/\mathbb{Z})^n \simeq (\mathbb{S}^1)^n,$$

i.e. in that case they correspond to (compact) n-dimensional tori. If in addition N is connected we get $N \simeq (\mathbb{S}^1)^n$.

Thus we see that a completely integrable system around a compact connected leaf is foliated into an n-paramater family of invariant tori.

1.3 What is done in this work

The main goal of this thesis is to work out the details that help us to identify the sinh-Gordon equation as a completely integrable system. In particular we will recognize the features that appeared in the simplest example $(\mathbb{R}^{2n}, \Omega, H)$. We now give a short overview of the content of the various chapters.

In the *second chapter* we are going through some notational conventions as well as the basic concepts of differential geometry such as the first and second fundamental form or equivalently the three quantities u, Q and H, that is the conformal factor u, the Hopf differential Q and the mean curvature H.

Since the hyperelliptic spectral curve Y can be compactified in the finite type setting this chapter also deals with compact Riemann surfaces and describes the Riemann-Roch Theorem in terms of divisors and sheafs. We also consider Lie groups and mainly reduce our attention to the Lie group $SU(2) \simeq \mathbb{S}^3$ and its Lie algebra $\mathfrak{su}(2)$. Moreover, the concept of moving frames and Lax pairs is elucidated and the relationship between solutions F of the system

$$F_z = FU, \quad F_{\bar{z}} = FV$$

with the compatibility condition $U_{\bar{z}} - V_z - [U, V] = 0$ and solutions u to the Gauss and Codazzi equations

$$2u_{z\bar{z}} + 2e^{2u}(1 + H^2) - \frac{1}{2}Q\overline{Q}e^{-2u} = 0, \quad Q_{\bar{z}} = 2H_z e^{2u}$$

for given Q and $H \equiv$ const is investigated. This leads to the introduction of a $\mathbb{C}^*$-family of flat connections $d + \alpha_\lambda$ and the question how the connection form α_λ behaves under certain parameter transformations.

The *third chapter* introduces spectral data (Y, D) for periodic finite type solutions of the sinh-Gordon equation consisting of a spectral curve Y and a divisor D. To do so we study the monodromy M_λ of the λ-dependent frame F_λ and consider its asymptotic expansion around the points $\lambda = 0$ and $\lambda = \infty$. At these points M_λ has essential singularities and it turns out that this expansion carries a lot of information concerning the solution of the sinh-Gordon equation.
Instead of taking a periodic u defined on $\mathbb{R}^2$ we will study a pair $(u, u_y) \in C^\infty(\mathbb{R}/\mathbf{p}) \times C^\infty(\mathbb{R}/\mathbf{p})$ with fixed period $\mathbf{p} \in \mathbb{R}$ that corresponds to u if one considers the coordinate y as a flow parameter. Setting $y = 0$ in $\alpha_\lambda(x, y)$ one obtains the matrix $U_\lambda(x)$. Now it is possible to define finite type Cauchy data $(u, u_y) \in C^\infty(\mathbb{R}/\mathbf{p}) \times C^\infty(\mathbb{R}/\mathbf{p})$ by introducing polynomial Killing fields and the appropriate space of potentials $\mathcal{P}_g$. These potentials will be used to parameterize the finite type solutions.
Since the monodromy M_λ and the initial value $\xi_\lambda \in \mathcal{P}_g$ of the corresponding polynomial Killing field ζ_λ commute, one can introduce two equivalent definitions of the spectral curve $Y(u, u_y)$ that encodes the eigenvalues μ of M_λ and ν of ξ_λ as functions on Y. In order to describe M_λ or ξ_λ completely one also has to encode the λ-dependent eigenlines of M_λ and ξ_λ. Since $[M_\lambda, \xi_\lambda] = 0$ one can find eigenlines that diagonalize M_λ and ξ_λ simultaneously. This will lead to the definition of the holomorphic eigenline bundle or equivalently to the divisor $D(u, u_y)$ on $Y(u, u_y)$.

In the *fourth chapter* we will focus on the inverse problem that yields a bijective map $(u, u_y) \mapsto (Y(u, u_y), D(u, u_y))$. In a first step we recall the Krichever construction that leads to linear flows on the Jacobi variety $\mathrm{Jac}(Y)$ of a spectral curve Y. It will be investigated how one can obtain periodic (isospectral) flows and if there exists a suitable basis of $H^1(Y, \mathcal{O})$, the Lie algebra of $\mathrm{Jac}(Y)$. Moreover, we will see which condition arises if one translates the reality condition on M_λ or equivalently on ξ_λ to this setting.

We will also investigate the Baker-Akhiezer function and its analytic properties in order to reconstruct the x-dependent eigenvectors of $M_\lambda(x) = F_\lambda^{-1}(x) M_\lambda F_\lambda(x)$ and $\zeta_\lambda(x)$. With this tool at hand we are able to reconstruct the Cauchy data (u, u_y) from the spectral data $(Y(u, u_y), D(u, u_y))$ and thus arrive at the bijective map

$$(u, u_y) \mapsto (Y(u, u_y), D(u, u_y))$$

that establishes a one-to-one correspondence between Cauchy data (u, u_y) and spectral data (Y, D).

The *fifth chapter* deals with isospectral and non-isospectral deformations of the spectral data (Y, D). On the one hand we study non-isospectral (but isoperiodic) deformations of spectral curves Y of genus g and will show that the space of such curves is a smooth g-dimensional manifold. This will lead to the conclusion that the space of Cauchy data (u, u_y) that leads to such smooth spectral curves Y is a smooth $2g$-dimensional manifold. Moreover, we will identify the space of such deformations with holomorphic one-forms on the spectral curve Y.
Since the map $(u, u_y) \mapsto (Y(u, u_y), D(u, u_y))$ is bijective we can fix Y and ask for Cauchy data (u, u_y) with $Y(u, u_y) = Y$. This leads to the isospectral set $\mathrm{Iso}(Y)$. By introducing an isospectral group action one can show that $\mathrm{Iso}(Y)$ is parameterized by a g-dimensional torus. This degree of freedom corresponds to the degree of freedom for the movement of the divisor D in the Jacobi variety $\mathrm{Jac}(Y)$. Moreover, the infinitesimal deformations of ξ_λ and U_λ that result from that isospectral group action are investigated.

The *sixth chapter* combines the third, fourth and fifth chapter and deals with the symplectic form Ω on the $2g$-dimensional phase space $M_g^{\mathbf{p}}$ as well as the Hamiltonian formalism for the sinh-Gordon hierarchy, that is induced by a Hamiltonian $H_2 : M_g^{\mathbf{p}} \to \mathbb{R}$.
As a first step we introduce the notion of a completely integrable Hamiltonian system and define the phase space $M_g^{\mathbf{p}}$ as the set of finite type Cauchy data $(u, u_y) \in C^\infty(\mathbb{R}/\mathbf{p}) \times C^\infty(\mathbb{R}/\mathbf{p})$ such that the resulting spectral curve $Y(u, u_y)$ obeys some special conditions. Moreover, we define a series of functions $(H_n)_{n\in\mathbb{N}_0}$ (compare with [44]) on the phase space that also contains the Hamiltonian $H_2 : M_g^{\mathbf{p}} \to \mathbb{R}$. We will relate this series to the series $(\omega_n)_{n\in\mathbb{N}_0}$ of solutions of the linearized sinh-Gordon equation that are obtained via the Pinkall-Sterling iteration (see [45] and [36]) and show that $(H_n)_{n\in\mathbb{N}_0}$ are involutive integrals of motion for the Hamiltonian system $(M_g^{\mathbf{p}}, \Omega, H_2)$.
Moreover, we introduce an inner product on the loop Lie algebra $\Lambda_r\mathfrak{sl}_2(\mathbb{C})$ and use this inner product to establish a connection between the symplectic form Ω and Serre duality as it was done in [47] for the non-linear Schrödinger operator. This part contains the main results of the thesis.

Chapter seven summarizes the most important results of this thesis and gives an outlook on possible interesting further research.

2 Preliminaries

2.1 Surface theory in $\mathbb{S}^3$

We want to recall some basic facts from surface theory and follow the terminology introduced in [22]. In the following, we will consider 2-dimensional submanifolds of

$$\mathbb{S}^3 = \{x \in \mathbb{R}^4 \mid \|x\| = 1\},$$

where $\mathbb{S}^3$ is equipped with the metric defined by restricting the metric $\langle \cdot, \cdot \rangle_{\mathbb{R}^4}$ of $\mathbb{R}^4$ to the 3-dimensional tangent spaces of $\mathbb{S}^3$. We will investigate conformal immersions

$$f : M \to \mathbb{S}^3,$$

where M is an arbitrary Riemann surface. The smooth function f will for now be considered as $\mathbb{R}^4$-valued with $\|f\|^2 = 1$.

Definition 2.1. *A Riemann surface is a pair (M, Σ), consisting of a connected two-dimensional manifold M with a complex structure Σ, that is an equivalence class of biholomorphic equivalent collections of charts, that cover M.*

In the following we will describe the intrinsic geometry of a surface by its first fundamental form and the extrinsic geometry of an immersed surface by its second fundamental form respectively.

Definition 2.2. *Let $f : M \to \mathbb{S}^3$ be an immersion. The induced metric $g : T_pM \times T_pM \to \mathbb{R}$ is defined by*

$$g(v, w) = \langle df(v), df(w) \rangle_{\mathbb{R}^4}$$

*and is called the **first fundamental form**. Both g and ds^2 are commonly used notations.*

If (x, y) is a coordinate for M and f is an immersion, a basis for T_pM can be chosen as

$$f_x = \left(\frac{\partial f}{\partial x}\right)_p, \quad f_y = \left(\frac{\partial f}{\partial y}\right)_p.$$

Then the metric g is represented by the matrix

$$g_p = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \langle f_x, f_x \rangle & \langle f_x, f_y \rangle \\ \langle f_y, f_x \rangle & \langle f_y, f_y \rangle \end{pmatrix}$$

and one has with the identification $T_pM \simeq \mathbb{R}^2$ via the basis (f_x, f_y)

$$g_p(v, w) = v^t \begin{pmatrix} \langle f_x, f_x \rangle & \langle f_x, f_y \rangle \\ \langle f_y, f_x \rangle & \langle f_y, f_y \rangle \end{pmatrix} w.$$

Remark 2.3. *The map $f : M \to \mathbb{S}^3$ is an immersion $\Leftrightarrow$ the matrix g_p has positive determinant for all p.*

Definition 2.4. *An immersion $f : M \to \mathbb{S}^3$ is **conformal** if there exists a function $u : M \to \mathbb{R}$, called the conformal factor, such that*

$$g(v,w) = v^t \begin{pmatrix} e^{2u} & 0 \\ 0 & e^{2u} \end{pmatrix} w = e^{2u}\langle v,w\rangle_{\mathbb{R}^2}, \quad v,w \in T_pM.$$

Now we turn to the extrinsic geometry of the immersed surface. The unit normal vector field to the surface is $N := \frac{\widetilde{N}}{\|\widetilde{N}\|}$, where $\widetilde{N}$ is given by

$$\widetilde{N} := \sum_{i=1}^{4} \det(e_i, f, f_x, f_y) \cdot e_i \text{ with an orthonormal basis } e_1, \dots, e_4 \text{ of } \mathbb{R}^4,$$

i.e. $\widetilde{N}$ is the vector in $\mathbb{R}^4$ that is perpendicular to f, f_x and f_y at every point of the surface. Note that N is a globally defined object because M and $\mathbb{S}^3$ are orientable manifolds.

Definition 2.5. *The symmetric bilinear map $b : T_pM \times T_pM \to \mathbb{R}$ defined by*

$$b(v,w) = \langle d^2 f(v,w), N\rangle_{\mathbb{R}^4}$$

*is called the **second fundamental form**.*

Due to the definition of N we get $\langle df, N\rangle = 0$ and therefore by Leibniz's rule

$$b = \langle d^2 f, N\rangle = -\langle df, dN\rangle.$$

Again the map b can locally be represented by a matrix

$$b_p = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} \langle f_{xx}, N\rangle & \langle f_{xy}, N\rangle \\ \langle f_{yx}, N\rangle & \langle f_{yy}, N\rangle \end{pmatrix} = -\begin{pmatrix} \langle f_x, N_x\rangle & \langle f_x, N_y\rangle \\ \langle f_y, N_x\rangle & \langle f_y, N_y\rangle \end{pmatrix}$$

Now let $z = x + iy$ and $\bar{z} = x - iy$ be local complex coordinates on M and define

$$\tfrac{\partial}{\partial z} := \tfrac{1}{2}\left(\tfrac{\partial}{\partial x} - i\tfrac{\partial}{\partial y}\right) \quad \text{and} \quad \tfrac{\partial}{\partial \bar{z}} := \tfrac{1}{2}\left(\tfrac{\partial}{\partial x} + i\tfrac{\partial}{\partial y}\right).$$

Let us rephrase the above objects with respect to these complex coordinates. In case of a *conformal immersion* f, one can write the first fundamental form as

$$g = e^{2u} dz d\bar{z} = 2\langle f_z, f_{\bar{z}}\rangle dz d\bar{z}$$

and for the second fundamental form one obtains (in general)

$$b = Q dz^2 + \widetilde{H} dz d\bar{z} + \overline{Q} d\bar{z}^2,$$

where Q is the complex-valued function

$$Q := \frac{1}{4}(b_{11} - b_{22} - ib_{12} - ib_{21}) = \langle f_{zz}, N\rangle$$

and $\widetilde{H}$ is the real valued function

$$\widetilde{H} := \frac{1}{2}(b_{11} + b_{22}) = 2\langle f_{z\bar{z}}, N\rangle.$$

Definition 2.6. *The quadratic differential Qdz^2 is called the* ***Hopf differential*** *of the immersion f.*

Definition 2.7. *The linear map $S : T_pM \to T_pM$ given by*

$$S := g_p^{-1} \cdot b_p$$

is called the ***shape operator*** *of the immersion f.*

The eigenvalues k_1, k_2 and corresponding eigenvectors of the shape operator S are the *principal curvatures* and *principal curvature directions* of the surface $f(M)$ at $f(p)$. We can now define the Gauss and mean curvature using the objects introduced above.

Definition 2.8. *Let $f : M \to \mathbb{S}^3$ be an immersion and $S = g^{-1}b$ the corresponding shape operator. The determinant $K := \det(S)$ of the shape operator S is the* ***Gauss curvature*** *and $H := \frac{1}{2}tr(S)$ is the* ***mean curvature*** *of the immersion. The immersion f is CMC (i.e. of constant mean curvature) if H is constant, i.e. $H \equiv const.$*

Remark 2.9. *In case of a conformal immersion one gets $H = e^{-2u}\widetilde{H}$.*

Definition 2.10. *Let M be a 2-dimensional manifold. The* ***umbilic points*** *of an immersion $f : M \to \mathbb{S}^3$ are the points where the two principal curvatures are equal.*

The Hopf differential Qdz^2 encodes some important information. Besides the fact that the investigated surface will be CMC if and only if Q is holomorphic, the Hopf differential can also be used to characterize the umbilic points of that surface.

Proposition 2.11. *If M is a Riemann surface and $f : M \to \mathbb{S}^3$ is a conformal immersion, then $p \in M$ is an umbilic point if and only if $Q_p = 0$.*

Proof. Omitting the subscript p the shape operator corresponding to the conformal immersion f is given by

$$S = g^{-1}b = \frac{1}{e^{2u}} \begin{pmatrix} H + Q + \bar{Q} & i(Q - \bar{Q}) \\ i(Q - \bar{Q}) & H - Q - \bar{Q} \end{pmatrix}$$

with respect to the basis f_x and f_y of each tangent space of $f(M)$. The two principal curvatures are then the two eigenvalues of this self-adjoint operator, i.e. the solutions of

$$\begin{aligned} e^{4u} \det(S - k\mathbb{1}) &= (H + Q + \bar{Q} - k)(H - Q - \bar{Q} - k) + (Q - \bar{Q})^2 \\ &= ((H - k) + (Q + \bar{Q}))((H - k) - (Q + \bar{Q})) - (Q - \bar{Q})^2 \\ &= (H - k)^2 - (Q + \bar{Q})^2 - (Q - \bar{Q})^2 \\ &= (H - k)^2 - 4|Q|^2 = 0, \end{aligned}$$

and thus one obtains

$$k_1 = H + 2|Q|, \;\; k_2 = H - 2|Q|.$$

Finally one gets $k_1 = k_2 \Leftrightarrow |Q| = 0 \Leftrightarrow Q = 0$ and the result follows. □

2.2 Compact Riemann surfaces

In this section we will focus on divisors and the Riemann-Roch Theorem for compact Riemann surfaces that will be useful tools in the following chapters. Most results and terminology are taken from [20] and [21].

Definition 2.12. *Let Y be a Riemann surface. A **divisor** on Y is a map*

$$D : Y \to \mathbb{Z}$$

such that for every compact subset $K \subset Y$ there are only finitely many points $y \in K$ with $D(y) \neq 0$. With respect to addition the set of all divisors on Y is an abelian group, denoted by $Div(Y)$.
For $D, D' \in Div(Y)$ we say $D \leq D'$ if $D(y) \leq D'(y)$ for every $y \in Y$.

For a Riemann surface Y let $\mathcal{M}(Y)$ denote the field of meromorphic functions on Y. Now suppose that U is an open subset of Y. For a meromorphic function $f \in \mathcal{M}(U)$ and $x \in U$ define

$$\mathrm{ord}_x(f) := \begin{cases} 0, & \text{if } f \text{ is holomorphic and non-zero at } x, \\ k, & \text{if } f \text{ has a zero of order } k \text{ at } x, \\ -k, & \text{if } f \text{ has a pole of order } k \text{ at } x, \\ \infty, & \text{if } f \text{ is identically zero in a neighborhood of } x. \end{cases}$$

Thus for any meromorphic function $f \in \mathcal{M}(Y)\backslash\{0\}$, the mapping $y \mapsto \mathrm{ord}_y(f)$ is a divisor on Y. It is called the *divisor of* f and will be denoted by (f).

The function f is said to be a *multiple* of the divisor D if $(f) \geq D$. Then f is holomorphic if and only if $(f) \geq 0$.

For a meromorphic 1-form ω one can define its order at a point $x \in U$ as follows. Choose a coordinate neighborhood (V, z) of x. Then on $V \cap U$ one has $\omega = f dz$, where f is a meromorphic function. Set $\mathrm{ord}_x(\omega) = \mathrm{ord}_x(f)$, this is independent of the choice of (V, z). Again the mapping $y \mapsto \mathrm{ord}_y(\omega)$ is a divisor on Y, denoted by (ω).

A divisor $D \in \mathrm{Div}(Y)$ is called a *principal divisor* if there exists a function $f \in \mathcal{M}(Y)\backslash\{0\}$ such that $D = (f)$. Two divisors $D, D' \in \mathrm{Div}(Y)$ are said to be *equivalent* if their difference $D - D'$ is a principal divisor. A *canonical divisor* is the divisor of a meromorphic 1-form ω.

Definition 2.13. *For a compact Riemann surface Y let*

$$\deg : Div(Y) \to \mathbb{Z}, \quad D \mapsto \sum_{y \in Y} D(y).$$

*For $D \in Div(Y)$ the integer $\deg(D)$ is called the **degree** of the divisor D.*

The map $\deg : \mathrm{Div}(Y) \to \mathbb{Z}$ is a group homomorphism and $\deg(f) = 0$ for any principal divisor (f) on a compact Riemann surface since a meromorphic function has as many zeros as poles.

Before we can state the Riemann-Roch Theorem we have to introduce the notion of a sheaf and of its corresponding cohomology.

Definition 2.14. *Suppose Y is a topological space and $\mathcal{I}$ is the system of open sets in Y. A **presheaf** of abelian groups on Y is a pair $(\mathcal{F}, \rho)$ consisting of*

1. *a family $\mathcal{F} = (\mathcal{F}(U))_{U \in \mathcal{I}}$ of abelian groups,*
2. *a family $\rho = (\rho_V^U)_{U,V \in \mathcal{I}, V \subset U}$ of group homomorphisms (called restriction homomorphisms)*
$$\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V), \text{ where } V \text{ is open in } U,$$

with the following properties:

$$\begin{aligned} \rho_U^U &= id_{\mathcal{F}(U)} \text{ for every } U \in \mathcal{I}, \\ \rho_W^V \circ \rho_V^U &= \rho_W^U \text{ for } W \subset V \subset U. \end{aligned}$$

Instead of $\rho_V^U(f)$ for $f \in \mathcal{F}(U)$ one writes $f|V$. We can now define a sheaf.

Definition 2.15. *A presheaf $\mathcal{F}$ on a topological space Y is called a **sheaf** if for every open set $U \subset Y$ and every family of open subsets $U_i \subset U, i \in I$, with $U = \bigcup_{i \in I} U_i$, the following conditions are satisfied:*

(S1) If $f, g \in \mathcal{F}(U)$ are elements such that $f|U_i = g|U_i$ for every $i \in I$, then $f = g$.

(S2) Given elements $f_i \in \mathcal{F}(U_i), i \in I$, obeying

$$f_i|U_i \cap U_j = f_j|U_i \cap U_j \text{ for all } i, j \in I,$$

then there exists $f \in \mathcal{F}(U)$ such that $f|U_i = f_i$ for every $i \in I$.

(S1) and (S2) are called the sheaf axioms.

Definition 2.16. *Let Y be a topological space and $\mathcal{F}$ a sheaf of abelian groups on Y. Let $\mathcal{U}$ be an open covering of Y, i.e. a family $\mathcal{U} = (U_i)_{i \in I}$ of open subsets of Y such that $\bigcup_{i \in I} U_i = Y$. For $q = 0, 1, 2, \ldots$ define the **qth cochain group** of $\mathcal{F}$, with respect to $\mathcal{U}$, as*

$$C^q(\mathcal{U}, \mathcal{F}) := \prod_{(i_0, \ldots, i_q) \in I^{q+1}} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_q}).$$

The elements of $C^q(\mathcal{U}, \mathcal{F})$ are called q-cochains.

Now define *coboundary operators*

$$\delta : C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$$
$$\delta : C^1(\mathcal{U}, \mathcal{F}) \to C^2(\mathcal{U}, \mathcal{F})$$

as follows:

1. For $(f_i)_{i \in I} \in C^0(\mathcal{U}, \mathcal{F})$ let $\delta((f_i)_{i\in I}) = (g_{ij})_{i,j\in I}$ where
$$g_{ij} := f_j - f_i \in \mathcal{F}(U_i \cap U_j).$$
2. For $(f_{ij})_{i,j \in I} \in C^1(\mathcal{U}, \mathcal{F})$ let $\delta((f_{ij})_{i,j\in I}) = (g_{ijk})$ where
$$g_{ijk} := f_{jk} - f_{ik} + f_{ij} \in \mathcal{F}(U_i \cap U_j \cap U_k).$$

These coboundary operators are group homomorphisms. Thus we arrive at

Definition 2.17. *Let*

$$\begin{aligned} Z^1(\mathcal{U}, \mathcal{F}) &:= Ker(C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^2(\mathcal{U}, \mathcal{F})), \\ B^1(\mathcal{U}, \mathcal{F}) &:= Im(C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathcal{F})). \end{aligned}$$

*The elements of $Z^1(\mathcal{U}, \mathcal{F})$ are called **1-cocycles** and those of $B^1(\mathcal{U}, \mathcal{F})$ are called **1-coboundaries**.*

Definition 2.18. *The quotient group*

$$H^1(\mathcal{U}, \mathcal{F}) := Z^1(\mathcal{U}, \mathcal{F})/B^1(\mathcal{U}, \mathcal{F})$$

*is called the **first cohomology group** with coefficients in $\mathcal{F}$ and with respect to the covering $\mathcal{U}$.*

An open covering $\mathcal{B} = (V_k)_{k\in K}$ is finer with respect to the covering $\mathcal{U} = (U_i)_{i\in I}$, denoted by $\mathcal{B} < \mathcal{U}$, if every V_k is contained in at least one U_i. Thus there is a mapping $\tau : K \to I$ such that
$$V_k \subset U_{\tau(k)} \text{ for every } k \in K.$$
We can now define a mapping

$$t^{\mathcal{U}}_{\mathcal{B}} : Z^1(\mathcal{U}, \mathcal{F}) \to Z^1(\mathcal{B}, \mathcal{F})$$

in the following way. For $(f_{ij}) \in Z^1(\mathcal{U}, \mathcal{F})$ let $t^{\mathcal{U}}_{\mathcal{B}}((f_{ij})) = (g_{kl})$ where

$$g_{kl} := f_{\tau(k),\tau(l)} | V_k \cap V_l \text{ for every } k, l \in K.$$

This mapping induces a homomorphism of the cohomology groups (also denoted by $t^{\mathcal{U}}_{\mathcal{B}}$) and we are finally ready to define $H^1(Y, \mathcal{F})$.

Definition 2.19. *Given three open coverings such that $\mathcal{W} < \mathcal{B} < \mathcal{U}$, one has*

$$t^{\mathcal{B}}_{\mathcal{W}} \circ t^{\mathcal{U}}_{\mathcal{B}} = t^{\mathcal{U}}_{\mathcal{W}}.$$

Now define the following equivalence relation $\sim$ on the disjoint union of the $H^1(\mathcal{U}, \mathcal{F})$, where $\mathcal{U}$ runs through all open coverings of Y, for two cohomology classes $\xi \in H^1(\mathcal{U}, \mathcal{F})$, $\eta \in H^1(\mathcal{U}', \mathcal{F})$ by

$$\xi \sim \eta \quad :\Leftrightarrow \quad \exists \text{ open covering } \mathcal{B} \text{ with } \mathcal{B} < \mathcal{U} \text{ and } \mathcal{B} < \mathcal{U}' \text{ such that } t^{\mathcal{U}}_{\mathcal{B}}(\xi) = t^{\mathcal{U}'}_{\mathcal{B}}(\eta).$$

*The set of equivalence classes is called the **first cohomology group** of Y with coefficients in the sheaf $\mathcal{F}$:*

$$H^1(Y, \mathcal{F}) = \left(\bigcup_{\mathcal{U}} H^1(\mathcal{U}, \mathcal{F}) \right) / \sim .$$

The following theorem shows how one obtains $H^1(Y, \mathcal{F})$ by using a single open covering of Y.

Theorem 2.20 (Leray, [21], Theorem 12.8)**.** *Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space Y and let $\mathcal{U} = (U_i)_{i \in I}$ be an open covering of Y such that $H^1(U_i, \mathcal{F}) = 0$ for all $i \in I$. Then one has*

$$H^1(Y, \mathcal{F}) \simeq H^1(\mathcal{U}, \mathcal{F}).$$

Such an open covering $\mathcal{U}$ is called a Leray covering with respect to $\mathcal{F}$.

Now suppose D is a divisor on the Riemann surface Y. For any open set $U \subset Y$ define $\mathcal{O}_D(U)$ to be the set of all meromorphic functions on U which are multiples of the divisor $-D$, i.e.

$$\mathcal{O}_D(U) := \{ f \in \mathcal{M}(U) \mid \mathrm{ord}_x(f) \geq -D(x) \text{ for every } x \in U \}.$$

Together with the natural restriction mappings, $\mathcal{O}_D$ is a sheaf. In the special case of the zero divisor $D \equiv 0$ one has $\mathcal{O}_0 =: \mathcal{O}$, the *sheaf of holomorphic functions.* Note that $H^1(Y, \mathcal{O}_D)$ and $H^0(Y, \mathcal{O}_D) := \mathcal{O}_D(Y)$ are vector spaces.

We recall the definition of the genus of a compact Riemann surface before we state the theorem that is central in the theory of compact Riemann surfaces.

Definition 2.21. *For a compact Riemann surface Y,*

$$g := \dim H^1(Y, \mathcal{O})$$

*is called the **genus** of Y.*

Theorem 2.22 (The Riemann-Roch Theorem, [21], Theorem 16.9)**.** *Suppose D is a divisor on a compact Riemann surface Y of genus g. Then $H^0(Y, \mathcal{O}_D)$ and $H^1(Y, \mathcal{O}_D)$ are finite dimensional vector spaces and*

$$\dim H^0(Y, \mathcal{O}_D) - \dim H^1(Y, \mathcal{O}_D) = 1 - g + \deg D.$$

Definition 2.23. *The non-negative integer*

$$i(D) := \dim H^1(Y, \mathcal{O}_D)$$

is called the ***index of speciality*** *of the divisor D.*

We can reformulate the Riemann-Roch Theorem in the following form

$$\dim H^0(Y, \mathcal{O}_D) = 1 - g + \deg D + i(D).$$

We will now state the Serre Duality Theorem that permits a simpler interpretation of the cohomology groups $H^1(Y, \mathcal{O}_D)$ in terms of differential forms.

For this purpose let Y be a compact Riemann surface. For any divisor $D \in \mathrm{Div}(Y)$ we denote by Ω_D the sheaf of meromorphic 1-forms which are multiples of $-D$. Thus for any open set $U \subset Y$ the linear space $\Omega_D(U)$ consists of all differential forms ω such that $\mathrm{ord}_x(\omega) \geq -D(x)$ for every $x \in U$.

Theorem 2.24 (The Duality Theorem of Serre, [21], Theorem 17.9)**.** *For any divisor D on a compact Riemann surface Y the map*

$$i_D : H^0(Y, \Omega_{-D}) \to H^1(Y, \mathcal{O}_D)^*, \ \omega \mapsto i_D(\omega)$$

with

$$i_D(\omega) : H^1(Y, \mathcal{O}_D) \to \mathbb{C}, \ \xi \mapsto i_D(\omega)(\xi) = Res(\xi\omega)$$

is an isomorphism of vector spaces, i.e. $H^0(Y, \Omega_{-D}) \simeq H^1(Y, \mathcal{O}_D)^*$.

Remark 2.25. *From the Serre Duality Theorem one immediately obtains*

$$\dim H^1(Y, \mathcal{O}_D) = \dim H^0(Y, \Omega_{-D}).$$

In particular for $D = 0$ one has

$$g = \dim H^1(Y, \mathcal{O}) = \dim H^0(Y, \Omega).$$

Thus the genus of a compact Riemann surface Y is equal to the maximum number of linearly independent holomorphic 1-forms on Y. One can now reformulate the Riemann-Roch Theorem as follows:

$$\dim H^0(Y, \mathcal{O}_{-D}) - \dim H^0(Y, \Omega_D) = 1 - g - \deg D.$$

Theorem 2.26 ([21], Theorem 17.12)**.** *The divisor of a non-vanishing meromorphic 1-form ω on a compact Riemann surface of genus g satisfies*

$$\deg(\omega) = 2g - 2.$$

Thus the canonical divisor K satisfies $\deg(K) = \deg(\omega) = 2g - 2$.

2.3 Lie groups

In order to understand the concept of moving frames and the following considerations, one has to recall some basic facts about Lie groups. We will also turn to the description of $\mathbb{S}^3$ via the Lie group SU(2).

Definition 2.27. *Let G be a Lie group. For $g \in G$ we consider the maps of left and right multiplication by g*

$$\begin{aligned} L_g : G \to G, \qquad h \mapsto gh, \\ R_g : G \to G, \qquad h \mapsto hg. \end{aligned}$$

*A vector field $X : G \to TG$ is called **left-invariant**, if*

$$dL_g \circ X = X \circ L_g \text{ for all } g \in G.$$

With the above definition it follows that left-invariant vector fields are uniquely determined through their values at the identity e, since

$$X(g) = d_e L_g X(e).$$

Denoting the linear space of **left-invariant vector fields** by $\Gamma_L(G)$ one obtains the following vector space isomorphism

$$\begin{array}{rcl} \Gamma_L(G) & \to & T_eG \\ X & \mapsto & X(e) \end{array}$$

with inverse map given by $T_eG \ni v_e \mapsto X \in \Gamma_L(G),\ X(g) := d_e L_g(v_e)$.

Definition 2.28. *The **Lie algebra** $\mathfrak{g}$ associated with a Lie group G is the tangent space of G at the identity e, i.e. $\mathfrak{g} = T_eG$, together with the Lie bracket operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ given by*

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \text{ for } X, Y \in \mathfrak{g} \text{ and smooth } f : G \to \mathbb{R}.$$

Here the vector field X acts on the function f by $X(f) := df(X)$.

Thus the left-invariant vector fields, equipped with the commutator $[\cdot, \cdot]$ correspond to $\mathfrak{g}$. Moreover, the tangent bundle of a Lie group is trivial:

$$\begin{array}{rcl} TG & \to & G \times \mathfrak{g} \\ v_g & \mapsto & (g, d_g L_g^{-1}(v_g)), \end{array}$$

where the inverse map of this isomorphism is given by $(g, v_e) \mapsto d_e L_g(v_e)$. We can now define the Maurer-Cartan form.

Definition 2.29. *The (left) Maurer-Cartan form is the $\mathfrak{g}$-valued 1-form $g \mapsto \theta_g$ with*

$$\begin{array}{rccl} \theta_g : & TG & \to & \mathfrak{g} \\ & v_g & \mapsto & d_g L_g^{-1}(v_g). \end{array}$$

This is often written as $\theta = g^{-1}dg$.

For two $\mathfrak{g}$-valued 1-forms α, β we define the $\mathfrak{g}$-valued 2-form $[\alpha \wedge \beta]$ by

$$[\alpha \wedge \beta](X, Y) := [\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)]$$

for vector fields X, Y. Now we arrive at the following well-known

Proposition 2.30. *The Maurer-Cartan form satisfies the equation*

$$2d\theta + [\theta \wedge \theta] = 0.$$

*It is called the structure equation of or the **Maurer-Cartan equation** on $\mathfrak{g}$.*

Proof. First we note that

$$d\theta = d(g^{-1}) \wedge dg.$$

To compute $d(g^{-1})$, consider the function e identically equal to the unit $e \in G$ and note that it equals the product of g and g^{-1}. Then we have

$$0 = d(e) = d(g^{-1}g) = d(g^{-1})g + g^{-1}dg.$$

So, $d(g^{-1}) = -g^{-1}(dg)g^{-1}$ and thus

$$d\theta = -g^{-1}(dg)g^{-1} \wedge dg = -(g^{-1}dg) \wedge (g^{-1}dg) = -\theta \wedge \theta =: -\frac{1}{2}[\theta \wedge \theta].$$

□

We state the following proposition that will be useful later on.

Proposition 2.31. *For a map $f : M \to G$, the pullback $\omega := f^\star\theta$ also satisfies the Maurer-Cartan equation, i.e.*

$$2d\omega + [\omega \wedge \omega] = 0.$$

Proof. A short calculation yields

$$\begin{aligned} 2d\omega + [\omega \wedge \omega] &= 2d(f^\star\theta) + [f^\star\theta \wedge f^\star\theta] = 2f^\star d\theta + f^\star[\theta \wedge \theta] \\ &= f^\star(2d\theta + [\theta \wedge \theta]) = 0 \end{aligned}$$

and thus the claim is proved. □

The Lie groups $SL(2,\mathbb{C})$ and $SU(2)$. Let us consider the Lie group $SL(2,\mathbb{C}) := \{A \in M_{2\times 2}(\mathbb{C}) \mid \det(A) = 1\}$. The Lie algebra $\mathfrak{sl}_2(\mathbb{C}) := \{B \in M_{2\times 2}(\mathbb{C}) \mid \mathrm{tr}(B) = 0\}$ of $SL(2,\mathbb{C})$ is spanned by $\epsilon_+, \epsilon_-, \epsilon$ with

$$\epsilon_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \epsilon_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Remark 2.32. *Another commonly used basis for $\mathfrak{sl}_2(\mathbb{C})$ is given by the Cartan-basis*

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It will be convenient to identify $\mathbb{S}^3$ with the Lie group

$$\begin{aligned} SU(2) &= \{A \in M_{2\times 2}(\mathbb{C}) \mid \det(A) = 1,\ \bar{A}^t = A^{-1}\} \\ &= \left\{ \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \,\middle|\, z, w \in \mathbb{C} \text{ with } |z|^2 + |w|^2 = 1 \right\} \simeq \mathbb{S}^3. \end{aligned}$$

The Lie algebra corresponding to $SU(2)$ is denoted by $\mathfrak{su}(2)$ and a direct computation shows that

$$\begin{aligned} \mathfrak{su}(2) &= \{B \in M_{2\times 2}(\mathbb{C}) \mid \mathrm{tr}(B) = 0,\ \bar{B}^t = -B\} \\ &= \left\{ \begin{pmatrix} ix_4 & -x_3 + ix_2 \\ x_3 + ix_2 & -ix_4 \end{pmatrix} \,\middle|\, x_2, x_3, x_4 \in \mathbb{R} \right\} \simeq \mathbb{R}^3. \end{aligned}$$

The identification $\mathbb{S}^3 \simeq SU(2)$ results from the following proposition (see [22], Chap. 5).

Proposition 2.33. *The map $\Phi : \mathbb{R}^4 \to \mathbb{R}^4_{Mat}$ given by*

$$x \mapsto \Phi(x) =: X = \begin{pmatrix} x_1 + ix_4 & -x_3 + ix_2 \\ x_3 + ix_2 & x_1 - ix_4 \end{pmatrix}$$

is an isometry, i.e. $\langle x, y\rangle_{\mathbb{R}^4} = \langle \Phi(x), \Phi(y)\rangle_{\mathbb{R}^4_{Mat}}$ for all $x, y \in \mathbb{R}^4$. Here the inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}^4_{Mat}}$ on $\mathbb{R}^4_{Mat}$ is given by

$$\langle X, Y\rangle_{\mathbb{R}^4_{Mat}} := \tfrac{1}{2} tr(X\overline{Y}^t), \quad X, Y \in \mathbb{R}^4_{Mat}.$$

Remark 2.34. *If we consider the complex bilinear extension of $\langle \cdot, \cdot \rangle_{\mathbb{R}^4}$ to $\mathbb{C}^4$ the inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}^4_{Mat}}$ is replaced by (compare [22], Chap. 5)*

$$\langle X, Y\rangle := \tfrac{1}{2} tr(X\sigma_2 Y^t \sigma_2) \quad \textit{with} \quad \sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right).$$

Proof. Consider $x, y \in \mathbb{R}^4$ and their images $X = \Phi(x), Y = \Phi(y) \in \mathbb{R}^4_{\mathrm{Mat}}$. Then we get

$$\begin{aligned} \langle \Phi(x), \Phi(y)\rangle_{\mathbb{R}^4_{\mathrm{Mat}}} &= \langle X, Y\rangle_{\mathbb{R}^4_{\mathrm{Mat}}} = \tfrac{1}{2}\mathrm{tr}(X\overline{Y}^t) \\ &= \frac{1}{2}\mathrm{tr}\left[\begin{pmatrix} x_1 + ix_4 & -x_3 + ix_2 \\ x_3 + ix_2 & x_1 - ix_4 \end{pmatrix} \begin{pmatrix} y_1 - iy_4 & y_3 - iy_2 \\ -y_3 - iy_2 & y_1 + iy_4 \end{pmatrix} \right] \\ &= x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 \\ &= \langle x, y\rangle_{\mathbb{R}^4}. \end{aligned}$$

This shows that Φ is an isometry and concludes the proof. □

Remark 2.35. *Restricting Φ to the 3-sphere $\mathbb{S}^3$ one obtains the following commutative diagram:*

$$\begin{tikzcd} T\mathbb{S}^3 \arrow[r, "d\Phi"] \arrow[d, "\pi_1"'] & TSU(2) \arrow[d, "\pi_2"] \\ \mathbb{S}^3 \arrow[r, "\Phi"] & SU(2) \end{tikzcd}$$

Here π_1, π_2 denote the projections of the tangent bundle of $\mathbb{S}^3$ and $SU(2)$ respectively. Moreover, $d\Phi$ respects the metrics on $T\mathbb{S}^3$ and $TSU(2)$ and there holds

$$\langle v, w\rangle_{\mathbb{R}^4} = \langle d\Phi(v), d\Phi(w)\rangle_{\mathbb{R}^4_{Mat}} = \tfrac{1}{2} tr(\alpha \cdot \overline{\beta}^t) = -\tfrac{1}{2} tr(\alpha \cdot \beta)$$

for $v, w \in T_x\mathbb{S}^3$ and $\alpha := d\Phi(v), \beta := d\Phi(w) \in \mathfrak{su}(2)$.

2.4 The concept of moving frames and Lax pairs

Given a Riemann surface M with coordinates x, y we can introduce the so-called extended frame for an immersion $f : M \to \mathbb{S}^3$. Again, we follow the terminology introduced in [22].

Definition 2.36. *Let $f : M \to \mathbb{S}^3 \subset \mathbb{R}^4$ be an immersion of a Riemann surface M. The map $\widetilde{\mathcal{F}} : M \to SO(4)$ given by*

$$p \mapsto \widetilde{\mathcal{F}}(p) := \left(f(p), \frac{f_x(p)}{\|f_x(p)\|}, \frac{f_y(p)}{\|f_y(p)\|}, N(p) \right)$$

is called the (normalized) ***extended moving frame****.*

Given an immersion $f : M \to \mathbb{S}^3 \subset \mathbb{R}^4$ of a Riemann surface with complex coordinates $z, \bar{z}$, we set $\mathcal{F} := (f, f_z, f_{\bar{z}}, N)$ and can check that $\mathcal{F}$ satisfies the following integrability conditions.

Proposition 2.37. *Let $f : M \to \mathbb{S}^3$ be a conformal immersion of a simply-connected Riemann surface M and set $\mathcal{F}$ as the matrix $\mathcal{F} = (f, f_z, f_{\bar{z}}, N)$. Then $\mathcal{F}$ is a solution of the system*

$$\mathcal{F}_z = \mathcal{F}\mathcal{U}, \quad \mathcal{F}_{\bar{z}} = \mathcal{F}\mathcal{V}$$

with

$$\mathcal{U} = \begin{pmatrix} 0 & 0 & -\frac{1}{2}e^{2u} & 0 \\ 1 & 2u_z & 0 & -H \\ 0 & 0 & 0 & -2Qe^{-2u} \\ 0 & Q & \frac{1}{2}He^{2u} & 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} 0 & -\frac{1}{2}e^{2u} & 0 & 0 \\ 0 & 0 & 0 & -2\overline{Q}e^{-2u} \\ 1 & 0 & 2u_{\bar{z}} & -H \\ 0 & \frac{1}{2}He^{2u} & \overline{Q} & 0 \end{pmatrix}.$$

The pair of matrices $(\mathcal{U}, \mathcal{V})$ is called the ***Lax pair*** *of the immersion f.*

Proof. The result is classical, but nevertheless we present a proof in order to fix notation. Since f maps to $\mathbb{S}^3$ we have $\langle f, f\rangle = 1$ and obtain $\langle f, f_z\rangle = 0 = \langle f, f_{\bar z}\rangle$ by Leibniz's rule. Moreover, the unit normal field N with $\langle N, N\rangle = 1$ satisfies $\langle f, N\rangle = 0$ as well as $\langle f_x, N\rangle = 0 = \langle f_y, N\rangle$ due to its definition and thus $\langle f_z, N\rangle = 0 = \langle f_{\bar z}, N\rangle$. Since f is conformal we have $\langle f_x, f_y\rangle = 0$ and $\langle f_x, f_x\rangle = e^{2u} = \langle f_y, f_y\rangle$. This leads to $\langle f_z, f_z\rangle = 0 = \langle f_{\bar z}, f_{\bar z}\rangle$ and $\langle f_z, f_{\bar z}\rangle = \frac{1}{2}e^{2u}$. Moreover, we get

$$\tfrac{d}{dz}\langle f, f_z\rangle = \langle f_z, f_z\rangle + \langle f, f_{zz}\rangle = \langle f, f_{zz}\rangle \overset{!}{=} 0$$

as well as $\langle f_{z\bar z}, f_z\rangle = 0 = \langle f_{\bar z z}, f_{\bar z}\rangle$ and $\langle f_{zz}, f_z\rangle = 0 = \langle f_{\bar z\bar z}, f_{\bar z}\rangle$ by taking the corresponding derivative of $\langle f_z, f_z\rangle = 0 = \langle f_{\bar z}, f_{\bar z}\rangle$. Differentiation of $\langle N, f\rangle = 0$ and $\langle N, N\rangle = 1$ leads to

$$\tfrac{d}{dz}\langle N, f\rangle = \langle N_z, f\rangle + \langle N, f_z\rangle = \langle N_z, f\rangle \overset{!}{=} 0$$

and $\langle N_z, N\rangle = 0 = \langle N_{\bar z}, N\rangle$. Thus the derivatives of the entries of the extended frame $\mathcal{F} = (f, f_z, f_{\bar z}, N)$ with respect to z are given by

$$\begin{aligned}
f_z &= \langle f_z, f\rangle f + \langle f_z, f_{\bar z}\rangle 2\frac{f_z}{e^{2u}} + \langle f_z, f_z\rangle 2\frac{f_{\bar z}}{e^{2u}} + \langle f_z, N\rangle N \\
&= \langle f_z, f_{\bar z}\rangle 2\frac{f_z}{e^{2u}}, \\
f_{zz} &= \langle f_{zz}, f\rangle f + \langle f_{zz}, f_{\bar z}\rangle 2\frac{f_z}{e^{2u}} + \langle f_{zz}, f_z\rangle 2\frac{f_{\bar z}}{e^{2u}} + \langle f_{zz}, N\rangle N \\
&= \langle f_{zz}, f_{\bar z}\rangle 2\frac{f_z}{e^{2u}} + \langle f_{zz}, N\rangle N, \\
f_{\bar z z} &= \langle f_{z\bar z}, f\rangle f + \langle f_{z\bar z}, f_{\bar z}\rangle 2\frac{f_z}{e^{2u}} + \langle f_{z\bar z}, f_z\rangle 2\frac{f_{\bar z}}{e^{2u}} + \langle f_{z\bar z}, N\rangle N \\
&= \langle f_{z\bar z}, f\rangle f + \langle f_{z\bar z}, N\rangle N, \\
N_z &= \langle N_z, f\rangle f + \langle N_z, f_{\bar z}\rangle 2\frac{f_z}{e^{2u}} + \langle N_z, f_z\rangle 2\frac{f_{\bar z}}{e^{2u}} + \langle N_z, N\rangle N \\
&= \langle N_z, f_{\bar z}\rangle 2\frac{f_z}{e^{2u}} + \langle N_z, f_z\rangle 2\frac{f_{\bar z}}{e^{2u}}.
\end{aligned}$$

Note that $f_{\bar z}, f_{\bar z\bar z}$ and $N_{\bar z}$ are obtained by complex conjugation. Recall that the Hopf differential Q and the mean curvature H are defined by

$$Q = \langle f_{zz}, N\rangle, \quad \tfrac{1}{2}He^{2u} = \langle f_{z\bar z}, N\rangle.$$

If we differentiate $\langle f_z, f\rangle = 0$ with respect to $\bar z$ we get $\langle f_{z\bar z}, f\rangle = -\langle f_z, f_{\bar z}\rangle = -\frac{1}{2}e^{2u}$ and differentiating the equation $\langle f_z, f_{\bar z}\rangle = \frac{1}{2}e^{2u}$ one obtains

$$\langle f_{zz}, f_{\bar z}\rangle = e^{2u}u_z \quad \text{and} \quad \langle f_{\bar z\bar z}, f_z\rangle = e^{2u}u_{\bar z}.$$

Moreover, differentiation of the equations $\langle N, f_z\rangle = 0$ and $\langle N, f_{\bar z}\rangle = 0$ leads to

$$\langle N_z, f_z\rangle = -\langle N, f_{zz}\rangle = -Q \quad \text{and} \quad \langle N_z, f_{\bar z}\rangle = -\langle N, f_{\bar z z}\rangle = -\tfrac{1}{2}He^{2u}.$$

Equipped with all these equations one can directly check that the matrices $\mathcal{U} = \mathcal{F}^{-1}\mathcal{F}_z$ and $\mathcal{V} = \mathcal{F}^{-1}\mathcal{F}_{\bar z}$ are of the form stated above. $\square$

Lax pairs in terms of 2×2 matrices. We will rework 4×4 Lax pairs into 2×2 Lax pairs and make the following observation: Obviously the matrix $\widetilde{\mathcal{F}} \in SO(4)$ acts on $\mathbb{R}^4$. Via the identification of $\mathbb{R}^4$ with $\mathbb{R}^4_{\mathrm{Mat}}$ (see Proposition 2.33) one obtains an identification of $\widetilde{\mathcal{F}}$ with $(F, G) \in SU(2) \times SU(2)$

$$x \mapsto \widetilde{\mathcal{F}}x \quad \longleftrightarrow \quad \Phi(x) \mapsto F\Phi(x)G^{-1} = FXG^{-1} \tag{2.4.1}$$

with

$$F_z = FU, \;\; F_{\bar{z}} = FV, \quad G_z = G\widetilde{U}, \;\; G_{\bar{z}} = G\widetilde{V} \tag{2.4.2}$$

or in a shorter form

$$dF = F\alpha, \quad dG = G\beta$$

with

$$\alpha = Udz + Vd\bar{z} = F^{-1}dF, \quad \beta = \widetilde{U}dz + \widetilde{V}d\bar{z} = G^{-1}dG.$$

With respect to the group action $X \mapsto FXG^{-1}$ the pair (F, G) is equivalent to the pair $(-F, -G)$ and therefore one obtains a double cover of $SO(4)$ by the group $SU(2) \times SU(2)$. This leads to the following commutative diagram:

$$\begin{tikzcd}
SO(4) \arrow[r] \arrow[d, "\phi_1"'] & SU(2) \times SU(2)/\pm(\mathbb{1}, \mathbb{1}) \arrow[d, "\phi_2"] \\
\mathbb{R}^4 \arrow[r, "\Phi"] & \mathbb{R}^4_{\mathrm{Mat}}
\end{tikzcd}$$

Here ϕ_1, ϕ_2 denote the group actions as stated in (2.4.1). Finally the above identification yields the map f via

$$f = \widetilde{\mathcal{F}}e_1 \quad \longleftrightarrow \quad f = F\Phi(e_1)G^{-1} = F\mathbb{1}G^{-1} = FG^{-1}.$$

We now calculate the new Lax pairs (U, V) and $(\widetilde{U}, \widetilde{V})$ (compare with [22], Section 3.2).

Lemma 2.38. *The double cover of $SO(4)$ is $SU(2) \times SU(2)$ via the group action*

$$X \mapsto FXG^{-1}$$

and the Lax pair $(\mathcal{U}, \mathcal{V})$ is transformed to

$$U = \frac{1}{2}\begin{pmatrix} u_z & (H+i)e^u \\ -2e^{-u}Q & -u_z \end{pmatrix}, \quad V = \frac{1}{2}\begin{pmatrix} -u_{\bar{z}} & 2e^{-u}\overline{Q} \\ -(H-i)e^u & u_{\bar{z}} \end{pmatrix},$$

$$\widetilde{U} = \frac{1}{2}\begin{pmatrix} u_z & (H-i)e^u \\ -2e^{-u}Q & -u_z \end{pmatrix}, \quad \widetilde{V} = \frac{1}{2}\begin{pmatrix} -u_{\bar{z}} & 2e^{-u}\overline{Q} \\ -(H+i)e^u & u_{\bar{z}} \end{pmatrix}.$$

Proof. Consider the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let F, $G \in SU(2)$ be the matrices that rotate $\Phi(e_1) = \mathbb{1}, \Phi(e_2) = i\sigma_1, \Phi(e_3) = -i\sigma_2$ and $\Phi(e_4) = i\sigma_3$ (see Proposition 2.33) to the (2×2)-matrix forms of f, $\frac{f_x}{|f_x|}$, $\frac{f_y}{|f_y|}$ and N via the group action (2.4.1) of $SU(2) \times SU(2)$ on $\mathbb{R}^4_{\mathrm{Mat}}$, i.e.

$$f = F\mathbb{1}G^{-1}, \quad \frac{f_x}{|f_x|} = F\,(i\sigma_1)\,G^{-1}, \quad \frac{f_y}{|f_y|} = F\,(-i\sigma_2)\,G^{-1}, \quad N = F\,(i\sigma_3)\,G^{-1}.$$

We now define

$$\begin{aligned} U &= \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} := F^{-1}F_z, & V &= \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} := F^{-1}F_{\bar z} \\ \widetilde{U} &= \begin{pmatrix} \widetilde{U}_{11} & \widetilde{U}_{12} \\ \widetilde{U}_{21} & \widetilde{U}_{22} \end{pmatrix} := G^{-1}G_z, & \widetilde{V} &= \begin{pmatrix} \widetilde{V}_{11} & \widetilde{V}_{12} \\ \widetilde{V}_{21} & \widetilde{V}_{22} \end{pmatrix} := G^{-1}G_{\bar z} \end{aligned}$$

and can compute $U, \widetilde{U}, V$ and $\widetilde{V}$ in terms of the conformal factor u, the mean curvature H and the Hopf differential Q. Making use of

$$\frac{f_x}{|f_x|} = \frac{f_x}{e^u} = F \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} G^{-1}, \quad \frac{f_y}{|f_y|} = \frac{f_y}{e^u} = F \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} G^{-1}$$

we get

$$f_z = ie^u F \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} G^{-1}, \quad f_{\bar z} = ie^u F \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} G^{-1}.$$

The entries of the matrices U, V and $\widetilde{U}, \widetilde{V}$ will be derived in the following.

Differentiating $f_{\bar z}$ with respect to z leads to

$$\begin{aligned} f_{\bar z z} &= u_z f_{\bar z} + ie^u \left(F_z \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} G^{-1} + F \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} (G^{-1})_z \right) \\ &= u_z f_{\bar z} + ie^u \left(FU \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} G^{-1} - F \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \widetilde{U} G^{-1} \right) \\ &= u_z f_{\bar z} + ie^u F \begin{pmatrix} U_{12} & 0 \\ U_{22} - \widetilde{U}_{11} & -\widetilde{U}_{12} \end{pmatrix} G^{-1}. \end{aligned}$$

We now differentiate f_z with respect to $\bar z$:

$$\begin{aligned} f_{z\bar z} &= u_{\bar z} f_z + ie^u \left(F_{\bar z} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} G^{-1} + F \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} (G^{-1})_{\bar z} \right) \\ &= u_{\bar z} f_z + ie^u \left(FV \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} G^{-1} - F \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \widetilde{V} G^{-1} \right) \\ &= u_{\bar z} f_z + ie^u F \begin{pmatrix} -\widetilde{V}_{21} & V_{11} - \widetilde{V}_{22} \\ 0 & V_{21} \end{pmatrix} G^{-1}. \end{aligned}$$

Since $f_{z\bar{z}} = f_{\bar{z}z}$ we therefore obtain

$$u_{\bar{z}} f_z + ie^u F \begin{pmatrix} -\widetilde{V}_{21} & V_{11} - \widetilde{V}_{22} \\ 0 & V_{21} \end{pmatrix} G^{-1} = u_z f_{\bar{z}} + ie^u F \begin{pmatrix} U_{12} & 0 \\ U_{22} - \widetilde{U}_{11} & -\widetilde{U}_{12} \end{pmatrix} G^{-1}$$

and thus

$$u_{\bar{z}} f_z - u_z f_{\bar{z}} = ie^u \left(F \begin{pmatrix} U_{12} & 0 \\ U_{22} - \widetilde{U}_{11} & -\widetilde{U}_{12} \end{pmatrix} G^{-1} - F \begin{pmatrix} -\widetilde{V}_{21} & V_{11} - \widetilde{V}_{22} \\ 0 & V_{21} \end{pmatrix} G^{-1} \right),$$

implying

$$u_{\bar{z}} f_z - u_z f_{\bar{z}} = ie^u F \begin{pmatrix} \widetilde{V}_{21} + U_{12} & -V_{11} + \widetilde{V}_{22} \\ U_{22} - \widetilde{U}_{11} & -V_{21} - \widetilde{U}_{12} \end{pmatrix} G^{-1}.$$

Writing out the left part of the above equation yields

$$\begin{aligned} u_{\bar{z}} f_z - u_z f_{\bar{z}} &= ie^u F \begin{pmatrix} 0 & u_{\bar{z}} \\ -u_z & 0 \end{pmatrix} G^{-1} \\ &= ie^u F \begin{pmatrix} \widetilde{V}_{21} + U_{12} & -V_{11} + \widetilde{V}_{22} \\ U_{22} - \widetilde{U}_{11} & -V_{21} - \widetilde{U}_{12} \end{pmatrix} G^{-1}. \end{aligned}$$

Hence we get

$$V_{11} - \widetilde{V}_{22} + u_{\bar{z}} = 0, \quad U_{22} - \widetilde{U}_{11} + u_z = 0, \quad V_{21} = -\widetilde{U}_{12}, \quad \widetilde{V}_{21} = -U_{12}. \tag{$*$}$$

Computing f_{zz} yields

$$\begin{aligned} f_{zz} &= u_z f_z + ie^u \left(F_z \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} G^{-1} + F \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} (G^{-1})_z \right) \\ &= u_z f_z + ie^u \left(FU \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} G^{-1} - F \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \widetilde{U} G^{-1} \right) \\ &= u_z f_z + ie^u F \begin{pmatrix} -\widetilde{U}_{21} & U_{11} - \widetilde{U}_{22} \\ 0 & U_{21} \end{pmatrix} G^{-1}. \end{aligned}$$

There holds $f_{zz} = 2u_z f_z + QN$ (see Proposition 2.37) and with the formula $N = F\,(i\sigma_3)\,G^{-1}$ we therefore obtain

$$2u_z f_z + QN = u_z f_z + ie^u F \begin{pmatrix} -\widetilde{U}_{21} & U_{11} - \widetilde{U}_{22} \\ 0 & U_{21} \end{pmatrix} G^{-1},$$

thus

$$\begin{aligned} ie^u F \begin{pmatrix} -\widetilde{U}_{21} & U_{11} - \widetilde{U}_{22} \\ 0 & U_{21} \end{pmatrix} G^{-1} &= u_z f_z + QN \\ &= ie^u F \begin{pmatrix} e^{-u} Q & u_z \\ 0 & -e^{-u} Q \end{pmatrix} G^{-1}. \end{aligned}$$

This gives

$$U_{21} = \widetilde{U}_{21} = -e^{-u}Q, \qquad U_{11} - \widetilde{U}_{22} - u_z = 0. \tag{$**$}$$

We consider $f_{\bar{z}z}$ again and obtain

$$\begin{aligned} f_{\bar{z}z} &= u_z f_{\bar{z}} + ie^u F \begin{pmatrix} U_{12} & 0 \\ U_{22} - \widetilde{U}_{11} & -\widetilde{U}_{12} \end{pmatrix} G^{-1} \\ &\overset{(*)}{=} ie^u F \begin{pmatrix} U_{12} & 0 \\ 0 & -\widetilde{U}_{12} \end{pmatrix} G^{-1}. \end{aligned}$$

With $N = F\,(i\sigma_3)\,G^{-1}$ and $f_{\bar{z}z} = -\frac{1}{2}e^{2u}f + \frac{1}{2}He^{2u}N$ (compare Proposition 2.37) we get

$$\begin{aligned} f_{\bar{z}z} &= -\tfrac{1}{2}e^{2u}f + \tfrac{1}{2}He^{2u}N \\ &= ie^u F \begin{pmatrix} \frac{1}{2}ie^u & 0 \\ 0 & \frac{1}{2}ie^u \end{pmatrix} G^{-1} + ie^u F \begin{pmatrix} \frac{1}{2}He^u & 0 \\ 0 & -\frac{1}{2}He^u \end{pmatrix} G^{-1} \end{aligned}$$

and thus $U_{12} = \frac{1}{2}(H+i)e^u$, $\widetilde{U}_{12} = \frac{1}{2}(H-i)e^u$. Considering $f_{\bar{z}\bar{z}}$ one obtains

$$\begin{aligned} f_{\bar{z}\bar{z}} &= 2u_{\bar{z}} f_{\bar{z}} + \overline{Q}N \\ &= u_{\bar{z}} f_{\bar{z}} + ie^u F \begin{pmatrix} V_{12} & 0 \\ V_{22} - \widetilde{V}_{11} & -\widetilde{V}_{12} \end{pmatrix} G^{-1} \end{aligned}$$

and therefore

$$V_{12} = \widetilde{V}_{12} = e^{-u}\overline{Q}, \qquad V_{22} - \widetilde{V}_{11} - u_{\bar{z}} = 0. \tag{$***$}$$

From the equations $(*)$, $(**)$ and $(***)$ we deduce that the Lax pairs in terms of 2×2-matrices are of the form

$$U = \frac{1}{2}\begin{pmatrix} u_z & (H+i)e^u \\ -2e^{-u}Q & -u_z \end{pmatrix}, \quad V = \frac{1}{2}\begin{pmatrix} -u_{\bar{z}} & 2e^{-u}\overline{Q} \\ -(H-i)e^u & u_{\bar{z}} \end{pmatrix},$$

$$\widetilde{U} = \frac{1}{2}\begin{pmatrix} u_z & (H-i)e^u \\ -2e^{-u}Q & -u_z \end{pmatrix}, \quad \widetilde{V} = \frac{1}{2}\begin{pmatrix} -u_{\bar{z}} & 2e^{-u}\overline{Q} \\ -(H+i)e^u & u_{\bar{z}} \end{pmatrix}.$$

□

2.5 The integrability condition and a $\mathbb{C}^*$-family of flat connections

We want to introduce a λ-dependent $\mathfrak{sl}_2(\mathbb{C})$-valued one-form α_λ with $\lambda \in \mathbb{C}^*$ following the exposition of Hitchin in [30] to obtain a $\mathbb{C}^*$-family of flat connections.

Assumption 2.39. *From now on we will assume that $H \equiv const$. Thus the corresponding surface will be of constant mean curvature (CMC).*

The equation $dF = F\alpha$ can be solved if and only if $\alpha = Udz + Vd\bar{z}$ satisfies a certain integrability condition (see [22], Section 3.1).

Theorem 2.40. *Let $O \subset \mathbb{C}$ be a convex open set containing 0. For $U, V : O \to \mathfrak{sl}_2(\mathbb{C})$ there exists a unique solution $F : O \to SL(2, \mathbb{C})$ of the Lax pair*

$$F_z = FU, \quad F_{\bar{z}} = FV \tag{$*$}$$

for an initial condition $F(0) \in SL(2, \mathbb{C})$ if and only if

$$U_{\bar{z}} - V_z - [U, V] = 0 \quad \textit{with } [U, V] = UV - VU.$$

Each pair of solutions $F, \widetilde{F}$ of $(*)$ differs only by multiplication with a constant matrix G, i.e. $\widetilde{F} = GF$.

Proof. Suppose there exists an invertible solution F. Since $F_{z\bar{z}} = F_{\bar{z}z}$ one obtains

$$0 = F_{z\bar{z}} - F_{\bar{z}z} = FU_{\bar{z}} - FV_z + F_{\bar{z}}U - F_zV$$

and therefore

$$0 = FU_{\bar{z}} - FV_z + FVU - FUV.$$

Thus $U_{\bar{z}} - V_z - [U, V] = 0$ must hold.

Now suppose that $U_{\bar{z}} - V_z - [U, V] = 0$ holds. Reworking this into the coordinates (x, y) we get

$$U_x + iU_y - V_x + iV_y - 2[U, V] = 0.$$

Then we can solve the ordinary differential equation

$$(F(x, 0))_x = F(x, 0)(U + V)(x, 0)$$

with initial condition $F(0, 0)$. For each fixed x_0 it remains to solve

$$(F(x_0, y))_y = F(x_0, y)i(U - V)(x_0, y)$$

with initial condition $F(x_0, 0)$. Hence $F(x, y)$ is defined and there holds $F_y = Fi(U - V)$ for all x, y. Since

$$(F_x - F(U + V))(x, y) = 0$$

if $y = 0$ and $F_{xy} = F_{yx}$, we have

$$\begin{aligned}
(F_x - F(U + V))_y &= F_{xy} - F_y(U + V) - F(U_y + V_y) \\
&= (Fi(U - V))_x - F_y(U + V) - F(U_y + V_y) \\
&= F_xi(U - V) + Fi(U_x - V_x) - F_y(U + V) - F(U_y + V_y) \\
&= F_xi(U - V) + Fi(2[U, V]) - F_y(U + V) \\
&= F_xi(U - V) + Fi(2[U, V]) - Fi(U - V)(U + V) \\
&= (F_x - F(U + V))i(U - V).
\end{aligned}$$

Set $G = F_x - F(U+V)$. G is a solution of $G_y = Gi(U-V)$ with initial condition $G(0) = 0$. By the uniqueness of the solution $G \equiv 0$ and therefore $F_x - F(U+V) \equiv 0$. Hence F is a solution to the Lax pair, since

$$F_z = \tfrac{1}{2}(F_x - iF_y) = FU, \quad F_{\bar{z}} = \tfrac{1}{2}(F_x + iF_y) = FV.$$

Considering

$$\det(F) \cdot \operatorname{tr}(F^{-1}F_z) = (\det(F))_z \text{ and } \det(F) \cdot \operatorname{tr}(F^{-1}F_{\bar{z}}) = (\det(F))_{\bar{z}}$$

with $U, V \in \mathfrak{sl}_2(\mathbb{C})$ we have

$$(\det(F))_z = (\det(F))_{\bar{z}} = 0$$

and it follows $\det(F) = 1$ since $\det(F(0)) = 1$. Now assume that there exists another solution $\widetilde{F}$ of $(*)$ and consider

$$\begin{aligned}(\widetilde{F}F^{-1})_z &= \widetilde{F}UF^{-1} - \widetilde{F}F^{-1}FUF^{-1} = 0,\\ (\widetilde{F}F^{-1})_{\bar{z}} &= \widetilde{F}VF^{-1} - \widetilde{F}F^{-1}FVF^{-1} = 0.\end{aligned}$$

Thus $G := \widetilde{F}F^{-1}$ is constant and therefore $\widetilde{F} = GF$. If we fix the initial condition by $F(0) = \widetilde{F}(0)$ the matrix G must be the identity $\mathbb{1}$ and $F = \widetilde{F}$. □

Corollary 2.41. *The matrices U, V from Lemma 2.38 obey the compatibility condition*

$$U_{\bar{z}} - V_z - [U, V] = 0,$$

if and only if

$$2u_{z\bar{z}} + 2e^{2u}(1 + H^2) - \frac{1}{2}Q\overline{Q}e^{-2u} = 0, \;\; Q_{\bar{z}} = 2H_z e^{2u}.$$

These are the ***Gauss*** *and* ***Codazzi equation*** *respectively.*

Remark 2.42. *Since $H \equiv const.$ due to Assumption 2.39, we get $Q_{\bar{z}} = 0$ and thus Q is holomorphic.*

A zero-curvature condition for the connection form α. We want to take another point of view and will treat the Gauss and Codazzi equations as a zero-curvature condition. For this purpose recall that for a map $F : \mathbb{R}^2 \simeq \mathbb{C} \to SL(2, \mathbb{C})$, the pullback $\alpha = F^*\theta$ of the Mauer-Cartan form θ also satisfies the Mauer-Cartan equation

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0$$

due to Proposition 2.31. Conversely, for every solution $\alpha = \alpha' dz + \alpha'' d\bar{z} \in \Omega^1(\mathbb{C}, \mathfrak{sl}_2(\mathbb{C}))$ of the Mauer-Cartan equation we have

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0 \iff \alpha'_{\bar{z}} - \alpha''_z - [\alpha', \alpha''] = 0,$$

and thus α integrates to a smooth map $F : \mathbb{C} \to SL(2, \mathbb{C})$ with $\alpha = F^*\theta$ due to Theorem 2.40. In particular one obtains the Gauss and Codazzi equations from the Maurer-Cartan equation for α on $\mathfrak{sl}_2(\mathbb{C})$.

If one thinks of α as a connection form, $d\alpha + \alpha \wedge \alpha = d\alpha + \frac{1}{2}[\alpha \wedge \alpha]$ is the corresponding curvature form. Thus the Maurer-Cartan equation is a *zero curvature condition* and the corresponding connection $\nabla = d - \alpha$ is *flat*.

A $\mathbb{C}^*$-family of flat connections. We want to introduce the so-called **spectral parameter** $\lambda \in \mathbb{C}^*$ that allows us to define a $\mathbb{C}^*$-family of flat $\mathfrak{sl}_2(\mathbb{C})$-connections $\nabla_\lambda := d - \omega_\lambda$. In order to achieve this we need some preparation.

For $\omega \in \Omega^1(\mathbb{C}, \mathfrak{sl}_2(\mathbb{C}))$ we perform a splitting into the $(1, 0)$-part ω' and the $(0, 1)$-part ω'', i.e.

$$\omega = \omega' + \omega'',$$

according to the decomposition of the tangent bundle $T\mathbb{C}$ with $d = \partial + \bar{\partial}$. Setting the $*$-operator on $\Omega^1(\mathbb{C}, \mathfrak{sl}_2(\mathbb{C}))$ to

$$* \, \omega = -i\omega' + i\omega''$$

one obtains the following

Lemma 2.43 ([49], Lemma 2)**.** *Let $f : \mathbb{C} \to SU(2) \simeq \mathbb{S}^3$ be a conformal immersion and $\omega = f^{-1}df$. The mean curvature H is given by*

$$2d * \omega = H[\omega \wedge \omega].$$

The trivialiuations of $TSU(2)$ that are induced by the left and right multiplication in $SU(2)$ lead to covariant derivatives ∇^L and ∇^R such that $(SU(2), \nabla^L)$ and $(SU(2), \nabla^R)$ are flat. Moreover, the Levi-Civita connection for $SU(2)$ is given by $\nabla = \frac{1}{2}(\nabla^R + \nabla^L)$. In [30] Hitchin investigates harmonic maps $f : M \to \mathbb{S}^3$ from the torus to the 3-sphere and uses the equations

$$d^\nabla(\omega) = 0, \quad d^\nabla(*\omega) = 0$$

to construct a $\mathbb{C}^*$-family of flat connections on M. Here $\omega = f^{-1}df$ and d^∇ is the exterior derivative with respect to a connection ∇. In particular there holds $d^{\nabla^L} = d$. We will now derive similar formulas with the help of Lemma 2.43 in order extend this ansatz to the present situation.

Lemma 2.44. *Let $f : \mathbb{C} \to SU(2) \simeq \mathbb{S}^3$ be a conformal immersion and set $\omega = f^{-1}df$. Then we have*

$$d^\nabla(\omega) = 0.$$

Beweis. Applying Cartan's formula for the exterior derivative with respect to ∇ we get

$$\begin{aligned} d^{\nabla}(\omega)(X_0,X_1) &= \nabla_{X_0}\omega(X_1) - \nabla_{X_1}\omega(X_0) - \omega([X_0,X_1]) \\ &= \nabla^L_{X_0}\omega(X_1) + \frac{1}{2}[\omega(X_0),\omega(X_1)] \\ &\quad -\nabla^L_{X_1}\omega(X_0) - \frac{1}{2}[\omega(X_1),\omega(X_0)] - \omega([X_0,X_1]) \\ &= d^{\nabla^L}(\omega)(X_0,X_1) + \frac{1}{2}[\omega\wedge\omega](X_0,X_1) \\ &= 0, \end{aligned}$$

since ω satisfies the Maurer-Cartan equation. □

Lemma 2.45. *Let $f : \mathbb{C} \to SU(2) \simeq \mathbb{S}^3$ be a conformal immersion and set $\omega = f^{-1}df$. Then we have*

$$d^{\nabla}(*\omega) = \frac{1}{2}H[\omega\wedge\omega].$$

In particular one obtains $d^{\nabla}(*\omega) = 0$ in case of a minimal surface.

Proof. Setting $\omega = f^{-1}df = G(\alpha-\beta)G^{-1}$ one immediately obtains

$$*\,\omega = N\omega = -\omega N \;\text{ with } N = G\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}G^{-1}.$$

Applying Cartan's formula for the exterior derivative with respect to ∇ we get

$$\begin{aligned} d^{\nabla}(*\omega)(X_0,X_1) &= \nabla_{X_0} *\omega(X_1) - \nabla_{X_1} *\omega(X_0) - *\omega([X_0,X_1]) \\ &= \nabla^L_{X_0} *\omega(X_1) + \frac{1}{2}[\omega(X_0),*\omega(X_1)] \\ &\quad -\nabla^L_{X_1} *\omega(X_0) - \frac{1}{2}[\omega(X_1),*\omega(X_0)] - *\omega([X_0,X_1]) \\ &= d^{\nabla^L}(*\omega)(X_0,X_1) + \frac{1}{2}[\omega(X_0),N\omega(X_1)] - \frac{1}{2}[\omega(X_0),N\omega(X_1)] \\ &= d^{\nabla^L}(*\omega)(X_0,X_1) \\ &= \frac{1}{2}H[\omega\wedge\omega](X_0,X_1) \end{aligned}$$

due to Lemma 2.43. For a minimal surface one has $H \equiv 0$ and therefore $d^{\nabla}(*\omega) = 0$. □

The following result is based on an observation by Uhlenbeck [50] and the calculation presented in [36], Section 1.1.

Lemma 2.46. *Let $f : \mathbb{C} \to SU(2)$ be a conformal immersion and $\omega = f^{-1}df$. For the $(1,0)$-part ω' of $\omega = \omega' + \omega''$ we have*

$$d''\omega' = -\frac{1}{2}(1-iH)[\omega'\wedge\omega'']$$

and for the $(0,1)$*-part* ω'' *of* ω *there holds*

$$d'\omega'' = -\frac{1}{2}(1+iH)[\omega' \wedge \omega''].$$

Proof. We know that ω satisfies the following equations

$$\begin{aligned} d\omega + \frac{1}{2}[\omega \wedge \omega] &= 0, \\ d * \omega &= \frac{1}{2}H[\omega \wedge \omega]. \end{aligned}$$

Combining these equations one obtains $d * \omega + H d\omega = 0$, or after expanding $*\omega$

$$(H-i)d\omega' + (H+i)d\omega'' = 0 \quad \text{and} \quad (H-i)d''\omega' + (H+i)d'\omega'' = 0.$$

Moreover, one has

$$\begin{aligned} 0 = d\omega + \frac{1}{2}[\omega \wedge \omega] = d'\omega'' + d''\omega' + [\omega' \wedge \omega''] \\ \Leftrightarrow d''\omega' = -d'\omega'' + [\omega'' \wedge \omega'] \end{aligned}$$

and therefore obtains

$$d''\omega' = -d'\omega'' + [\omega'' \wedge \omega'] = \frac{H-i}{H+i}d''\omega' + [\omega'' \wedge \omega'],$$

i.e.

$$d''\omega' = \frac{1}{2}(1-iH)[\omega'' \wedge \omega'] = -\frac{1}{2}(1-iH)[\omega' \wedge \omega''].$$

An analogous calculation shows the equation for the $(0,1)$-part ω''. □

Proposition 2.47. *Let* ω_λ *be defined by*

$$\omega_\lambda := \frac{1}{2}(1+\lambda^{-1})(1+iH)\omega' + \frac{1}{2}(1+\lambda)(1-iH)\omega'' \quad \textit{for} \quad \lambda \in \mathbb{C}^*.$$

Then there holds

$$d\omega_\lambda + \frac{1}{2}[\omega_\lambda \wedge \omega_\lambda] = 0 \ \ \forall\, \lambda \in \mathbb{C}^*,$$

i.e. for every $\lambda \in \mathbb{C}^*$ *the form* ω_λ *is the connection form of a flat connection.*

Proof. By applying the results of Lemma 2.46 a straightforward calculation shows

$$\begin{aligned} d\omega_\lambda + \frac{1}{2}[\omega_\lambda \wedge \omega_\lambda] &= d'\omega''_\lambda + d''\omega'_\lambda + [\omega'_\lambda \wedge \omega''_\lambda] \\ &= \frac{1}{2}(1+\lambda)(1-iH)d'\omega'' + \frac{1}{2}(1+\lambda^{-1})(1+iH)d''\omega' \\ &\quad + \frac{1}{4}(1+\lambda^{-1})(1+\lambda)(1+iH)(1-iH)[\omega' \wedge \omega''] \\ &= \frac{1}{4}\left(-2-\lambda^{-1}-\lambda+(1+\lambda^{-1})(1+\lambda)\right) \\ &\quad \cdot (1+iH)(1-iH)[\omega' \wedge \omega''] \\ &= 0 \end{aligned}$$

since $(1+\lambda^{-1})(1+\lambda) = 2+\lambda^{-1}+\lambda$. □

Proposition 2.48. *Let $G \in SU(2)$ be the solution of $dG = G\beta$. By performing a gauge transformation with G the $\mathfrak{sl}_2(\mathbb{C})$-valued form*

$$\omega_\lambda = \frac{1}{2}(1+\lambda^{-1})(1+iH)\omega' + \frac{1}{2}(1+\lambda)(1-iH)\omega''$$

is transformed into

$$\alpha_\lambda = \frac{1}{2}\begin{pmatrix} u_z dz - u_{\bar z} d\bar z & i\lambda^{-1}(1+iH)e^u dz + 2e^{-u}\overline{Q} d\bar z \\ -2e^{-u}Q dz + i\lambda(1-iH)e^u d\bar z & -u_z dz + u_{\bar z} d\bar z \end{pmatrix}.$$

Setting $\lambda_0 = -1$ and $\lambda_1 = \frac{1+iH}{1-iH}$ one obtains $\beta, \alpha \in \Omega^1(\mathbb{C}, \mathfrak{su}_2(\mathbb{C}))$ resulting from the Lax pairs $(\widetilde{U}, \widetilde{V})$ and (U, V) respectively, i.e. $\alpha_{\lambda_0} = \beta$ and $\alpha_{\lambda_1} = \alpha$.

Proof. By considering

$$\omega = f^{-1} df = G(\alpha - \beta)G^{-1} = \frac{1}{2} G \begin{pmatrix} 0 & 2ie^u\, dz \\ 2ie^u\, d\bar z & 0 \end{pmatrix} G^{-1},$$

we obtain for $\omega_\lambda = \frac{1}{2}(1+\lambda^{-1})(1+iH)\omega' + \frac{1}{2}(1+\lambda)(1-iH)\omega''$

$$\omega_\lambda = \frac{1}{2} G \begin{pmatrix} 0 & i(1+\lambda^{-1})(1+iH)e^u\, dz \\ i(1+\lambda)(1-iH)e^u\, d\bar z & 0 \end{pmatrix} G^{-1}.$$

Gauging ω_λ with G leads to

$$\begin{aligned}
\alpha_\lambda &= G^{-1}\omega_\lambda G + G^{-1} dG \\
&= \frac{1}{2}\begin{pmatrix} 0 & i(1+\lambda^{-1})(1+iH)e^u\, dz \\ i(1+\lambda)(1-iH)e^u\, d\bar z & 0 \end{pmatrix} \\
&\quad + \frac{1}{2}\begin{pmatrix} u_z dz - u_{\bar z} d\bar z & -i(1+iH)e^u dz + 2e^{-u}\overline{Q} d\bar z \\ -2e^{-u}Q dz - i(1-iH)e^u d\bar z & -u_z dz + u_{\bar z} d\bar z \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} u_z dz - u_{\bar z} d\bar z & i\lambda^{-1}(1+iH)e^u dz + 2e^{-u}\overline{Q} d\bar z \\ -2e^{-u}Q dz + i\lambda(1-iH)e^u d\bar z & -u_z dz + u_{\bar z} d\bar z \end{pmatrix}
\end{aligned}$$

and the claim is proved. □

Remark 2.49. *Solving $dF_\lambda = F_\lambda \alpha_\lambda$ yields $G = F_{\lambda_0}$ and $F = F_{\lambda_1}$ respectively and therefore*

$$f = FG^{-1} = F_{\lambda_1} F_{\lambda_0}^{-1}.$$

A transformation of the form α_λ. Finally, we want to relate the above α_λ to the representation that has been introduced in [35, 36]. Rescaling of the conformal factor e^u and the Hopf differential Q and after performing a Möbius transformation (with respect to the spectral parameter λ) we obtain

$$v := e^u\sqrt{H^2+1}, \quad \widetilde{Q} := 2iQ\sqrt{H^2+1}$$

and

$$\lambda \mapsto \widetilde{\lambda} := \lambda \frac{1 - iH}{\sqrt{H^2+1}}, \quad \text{i.e. } \widetilde{\lambda}^{-1} = \lambda^{-1} \frac{1 + iH}{\sqrt{H^2+1}},$$

and therefore

$$\alpha_\lambda \mapsto \widetilde{\alpha}_\lambda = \frac{1}{2} \begin{pmatrix} v^{-1} v_z dz - v^{-1} v_{\bar{z}} d\bar{z} & i\widetilde{\lambda}^{-1} v dz + i\overline{\widetilde{Q}} v^{-1} d\bar{z} \\ i\widetilde{Q} v^{-1} dz + i\widetilde{\lambda} v d\bar{z} & -v^{-1} v_z dz + v^{-1} v_{\bar{z}} d\bar{z} \end{pmatrix}.$$

The points $\lambda_0, \lambda_1 \in \mathbb{S}^1$ are transformed into $\widetilde{\lambda}_0 = \frac{-1+iH}{\sqrt{H^2+1}}$ and $\widetilde{\lambda}_1 = \frac{1+iH}{\sqrt{H^2+1}}$ respectively. The mean curvature H is now given in terms of $\widetilde{\lambda}_0$ and $\widetilde{\lambda}_1$, i.e. one has

$$H = i \frac{\widetilde{\lambda}_0 + \widetilde{\lambda}_1}{\widetilde{\lambda}_0 - \widetilde{\lambda}_1}.$$

In the following we return to the notation (e^u, Q, λ) for the transformed quantities $(v, \widetilde{Q}, \widetilde{\lambda})$ and consider the "inverse" situation. In this general case $\lambda_0 \neq \lambda_1$ will not be symmetric with respect to the imaginary axis as in the preceeding construction. We obtain the following version of a result by Bobenko [9] (compare with [36], Theorem 1.1).

Theorem 2.50. *Let $u : \mathbb{C} \to \mathbb{R}$ and $Q : \mathbb{C} \to \mathbb{C}$ be smooth functions and define*

$$\alpha_\lambda = \frac{1}{2} \begin{pmatrix} u_z dz - u_{\bar{z}} d\bar{z} & i\lambda^{-1} e^u dz + i\overline{Q} e^{-u} d\bar{z} \\ iQ e^{-u} dz + i\lambda e^u d\bar{z} & -u_z dz + u_{\bar{z}} d\bar{z} \end{pmatrix}.$$

Then $2d\alpha_\lambda + [\alpha_\lambda \wedge \alpha_\lambda] = 0$ if and only if Q is holomorphic, i.e. $Q_{\bar{z}} = 0$, and u is a solution of the reduced Gauss equation

$$2u_{z\bar{z}} + \frac{1}{2}(e^{2u} - Q\overline{Q} e^{-2u}) = 0.$$

For any solution u of the above equation and corresponding extended frame F_λ, and $\lambda_0, \lambda_1 \in \mathbb{S}^1, \lambda_0 \neq \lambda_1$, i.e. $\lambda_k = e^{it_k}$ the map defined by the Sym-Bobenko-formula

$$f = F_{\lambda_1} F_{\lambda_0}^{-1}$$

is a conformal immersion $f : \mathbb{C} \to SU(2) \simeq \mathbb{S}^3$ with constant mean curvature

$$H = i \frac{\lambda_0 + \lambda_1}{\lambda_0 - \lambda_1} = \cot(t_0 - t_1),$$

conformal factor $v = e^u / \sqrt{H^2+1}$, and Hopf differential $\widetilde{Q} dz^2$ with $\widetilde{Q} = -\frac{i}{4}(\lambda_1^{-1} - \lambda_0^{-1}) Q$.

Proof. We adapt the proof of [35], Theorem 1.1. Decomposing α_λ into the $(1,0)$- and $(0,1)$-parts $\alpha_\lambda = \alpha'_\lambda dz + \alpha''_\lambda d\bar{z}$ we get

$$\begin{aligned}
\bar{\partial} \alpha'_\lambda &= \frac{1}{2} \begin{pmatrix} u_{z\bar{z}} & i\lambda^{-1} u_{\bar{z}} e^u \\ -iu_{\bar{z}} e^{-u} Q + ie^{-u} Q_{\bar{z}} & -u_{z\bar{z}} \end{pmatrix}, \\
\partial \alpha''_\lambda &= \frac{1}{2} \begin{pmatrix} -u_{z\bar{z}} & -iu_z e^{-u} \overline{Q} + ie^{-u} \overline{Q}_z \\ i\lambda u_z e^u & u_{z\bar{z}} \end{pmatrix}, \\
[\alpha'_\lambda, \alpha''_\lambda] &= \frac{1}{4} \begin{pmatrix} -e^{2u} + Q\overline{Q} e^{-2u} & 2iu_{\bar{z}} \lambda^{-1} e^u + 2iu_z e^{-u} \overline{Q} \\ -2i\lambda u_z e^u - 2iu_{\bar{z}} e^{-u} Q & e^{2u} - Q\overline{Q} e^{-2u} \end{pmatrix}.
\end{aligned}$$

Since $2d\alpha_\lambda + [\alpha_\lambda \wedge \alpha_\lambda] = 0$ is equivalent to $\bar\partial\alpha'_\lambda - \partial\alpha''_\lambda = [\alpha'_\lambda, \alpha''_\lambda]$ we see that u must fulfill the reduced Gauss equation and $Q_{\bar z} = 0$.

Now let u be a solution of the above equation and consider for $\lambda_0, \lambda_1 \in \mathbb{S}^1$, $\lambda_0 \neq \lambda_1$ the map $f = F_{\lambda_1} F_{\lambda_0}^{-1}$ defined by the Sym-Bobenko-formula. Setting $\omega = f^{-1}df = F_{\lambda_0}(\alpha_{\lambda_1} - \alpha_{\lambda_0})F_{\lambda_0}^{-1}$ one has

$$\begin{aligned}
\omega' = f^{-1}\partial f &= F_{\lambda_0} F_{\lambda_1}^{-1}\left((\partial F_{\lambda_1})F_{\lambda_0}^{-1} + F_{\lambda_1}(\partial F_{\lambda_0}^{-1})\right)\\
&= F_{\lambda_0} F_{\lambda_1}^{-1}\left(F_{\lambda_1}\alpha'_{\lambda_1}F_{\lambda_0}^{-1} - F_{\lambda_1}F_{\lambda_0}^{-1}(\partial F_{\lambda_0})F_{\lambda_0}^{-1}\right)\\
&= F_{\lambda_0}\left(\alpha'_{\lambda_1} - \alpha'_{\lambda_0}\right)F_{\lambda_0}^{-1}
\end{aligned}$$

and therefore

$$f^{-1}\partial f = \frac{1}{2} i e^u (\lambda_1^{-1} - \lambda_0^{-1}) F_{\lambda_0}\epsilon_+ F_{\lambda_0}^{-1}.$$

A similar calculation reveals $f^{-1}\bar\partial f = \frac{1}{2} i e^u(\lambda_1 - \lambda_0)F_{\lambda_0}\epsilon_- F_{\lambda_0}^{-1}$ (recall that $\epsilon_- = \epsilon_+^t = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0\end{smallmatrix}\right)$) and it is clear that $\langle f^{-1}\partial f, f^{-1}\partial f\rangle = \langle f^{-1}\bar\partial f, f^{-1}\bar\partial f\rangle = 0$. For the conformal factor one has to calculate

$$v^2 = 2\langle f^{-1}\partial f, f^{-1}\bar\partial f\rangle = \frac{1}{4} e^{2u}(\lambda_1^{-1} - \lambda_0^{-1})(\lambda_1 - \lambda_0).$$

For $\omega = f^{-1}df = F_{\lambda_0}(\alpha_{\lambda_1} - \alpha_{\lambda_0})F_{\lambda_0}^{-1}$ one has the splitting

$$\begin{aligned}
\omega &= \frac{1}{2} i F_{\lambda_0}\left((\lambda_1^{-1} - \lambda_0^{-1})e^u\epsilon_+ dz + (\lambda_1 - \lambda_0)e^u\epsilon_- d\bar z\right)F_{\lambda_0}^{-1}\\
&= \frac{1}{2} i F_{\lambda_0}\left((\lambda_1^{-1} - \lambda_0^{-1})\zeta' + (\lambda_1 - \lambda_0)\zeta''\right)F_{\lambda_0}^{-1},
\end{aligned}$$

where we set $\zeta' := e^u\epsilon_+ dz$ and $\zeta'' := e^u\epsilon_- d\bar z$. Then another calculation shows

$$d * \omega = \frac{1}{4} i(\lambda_1\lambda_0^{-1} - \lambda_0\lambda_1^{-1})F_{\lambda_0}[\zeta' \wedge \zeta'']F_{\lambda_0}^{-1}$$

and

$$[\omega \wedge \omega] = \frac{1}{2}(1 - \lambda_1\lambda_0^{-1})(1 - \lambda_0\lambda_1^{-1})F_{\lambda_0}[\zeta' \wedge \zeta'']F_{\lambda_0}^{-1},$$

and therefore $H = i\frac{\lambda_0 + \lambda_1}{\lambda_0 - \lambda_1}$ is the mean curvature for f. From this formula we obtain

$$(H^2 + 1)(\lambda_1^{-1} - \lambda_0^{-1})(\lambda_1 - \lambda_0) = 4$$

and thus $v^2 = e^{2u}/(H^2 + 1)$. Finally we want to determine the Hopf differential and consider the normal $N = F_{\lambda_1}\epsilon F_{\lambda_0}^{-1}$ with $\epsilon = \left(\begin{smallmatrix} i & 0 \\ 0 & -i\end{smallmatrix}\right)$. Similar to the above calculations one obtains $\partial N = F_{\lambda_1}(\alpha'_{\lambda_1}\epsilon - \epsilon\alpha'_{\lambda_0})F_{\lambda_0}^{-1}$ with

$$\alpha'_{\lambda_1}\epsilon - \epsilon\alpha'_{\lambda_0} = \begin{pmatrix} 0 & \frac{1}{2}e^u(\lambda_1^{-1} + \lambda_0^{-1}) \\ -Qe^{-u} & 0 \end{pmatrix}.$$

Thus one has

$$\begin{aligned}\widetilde{Q} &= \langle \partial\partial f, N\rangle = -\langle \partial f, \partial N\rangle = -\langle F_{\lambda_1}^{-1}\partial f F_{\lambda_0}, F_{\lambda_1}^{-1}\partial N F_{\lambda_0}\rangle \\ &\overset{2.34}{=} -\frac{1}{2}\mathrm{tr}\left[\begin{pmatrix} 0 & \frac{i}{2}(\lambda_1^{-1}-\lambda_0^{-1})e^u \\ 0 & 0\end{pmatrix}\sigma_2\begin{pmatrix} 0 & \frac{1}{2}e^u(\lambda_1^{-1}+\lambda_0^{-1}) \\ -Qe^{-u} & 0\end{pmatrix}^t\sigma_2\right] \\ &= -\frac{1}{2}\mathrm{tr}\left[\begin{pmatrix} 0 & \frac{i}{2}(\lambda_1^{-1}-\lambda_0^{-1})e^u \\ 0 & 0\end{pmatrix}\begin{pmatrix} 0 & -\frac{1}{2}e^u(\lambda_1^{-1}+\lambda_0^{-1}) \\ Qe^{-u} & 0\end{pmatrix}\right] \\ &= -\frac{i}{4}(\lambda_1^{-1}-\lambda_0^{-1})Q\end{aligned}$$

and the claim is proved. □

Remark 2.51. *Since λ_0, λ_1 are not symmetric with respect to the imaginary axis in general the formula $\widetilde{Q} = -\frac{i}{2\sqrt{H^2+1}}Q$ is not valid and one obtains the more general formula $\widetilde{Q} = -\frac{i}{4}(\lambda_1^{-1}-\lambda_0^{-1})Q$ from Theorem 2.50.*

2.6 Transformation rules for α_λ

We want to investigate how the connection form α_λ behaves under certain parameter transformations.

Lemma 2.52. *Holomorphic transformations of the parameter z of the form $z \mapsto w = w(z)$ leave the Gauss and Codazzi equations invariant away from the zeros of $\frac{dw}{dz}$.*

Proof. The Gauss and Codazzi equations for $\mathbb{S}^3$ are given by

$$2u_{z\bar{z}} + 2e^{2u}(1+H^2) - \frac{1}{2}Q\overline{Q}e^{-2u} = 0, \quad Q_{\bar{z}} = 2H_z e^{2u}.$$

Since $H \equiv \mathrm{const}$ we only have to consider the first equation and investigate the transformation of the corresponding terms resulting from this mapping. First we observe that from the equation

$$e^{2\widetilde{u}(w,\bar{w})}dw d\bar{w} = e^{2u(z,\bar{z})}dz d\bar{z}$$

we get

$$e^{2u(z,\bar{z})} = e^{2\widetilde{u}(w,\bar{w})}\left(\frac{dw}{dz}\right)\overline{\left(\frac{dw}{dz}\right)}$$

and therefore

$$u(z,\bar{z}) = \widetilde{u}(w,\bar{w}) + \ln\left(\left|\frac{dw}{dz}\right|\right).$$

Differentiation yields

$$\begin{aligned}2u(z,\bar{z})_{z\bar{z}} &= \left|\frac{dw}{dz}\right|^2 2\widetilde{u}(w,\bar{w})_{w\bar{w}} + \ln\left(\frac{dw}{dz}\right)_{z\bar{z}} + \ln\left(\frac{d\bar{w}}{d\bar{z}}\right)_{z\bar{z}} \\ &= \left|\frac{dw}{dz}\right|^2 2\widetilde{u}(w,\bar{w})_{w\bar{w}} + \left(\frac{1}{\left(\frac{dw}{dz}\right)}\cdot\frac{d^2w}{dz^2}\right)_{\bar{z}} + \left(\frac{1}{\left(\frac{d\bar{w}}{d\bar{z}}\right)}\cdot\frac{d^2\bar{w}}{d\bar{z}dz}\right)_{\bar{z}}.\end{aligned}$$

Since w is holomorphic and $\bar{w}$ anti-holomorphic we get

$$\begin{aligned} 2u(z,\bar{z})_{z\bar{z}} &= \left|\frac{dw}{dz}\right|^2 2\widetilde{u}(w,\bar{w})_{w\bar{w}} - \frac{1}{\left(\frac{dw}{dz}\right)^2}\cdot\frac{d^2w}{dzd\bar{z}}\cdot\frac{d^2w}{dzdz} + \frac{1}{\left(\frac{dw}{dz}\right)}\cdot\frac{d^3w}{dz^2d\bar{z}} + 0 \\ &= \left|\frac{dw}{dz}\right|^2 2\widetilde{u}(w,\bar{w})_{w\bar{w}}. \end{aligned}$$

Now consider the quadratic Hopf differential and its transformation rule for a given mapping, namely

$$Qdz^2 = \widetilde{Q}dw^2 \Longleftrightarrow Q = \widetilde{Q}\left(\frac{dw}{dz}\right)^2.$$

For a mapping of the above form the Gauss and Codazzi equations are transformed into

$$\left|\frac{dw}{dz}\right|^2 2\widetilde{u}_{w\bar{w}} + \left|\frac{dw}{dz}\right|^2 2e^{2\widetilde{u}}(1+H^2) - \frac{1}{2}\left(\frac{dw}{dz}\right)^2\overline{\left(\frac{dw}{dz}\right)^2}\widetilde{Q}\overline{\widetilde{Q}}\frac{1}{\left(\frac{dw}{dz}\right)\overline{\left(\frac{dw}{dz}\right)}}e^{-2\widetilde{u}}$$

and therefore one obtains

$$\left|\frac{dw}{dz}\right|^2\left(2\widetilde{u}_{w\bar{w}} + 2e^{2\widetilde{u}}(1+H^2) - \frac{1}{2}\widetilde{Q}\overline{\widetilde{Q}}e^{-2\widetilde{u}}\right) = 0.$$

From the above considerations we see that the Gauss and Codazzi equations are left invariant away from the zeros of $\frac{dw}{dz}$ and behave singular at these points. $\square$

Remark 2.53. *Since*

$$u(z,\bar{z}) = \widetilde{u}(w,\bar{w}) + \ln\left(\left|\frac{dw}{dz}\right|\right)$$

we see that the conformal factor u has a singularity at the zeros of $\frac{dw}{dz}$.

We obtain the following version of a result by Bobenko (see [10], Section 2.3).

Theorem 2.54. *The frame F_λ and the $\mathfrak{sl}_2(\mathbb{C})$-valued 1-form α_λ transform as follows under a holomorphic mapping of the form $z \mapsto w(z)$:*

$$F_\lambda \mapsto F_\lambda\cdot B_w \quad \textit{and} \quad \alpha_\lambda \mapsto w^*\alpha_\lambda = B_w^{-1}\alpha_\lambda B_w + B_w^{-1}dB_w$$

with

$$B_w = \begin{pmatrix} \frac{\sqrt[4]{\frac{d\bar{w}}{d\bar{z}}}}{\sqrt[4]{\frac{dw}{dz}}} & 0 \\ 0 & \frac{\sqrt[4]{\frac{dw}{dz}}}{\sqrt[4]{\frac{d\bar{w}}{d\bar{z}}}} \end{pmatrix}$$

Proof. Considering

$$\alpha_\lambda = \frac{1}{2}\begin{pmatrix} u_z dz - u_{\bar z} d\bar z & i\lambda^{-1}e^u dz + i\overline{Q}e^{-u}d\bar z \\ iQe^{-u}dz + i\lambda e^u d\bar z & -u_z dz + u_{\bar z} d\bar z \end{pmatrix}$$

one has to investigate the transformations of the quantities appearing in this matrix. According to Lemma 2.52 one gets

$$\begin{aligned} Q = \left(\tfrac{dw}{dz}\right)^2 \widetilde{Q}, &\quad \overline{Q} = \left(\tfrac{d\bar w}{d\bar z}\right)^2 \overline{\widetilde{Q}}, \\ e^u = e^{\widetilde{u}}\sqrt{\frac{dw}{dz}}\sqrt{\frac{d\bar w}{d\bar z}}, &\quad e^{-u} = e^{-\widetilde{u}}\frac{1}{\sqrt{\frac{dw}{dz}}\sqrt{\frac{d\bar w}{d\bar z}}}, \\ u_z = \widetilde{u}_w \frac{dw}{dz} + \frac{1}{2}\frac{1}{\frac{dw}{dz}}\cdot\frac{d^2 w}{dz^2}, &\quad u_{\bar z} = \widetilde{u}_{\bar w}\frac{d\bar w}{d\bar z} + \frac{1}{2}\frac{1}{\frac{d\bar w}{d\bar z}}\cdot\frac{d^2\bar w}{d\bar z^2}. \end{aligned}$$

By first gauging with $C = \begin{pmatrix} e^{-u/2} & 0 \\ 0 & e^{-u/2}\end{pmatrix}$ one has

$$\begin{aligned} C^{-1}dC &= -\frac{1}{2}\begin{pmatrix} e^{u/2} & 0 \\ 0 & e^{u/2}\end{pmatrix}\begin{pmatrix} e^{-u/2}(u_z dz + u_{\bar z}d\bar z) & 0 \\ 0 & e^{-u/2}(u_z dz + u_{\bar z} d\bar z)\end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} -u_z dz - u_{\bar z}d\bar z & 0 \\ 0 & -u_z dz - u_{\bar z}d\bar z\end{pmatrix} \end{aligned}$$

and therefore

$$\widetilde{\alpha}_\lambda := C^{-1}\alpha_\lambda C + C^{-1}dC = \frac{1}{2}\begin{pmatrix} -2u_{\bar z}d\bar z & i\lambda^{-1}e^u dz + i\overline{Q}e^{-u}d\bar z \\ iQe^{-u}dz + i\lambda e^u d\bar z & -2u_z dz\end{pmatrix}.$$

By performing a gauge with $D = \begin{pmatrix} \sqrt{\frac{d\bar w}{d\bar z}} & 0 \\ 0 & \sqrt{\frac{dw}{dz}}\end{pmatrix}$ one obtains

$$\begin{aligned} D^{-1}\widetilde{\alpha}_\lambda D &= \frac{1}{2}\begin{pmatrix} -2u_{\bar z}d\bar z & i\lambda^{-1}e^u\sqrt{\frac{dw}{dz}}\cdot\frac{1}{\sqrt{\frac{d\bar w}{d\bar z}}}dz + i\overline{Q}e^{-u}\sqrt{\frac{dw}{dz}}\cdot\frac{1}{\sqrt{\frac{d\bar w}{d\bar z}}}d\bar z \\ iQe^{-u}\sqrt{\frac{d\bar w}{d\bar z}}\cdot\frac{1}{\sqrt{\frac{dw}{dz}}}dz + i\lambda e^u\sqrt{\frac{d\bar w}{d\bar z}}\cdot\frac{1}{\sqrt{\frac{dw}{dz}}}d\bar z & -2u_z dz\end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} -2u_{\bar z}d\bar z & i\lambda^{-1}e^{\widetilde{u}}dw + i\overline{\widetilde{Q}}e^{-\widetilde{u}}d\bar w \\ i\widetilde{Q}e^{-\widetilde{u}}dw + i\lambda e^{\widetilde{u}}d\bar w & -2u_z dz\end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} D^{-1}dD &= \frac{1}{2}\begin{pmatrix} \frac{1}{\sqrt{\frac{d\bar w}{d\bar z}}} & 0 \\ 0 & \frac{1}{\sqrt{\frac{dw}{dz}}}\end{pmatrix}\cdot\begin{pmatrix} \frac{1}{\sqrt{\frac{d\bar w}{d\bar z}}}\frac{d^2\bar w}{d\bar z^2}d\bar z & 0 \\ 0 & \frac{1}{\sqrt{\frac{dw}{dz}}}\frac{d^2 w}{dz^2}dz\end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \frac{1}{\frac{d\bar w}{d\bar z}}\frac{d^2\bar w}{d\bar z^2}d\bar z & 0 \\ 0 & \frac{1}{\frac{dw}{dz}}\frac{d^2 w}{dz^2}dz\end{pmatrix}. \end{aligned}$$

This yields

$$
\begin{aligned}
\widetilde{\widetilde{\alpha}}_\lambda &= D^{-1}\widetilde{\alpha}_\lambda D + D^{-1}dD \\
&= \frac{1}{2}\begin{pmatrix} -2u_{\bar z}d\bar z + \frac{1}{\frac{d\bar w}{d\bar z}}\frac{d^2\bar w}{d\bar z^2}d\bar z & i\lambda^{-1}e^{\widetilde{u}}dw + i\overline{\widetilde{Q}}e^{-\widetilde{u}}d\bar w \\ i\widetilde{Q}e^{-\widetilde{u}}dw + i\lambda e^{\widetilde{u}}d\bar w & -2u_z dz + \frac{1}{\frac{dw}{dz}}\frac{d^2 w}{dz^2}dz \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} -2\widetilde{u}_{\bar w}d\bar w & i\lambda^{-1}e^{\widetilde{u}}dw + i\overline{\widetilde{Q}}e^{-\widetilde{u}}d\bar w \\ i\widetilde{Q}e^{-\widetilde{u}}dw + i\lambda e^{\widetilde{u}}d\bar w & -2\widetilde{u}_w dw \end{pmatrix}.
\end{aligned}
$$

A similar calculation for the gauge with $E = \begin{pmatrix} e^{\widetilde{u}/2} & 0 \\ 0 & e^{\widetilde{u}/2}\end{pmatrix}$ transforms $\widetilde{\widetilde{\alpha}}_\lambda$ into

$$
\widehat{\alpha}_\lambda = \frac{1}{2}\begin{pmatrix} \widetilde{u}_w dw - \widetilde{u}_{\bar w}d\bar w & i\lambda^{-1}e^{\widetilde{u}}dw + i\overline{\widetilde{Q}}e^{-\widetilde{u}}d\bar w \\ i\widetilde{Q}e^{-\widetilde{u}}dw + i\lambda e^{\widetilde{u}}d\bar w & \widetilde{u}_{\bar w}d\bar w - \widetilde{u}_w dw \end{pmatrix},
$$

thus

$$
B_w := CDE = \begin{pmatrix} e^{(\widetilde{u}-u)/2}\sqrt{\frac{d\bar w}{d\bar z}} & 0 \\ 0 & e^{(\widetilde{u}-u)/2}\sqrt{\frac{dw}{dz}} \end{pmatrix}
$$

is the corresponding gauge for a mapping $z \mapsto w = w(z)$. Since

$$
\tfrac{1}{2}(\widetilde{u} - u) = -\tfrac{1}{4}\log\left(\tfrac{dw}{dz}\cdot\tfrac{d\bar w}{d\bar z}\right),
$$

the gauge B_w is of the desired form. This concludes the proof. $\square$

Remark 2.55.

1. *From Theorem 2.54 we get the formula* $e^{-u/2}F_\lambda D = e^{-\widetilde{u}/2}\widetilde{F}_\lambda$ *for the transformed frame* $\widetilde{F}_\lambda$ *and therefore*

$$
e^{-u}\left(e^{u/2}F_\lambda\right)D = e^{-\widetilde{u}}\left(e^{\widetilde{u}/2}\widetilde{F}_\lambda\right) \iff \left(e^{u/2}F_\lambda\right)\widehat{D} = \left(e^{\widetilde{u}/2}\widetilde{F}_\lambda\right)
$$

with

$$
\widehat{D} = e^{\widetilde{u}-u}D = \begin{pmatrix} \left(\sqrt{\frac{dw}{dz}}\right)^{-1} & 0 \\ 0 & \left(\sqrt{\frac{d\bar w}{d\bar z}}\right)^{-1} \end{pmatrix}.
$$

In particular the modified frame $F_\lambda^u := e^{u/2}F_\lambda$ *satisfies* $\widetilde{F}_\lambda^{\widetilde{u}} = F_\lambda^u\widehat{D}$, *i.e.* $F_\lambda^u\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ *defines the inverse* S^{-1} *of a spin-bundle* S *(compare with [10], Section 2.3).*

2. *Setting* $\nabla := d+\alpha_\lambda$ *the inverse frame* F_λ^{-1} *may be regarded as a* ∇*-horizontal section of the trivial* $\mathbb{C}^2$*-bundle* $V := M \times \mathbb{C}^2 \to M$ *for a compact Riemann surface* M *of genus* g, *i.e.* $\nabla F_\lambda^{-1} = 0$. *We consider* α_λ *as the gauged connection form for the*

associated family f_λ *starting from an immersion* $f : M \to SU(2)$*. Due to Hitchin [30] the* $(1,0)$*-part* $\omega' = f^{-1}\partial f = \frac{1}{2}ie^u(\lambda_1^{-1} - \lambda_0^{-1})F_{\lambda_0}\epsilon_+ F_{\lambda_0}^{-1}$ *of* $\omega = f^{-1}df$ *gives rise to a line bundle*

$$L := ker(\omega') \quad with \quad ker(\omega') = span\{F_{\lambda_0}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\} = span\{F^u_{\lambda_0}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\}$$

of degree $1-g$*. It turns out that* $L \otimes L \simeq K^{-1}$*, where* K *is the canonical bundle, i.e.* $S := L^{-1}$ *is a spin-bundle since* $L^{-1} = \sqrt{K}$*. Moreover, one obtains* $V = L \oplus \bar{L}$*. From the proof of Theorem 2.54 we see that the transformation of* F^u_λ *exactly reflects this decomposition.*

3 Spectral data for periodic solutions of the sinh-Gordon equation

We will now derive spectral data (Y, D) for periodic finite type solutions of the sinh-Gordon equation.

3.1 The monodromy and its expansion

The central object for the forthcoming considerations is contained in

Definition 3.1. *Let F_λ be an extended frame assume that $\alpha_\lambda = F_\lambda^{-1} dF_\lambda$ has period τ, i.e. $\alpha_\lambda(z+\tau) = \alpha_\lambda(z)$. Then the* ***monodromy*** *of the frame F_λ with respect to the period τ is given by*

$$M_\lambda^\tau := F_\lambda(z+\tau) F_\lambda^{-1}(z).$$

Note that we have

$$\begin{aligned} dM_\lambda^\tau &= F_\lambda(z+\tau)\alpha_\lambda(z+\tau)F_\lambda^{-1}(z) - F_\lambda(z+\tau)\alpha_\lambda(z)F_\lambda^{-1}(z) \\ &= 0, \end{aligned}$$

since $\alpha_\lambda(z+\tau) = \alpha_\lambda(z)$ and thus M_λ^τ does not depend on z. Setting the period to $\mathbf{p} \in \mathbb{C}$ and $F_\lambda(0) = \mathbb{1}$ we get

$$M_\lambda := M_\lambda^{\mathbf{p}} = F_\lambda(\mathbf{p}) F_\lambda^{-1}(0) = F_\lambda(\mathbf{p}).$$

Assumption 3.2. *Let us assume that the Hopf differential Q is constant with $|Q| = 1$.*

We can rotate the coordinate z by the map $z \mapsto w(z) = e^{i\varphi} z$ in such a way that we can assume a real period $\mathbf{p} \in \mathbb{R}$ due to Remark 2.53. Since

$$u(z) = \widetilde{u}(e^{i\varphi} z) + \ln(|\tfrac{d}{dz}(e^{i\varphi} z)|) = \widetilde{u}(e^{i\varphi} z)$$

we get

$$\widetilde{u}(e^{i\varphi} z) = u(z) = u(z + \widetilde{\mathbf{p}}) = \widetilde{u}(e^{i\varphi}(z + \widetilde{\mathbf{p}})) = \widetilde{u}(e^{i\varphi} z + \mathbf{p})$$

for a suitable $\varphi \in [0, 2\pi)$ and $\mathbf{p} = e^{i\varphi}\widetilde{\mathbf{p}}$. For the corresponding B_w (see Theorem 2.54) there holds

$$B_w = \begin{pmatrix} \delta^{-1/2} & 0 \\ 0 & \delta^{1/2} \end{pmatrix}$$

with $\delta = e^{i\varphi} \in \mathbb{S}^1$. This corresponds to the isometric normalization described in [27], Remark 1.5, and the corresponding gauged α_λ is of the form

$$\alpha_\lambda = \frac{1}{2}\begin{pmatrix} u_z dz - u_{\bar z} d\bar z & i\lambda^{-1}\delta e^u dz + i\overline{\gamma} e^{-u} d\bar z \\ i\gamma e^{-u} dz + i\lambda\bar\delta e^u d\bar z & -u_z dz + u_{\bar z} d\bar z \end{pmatrix},$$

where the constant $\gamma \in \mathbb{S}^1$ is given by $\gamma = \delta^{-1} Q = \bar\delta Q$.

The sinh-Gordon equation. We can normalize the above parametrization with $\delta = 1$ and $|\gamma| = 1$ by choosing the appropriate value for $Q \in \mathbb{S}^1$. Then we can consider the system

$$dF_\lambda = F_\lambda \alpha_\lambda \quad \text{with} \quad F_\lambda(0) = \mathbb{1}$$

for

$$F(z,\lambda) : \mathbb{C} \times \mathbb{C}^* \to SL(2,\mathbb{C})$$

and

$$\alpha_\lambda = \frac{1}{2}\begin{pmatrix} u_z dz - u_{\bar z} d\bar z & i\lambda^{-1} e^u dz + i\overline{\gamma} e^{-u} d\bar z \\ i\gamma e^{-u} dz + i\lambda e^u d\bar z & -u_z dz + u_{\bar z} d\bar z \end{pmatrix}.$$

Since $|\gamma| = 1$, wee see that the compatibility condition $2d\alpha_\lambda + [\alpha_\lambda \wedge \alpha_\lambda] = 0$ from Theorem 2.50 holds if and only if

$$2u_{z\bar z} + \frac{1}{2}(e^{2u} - \gamma\overline{\gamma} e^{-2u}) = 2u_{z\bar z} + \sinh(2u) = 0.$$

Thus the reduced Gauss equation turns into the **sinh-Gordon equation** in that situation. The monodromy of F is then $M_\lambda = F(\mathbf{p},\lambda)$ for a period $\mathbf{p}$ of the solution u of the sinh-Gordon equation. For the following we make the additional

Assumption 3.3. *Let $\gamma = \delta = 1$. This yields*

$$\alpha_\lambda = \frac{1}{2}\begin{pmatrix} u_z dz - u_{\bar z} d\bar z & i\lambda^{-1} e^u dz + i e^{-u} d\bar z \\ i e^{-u} dz + i\lambda e^u d\bar z & -u_z dz + u_{\bar z} d\bar z \end{pmatrix}.$$

If we evaluate α_λ along the vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ we obtain

$$U_\lambda := \alpha_\lambda(\tfrac{\partial}{\partial x}), \quad V_\lambda := \alpha_\lambda(\tfrac{\partial}{\partial y}).$$

These matrices will be important for the upcoming considerations. In particular U_λ reads

$$U_\lambda = \frac{1}{2}\begin{pmatrix} -iu_y & i\lambda^{-1} e^u + i e^{-u} \\ i\lambda e^u + i e^{-u} & iu_y \end{pmatrix}.$$

Remark 3.4. *Due to the one-to-one correspondence $(u, u_y) \mapsto U_\lambda$ with*

$$U_\lambda = \frac{1}{2}\begin{pmatrix} -iu_y & i\lambda^{-1} e^u + i e^{-u} \\ i\lambda e^u + i e^{-u} & iu_y \end{pmatrix}$$

we can identify the tuple (u, u_y) with the matrix U_λ.

We will now investigate the **Lax operator** $L_\lambda(x) := \frac{d}{dx} + U_\lambda(x,0)$ and take a closer look at solutions $F_\lambda : \mathbb{R} \to SL(2,\mathbb{C})$ of $\frac{d}{dx}F_\lambda(x) = F_\lambda(x)U_\lambda(x,0)$. Given such a solution there holds

$$\begin{aligned} L_\lambda(x)F_\lambda^{-1}(x) &= \tfrac{d}{dx}F_\lambda^{-1}(x) + U_\lambda(x,0)F_\lambda^{-1}(x) \\ &= -F_\lambda^{-1}(x)(\tfrac{d}{dx}F_\lambda(x))F_\lambda^{-1}(x) + U_\lambda(x,0)F_\lambda^{-1}(x) \\ &= -F_\lambda^{-1}(x)F_\lambda(x)U_\lambda(x,0)F_\lambda^{-1}(x) + U_\lambda(x,0)F_\lambda^{-1}(x) \\ &= 0, \end{aligned}$$

i.e. one obtains a solution for the Lax operator $L_\lambda(x)$. The next lemma shows how α can be integrated to obtain a solution F.

Lemma 3.5. *Let $\alpha : [0,\mathbf{p}] \to M_{2\times 2}(\mathbb{C})$ be smooth. Then the map*

$$x \mapsto \mathbb{1} + \sum_{n=1}^{\infty} \int_0^x \int_0^{t_n} \cdots \int_0^{t_2} \alpha(t_1)\alpha(t_2)\cdots\alpha(t_n)dt_1\cdots dt_n$$

converges to the solution of $\frac{d}{dx}F = F\alpha$ with $F(0) = \mathbb{1}$. The map F is the so-called fundamental solution.

Proof. The series converges absolutely, since for each summand of the above sum one has

$$\begin{aligned} &\left\| \int_0^x \int_0^{t_n} \cdots \int_0^{t_2} \alpha(t_1)\alpha(t_2)\cdots\alpha(t_n)dt_1\cdots dt_n \right\| \\ \leq\ & \int_0^x \int_0^{t_n} \cdots \int_0^{t_2} \|\alpha(t_1)\|\|\alpha(t_2)\|\cdots\|\alpha(t_n)\|dt_1\cdots dt_n \\ \leq\ & \frac{1}{n!}\int_0^x \int_0^{x} \cdots \int_0^{x} \|\alpha(t_1)\|\|\alpha(t_2)\|\cdots\|\alpha(t_n)\|dt_1\cdots dt_n \\ \leq\ & \frac{1}{n!}\left(\int_0^x \|\alpha(t)\|dt\right)^n. \end{aligned}$$

Therefore $\exp(\int_0^x \|\alpha(t)\|dt)$ is a majorant for this series and the claim is proved. ☐

Remark 3.6. *Since $M_\lambda = F_\lambda(\mathbf{p})$ we see that the map $\lambda \mapsto M_\lambda$ is holomorphic for $\lambda \in \mathbb{C}^*$ and has essential singularities at $\lambda = 0$ and $\lambda = \infty$.*

An asymptotic expansion of the monodromy M_λ. We seek an asymptotic expansion of the monodromy M_λ corresponding to the frame F_λ. The following lemma will be useful for the asymptotic analysis at $\lambda = 0$.

Lemma 3.7. *By performing a gauge transformation with*

$$g_\lambda(z) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{\frac{u}{2}} & 0 \\ 0 & \sqrt{\lambda}e^{-\frac{u}{2}} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

the frame $F_\lambda(z)$ is transformed into $F_\lambda(z)g_\lambda(z)$ and the map $G_\lambda(z) := g_\lambda(0)^{-1}F_\lambda(z)g_\lambda(z)$ solves

$$dG_\lambda = G_\lambda\beta_\lambda \;\; \textit{with } \beta_\lambda = g_\lambda^{-1}\alpha_\lambda g_\lambda + g_\lambda^{-1}dg_\lambda \textit{ and } G_\lambda(0) = \mathbb{1}.$$

Evaluating the form β_λ along the vector field $\frac{\partial}{\partial x}$ and setting $y = 0$ yields $\beta_\lambda(\frac{\partial}{\partial x}) = \frac{1}{\sqrt{\lambda}}\beta_{-1} + \beta_0 + \sqrt{\lambda}\beta_1$ with

$$\beta_{-1} = \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix}, \;\; \beta_0 = \begin{pmatrix} 0 & -u_z \\ -u_z & 0 \end{pmatrix}, \;\; \beta_1 = \begin{pmatrix} \frac{i}{2}\cosh(2u) & -\frac{i}{2}\sinh(2u) \\ \frac{i}{2}\sinh(2u) & -\frac{i}{2}\cosh(2u) \end{pmatrix}.$$

Proof. Considering

$$g_\lambda(z) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{\frac{u}{2}} & 0 \\ 0 & \sqrt{\lambda}e^{-\frac{u}{2}} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

one obtains for $g_\lambda^{-1}\alpha_\lambda g_\lambda$ the matrix

$$\frac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u_z dz - u_{\bar z}d\bar z & i\frac{1}{\sqrt{\lambda}}dz + ie^{-2u}\sqrt{\lambda}d\bar z \\ i\frac{1}{\sqrt{\lambda}}dz + ie^{2u}\sqrt{\lambda}d\bar z & -u_z dz + u_{\bar z}d\bar z \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Moreover,

$$g_\lambda^{-1}dg_\lambda = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u_z dz + u_{\bar z}d\bar z & 0 \\ 0 & -u_z dz - u_{\bar z}d\bar z \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

and thus

$$\begin{aligned} \beta_\lambda &= g_\lambda^{-1}\alpha_\lambda g_\lambda + g_\lambda^{-1}dg_\lambda \\ &= \frac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2u_z dz & i\frac{1}{\sqrt{\lambda}}dz + ie^{-2u}\sqrt{\lambda}d\bar z \\ i\frac{1}{\sqrt{\lambda}}dz + ie^{2u}\sqrt{\lambda}d\bar z & -2u_z dz \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} i\frac{1}{\sqrt{\lambda}}dz + i\cosh(2u)\sqrt{\lambda}d\bar z & -2u_z dz - i\sinh(2u)\sqrt{\lambda}d\bar z \\ -2u_z dz + i\sinh(2u)\sqrt{\lambda}d\bar z & -i\frac{1}{\sqrt{\lambda}}dz - i\cosh(2u)\sqrt{\lambda}d\bar z \end{pmatrix} \\ &= \tfrac{1}{\sqrt{\lambda}}\beta_{-1}dz + \beta_0 dz + \sqrt{\lambda}\beta_1 d\bar z \end{aligned}$$

with

$$\beta_{-1} = \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix}, \;\; \beta_0 = \begin{pmatrix} 0 & -u_z \\ -u_z & 0 \end{pmatrix}, \;\; \beta_1 = \begin{pmatrix} \frac{i}{2}\cosh(2u) & -\frac{i}{2}\sinh(2u) \\ \frac{i}{2}\sinh(2u) & -\frac{i}{2}\cosh(2u) \end{pmatrix}.$$

Evaluating the form β_λ along the vector field $\frac{\partial}{\partial x}$ and setting $y = 0$ yields $\beta_\lambda(\frac{\partial}{\partial x}) = \frac{1}{\sqrt{\lambda}}\beta_{-1} + \beta_0 + \sqrt{\lambda}\beta_1$ with β_{-1}, β_0 and β_1 as given above. □

Since F_λ solves $\frac{d}{dx}F_\lambda = F_\lambda U_\lambda(\cdot, 0)$ we assume that y is set to zero in the following considerations. Let us consider the simplest case with $u \equiv 0$.

Lemma 3.8. *Setting $u \equiv 0$ in the Maurer-Cartan form α_λ yields a frame $F_{\lambda,0}(x)$ with monodromy*

$$M_{\lambda,0} = g_\lambda(0) \begin{pmatrix} \exp(\frac{i\mathbf{p}}{2}(\frac{1}{\sqrt{\lambda}} + \sqrt{\lambda})) & 0 \\ 0 & \exp(\frac{i\mathbf{p}}{2}(-\sqrt{\lambda} - \frac{1}{\sqrt{\lambda}})) \end{pmatrix} g_\lambda(0)^{-1}.$$

Moreover, $M_{\lambda,0} = \pm\mathbb{1}$ holds if and only if $\lambda \in \mathcal{D}_0$ with

$$\mathcal{D}_0 = \{\lambda \in \mathbb{C}^* \mid M_{\lambda,0} = \pm\mathbb{1}\} = \{\lambda = \tfrac{2\pi^2k^2 - \mathbf{p}^2}{\mathbf{p}^2} \pm \tfrac{2\pi k\sqrt{\pi^2k^2 - \mathbf{p}^2}}{\mathbf{p}^2} \mid k \in \mathbb{N}_0\}.$$

Proof. Obviously the solution of

$$\tfrac{d}{dx} G_{\lambda,0} = G_{\lambda,0}\beta_\lambda \quad \text{with } u \equiv 0 \text{ and } G_{\lambda,0}(0) = \mathbb{1}$$

is given by

$$G_{\lambda,0}(x) = \begin{pmatrix} \exp(x\frac{i}{2}(\frac{1}{\sqrt{\lambda}} + \sqrt{\lambda})) & 0 \\ 0 & \exp(x\frac{i}{2}(-\sqrt{\lambda} - \frac{1}{\sqrt{\lambda}})) \end{pmatrix}.$$

Thus one obtains for $x = \mathbf{p}$ (recall that $G_{\lambda,0}(x) = g_\lambda(0)^{-1}F_{\lambda,0}(x)g_\lambda(x)$)

$$\begin{aligned} M_{\lambda,0} &= g_\lambda(0)G_\lambda(\mathbf{p})g_\lambda(\mathbf{p})^{-1} = g_\lambda(0)G_\lambda(\mathbf{p})g_\lambda(0)^{-1} \\ &= g_\lambda(0) \begin{pmatrix} \exp(\frac{i\mathbf{p}}{2}(\frac{1}{\sqrt{\lambda}} + \sqrt{\lambda})) & 0 \\ 0 & \exp(\frac{i\mathbf{p}}{2}(-\sqrt{\lambda} - \frac{1}{\sqrt{\lambda}})) \end{pmatrix} g_\lambda(0)^{-1} \end{aligned}$$

and finally a direct calculation shows $M_{\lambda,0} = \pm\mathbb{1} \iff \sqrt{\lambda} + \frac{1}{\sqrt{\lambda}} \in \frac{2\pi}{\mathbf{p}}\mathbb{Z} \iff \lambda \in \mathcal{D}_0$. $\square$

Remark 3.9. *Denoting by $(\lambda_1(k))_{k\in\mathbb{N}_0}$ and $(\lambda_2(k))_{k\in\mathbb{N}_0}$ the sequences given by*

$$\lambda_1(k) = \tfrac{2\pi^2k^2 - \mathbf{p}^2 + 2\pi k\sqrt{\pi^2k^2 - \mathbf{p}^2}}{\mathbf{p}^2}, \qquad \lambda_2(k) = \tfrac{2\pi^2k^2 - \mathbf{p}^2 - 2\pi k\sqrt{\pi^2k^2 - \mathbf{p}^2}}{\mathbf{p}^2}$$

we have the following limits for $k \to \infty$

$$\lim_{k\to\infty} \lambda_1(k) = \infty, \qquad \lim_{k\to\infty} \lambda_2(k) = 0,$$

i.e. $\mathcal{D}_0 \subset \mathbb{R}$ has the accumulation points 0 and ∞.

Let us relate the monodromy M_λ of the frame $F_\lambda(x)$ to that of the "vacuum" monodromy $M_{\lambda,0}$. Since β_{-1} does not depend on x we get from the theorem about variation of parameters that the unique solution of

$$\frac{d}{dx} G_\lambda(x) = G_\lambda(x) \left(\frac{1}{\sqrt{\lambda}}\beta_{-1} + \beta_0(x) + \sqrt{\lambda}\beta_1(x) \right)$$

with $G_\lambda(x) = g_\lambda(0)^{-1}F_\lambda(x)g_\lambda(x)$ and $G_\lambda(0) = \mathbb{1}$ is given by

$$G_\lambda(x) = \sum_{n=0}^{\infty} \widehat{G}_n(x) \text{ with } \widehat{G}_0(x) = \exp(\tfrac{x}{\sqrt{\lambda}}\beta_{-1}) \text{ and}$$
$$\widehat{G}_{n+1}(x) = \int_0^x \widehat{G}_n(t)(\beta_0(t) + \sqrt{\lambda}\beta_1(t))\exp(\tfrac{x-t}{\sqrt{\lambda}}\beta_{-1})\,dt.$$

Following this ansatz a careful asymptotic analysis [37] shows that for every $\varepsilon > 0$ one can choose an appropriate neighborhood around $\lambda = 0$ such that the inequality

$$\|g_\lambda(0)^{-1}M_\lambda g_\lambda - g_\lambda(0)^{-1}M_{\lambda,0}g_\lambda(0)\| \leq \varepsilon\|g_\lambda(0)^{-1}M_{\lambda,0}g_\lambda(0)\|$$

holds for $|\lambda|$ small enough and a similar inequality also holds around $\lambda = \infty$. Moreover, we can deduce from that inequality that the so-called double points $\mathcal{D}$ for a general u lie very close to the points $\mathcal{D}_0$ from Remark 3.9 around $\lambda = 0$ and $\lambda = \infty$.

A formal diagonalization of the monodromy M_λ. We want to diagonalize the monodromy M_λ and therefore need to diagonalize α_λ. A diagonalization for the Schrödinger-operator is done in [47] based on a result from [26]. In order to adapt the techniques applied there we search for a λ-dependent periodic formal power series $\widehat{g}_\lambda(x)$ such that

$$\widehat{\beta}_\lambda = \widehat{g}_\lambda^{-1}\alpha_\lambda\widehat{g}_\lambda + \widehat{g}_\lambda^{-1}\tfrac{d}{dx}\widehat{g}_\lambda$$

is a diagonal matrix, i.e.

$$\widehat{\beta}_\lambda(x) = \begin{pmatrix} \sum_m(\sqrt{\lambda})^m b_m(x) & 0 \\ 0 & -\sum_m(\sqrt{\lambda})^m b_m(x) \end{pmatrix}$$

with $m \geq -1$. Since $F_\lambda(x) = \widehat{g}_\lambda(0)\widehat{G}_\lambda(x)\widehat{g}_\lambda(x)^{-1}$ (where $\widehat{G}_\lambda$ solves $\frac{d}{dx}\widehat{G}_\lambda(x) = \widehat{G}_\lambda(x)\widehat{\beta}_\lambda(x)$ with $\widehat{G}_\lambda(0) = \mathbb{1}$) we get

$$M_\lambda = F_\lambda(\mathbf{p}) = \widehat{g}_\lambda(0)\widehat{G}_\lambda(\mathbf{p})\widehat{g}_\lambda(\mathbf{p})^{-1} = \widehat{g}_\lambda(0)\widehat{G}_\lambda(\mathbf{p})\widehat{g}_\lambda(0)^{-1}$$

and due to

$$\widehat{G}_\lambda(x) = \begin{pmatrix} \exp\left(\int_0^x \sum_m(\sqrt{\lambda})^m b_m(t)\,dt\right) & 0 \\ 0 & \exp\left(-\int_0^x \sum_m(\sqrt{\lambda})^m b_m(t)\,dt\right) \end{pmatrix}$$

one has

$$\widehat{G}_\lambda(\mathbf{p}) = \begin{pmatrix} \exp\left(\int_0^{\mathbf{p}} \sum_m(\sqrt{\lambda})^m b_m(t)\,dt\right) & 0 \\ 0 & \exp\left(-\int_0^{\mathbf{p}} \sum_m(\sqrt{\lambda})^m b_m(t)\,dt\right) \end{pmatrix}.$$

The conjugation with the matrix $\widehat{g}_\lambda(0)$ leaves the eigenvalues $\mu, \frac{1}{\mu}$ of M_λ invariant and thus we obtain

$$\mu = \exp\left(\int_0^{\mathbf{p}} \sum_m (\sqrt{\lambda})^m b_m(t)\, dt\right)$$

or equivalently

$$\ln \mu = \sum_m (\sqrt{\lambda})^m \int_0^{\mathbf{p}} b_m(t)\, dt.$$

From the following theorem we obtain a periodic formal power series $\widetilde{g}_\lambda(x) = \mathbb{1} + \sum_{m\geq 1} a_m(x)(\sqrt{\lambda})^m$ such that $\widehat{g}_\lambda(x) := g_\lambda(x)\widetilde{g}_\lambda(x)$ (with $g_\lambda(x)$ defined in Lemma 3.7) yields the desired result around $\lambda = 0$.

Theorem 3.10. *Let $(u, u_y) \in C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R})$ be periodic with period $\mathbf{p}$. Then there exist two series*

$$\begin{aligned} a_1(x), a_2(x), \ldots \quad &\in \mathit{span}\{\epsilon_+, \epsilon_-\} \text{ of periodic off-diagonal matrices and} \\ b_1(x), b_2(x), \ldots \quad &\in \mathit{span}\{\epsilon\} \text{ of periodic diagonal matrices, respectively} \end{aligned}$$

such that $a_{m+1}(x)$ and $b_m(x)$ are differential polynomials in u and u_y with derivatives of order m at most and the following equality for formal power series holds asympotically around $\lambda = 0$:

$$\beta_\lambda(x)\left(\mathbb{1} + \sum_{m\geq 1} a_m(x)(\sqrt{\lambda})^m\right) + \sum_{m\geq 1} \frac{d}{dx} a_m(x)(\sqrt{\lambda})^m = \left(\mathbb{1} + \sum_{m\geq 1} a_m(x)(\sqrt{\lambda})^m\right) \sum_{m\geq -1} b_m(x)(\sqrt{\lambda})^m. \qquad (*)$$

Here $b_{-1}(x)$ and $b_0(x)$ are given by $b_{-1}(x) \equiv \beta_{-1} = \frac{i}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $b_0(x) \equiv \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

Remark 3.11. *Since we are interested in so-called finite type solutions we can guarantee that the power series in $(*)$ indeed are convergent, see Theorem 4.35.*

Proof. We start the iteration with $b_0(x) \equiv \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and will inductively solve the given ansatz in all powers of $\sqrt{\lambda}$:

1. $(\sqrt{\lambda})^{-1}$: $\beta_{-1} = \beta_{-1}$. ✓

2. $(\sqrt{\lambda})^0$: $\beta_{-1}a_1(x) + \beta_0(x) = b_0(x) = 0$ and thus

$$a_1(x) = -\beta_{-1}^{-1}\beta_0(x) = \begin{pmatrix} 0 & -i\partial u \\ i\partial u & 0 \end{pmatrix}.$$

3. $(\sqrt{\lambda})^1$: $\beta_{-1}a_2(x) + \beta_0(x)a_1(x) + \beta_1(x) + \frac{d}{dx}a_1(x) = b_1(x) + a_2(x)\beta_{-1} + a_1(x)b_0(x)$. Rearranging terms and sorting with respect to diagonal (d) and off-diagonal (off) matrices we get two equations:

$$\begin{aligned} b_1(x) &= \beta_0(x)a_1(x) + \beta_{1,\mathrm{d}}(x) \\ &= \begin{pmatrix} -i(\partial u)^2 + \frac{i}{2}\cosh(2u) & 0 \\ 0 & i(\partial u)^2 - \frac{i}{2}\cosh(2u) \end{pmatrix}, \\ [\beta_{-1}, a_2(x)] &= -\beta_{1,\mathrm{off}}(x) - \tfrac{d}{dx}a_1(x). \end{aligned}$$

In order to solve the second equation for $a_2(x)$ we make the following observation: Since $[\epsilon, \epsilon_+] = 2i\epsilon_+$ and $[\epsilon, \epsilon_-] = -2i\epsilon_-$ we get for $a(x) = a_+(x)\epsilon_+ + a_-(x)\epsilon_-$

$$\begin{aligned} \phi(a(x)) &:= [\beta_{-1}, a(x)] = [\tfrac{1}{2}\epsilon, a(x)] \\ &= ia_+(x)\epsilon_+ - ia_-(x)\epsilon_- \in \mathrm{span}\{\epsilon_+, \epsilon_-\}. \end{aligned}$$

This defines a linear map $\phi : \mathrm{span}\{\epsilon_+, \epsilon_-\} \to \mathrm{span}\{\epsilon_+, \epsilon_-\}$. Obviously $\ker(\phi) = \{0\}$ and thus ϕ is an isomorphism. Therefore we can uniquely solve the equation $[\beta_{-1}, a_2(x)] = -\beta_{1,\mathrm{off}}(x) - \frac{d}{dx}a_1(x)$ and obtain $a_2(x)$.

We now proceed inductively for $m \geq 2$ and assume that we already found $a_m(x)$ and $b_{m-1}(x)$. Consider the equation

$$\begin{aligned} &\beta_{-1}a_{m+1}(x) + \beta_0(x)a_m(x) + \beta_1(x)a_{m-1}(x) + \tfrac{d}{dx}a_m(x) = \\ &\qquad\qquad b_m(x) + a_{m+1}(x)\beta_{-1} + \sum_{i=1}^{m} a_i(x)b_{m-i}(x) \end{aligned}$$

for the power $(\sqrt{\lambda})^m$. Rearranging terms and after decomposition in the diagonal (d) and off-diagonal (off) part we get

$$\begin{aligned} b_m(x) &= \beta_0(x)a_m(x) + \beta_{1,\mathrm{off}}(x)a_{m-1}(x), \\ [\beta_{-1}, a_{m+1}(x)] &= -\beta_{1,\mathrm{d}}(x)a_{m-1}(x) - \tfrac{d}{dx}a_m(x) + \sum_{i=1}^{m} a_i(x)b_{m-i}(x). \end{aligned}$$

From the discussion above we see that these equations can uniquely be solved and one obtains $a_{m+1}(x)$ and $b_m(x)$. By induction one therefore obtains a unique formal solution of $(*)$ with the desired properties. □

With the help of Theorem 3.10 we can reproduce Proposition 3.6 presented in [36].

Corollary 3.12. *The logarithm* $\ln\mu$ *of the eigenvalue* μ *of the monodromy* M_λ *has the following asymptotic expansions*

$$\begin{aligned} \ln\mu &= \tfrac{1}{\sqrt{\lambda}}\tfrac{i\mathbf{p}}{2} + \sqrt{\lambda}\int_0^{\mathbf{p}} \left(-i(\partial u)^2 + \tfrac{i}{2}\cosh(2u)\right) dt + O(\lambda) \quad \text{at } \lambda = 0, \\ \ln\mu &= \sqrt{\lambda}\tfrac{i\mathbf{p}}{2} + \tfrac{1}{\sqrt{\lambda}}\int_0^{\mathbf{p}} \left(-i(\bar\partial u)^2 + \tfrac{i}{2}\cosh(2u)\right) dt + O(\lambda^{-1}) \quad \text{at } \lambda = \infty. \end{aligned}$$

Proof. From Theorem 3.10 we know that at $\lambda = 0$ we have

$$\begin{aligned}\ln\mu &= \tfrac{1}{\sqrt{\lambda}}\tfrac{i\mathbf{p}}{2} + \sqrt{\lambda}\int_0^{\mathbf{p}} b_1(t)\,dt + \sum_{m\geq 2}(\sqrt{\lambda})^m\int_0^{\mathbf{p}} b_m(t)\,dt \\ &= \tfrac{1}{\sqrt{\lambda}}\tfrac{i\mathbf{p}}{2} + \sqrt{\lambda}\int_0^{\mathbf{p}}\left(-i(\partial u)^2 + \tfrac{i}{2}\cosh(2u)\right)\,dt + O(\lambda).\end{aligned}$$

The equation $M_\lambda = \left(\overline{M}^t_{\bar{\lambda}^{-1}}\right)^{-1}$ implies $\mu(\lambda) = \bar{\mu}^{-1}(\bar{\lambda}^{-1})$. Thus the expansion of $\ln\mu(\lambda)$ at $\lambda = \infty$ is equal to the expansion of $-\overline{\ln\mu(\bar{\lambda}^{-1})}$ at $\lambda = 0$ and one obtains

$$\ln\mu = \sqrt{\lambda}\tfrac{i\mathbf{p}}{2} + \tfrac{1}{\sqrt{\lambda}}\int_0^{\mathbf{p}}\left(-i(\bar{\partial}u)^2 + \tfrac{i}{2}\cosh(2u)\right)\,dt + O(\lambda^{-1}) \text{ at } \lambda = \infty.$$

$\square$

3.2 Polynomial Killing fields for finite type solutions

In the following we will consider the variable y as a flow paramater. Expanding the matrices U_λ and V_λ with respect to this flow parameter y we get for U_λ

$$\begin{aligned}U_\lambda(x,y) &= U_\lambda(x,0) + y\delta U_\lambda(x) + O(y^2),\\ \tfrac{d}{dx}U_\lambda(x,y) &= \tfrac{d}{dx}U_\lambda(x,0) + y\tfrac{d}{dx}\delta U_\lambda(x) + O(y^2),\\ \tfrac{d}{dy}U_\lambda(x,y) &= \delta U_\lambda(x) + O(y)\end{aligned}$$

and for V_λ the equations

$$\begin{aligned}V_\lambda(x,y) &= V_\lambda(x,0) + y\delta V_\lambda(x) + O(y^2),\\ \tfrac{d}{dx}V_\lambda(x,y) &= \tfrac{d}{dx}V_\lambda(x,0) + y\tfrac{d}{dx}\delta V_\lambda(x) + O(y^2).\end{aligned}$$

Plugging these equations into the zero-curvature condition

$$\tfrac{d}{dy}U_\lambda - \tfrac{d}{dx}V_\lambda - [U_\lambda, V_\lambda] = 0$$

we obtain the following equation with respect to the constant term $y = 0$

$$\delta U_\lambda(x) - \tfrac{d}{dx}V_\lambda(x,0) - [U_\lambda(x,0), V_\lambda(x,0)] = 0$$

and therefore with the Lax operator $L_\lambda(x) := \frac{d}{dx} + U_\lambda(x,0)$

$$\delta L_\lambda(x) = \delta U_\lambda(x) = \tfrac{d}{dx}V_\lambda(x,0) + [U_\lambda(x,0), V_\lambda(x,0)] = [L_\lambda(x), V_\lambda(x,0)].$$

If we replace $V_\lambda(x,0)$ by a map $W_\lambda(x)$ solving $\frac{d}{dx}W_\lambda(x) = [W_\lambda(x), U_\lambda(x,0)]$ we get for the constant term

$$\delta L_\lambda = \delta U_\lambda(x,0) = [L_\lambda, W_\lambda(x)] = \tfrac{d}{dx}W_\lambda(x) + [U_\lambda(x,0), W_\lambda(x)] \equiv 0,$$

i.e. the corresponding Lax equation $\delta L_\lambda = [L_\lambda, W_\lambda]$ is stationary. Let us adapt Definition 2.1 in [27] to obtain the following

Definition 3.13. *A pair* $(u, u_y) \simeq U_\lambda(\cdot, 0)$ *corresponding to a periodic solution of the* sinh-*Gordon equation is of* ***finite type*** *if there exists* $g \in \mathbb{N}_0$ *such that*

$$\Phi_\lambda(x) = \frac{\lambda^{-1}}{2}\begin{pmatrix} 0 & ie^u \\ 0 & 0 \end{pmatrix} + \sum_{n=0}^{g} \lambda^n \begin{pmatrix} \omega_n & e^u \tau_n \\ e^u \sigma_n & -\omega_n \end{pmatrix}$$

is a solution of the Lax equation

$$\frac{d}{dx}\Phi_\lambda = [\Phi_\lambda, U_\lambda(\cdot, 0)]$$

for some periodic functions $\omega_n, \tau_n, \sigma_n : \mathbb{R}/\mathbf{p} \to \mathbb{C}$.

Given a map $\widetilde{\Phi}_\lambda$ of the form

$$\widetilde{\Phi}_\lambda(z) = \frac{\lambda^{-1}}{2}\begin{pmatrix} 0 & ie^u \\ 0 & 0 \end{pmatrix} + \sum_{n=0}^{g} \lambda^n \begin{pmatrix} \widetilde{\omega}_n & e^u \widetilde{\tau}_n \\ e^u \widetilde{\sigma}_n & -\widetilde{\omega}_n \end{pmatrix}$$

with expansion

$$\begin{aligned} \widetilde{\Phi}_\lambda(x,y) &= \widetilde{\Phi}_\lambda(x,0) + y\delta\widetilde{\Phi}_\lambda(x) + O(y^2) \\ \tfrac{d}{dx}\widetilde{\Phi}_\lambda(x,y) &= \tfrac{d}{dx}\widetilde{\Phi}_\lambda(x,0) + y\tfrac{d}{dx}\delta\widetilde{\Phi}_\lambda(x) + O(y^2) \\ \tfrac{d}{dy}\widetilde{\Phi}_\lambda(x,y) &= \delta\widetilde{\Phi}_\lambda(x) + O(y) \end{aligned}$$

that is a solution of the Lax equation

$$d\widetilde{\Phi}_\lambda = [\widetilde{\Phi}_\lambda, \alpha_\lambda] \Longleftrightarrow \begin{cases} \tfrac{d}{dx}\widetilde{\Phi}_\lambda(x,y) = [\widetilde{\Phi}_\lambda(x,y), U_\lambda(x,y)] \\ \tfrac{d}{dy}\widetilde{\Phi}_\lambda(x,y) = [\widetilde{\Phi}_\lambda(x,y), V_\lambda(x,y)] \end{cases}$$

we obtain a map Φ_λ as in Definition 3.13 by setting

$$\Phi_\lambda(x) := \widetilde{\Phi}_\lambda(x, 0).$$

In order to obtain a map $\widetilde{\Phi}_\lambda$ we recall the purely geometric approach discovered by Pinkall and Sterling [45] and adapted to the case of $\mathbb{S}^3$ by Kilian and Schmidt [36].

The Pinkall-Sterling iteration for $\mathbb{S}^3$. We consider a normal variation of a conformal CMC-immersion $f : \mathbb{C} \to SU(2) \simeq \mathbb{S}^3$, i.e.

$$\dot{f} = \tfrac{d}{dt}f|_{t=0} = \omega \cdot N, \tag{$*$}$$

where the smooth function $\omega : \mathbb{C} \to \mathbb{R}$ represents the infinitesimal change of the surface in the direction of the normal N. In general this variation will not lead to conformal immersions, therefore we have to extend $(*)$ to

$$\dot{f} := \tau\partial f + \sigma\bar{\partial} f + \omega N \tag{$**$}$$

with smooth functions $\tau, \sigma : \mathbb{C} \to \mathbb{C}$ obeying $\bar{\tau} = \sigma$ and some differential equations. By differentiating the sinh-Gordon equation we obtain the *linearized* sinh-Gordon equation

$$\bar{\partial}\partial\dot{u} + \cosh(2u)\dot{u} = \left(\tfrac{1}{4}\Delta + \cosh(2u)\right)\dot{u} = 0. \tag{3.2.1}$$

Equation (3.2.1) is called the homogeneous **Jacobi equation** and we see that $\dot{u}$ is a solution of this equation. The following proposition shows that the situation for the function $\omega : \mathbb{C} \to \mathbb{R}$ is quite similar.

Proposition 3.14 ([36], Proposition 2.1)**.** *Every Jacobi field ωN along f can be supplemented by a tangential component $\tau\partial f + \sigma\bar{\partial} f$ to yield a parametric Jacobi field. Further, if $\tau\partial f + \sigma\bar{\partial} f + \omega N$ is a parametric Jacobi field, then ω solves the inhomogeneous Jacobi equation*

$$\bar{\partial}\partial\omega + \cosh(2u)\omega = \frac{\dot{H}e^{2u}}{2(H^2+1)}. \tag{3.2.2}$$

We want to present a formula that comes up in the proof of the above proposition:

$$\dot{u} = \frac{1}{2}\partial\tau + \tau\partial u + \frac{1}{2}\bar{\partial}\sigma + \sigma\bar{\partial}u - \omega H + \frac{H\dot{H}}{H^2+1}.$$

A parametric Jacobi field is called a **Killing field**, if it is induced by an infinitesimal isometry of $\mathbb{S}^3$.

Proposition 3.15 ([36], Proposition 2.2)**.** *A parametric Jacobi field is a Killing field if and only if $\dot{u} = 0$.*

Remark 3.16. *Since the derivative of α_λ with respect to t is of the form*

$$\dot{\alpha}_\lambda = \frac{1}{2}\begin{pmatrix} \dot{u}_z dz - \dot{u}_{\bar{z}} d\bar{z} & i\lambda^{-1}\dot{u}e^{u}dz - i\dot{u}e^{-u}d\bar{z} \\ -i\dot{u}e^{-u}dz + i\lambda\dot{u}e^{u}d\bar{z} & -\dot{u}_z dz + \dot{u}_{\bar{z}} d\bar{z} \end{pmatrix}$$

we see that $\dot{u} = 0 \Leftrightarrow \dot{\alpha}_\lambda = 0$.

From the Sym-Bobenko formula we know that a conformal CMC-immersion $f : \mathbb{C} \to SU(2)$ can be written as $f = F_1 F_0^{-1}$. Taking the derivative with respect to t we obtain

$$\dot{f} = (F_1 F_0^{-1})^{\cdot} = F_1(W_1 - W_0)F_0^{-1} = \tau\partial f + \sigma\bar{\partial} f + \omega N,$$

where W_i are given by $W_i := F_i^{-1}\dot{F}_i$ for $i = 0, 1$. In case $\dot{f}$ is a Killing field, i.e. $\dot{u} = 0$, these maps obey

$$\begin{aligned} dW_i &= d\left(F_i^{-1}\dot{F}_i\right) = -F_i^{-1}(dF_i)F_i^{-1}\dot{F}_i + F_i^{-1}d\dot{F}_i \\ &= -\alpha_i W_i + F_i^{-1}\dot{F}_i\alpha_i + F_i^{-1}F_i\dot{\alpha}_i \\ &= [W_i, \alpha_i] + \dot{\alpha}_i \\ &\overset{\dot{u}=0}{=} [W_i, \alpha_i]. \end{aligned}$$

Now we search for a λ-dependent map $\widetilde{\Phi}_\lambda$ such that the equation $d\widetilde{\Phi}_\lambda = [\widetilde{\Phi}_\lambda, \alpha_\lambda]$ holds for all $\lambda \in \mathbb{C}^*$. Let us omit the tilde in the following

Proposition 3.17 ([27], Proposition 2.2). *Suppose Φ_λ is of the form*

$$\Phi_\lambda(z) = \frac{\lambda^{-1}}{2}\begin{pmatrix} 0 & ie^u \\ 0 & 0 \end{pmatrix} + \sum_{n=0}^{g} \lambda^n \begin{pmatrix} \omega_n & e^u\tau_n \\ e^u\sigma_n & -\omega_n \end{pmatrix}$$

for some $u : \mathbb{C} \to \mathbb{R}$, and that Φ_λ solves the Lax equation $d\Phi_\lambda = [\Phi_\lambda, \alpha_\lambda]$. Then:

(i) The function u is a solution of the sinh-*Gordon equation, i.e.* $\Delta u + 2\sinh(2u) = 0$.

(ii) The functions ω_n are solutions of the homogeneous Jacobi equation (3.2.1).

(iii) The following iteration gives a formal solution of $d\Phi_\lambda = [\Phi_\lambda, \alpha_\lambda]$. Let $\omega_n, \sigma_n, \tau_{n-1}$ with a solution ω_n of (3.2.1) *be given. Now solve the system*

$$\tau_{n,\bar z} = ie^{-2u}\omega_n, \quad \tau_{n,z} = 2iu_z\omega_{n,z} - i\omega_{n,zz}$$

for $\tau_{n,z}$ and $\tau_{n,\bar z}$. Then define ω_{n+1} and σ_{n+1} by

$$\omega_{n+1} := -i\tau_{n,z} - 2iu_z\tau_n, \quad \sigma_{n+1} := e^{2u}\tau_n + 2i\omega_{n+1,\bar z}.$$

(iv) Each τ_n is defined up to a complex constant c_n, so ω_{n+1} is defined up to $-2ic_nu_z$.

(v) $\omega_0 = u_z, \omega_{g-1} = cu_{\bar z}$ for some $c \in \mathbb{C}$, and $\lambda^g \overline{\Phi_{1/\bar\lambda}}^t$ also solves $d\Phi_\lambda = [\Phi_\lambda, \alpha_\lambda]$.

Proof. Let us sketch some parts of the proof since we use different normalizations for α_λ and Φ_λ. Set $\Phi_\lambda = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & -\Phi_{11} \end{pmatrix}$ and consider the equation

$$d\Phi_\lambda = [\Phi_\lambda, \alpha_\lambda].$$

After decomposition into the $(1,0)$- and $(0,1)$-part we get

$$\begin{aligned} 2\partial \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & -\Phi_{11} \end{pmatrix} &= \left[\begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & -\Phi_{11} \end{pmatrix}, \begin{pmatrix} \partial u & i\lambda^{-1}e^u \\ ie^{-u} & -\partial u \end{pmatrix}\right] \\ &= \begin{pmatrix} \Phi_{12}ie^{-u} - \Phi_{21}i\lambda^{-1}e^u & (\Phi_{11}+\Phi_{11})i\lambda^{-1}e^u - 2\Phi_{12}\partial u \\ 2\Phi_{21}\partial u + (-\Phi_{11}-\Phi_{11})ie^{-u} & \Phi_{21}i\lambda^{-1}e^u - \Phi_{12}ie^{-u} \end{pmatrix} \\ &= \begin{pmatrix} \Phi_{12}ie^{-u} - \Phi_{21}i\lambda^{-1}e^u & 2\Phi_{11}i\lambda^{-1}e^u - 2\Phi_{12}\partial u \\ 2\Phi_{21}\partial u - 2\Phi_{11}ie^{-u} & \Phi_{21}i\lambda^{-1}e^u - \Phi_{12}ie^{-u} \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} 2\bar\partial \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & -\Phi_{11} \end{pmatrix} &= \left[\begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & -\Phi_{11} \end{pmatrix}, \begin{pmatrix} -\bar\partial u & ie^{-u} \\ i\lambda e^u & \bar\partial u \end{pmatrix}\right] \\ &= \begin{pmatrix} \Phi_{12}i\lambda e^u - \Phi_{21}ie^{-u} & (\Phi_{11}+\Phi_{11})ie^{-u} + \Phi_{12}\bar\partial u \\ -2\Phi_{21}\bar\partial u + (-\Phi_{11}-\Phi_{11})i\lambda e^u & \Phi_{21}ie^{-u} - \Phi_{12}i\lambda e^u \end{pmatrix} \\ &= \begin{pmatrix} \Phi_{12}i\lambda e^u - \Phi_{21}ie^{-u} & 2\Phi_{11}ie^{-u} + 2\Phi_{12}\bar\partial u \\ -2\Phi_{21}\bar\partial u - 2\Phi_{11}i\lambda e^u & \Phi_{21}ie^{-u} - \Phi_{12}i\lambda e^u \end{pmatrix}. \end{aligned}$$

Comparing coefficients we arrive at the equations

$$\begin{aligned}
2\omega_{n,z} - i\tau_n + ie^{2u}\sigma_{n+1} &= 0, && (3.2.3)\\
2\omega_{n,\bar z} + i\sigma_n - ie^{2u}\tau_{n-1} &= 0, && (3.2.4)\\
2\tau_n u_z + \tau_{n,z} - i\omega_{n+1} &= 0, && (3.2.5)\\
\tau_{n,\bar z} - ie^{-2u}\omega_n &= 0, && (3.2.6)\\
\sigma_{n,z} + ie^{-2u}\omega_n &= 0, && (3.2.7)\\
2\sigma_n u_{\bar z} + \sigma_{n,\bar z} + i\omega_{n-1} &= 0. && (3.2.8)
\end{aligned}$$

We only outline the proof of (iii), since this is the part where the different normalizations for α_λ and Φ_λ take effect. The equation for $\tau_{n,\bar z}$ is given by (3.2.6). Taking the z-derivative of equation (3.2.3) we get

$$2\omega_{n,zz} - i\tau_{n,z} + 2iu_z e^{2u}\sigma_{n+1} + ie^{2u}\sigma_{n+1,z} = 0.$$

Rearranging terms and applying equations (3.2.5) and (3.2.7) leads to

$$\begin{aligned}
\tau_{n,z} &= -2i\omega_{n,zz} + 2u_z e^{2u}\sigma_{n+1} + e^{2u}\sigma_{n+1,z}\\
&\overset{(3.2.7)}{=} -2i\omega_{n,zz} + 2u_z e^{2u}\sigma_{n+1} - i\omega_{n+1}\\
&\overset{(3.2.5)}{=} -2i\omega_{n,zz} + 2u_z e^{2u}\sigma_{n+1} - 2\tau_n u_z - \tau_{n,z}
\end{aligned}$$

and thus

$$\tau_{n,z} = -i\omega_{n,zz} + iu_z(-ie^{2u}\sigma_{n+1} + i\tau_n).$$

Now equation (3.2.3) gives

$$\tau_{n,z} = 2iu_z\omega_{n,z} - i\omega_{n,zz}.$$

The equations for ω_{n+1} and σ_{n+1} are given by (3.2.5) and (3.2.4) respectively. □

In [45] Pinkall-Sterling construct a series of solutions for the induction introduced in Proposition 3.17, (iii). From this **Pinkall-Sterling iteration** we obtain for the first terms of $\omega = \sum_{n\geq -1}\lambda^n\omega_n$

$$\begin{gathered}
\omega_{-1} = 0,\;\; \omega_0 = u_z = \tfrac{1}{2}(u_x - iu_y),\;\; \omega_1 = u_{zzz} - 2(u_z)^3,\\
\omega_2 = u_{zzzzz} - 10u_{zzz}(u_z)^3 - 10(u_{zz})^2u_z + 6(u_z)^5,\; \ldots
\end{gathered}$$

Potentials and polynomial Killing fields. We follow the exposition given in [27], Section 2. For $g \in \mathbb{N}_0$ consider the $3g+1$-dimensional real vector space

$$\Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C}) = \left\{\xi_\lambda = \sum_{n=-1}^{g}\lambda^n\widehat{\xi}_n \,\middle|\, \widehat{\xi}_{-1} \in i\mathbb{R}\epsilon_+, \widehat{\xi}_n = -\overline{\widehat{\xi}_{g-1-n}}^{\,t} \in \mathfrak{sl}_2(\mathbb{C}) \text{ for } n = -1,\ldots,g\right\}$$

and define an open subset of $\Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})$ by

$$\mathcal{P}_g := \{\xi_\lambda \in \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C}) \,|\, \widehat{\xi}_{-1} \in i\mathbb{R}^+\epsilon_+, \operatorname{tr}(\widehat{\xi}_{-1}\widehat{\xi}_0) \neq 0\}.$$

Every $\xi_\lambda \in \mathcal{P}_g$ satisfies the so-called **reality condition**

$$\lambda^{g-1}\overline{\xi_{1/\bar{\lambda}}}^t = -\xi_\lambda.$$

Definition 3.18. *A **polynomial Killing field** is a map $\zeta_\lambda : \mathbb{R} \to \mathcal{P}_g$ which solves*

$$\frac{d}{dx}\zeta_\lambda = [\zeta_\lambda, U_\lambda(\cdot,0)] \quad \textit{with} \quad \zeta_\lambda(0) = \xi_\lambda \in \mathcal{P}_g.$$

For each initial value $\xi_\lambda \in \mathcal{P}_g$, there exists a unique polynomial Killing field given by

$$\zeta_\lambda(x) := F_\lambda^{-1}(x)\xi_\lambda F_\lambda(x)$$

with $\frac{d}{dx}F_\lambda(x) = F_\lambda(x)U_\lambda(x,0)$, since there holds

$$\begin{aligned}
\tfrac{d}{dx}\zeta_\lambda &= \tfrac{d}{dx}\left(F_\lambda^{-1}\xi_\lambda F_\lambda\right) = -F_\lambda^{-1}(\tfrac{d}{dx}F_\lambda)F_\lambda^{-1}\xi_\lambda F_\lambda + F_\lambda^{-1}\xi_\lambda(\tfrac{d}{dx}F_\lambda)\\
&= -U_\lambda(\cdot,0)F_\lambda^{-1}\xi_\lambda F_\lambda + F_\lambda^{-1}\xi_\lambda F_\lambda U_\lambda(\cdot,0)\\
&= [\zeta_\lambda, U_\lambda(\cdot,0)].
\end{aligned}$$

In order to obtain a *periodic* polynomial Killing field $\zeta_\lambda : \mathbb{R}/\mathbf{p} \to \mathcal{P}_g$ from a pair $(u, u_y) \simeq U_\lambda(\cdot,0)$ of finite type we set

$$\zeta_\lambda(x) := \Phi_\lambda(x) - \lambda^{g-1}\overline{\Phi_{1/\bar{\lambda}}}^t(x) \quad \text{and} \quad \zeta_\lambda(0) =: \xi_\lambda = \Phi_\lambda(0) - \lambda^{g-1}\overline{\Phi_{1/\bar{\lambda}}}^t(0).$$

Suppose we have a polynomial Killing field

$$\zeta_\lambda(x) = \begin{pmatrix} 0 & \beta_{-1}(x) \\ 0 & 0 \end{pmatrix}\lambda^{-1} + \begin{pmatrix} \alpha_0(x) & \beta_0(x) \\ \gamma_0(x) & -\alpha_0(x) \end{pmatrix}\lambda^0 + \ldots + \begin{pmatrix} \alpha_g(x) & \beta_g(x) \\ \gamma_g(x) & -\alpha_g(x) \end{pmatrix}\lambda^g.$$

Then one can associate a matrix-valued form $U(\zeta_\lambda)$ to ζ_λ defined by

$$U(\zeta_\lambda) = \begin{pmatrix} \alpha_0(x) - \overline{\alpha}_0(x) & \lambda^{-1}\beta_{-1}(x) - \overline{\gamma}_0(x) \\ -\lambda\overline{\beta}_{-1}(x) + \gamma_0(x) & -\alpha_0(x) + \overline{\alpha}_0(x) \end{pmatrix} dx.$$

Remark 3.19. *One can show that for every $\xi_\lambda \in \mathcal{P}_g$ there exists a unique polynomial Killing field $\zeta_\lambda : \mathbb{R} \to \mathcal{P}_g$ that solves*

$$\frac{d}{dx}\zeta_\lambda = [\zeta_\lambda, U(\zeta_\lambda)] \quad \textit{with} \quad \zeta_\lambda(0) = \xi_\lambda.$$

3.3 The spectral curve

In this section we want to introduce the Riemann surface associated to the monodromy matrix M_λ of the frame $F_\lambda(x)$. We start with the following

Definition 3.20. *Let $\widehat{Y}$ be defined by*

$$\widehat{Y} = \{(\lambda,\mu) \in \mathbb{C}^* \times \mathbb{C}^* \mid R(\lambda,\mu) = \det(\mu\mathbb{1} - M(\lambda)) = 0\}.$$

*$\widehat{Y}$ is an open Riemann surface called **multiplier curve**.*

Considering the eigenvalue equation of the monodromy matrix we get $\widehat{Y} = \{(\lambda, \mu) \in \mathbb{C}^* \times \mathbb{C}^* \mid R(\lambda, \mu) = \mu^2 - \Delta(\lambda)\mu + 1 = 0\}$ and thus see that the eigenvalues of $M(\lambda)$ are given by

$$\mu_{1,2} = \frac{1}{2}\left[\Delta(\lambda) \pm \sqrt{\Delta(\lambda)^2 - 4}\right], \quad \Delta(\lambda) = \mathrm{tr}(M(\lambda)).$$

The branch points of the 2-valued function $\mu : \mathbb{C}^* \to \mathbb{C}^*$ are given by the zeros of odd order of $\Delta(\lambda)^2 - 4$.

Assumption 3.21. *In the following we will assume that the function $\lambda \mapsto \Delta(\lambda)^2 - 4$ has only zeros up to order two and that there are only finitely many zeros of order one.*

Due to Assumption 3.21 the function μ defines a hyperelliptic curve with branch points at the simple zeros of $\Delta(\lambda)^2 - 4$.

Considering the differential of $R(\lambda, \mu) = 0$ we get

$$(2\mu - \Delta(\lambda))\, d\mu - \Delta'(\lambda)\mu\, d\lambda = 0$$

We say that $\widehat{Y}$ has an **ordinary double point** at (λ_0, μ_0) if and only if the above differential vanishes at (λ_0, μ_0), i.e.

$$2\mu_0 - \Delta(\lambda_0) = 0 \quad \text{and} \quad \Delta'(\lambda_0)\mu_0 = 0.$$

The first condition is equivalent to $\Delta(\lambda_0)^2 - 4 = 0$ and therefore $\Delta(\lambda_0) = \pm 2 \iff \mu_0 = \pm 1$. Taking the derivative of $\Delta(\lambda)^2 - 4$ with respect to λ we get

$$\frac{d}{d\lambda}\left(\Delta(\lambda)^2 - 4\right) = 2\Delta(\lambda)\Delta'(\lambda)$$

and therefore $\frac{d}{d\lambda}\left(\Delta(\lambda)^2 - 4\right)|_{\lambda=\lambda_0} = 2\Delta(\lambda_0)\Delta'(\lambda_0) = 0$. Thus the double points of $\widehat{Y}$ correspond to the zeros of $\Delta(\lambda)^2 - 4$ of order two.

There are infinitely many double points (denoted by $\mathcal{D}$) on the multiplier curve $\widehat{Y}$ (recall that for $u \equiv 0$ the set of double points $\mathcal{D}_0$ is given by Remark 3.9). Nethertheless we can consider its normalization, i.e. a covering $\pi : \widetilde{Y} \to \widehat{Y}$ with a smooth Riemann surface $\widetilde{Y}$ such that $\pi|_{\widetilde{Y}\setminus\pi^{-1}(\mathcal{D})} : \widetilde{Y}\setminus\pi^{-1}(\mathcal{D}) \to \widehat{Y}\setminus\mathcal{D}$ is biholomorphic.

Definition 3.22. *Consider the normalization $\pi : \widetilde{Y} \to \widehat{Y}$ of the multiplier curve $\widehat{Y}$. By declaring $\lambda = 0, \infty$ to be two additional branch points, one obtains a compact hyperelliptic curve Y that is called the* ***spectral curve****. The simple zeros of $\frac{\partial R(\lambda,\mu)}{d\mu} = 2\mu - \Delta(\lambda)$ together with the points y_0, y_∞ corresponding to $\lambda = 0$ and $\lambda = \infty$ define the* ***branching divisor*** *b of Y.*

We now want to derive some properties of Y and first study its involutions. These result from the well-known transformation properties of the monodromy $M(\lambda)$.

Proposition 3.23. *The monodromy satisfies*

$$M(\bar{\lambda}^{-1}) = (\overline{M}^t(\lambda))^{-1}.$$

Proof. First we have to show that $\alpha_\lambda = -\overline{\alpha}^t_{\bar{\lambda}^{-1}}$ holds. Obviously one has

$$\begin{aligned}
\overline{\alpha}_\lambda &= \frac{1}{2}\begin{pmatrix} u_{\bar{z}}d\bar{z} - u_z dz & -i\bar{\lambda}^{-1}e^u d\bar{z} - ie^{-u}dz \\ -ie^{-u}d\bar{z} - i\bar{\lambda}e^u dz & -u_{\bar{z}}d\bar{z} + u_z dz \end{pmatrix}, \\
\overline{\alpha}^t_\lambda &= \frac{1}{2}\begin{pmatrix} u_{\bar{z}}d\bar{z} - u_z dz & -ie^{-u}d\bar{z} - i\bar{\lambda}e^u dz \\ -i\bar{\lambda}^{-1}e^u d\bar{z} - ie^{-u}dz & -u_{\bar{z}}d\bar{z} + u_z dz \end{pmatrix}.
\end{aligned}$$

Inserting $\bar{\lambda}^{-1}$ into $\overline{\alpha}^t_\lambda$ one gets

$$\begin{aligned}
\overline{\alpha}^t_{\bar{\lambda}^{-1}} &= \frac{1}{2}\begin{pmatrix} u_{\bar{z}}d\bar{z} - u_z dz & -ie^{-u}d\bar{z} - i\lambda^{-1}e^u dz \\ -i\lambda e^u d\bar{z} - ie^{-u}dz & -u_{\bar{z}}d\bar{z} + u_z dz \end{pmatrix} \\
&= -\alpha_\lambda.
\end{aligned}$$

Since $dF_\lambda = F_\lambda\alpha_\lambda$ we have $d\overline{F}^t_{\bar{\lambda}^{-1}} = \overline{\alpha}^t_{\bar{\lambda}^{-1}} \cdot \overline{F}^t_{\bar{\lambda}^{-1}}$ and therefore

$$\begin{aligned}
d\left(\overline{F}^t_{\bar{\lambda}^{-1}}\right)^{-1} &= -\left(\overline{F}^t_{\bar{\lambda}^{-1}}\right)^{-1}\left(d\overline{F}^t_{\bar{\lambda}^{-1}}\right)\left(\overline{F}^t_{\bar{\lambda}^{-1}}\right)^{-1} \\
&= \left(\overline{F}^t_{\bar{\lambda}^{-1}}\right)^{-1}\alpha_\lambda \cdot \overline{F}^t_{\bar{\lambda}^{-1}}\left(\overline{F}^t_{\bar{\lambda}^{-1}}\right)^{-1} \\
&= \left(\overline{F}^t_{\bar{\lambda}^{-1}}\right)^{-1}\alpha_\lambda
\end{aligned}$$

with $\left(\overline{F}^t_{\bar{\lambda}^{-1}}\right)^{-1}(z_0) = \mathbb{1}$. Since the initial value problem

$$dF_\lambda = F_\lambda\alpha_\lambda, \quad F_\lambda(z_0) = \mathbb{1}$$

has a unique solution, one has $F_\lambda = \left(\overline{F}^t_{\bar{\lambda}^{-1}}\right)^{-1}$ and hence the result follows from the definition of the monodromy. □

Proposition 3.24. *For the Pauli matrix $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ one has*

(i) $\sigma_2 M(\lambda)\sigma_2 = (M^t(\lambda))^{-1}$,

(ii) $\sigma_2\overline{M(\bar{\lambda}^{-1})}\sigma_2 = M(\lambda)$.

Proof.

(ii) A short calculation yields

$$\sigma_2\overline{\alpha}_{\bar{\lambda}^{-1}}\sigma_2 = \alpha_\lambda$$

and thus

$$\begin{aligned}
d\left(\sigma_2\overline{F_{\bar{\lambda}^{-1}}}\sigma_2\right) &= \sigma_2\overline{F_{\bar{\lambda}^{-1}}}\,\overline{\alpha}_{\bar{\lambda}^{-1}}\sigma_2 = \sigma_2\overline{F_{\bar{\lambda}^{-1}}}\sigma_2\sigma_2\overline{\alpha}_{\bar{\lambda}^{-1}}\sigma_2 \\
&= \sigma_2\overline{F_{\bar{\lambda}^{-1}}}\sigma_2\alpha_\lambda.
\end{aligned}$$

This gives $F_\lambda = \sigma_2\overline{F_{\bar{\lambda}^{-1}}}\sigma_2$ and the claim follows.

(i) Again we are considering α_λ and deduce from the previous proposition the equation

$$\alpha^t_\lambda = -\overline{\alpha}_{\bar\lambda^{-1}}$$

and thus

$$\begin{aligned} d\sigma_2 \left(F^t_\lambda\right)^{-1} \sigma_2 &= \sigma_2 d \left(F^t_\lambda\right)^{-1} \sigma_2 \\ &= \sigma_2 \left(F^t_\lambda\right)^{-1} \overline{\alpha}_{\bar\lambda^{-1}} F^t_\lambda \left(F^t_\lambda\right)^{-1} \sigma_2 \\ &= \sigma_2 \left(F^t_\lambda\right)^{-1} \sigma_2 \cdot \sigma_2 \overline{\alpha}_{\bar\lambda^{-1}} \sigma_2 \\ &\overset{(ii)}{=} \sigma_2 \left(F^t_\lambda\right)^{-1} \sigma_2 \alpha_\lambda. \end{aligned}$$

Summing up we get

$$F_\lambda = \sigma_2 \left(F^t_\lambda\right)^{-1} \sigma_2$$

by the same argument as in the preceeding proposition. This concludes the proof.

□

Lemma 3.25. *There are three involutions on the spectral curve Y given by*

$$\begin{aligned} \sigma: &\quad (\lambda,\mu) \mapsto (\lambda, 1/\mu) \\ \rho: &\quad (\lambda,\mu) \mapsto (1/\bar\lambda, 1/\bar\mu) \\ \eta: &\quad (\lambda,\mu) \mapsto (1/\bar\lambda, \bar\mu) \end{aligned}$$

and the involution η has no fixed points on the spectral curve Y.

Remark 3.26. *Note that σ is the holomorphic hyperelliptic involution and ρ, η are anti-holomorphic involutions that arise for real u.*

Proof. With the help of Proposition 3.23 and 3.24 we compute

$$\begin{aligned} R(\lambda,\mu^{-1}) &= \det(\mu^{-1}\mathbb{1} - M(\lambda)) = \det(\mu^{-1}\mathbb{1} - \sigma_2(M^t(\lambda))^{-1}\sigma_2) \\ &= \det(\mu^{-1}\mathbb{1} - (M^t(\lambda))^{-1}) = \det(\mu^{-1}M^t(\lambda)^{-1}(\mu\mathbb{1} - M^t(\lambda))) \\ &= \frac{\det(\mu\mathbb{1} - M^t(\lambda))}{\mu^2\det(M^t(\lambda))} = \frac{R(\lambda,\mu)}{\mu^2}, \\ \overline{R(\bar\lambda^{-1},\bar\mu^{-1})} &= \overline{\det(\bar\mu^{-1}\mathbb{1} - M(\bar\lambda^{-1}))} = \det(\mu^{-1}\mathbb{1} - \overline{M}(\bar\lambda^{-1})) \\ &= \det(\mu^{-1}\mathbb{1} - (M^t(\lambda))^{-1}) = \frac{R(\lambda,\mu)}{\mu^2}, \\ \overline{R(\bar\lambda^{-1},\bar\mu)} &= \overline{\det(\bar\mu\mathbb{1} - M(\bar\lambda^{-1}))} = \det(\mu\mathbb{1} - \overline{M}(\bar\lambda^{-1})) \\ &= \det(\mu\mathbb{1} - \sigma_2 M(\lambda)\sigma_2) = R(\mu,\lambda) \end{aligned}$$

and therefore obtain the existence of the three involutions. To complete the proof we have to check that η has no fixed points: If $v(\lambda,\mu)$ is an eigenvector of M_λ for the eigenvalue μ then $\bar v(\bar\lambda^{-1},\bar\mu)$ is an eigenvector of $\overline{M}_{\bar\lambda^{-1}}$ for the eigenvalue μ since

$$\overline{M}_{\bar\lambda^{-1}}\bar v(\bar\lambda^{-1},\bar\mu) = \overline{M_{\bar\lambda^{-1}} v(\bar\lambda^{-1},\bar\mu)} = \overline{\bar\mu v(\bar\lambda^{-1},\bar\mu)} = \mu \bar v(\bar\lambda^{-1},\bar\mu).$$

With $\overline{M}_{\bar{\lambda}^{-1}} = \sigma_2 M_\lambda \sigma_2$ we get

$$\begin{aligned} & \overline{M}_{\bar{\lambda}^{-1}} \bar{v}(\bar{\lambda}^{-1}, \bar{\mu}) = \mu \bar{v}(\bar{\lambda}^{-1}, \bar{\mu}) \\ \Leftrightarrow \quad & \sigma_2 M_\lambda \sigma_2 \bar{v}(\bar{\lambda}^{-1}, \bar{\mu}) = \mu \bar{v}(\bar{\lambda}^{-1}, \bar{\mu}) \\ \Leftrightarrow \quad & M_\lambda \sigma_2 \bar{v}(\bar{\lambda}^{-1}, \bar{\mu}) = \mu \sigma_2 \bar{v}(\bar{\lambda}^{-1}, \bar{\mu}) \end{aligned}$$

and therefore $\sigma_2 \bar{v}(\bar{\lambda}^{-1}, \bar{\mu})$ is an eigenvector of M_λ for the eigenvalue μ. If η would have fixed points, one would obtain the following identity

$$(\lambda, \mu) \stackrel{!}{=} (1/\bar{\lambda}, \bar{\mu})$$

and the eigenvectors $v(\lambda, \mu) =: v$ and $\sigma_2 \bar{v}(\bar{\lambda}^{-1}, \bar{\mu}) \stackrel{!}{=} \sigma_2 \bar{v}(\lambda, \mu) = \sigma_2 \bar{v}$ of M_λ would linearly depend on each other, i.e. $\sigma_2 \bar{v} = \gamma v$. But this would imply

$$- \bar{v} = \overline{\sigma_2} \sigma_2 \bar{v} = \gamma \overline{\sigma_2} v = \gamma \overline{(\sigma_2 \bar{v})} = \gamma \bar{\gamma} \bar{v}$$

and therefore $\gamma \bar{\gamma} = -1$, which is a contradiction. Hence the eigenvectors are linearly independent and η has no fixed points. $\square$

Spectral curves defined by $\xi_\lambda \in \mathcal{P}_g$. Let us introduce an equivalent definition for the spectral curve Y that results from a periodic polynomial Killing field. For this, following the exposition of [27], let $\zeta_\lambda : \mathbb{R}/\mathbf{p} \to \mathcal{P}_g$ be a periodic polynomial Killing field, i.e. $\zeta_\lambda(x + \mathbf{p}) = \zeta_\lambda(x)$ for all $x \in \mathbb{R}$. Then $U_\lambda(\zeta_\lambda)$ is periodic as well with period $\mathbf{p}$. For $F_\lambda : \mathbb{R} \to SL(2, \mathbb{C})$ that is a solution of $\frac{d}{dx} F_\lambda = F_\lambda U_\lambda(\zeta_\lambda)$ with $F_\lambda(0) = \mathbb{1}$ we get

$$\xi_\lambda = \zeta_\lambda(0) = \zeta_\lambda(\mathbf{p}) = F_\lambda^{-1}(\mathbf{p}) \xi_\lambda F_\lambda(\mathbf{p}) = M_\lambda^{-1} \xi_\lambda M_\lambda$$

and therefore

$$[M_\lambda, \xi_\lambda] = 0.$$

Since M_λ and ξ_λ commute it is possible to diagonalize them simultaneously away from the branch points (see Proposition 3.31). Note that $\operatorname{tr}(\xi_\lambda) = 0$ and thus the eigenvalues of ξ_λ are given by $\nu^2 = -\det \xi_\lambda$. Then one obtains

$$M_\lambda = f(\lambda) \xi_\lambda + g(\lambda) \mathbb{1} \iff \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} = \begin{pmatrix} f(\lambda)\nu + g(\lambda) & 0 \\ 0 & -f(\lambda)\nu + g(\lambda) \end{pmatrix}$$

for $\nu \neq 0, \mu \neq \pm 1$. In particular we get the equation

$$\mu = f(\lambda)\nu + g(\lambda) = \tfrac{1}{2} (\tfrac{\mu - \sigma^* \mu}{\nu}) \nu + \tfrac{1}{2} (\mu + \sigma^* \mu) \tag{$*$}$$

with

$$f(\lambda) = \tfrac{1}{2} (\tfrac{\mu - \sigma^* \mu}{\nu}) = \tfrac{1}{2} \quad \text{and} \quad g(\lambda) = \tfrac{1}{2} (\mu + \sigma^* \mu) = \tfrac{1}{2} \Delta(\lambda).$$

Thus $\mu = \frac{1}{2} (\Delta(\lambda) + \nu)$ with $\nu = \sqrt{\Delta(\lambda)^2 - 4}$ away from the branch points. With the help of Theorem 8.2 in [21] the functions f, g in equation $(*)$ extend to holomorphic functions on $\mathbb{C}^*$. Summing up one can consider μ and ν as two different functions on the same Riemann surface Y.

Remark 3.27. *From the previous considerations we see that μ is a non-zero holomorphic map on Y^*.*

Definition 3.28. *Let Y^* be defined by*

$$Y^* = \{(\lambda, \widetilde{\nu}) \in \mathbb{C}^* \times \mathbb{C}^* \mid \det(\widetilde{\nu}\mathbb{1} - \zeta_\lambda) = \widetilde{\nu}^2 + \det(\xi_\lambda) = 0\}$$

and suppose that the polynomial $a(\lambda) = -\lambda \det(\xi_\lambda)$ has $2g$ pairwise distinct roots. By declaring $\lambda = 0, \infty$ to be two additional branch points and setting $\nu = \widetilde{\nu}\lambda$ one obtains that

$$Y := \{(\lambda, \nu) \in \mathbb{CP}^1 \times \mathbb{CP}^1 \mid \nu^2 = \lambda a(\lambda)\}$$

defines a compact hyperelliptic curve Y of genus g, the ***spectral curve****. The genus g is called the* ***spectral genus****.*

Remark 3.29. *Note that the eigenvalue $\widetilde{\nu}$ of ξ_λ is given by $\widetilde{\nu} = \frac{\nu}{\lambda}$.*

We obtain two different parametrizations of the spectral curve Y, namely with coordinates (λ, μ) or coordinates (λ, ν). In order to translate the involutions σ, ρ, η to the coordinates (λ, ν) one has to consider the realization of $\xi_\lambda \in \mathcal{P}_g$.

Remark 3.30. *Since $\lambda^{g-1}\overline{\xi_{1/\bar{\lambda}}}^t = -\xi_\lambda$, the polynomial $a(\lambda) = -\lambda \det(\xi_\lambda)$ satisfies the reality condition*

$$\lambda^{2g}\overline{a(1/\bar{\lambda})} = a(\lambda)$$

and therefore the involutions σ, ρ, η with respect to (λ, ν) are given by

$$\sigma : (\lambda, \nu) \mapsto (\lambda, -\nu), \quad \rho : (\lambda, \nu) \mapsto (\bar{\lambda}^{-1}, -\bar{\lambda}^{-1-g}\bar{\nu}), \quad \eta : (\lambda, \nu) \mapsto (\bar{\lambda}^{-1}, \bar{\lambda}^{-1-g}\bar{\nu}).$$

Note that $\lambda^{g-1}\overline{\xi_{1/\bar{\lambda}}}^t = -\xi_\lambda$ implies $\lambda^{\frac{g-1}{2}}\overline{\xi_{1/\bar{\lambda}}}^t = -\lambda^{\frac{1-g}{2}}\xi_\lambda$ and thus the matrix $\lambda^{\frac{1-g}{2}}\xi_\lambda$ lies in $\mathfrak{su}(2)$ for $\lambda \in \mathbb{S}^1$. On $\mathfrak{su}(2)$ the determinant is the square of a norm and therefore

$$0 \leq \det(\lambda^{\frac{1-g}{2}}\xi_\lambda) = \lambda^{1-g}\det(\xi_\lambda) = -\lambda^{-g}a(\lambda)$$

holds for $\lambda \in \mathbb{S}^1$. Moreover, $a(\lambda)$ has distinct roots and thus $\lambda^{-g}a(\lambda) < 0$ holds on $\mathbb{S}^1$. Let us show that η has no fixed points. Suppose (λ, ν) is a point on Y such that $(\bar{\lambda}^{-1}, \bar{\lambda}^{-1-g}\bar{\nu}) = (\lambda, \nu)$. Then $\lambda \in \mathbb{S}^1$ and

$$\bar{\lambda}^{-1-g}\bar{\nu} = \nu \iff \bar{\lambda}^{-g}\overline{a(\lambda)} = \bar{\lambda}^{-1-g}\bar{\nu}^2 = |\nu|^2 \geq 0,$$

which contradicts the previous inequality. This proves that η has no fixed points.

3.4 The eigenline bundle

We want to establish a 1:1-correspondence between pairs (u, u_y) that originate from solutions of the sinh-Gordon equation and the so-called spectral data $(Y(u, u_y), D(u, u_y))$ consisting of the spectral curve $Y(u, u_y)$ and a divisor $D(u, u_y)$ on $Y(u, u_y)$. Let us investigate how this divisor can be constructed from a pair (u, u_y). We denote the eigenvalue $\widetilde{\nu}$ of ξ_λ by ν in the following.

Proposition 3.31 ([27], Proposition 5.1)**.** *Consider the monodromy M_λ that satisfies $\overline{M_{1/\bar\lambda}}^t = M_\lambda^{-1}$ and $\xi_\lambda = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \in \mathcal{P}_g$ with $[M_\lambda, \xi_\lambda] = 0$. Assume $\nu \neq 0$ and $\mu^2 \neq 1$. Then M_λ and ξ_λ have the same eigenvectors $v_+ = (1, (\nu - \alpha)/\beta)^t$, $v_- = (1, -(\nu + \alpha)/\beta)^t$ with*

$$\begin{aligned} \xi_\lambda v_+ &= \nu v_+ \qquad &&\text{and} \qquad && M_\lambda v_+ = \mu v_+, \\ \xi_\lambda v_- &= -\nu v_- \qquad &&\text{and} \qquad && M_\lambda v_- = \mu^{-1} v_-. \end{aligned}$$

The same argumentation as in Proposition 3.31 yields for the eigenvectors of M_λ^t and ξ_λ^t

Remark 3.32. *Consider the monodromy M_λ that satisfies $\overline{M_{1/\bar\lambda}}^t = M_\lambda^{-1}$ and $\xi_\lambda = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \in \mathcal{P}_g$ with $[M_\lambda, \xi_\lambda] = 0$. Assume $\nu \neq 0$ and $\mu^2 \neq 1$. Then M_λ^t and ξ_λ^t have the same eigenvectors $w_+ = (1, \beta/(\nu + \alpha))^t$, $w_- = (1, \beta/(-\nu + \alpha))^t$ with*

$$\begin{aligned} \xi_\lambda^t w_+ &= \nu w_+ \qquad &&\text{and} \qquad && M_\lambda^t w_+ = \mu w_+, \\ \xi_\lambda^t w_- &= -\nu w_- \qquad &&\text{and} \qquad && M_\lambda^t w_- = \mu^{-1} w_-. \end{aligned}$$

Lemma 3.33. *The eigenvectors v_+ and v_- have the asymptotic expansions*

$$v_\pm = \begin{cases} \begin{pmatrix} 1 \\ \pm e^{-u(0)}\sqrt{\lambda} \end{pmatrix} + O(\lambda) & \text{at } \lambda = 0, \\ \begin{pmatrix} 1 \\ \pm e^{u(0)}\sqrt{\lambda} \end{pmatrix} + O(\lambda^{-1}) & \text{at } \lambda = \infty. \end{cases}$$

Proof. Since $\nu^2 = -\det(\xi_\lambda) = -\frac{1}{4\lambda} + O(1)$ around $\lambda = 0$ we obtain $\nu = \frac{i}{2\sqrt{\lambda}}$ around $\lambda = 0$. Moreover, there holds

$$\zeta_\lambda(x) = \begin{pmatrix} 0 & \beta_{-1}(x) \\ 0 & 0 \end{pmatrix} \lambda^{-1} + \begin{pmatrix} \alpha_0(x) & \beta_0(x) \\ \gamma_0(x) & -\alpha_0(x) \end{pmatrix} \lambda^0 + \ldots + \begin{pmatrix} \alpha_g(x) & \beta_g(x) \\ \gamma_g(x) & -\alpha_g(x) \end{pmatrix} \lambda^g$$

and thus with $\xi_\lambda = \zeta_\lambda(0)$ we get the expansion

$$\frac{\nu - \alpha(0)}{\beta(0)} = \frac{\frac{i}{2}\sqrt{\lambda}^{-1} - (\alpha_0(0) + O(\lambda))}{\beta_{-1}(0)\lambda^{-1} + O(1)} = \frac{\frac{i}{2} + O(\sqrt{\lambda})}{\beta_{-1}(0)\sqrt{\lambda}^{-1} + O(\sqrt{\lambda})} = \frac{i\sqrt{\lambda}}{2\beta_{-1}(0)} + O(1)$$

around $\lambda = 0$. Now $\beta_{-1}(x) = \frac{i}{2}e^u$ yields the claim for $\lambda = 0$. A similar consideration gives the expansion around $\lambda = \infty$. □

If we adapt the proof of Lemma 3.33 for w_+ and w_- we arrive at

Corollary 3.34. *The transposed eigenvectors w_+ and w_- have the asymptotic expansions*

$$w_\pm = \begin{cases} \begin{pmatrix} 1 \\ \pm e^{u(0)}/\sqrt{\lambda} \end{pmatrix} + O(\lambda) & \text{at } \lambda = 0, \\ \begin{pmatrix} 1 \\ \pm e^{-u(0)}/\sqrt{\lambda} \end{pmatrix} + O(\lambda^{-1}) & \text{at } \lambda = \infty. \end{cases}$$

In the following we will see that the functions v_+, v_- and w_+, w_- define meromorphic functions on the spectral curve Y.

Lemma 3.35. *On the spectral curve Y there exist unique meromorphic maps $v(\lambda,\mu)$ and $w(\lambda,\mu)$ from Y to $\mathbb{C}^2$ such that*

(i) For all $(\lambda,\mu) \in Y^*$ the value of $v(\lambda,\mu)$ is an eigenvector of M_λ with eigenvalue μ and $w(\lambda,\mu)$ is an eigenvector of M_λ^t with eigenvalue μ, i.e.

$$M_\lambda v(\lambda,\mu) = \mu v(\lambda,\mu), \quad M_\lambda^t w(\lambda,\mu) = \mu w(\lambda,\mu).$$

(ii) The first component of $v(\lambda,\mu)$ and $w(\lambda,\mu)$ is equal to 1, i.e. $v(\lambda,\mu) = (1, v_2(\lambda,\mu))^t$ and $w(\lambda,\mu) = (1, w_2(\lambda,\mu))^t$ on Y.

Proof. Since the form α_λ satisfies $\overline{\alpha_{1/\bar\lambda}} = -\alpha_\lambda^t$ the monodromy satisfies

$$\overline{M_{1/\bar\lambda}} = M_\lambda^{t\,-1}$$

and is therefore of the form

$$M_\lambda = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix},$$

where we set $f^*(\lambda) = \overline{f(1/\bar\lambda)}$ for holomorphic functions $f : \mathbb{C}^* \to \mathbb{C}$. In analogy to the proof of Proposition 3.31 we obtain

$$v(\lambda,\mu) = \begin{pmatrix} 1 \\ \frac{\mu-a}{b} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{b^*}{a^*-\mu} \end{pmatrix} \quad \text{and} \quad w(\lambda,\mu) = \begin{pmatrix} 1 \\ \frac{a-\mu}{b^*} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{b}{\mu-a^*} \end{pmatrix}.$$

If the denominators do not vanish identically, this gives vector-valued functions of the desired form on Y^*. From Lemma 3.33 and Corollary 3.34 we know that

$$v_\pm = \begin{cases} \begin{pmatrix} 1 \\ \pm e^{-u(0)}\sqrt{\lambda} \end{pmatrix} + O(\lambda) & \text{at } \lambda = 0, \\ \begin{pmatrix} 1 \\ \pm e^{u(0)}\sqrt{\lambda} \end{pmatrix} + O(\lambda^{-1}) & \text{at } \lambda = \infty. \end{cases}$$

and

$$w_\pm = \begin{cases} \begin{pmatrix} 1 \\ \pm e^{u(0)}/\sqrt{\lambda} \end{pmatrix} + O(\lambda) & \text{at } \lambda = 0, \\ \begin{pmatrix} 1 \\ \pm e^{-u(0)}/\sqrt{\lambda} \end{pmatrix} + O(\lambda^{-1}) & \text{at } \lambda = \infty. \end{cases}$$

Therefore the maps v, w are globally defined on Y. Now the subspace

$$E_{(\lambda_0,\mu_0)} = \{v \in \mathbb{C}^2 \mid M_{\lambda_0} v = \mu_0 v\}$$

for $(\lambda_0,\mu_0) \in Y$ is at least one-dimensional, that is $\dim(E_{(\lambda_0,\mu_0)}) \geq 1$ for all $(\lambda_0,\mu_0) \in Y$. Since Y is non-singular the set

$$Y' := \{(\lambda_0,\mu_0) \in Y \mid \dim(E_{(\lambda_0,\mu_0)}) = 2\}$$

is empty. Moreover, the set

$$Y'' := \{(\lambda_0, \mu_0) \in Y \mid \nexists\, v = \left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right) \in E_{(\lambda_0,\mu_0)} : v_1 \neq 0\}$$

which is equal to $(Y^*)'' := \{(\lambda_0, \mu_0) \in Y^* \mid b(\lambda_0) = 0 \text{ and } a^*(\lambda_0) = \mu_0\}$ for $(\lambda_0, \mu_0) \in Y^*$ is a subvariety of Y and therefore either discrete or equal to Y. Due to Lemma 3.33 the point $y_\infty \in Y''$ and $y_0 \notin Y''$. Thus $Y'' \neq Y$ is a discrete subset of Y and therefore finite. In particular $(Y^*)''$ is finite as well. A similar reasoning holds for w. This implies that v, w can be extended uniquely to meromorphic maps from Y to $\mathbb{C}^2$ and concludes the proof. □

The projector P. We will use the meromorphic maps $v : Y \to \mathbb{C}^2$ and $w^t : Y \to \mathbb{C}^2$ to define a matrix-valued meromorphic function on Y by setting $P := \frac{vw^t}{w^t v}$. Given a meromorphic map f on Y we also define

$$P(f) := \frac{vfw^t}{w^t v}.$$

It turns out that P is a projector and has the following properties (see [47], Lemma 3.5).

Lemma 3.36.

(i) $P^2 = P$

(ii) $P \cdot M_\lambda = M_\lambda \cdot P = \mu P$

(iii) $\sum_{i=1}^2 \frac{v_i w_i^t}{w_i^t v_i} = \mathbb{1}$*, where* v_1, w_1^t *are the eigenvectors for* μ *and* v_2, w_2^t *the corresponding eigenvectors for* $\frac{1}{\mu}$.

(iv) The divisor of P *is* $-b$*, where* b *is the branching divisor of* Y*, see Def. 3.22.*

Proof.

(i) First we note that P is independent of the choice of v, w^t, since for $\widetilde{v} = fv$ and $\widetilde{w}^t = gw^t$ with meromorphic functions f, g one gets

$$\frac{\widetilde{v}\widetilde{w}^t}{\widetilde{w}^t\widetilde{v}} = \frac{vw^t}{w^t v} = P.$$

Therefore we may assume that locally v, w^t have neither poles nor zeros. From the definition of P one obtains

$$P^2 = \frac{(vw^t)(vw^t)}{(w^t v)(w^t v)} = \frac{v(w^t v)w^t}{(w^t v)(w^t v)} = \frac{vw^t}{w^t v} = P.$$

(ii) A direct calculation gives $P \cdot M_\lambda = \mu P = M_\lambda \cdot P$.

(iii) Away from the branch points v_1, v_2 and w_1, w_2 are bases of $\mathbb{C}^2$. Moreover, the equations $M_\lambda v_1 = \mu v_1$ and $M_\lambda v_2 = \frac{1}{\mu} v_2$ imply

$$w_2^t M_\lambda v_1 = \tfrac{1}{\mu} w_2^t v_1 = \mu w_2^t v_1 \quad \text{and} \quad w_1^t M_\lambda v_2 = \mu w_2^t v_1 = \tfrac{1}{\mu} w_2^t v_1.$$

Therefore $w_i^t v_j = 0$ if $i \neq j$. This shows that up to a factor v_1, v_2 and w_1, w_2 are dual bases of $\mathbb{C}^2$ and thus $\sum_{i=1}^2 \frac{v_i w_i^t}{w_i^t v_i} = \mathbb{1}$ holds.

(iv) From the construction of P we see that

$$\begin{aligned} P(\lambda_0, \mu_0) &= \left(\frac{vw^t}{w^t v}\right)(\lambda_0, \mu_0) \\ &= \left(\frac{1}{2\mu - \Delta(\lambda)} \begin{pmatrix} \mu - a^* & b \\ -b^* & \mu - a \end{pmatrix}\right)(\lambda_0, \mu_0) \end{aligned}$$

and therefore P can only have poles and those occur at the points where $2\mu - \Delta(\lambda)$ vanishes. From Lemma 3.33 and Corollary 3.34 we see that $w^t v = 2$ at $\lambda = 0$ and at $\lambda = \infty$. Moreover, we have

$$vw^t = \begin{pmatrix} 1 & \frac{e^{u(0)}}{\sqrt{\lambda}} \\ e^{-u(0)}\sqrt{\lambda} & 1 \end{pmatrix} + O(\lambda) \text{ at } \lambda = 0$$

and

$$vw^t = \begin{pmatrix} 1 & \frac{e^{-u(0)}}{\sqrt{\lambda}} \\ e^{u(0)}\sqrt{\lambda} & 1 \end{pmatrix} + O(\lambda^{-1}) \text{ at } \lambda = \infty.$$

Thus P has a pole at $\lambda = 0$ and $\lambda = \infty$. This shows that the divisor of P is the negative branching divisor $-b$ and concludes the proof.

□

Remark 3.37. *Considering $\lambda : Y \to \mathbb{CP}^1$ as a holomorphic map from Y to $\mathbb{CP}^1$ and denoting the $2g$ simple zeros of $2\mu - \Delta(\lambda)$ by $\alpha_1, \ldots, \alpha_{2g}$ we get the following divisors*

$$(\lambda) = 2y_0 - 2y_\infty, \quad (d\lambda) = \alpha_1 + \ldots + \alpha_{2g} + y_0 - 3y_\infty.$$

Thus Lemma 3.36 (iv) implies that $P\frac{d\lambda}{\lambda}$ is a holomorphic 1-form on Y^* since the branching divisor b is given by $b = \alpha_1 + \ldots + \alpha_{2g} + y_0 + y_\infty$.

Remark 3.38. *Note that we have $M_\lambda = P(\mu) + \sigma^* P(\mu)$ and $\xi_\lambda = P(\nu) + \sigma^* P(\nu)$.*

The holomorphic maps $v : Y \to \mathbb{CP}^1$ and $w : Y \to \mathbb{CP}^1$ from Lemma 3.35 motivate the following

Definition 3.39. *Set $D(u, u_y)$ and $D^t(u, u_y)$ as*

$$D(u, u_y) = -(v(\lambda, \mu)) \quad \textit{and} \quad D^t(u, u_y) = -(w(\lambda, \mu))$$

*and denote by $E(u, u_y) := \mathcal{O}_{D(u,u_y)}$ and $E^t(u, u_y) := \mathcal{O}_{D^t(u,u_y)}$ the corresponding holomorphic line bundles. Then $E(u, u_y)$ is called the **eigenline bundle** and $E^t(u, u_y)$ is called the transposed eigenline bundle.*

Since there holds

$$\{(\lambda_i,\mu_i)\,|\,b(\lambda_i)=0 \text{ and } a(\lambda_i)\neq\mu_i\}=\{(\lambda_i,\mu_i)\,|\,b(\lambda_i)=0 \text{ and } a^*(\lambda_i)=\mu_i\}$$

we get

$$D(u,u_y)=\sum(\lambda_i,\mu_i) \text{ with } b(\lambda_i)=0 \text{ and } a^*(\lambda_i)=\mu_i$$

and analogously

$$D^t(u,u_y)=\sum(\widetilde{\lambda}_i,\widetilde{\mu}_i) \text{ with } b^*(\widetilde{\lambda}_i)=0 \text{ and } a(\widetilde{\lambda}_i)=\widetilde{\mu}_i$$

on Y^*. In the proof of Lemma 3.35 we saw that the set

$$\begin{aligned}(Y^*)'' &= \left\{(\lambda_0,\mu_0)\in Y^*\,|\,\nexists\, v=\left(\begin{smallmatrix}v_1\\ v_2\end{smallmatrix}\right)\in E_{(\lambda_0,\mu_0)}\,:\,v_1\neq 0\right\}\\ &= \{(\lambda_0,\mu_0)\in Y^*\,|\,b(\lambda_0)=0 \text{ and } a^*(\lambda_0)=\mu_0\}\end{aligned}$$

is finite, therefore $D(u,u_y)$ and $D^t(u,u_y)$ indeed define divisors on Y.

Remark 3.40. *The sections v_1,v_2 and w_1,w_2 span the space of global sections of $\mathcal{O}_{D(u,u_y)}$ and $\mathcal{O}_{D^t(u,u_y)}$. Moreover, v and $\psi=F_\lambda^{-1}v$ define linear equivalent divisors $D\simeq D'$.*

We want to understand how the involutions of the spectral curve Y act on the divisors $D(u,u_y)$, $D^t(u,u_y)$ described above and therefore investigate how the involutions act on the eigenvectors v,w^t of the monodromy M_λ.

Lemma 3.41. *Let $\sigma_2=\left(\begin{smallmatrix}0&-i\\ i&0\end{smallmatrix}\right)$. The eigenvectors v,w^t transform as follows under the involutions of the spectral curve:*

$$\sigma^*v\sim\sigma_2 w,\quad \rho^*v\sim\bar{w},\quad \eta^*v\sim\sigma_2\bar{v}.$$

Proof. If $v(\lambda,\mu^{-1})$ is an eigenvector of M_λ then $v(\lambda,\mu^{-1})$ is also an eigenvector of M_λ^{-1} since

$$M_\lambda v(\lambda,\mu^{-1})=\tfrac{1}{\mu}v(\lambda,\mu^{-1})\Longleftrightarrow M_\lambda^{-1}v(\lambda,\mu^{-1})=\mu v(\lambda,\mu^{-1}).$$

With $M_\lambda^{-1}=\sigma_2 M_\lambda^t\sigma_2$ we get

$$\begin{aligned}&M_\lambda^{-1}v(\lambda,\mu^{-1})=\mu v(\lambda,\mu^{-1})\\ \Leftrightarrow\quad &\sigma_2 M_\lambda^t\sigma_2 v(\lambda,\mu^{-1})=\mu v(\lambda,\mu^{-1})\\ \Leftrightarrow\quad &M_\lambda^t\sigma_2 v(\lambda,\mu^{-1})=\mu\sigma_2 v(\lambda,\mu^{-1})\end{aligned}$$

and therefore $\sigma_2 v(\lambda,\mu^{-1})$ is an eigenvector of M_λ^t. Since

$$w^tM_\lambda=\mu w^t\Longleftrightarrow M_\lambda^t w=\mu w$$

we see that $\sigma_2 v(\lambda,\mu^{-1})$ must be a multiple of w and the first claim is proved.

With an analogous argument one can see that $\bar{v}(\bar{\lambda}^{-1}, \bar{\mu}^{-1})$ is an eigenvector of $\overline{M}^{-1}_{\bar{\lambda}^{-1}}$ since

$$\overline{M}^{-1}_{\bar{\lambda}^{-1}} \bar{v}(\bar{\lambda}^{-1}, \bar{\mu}^{-1}) = \overline{M^{-1}_{\bar{\lambda}^{-1}} v(\bar{\lambda}^{-1}, \bar{\mu}^{-1})} = \overline{\bar{\mu} v(\bar{\lambda}^{-1}, \bar{\mu}^{-1})} = \mu \bar{v}(\bar{\lambda}^{-1}, \bar{\mu}^{-1}).$$

With $\overline{M}^{-1}_{\bar{\lambda}^{-1}} = M^t_\lambda$ we get

$$\begin{aligned} & \overline{M}^{-1}_{\bar{\lambda}^{-1}} \bar{v}(\bar{\lambda}^{-1}, \bar{\mu}^{-1}) = \mu \bar{v}(\bar{\lambda}^{-1}, \bar{\mu}^{-1}) \\ \Leftrightarrow \quad & M^t_\lambda \bar{v}(\bar{\lambda}^{-1}, \bar{\mu}^{-1}) = \mu \bar{v}(\bar{\lambda}^{-1}, \bar{\mu}^{-1}) \end{aligned}$$

and therefore $\bar{v}(\bar{\lambda}^{-1}, \bar{\mu}^{-1})$ is an eigenvector of M^t_λ and a multiple of w. Finally the last claim follows directly from the above lemma. $\square$

Lemma 3.42. *The divisors $D(u, u_y)$ and $D^t(u, u_y)$ transform as follows under the involutions of the spectral curve:*

$$\begin{gathered} \sigma \circ D(u, u_y) = D^t(u, u_y) + \rho \circ (f), \quad \rho \circ D(u, u_y) = D^t(u, u_y), \\ D(u, u_y) - \eta \circ D(u, u_y) = (f) \textit{ for a merom. } f \textit{ with } f\eta^* \bar{f} = -1. \end{gathered}$$

Proof. From Lemma 3.41 we know that $w \sim \rho^* \bar{v}$ and due to the required normalization $v_1 = 1 = w_1$ even

$$w = \rho^* \bar{v}.$$

Now we get $\rho \circ D(u, u_y) = D^t(u, u_y)$. From Lemma 3.41 we also know that $v = f \sigma_2 \eta^* \bar{v}$ and thus get

$$D(u, u_y) = (f) + \eta \circ D(u, u_y).$$

Moreover, applying the equation $v = f \sigma_2 \eta^* \bar{v}$, one can compute

$$-v = \bar{\sigma}_2 \sigma_2 v = \bar{\sigma}_2 f \eta^* \bar{v} = f \eta^* (\overline{\sigma_2 v}) = f \eta^* (\overline{f \eta^* \bar{v}}) = f \eta^* (\bar{f}) \eta^* (\eta^* v) = f \eta^* (\bar{f}) v$$

and therefore $f \eta^* \bar{f} = -1$. This yields the last equation. In order to obtain the first equation we calculate

$$\sigma(D(u, u_y)) = (\rho \circ \eta)(D(u, u_y)) = \rho(D(u, u_y) + (f)) = D^t(u, u_y) + \rho \circ (f).$$

This equation yields the desired result and concludes the proof. $\square$

Remark 3.43. *The meromorphic function $f = \frac{\mu - a}{b}$ satisfies $f \eta^* \bar{f} = -1$ and there holds*

$$\eta(D(u, u_y)) - D(u, u_y) = (f).$$

Proof. A direct calculation shows $\eta^* \bar{f} = \frac{\mu - a^*}{b^*}$ and thus

$$f \eta^* \bar{f} = \frac{\mu - a}{b} \cdot \frac{\mu - a^*}{b^*} = -1 \Longleftrightarrow \mu^2 - (a + a^*)\mu + aa^* + bb^* = 0.$$

The second statement follows directly from the definition of the divisor $D(u, u_y)$. $\square$

We are now able to calculate the degrees of the line bundles $E(u, u_y) = \mathcal{O}_{D(u,u_y)}$ and $E^t(u, u_y) = \mathcal{O}_{D^t(u,u_y)}$ (compare with [47], Theorem 3.6).

Theorem 3.44. *The divisors $D(u, u_y)$ and $D^t(u, u_y)$ have the degree $g+1$, where g is the genus of the spectral curve Y. For solutions v, w^t of Lemma 3.35 the following equation for divisors holds:*

$$D(u, u_y) + D^t(u, u_y) + (w^t \cdot v) = b,$$

where b is the branching divisor of the spectral curve Y.

Proof. From the definition of $D(u, u_y)$ resp. $D^t(u, u_y)$ we get

$$(vw^t) = -D(u, u_y) - D^t(u, u_y).$$

Now the equation for divisors follows directly from Lemma 3.36, since

$$(P) = \left(\frac{vw^t}{w^t v}\right) = (vw^t) - (w^t v) = -D(u, u_y) - D^t(u, u_y) - (w^t v) = -b,$$

i.e.

$$D(u, u_y) + D^t(u, u_y) + (w^t \cdot v) = b.$$

In order to prove the first part of the claim we first note that $D^t(u, u_y) = \rho \circ D(u, u_y)$ implies

$$2 \deg D(u, u_y) = \deg b - \deg(w^t \cdot v) = \deg b$$

since $w^t v$ is a meromorphic function on Y. Now $\deg b = 2g + 2$ yields the claim. □

We want to consider the monodromy $M_\lambda(z_1)$ of a frame G_λ with a different basepoint z_1 with $G_\lambda(z_1) = \mathbb{1}$.

Lemma 3.45. *Consider the two fundamental solutions $F_\lambda, G_\lambda \in SL(2, \mathbb{C})$ of*

$$\begin{aligned} dF_\lambda &= F_\lambda \alpha_\lambda, & F_\lambda(z_0) &= \mathbb{1} \\ dG_\lambda &= G_\lambda \alpha_\lambda, & G_\lambda(z_1) &= \mathbb{1} \end{aligned}$$

for periodic α_λ with period $\mathbf{p}$. Then the monodromies $M_\lambda(z_0)$ and $M_\lambda(z_1)$ for the frames F_λ and G_λ satisfy the following equation

$$M_\lambda(z_1) = F_\lambda^{-1}(z_1) M_\lambda(z_0) F_\lambda(z_1).$$

Proof. Consider the system

$$dG_\lambda = G_\lambda \alpha_\lambda \text{ with } G_\lambda(z_0) =: G_0$$

Then one obtains

$$G_\lambda(z) = G_\lambda(z_0) \cdot F_\lambda(z) \quad \forall z$$

since $G_\lambda(z_0) \cdot F_\lambda(z)$ is also a solution of the above system with the same initial value G_0. In particular one has

$$G_\lambda(z_1) = \mathbb{1} = G_\lambda(z_0) \cdot F_\lambda(z_1)$$

and therefore $G_\lambda(z_0) = F_\lambda^{-1}(z_1)$. Since $G_\lambda(z_1) = \mathbb{1}$ we get

$$\begin{aligned} M_\lambda(z_1) &= G_\lambda(z_1 + \mathbf{p}) = G_\lambda(z_0) F_\lambda(z_1 + \mathbf{p}) \\ &= G_\lambda(z_0) M_\lambda(z_0) F_\lambda(z_1) \\ &= F_\lambda^{-1}(z_1) M_\lambda(z_0) F_\lambda(z_1) \end{aligned}$$

and the claim follows. □

Remark 3.46. *If v is an eigenvector for $M_\lambda(z_0)$ with eigenvalue μ then $\widetilde{v} = F_\lambda^{-1}(z_1)v$ is an eigenvector for the conjugated monodromy $M_\lambda(z_1) = F_\lambda^{-1}(z_1) M_\lambda(z_0) F_\lambda(z_1)$.*

If we replace z_1 by the variable z we see that the basepoint-dependent monodromy $M_\lambda(z) = F_\lambda^{-1}(z) M_\lambda(z_0) F_\lambda(z)$ satisfies

$$\begin{aligned} dM_\lambda(z) &= -F_\lambda^{-1}(z)\,(dF_\lambda(z))\,F_\lambda^{-1}(z) M_\lambda(z_0) F_\lambda(z) + F_\lambda^{-1}(z) M_\lambda(z_0)\,(dF_\lambda(z)) \\ &= [M_\lambda(z), \alpha_\lambda(z)]. \end{aligned}$$

The result of Lemma 3.35 can be transfered to the situation where the monodromy depends on the basepoint.

Proposition 3.47. *Consider the monodromy M_λ that satisfies $\overline{M_{1/\bar{\lambda}}}^t = M_\lambda^{-1}$ and $\xi_\lambda \in \mathcal{P}_g$ with $[M_\lambda, \xi_\lambda] = 0$. Assume $\nu \neq 0$ and $\mu^2 \neq 1$. Then $M_\lambda(x) = F_\lambda^{-1}(x) M_\lambda F_\lambda(x)$ and $\zeta_\lambda(x) = F_\lambda^{-1}(x)\xi_\lambda F_\lambda(x) = \begin{pmatrix} \alpha(x) & \beta(x) \\ \gamma(x) & -\alpha(x) \end{pmatrix}$ have the same eigenvectors $v_+(x) = (1, (\nu - \alpha(x))/\beta(x))^t$, $v_-(x) = (1, -(\nu + \alpha(x))/\beta(x))^t$ with*

$$\begin{aligned} \zeta_\lambda(x) v_+(x) &= \nu v_+(x) \qquad \textit{and} \qquad M_\lambda(x) v_+(x) = \mu v_+(x), \\ \zeta_\lambda(x) v_-(x) &= -\nu v_-(x) \qquad \textit{and} \qquad M_\lambda(x) v_-(x) = \mu^{-1} v_-(x). \end{aligned}$$

Remark 3.48. *Consider the monodromy M_λ that satisfies $\overline{M_{1/\bar{\lambda}}}^t = M_\lambda^{-1}$ and $\xi_\lambda \in \mathcal{P}_g$ with $[M_\lambda, \xi_\lambda] = 0$. Assume $\nu \neq 0$ and $\mu^2 \neq 1$. Then $M_\lambda^t(x)$ and $\zeta_\lambda^t(x)$ have the same eigenvectors $w_+(x) = (1, \beta(x)/(\nu + \alpha(x)))^t$, $w_- = (1, \beta(x)/(-\nu + \alpha(x)))^t$ with*

$$\begin{aligned} \zeta_\lambda^t(x) w_+(x) &= \nu w_+(x) \qquad \textit{and} \qquad M_\lambda^t(x) w_+(x) = \mu w_+(x), \\ \zeta_\lambda^t(x) w_-(x) &= -\nu w_-(x) \qquad \textit{and} \qquad M_\lambda^t(x) w_-(x) = \mu^{-1} w_-(x). \end{aligned}$$

Theorem 3.49. *On the spectral curve Y there exist unique holomorphic x-dependent maps $v((\lambda,\mu),x)$ and $w((\lambda,\mu),x)$ from $Y \times \mathbb{R}$ to $\mathbb{C}^2$ such that*

(i) For all $(\lambda,\mu) \in Y^*$ and all $x \in \mathbb{R}$ the value of $v((\lambda,\mu),x)$ is an eigenvector of $M_\lambda(x)$ with eigenvalue μ and $w((\lambda,\mu),x)$ is an eigenvector of $M_\lambda^t(x)$ with eigenvalue μ, i.e.

$$M_\lambda(x) v((\lambda,\mu),x) = \mu v((\lambda,\mu),x), \quad M_\lambda^t(x) w((\lambda,\mu),x) = \mu w((\lambda,\mu),x).$$

(ii) The first component of $v((\lambda,\mu),x)$ and $w((\lambda,\mu),x)$ is equal to 1, i.e. $v((\lambda,\mu),x) = (1, v_2((\lambda,\mu),x))^t$ and $w((\lambda,\mu),x) = (1, w_2((\lambda,\mu),x))^t$ on $Y \times \mathbb{R}$.

Proof. Again we can put together $v_+(x), v_-(x)$ and $w_+(x), w_-(x)$ to obtain maps $v((\lambda,\mu),x)$ and $w((\lambda,\mu),x)$. Adapting the proofs of Lemma 3.33 and Corollary 3.34 we get

$$v_\pm(x) = \begin{cases} \begin{pmatrix} 1 \\ \pm e^{-u(x)}\sqrt{\lambda} \end{pmatrix} + O(\lambda) & \text{at } \lambda = 0, \\ \begin{pmatrix} 1 \\ \pm e^{u(x)}\sqrt{\lambda} \end{pmatrix} + O(\lambda^{-1}) & \text{at } \lambda = \infty. \end{cases}$$

and

$$w_\pm(x) = \begin{cases} \begin{pmatrix} 1 \\ \pm e^{u(x)}/\sqrt{\lambda} \end{pmatrix} + O(\lambda) & \text{at } \lambda = 0, \\ \begin{pmatrix} 1 \\ \pm e^{-u(x)}/\sqrt{\lambda} \end{pmatrix} + O(\lambda^{-1}) & \text{at } \lambda = \infty. \end{cases}$$

The remaining part of the proof coincides with the proof of Lemma 3.35. □

Remark 3.50. *On Y^* these maps are given by*

$$v((\lambda,\mu),x) = \frac{\psi((\lambda,\mu),x)}{\psi_1((\lambda,\mu),x)} \quad \textit{and} \quad w((\lambda,\mu),x) = \frac{\varphi((\lambda,\mu),x)}{\varphi_1((\lambda,\mu),x)}$$

where

$$\psi((\lambda,\mu),x) = F_\lambda^{-1}(x)v(\lambda,\mu) \quad \textit{and} \quad \varphi((\lambda,\mu),x) = F_\lambda^t(x)w(\lambda,\mu).$$

Obviously there holds $v((\lambda,\mu),x+\mathbf{p}) = v((\lambda,\mu),x)$ and $w((\lambda,\mu),x+\mathbf{p}) = w((\lambda,\mu),x)$. In particular $v((\lambda,\mu),\mathbf{p}) = v((\lambda,\mu),0) = v(\lambda,\mu)$.

Lemma 3.51. *The map $\psi_+ = F_\lambda^{-1}v_+$ has the asymptotic expansions*

$$\psi_+ = \begin{cases} \exp(\frac{-ix}{2\sqrt{\lambda}}) \left(\begin{pmatrix} 1 \\ e^{-u(x)}\sqrt{\lambda} \end{pmatrix} + O(\lambda) \right) & \textit{at } \lambda = 0, \\ \exp(\frac{-ix\sqrt{\lambda}}{2}) \left(\begin{pmatrix} 1 \\ e^{u(x)}\sqrt{\lambda} \end{pmatrix} + O(\lambda^{-1}) \right) & \textit{at } \lambda = \infty. \end{cases}$$

Proof. We follow the proof of [27], Lemma 5.2. By Proposition 3.47 an eigenvector of $\zeta_\lambda(x)$ and $M_\lambda(x)$ for the eigenvalues ν and μ is given by $v_+(x) = (1, (\nu - \alpha(x))/\beta(x))^t$. Since $\zeta_\lambda(x) = F_\lambda^{-1}(x)\xi_\lambda F_\lambda(x)$ we see that $\psi_+(x) = F_\lambda^{-1}(x)v_+$ is an eigenvector of $\zeta_\lambda(x)$ and it is collinear to $v_+(x)$. This defines a function $f(\lambda,x)$ such that

$$f(\lambda,x)v_+(x) = \psi_+(x) \tag{3.4.1}$$

holds. Differentiating equation (3.4.1) we obtain

$$(\tfrac{d}{dx}f)v_+ + f(\tfrac{d}{dx}v_+) = -U_\lambda\psi_+$$

and thus

$$f^{-1}(\tfrac{d}{dx}f)v_+ = -U_\lambda v_+ - \tfrac{d}{dx}v_+.$$

Moreover, considering the first entry of the last vector equation we get

$$f^{-1}(\tfrac{d}{dx}f) = -(\alpha_0 - \bar{\alpha}_0) - (\lambda^{-1}\beta_{-1} - \bar{\gamma}_0)\tfrac{\nu-\alpha}{\beta}. \tag{3.4.2}$$

Inserting the results from Lemma 3.33 and Corollary 3.34 into equation (3.4.2) we obtain

$$f^{-1}(\tfrac{d}{dx}f) = \begin{cases} -\lambda^{-1}\beta_{-1}\frac{i\sqrt{\lambda}}{2\beta_{-1}} + O(1) = -\frac{i}{2\sqrt{\lambda}} + O(1) & \text{at } \lambda = 0, \\ -\bar{\gamma}_0\frac{i\sqrt{\lambda}}{2\bar{\gamma}_0} + O(1) = -\frac{i\sqrt{\lambda}}{2} + O(1) & \text{at } \lambda = \infty. \end{cases}$$

Now integration of $f^{-1}(\frac{d}{dx}f)$ yields the claim, since $\psi_+(x) = f(\lambda, x)v_+(x)$. □

With the help of Lemma 3.51 we obtain the following version of Theorem 3.10 in [47].

Corollary 3.52. *For the eigenline bundle $E(T_x(u, u_y))$ of the translated Cauchy data $T_x(u, u_y) = (u(\cdot + x), u_y(\cdot + x))$ we have*

$$E(T_x(u, u_y)) \simeq E(u, u_y) \otimes L(x),$$

where the holomorphic line bundle $L(x)$ with $\deg(L(x)) = 0$ for all $x \in \mathbb{R}$ is defined by the cocycles $\phi_0 := \exp(\frac{1}{\sqrt{\lambda}}\frac{ix}{2})$ on $U_0\backslash\{y_0\}$ and $\phi_\infty := \exp(\sqrt{\lambda}\frac{ix}{2})$ on $U_\infty\backslash\{y_\infty\}$.

Proof. Translating by x results in the x-dependent monodromy $M_\lambda(x)$. From Corollary 3.12 we know that the asymptotic expansion of $\ln\mu$ is given by

$$\ln\mu = \frac{i\mathbf{p}}{2\sqrt{\lambda}} + O(\sqrt{\lambda}) \text{ at } \lambda = 0 \quad\text{and}\quad \ln\mu = \frac{i\mathbf{p}\sqrt{\lambda}}{2} + O(1/\sqrt{\lambda}) \text{ at } \lambda = \infty.$$

Thus Lemma 3.51 shows that the formula

$$v((\lambda,\mu), x) = e^{\frac{x}{\mathbf{p}}\ln\mu}F_\lambda^{-1}(x)v(\lambda,\mu)$$

holds around $\lambda = 0$ and $\lambda = \infty$. Moreover, the map $F_\lambda^{-1}(x)v(\lambda,\mu)$ defines a holomorphic line bundle that is isomorphic to $E(u, u_y)$ on Y^*. This proves the claim. □

Remark 3.53. *Due to Theorem 3.49 it is possible to extend the definition of the projector P to a projector P_x that is defined by*

$$P_x(f) := \frac{v(x) f w(x)^t}{w(x)^t v(x)} = F_\lambda^{-1}(x)P(f)F_\lambda(x).$$

Moreover, there holds $M_\lambda(x) = P_x(\mu) + \sigma^* P_x(\mu)$ and $\zeta_\lambda(x) = P_x(\nu) + \sigma^* P_x(\nu)$.

Remark 3.54. *Given a doubly periodic solution of the* sinh-*Gordon equation with respect to the lattice $\Gamma = \mathbb{Z}\tau_1 \oplus \mathbb{Z}\tau_2 \subset \mathbb{C}$ we see that for $\psi := F_\lambda^{-1}v$ we obtain the equation*

$$\psi(z + a\tau_1 + b\tau_2) = e^{-a\ln\mu_1 - b\ln\mu_2}\psi(z) \quad \textit{for } (a,b) \in \mathbb{Z}^2.$$

Here μ_1, μ_2 are the eigenvalues of the monodromies $M_\lambda^{\tau_1} = F_\lambda(\tau_1)$ and $M_\lambda^{\tau_2} = F_\lambda(\tau_2)$ respectively. In order to obtain a doubly periodic $\widetilde{\psi}$ we make the ansatz

$$\widetilde{\psi} = e^{cz+d\bar{z}}\psi$$

and arrive at the equation

$$\widetilde{\psi}(z + a\tau_1 + b\tau_2) = e^{cz+d\bar{z}}e^{a(c\tau_1+d\bar{\tau}_1)+b(c\tau_2+d\bar{\tau}_2)}\psi(z + a\tau_1 + b\tau_2).$$

In particular $\widetilde{\psi}(z + a\tau_1 + b\tau_2) = \widetilde{\psi}(z)$ for all $(a, b) \in \mathbb{Z}^2$ if and only if

$$e^{a(c\tau_1+d\bar{\tau}_1)+b(c\tau_2+d\bar{\tau}_2)} = e^{a\ln\mu_1+b\ln\mu_2}.$$

A direct calculation gives

$$c = \frac{\bar{\tau}_2\ln\mu_1 - \bar{\tau}_1\ln\mu_2}{\tau_1\bar{\tau}_2 - \bar{\tau}_1\tau_2}, \quad d = \frac{\tau_1\ln\mu_2 - \tau_2\ln\mu_1}{\tau_1\bar{\tau}_2 - \bar{\tau}_1\tau_2}$$

and

$$\ln\mu_i = \tau_i c + \bar{\tau}_i d \quad \textit{for} \;\; i = 1, 2.$$

Thus

$$\widetilde{\psi} = \exp\left(\frac{\bar{\tau}_2\ln\mu_1 - \bar{\tau}_1\ln\mu_2}{\tau_1\bar{\tau}_2 - \bar{\tau}_1\tau_2}z + \frac{\tau_1\ln\mu_2 - \tau_2\ln\mu_1}{\tau_1\bar{\tau}_2 - \bar{\tau}_1\tau_2}\bar{z}\right)\psi$$

is periodic with respect to the lattice $\Gamma \subset \mathbb{C}$. The asymptotic expansions of $M_\lambda^{\tau_1}$ and $M_\lambda^{\tau_2}$ show

$$\ln\mu_i \sim \frac{\tau_i}{\sqrt{\lambda}} \;\; \textit{at} \;\; \lambda = 0 \;\; \textit{and} \;\; \ln\mu_i \sim \bar{\tau}_i\sqrt{\lambda} \;\; \textit{at} \;\; \lambda = \infty \;\; \textit{for} \;\; i = 1, 2.$$

Therefore $c \sim \frac{1}{\sqrt{\lambda}}$ at $\lambda = 0$ and $c = 0$ at $\lambda = \infty$. Moreover, $d = 0$ at $\lambda = 0$ and $d \sim \sqrt{\lambda}$ at $\lambda = \infty$. From this we deduce that c is holomorphic at $\lambda = \infty$ and d is holomorphic at $\lambda = 0$.

3.5 The associated spectral data

In this section we want to summarize the description for periodic finite type solutions of the sinh-Gordon equation. Given a hyperelliptic Riemann surface Y with branch points over $\lambda = 0$ (y_0) and $\lambda = \infty$ (y_∞) we can deduce conditions such that Y is the spectral curve of a periodic finite type solution of the sinh-Gordon equation.

Let us recall the well-known characterization of such spectral curves (compare with [36], Section 1.2, in the case of immersed CMC tori in $\mathbb{S}^3$).

Theorem 3.55. *Let Y be a hyperelliptic Riemann surface with branch points over $\lambda = 0$ (y_0) and $\lambda = \infty$ (y_∞). Then Y is the spectral curve of a periodic real finite type solution of the* sinh-*Gordon equation if and only if the following three conditions hold:*

(i) *Besides the hyperelliptic involution σ the Riemann surface Y has two further anti-holomorphic involutions η and $\rho = \eta \circ \sigma$. Moreover, η has no fixed points and $\eta(y_0) = y_\infty$.*

(ii) *There exists a non-zero holomorphic function μ on $Y \backslash \{y_0, y_\infty\}$ that obeys*

$$\sigma^* \mu = \mu^{-1}, \quad \eta^* \bar{\mu} = \mu, \quad \rho^* \bar{\mu} = \mu^{-1}.$$

(iii) *The form $d \ln \mu$ is a meromorphic differential of the second kind with double poles at y_0 and y_∞ only.*

Proof. We first consider the "if"-part "$\Rightarrow$" and get the conditions (i) and (ii) from Remark 3.30 and Remark 3.27 together with Lemma 3.25. From Corollary 3.12 we also have

$$\begin{aligned} \ln \mu &= \tfrac{1}{\sqrt{\lambda}} \tfrac{i\mathbf{p}}{2} + \sqrt{\lambda} \int_0^{\mathbf{p}} \left(-i(\partial u)^2 + \tfrac{i}{2} \cosh(2u)\right) dt + O(\lambda) \text{ at } \lambda = 0, \\ \ln \mu &= \sqrt{\lambda} \tfrac{i\mathbf{p}}{2} + \tfrac{1}{\sqrt{\lambda}} \int_0^{\mathbf{p}} \left(-i(\bar{\partial} u)^2 + \tfrac{i}{2} \cosh(2u)\right) dt + O(\lambda^{-1}) \text{ at } \lambda = \infty \end{aligned}$$

and therefore get in the $\sqrt{\lambda}$-chart around $\lambda = 0$ and the $(1/\sqrt{\lambda})$-chart around $\lambda = \infty$

$$\begin{aligned} d \ln \mu &= d\sqrt{\lambda} \left(-\tfrac{1}{\lambda} \tfrac{i\mathbf{p}}{2} + \int_0^{\mathbf{p}} \left(-i(\partial u)^2 + \tfrac{i}{2} \cosh(2u)\right) dt + O(\sqrt{\lambda}) \right) \text{ at } \lambda = 0, \\ d \ln \mu &= \frac{d}{\sqrt{\lambda}} \left(-\lambda \tfrac{i\mathbf{p}}{2} + \int_0^{\mathbf{p}} \left(-i(\bar{\partial} u)^2 + \tfrac{i}{2} \cosh(2u)\right) dt + O(1/\sqrt{\lambda}) \right) \text{ at } \lambda = \infty. \end{aligned}$$

This implies condition (iii). The "only if"-part "$\Leftarrow$" follows from Proposition 4.34 in Chapter 4. □

Remark 3.56. *Since $\sigma^* d \ln \mu = d \ln(1/\mu) = -d \ln \mu$, we see that $d \ln \mu$ changes its sign under the hyperelliptic involution σ.*

Following the terminology of [27, 35, 36], we will describe spectral curves of periodic real finite type solutions of the sinh-Gordon equation via hyperelliptic curves of the form

$$\nu^2 = \lambda\, a(\lambda) = -\lambda^2 \det(\xi_\lambda) = (\lambda \widetilde{\nu})^2.$$

Here $\widetilde{\nu}$ is the eigenvalue of ξ_λ and $\lambda : Y \to \mathbb{CP}^1$ is chosen in a way such that y_0 and y_∞ correspond to $\lambda = 0$ and $\lambda = \infty$ with

$$\sigma^* \lambda = \lambda, \quad \eta^* \bar{\lambda} = \lambda^{-1}, \quad \rho^* \bar{\lambda} = \lambda^{-1}.$$

Note that the function $\lambda : Y \to \mathbb{CP}^1$ is fixed only up to a Möbius transformations of the form $\lambda \mapsto e^{2i\varphi} \lambda$. Moreover, $d \ln \mu$ is of the form

$$d \ln \mu = \frac{b(\lambda)}{\nu} \frac{d\lambda}{\lambda},$$

where b is a polynomial of degree $g+1$ with $\lambda^{g+1} \overline{b(\bar{\lambda}^{-1})} = -b(\lambda)$. This yields the following

Definition 3.57. *The* ***spectral curve data*** *of a periodic real finite type solution of the* sinh-*Gordon equation is a pair* $(a, b) \in \mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda]$ *such that*

(i) $\lambda^{2g}\overline{a(\bar{\lambda}^{-1})} = a(\lambda)$ *and* $\lambda^{-g}a(\lambda) \leq 0$ *for all* $\lambda \in \mathbb{S}^1$ *and* $|a(0)| = 1$.

(ii) On the hyperelliptic curve $\nu^2 = \lambda a(\lambda)$ *there is a single-valued holomorphic function* μ *with essential singularities at* $\lambda = 0$ *and* $\lambda = \infty$ *with logarithmic differential*

$$d \ln \mu = \frac{b(\lambda)}{\nu} \frac{d\lambda}{\lambda}$$

with $b(0) = i\frac{\sqrt{a(0)}}{2}\mathbf{p}$ *that transforms under the three involutions*

$$\sigma : (\lambda, \nu) \mapsto (\lambda, -\nu), \quad \rho : (\lambda, \nu) \mapsto (\bar{\lambda}^{-1}, -\bar{\lambda}^{-1-g}\bar{\nu}), \quad \eta : (\lambda, \nu) \mapsto (\bar{\lambda}^{-1}, \bar{\lambda}^{-1-g}\bar{\nu})$$

according to $\sigma^*\mu = \mu^{-1}$, $\rho^*\mu = \bar{\mu}^{-1}$ *and* $\eta^*\mu = \bar{\mu}$.

Remark 3.58. *The conditions (i) and (ii) from Definition 3.57 are equivalent to the following conditions (compare with Definition 5.10 in [27]):*

(i) $\lambda^{2g}\overline{a(\bar{\lambda}^{-1})} = a(\lambda)$ *and* $\lambda^{-g}a(\lambda) \leq 0$ *for all* $\lambda \in \mathbb{S}^1$ *and* $|a(0)| = 1$.

(ii) $\lambda^{g+1}\overline{b(\bar{\lambda}^{-1})} = -b(\lambda)$ *and* $b(0) = i\frac{\sqrt{a(0)}}{2}\mathbf{p}$.

(iii) $\int_{\alpha_i}^{1/\bar{\alpha}_i} \frac{b(\lambda)}{\nu}\frac{d\lambda}{\lambda} = 0$ *for all roots* α_i *of* a.

(iv) The unique function $h : \widetilde{Y} \to \mathbb{C}$, *where* $\widetilde{Y} = Y \setminus \bigcup \gamma_i$ *and* γ_i *are closed cycles over the straight lines connecting* α_i *and* $1/\bar{\alpha}_i$, *obeying* $\sigma^*h = -h$ *and* $dh = \frac{b(\lambda)}{\nu}\frac{d\lambda}{\lambda}$, *satisfies* $h(\alpha_i) \in \pi i\mathbb{Z}$ *for all roots* α_i *of* a.

Since a Möbius transformation of the form $\lambda \mapsto e^{2i\varphi}\lambda$ changes the spectral curve data (a, b) but does not change the corresponding periodic solution of the sinh-Gordon equation we introduce the following

Definition 3.59. *For all* $g \in \mathbb{N}_0$ *let* $\mathcal{M}_g(\mathbf{p})$ *be the space of equivalence classes of spectral curve data* (a, b) *from Definition 3.57 with respect to the action of* $\lambda \mapsto e^{2i\varphi}\lambda$ *on* (a, b). $\mathcal{M}_g(\mathbf{p})$ *is called the* ***moduli space*** *of spectral curve data for Cauchy data* (u, u_y) *of periodic real finite type solutions of the* sinh-*Gordon equation.*

Each pair of polynomials $(a, b) \in \mathcal{M}_g(\mathbf{p})$ represents a spectral curve $Y_{(a,b)}$ for Cauchy data (u, u_y) of a periodic real finite type solution of the sinh-Gordon equation.

Definition 3.60. *Let*

$$\mathcal{M}_g^1(\mathbf{p}) \;:=\; \{(a, b) \in \mathcal{M}_g(\mathbf{p}) \mid a \text{ has } 2g \text{ pairwise distinct roots and } (a, b) \text{ have no common roots}\}$$

be the moduli space of non-degenerated smooth spectral curve data for Cauchy data (u, u_y) *of periodic real finite type solutions of the* sinh-*Gordon equation.*

The term "non-degenerated" in Definition 3.60 reflects the following fact (compare with [28], Section 9): If one considers deformations of spectral curve data (a, b), the corresponding integral curves have possible singularities, if a and b have common roots. By excluding the case of common roots of (a, b), one can avoid that situation and identify the space of such deformations with certain polynomials $c \in \mathbb{C}^{g+1}[\lambda]$ (see Chapter 5).

Remark 3.61. *By studying Cauchy data (u, u_y) whose spectral curve $Y(u, u_y)$ corresponds to $(a, b) \in \mathcal{M}_g^1(\mathbf{p})$, we have the following benefits:*

1. *Since $(a, b) \in \mathcal{M}_g^1(\mathbf{p})$ correspond to Cauchy data (u, u_y) of finite type, we can avoid difficult functional analytic methods for the asymptotic analysis of the spectral curves Y at $\lambda = 0$ and $\lambda = \infty$.*

2. *Since $(a, b) \in \mathcal{M}_g^1(\mathbf{p})$ have no common roots, we obtain non-singular smooth spectral curves Y and can apply the standard tools from complex analysis for their investigation.*

Note, that these assumptions can be dropped in order to extend the results from this thesis to the more general setting. This was done in [47] for the case of the non-linear Schrödinger operator, for example.

Definition 3.62. *The **spectral data** of a periodic real finite type solution of the* sinh-*Gordon equation is a pair $(Y(u, u_y), D(u, u_y))$ such that $Y(u, u_y)$ is a hyperelliptic Riemann surface of genus g that obeys the conditions from Theorem 3.55 and $D(u, u_y)$ is a divisor of degree $g + 1$ on $Y(u, u_y)$ that obeys $\eta(D) - D = (f)$ for a meromorphic f with $f\eta^*\bar{f} = -1$.*

Remark 3.63. *In the following chapter we will treat the inverse problem, that is, we will associate a periodic real finite type solution of the* sinh-*Gordon equation to given spectral data (Y, D) and thus show that the correspondence between such solutions and the spectral data is bijective.*

4 The inverse problem

The Picard group $Pic(Y(u,u_y)) \simeq H^1(Y(u,u_y),\mathcal{O}^*)$ is the space of isomorphy classes of holomorphic line bundles. We want to construct linear flows on $Pic^{\mathbb{R}}_{g+1}(Y(u,u_y))$, i.e. on the connected component of $Pic(Y(u,u_y))$ of holomorphic line bundles of degree $g+1$ obeying some reality condition.

The following two sections are based on notes from the lecture "Geometrical applications of integrable systems" given by Martin Kilian and Martin U. Schmidt at the university of Mannheim in 2005.

4.1 The space $\mathfrak{h}^-_{\mathbf{finite}}$ and Mittag-Leffler distributions

On the compact hyperelliptic Riemann surface $Y(u,u_y)$ there lie two distinguished points y_0 and y_∞ that correspond to the points lying above $\lambda = 0$ and $\lambda = \infty$ respectively.

Definition 4.1. *Let $\mathfrak{h}$ be the algebra of germs of functions that are holomorphic in a punctered neighborhood of $0 \in \mathbb{C}$, i.e.*

$$\mathfrak{h} := \{(U,h) \mid 0 \in U \subset \mathbb{C} \text{ open and connected, } h : U\backslash\{0\} \to \mathbb{C} \text{ holomorphic}\}/\sim$$

where $(U,h) \sim (U',h')$ if $h|_{(U\cap U')_0\backslash\{0\}} = h'|_{(U\cap U')_0\backslash\{0\}}$ (with $(U\cap U')_0$ the connected component of 0). Furthermore define the following subsets

$$\begin{aligned}
\mathfrak{h}^+ &:= \{(U,h) \in \mathfrak{h} \mid h \text{ extends holomorphically to } 0\}\\
\mathfrak{h}^- &:= \{(U,h) \in \mathfrak{h} \mid h \text{ extends holomorphically to } \mathbb{CP}^1\backslash\{0\} \text{ with } h(\infty) = 0\}\\
\mathfrak{h}^-_{finite} &:= \{(U,h) \in \mathfrak{h}^- \mid h \text{ has a pole at } 0\}.
\end{aligned}$$

The following lemma provides us with a decomposition of $\mathfrak{h}$ that is analogous to the Birkhoff factorization.

Lemma 4.2. *There holds $\mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{h}^-$.*

Proof. For any $h \in \mathfrak{h}$ let $h^+(z) = \frac{1}{2\pi i}\oint \frac{h(\tilde{z})d\tilde{z}}{z-\tilde{z}}$. Here the integral is taken along a path in the domain of definition of h around z and 0 in the anti-clockwise direction. Moreover, let $h^-(z) = \frac{1}{2\pi i}\oint \frac{h(\tilde{z})d\tilde{z}}{z-\tilde{z}}$, where this integral is taken along a path in the domain of definition of h around 0, but not around z, in the anti-clockwise direction. Since the form $\frac{h(\tilde{z})d\tilde{z}}{z-\tilde{z}}$ is closed, these integrals do not depend on the choice of the path of integration. Due to Cauchy's integral formula we have $h = h^+ + h^-$. Moreover, h^+ is holomorphic in a neighborhood of 0 and h^- is holomorphic on $\mathbb{CP}^1\backslash\{0\}$ and vanishes at ∞. $\square$

We will construct linear flows on $\mathrm{Pic}_0(Y(u, u_y))$ by means of the so-called *Krichever construction* and therefore recall the following

Lemma 4.3. *The cohomology group $H^1(Y(u, u_y), \mathcal{O})$ is the Lie algebra of $Pic_0(Y(u, u_y)) \simeq Jac(Y(u, u_y))$.*

Proof. Set $Y := Y(u, u_y)$. The long cohomology exact sequence corresponding to $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 1$ is

$$0 \to H^1(Y, \mathbb{Z}) \to H^1(Y, \mathcal{O}) \xrightarrow{\exp(2\pi i \cdot)} H^1(Y, \mathcal{O}^*) \xrightarrow{\deg} H^2(Y, \mathbb{Z}) \to 0.$$

Restricting ourselves to line bundles of degree 0, we see that the map $H^1(Y, \mathcal{O}) \xrightarrow{\exp(2\pi i \cdot)} H^1(Y, \mathcal{O}^*)$ is surjective and therefore

$$\mathrm{Pic}_0(Y) \simeq \mathrm{Jac}(Y) \simeq H^1(Y, \mathcal{O})/H^1(Y, \mathbb{Z}).$$

It is well-known that for G_0 (the connected component of the unit e of a Lie group G) one has $G_0 \simeq \mathfrak{g}/\ker(\exp)$ via the exponential map. Since $H^1(Y, \mathbb{Z}) = \ker(\exp)$ the claim follows. $\square$

The maps L and φ. We will focus on the following diagram

$$\begin{tikzcd} \varphi : \mathfrak{h}^2 \arrow[r] \arrow[d, "\exp(\cdot)"'] & H^1(Y, \mathcal{O}) \arrow[d, "\exp(2\pi i \cdot)"] \\ L : H^2 \arrow[r] & H^1(Y, \mathcal{O}^*) \end{tikzcd}$$

and describe the maps $L : H^2 \to H^1(Y, \mathcal{O}^*)$ and $\varphi : \mathfrak{h}^2 \to H^1(Y, \mathcal{O})$ in more detail. Here H denotes the Lie group of all non-vanishing holomorphic functions $g = \exp(h)$ defined on $U \backslash \{0\}$, where U is some neighborhood of 0. The group multiplication is given by multiplication of functions. In particular $\mathfrak{h}$ is the Lie algebra of H.

Let $k = \sqrt{\lambda}$ be a local parameter such that $k(y_0) = 0$ and $\widetilde{k} = 1/\sqrt{\lambda}$ be a local parameter such that $\widetilde{k}(y_\infty) = 0$. We will describe the map $\varphi : \mathfrak{h}^2 \to H^1(Y, \mathcal{O})$, $(h_0, h_\infty) \mapsto \varphi(h_0, h_\infty)$ and therefore consider disjoint open simply-connected neighborhoods U_0, U_∞ of y_0, y_∞ such that $k^* h_0$ and $\widetilde{k}^* h_\infty$ are defined on $U_0 \backslash \{y_0\}$ and $U_\infty \backslash \{y_\infty\}$ respectively.

Setting $U := Y \backslash \{y_0, y_\infty\}$ we get a cover $\mathcal{U} := \{U_0, U, U_\infty\}$ of Y. The only non-empty intersections of neighborhoods from $\mathcal{U}$ are $U_0 \backslash \{y_0\}$ and $U_\infty \backslash \{y_\infty\}$. Thus $k^* h_0$ and $\widetilde{k}^* h_\infty$ are cocycles for this cover and induce a cohomology-class in $H^1(Y, \mathcal{O})$. Since U_0 and U_∞ are simply connected we have $H^1(U_0, \mathcal{O}) = 0 = H^1(U_\infty, \mathcal{O})$. Moreover, $H^1(U, \mathcal{O}) = 0$ since U is a non-compact Riemann surface. This shows that $\mathcal{U}$ is a Leray cover (see Theorem 2.20) and therefore $H^1(Y, \mathcal{O}) = H^1(\mathcal{U}, \mathcal{O})$. Summing up we get a surjective map

$$\varphi : \mathfrak{h}^2 \to H^1(Y, \mathcal{O}).$$

Moreover, the map $L : H^2 \to H^1(Y, \mathcal{O}^*)$ is given by the element that corresponds to the line bundle $L(g_0, g_\infty)$ that is induced the the cocycles $k^* g_0 = k^* \exp(h_0)$ over U_0 and $\widetilde{k}^* g_\infty = \widetilde{k}^* \exp(h_\infty)$ over U_∞. Now we have the following

Lemma 4.4.

(i) The kernel of $\varphi : \mathfrak{h}^2 \to H^1(Y, \mathcal{O})$ consists of those $(h_0, h_\infty) \in \mathfrak{h}^2$ that admit a holomorphic function h on $Y \backslash \{y_0, y_\infty\}$ such that $h - k^* h_0$ and $h - \widetilde{k}^* h_\infty$ are holomorphic at y_0 and y_∞ respectively. In particular one has $(\mathfrak{h}^+)^2 \subset \ker(\varphi)$.

(ii) $\varphi[(\mathfrak{h}^-_{finite})^2] = H^1(Y, \mathcal{O})$.

Proof. From the Serre Duality Theorem we know that the pairing

$$\langle \omega, \varphi(h_0, h_\infty) \rangle = \mathrm{Res}_{y_0}(k^* h_0 \omega) + \mathrm{Res}_{y_\infty}(\widetilde{k}^* h_\infty \omega) \qquad (*)$$

is non-degenerate for $\omega \in H^0(Y, \Omega)$ and $\varphi(h_0, h_\infty) \in H^1(Y, \mathcal{O})$, where $\mathrm{Res}_{y_0}(k^* h_0 \omega)$ and $\mathrm{Res}_{y_\infty}(\widetilde{k}^* h_\infty \omega)$ are defined via integrals over small paths around y_0 and y_∞ respectively.

(i) If $(k^* h_0, \widetilde{k}^* h_\infty)$ is a co-boundary, there exist holomorphic functions g_0, g_∞ on U_0 and U_∞ and a holomorphic h on $Y \backslash \{y_0, y_\infty\}$ such that $k^* h_0 = h - g_0$ on $U_0 \backslash \{y_0\}$ and $\widetilde{k}^* h_\infty = h - g_\infty$ on $U_\infty \backslash \{y_\infty\}$. Thus the 1-form $h\omega$ is holomorphic on $Y \backslash \{y_0, y_\infty\}$ for all $\omega \in H^0(Y, \Omega)$ and due to the Residue Theorem [21] the equation

$$\begin{aligned} \langle \omega, \varphi(h_0, h_\infty) \rangle &= \mathrm{Res}_{y_0}(k^* h_0 \omega) + \mathrm{Res}_{y_\infty}(\widetilde{k}^* h_\infty \omega) \\ &= \mathrm{Res}_{y_0}(h\omega) + \mathrm{Res}_{y_\infty}(h\omega) \\ &= 0 \end{aligned}$$

holds for all $\omega \in H^0(Y, \Omega)$. Since $(*)$ is non-degenerate one obtains $\varphi(h_0, h_\infty) = 0$. Conversely $\varphi(h_0, h_\infty) = 0 \in H^1(Y, \mathcal{O})$ implies that $(k^* h_0, \widetilde{k}^* h_\infty)$ is a co-boundary.

(ii) Since $\mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{h}^-$ by Lemma 4.2 and $\varphi[(\mathfrak{h}^+)^2] = 0$ by (i) we have $\varphi[(\mathfrak{h}^-)^2] = H^1(Y, \mathcal{O})$. Denote by N the highest possible vanishing order of differentials $\omega \in H^0(Y, \Omega)$ at the points y_0, y_∞. For $(h_0, h_\infty) \in (\mathfrak{h}^-)^2$ define $(\widetilde{h}_0, \widetilde{h}_\infty)$ to be the Taylor polynomials of (h_0, h_∞) at ∞ of order $N + 1$. Then we get

$$\mathrm{Res}_{y_0}(k^*(h_0 - \widetilde{h}_0)\omega) = 0 = \mathrm{Res}_{y_\infty}(\widetilde{k}^*(h_\infty - \widetilde{h}_\infty)\omega)$$

since there only appear poles of order greater or equal 2. Serre duality $(*)$ gives

$$\begin{aligned} \langle \omega, \varphi(h_0 - \widetilde{h}_0, h_\infty - \widetilde{h}_\infty) \rangle &= \mathrm{Res}_{y_0}(k^*(h_0 - \widetilde{h}_0)\omega) + \mathrm{Res}_{y_\infty}(\widetilde{k}^*(h_\infty - \widetilde{h}_\infty)\omega) \\ &= 0 \end{aligned}$$

for all $\omega \in H^0(Y, \Omega)$. Therefore $(h_0 - \widetilde{h}_0, h_\infty - \widetilde{h}_\infty) \in \ker(\varphi)$ and $(h_0 - \widetilde{h}_0, h_\infty - \widetilde{h}_\infty) \in (\mathfrak{h}^-)^2 \backslash (\mathfrak{h}^-_{\mathrm{finite}})^2$. This shows $\varphi[(\mathfrak{h}^-_{\mathrm{finite}})^2] = H^1(Y, \mathcal{O})$ and concludes the proof.

□

Mittag-Leffler distributions. Each element $(h_0, h_\infty) \in (\mathfrak{h}^-_{\text{finite}})^2$ defines a *Mittag-Leffler distribution* (M-L distribution) $\xi(h_0, h_\infty) = (f_0, f, f_\infty) := (k^* h_0, 0, \widetilde{k}^* h_\infty)$ on Y with respect to the cover $\mathcal{U}$. A solution of such a M-L distribution is a meromorphic function h on Y that is holomorphic on Y^* such that $h - k^* h_0$ and $h - \widetilde{k}^* h_\infty$ are holomorphic on U_0 and U_∞ respectively. Thus Lemma 4.4 implies that the M-L distribution induced by $(h_0, h_\infty) \in (\mathfrak{h}^-_{\text{finite}})^2$ has a solution if and only if $\varphi(h_0, h_\infty) = 0$, i.e.

$$\langle \omega, \varphi(h_0, h_\infty) \rangle = \operatorname{Res}_{y_0}(k^* h_0 \omega) + \operatorname{Res}_{y_\infty}(\widetilde{k}^* h_\infty \omega) = 0$$

for all $\omega \in H^0(Y, \Omega)$. Since $H^1(Y, \mathcal{M}) = 0$ (see [21], Corollary 17.17), for every element $[f] \in H^1(Y, \mathcal{O})$ there exists a Mittag-Leffler distribution $\xi = (f_0, f, f_\infty) \in C^0(\mathcal{U}, \mathcal{M})$ such that $[\delta\xi] = [f]$. Recall that $\delta : C^0(\mathcal{U}, \mathcal{M}) \to Z^1(\mathcal{U}, \mathcal{O})$ is given by

$$\delta(f_0, f, f_\infty) = (f_0 - f, f_\infty - f).$$

In our situation we have $\xi(h_0, h_\infty) = (f_0, f, f_\infty) = (k^* h_0, 0, \widetilde{k}^* h_\infty)$ and therefore

$$\delta(\xi(h_0, h_\infty)) = (k^* h_0, \widetilde{k}^* h_\infty).$$

Thus we arrive at the following diagram:

$$\begin{tikzcd}
(\mathfrak{h}^-_{\text{finite}})^2 \arrow[r, "\xi"] \arrow[dr, "\varphi"] & \xi[(\mathfrak{h}^-_{\text{finite}})^2] \arrow[d, "{[\delta]}"] \\
& H^1(Y, \mathcal{O})
\end{tikzcd}$$

Since $\varphi = [\delta] \circ \xi$ is surjective we see that $[\delta] : \xi[(\mathfrak{h}^-_{\text{finite}})^2] \subset C^0(\mathcal{U}, \mathcal{M}) \to H^1(Y, \mathcal{O})$ is surjective as well. We can introduce a "basis" for $\xi[(\mathfrak{h}^-_{\text{finite}})^2] \subset C^0(\mathcal{U}, \mathcal{M})$ for hyperelliptic Y since every meromorphic function f can be written as

$$f = r(\lambda) + \nu s(\lambda)$$

in that situation. Here r, s are rational functions with respect to λ and Y is given by $\nu^2 = \lambda a(\lambda)$. Considering some element $(f_0, 0, f_\infty) \in \xi[(\mathfrak{h}^-_{\text{finite}})^2] \subset C^0(\mathcal{U}, \mathcal{M})$ we demand that f_0 has a pole at y_0 and that f_∞ has a pole at y_∞. If f_0 and f_∞ were of the form $f_0 = r_0(\lambda)$ and $f_\infty = r_\infty(\lambda)$ we could reduce the question to a consideration on $\mathbb{CP}^1$ and hence $\delta((f_0, 0, f_\infty)) = 0$ since $H^1(\mathbb{CP}^1, \mathcal{O}) = 0$ in that case. Since δ is a group homomorphism we get

$$\begin{aligned}
\delta(f_0, 0, f_\infty) &= \delta(r_0(\lambda) + \nu s_0(\lambda), 0, r_\infty(\lambda) + \nu s_\infty(\lambda)) \\
&= \delta(r_0(\lambda), 0, r_\infty(\lambda)) + \delta(\nu s_0(\lambda), 0, \nu s_\infty(\lambda)) \\
&= \delta(\nu s_0(\lambda), 0, \nu s_\infty(\lambda))
\end{aligned}$$

for general $(f_0, 0, f_\infty)$. Therefore we can restrict ourselves to the case where f_0 is given by $f_0 = \nu s_0(\lambda)$ and likewise $f_\infty = \nu s_\infty(\lambda)$. The following lemma provides us with a possible choice for a basis we are looking for.

Lemma 4.5. *The equivalence classes $[h_i]$ of the g tuples $h_i := (f_0^i, f_\infty^i)$ given by $h_i = (\nu\lambda^{-i}, -\nu\lambda^{-i})$ for $i = 1, \dots, g$ are a basis of $H^1(Y, \mathcal{O})$.*

Proof. We know that for $i = 1, \dots, g$ the differentials $\omega_i = \frac{\lambda^{i-1} d\lambda}{\nu}$ span a basis for $H^0(Y, \Omega)$ and that the pairing $\langle \cdot, \cdot \rangle : H^0(Y, \Omega) \times H^1(Y, \mathcal{O}) \to \mathbb{C}$ given by $(\omega, [h]) \mapsto \mathrm{Res}(h\omega)$ is non-degenerate due to Serre duality 2.24. Therefore we can calculate the dual basis of the ω_i with respect to this pairing and see

$$\begin{aligned}
\langle \omega_i, [h_j] \rangle &= \mathrm{Res}_{\lambda=0} f_0^j \omega_i + \mathrm{Res}_{\lambda=\infty} f_\infty^j \omega_i \\
&= \mathrm{Res}_{\lambda=0} \lambda^{i-j-1} d\lambda - \mathrm{Res}_{\lambda=\infty} \lambda^{i-j-1} d\lambda \\
&= \mathrm{Res}_{\lambda=0} \lambda^{i-j-1} d\lambda + \mathrm{Res}_{\lambda=0} \lambda^{-i+j+1} \frac{d\lambda}{\lambda^2} \\
&= \mathrm{Res}_{\lambda=0} (\lambda^{i-j-1} + \lambda^{-i+j-1}) d\lambda \\
&= 2 \cdot \delta_{ij}.
\end{aligned}$$

This shows $\mathrm{span}_{\mathbb{C}}\{[h_1], \dots, [h_g]\} = H^1(Y, \mathcal{O})$ and concludes the proof. □

As a direct consequence of Lemma 4.5 we get

Corollary 4.6. *Any element $[f] \in H^1(Y, \mathcal{O})$ can be represented by a function $f_0(\lambda, \nu)$ that is given by*

$$f_0(\lambda, \nu) = \sum_{i=0}^{g-1} c_i \lambda^{-i-1} \nu \quad \textit{with} \quad (c_0, \dots, c_{g-1}) \in \mathbb{C}^g$$

and thus $[f] \stackrel{!}{=} [(f_0, -f_0)] = [(\sum_{i=0}^{g-1} c_i \lambda^{-i-1} \nu, -\sum_{i=0}^{g-1} c_i \lambda^{-i-1} \nu)]$.

Remark 4.7. *Recall the following divisor equations (written in multiplicative form)*

$$(\lambda) = \frac{y_0^2}{y_\infty^2}, \quad (d\lambda) = \frac{\alpha_1 \cdot \ldots \cdot \alpha_{2g} y_0}{y_\infty^3}, \quad (\nu) = \frac{\alpha_1 \cdot \ldots \cdot \alpha_{2g} y_0}{y_\infty^{2g+1}}.$$

Considering the function $\lambda^{-i-1}\nu$ we get for $0 \le i \le g-1$

$$\left(\lambda^{-i-1}\nu\right) = \frac{\alpha_1 \cdot \ldots \cdot \alpha_{2g} y_0 y_\infty^{2i+2}}{y_\infty^{2g+1} y_0^{2i+2}} = \frac{\alpha_1 \cdot \ldots \cdot \alpha_{2g}}{y_\infty^{2g-2i-1} y_0^{2i+1}}$$

and thus the expressions in $(f_0, f_\infty) = (\sum_{i=0}^{g-1} c_i \lambda^{-i-1}\nu, -\sum_{i=0}^{g-1} c_i \lambda^{-i-1}\nu)$ have the right behaviour with respect to the poles at $\lambda = 0$ and $\lambda = \infty$. Summing up we see that $\xi[(\mathfrak{h}^-_{finite})^2] \subset C^0(\mathcal{U}, \mathcal{M})$ can be parametrized by

$$f = (f_0, 0, f_\infty) = \left(\sum_{i=0}^{g-1} c_i \lambda^{-i-1} \nu, 0, -\sum_{i=0}^{g-1} c_i \lambda^{-i-1} \nu \right)$$

with $(c_0, \dots, c_{g-1}) \in \mathbb{C}^g$.

Remark 4.8. *Similarily as in Lemma 4.5 all equivalence classes $[\widetilde{h}_i]$ of the g tuples $\widetilde{h}_i := (\widetilde{f}_0^i, \widetilde{f}_\infty^i)$ given by $\widetilde{h}_i = (\nu\lambda^{-i}, \nu\lambda^{-i})$ are equal to $[0] \in H^1(Y, \mathcal{O})$ since the $\widetilde{h}_i$ correspond to solvable Mittag-Leffler distributions $(\nu\lambda^{-i}, 0, \nu\lambda^{-i})$ that are solved by the global meromorphic function $\nu\lambda^{-i}$. In particular we see that the equivalence classes of*

$$\widehat{h}_i := h_i + \widetilde{h}_i = (2\nu\lambda^{-i}, 0) \quad \textit{for } i = 1, \dots, g$$

also define a basis of $H^1(Y, \mathcal{O})$ since $[\widetilde{h}_i] = [0]$ and thus $[\widehat{h}_i] = [h_i] + [\widetilde{h}_i] = [h_i]$.

4.2 The Krichever construction

The construction procedure for linear flows on $\mathrm{Pic}_0(Y)$ is due to Krichever [38] and shall be described in the following. In [43] McIntosh desribes the Krichever construction for finite type solutions of the sinh-Gordon equation. Every $h = (h_0, h_\infty) \in (\mathfrak{h}^-_{\text{finite}})^2$ defines a one-parameter family $L_h(t)$ in $\mathrm{Pic}_0(Y)$ with the cocycles $k^* \exp(th_0)$ and $\widetilde{k}^* \exp(th_\infty)$ over $U_0 \backslash \{y_0\}$ and $U_\infty \backslash \{y_\infty\}$ respectively. This corresponds to the assignment

$$\mathbb{C} \ni t \mapsto \exp(2\pi i \varphi(th)) =: L_h(t) \in H^1(Y, \mathcal{O}^*)$$

since $\varphi : (\mathfrak{h}^-_{\text{finite}})^2 \to H^1(Y, \mathcal{O})$ maps into the Lie algebra of $\mathrm{Pic}(Y) \simeq H^1(Y, \mathcal{O}^*)$. In particular one has $L_h(t + t') = L_h(t) \otimes L_h(t')$ and therefore

$$\deg(L_h(t + t')) = \deg(L_h(t) \otimes L_h(t')) = \deg(L_h(t)) + \deg(L_h(t')).$$

Since $\deg(L_h(0)) = 0$ this flow stays in $\mathrm{Pic}_0(Y)$, i.e. $\deg(L_h(t)) = 0$ for all t, and defines a one-parameter group. Conversely every one-parameter group in $\mathrm{Pic}_0(Y)$ is obtained that way.

The following lemma describes the relationship between these flows and Mittag-Leffler distributions.

Lemma 4.9.

(i) An element $h = (h_0, h_\infty) \in (\mathfrak{h}^-_{\mathit{finite}})^2$ induces a trivial flow, i.e. $L_h(t)$ is trivial for all $t \in \mathbb{C}$, if and only if the corresponding M-L distribution is solvable.

(ii) An element $h = (h_0, h_\infty) \in (\mathfrak{h}^-_{\mathit{finite}})^2$ induces a periodic flow, i.e. $L_h(\mathbf{p}) \cong \mathbb{1}$, if and only if the M-L distribution can be solved by means of a multi-valued meromorphic function p whose values at a point differ by an element of $\frac{2\pi i}{\mathbf{p}}\mathbb{Z}$. In particular dp is an Abelian differential of the second kind with $\int_\gamma dp \in \frac{2\pi i}{\mathbf{p}}\mathbb{Z}$ for all $\gamma \in H_1(Y, \mathbb{Z})$.

Proof.

(i) The bundle $L_h(t)$ is trivial for all $t \in \mathbb{C}$ if and only if

$$H^1(Y, \mathcal{O}^*) \ni 1 = \exp(2\pi i \varphi(th)).$$

Since the kernel of $\exp(2\pi i\cdot)$ is discrete, this is equivalent to $th \in \ker(\varphi)$ for all $t \in \mathbb{C}$. Now Lemma 4.4 implies that $h \in \ker(\varphi)$ if and only if the M-L distribution is solvable.

(ii) The condition $L_h(\mathbf{p}) \cong \mathbb{1}$ (and thus that $L_h(\mathbf{p})$ is a trivial holomorphic line bundle) is equivalent to the existence of a nowhere vanishing holomorphic section, i.e. there exist nowhere vanishing holomorphic functions g on $Y\backslash\{y_0, y_\infty\}$ and g_0, g_∞ on U_0, U_∞ such that

$$g = k^* \exp(\mathbf{p} \cdot h_0) g_0 \text{ on } U_0\backslash\{y_0\}$$

and

$$g = \widetilde{k}^* \exp(\mathbf{p} \cdot h_\infty) g_\infty \text{ on } U_\infty\backslash\{y_\infty\}.$$

Then $p := \frac{1}{\mathbf{p}} \ln g$ is a multi-valued meromorphic function that satisfies

$$dp = \frac{1}{\mathbf{p}} \frac{dg}{g} = \frac{d}{dk} h_0(k) + \frac{1}{\mathbf{p}} \frac{dg_0}{g_0} \text{ on } U_0\backslash\{y_0\}$$

and a similar equation holds around y_∞. Since h_0 and h_∞ have poles of order at least one we see that dp is an Abelian differential of the second kind with the desired properties.

$\square$

4.3 A reality condition on $H^1(Y,\mathcal{O})$

Since we are dealing with real Cauchy data (u, u_y) the spectral curve $Y(u, u_y)$ has an anti-holomorphic involution η and we may ask which conditions are imposed by η on the different objects we are dealing with. We start with the following

Definition 4.10. *Let $H^1_{\mathbb{R}}(Y,\mathcal{O}) := \{[f] \in H^1(Y,\mathcal{O}) \mid \overline{\eta^*[f]} = [f]\}$ be the real part of $H^1(Y,\mathcal{O})$ with respect to the involution η and $Pic_0^{\mathbb{R}}(Y) := \{[g] \in Pic_0(Y) \mid [\overline{\eta^* g}] = [g]\}$ the corresponding real part of $Pic_0(Y)$.*

Lemma 4.11.

(i) For the line bundle $L(x)$ defined by $E(T_x(u, u_y)) \simeq E(u, u_y) \otimes L(x)$ we have $L(x) \in Pic_0^{\mathbb{R}}(Y(u, u_y))$ for all $x \in \mathbb{R}$.

(ii) The cocycle $[f]$ corresponding to the line bundle $L(x)$ lies in the real part of $H^1(Y,\mathcal{O})$ with respect to the involution η, i.e. there holds $[f] \in H^1_{\mathbb{R}}(Y(u, u_y),\mathcal{O})$.

Proof.

(i) From the proof of Corollary 3.52 we know that

$$v((\lambda,\mu), x) = e^{\frac{x}{\mathbf{p}} \ln \mu} F_\lambda^{-1}(x) v(\lambda,\mu)$$

around $\lambda = 0$ and $\lambda = \infty$. The cocycles $(g_0, g_\infty) \in H^1(Y,\mathcal{O}^*)$ for $L(x)$ are given by $((e^{\frac{x}{\mathbf{p}} \ln \mu}, U_0), (e^{\frac{x}{\mathbf{p}} \ln \mu}, U_\infty))$. Since $\eta^* \overline{\ln \mu} = \ln \mu$ the claim follows.

(ii) Due to (i) we know that $L(x) \in \mathrm{Pic}_0^{\mathbb{R}}(Y(u,u_y))$. Applying the inverse of the exponential map we see that for the cocycle $[f]$ corresponding to the line bundle $L(x)$ there holds $[f] \in H^1_{\mathbb{R}}(Y(u,u_y),\mathcal{O})$.

□

Remark 4.12. *We want to state an important remark concerning the involutions σ, ρ, η on Y and their behaviour on U_0, U_∞, i.e. around the distinguished points $\lambda = 0$ and $\lambda = \infty$. From Corollary 3.12 we know that*

$$\ln\mu \sim \begin{cases} \frac{i}{\sqrt{\lambda}} = \frac{i}{k} & \text{around } \lambda = 0, \\ i\sqrt{\lambda} = \frac{i}{\widetilde{k}} & \text{around } \lambda = \infty. \end{cases}$$

On the other hand there holds

$$\sigma^* \ln\mu = -\ln\mu, \quad \overline{\rho^* \ln\mu} = -\ln\mu, \quad \overline{\eta^* \ln\mu} = \ln\mu.$$

Inserting the local expressions for $\ln\mu$ with respect to the charts $k, \widetilde{k}$ we obtain

$$\sigma^* k = -k, \quad \sigma^* \widetilde{k} = -\widetilde{k}, \quad \overline{\rho^* k} = \widetilde{k}, \quad \overline{\eta^* k} = -\widetilde{k}.$$

The above formulas will be important in the following lemma.

Lemma 4.13. *The tuple $h = (h_0, h_\infty) \in (\mathfrak{h}^-_{\mathit{finite}})^2$ corresponding to the function $\ln\mu$ satisfies the reality condition $h_\infty(\widetilde{k}) = \eta^* \overline{h_0(k)}$ with respect to the involution η.*

Proof. From the previous Remark 4.12 we know that $\overline{\eta^* k} = -\widetilde{k}$. From Corollary 3.12 we also obtain

$$\begin{aligned} \ln\mu &= \frac{1}{\sqrt{\lambda}}\frac{i\mathbf{p}}{2} + \sqrt{\lambda}\int_0^{\mathbf{p}} \left(-i(\partial u)^2 + \tfrac{i}{2}\cosh(2u)\right) dt + O(\lambda) \quad \text{at } \lambda = 0, \\ \ln\mu &= \sqrt{\lambda}\frac{i\mathbf{p}}{2} + \frac{1}{\sqrt{\lambda}}\int_0^{\mathbf{p}} \left(-i(\bar\partial u)^2 + \tfrac{i}{2}\cosh(2u)\right) dt + O(\lambda^{-1}) \quad \text{at } \lambda = \infty. \end{aligned}$$

Since $\mathbf{p} \in \mathbb{R}$ we get for the leading terms of this expansion

$$\mathbf{p} \cdot h_\infty(\widetilde{k}) = \frac{1}{\widetilde{k}}\frac{i\mathbf{p}}{2} = \eta^* \overline{\frac{1}{k}\frac{i\mathbf{p}}{2}} = \mathbf{p}(\eta^* \overline{h_0(k)})$$

and the claim is proved. In particular we have $h_0(k) = \frac{1}{k}\frac{i}{2}$ and $h_\infty(\widetilde{k}) = \frac{1}{\widetilde{k}}\frac{i}{2}$. □

The following lemma shows which conditions are imposed on an element $[f] \in H^1_{\mathbb{R}}(Y,\mathcal{O})$ and that $\dim_{\mathbb{R}} H^1_{\mathbb{R}}(Y,\mathcal{O}) = g$.

Lemma 4.14. *An element $[f] = [(f_0, f_\infty)] \in H^1(Y,\mathcal{O})$ satisfies $\overline{\eta^*[f]} = [f]$ if and only if $f_\infty = \eta^* \bar f_0$. The corresponding representative $f_\infty \overset{!}{=} -f_0 = -\sum_{i=0}^{g-1} c_i \lambda^{-i-1}\nu$ satisfies $\bar c_i = -c_{g-1-i}$ for $i = 0, \ldots, g-1$. This defines a real g-dimensional subspace of $\mathbb{C}^g$. In particular $\dim_{\mathbb{R}} H^1_{\mathbb{R}}(Y,\mathcal{O}) = g$.*

Proof. The first part of the lemma is obvious. Now a direct calculation gives

$$\begin{aligned}-\eta^*(\sum_{i=0}^{g-1}\overline{c_i\lambda^{-i-1}\nu}) &= -\sum_{i=0}^{g-1}\bar{c}_i\lambda^{i+1}\lambda^{-g-1}\nu = -\sum_{i=0}^{g-1}\bar{c}_i\lambda^{i-g}\nu \\ &\overset{j:=g-i-1}{=} -\sum_{j=0}^{g-1}\bar{c}_{g-1-j}\lambda^{-j-1}\nu \overset{!}{=} \sum_{i=0}^{g-1}c_i\lambda^{-i-1}\nu\end{aligned}$$

if and only if $\bar{c}_i = -c_{g-1-i}$ for $i=0,\dots,g-1$. The subspace of elements $(c_0,\dots,c_{g-1})\in\mathbb{C}^g$ that obey these conditions is a real g-dimensional subspace. $\square$

Corollary 4.15. *Any element $[f]=[(f_0,\eta^*\bar{f}_0)]\in H^1_{\mathbb{R}}(Y,\mathcal{O})$ can be represented by $f_0(\lambda,\nu)$ with*

$$f_0(\lambda,\nu)=\sum_{i=0}^{g-1}c_i\lambda^{-i-1}\nu$$

and $\bar{c}_i=-c_{g-1-i}$ for $i=0,\dots,g-1$. In that case the function $f_0(\lambda,\nu)$ can be written as

$$f_0(\lambda,\nu)=\frac{1}{2}\left(\sum_{i=0}^{g-1}c_i\lambda^{-i-1}\nu-\eta^*(\sum_{i=0}^{g-1}\overline{c_i\lambda^{-i-1}\nu})\right).$$

Remark 4.16. *We already saw that the equivalence class of*

$$\widetilde{f}=(\widetilde{f}_0,\widetilde{f}_\infty)\overset{!}{=}(\widetilde{f}_0,\widetilde{f}_0)=\left(\sum_{i=0}^{g-1}\widetilde{c}_i\lambda^{-i-1}\nu,\sum_{i=0}^{g-1}\widetilde{c}_i\lambda^{-i-1}\nu\right)$$

is equal to $[0]\in H^1(Y,\mathcal{O})$ since $\widetilde{f}$ corresponds to a solvable Mittag-Leffler distribution. From Lemma 4.14 we know that $\eta^*[\overline{\widetilde{f}}]=[\widetilde{f}]$ if and only if $\widetilde{f}_\infty=\eta^*\overline{\widetilde{f}_0}$. The corresponding representative $\widetilde{f}_\infty\overset{!}{=}\widetilde{f}_0=\sum_{i=0}^{g-1}\widetilde{c}_i\lambda^{-i-1}\nu$ satisfies $\bar{\widetilde{c}}_i=\widetilde{c}_{g-1-i}$ for $i=0,\dots,g-1$. By choosing $\widetilde{c}_i:=c_i$ for $i=0,\dots,g-1$ we can deduce that any element of $H^1_{\mathbb{R}}(Y,\mathcal{O})$ can be respresented by

$$\widehat{f}=f+\widetilde{f}=\left(2\sum_{i=0}^{\frac{g}{2}-1}c_i\lambda^{-i-1}\nu,2\sum_{i=0}^{\frac{g}{2}-1}\bar{c}_i\lambda^{i-g}\nu\right)$$

since $[\widetilde{f}]=[0]$ and thus $[\widehat{f}]=[f]+[\widetilde{f}]=[f]$.

From the previous discussions about Krichever's construction procedure for one-dimensional subgroups of $\mathrm{Pic}_0^{\mathbb{R}}(Y)$ we immediately get (see [35], Proposition 2.8)

Proposition 4.17. *Cauchy data (u,u_y) of finite type correspond to a periodic solution of the* sinh-*Gordon equation if and only if*

(i) There exists a meromorphic differential $d\ln\mu$ of the second kind on the spectral curve Y with second order poles at the points y_0 and y_∞.

(ii) The differential $d\ln\mu$ is the logarithmic derivative of a function μ on Y which transforms under the involutions in the following way:

$$\sigma^*\mu = \mu^{-1}, \ \ \rho^*\mu = \bar{\mu}^{-1}, \ \ \eta^*\mu = \bar{\mu}.$$

Three classes of integrable systems. The Krichever construction provides us with a way to distinguish three classes of integrable systems with respect to the following data: a compact Riemann surface Y, the spectral curve; points y_0 and y_∞ on Y; and conformal charts $k, \widetilde{k}$ that are centered at these points together with two elements $h = (h_0, h_\infty)$ and $\widetilde{h} = (\widetilde{h}_0, \widetilde{h}_\infty) \in (\mathfrak{h}^-_{\text{finite}})^2$.

(I) (*Finite dimensional case*) The first class corresponds to finite dimensional integrable systems. It is characterized by the property that both flows that are induced by h and $\widetilde{h}$ are trivial. The corresponding Lax operators are matrices.

(II) (*Simply periodic case*) The second class is characterized by a trivial flow induced by $\widetilde{h}$ and a periodic flow induced by h. The corresponding Lax operators are ordinary differential operators.

(III) (*Doubly periodic case*) The third class is characterized by the property that both flows induced by h and $\widetilde{h}$ are periodic. The corresponding Lax operators are partial differential operators.

Remark 4.18. *In the present case we have $\widetilde{h} = (\frac{1}{z^2}, \frac{1}{z^2})$ and $h = (\frac{i}{2z}, \frac{i}{2z})$. This corresponds to the second class (II). Here the function λ corresponds to $\widetilde{h}$ and $\ln\mu$ corresponds to h. In the present notation h corresponds to the Mittag-Leffler distribution $\xi = (\frac{\mathbf{p}\nu}{\lambda}, 0, \frac{\mathbf{p}\nu}{\lambda^g})$. Then*

$$\delta\xi = (\tfrac{\mathbf{p}\nu}{\lambda} - 0, \tfrac{\mathbf{p}\nu}{\lambda^g} - 0) = (\tfrac{\mathbf{p}\nu}{\lambda}, \tfrac{\mathbf{p}\nu}{\lambda^g}) \in H^1_{\mathbb{R}}(Y, \mathcal{O}).$$

Moreover, $\ln\mu - \frac{\mathbf{p}\nu}{\lambda}$ is holomorphic on U_0 and $\ln\mu - \eta^*\overline{(\frac{\mathbf{p}\nu}{\lambda})} = \ln\mu - \frac{\mathbf{p}\nu}{\lambda^g}$ is holomorphic on U_∞. Thus h induces a periodic flow in $Pic^{\mathbb{R}}_0(Y)$.

4.4 The Baker-Akhiezer function

In order to tackle the inverse problem one has to implement a procedure that yields Cauchy data (u, u_y) of a periodic finite type solution of the sinh-Gordon equation from given spectral data (Y, D) with certain properties. The function $\lambda : Y \to \mathbb{CP}^1$ corresponds to the trivial flow $L_{\widetilde{h}}(t)$ on $\mathrm{Pic}_0(Y)$ that is induced by the solvable Mittag-Leffler distribution $\widetilde{h} = (\widetilde{h}_0, \widetilde{h}_\infty) = (\frac{1}{z^2}, \frac{1}{z^2})$.

By choosing a second element $h = (h_0, h_\infty) \in (\mathfrak{h}^-_{\text{finite}})^2$ that corresponds to a periodic flow it is possible to determine a map that has the same analytic properties as the map

$\psi((\lambda,\mu),x) = F_\lambda^{-1}(x)v(\lambda,\mu)$ completely.

Recall that Lemma 3.51 provides us with the asymptotic expansions

$$\psi = \exp\left(\frac{-ix}{2\sqrt{\lambda}}\right)\left(\begin{pmatrix}1\\ e^{-u(x)}\sqrt{\lambda}\end{pmatrix} + O(\lambda)\right) \text{ at } \lambda = 0$$

and

$$\psi = \exp\left(\frac{-ix\sqrt{\lambda}}{2}\right)\left(\begin{pmatrix}1\\ e^{u(x)}\sqrt{\lambda}\end{pmatrix} + O(\lambda^{-1})\right) \text{ at } \lambda = \infty.$$

The modified Baker-Akhiezer function. We will now modify ψ in order to obtain an object that will play the role of the Baker-Akhiezer function in the following. Setting

$$\psi^0 = \exp\left(\frac{ix}{2\sqrt{\lambda}}\right)\psi \text{ at } \lambda = 0 \quad\text{and}\quad \psi^\infty = \exp\left(\frac{ix\sqrt{\lambda}}{2}\right)\psi \text{ at } \lambda = \infty$$

we see from the above expansions that ψ has two disadvantages: On the one hand ψ^0 and ψ^∞ are x-dependent at $\lambda = 0$ and $\lambda = \infty$ respectively. On the other hand ψ^∞ is not holomorphic around $\lambda = \infty$ in contrast to the holomorphic ψ^0 (around $\lambda = 0$). By gauging the frame F_λ with the constant matrix $T = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix}1 & -1\\ 1 & 1\end{smallmatrix}\right)$ we get

$$\widetilde{U}_\lambda = T^{-1}U(\zeta_\lambda)T \quad\text{and}\quad \widetilde{\zeta}_\lambda = T^{-1}\zeta_\lambda T$$

for the corresponding Killing field that solves $\frac{d}{dx}\widetilde{\zeta}_\lambda = [\widetilde{\zeta}_\lambda, \widetilde{U}_\lambda]$. Setting $\zeta_\lambda = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & -\alpha\end{smallmatrix}\right)$ we obtain

$$\widetilde{\zeta}_\lambda = \begin{pmatrix}\widetilde{\alpha} & \widetilde{\beta}\\ \widetilde{\gamma} & -\widetilde{\alpha}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\beta+\gamma & -2\alpha+\beta-\gamma\\ -2\alpha-\beta+\gamma & -\beta-\gamma\end{pmatrix}.$$

Denoting the eigenvalue $\widetilde{\nu} = \lambda^{-1}\nu$ of ξ_λ by ν, we have

$$\widetilde{v}_+(x) = \begin{pmatrix}1\\ \frac{\nu-\widetilde{\alpha}}{\widetilde{\beta}}\end{pmatrix} = \begin{pmatrix}1\\ \frac{2\nu-\beta-\gamma}{-2\alpha+\beta-\gamma}\end{pmatrix}$$

and the expansion

$$\frac{2\nu-\beta-\gamma}{-2\alpha+\beta-\gamma} = \begin{cases}\frac{-\beta_{-1}+i\sqrt{\lambda}+O(\lambda)}{\beta_{-1}+O(\lambda)} = -1 + \frac{i}{\beta_{-1}}\sqrt{\lambda} + O(\lambda) & \text{at } \lambda = 0,\\ \frac{\bar{\gamma}_0+i/\sqrt{\lambda}+O(1/\lambda)}{\bar{\gamma}_0+O(1/\lambda)} = 1 + \frac{i}{\bar{\gamma}_0\sqrt{\lambda}} + O(1/\lambda) & \text{at } \lambda = \infty.\end{cases} \tag{4.4.1}$$

If we repeat the steps from the proof of Lemma 3.51 we arrive at the following

Lemma 4.19. *The map $\widetilde{\psi}_+ := T^{-1}F_\lambda^{-1}(x)\widetilde{v}_+(0)$ has the expansions*

$$\widetilde{\psi}_+ = \begin{cases}\exp\left(\frac{-ix}{2\sqrt{\lambda}}\right)\begin{pmatrix}1\\ -1+O(\sqrt{\lambda})\end{pmatrix} & \textit{at } \lambda = 0,\\ \exp\left(\frac{-ix\sqrt{\lambda}}{2}\right)\begin{pmatrix}1\\ 1+O(1/\sqrt{\lambda})\end{pmatrix} & \textit{at } \lambda = \infty.\end{cases}$$

Proof. Obviously $\widetilde{\psi}_+(x)$ is an eigenvector of $\widetilde{\zeta}_\lambda(x) = T^{-1}\zeta_\lambda(x)T$ and it is collinear to $\widetilde{v}_+(x)$. This defines a function $f(\lambda, x)$ such that

$$f(\lambda, x)\widetilde{v}_+(x) = \widetilde{\psi}_+(x) \tag{4.4.2}$$

holds. Differentiating equation (4.4.2) we obtain

$$(\tfrac{d}{dx}f)\widetilde{v}_+ + f(\tfrac{d}{dx}\widetilde{v}_+) = -\widetilde{U}_\lambda\widetilde{\psi}_+$$

with $\widetilde{U}_\lambda = T^{-1}U(\zeta_\lambda)T$ and thus

$$f^{-1}(\tfrac{d}{dx}f)\widetilde{v}_+ = -\widetilde{U}_\lambda\widetilde{v}_+ - \tfrac{d}{dx}\widetilde{v}_+.$$

Moreover, setting $U(\zeta_\lambda) = \left(\begin{smallmatrix} U_{11} & U_{12} \\ U_{21} & -U_{11} \end{smallmatrix}\right)$ and considering the first entry of the last vector equation we obtain

$$f^{-1}(\tfrac{d}{dx}f) = -\tfrac{1}{2}(U_{12} + U_{21}) - \tfrac{1}{2}(-2U_{11} + U_{12} - U_{21})\tfrac{\nu - \widetilde{\alpha}}{\widetilde{\beta}}. \tag{4.4.3}$$

Writing out equation (4.4.3) we get

$$f^{-1}(\tfrac{d}{dx}f) = -\tfrac{1}{2}(\lambda^{-1}\beta_{-1} + \gamma_0 - \bar{\gamma}_0 - \lambda\bar{\beta}_{-1}) - \tfrac{1}{2}(2\bar{\alpha}_0 - 2\alpha_0 + \lambda^{-1}\beta_{-1} - \gamma_0 - \bar{\gamma}_0 + \lambda\bar{\beta}_{-1})\tfrac{\nu - \widetilde{\alpha}}{\widetilde{\beta}}$$

and thus inserting the expansion (4.4.1) into equation (4.4.3) gives

$$f^{-1}(\tfrac{d}{dx}f) = \begin{cases} -\lambda^{-1}\beta_{-1}\frac{i\sqrt{\lambda}}{2\beta_{-1}} + O(1) = -\frac{i}{2\sqrt{\lambda}} + O(1) & \text{at } \lambda = 0, \\ -\bar{\gamma}_0\frac{i\sqrt{\lambda}}{2\bar{\gamma}_0} + O(1) = -\frac{i\sqrt{\lambda}}{2} + O(1) & \text{at } \lambda = \infty. \end{cases}$$

Now integration of $f^{-1}(\frac{d}{dx}f)$ yields the claim, since $\widetilde{\psi}_+(x) = f(\lambda, x)\widetilde{v}_+(x)$. □

Remark 4.20. *Since T is invertible we see that the divisor $\widetilde{D}(x)$ corresponding to $\widetilde{\psi}(x)$ is equivalent to the divisor $D(x)$ that corresponds to $\psi(x)$. In particular $\deg(\widetilde{D}(x)) = \deg(D(x)) = g+1$ for all $x \in \mathbb{R}$. Moreover, the above gauge leads to constant values of $\widetilde{\psi}^0$ and $\widetilde{\psi}^\infty$ at y_0 and y_∞ and has in addition the advantage that $y_0, y_\infty \notin \widetilde{D}(x)$ for all $x \in \mathbb{R}$.*

We omit the tilde in the following and arrive at the following characterization of the entries ψ_i appearing in $\psi = (\psi_1, \psi_2)^t$:

(i) For fixed $x \in \mathbb{R}$ there holds $(\psi_i) \geq -D$ on $Y^* = Y \backslash \{y_0, y_\infty\}$.

(ii) $\psi_i^0 = \exp\left(\frac{ix}{2\sqrt{\lambda}}\right)\psi_i$ is holomorphic at $\lambda = 0$ and $\psi_i^\infty = \exp\left(\frac{ix\sqrt{\lambda}}{2}\right)\psi_i$ is holomorphic at $\lambda = \infty$.

For fixed $x \in \mathbb{R}$, conditions (i) and (ii) imply that ψ is a holomorphic section of the bundle $\mathcal{O}_D$ for some divisor D of degree $g+1$.

Lemma 4.21. *Let D be a divisor of degree $g+1$ on Y and assume* $\dim H^0(Y, \mathcal{O}_{D-y_0-y_\infty}) = 0$ *or* $\dim H^0(Y, \mathcal{O}_{D-2\cdot y_0}) = 0$. *Then* $\dim H^0(Y, \mathcal{O}_{D-y_0}) = 1 = \dim H^0(Y, \mathcal{O}_{D-y_\infty})$.

Remark 4.22. *Note that $D-2y_0$ and $D-2y_\infty$ are equivalent divisors, that is $D-2y_\infty-(D-2y_0) = 2y_0 - 2y_\infty = (\lambda)$ and thus*

$$H^0(Y, \mathcal{O}_{D-2\cdot y_0}) = 0 \iff H^0(Y, \mathcal{O}_{D-2\cdot y_\infty}) = 0.$$

Proof of Lemma 4.21. First assume that $\dim H^0(Y, \mathcal{O}_{D-y_0-y_\infty}) = 0$ holds. Due to the Riemann-Roch Theorem 2.22 we get

$$\dim H^0(Y, \mathcal{O}_{D-y_0}) = 1 - g + \deg(D-y_0) + i(D-y_0) = 1 + i(D-y_0) \geq 1.$$

Suppose $\dim H^0(Y, \mathcal{O}_{D-y_0}) > 1$. Then there exist two linearly independent sections $\psi_1, \psi_2 \in H^0(Y, \mathcal{O}_{D-y_0})$ with $\psi_1(y_0) = 0 = \psi_2(y_0)$. Since $\dim H^0(Y, \mathcal{O}_{D-y_0-y_\infty}) = 0$ there holds $a = \psi_1(y_\infty) \neq 0$ and $b = \psi_2(y_\infty) \neq 0$. But then $(b\psi_1 - a\psi_2)(y_\infty) = 0$, i.e. $b\psi_1 - a\psi_2 \in H^0(Y, \mathcal{O}_{D-y_0-y_\infty}) \stackrel{!}{=} 0$. This implies that $b\psi_1 - a\psi_2 \equiv 0$ and thus $\psi_2 = \frac{b}{a}\psi_1$ - a contradiction. This implies $\dim H^0(Y, \mathcal{O}_{D-y_0}) = 1$. The same reasoning leads to $\dim H^0(Y, \mathcal{O}_{D-y_\infty}) = 1$.

Now let us assume that $\dim H^0(Y, \mathcal{O}_{D-2\cdot y_0}) = 0$ holds. Suppose again that we have $\dim H^0(Y, \mathcal{O}_{D-y_0}) > 1$ and that there exist two linearly independent sections $\psi_1, \psi_2 \in H^0(Y, \mathcal{O}_{D-y_0})$ with $\psi_1(y_0) = 0 = \psi_2(y_0)$. Since $\dim H^0(Y, \mathcal{O}_{D-2\cdot y_0}) = 0$ the derivatives of ψ_1 and ψ_2 at y_0 are given by $a = \psi_1'(y_0) \neq 0$ and $b = \psi_2'(y_0) \neq 0$. But then

$$(b\psi_1 - a\psi_2)'(y_0) = (b\psi_1' - a\psi_2')(y_0) = 0,$$

i.e. $b\psi_1 - a\psi_2 \in H^0(Y, \mathcal{O}_{D-2y_0}) \stackrel{!}{=} 0$. This implies that $b\psi_1 - a\psi_2 \equiv 0$ and thus $\psi_2 = \frac{b}{a}\psi_1$ - a contradiction. This implies $\dim H^0(Y, \mathcal{O}_{D-y_0}) = 1$. Due to Remark 4.22 there holds $H^0(Y, \mathcal{O}_{D-2\cdot y_0}) = 0$ if and only if $H^0(Y, \mathcal{O}_{D-2\cdot y_\infty}) = 0$. Then an analogous argumentation leads to $\dim H^0(Y, \mathcal{O}_{D-y_\infty}) = 1$. $\square$

Thus for a fixed value of $x \in \mathbb{R}$ the map ψ_i obeying conditions (i) and (ii) is uniquely determined by either

(1) the value of ψ_i^0 at y_0 and the value of ψ_i^∞ at y_∞ or

(2) the value of ψ_i^0 at y_0 and its first derivative at y_0 or

(3) the value of ψ_i^∞ at y_∞ and its first derivative at y_∞.

Remark 4.23. *The gauge with $T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ leads to $\psi = T^{-1}F_\lambda^{-1}(x)\widetilde{v}(0)$ and its entries ψ_i are uniquely determined by the value of ψ_i^0 at y_0 and the value of ψ_i^∞ at y_∞. This corresponds to condition (1). By choosing another gauge one can find normalizations that correspond to the conditions (2) and (3).*

Definition 4.24. *Let Pic(Y) be the Picard variety of Y and let*

$$Pic^{\mathbb{R}}(Y) := \{D \in Pic(Y) \,|\, \eta(D) - D = (f) \text{ for a merom. } f \text{ with } f\eta^*\bar{f} = -1\}$$

be the set of ***quaternionic divisors*** *with respect to the involution η.*

Note that the condition on D from Definition 4.24 arises from the reality condition on α_λ (or equivalently ξ_λ). Nevertheless Proposition 4.25 justifies the notion "quaternionic divisor". With this definition at hand we see that the violation of condition (1) corresponds to divisors D that are contained in the set

$$S_1 := \{D \in \mathrm{Pic}_{g+1}(Y) \cap \mathrm{Pic}^{\mathbb{R}}(Y) \,|\, \dim H^0(Y, \mathcal{O}_{D-y_0-y_\infty}) \neq 0\}.$$

Here $\mathrm{Pic}_{g+1}(Y)$ is the connected component of divisors D in $\mathrm{Pic}(Y)$ with $\deg(D) = g+1$. Likewise the violations of conditions (2) and (3) are described by

$$S_2 := \{D \in \mathrm{Pic}_{g+1}(Y) \cap \mathrm{Pic}^{\mathbb{R}}(Y) \,|\, \dim H^0(Y, \mathcal{O}_{D-2\cdot y_0}) \neq 0\}$$

and

$$S_3 := \{D \in \mathrm{Pic}_{g+1}(Y) \cap \mathrm{Pic}^{\mathbb{R}}(Y) \,|\, \dim H^0(Y, \mathcal{O}_{D-2\cdot y_\infty}) \neq 0\}.$$

We need some preparation in order to prove that $S_i = \emptyset$ for $i = 1, 2, 3$ in our situation.

Proposition 4.25. *Let D be a divisor on Y that satisfies $D - \eta(D) = (f)$ for a meromorphic function f with $f\overline{\eta^* f} = -1$. Then the space of holomorphic sections $H^0(Y, \mathcal{O}_D)$ is a quaternionic vector space with quaternionic structure given by $j : h \mapsto j(h) = \frac{1}{f}\eta^*\bar{h}$. In particular there holds $\dim H^0(Y, \mathcal{O}_D) = 2n$ with $n \in \mathbb{N}_0$.*

Proof. We only have to verify that the anti-linear map $j : H^0(Y, \mathcal{O}_D) \to H^0(Y, \mathcal{O}_D)$, $h \mapsto j(h) = \frac{1}{f}\eta^*\bar{h}$ (cf. [30]) satisfies $j^2 = -\mathrm{id}$. A direct calculation gives

$$j^2(h) = \tfrac{1}{f}\eta^*\overline{\tfrac{1}{f}\overline{\eta^* h}} = \tfrac{1}{f\overline{\eta^* f}}h = -h.$$

This shows $j^2 = -\mathrm{id}$ and concludes the proof. □

Lemma 4.26. *Let Y be a hyperelliptic Riemann surface of genus g with $\lambda : Y \to \mathbb{CP}^1$ of degree 2 and branch points over $\lambda = 0$ (y_0) and $\lambda = \infty$ (y_∞). Let η be an anti-holomorphic involution on Y without fixed points such that $\eta^*\bar{\lambda} = \lambda^{-1}$. Moreover, let D be a divisor of degree $g-1$ on Y with $\eta(D) - D = (f)$ for a meromorphic function f obeying $f\eta^*\bar{f} = -1$. Then one has $H^0(Y, \mathcal{O}_D) = 0$.*

Proof. First we prove that $\deg(f) > g$ holds. For this let us assume that $\deg(f) \leq g$ and show that this yields a contradiction. Due to Proposition III.7.10 in [20] the function f must be of even degree. Moreover the proof of that proposition shows that f is a rational function of λ. But then the condition $f\eta^*\bar{f} = -1$ is violated and thus $\deg(f) > g$ must hold. Now suppose that $\dim H^0(Y, \mathcal{O}_D) > 0$, i.e. $\dim H^0(Y, \mathcal{O}_D) \geq 2$ due to

Proposition 4.25. Then there exists a meromorphic function h that satisfies $(h) \geq -D$ and an equivalent effective divisor $D' = D + (h) \geq 0$ with $\deg(D') = \deg(D) = g - 1$. Moreover, there exists a corresponding meromorphic function f' such that

$$\eta(D') - D' = (f') \text{ with } f'\eta^*\bar{f}' = -1.$$

Since $D' \geq 0$ we get $\deg(f') \leq g - 1 < g$, i.e. a contradiction! Thus there must hold $H^0(Y, \mathcal{O}_D) = 0$ and the claim is proved. □

With the help of Lemma 4.26 we get

Proposition 4.27. *There holds $S_i = \emptyset$ for $i = 1, 2, 3$.*

Proof. Let $D \in \mathrm{Pic}_{g+1}(Y) \cap \mathrm{Pic}^{\mathbb{R}}(Y)$ be an arbitrary quaternionic divisor of degree $g+1$. We make the observation that

$$\eta(D - y_0 - y_\infty) - (D - y_0 - y_\infty) = \eta(D) - D = (f) \text{ with } f\eta^*\bar{f} = -1$$

holds. Thus $D_1 := D - y_0 - y_\infty$ is a quaternionic divisor with $\deg(D_1) = g - 1$ that fulfills all requirements of Lemma 4.26. This shows $\dim H^0(Y, \mathcal{O}_{D-y_0-y_\infty}) = 0$ and thus $S_1 = \emptyset$. Now consider the equation

$$\eta(D - 2y_0) - (D - 2y_0) = \eta(D) - D + 2y_0 - 2y_\infty = \eta(D) - D + (\lambda) = (\lambda f)$$

and set $\widetilde{f} := \lambda f$. Then $\widetilde{f}\eta^*\overline{\widetilde{f}} = f\overline{\eta^* f} = -1$ and thus $D_2 := D - 2y_0$ is a quaternionic divisor with $\deg(D_2) = g - 1$ that fulfills all requirements of Lemma 4.26. This shows $\dim H^0(Y, \mathcal{O}_{D-2\cdot y_0}) = 0$ and thus $S_2 = \emptyset$. Finally $S_3 = \emptyset$ follows from the same argumentation since

$$\eta(D - 2y_\infty) - (D - 2y_\infty) = \eta(D) - D - 2y_0 + 2y_\infty = \eta(D) - D + (\tfrac{1}{\lambda}) = (\tfrac{1}{\lambda}f)$$

and $\widehat{f} = \frac{1}{\lambda}f$ also fulfills $\widehat{f}\eta^*\overline{\widehat{f}} = -1$. Thus $D_3 := D - 2y_\infty$ is a quaternionic divisor with $\deg(D_3) = g - 1$ that fulfills all requirements from Lemma 4.26. This yields $\dim H^0(Y, \mathcal{O}_{D-2\cdot y_\infty}) = 0$ and concludes the proof. □

Remark 4.28. *The situation changes significantly if one considers real solutions $u : \mathbb{R}^2 \to \mathbb{R}$ of the* cosh-*Gordon equation $\Delta u = \cosh(u)$, for example. Then the sets S_i are in general no longer empty and one obtains singularities of the solutions u (see [4],[6]).*

Definition 4.29. *Let Y be a spectral curve with distinguished points y_0, y_∞ and charts $k, \widetilde{k}$ centered at y_0 and y_∞ respectively. Let $D \geq 0$ be an effective divisor on Y with $\deg(D) = g + 1$ and $y_0, y_\infty \notin D$ that satisfies the reality condition $\eta(D) - D = (f)$ for a meromorphic function f with $f\eta^*\bar{f} = -1$. Moreover, let $h = (h_0, h_\infty) \in (\mathfrak{h}^-_{finite})^2$ be given. A **Baker-Akhiezer function** $\psi : Y \times \mathbb{R} \to \mathbb{C}^2$ is a vector-valued map with the following properties:*

(i) For fixed $x \in \mathbb{R}$ the entries ψ_i of $\psi(x)$ satisfy $(\psi_i) \geq -D$ on $Y^* := Y \backslash \{y_0, y_\infty\}$.

(ii) The map $\psi^0 = k^* \exp(x \cdot h_0)\psi$ is holomorphic in a neighborhood of y_0 and the map $\psi^\infty = \widetilde{k}^* \exp(x \cdot h_\infty)\psi$ is holomorphic in a neighborhood of y_∞.

*We call a Baker-Akhiezer function ψ **normalized** if in addition there holds:*

(iii) The maps ψ^0 and ψ^∞ from (ii) are normalized by

$$\psi^0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + O(\sqrt{\lambda}) \quad \text{at } y_0$$

and

$$\psi^\infty = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + O(1/\sqrt{\lambda}) \quad \text{at } y_\infty.$$

Remark 4.30. *Conditions (i) and (ii) imply that ψ is a holomorphic section of the bundle $\mathcal{O}_D \otimes L_h(x)$. Here the cocycles of $L_h(x)$ are given by $k^* \exp(x \cdot h_0)$ at y_0 and $\widetilde{k}^* \exp(x \cdot h_\infty)$ at y_∞.*

The following theorem shows that $\psi : Y \times \mathbb{R} \to \mathbb{C}^2$ is uniquely determined by the analytic properties stated in Definition 4.29.

Theorem 4.31. *Let Y be a spectral curve with distinguished points y_0, y_∞ and charts $k, \widetilde{k}$ centered at y_0 and y_∞. Let $D \geq 0$ be an effective divisor on Y with $\deg(D) = g + 1$ and $y_0, y_\infty \notin D$ that satisfies the reality condition $\eta(D) - D = (f)$ for a meromorphic function f with $f\eta^*\bar{f} = -1$. Moreover, let $h = (h_0, h_\infty) \in (\mathfrak{h}^-_{\mathit{finite}})^2$ be given. Then there exists a unique Baker-Akhiezer function ψ that satisfies properties (i) - (iii) from Defintion 4.29.*

Proof. For every $x \in \mathbb{R}$ the bundle $L_h(x)$ is a real line bundle (compare with Hitchin's terminology introduced in [30]) with $\deg(L_h(x)) = 0$. Thus for a given quaternionic divisor D and fixed $x \in \mathbb{R}$ the divisor corresponding to the holomorphic line bundle $\mathcal{O}_D \otimes L_h(x)$ is quaternionic as well. Denote this divisor by $D(x)$, i.e. $D(x) \in \mathrm{Pic}_{g+1}(Y) \cap \mathrm{Pic}^{\mathbb{R}}(Y)$ for every fixed $x \in \mathbb{R}$. First we observe that $\dim H^0(Y, \mathcal{O}_{D(x)}) \geq 2$ for every $x \in \mathbb{R}$ since by the Riemann-Roch Theorem 2.22 we have

$$\dim H^0(Y, \mathcal{O}_{D(x)}) = 1 - g + \deg(D(x)) + i(D(x)) = 2 + i(D(x)) \geq 2.$$

Moreover, there holds $H^0(Y, \mathcal{O}_{D(x)-y_0-y_\infty}) = 0$ for every $x \in \mathbb{R}$ due to Proposition 4.27 and thus $\dim H^0(Y, \mathcal{O}_{D(x)}) = 2$ for every $x \in \mathbb{R}$, i.e. the space of sections of the corresponding bundle is 2-dimensional. This guarantees the existence of the functions ψ_1, ψ_2. We now prove the uniqueness of the functions ψ_1, ψ_2 and thus the uniqueness of ψ. Therefore we suppose that there exist other functions $\widetilde{\psi}_1$ and $\widetilde{\psi}_2$ that obey conditions (i) to (iii) from Definition 4.29. But then Lemma 4.21 shows that $\widetilde{\psi}_i = \psi_i$ for $i = 1, 2$ since $S_1 = \emptyset$ due to Proposition 4.27. This implies the uniqueness of the normalized Baker-Akhiezer function ψ and concludes the proof. □

Remark 4.32. *Since the entries of the Baker-Akhiezer function ψ can be written in terms of Riemann's theta functions, we can deduce that ψ is differentiable with respect to the spatial variable x.*

4.5 The reconstruction of (u, u_y)

We will now describe the reconstruction procedure that allows us to identify Cauchy data (u, u_y) of periodic real finite type solution of the sinh-Gordon equation with their spectral data $(Y(u, u_y), D(u, u_y))$ and vice versa.

Proposition 4.33. *Let $\widetilde{\psi}_\pm(x) = (\widetilde{\psi}_1^\pm(x), \widetilde{\psi}_2^\pm(x))^t = T^{-1}F_\lambda^{-1}(x)\widetilde{v}_\pm(0)$ and set*

$$\Psi(x) := T \begin{pmatrix} \widetilde{\psi}_1^+(x) & \widetilde{\psi}_1^-(x) \\ \widetilde{\psi}_2^+(x) & \widetilde{\psi}_2^-(x) \end{pmatrix}.$$

Then the frame F_λ with $F_\lambda(0) = \mathbb{1}$ is given by

$$F_\lambda(x) = \Psi(0)\Psi^{-1}(x).$$

Proof. Since $\frac{d}{dx}\Psi = -U_\lambda\Psi$ with we get

$$\frac{d}{dx}\Psi^{-1} = \Psi^{-1}U_\lambda.$$

Then $F_\lambda(x) := \Psi(0)\Psi^{-1}(x)$ is the unique solution for $\frac{d}{dx}F_\lambda = F_\lambda U_\lambda$ with $F_\lambda(0) = \mathbb{1}$. $\square$

Proposition 4.34. *Let Y be a hyperelliptic Riemann surface of genus g with branch points over $\lambda = 0$ (y_0) and $\lambda = \infty$ (y_∞) and the following properties:*

(i) Besides the hyperelliptic involution σ the Riemann surface Y has two additional anti-holomorphic involutions η and $\rho = \eta \circ \sigma$. Moreover, η has no fixed points and $\eta(y_0) = y_\infty$.

(ii) There exists a non-zero holomorphic function μ on $Y \backslash \{y_0, y_\infty\}$ that obeys

$$\sigma^*\mu = \mu^{-1}, \quad \eta^*\bar{\mu} = \mu, \quad \rho^*\bar{\mu} = \mu^{-1}.$$

(iii) The form $d\ln\mu$ is a meromorphic differential of the second kind with double poles at y_0 and y_∞.

Moreover, let $D \geq 0$ be an effective divisor of degree $g+1$ on Y with $y_0, y_\infty \notin D$ and $\eta(D) - D = (f)$ for a meromorphic function f obeying $f\eta^*\bar{f} = -1$. Then there exist unique Cauchy data $(u, u_y) \simeq U_\lambda$ of a periodic real finite type solution of the sinh-*Gordon equation such that $Y(\widetilde{u}, \widetilde{u}_y) = Y$ and $D(\widetilde{u}, \widetilde{u}_y) = D$ with $(\widetilde{u}, \widetilde{u}_y) \simeq \widetilde{U}_\lambda = T^{-1}U_\lambda T$.*

Conversely, let $(u, u_y) \simeq U_\lambda$ be Cauchy data of a periodic real finite type solution of the sinh-*Gordon equation and consider $(\widetilde{u}, \widetilde{u}_y) \simeq \widetilde{U}_\lambda = T^{-1}U_\lambda T$. Then there exists a pair (Y, D) of spectral data with the above properties such that the associated Cauchy data $(\widetilde{u}, \widetilde{u}_y)(Y, D)$ satisfy $(\widetilde{u}, \widetilde{u}_y)(Y, D) = (\widetilde{u}, \widetilde{u}_y)$.*

Proof. For the first claim consider the unique Baker-Akhiezer function $\psi = (\psi_1, \psi_2)^t$ for $h = (\frac{iz}{2}, \frac{iz}{2})$ given by Theorem 4.31. In analogy to the proof of Proposition 4.33 we set

$$\Psi(x) := \begin{pmatrix} \psi_1(x) & \sigma^*\psi_1(x) \\ \psi_2(x) & \sigma^*\psi_2(x) \end{pmatrix}.$$

Then $G_\lambda(x) := \Psi(0)\Psi^{-1}(x)$ satisfies $\overline{G_{1/\bar\lambda}}^t = G_\lambda^{-1}$ due to Lemma 3.41 and Lemma 3.42. Moreover, we have $G_\lambda(0) = \Psi(0)\Psi(0)^{-1} = \mathbb{1}$. From the asymptotic expansion of ψ around $\lambda = 0$ we obtain the expansion

$$\Psi(x)\exp\begin{pmatrix} \frac{ix}{2\sqrt{\lambda}} & 0 \\ 0 & -\frac{ix}{2\sqrt{\lambda}} \end{pmatrix} = B_\lambda(x)\Psi(0)$$

around $\lambda = 0$, where $B_\lambda(x)$ is of the form $B_\lambda(x) = \sum_{i\geq 0}\lambda^i B_i(x)$, and thus

$$G_\lambda(x) = \Psi(0)\exp\begin{pmatrix} \frac{ix}{2\sqrt{\lambda}} & 0 \\ 0 & -\frac{ix}{2\sqrt{\lambda}} \end{pmatrix}\Psi^{-1}(0)B_\lambda^{-1}(x)$$

holds around $\lambda = 0$. Taking the derivative yields

$$\begin{aligned} \frac{d}{dx}G_\lambda(x) &= \Psi(0)\begin{pmatrix} \frac{i}{2\sqrt{\lambda}} & 0 \\ 0 & -\frac{i}{2\sqrt{\lambda}} \end{pmatrix}\exp\begin{pmatrix} \frac{ix}{2\sqrt{\lambda}} & 0 \\ 0 & -\frac{ix}{2\sqrt{\lambda}} \end{pmatrix}\Psi^{-1}(0)B_\lambda^{-1}(x) \\ &\quad - \Psi(0)\exp\begin{pmatrix} \frac{ix}{2\sqrt{\lambda}} & 0 \\ 0 & -\frac{ix}{2\sqrt{\lambda}} \end{pmatrix}\Psi^{-1}(0)B_\lambda^{-1}(x)\left(\frac{d}{dx}B_\lambda(x)\right)B_\lambda^{-1}(x) \end{aligned}$$

and therefore

$$G_\lambda^{-1}(x)\frac{d}{dx}G_\lambda(x) = B_\lambda(x)\Psi(0)\begin{pmatrix} \frac{i}{2\sqrt{\lambda}} & 0 \\ 0 & -\frac{i}{2\sqrt{\lambda}} \end{pmatrix}\Psi^{-1}(0)B_\lambda^{-1}(x) - \left(\frac{d}{dx}B_\lambda(x)\right)B_\lambda^{-1}(x)$$

holds around $\lambda = 0$. Gauging with $T^{-1} = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ yields

$$\begin{aligned} TG_\lambda^{-1}(x)\left(\frac{d}{dx}G_\lambda(x)\right)T^{-1} &= TB_\lambda(x)\Psi(0)\begin{pmatrix} \frac{i}{2\sqrt{\lambda}} & 0 \\ 0 & -\frac{i}{2\sqrt{\lambda}} \end{pmatrix}\Psi^{-1}(0)B_\lambda^{-1}(x)T^{-1} \\ &\quad - T\left(\frac{d}{dx}B_\lambda(x)\right)B_\lambda^{-1}(x)T^{-1}. \end{aligned}$$

From the proof of Lemma 3.36 we know that

$$v_+w_+^t = \begin{pmatrix} 1 & \frac{e^{u(0)}}{\sqrt{\lambda}} \\ e^{-u(0)}\sqrt{\lambda} & 1 \end{pmatrix} + O(\lambda) \text{ at } \lambda = 0$$

and $w_+^t v_+ = 2$ at $\lambda = 0$. Setting $\widetilde{\Psi}(0) = T\Psi(0)$ and applying the projector P we get

$$\begin{aligned}
\widetilde{\Psi}(0)\begin{pmatrix} \frac{i}{2\sqrt{\lambda}} & 0 \\ 0 & -\frac{i}{2\sqrt{\lambda}} \end{pmatrix}\widetilde{\Psi}^{-1}(0) &= P\left(\tfrac{i}{2\sqrt{\lambda}}\right) + \sigma^* P\left(\tfrac{i}{2\sqrt{\lambda}}\right) \\
&= \begin{pmatrix} \frac{i}{2\sqrt{\lambda}} & \frac{e^{u(0)}}{\sqrt{\lambda}}\frac{i}{2\sqrt{\lambda}} \\ e^{-u(0)}\sqrt{\lambda}\frac{i}{2\sqrt{\lambda}} & \frac{i}{2\sqrt{\lambda}} \end{pmatrix} + \begin{pmatrix} -\frac{i}{2\sqrt{\lambda}} & \frac{e^{u(0)}}{\sqrt{\lambda}}\frac{i}{2\sqrt{\lambda}} \\ e^{-u(0)}\sqrt{\lambda}\frac{i}{2\sqrt{\lambda}} & -\frac{i}{2\sqrt{\lambda}} \end{pmatrix} \\
&= \begin{pmatrix} 0 & ie^{u(0)}\lambda^{-1} \\ ie^{-u(0)} & 0 \end{pmatrix} + O(1)
\end{aligned}$$

around $\lambda = 0$. Define $U_\lambda := TG_\lambda^{-1}(x)\left(\frac{d}{dx}G_\lambda(x)\right)T^{-1}$. Then the previous considerations yield $U_\lambda = \sum_{i \geq -1} U_i \lambda^i$. Moreover, there holds $\overline{U_{1/\bar{\lambda}}}^t = -U_\lambda$ and thus we get

$$U_\lambda = U_{-1}\lambda^{-1} + U_0 + U_1\lambda$$

with $U_{-k} = -\overline{U}_k^t$. Note that $F_\lambda := TG_\lambda T^{-1}$ (with monodromy $M_\lambda = TG_\lambda(\mathbf{p})T^{-1}$) solves

$$\frac{d}{dx}F_\lambda(x) = F_\lambda(x)U_\lambda \quad \text{with} \quad F_\lambda(0) = \mathbb{1}.$$

Then the map

$$\zeta_\lambda(x) := T(P_x(\lambda^{-1}\nu) + \sigma^* P_x(\lambda^{-1}\nu))T^{-1} = T\left(\frac{\psi(\lambda^{-1}\nu)\varphi^t}{\varphi^t\psi} + \sigma^*\left(\frac{\psi(\lambda^{-1}\nu)\varphi^t}{\varphi^t\psi}\right)\right)T^{-1}$$

solves $\frac{d}{dx}\zeta_\lambda = [\zeta_\lambda, U_\lambda]$. Moreover, since $\lambda^{-1}\nu = \frac{i}{2\sqrt{\lambda}}$ around $\lambda = 0$, the asymptotic expansion of $P\left(\lambda^{-1}\nu\right) + \sigma^* P\left(\lambda^{-1}\nu\right)$ shows $\xi_\lambda := \zeta_\lambda(0) \in \mathcal{P}_g$. Thus $U_\lambda \simeq (u, u_y)$ is of finite type. The map $\widetilde{\psi} = T^{-1}F_\lambda^{-1}T\widetilde{v}$, where $\widetilde{v}$ is an eigenvector of $\widetilde{M}_\lambda = T^{-1}M_\lambda T$, satisfies

$$\frac{d}{dx}\widetilde{\psi} = -\widetilde{U}_\lambda\widetilde{\psi} \quad \text{with} \quad \widetilde{U}_\lambda = T^{-1}U_\lambda T = G_\lambda^{-1}\frac{d}{dx}G_\lambda \quad \text{and} \quad \widetilde{M}_\lambda\widetilde{\psi}(0) = \mu\widetilde{\psi}(0).$$

Moreover, $\widetilde{\psi}$ fulfills all requirements from Definition 4.29. Due to the uniqueness of the Baker-Akhiezer function from Theorem 4.31 we get $\widetilde{\psi} = \psi$. This proves the first claim.

Conversely, let $(u, u_y) \simeq U_\lambda$ be of finite type and consider the frame F_λ, that is a solution of $\frac{d}{dx}F_\lambda = F_\lambda U_\lambda$ with $F_\lambda(0) = \mathbb{1}$, together with the corresponding monodromy $M_\lambda = F_\lambda(\mathbf{p})$. Gauging with T yields $\widetilde{F}_\lambda = T^{-1}F_\lambda T$ and $\widetilde{U}_\lambda = T^{-1}U_\lambda T$. Then the map $\widetilde{\psi} = T^{-1}F_\lambda^{-1}T\widetilde{v}$, where $\widetilde{v}$ is an eigenvector of $\widetilde{M}_\lambda = T^{-1}M_\lambda T$, satisfies

$$\frac{d}{dx}\widetilde{\psi} = -\widetilde{U}_\lambda\widetilde{\psi} \quad \text{with} \quad \widetilde{U}_\lambda = T^{-1}U_\lambda T \quad \text{and} \quad \widetilde{M}_\lambda\widetilde{\psi}(0) = \mu\widetilde{\psi}(0)$$

and fulfills all requirements from Definition 4.29. From Chapter 3 we know that $\widetilde{U}_\lambda$ yields a spectral curve Y and a divisor D with the properties stated above. On Y there exists a unique Baker-Akhiezer function ψ. Due to the uniqueness we get $\widetilde{\psi} = \psi$ and the second claim is proved. $\square$

With the help of Proposition 4.34 we can prove the following

Theorem 4.35. *The power series that appear in the asymptotic expansion*

$$\beta_\lambda(x)\left(\mathbb{1}+\sum_{m\geq 1}a_m(x)(\sqrt{\lambda})^m\right)+\sum_{m\geq 1}\frac{d}{dx}a_m(x)(\sqrt{\lambda})^m = \left(\mathbb{1}+\sum_{m\geq 1}a_m(x)(\sqrt{\lambda})^m\right)\sum_{m\geq -1}b_m(x)(\sqrt{\lambda})^m \qquad (*)$$

around $\lambda=0$ from Theorem 3.10 are convergent if and only if the pair (u,u_y) is of finite type. Here $b_{-1}(x)$ and $b_0(x)$ are given by $b_{-1}(x)\equiv\beta_{-1}=\frac{i}{2}\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ and $b_0(x)\equiv\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$.

Proof. Let (u,u_y) be of finite type. For $g_\lambda(x):=\mathbb{1}+\sum_{m\geq 1}a_m(x)(\sqrt{\lambda})^m$ consider the systems $\frac{d}{dx}G_\lambda=G_\lambda\beta_\lambda$ and

$$\frac{d}{dx}\widehat{G}_\lambda=\widehat{G}_\lambda\widehat{\beta}_\lambda \quad\text{with}\quad \widehat{G}_\lambda(x)=g_\lambda(0)^{-1}G_\lambda(x)g_\lambda(x)=\exp\left(\int_0^x\widehat{\beta}_\lambda(t)\,dt\right).$$

Note that $\widehat{\beta}_\lambda(x)=\sum_{m\geq -1}b_m(x)(\sqrt{\lambda})^m$ is a diagonal matrix. Then the map

$$\begin{aligned}
\exp\left(\frac{ix}{2\sqrt{\lambda}}\right)\psi &= \exp\left(\frac{ix}{2\sqrt{\lambda}}\right)G_\lambda^{-1}(x)v=\exp\left(\frac{ix}{2\sqrt{\lambda}}\right)g_\lambda(x)\widehat{G}_\lambda^{-1}(x)g_\lambda(0)^{-1}v\\
&= \exp\left(\frac{ix}{2\sqrt{\lambda}}\right)g_\lambda(x)\widehat{G}_\lambda^{-1}(x)e_1\\
&= \exp\left(\frac{ix}{2\sqrt{\lambda}}-\frac{ix}{2\sqrt{\lambda}}-\sum_{m\geq 1}(\sqrt{\lambda})^m\int_0^x b_m(t)e_1\,dt\right)g_\lambda(x)\\
&= \exp\left(-\sum_{m\geq 1}(\sqrt{\lambda})^m\int_0^x b_m(t)e_1\,dt\right)g_\lambda(x)
\end{aligned}$$

is a convergent power series around $\lambda=0$ due to Lemma 3.51 and thus $g_\lambda(x)$ also converges on a neighborhood of $\lambda=0$. This shows that the power series in $(*)$ are convergent around $\lambda=0$.

Now suppose that the power series in $(*)$ are convergent on a small neighborhood of $\lambda=0$. Since $\widehat{G}_\lambda(x)=\exp(\int_0^x\beta_\lambda(t)\,dt)$ and $G_\lambda(\mathbf{p})=g_\lambda(0)\widehat{G}_\lambda(\mathbf{p})g_\lambda(0)^{-1}$ we see that

$$\ln\mu=\sum_{m\geq -1}(\sqrt{\lambda})^m\int_0^{\mathbf{p}}b_m(t)e_1\,dt$$

is a meromorphic function around $\lambda=0$ with a simple pole at $\lambda=0$ and a similar statement also holds around $\lambda=\infty$. Thus the normalization of the multiplier curve has finite genus g. Due to the equation

$$\psi = G_\lambda^{-1}(x)v = g_\lambda(x)\widehat{G}_\lambda^{-1}g_\lambda(0)^{-1}v$$

we see that the corresponding Baker-Akhiezer function has only finitely many poles around $\lambda = 0$ (and $\lambda = \infty$) and consequently the corresponding divisor D satisfies $\deg(D) < \infty$. Now Proposition 4.34 shows that the corresponding pair (u, u_y) is of finite type. $\square$

5 Isospectral and non-isospectral deformations

5.1 Deformations of spectral curves

We start this chapter with non-isospectral deformations, i.e. deformations that change the spectral curve Y. First we need some definitions.

Definition 5.1. *The group of homotopic cycles on a Riemann surface Y is called fundamental group and is denoted by $\pi_1(Y)$. The abelianization of $\pi_1(Y)$ is the* ***homology group*** *$H_1(Y, \mathbb{Z})$.*

Definition 5.2. *A basis $(A, B) = (a_1, \ldots, a_g, b_1, \ldots, b_g)$ of $H_1(Y, \mathbb{Z})$ is called canonical if the intersection numbers of this basis are given by*

$$\begin{aligned} a_j \circ a_k &= 0, \\ b_j \circ b_k &= 0, \\ a_j \circ b_k &= \delta_{jk} \end{aligned}$$

with $j, k \in \{1, \ldots, g\}$.

Consider the subset $A_\eta \subset H_1(Y, \mathbb{Z})$ of anti-invariant cycles with respect to the involution η, i.e.

$$A_\eta = \{c \in H_1(Y, \mathbb{Z}) \mid \eta^* c \equiv -c\}.$$

Then A_η is a sub-module of $H_1(Y, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$ of rank g and the same holds for the subset of invariant cycles

$$I_\eta = \{c \in H_1(Y, \mathbb{Z}) \mid \eta^* c \equiv c\}.$$

Since the short exact sequence

$$0 \to I_\eta \to H_1(Y, \mathbb{Z}) \to H_1(Y, \mathbb{Z})/I_\eta \simeq A_\eta \to 0$$

does not generally split we have $H_1(Y, \mathbb{Z}) \neq I_\eta \oplus A_\eta$ in general. Choosing a-cycles from $A_\eta \simeq \mathbb{Z}^g$ (i.e. $\eta^* a_i \equiv -a_i$ for $i = 1, \ldots, g$) the dual basis of b-cycles obeys

$$\eta^* b_i \equiv b_i \mod \langle a_1, \ldots, a_g \rangle.$$

With this observation in mind we choose the following homology basis (A, B) of $H_1(Y, \mathbb{Z})$ for the upcoming considerations:

For $i = 1, \ldots, g$ let a_i be defined as the closed cycle surrounding the line segment $[\alpha_i, 1/\bar{\alpha}_i]$ where $\alpha_1, \ldots, \alpha_g \in \mathbb{D}$ are the (simple) zeros of the polynomial $a(\lambda)$ in the open disk $\mathbb{D}$. Now the b-cycles again are chosen in a way such that

$$\eta^* b_i \equiv b_i \bmod \langle a_1, \ldots, a_g \rangle$$

holds for $i = 1, \ldots, g$. The resulting basis (A, B) for $H_1(Y, \mathbb{Z})$ is called an *adapted canonical basis* in analogy to the terminology introduced in [14].

5.1.1 Infinitesimal deformations of spectral curves

Definition 5.3. *Let $\Sigma_g^{\mathbf{p}}$ denote the space of smooth hyperelliptic Riemann surfaces Y of genus g with the properties described in Theorem 3.55, such that $d \ln \mu$ has no roots at the branchpoints of Y.*

We will now investigate deformations of $\Sigma_g^{\mathbf{p}} \simeq \mathcal{M}_g^1(\mathbf{p})$. For this, following the expositions of [24, 28, 35], we derive vector fields on open subsets of $\mathcal{M}_g^1(\mathbf{p})$ and parametrize the corresponding deformations by a parameter $t \in [0, \varepsilon)$. We consider deformations of $Y(u, u_y)$ that preserve the periods of $d \ln \mu$. We already know that

$$\int_{a_i} d \ln \mu = 0 \quad \text{and} \quad \int_{b_i} d \ln \mu \in 2\pi i \mathbb{Z} \quad \text{for } i = 1, \ldots, g.$$

Considering the Taylor expansion of $\ln \mu$ with respect to t we get

$$\ln \mu(t) = \ln \mu(0) + t\, \partial_t \ln \mu(0) + O(t^2)$$

and thus

$$d \ln \mu(t) = d \ln \mu(0) + t\, d\partial_t \ln \mu(0) + O(t^2).$$

Given a closed cycle $c \in H_1(Y, \mathbb{Z})$ we have

$$\int_c d \ln \mu(t) = \int_c d \ln \mu(0) + t \int_c d\partial_t \ln \mu(0) + O(t^2)$$

and therefore

$$\frac{d}{dt} \left(\int_c d \ln \mu(t) \right) \bigg|_{t=0} = \int_c d\partial_t \ln \mu(0) = 0.$$

This shows that deformations resulting from the prescription of $\partial_t \ln \mu|_{t=0}$ at $t = 0$ preserve the periods of $d \ln \mu$ infinitesimally along the deformation and thus are *isoperiodic*. If we set $\dot{\lambda} = 0$ we can consider $\ln \mu$ as a function of λ and t and get

$$\ln \mu(\lambda, t) = \ln \mu(\lambda, 0) + t\, \partial_t \ln \mu(\lambda, 0) + O(t^2)$$

as well as

$$\partial_\lambda \ln \mu(\lambda, t) = \partial_\lambda \ln \mu(\lambda, 0) + t\, \partial^2_{\lambda t} \ln \mu(\lambda, 0) + O(t^2).$$

We know that $\widehat{Y} = \{(\lambda, \mu) \in \mathbb{C}^* \times \mathbb{C}^* \mid R(\lambda, \mu) = \mu^2 - \Delta(\lambda)\mu + 1 = 0\}$. Differentiating the expression $R(\lambda, \mu) = 0$ with respect to λ we get

$$2\mu\partial_\lambda\mu - \Delta'(\lambda)\mu - \Delta(\lambda)\partial_\lambda\mu = 2\mu^2\frac{\partial_\lambda\mu}{\mu} - \Delta'(\lambda)\mu - \mu\Delta(\lambda)\frac{\partial_\lambda\mu}{\mu} = 0$$

and therefore

$$\partial_\lambda \ln\mu = \frac{\Delta'(\lambda)\mu}{2\mu^2 - \mu\Delta(\lambda)} = \frac{\Delta'(\lambda)}{2\mu - \Delta(\lambda)}$$

on $\widehat{Y}$. On the compact spectral curve Y the function $\partial_\lambda \ln\mu$ is therefore given by

$$\partial_\lambda \ln\mu = \frac{b(\lambda)}{\lambda\,\nu}$$

and the compatibility condition $\partial^2_{t\lambda} \ln\mu|_{t=0} = \partial^2_{\lambda t} \ln\mu|_{t=0}$ will lead to a deformation of the spectral data (a, b) or equivalently to a deformation of the spectral curve Y with its differential $d\ln\mu$. Therefore this deformation is *non-isospectral*. In the following we will investigate which conditions $\delta\ln\mu := (\partial_t \ln\mu)|_{t=0}$ has to obey in order obtain such a deformation.

The Whitham deformation. Following the ansatz given in [28] we consider the function $\ln\mu$ as a function of λ and t and write $\ln\mu$ locally as

$$\ln\mu = \begin{cases} f_{\alpha_i}(\lambda)\sqrt{\lambda - \alpha_i} + \pi i n_i & \text{at a zero } \alpha_i \text{ of } a, \\ f_0(\lambda)\lambda^{-1/2} + \pi i n_0 & \text{at } \lambda = 0, \\ f_\infty(\lambda)\lambda^{1/2} + \pi i n_\infty & \text{at } \lambda = \infty. \end{cases}$$

Here we choose small neighborhoods around the branch points such that each neighborhood contains at most one branch point. Moreover, the functions $f_{\alpha_i}, f_0, f_\infty$ do not vanish at the corresponding branch points. If we write $\dot{g}$ for $(\partial_t g)|_{t=0}$ we get

$$(\partial_t \ln\mu)|_{t=0} = \begin{cases} \dot{f}_{\alpha_i}(\lambda)\sqrt{\lambda - \alpha_i} - \frac{\dot{\alpha}_i f_{\alpha_i}(\lambda)}{2\sqrt{\lambda-\alpha_i}} & \text{at a zero } \alpha_i \text{ of } a, \\ \dot{f}_0(\lambda)\lambda^{-1/2} & \text{at } \lambda = 0, \\ \dot{f}_\infty(\lambda)\lambda^{1/2} & \text{at } \lambda = \infty. \end{cases}$$

Since the branches of $\ln\mu$ differ from each other by an element in $2\pi i\mathbb{Z}$ we see that $\delta\ln\mu = (\partial_t \ln\mu)|_{t=0}$ is a single-valued meromorphic function on Y with poles at the branch points of Y, i.e. the poles of $\delta\ln\mu$ are located at the zeros of a and at $\lambda = 0$ and $\lambda = \infty$. Thus we have

$$\delta\ln\mu = \frac{c(\lambda)}{\nu}$$

with a polynomial c of degree at most $g + 1$. Since $\eta^*\delta\ln\mu = \delta\ln\bar{\mu}$ and $\eta^*\nu = \bar{\lambda}^{-g-1}\bar{\nu}$ the polynomial c obeys the reality condition

$$\lambda^{g+1}\overline{c(\bar{\lambda}^{-1})} = c(\lambda). \tag{5.1.1}$$

Differentiating $\nu^2 = \lambda a$ with respect to t we get $2\nu\dot{\nu} = \lambda\dot{a}$. The same computation for the derivative with respect to λ gives $2\nu\nu' = a + \lambda a'$. Now a direct calculation shows

$$\begin{aligned}
\partial^2_{t\lambda} \ln\mu|_{t=0} &= \partial_t \left(\frac{b}{\lambda\,\nu}\right)\bigg|_{t=0} = \frac{\dot{b}\lambda\nu - b\lambda\dot{\nu}}{\lambda^2\nu^2} = \frac{2\dot{b}a - b\dot{a}}{2\nu^3}, \\
\partial^2_{\lambda t} \ln\mu|_{t=0} &= \partial_\lambda \left(\frac{c}{\nu}\right) = \frac{c'\nu - c\nu'}{\nu^2} = \frac{2c'\nu^2 - 2c\nu\nu'}{2\nu^3} = \frac{2c'\lambda a - ca - c\lambda a'}{2\nu^3}.
\end{aligned}$$

The compatibility condition $\partial^2_{t\lambda} \ln\mu|_{t=0} = \partial^2_{\lambda t} \ln\mu|_{t=0}$ holds if and only if

$$-2\dot{b}a + b\dot{a} = -2\lambda ac' + ac + \lambda a'c. \tag{5.1.2}$$

Both sides of this equation are polynomials of degree at most $3g+1$ and therefore describe relations for $3g+2$ coefficients. If we choose a polynomial c that obeys the reality condition (5.1.1) we obtain a vector field on $\mathcal{M}^1_g(\mathbf{p})$. Since $(a,b) \in \mathcal{M}^1_g(\mathbf{p})$ have no common roots, the polynomials a, b, c in equation (5.1.2) uniquely define a tangent vector $(\dot{a}, \dot{b})$ (see [28], Section 9). An application of these techniques to study CMC tori in $\mathbb{S}^3$ and $\mathbb{H}^3$ can be found in [34] and [40]. In the following we will specify such polynomials c that lead to deformations which do not change the period $\mathbf{p}$ of (u, u_y) (compare with [28], Section 9).

Preserving the period p along the deformation. If we evaluate the compatibility equation (5.1.2) at $\lambda = 0$ we get

$$-2\dot{b}(0)a(0) + b(0)\dot{a}(0) = a(0)c(0).$$

Moreover,

$$\dot{\mathbf{p}} = 2\frac{d}{dt}\left(\frac{b(0)}{i\sqrt{a(0)}}\right)\bigg|_{t=0} = \frac{-2\dot{b}(0)a(0) + b(0)\dot{a}(0)}{-i(a(0))^{3/2}} = i\frac{c(0)}{\sqrt{a(0)}}.$$

This proves

Lemma 5.4. *Vector fields on $\mathcal{M}^1_g(\mathbf{p})$ that are induced by polynomials c obeying* (5.1.1) *preserve the period* $\mathbf{p}$ *of (u, u_y) if and only if $c(0) = 0$.*

Let us take a closer look at the space of polynomials c that induce a Whitham deformation.

Lemma 5.5. *For the coefficients of the polynomial $c(\lambda) = \sum_{i=0}^{g+1} c_i\lambda^i$ obeying* (5.1.1) *there holds $c_i = \bar{c}_{g+1-i}$ for $i = 0, \ldots, g+1$.*

Proof. Inserting $c(\lambda) = \sum_{i=0}^{g+1} c_i\lambda^i$ into (5.1.1) yields

$$\lambda^{g+1}\sum_{i=0}^{g+1}\bar{c}_i\lambda^{-i} = \sum_{i=0}^{g+1}\bar{c}_i\lambda^{g+1-i} = \sum_{i=0}^{g+1}\bar{c}_{g+1-i}\lambda^i \stackrel{!}{=} \sum_{i=0}^{g+1}c_i\lambda^i.$$

Equating the coefficients shows $c_i = \bar{c}_{g+1-i}$ for $i = 0, \ldots, g+1$ and concludes the proof. $\square$

Proposition 5.6. *The space of polynomials c corresponding to deformations of spectral curve data $(a,b) \in \mathcal{M}_g^1(\mathbf{p})$ (with fixed period $\mathbf{p}$) is g-dimensional.*

Proof. The space of polynomials c of degree at most $g+1$ obeying the reality condition (5.1.1) is $(g+2)$-dimensional. From Lemma 5.4 we know that $\dot{p} = 0$ if and only if $c(0) = 0$. This yields the claim. $\square$

Note that c corresponds to a transformation $\lambda \mapsto e^{i\varphi}\lambda$ if and only if $(\exists f \in i\mathbb{R} : b = fc)$. This equivalence can be deduced as follows. Consider $\lambda(t) = e^{i\varphi t}\lambda$ with $\dot{\lambda}(t) = i\varphi\lambda$. Then

$$\frac{\partial}{\partial t} \ln \mu(\lambda(t))|_{t=0} = \frac{\partial}{\partial \lambda} \ln \mu \cdot \dot{\lambda}(t) = \frac{b}{\lambda \nu} i\varphi\lambda \overset{!}{=} \frac{c}{\nu}$$

and thus $c = i\varphi b$. We show that $(\exists f \in i\mathbb{R} : b = fc) \Longrightarrow (\deg(c) = g{+}1$ and c obeys (5.1.1)). Since b has degree $g{+}1$ there also holds $\deg(c) = g{+}1$. Moreover, b satisfies $\lambda^{g+1}\overline{b(1/\bar{\lambda})} = -b(\lambda)$. Since $f \in i\mathbb{R}$ we get

$$\lambda^{g+1}\overline{b(1/\bar{\lambda})} = -f\lambda^{g+1}\overline{c(1/\bar{\lambda})} \overset{!}{=} -fc(\lambda)$$

and thus c obeys (5.1.1). Finally there holds

$$\dot{\mathbf{p}} \neq 0 \overset{\text{Lem. 5.4}}{\Longleftrightarrow} c(0) \neq 0 \overset{\text{Lem. 5.5}}{\Longleftrightarrow} (\deg(c) = g+1 \text{ and } c \text{ obeys (5.1.1)})$$

and thus $\dot{\mathbf{p}} = 0 \Longleftrightarrow \neg(\deg(c) = g+1$ and c obeys (5.1.1)). Moreover,

$$\neg(\deg(c) = g+1 \text{ and } c \text{ obeys (5.1.1)}) \Longrightarrow \neg(\exists f \in i\mathbb{R} : b = fc)$$

and therefore infinitesimal Möbius transformations of the form $\lambda \mapsto e^{i\varphi}\lambda$ are excluded in the case of deformations that fix the period $\mathbf{p}$.

5.1.2 $\mathcal{M}_g^1(\mathbf{p})$ is a smooth g-dimensional manifold

From Proposition 5.6 we know that the space of polynomials c corresponding to deformations of $\mathcal{M}_g^1(\mathbf{p})$ with fixed period $\mathbf{p}$ is g-dimensional. In the following we want to show that $\mathcal{M}_g^1(\mathbf{p})$ is a real g-dimensional manifold. For this, we follow the terminology introduced by Carberry and Schmidt in [14].

Let us recall the conditions that characterize a representative $(a,b) \in \mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda]$ of an element in $\mathcal{M}_g(\mathbf{p})$:

(i) $\lambda^{2g}\overline{a(\bar{\lambda}^{-1})} = a(\lambda)$ and $\lambda^{-g}a(\lambda) < 0$ for all $\lambda \in \mathbb{S}^1$ and $|a(0)| = 1$.

(ii) $\lambda^{g+1}\overline{b(\bar{\lambda}^{-1})} = -b(\lambda)$.

(iii) $f_i(a,b) := \int_{\alpha_i}^{1/\bar{\alpha}_i} \frac{b}{\nu}\frac{d\lambda}{\lambda} = 0$ for the roots α_i of a in the open disk $\mathbb{D} \subset \mathbb{C}$.

(iv) The unique function $h : \widetilde{Y} \to \mathbb{C}$ with $\sigma^* h = -h$ and $dh = \frac{b}{\nu}\frac{d\lambda}{\lambda}$ satisfies $h(\alpha_i) \in \pi i \mathbb{Z}$ for all roots α_i of a.

Definition 5.7. *Let $\mathcal{H}^g$ be the set of polynomials $a \in \mathbb{C}^{2g}[\lambda]$ that satisfy condition (i) and whose roots are pairwise distinct.*

Every $a \in \mathcal{H}^g$ corresponds to a smooth spectral curve. Moreover, every $a \in \mathcal{H}^g$ is uniquely determined by its roots.

Definition 5.8. *For every $a \in \mathcal{H}^g$ let the space $\mathcal{B}_a$ be given by*

$$\mathcal{B}_a := \{b \in \mathbb{C}^{g+1}[\lambda] \mid b \text{ satisfies conditions (ii) and (iii)}\}.$$

Since (iii) imposes g linearly independent constraints on the $(g+2)$-dimensional space of polynomials $b \in \mathbb{C}^{g+1}[\lambda]$ obeying the reality condition (ii) we get

Proposition 5.9. $\dim_{\mathbb{R}} \mathcal{B}_a = 2$. *In particular every $b_0 \in \mathbb{C}$ uniquely determines an element $b \in \mathcal{B}_a$ with $b(0) = b_0$.*

Now we arrive at

Proposition 5.10. *The set*

$$M := \{(a,b) \in \mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda] \mid a \in \mathcal{H}^g, (a,b) \text{ have no common roots and } b \text{ satisfies (ii)}\}$$

is an open subset of a $(3g+2)$-dimensional real vector space. Moreover, the set

$$N := \{(a,b) \in M \mid f_i(a,b) = 0 \text{ for } i = 1,\dots,g\}$$

defines a real submanifold of M of dimension $2g+2$ that is parameterized by $(a, b(0))$. If $b(0) = b_0$ is fixed we get a real submanifold of dimension $2g$.

Proof. Consider the map $f = (f_1,\dots,f_g) : \mathbb{R}^{2g+2} \times \mathbb{R}^g \to \mathbb{R}^g$ given by

$$((a,b_0),(b_1,\dots,b_{(g+1)/2})) \mapsto f(a,b) := (f_1(a,b),\dots,f_g(a,b)).$$

If we choose $b \in \mathcal{B}_a$ with $b(0) = b_0$ we get $f(a,b) = 0$ due to Proposition 5.9. Moreover, f is linear with respect to $(b_1,\dots,b_{(g+1)/2})$ and thus $\frac{\partial(f_1,\dots,f_g)}{\partial(b_1,\dots,b_{(g+1)/2})}$ is invertible at (a,b). Now we can apply the Implicit Function Theorem and see that there exist neighborhoods $U \subset \mathbb{R}^{2g+2}$ and $V \subset \mathbb{R}^g$ with $(a,b_0) \in U$ and $(b_1,\dots,b_{(g+1)/2}) \in V$ and a smooth map $g : U \to V$ with $g(a,b_0) = (b_1,\dots,b_{(g+1)/2})$ such that

$$f((a,b_0),g(a,b_0)) = 0 \text{ for all } (a,b_0) \in U.$$

Therefore $N = f^{-1}[0]$ defines a real submanifold of M of dimension $2g+2$ that is parameterized by $(a, b(0))$. $\square$

The results in [27, 28] yield the following theorem (compare with Lemma 5.3 in [29]).

Theorem 5.11. *For a fixed choice* $n_1, \ldots, n_g \in \mathbb{Z}$ *the map* $h = (h_1, \ldots, h_g) : N \to (i\mathbb{R}/2\pi i\mathbb{Z})^g \simeq (\mathbb{S}^1)^g$ *with*

$$h_j : N \to i\mathbb{R}/2\pi i\mathbb{Z}, \;\; (a, b) \mapsto h_j(a, b) := \ln \mu(\alpha_j) - \pi i n_j$$

is smooth and its differential dh *has full rank. In particular* $\mathcal{M}^1_g(\mathbf{p}) = h^{-1}[0]$ *defines a real submanifold of dimension* g*. Here we consider* $b(0) = b_0$ *as fixed, i.e.* $\dim_{\mathbb{R}}(N) = 2g$.

Proof. Let us consider an integral curve $(a(t), b(t))$ for the vectorfield X_c that corresponds to a Whitham deformation that is induced by a polynomial c obeying the reality condition (5.1.1). Then there holds $h(a(t), b(t)) \equiv \text{const.}$ along this deformation and therefore

$$dh(a(t), b(t)) \cdot (\dot{a}(t), \dot{b}(t)) = dh(a(t), b(t)) \cdot X_c(a(t), b(t)) = 0.$$

From Proposition 5.6 we know that the space of polynomials c that correspond to a deformation with fixed period $\mathbf{p}$ is g-dimensional. We will now show that the map

$$c \mapsto X_c(a(t), b(t)) = (\dot{a}(t), \dot{b}(t)) \;\; \text{with} \;\; h(a(t), b(t)) \equiv \text{const.} \tag{5.1.3}$$

is one-to-one and onto. The first part of the claim is obvious. For the second part consider the functions

$$f_{b_j}(a, b) := \int_{b_j} d \ln \mu = \int_{b_j} \frac{b(\lambda)}{\nu} \frac{d\lambda}{\lambda} = \ln \mu(\alpha_j) = \pi i n_j \in \pi i \mathbb{Z}$$

along $(a(t), b(t))$, where the b_j are the b-cycles of Y. Taking the derivative yields

$$\frac{d}{dt} f_{b_j}(a, b)\big|_{t=0} = \int_{b_j} \frac{d}{dt}\left(\frac{b(\lambda)}{\nu}\right)\Big|_{t=0} \frac{d\lambda}{\lambda} = \int_{b_j} \partial_t(\partial_\lambda \ln \mu)|_{t=0} \, d\lambda = 0 = 0.$$

Morover, for the a-cycles a_j we have

$$f_{a_j}(a, b) = f_j(a, b) = \int_{a_j} d \ln \mu = \int_{a_j} \frac{b(\lambda)}{\nu} \frac{d\lambda}{\lambda} = 0$$

and consequently

$$\frac{d}{dt} f_{a_j}(a, b)\big|_{t=0} = \int_{a_j} \frac{d}{dt}\left(\frac{b(\lambda)}{\nu}\right)\Big|_{t=0} \frac{d\lambda}{\lambda} = \int_{a_j} \partial_t(\partial_\lambda \ln \mu)|_{t=0} \, d\lambda = 0.$$

Since all integrals of $\partial_t(\partial_\lambda \ln \mu)|_{t=0}$ vanish, there exists a meromorphic function ϕ with

$$d\phi = \partial_t(\partial_\lambda \ln \mu)|_{t=0} \, d\lambda.$$

Due to the Whitham equation (5.1.2) this function is given by $\phi = (\partial_t \ln \mu)|_{t=0} = \frac{c}{\nu}$. Thus the map in (5.1.3) is bijective. This shows $\dim(\ker dh) = g$ and consequently $\dim(\operatorname{im} dh) = g$ as well. Therefore $dh : \mathbb{R}^{2g} \to \mathbb{R}^g$ has full rank and the claim follows. $\square$

5.2 Deformations of the eigenline bundle

We want to consider isospectral deformations and therefore state the following lemma, that is motivated by the results presented in [47], Chapter 7.

Lemma 5.12. *Let v_1, w_1^t be the eigenvectors for μ and v_2, w_2^t the corresponding eigenvectors for $\frac{1}{\mu}$ of $M(\lambda)$. Then*

$$\delta M(\lambda)v_1 + M(\lambda)\delta v_1 = \mu\delta v_1 \text{ and } \delta M(\lambda)v_2 + M(\lambda)\delta v_2 = \tfrac{1}{\mu}\delta v_2 \tag{$*$}$$

if and only if

$$\delta M(\lambda) = \left[\sum_{i=1}^{2} \frac{(\delta v_i)w_i^t}{w_i^t v_i}, M(\lambda)\right]. \tag{$**$}$$

Proof. A direct calculation shows

$$\begin{aligned}
\left[\sum_{i=1}^{2} \frac{(\delta v_i)w_i^t}{w_i^t v_i}, M(\lambda)\right] v_1 &= \left(\sum_{i=1}^{2} \frac{(\delta v_i)w_i^t}{w_i^t v_i}\right) M(\lambda)v_1 - M(\lambda)\left(\sum_{i=1}^{2} \frac{(\delta v_i)w_i^t}{w_i^t v_i}\right) v_1 \\
&= \left(\sum_{i=1}^{2} \frac{(\delta v_i)w_i^t}{w_i^t v_i}\right) \mu v_1 - M(\lambda)\delta v_1 \\
&= \mu\delta v_1 - M(\lambda)\delta v_1 \\
&\overset{(*)}{=} \delta M(\lambda)v_1.
\end{aligned}$$

An analogous calculation for v_2 gives

$$\delta M(\lambda)v_2 = \left[\sum_{i=1}^{2} \frac{(\delta v_i)w_i^t}{w_i^t v_i}, M(\lambda)\right] v_2$$

and the claim is proved. ☐

5.2.1 Isospectral group action and loop groups

The one-to-one correspondence between Cauchy data (u, u_y) of periodic real finite type solutions of the sinh-Gordon equation and their spectral data $(Y(u, u_y), D(u, u_y))$ established in Proposition 4.34 allows us to deduce the following conclusions.

Definition 5.13. *Let $Iso(Y) := \{(u, u_y) \text{ of finite type } \mid Y(u, u_y) = Y\}$ be the set of finite type Cauchy data (u, u_y) whose spectral curve $Y(u, u_y)$ equals a given $Y \in \Sigma_g^{\mathbf{p}} \simeq \mathcal{M}_g^1(\mathbf{p})$.*

Definition 5.14. *Let $Pic_{g+1}^{\mathbb{R}}(Y) := Pic_{g+1}(Y) \cap Pic^{\mathbb{R}}(Y)$ be the real part of $Pic_{g+1}(Y)$ with respect to the involution η (compare with Definition 4.24).*

If we define (compare with Definition 3.39)

$$E : \mathrm{Iso}(Y) \to \mathrm{Pic}_{g+1}^{\mathbb{R}}(Y),\ (u, u_y) \mapsto E(u, u_y),$$

Proposition 4.34 gives

Proposition 5.15. *The map E induces a homeomorphism $Iso(Y) \simeq Pic^{\mathbb{R}}_{g+1}(Y)$.*

Before we can define the isospectral group action we need the following

Lemma 5.16. *The real part $Pic^{\mathbb{R}}_{g+1}(Y)$ of $Pic_{g+1}(Y)$ is connected.*

Proof. Let $\mathrm{Jac}_{\mathbb{R}}(Y) \simeq \mathrm{Pic}^{\mathbb{R}}_0(Y)$ denote the real part of the Jacobian $\mathrm{Jac}(Y) \simeq \mathrm{Pic}_0(Y)$ of Y with respect to the involution η and let $n(\mathrm{Jac}_{\mathbb{R}}(Y))$ be the number of connected components of $\mathrm{Jac}_{\mathbb{R}}(Y)$. Since the quotient Y/η is a connected manifold with zero boundary components we can deduce from the proof of Proposition 4.4 in [25] that

$$n(\mathrm{Jac}_{\mathbb{R}}(Y)) = \begin{cases} 1 & \text{if } g \equiv 0 \pmod 2, \\ 2 & \text{if } g \equiv 1 \pmod 2. \end{cases}$$

Since $\mathrm{Jac}(Y) \simeq \mathrm{Pic}_0(Y)$ is the Lie algebra of $\mathrm{Pic}_{g+1}(Y)$ we see that the claim follows immediately for $g \equiv 0 \pmod 2$. For the case $g \equiv 1 \pmod 2$ we have to exclude one component in order to obtain our result. For this, consider the divisor

$$D := \sum_{i=1}^{(g+1)/2} y_i + \eta(y_i)$$

with $\deg(D) = g+1$ and $\eta(D) - D = 0$, i.e. D is not quaternionic. Thus only one of the connected components of $\mathrm{Jac}_{\mathbb{R}}(Y)$ corresponds to the Lie algebra of $\mathrm{Pic}^{\mathbb{R}}_{g+1}(Y)$ and therefore $\mathrm{Pic}^{\mathbb{R}}_{g+1}(Y)$ is connected. □

One gets an action on $\mathrm{Pic}^{\mathbb{R}}_{g+1}(Y)$ by the following

Theorem 5.17. *The action of the tensor product on holomorphic line bundles induces a continuous commutative and transitive action of $\mathbb{R}^g$ on $Pic^{\mathbb{R}}_{g+1}(Y)$, which is denoted by*

$$\pi : \mathbb{R}^g \times Pic^{\mathbb{R}}_{g+1}(Y) \to Pic^{\mathbb{R}}_{g+1}(Y), \quad ((t_0, \dots, t_{g-1}), E) \mapsto \pi(t_0, \dots, t_{g-1})(E).$$

Proof. Since the map $\varphi : (\mathfrak{h}^-_{\text{finite}})^2 \to H^1_{\mathbb{R}}(Y, \mathcal{O})$ is onto there exist elements

$$(h^0_0, h^0_\infty), \dots, (h^{g-1}_0, h^{g-1}_\infty) \in (\mathfrak{h}^-_{\text{finite}})^2$$

such that

$$\mathrm{span}\{\varphi(h^0_0, h^0_\infty), \dots, \varphi(h^{g-1}_0, h^{g-1}_\infty)\} = H^1_{\mathbb{R}}(Y, \mathcal{O}).$$

Denote by $L(t) = L(t_0, \dots, t_{g-1})$ the family in $\mathrm{Pic}^{\mathbb{R}}_0(Y)$ which is obtained by applying Krichever's construction procedure to $(h_0, h_\infty) := \sum_{i=0}^{g-1} c_i(h^i_0, h^i_\infty)$ such that the cocycle of $L(t)$ around y_0 is given by $k^* \exp(\beta_0(t))$ with

$$\beta_0(t) = \beta_0(t_0, \dots, t_{g-1}) = \sum_{i=0}^{g-1} c_i t_i h^i_0.$$

Then the group action

$$\pi : \mathbb{R}^g \times \mathrm{Pic}^{\mathbb{R}}_{g+1}(Y) \to \mathrm{Pic}^{\mathbb{R}}_{g+1}(Y)$$

is given by

$$((t_0, \dots, t_{g-1}), E) \mapsto \pi(t_0, \dots, t_{g-1})(E) = E \otimes L(t_0, \dots, t_{g-1}).$$

Obviously this continuous action is commutative since the cocycles of the bundle $L(t)$ are of the form $\exp(\sum_{i=0}^{g-1} t_i c_i \varphi_i)$. Moreover, Lemma 5.16 ensures that $\mathrm{Pic}^{\mathbb{R}}_{g+1}(Y)$ and $\mathrm{Pic}^{\mathbb{R}}_{0}(Y)$ are connected and thus for every $E' \in \mathrm{Pic}^{\mathbb{R}}_{g+1}(Y)$ there exists $t = (t_0, \dots, t_{g-1}) \in \mathbb{R}^g$ such that $E' = \pi(t_0, \dots, t_{g-1})(E)$. This shows that the action $\pi : \mathbb{R}^g \times \mathrm{Pic}^{\mathbb{R}}_{g+1}(Y) \to \mathrm{Pic}^{\mathbb{R}}_{g+1}(Y)$ is transitive and concludes the proof. □

Corollary 5.18. *For the isospectral set Iso(Y) there holds Iso$(Y) \simeq \left(\mathbb{S}^1\right)^g$.*

Proof. Since the action $\pi : \mathbb{R}^g \times \mathrm{Pic}^{\mathbb{R}}_{g+1}(Y) \to \mathrm{Pic}^{\mathbb{R}}_{g+1}(Y)$ acts transitively one has $\mathrm{Pic}^{\mathbb{R}}_{g+1}(Y) = \pi(\mathbb{R}^g)(E)$ for some $E \in \mathrm{Pic}^{\mathbb{R}}_{g+1}(Y)$. Moreover, the stabilizer subgroup

$$\Gamma_E = \{(t_0, \dots, t_{g-1}) \in \mathbb{R}^g \mid \pi(t_0, \dots, t_{g-1})(E) = E\}$$

is a discrete lattice in $\mathbb{R}^g$ of full rank that is isomorphic to $\mathbb{Z}^g$ and therefore

$$\left(\mathbb{S}^1\right)^g = \mathbb{R}^g/\mathbb{Z}^g \simeq \mathbb{R}^g/\Gamma(E) \simeq \pi(\mathbb{R}^g)(E) = \mathrm{Pic}^{\mathbb{R}}_{g+1}(Y).$$

Thus we get $\mathrm{Iso}(Y) \simeq \left(\mathbb{S}^1\right)^g$ due to Proposition 5.15 and the claim is proved. □

Loop groups and the Iwasawa decomposition. For real $r \in (0, 1]$, denote the circle with radius r by $\mathbb{S}_r = \{\lambda \in \mathbb{C} \mid |\lambda| = r\}$ and the open disk with boundary $\mathbb{S}_r$ by $I_r = \{\lambda \in \mathbb{C} \mid |\lambda| < r\}$. Moreover, the open annulus with boundaries $\mathbb{S}_r$ and $\mathbb{S}_{1/r}$ is given by $A_r = \{\lambda \in \mathbb{C} \mid r < |\lambda| < 1/r\}$ for $r \in (0, 1)$. For $r = 1$ we set $A_1 := \mathbb{S}^1$. The loop group $\Lambda_r SL(2, \mathbb{C})$ of $SL(2, \mathbb{C})$ is the infinite dimensional Lie group of analytic maps from $\mathbb{S}_r$ to $SL(2, \mathbb{C})$, i.e.

$$\Lambda_r SL(2, \mathbb{C}) = \mathcal{O}(\mathbb{S}_r, SL(2, \mathbb{C})).$$

We will also need the following two subgroups of $\Lambda_r SL(2, \mathbb{C})$: First let

$$\Lambda_r SU(2) = \{F \in \mathcal{O}(A_r, SL(2, \mathbb{C})) \mid F|_{\mathbb{S}^1} \in SU(2)\}.$$

Thus we have

$$F_\lambda \in \Lambda_r SU(2) \iff \overline{F_{1/\bar\lambda}}^{\,t} = F_\lambda^{-1}.$$

The second subgroup is given by

$$\Lambda_r^+ SL(2, \mathbb{C}) = \{B \in \mathcal{O}(I_r \cup \mathbb{S}_r, SL(2, \mathbb{C})) \mid B(0) = \left(\begin{smallmatrix} \rho & c \\ 0 & 1/\rho \end{smallmatrix}\right) \text{ for } \rho \in \mathbb{R}^+ \text{ and } c \in \mathbb{C}\}.$$

The normalization $B(0) = B_0$ ensures that

$$\Lambda_r SU(2) \cap \Lambda_r^+ SL(2, \mathbb{C}) = \{\mathbb{1}\}.$$

Now we have the following important result due to Pressley-Segal [46] that has been generalized by McIntosh [42].

Theorem 5.19. *The multiplication $\Lambda_r SU(2) \times \Lambda_r^+ SL(2,\mathbb{C}) \to \Lambda_r SL(2,\mathbb{C})$ is a surjective real analytic diffeomorphism. The unique splitting of an element $\phi_\lambda \in \Lambda_r SL(2,\mathbb{C})$ into*

$$\phi_\lambda = F_\lambda B_\lambda$$

*with $F_\lambda \in \Lambda_r SU(2)$ and $B_\lambda \in \Lambda_r^+ SL(2,\mathbb{C})$ is called **r-Iwasawa decomposition** of ϕ_λ or **Iwasawa decomposition** if $r = 1$.*

Remark 5.20. *The r-Iwasawa decomposition also holds on the Lie algebra level, i.e. $\Lambda_r \mathfrak{sl}_2(\mathbb{C}) = \Lambda_r \mathfrak{su}_2 \oplus \Lambda_r^+ \mathfrak{sl}_2(\mathbb{C})$. This decomposition will play a very important role in the following.*

In the finite type situation we can consider the following r-Iwasawa decomposition for $t = (t_0, \dots, t_{g-1}) \in \mathbb{C}^g$

$$\exp\left(\xi_\lambda \sum_{i=0}^{g-1} \lambda^{-i} t_i\right) = F_\lambda(t) B_\lambda(t).$$

Since $[M_\lambda, \xi_\lambda] = 0$ the eigenvectors for M_λ also diagonalize $\xi_\lambda = B \begin{pmatrix} \lambda^{-1}\nu & 0 \\ 0 & -\lambda^{-1}\nu \end{pmatrix} B^{-1}$ and we get

$$\exp\left(\xi_\lambda \sum_{i=0}^{g-1} \lambda^{-i} t_i\right) = B \exp\begin{pmatrix} \sum_{i=0}^{g-1} t_i \lambda^{-i-1}\nu & 0 \\ 0 & -\sum_{i=0}^{g-1} t_i \lambda^{-i-1}\nu \end{pmatrix} B^{-1}$$

and therefore

$$F_\lambda^{-1}(t) B \exp\begin{pmatrix} \sum_{i=0}^{g-1} t_i \lambda^{-i-1}\nu & 0 \\ 0 & -\sum_{i=0}^{g-1} t_i \lambda^{-i-1}\nu \end{pmatrix} = B_\lambda(t) B.$$

Due to $Be_i = v_i$ we see

$$\exp(\sum_{i=0}^{g-1} t_i \lambda^{-i-1}\nu) F_\lambda^{-1}(t) v_1 = B_\lambda(t) v_1.$$

In particular we obtain for $t = (x, 0, \dots, 0)$ the equation

$$\exp(x\lambda^{-1}\nu) F_\lambda^{-1}(x) v_1 = B_\lambda(x) v_1.$$

For $\psi = F_\lambda^{-1} v_1$ and $\lambda^{-1}\nu = \frac{i}{2\sqrt{\lambda}} + O(1)$ around $\lambda = 0$ we see again that $\psi^0 = \exp(\frac{ix}{2\sqrt{\lambda}})\psi$ is holomorphic at $\lambda = 0$.

An equivariant mapping. For $t = (t_0, \dots, t_{g-1})$ consider the map $\zeta_\lambda(t) \in \Lambda_r \mathfrak{sl}_2(\mathbb{C})$ that is given by

$$\begin{aligned} \zeta_\lambda(t) &:= P_t(\widetilde{\nu}) + \sigma^* P_t(\widetilde{\nu}) = F_\lambda^{-1}(t) \left(P(\widetilde{\nu}) + \sigma^* P(\widetilde{\nu})\right) F_\lambda(t) \\ &= F_\lambda^{-1}(t) \xi_\lambda F_\lambda(t) \end{aligned}$$

with $\widetilde{\nu} = \frac{\nu}{\lambda}$. Then $\zeta_\lambda(t)$ obviously satisfies

$$\frac{d}{dt_i}\zeta_\lambda(t) = [\zeta_\lambda(t), U_\lambda(t_i)] \text{ with } U_\lambda(t_i) := F_\lambda(t)^{-1}\left(\tfrac{d}{dt_i}F_\lambda(t)\right).$$

For $i \neq 0$ the form $U_\lambda(t_i)$ corresponds to a higher flow in the sinh-Gordon hierarchy. Recall that the spectral curve $Y \in \Sigma_g^{\mathbf{p}} \simeq \mathcal{M}_g^1(\mathbf{p})$ is the compactification of $Y^* = \{(\lambda,\nu) \in \mathbb{C}^* \times \mathbb{C}^* \mid \nu^2 = -\det(\xi_\lambda)\}$. Due to Proposition 4.33 we can assign to every $E \in \mathrm{Pic}^{\mathbb{R}}_{g+1}(Y)$ a projector $P(E)$ such that $P(E)(\lambda^{-1}\nu) + \sigma^* P(E)(\lambda^{-1}\nu) = \xi_\lambda$.

Proposition 5.21. *The homeomorphism*

$$\xi_\lambda : Pic^{\mathbb{R}}_{g+1}(Y) \to \{\xi_\lambda \in \mathcal{P}_g \mid \det(\xi_\lambda) = -\lambda^{-1}a(\lambda)\},\ E \mapsto \xi_\lambda(E)$$

with

$$\xi_\lambda(E) := P(E)(\lambda^{-1}\nu) + \sigma^* P(E)(\lambda^{-1}\nu)$$

is an equivariant mapping for the action $\pi : \mathbb{R}^g \times Pic^{\mathbb{R}}_{g+1}(Y) \to Pic^{\mathbb{R}}_{g+1}(Y)$ introduced in Theorem 5.17 and the commutative and transitive group action (compare [27]) given by

$$\pi(t_0,\ldots,t_{g-1})(\xi_\lambda) := \phi_\lambda(t) = F_\lambda^{-1}(t)\xi_\lambda F_\lambda(t).$$

This action respects the Iwasawa decomposition for $t = (t_0,\ldots,t_{g-1})$ that is given by

$$\exp\left(\xi_\lambda \sum_{i=0}^{g-1}\lambda^{-i}t_i\right) = F_\lambda(t_0,\ldots,t_{g-1})B_\lambda(t_0,\ldots,t_{g-1}).$$

Proof. Setting $\widetilde{\nu} = \frac{\nu}{\lambda}$, a direct calculation shows

$$\begin{aligned}
\xi_\lambda(\pi(t_0,\ldots,t_{g-1})(E)) &= \xi_\lambda(E \otimes L(t_0,\ldots,t_{g-1})) = P_t(E)(\widetilde{\nu}) + \sigma^* P_t(E)(\widetilde{\nu}) \\
&= F_\lambda^{-1}(t)\left(P(E)(\widetilde{\nu}) + \sigma^* P(E)(\widetilde{\nu})\right) F_\lambda(t) \\
&= F_\lambda^{-1}(t)\xi_\lambda(E)F_\lambda(t) \\
&= \pi(t_0,\ldots,t_{g-1})(\xi_\lambda(E))
\end{aligned}$$

and thus the claim is proved. □

Considering $\zeta_\lambda(x) := \phi_\lambda(t) \in \Lambda_r\mathfrak{sl}_2(\mathbb{C})$ for $t = (x,0,\ldots,0)$ we obtain

$$\zeta_\lambda(x) = F_\lambda^{-1}(x)\xi_\lambda F_\lambda(x).$$

Then $\zeta_\lambda(x)$ satisfies the equation

$$\frac{d}{dx}\zeta_\lambda(x) = [\zeta_\lambda(x), U_\lambda(x)],$$

i.e. $\zeta_\lambda(x)$ is the polynomial Killing field with $\zeta_\lambda(0) = \xi_\lambda$. Moreover, the r-Iwasawa decomposition

$$\exp(x\,\xi_\lambda) = F_\lambda(x)B_\lambda(x)$$

holds in that situation.

5.2.2 Infinitesimal deformations of ξ_λ and U_λ

In order to describe the infinitesimal deformations of ξ_λ and U_λ, we follow the exposition of [2], IV.2.e. Transfering those methods to our situation yields

Theorem 5.22. *Let $f_0(\lambda,\nu) = \sum_{i=0}^{g-1} c_i \lambda^{-i-1}\nu$ be the representative of $[(f_0, \eta^*\bar{f}_0)] \in H^1_{\mathbb{R}}(X,\mathcal{O})$ and let*

$$A_{f_0} := P(f_0) + \sigma^* P(f_0) = \sum_{i=0}^{g-1} c_i \lambda^{-i}(P(\lambda^{-1}\nu) + \sigma^* P(\lambda^{-1}\nu)) = \sum_{i=0}^{g-1} c_i \lambda^{-i}\xi_\lambda$$

be the induced element in $\Lambda_r\mathfrak{sl}_2(\mathbb{C})$. Then the vector field of the isospectral action $\pi : \mathbb{R}^g \times Pic^{\mathbb{R}}_{g+1}(Y) \to Pic^{\mathbb{R}}_{g+1}(Y)$ at E takes the value

$$\dot{\xi}_\lambda = [A^+_{f_0}, \xi_\lambda] = -[\xi_\lambda, A^-_{f_0}]$$

under the equivariant map $E \mapsto \xi_\lambda(E)$ from Proposition 5.21. Here $A_{f_0} = A^+_{f_0} + A^-_{f_0}$ is the Lie algebra decomposition of the Iwasawa decomposition.

Proof. We write ν for $\widetilde{\nu}$. Obviously there holds $A_{f_0}v = f_0 v$. If we write $\widetilde{v} = e^{\beta_0(t)}v$ for local sections $\widetilde{v}$ of $\mathcal{O}_D \otimes L(t_0,\dots,t_{g-1})$ and v of $\mathcal{O}_D$ with $\beta_0(t) = \sum_{i=0}^{g-1} c_i t_i \lambda^{-i-1}\nu$ we get

$$\begin{aligned}
\delta\widetilde{v} &= \dot{\beta}_0(t)\widetilde{v} + e^{\beta_0(t)}\delta v = f_0\widetilde{v} + e^{\beta_0(t)}\delta v \\
&= A_{f_0}\widetilde{v} + e^{\beta_0(t)}\delta v \\
&= e^{\beta_0(t)}(A_{f_0}v + \delta v).
\end{aligned}$$

Moreover, $(f_0) \geq 0$ on Y^*. This shows that $A_{f_0}v + \delta v$ is a vector-valued section of $\mathcal{O}_D$ on Y^* and defines a map $A^+_{f_0}$ such that

$$A_{f_0}v + \delta v = A^+_{f_0}v$$

holds. Since $A_{f_0}v = f_0 v$ we also obtain the equations

$$\begin{cases} \xi_\lambda v = \nu v, \\ \xi_\lambda(A^+_{f_0}v - \delta v) = \nu(A^+_{f_0}v - \delta v). \end{cases}$$

This implies

$$\xi_\lambda \delta v + [A^+_{f_0}, \xi_\lambda]v = \nu\delta v.$$

Differentiating the equation $\xi_\lambda v = \nu v$ we additionally obtain

$$\dot{\xi}_\lambda v + \xi_\lambda \delta v = \dot{\nu} v + \nu\delta v = \nu\delta v.$$

Combining the last two equations yields

$$\dot{\xi}_\lambda v = [A^+_{f_0}, \xi_\lambda]v$$

and concludes the proof since this equation holds for a basis of eigenvectors. ☐

Remark 5.23. *The decomposition of $A_f \in \Lambda_r\mathfrak{sl}(2,\mathbb{C}) = \Lambda_r\mathfrak{su}_2 \oplus \Lambda_r^+\mathfrak{sl}_2(\mathbb{C})$ yields $A_{f_0} = A_{f_0}^+ + A_{f_0}^-$ and therefore $A_{f_0}v + \delta v = A_{f_0}^+ v$ implies*

$$\delta v = -A_{f_0}^- v.$$

In particular $A_{f_0}^-$ is given by $A_{f_0}^- = -\sum \frac{\delta v w^t}{w^t v}$.

We want to extend Theorem 5.22 to obtain the value of the vectorfield induced by $\pi : \mathbb{R}^g \times \mathrm{Pic}_{g+1}^{\mathbb{R}}(Y) \to \mathrm{Pic}_{g+1}^{\mathbb{R}}(Y)$ at U_λ.

Theorem 5.24. *Let $f_0(\lambda,\nu) = \sum_{i=0}^{g-1} c_i\lambda^{-i-1}\nu$ be the representative of $[(f_0, \eta^*\bar{f}_0)] \in H^1_{\mathbb{R}}(X,\mathcal{O})$ and let*

$$A_{f_0}(x) := P_x(f_0) + \sigma^* P_x(f_0) = \sum_{i=0}^{g-1} c_i\lambda^{-i}(P_x(\lambda^{-1}\nu) + \sigma^* P_x(\lambda^{-1}\nu)) = \sum_{i=0}^{g-1} c_i\lambda^{-i}\zeta_\lambda(x)$$

be the induced map $A_{f_0} : \mathbb{R} \to \Lambda_r\mathfrak{sl}_2(\mathbb{C})$. Then the vector field of the isospectral action $\pi : \mathbb{R}^g \times Pic_{g+1}^{\mathbb{R}}(Y) \to Pic_{g+1}^{\mathbb{R}}(Y)$ at E takes the value

$$\delta U_\lambda(x) = [A_{f_0}^+(x), L_\lambda(x)] = [L_\lambda(x), A_{f_0}^-(x)].$$

Here $A_{f_0}(x) = A_{f_0}^+(x) + A_{f_0}^-(x)$ is the Lie algebra decomposition of the Iwasawa decomposition.

Proof. Obviously $A_{f_0}(x)v(x) = f_0 v(x)$. In analogy to the proof of Theorem 5.22 we obtain a map $A_{f_0}^+(x)$ such that

$$A_{f_0}(x)v(x) + \delta v(x) = A_{f_0}^+(x)v(x)$$

holds. Since $A_{f_0}(x)v(x) = f_0 v(x)$ we also obtain the equations

$$\begin{cases} L_\lambda(x)v(x) = \frac{\ln\mu}{\mathbf{p}}v(x), \\ L_\lambda(x)(A_{f_0}^+(x)v(x) - \delta v(x)) = \frac{\ln\mu}{\mathbf{p}}(A_{f_0}^+(x)v(x) - \delta v(x)) \end{cases}$$

around $\lambda = 0$. This implies

$$L_\lambda(x)\delta v(x) + [A_{f_0}^+(x), L_\lambda(x)]v(x) = \frac{\ln\mu}{\mathbf{p}}\delta v(x).$$

Differentiating the equation $L_\lambda(x)v(x) = \frac{\ln\mu}{\mathbf{p}}v(x)$ we additionally obtain

$$\delta L_\lambda(x)v(x) + L_\lambda(x)\delta v(x) = \frac{\delta\ln\mu}{\mathbf{p}}v(x) + \frac{\ln\mu}{\mathbf{p}}\delta v(x) = \frac{\ln\mu}{\mathbf{p}}\delta v(x).$$

Combining the last two equations yields

$$\delta L_\lambda(x)v(x) = \delta U_\lambda(x)v(x) = [A_{f_0}^+(x), L_\lambda(x)]v(x)$$

and concludes the proof since this equation holds for a basis of eigenvectors. □

5.3 General deformations of M_λ and U_λ

In the next lemma we consider the situation of a general variation with isospectral and non-isospectral parts.

Lemma 5.25. *Let v_1, w_1^t be the eigenvectors for μ and v_2, w_2^t the corresponding eigenvectors for $\frac{1}{\mu}$ of $M(\lambda)$. Then*

$$\delta M(\lambda)v_1 + M(\lambda)\delta v_1 = (\delta\mu)v_1 + \mu\delta v_1 \quad \textit{and} \quad \delta M(\lambda)v_2 + M(\lambda)\delta v_2 = \delta(\tfrac{1}{\mu})v_2 + \tfrac{1}{\mu}\delta v_2 \qquad (*)$$

if and only if

$$\delta M(\lambda) = \left[\sum_{i=1}^{2} \frac{(\delta v_i)w_i^t}{w_i^t v_i}, M(\lambda)\right] + (P(\delta\mu) + \sigma^* P(\delta\mu)). \qquad (**)$$

Proof. A direct calculation shows

$$\begin{aligned}\left(\left[\sum_{i=1}^{2} \frac{(\delta v_i)w_i^t}{w_i^t v_i}, M(\lambda)\right] + (P(\delta\mu) + \sigma^* P(\delta\mu))\right) v_1 &= \left(\sum_{i=1}^{2} \frac{(\delta v_i)w_i^t}{w_i^t v_i}\right)\mu v_1 - M(\lambda)\delta v_1 + (\delta\mu)v_1 \\ &= \mu\delta v_1 - M(\lambda)\delta v_1 + (\delta\mu)v_1 \\ &\overset{(*)}{=} \delta M(\lambda)v_1.\end{aligned}$$

An analogous calculation for v_2 gives

$$\delta M(\lambda)v_2 = \left(\left[\sum_{i=1}^{2} \frac{(\delta v_i)w_i^t}{w_i^t v_i}, M(\lambda)\right] + (P(\delta\mu) + \sigma^* P(\delta\mu))\right) v_2$$

and the claim is proved. □

Since the arguments from the previous proof carry over to the equation $M_\lambda(x)v(x) = \mu v(x)$ we get the following

Corollary 5.26. *For the x-dependent monodromy $M_\lambda(x)$ a general variation is given by*

$$\delta M_\lambda(x) = \left[\sum_{i=1}^{2} \frac{(\delta v_i(x))w_i^t(x)}{w_i^t(x)v_i(x)}, M_\lambda(x)\right] + (P_x(\delta\mu) + \sigma^* P_x(\delta\mu)).$$

The above considerations also apply for the equation $L_\lambda(x)v(x) = (\frac{d}{dx} + U_\lambda)v(x) = \frac{\ln\mu}{\mathbf{p}} \cdot v(x)$ around $\lambda = 0$ and yield

Lemma 5.27. *Let $v_1(x), w_1^t(x)$ be the eigenvectors for μ and $v_2(x), w_2^t(x)$ the corresponding eigenvectors for $\frac{1}{\mu}$ of $M_\lambda(x)$ and $M_\lambda^t(x)$ respectively. Then*

$$\delta U_\lambda v_1(x) + L_\lambda(x)\delta v_1(x) = (\tfrac{\delta \ln\mu}{\mathbf{p}})v_1(x) + \tfrac{\ln\mu}{\mathbf{p}}\delta v_1(x) \qquad (*)$$

and

$$\delta U_\lambda v_2(x) + L_\lambda(x)\delta v_2(x) = -(\tfrac{\delta \ln \mu}{\mathbf{p}})v_2(x) - \tfrac{\ln \mu}{\mathbf{p}}\delta v_2(x) \tag{$*$}$$

around $\lambda = 0$ *if and only if*

$$\delta U_\lambda = \left[L_\lambda(x), -\sum_{i=1}^{2} \frac{(\delta v_i(x))w_i^t(x)}{w_i^t(x)v_i(x)}\right] + (P_x(\tfrac{\delta \ln \mu}{\mathbf{p}}) + \sigma^* P_x(\tfrac{\delta \ln \mu}{\mathbf{p}})). \tag{$**$}$$

Proof. Following the steps from the proof of Lemma 5.25 and keeping in mind that $\delta\mathbf{p} = 0$ in our situation yields the claim. □

Remark 5.28. *The above formula* $(**)$ *reflects the decomposition of the tangent space into isospectral and non-isospectral deformations.*

6 Hamiltonian formalism and the symplectic form

6.1 Completely integrable Hamiltonian systems

In the following we will show that periodic finite type solutions of the sinh-Gordon equation can be considered as a completely integrable system (compare with [15, 16] in the context of classical string theory). We refer to [19, 31, 32] for further reading. A nice introduction to the subject can also be found in [33]. We start with some definitions.

Definition 6.1. *A bilinear mapping*

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M), \quad (f,g) \mapsto \{f,g\}$$

on a differentiable manifold M is called a ***Poisson bracket*** *if it satisfies*

(i) (Anti-symmetry) $\{f,g\} = -\{g,f\}$,

(ii) (Jacobi identity) $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$ and

(iii) (Leibniz rule) $\{f,gh\} = \{f,g\}h + h\{f,h\}$.

A ***Poisson manifold*** *$(M,\{\cdot,\cdot\})$ is a differentiable manifold M with a Poisson bracket $\{\cdot,\cdot\}$ on M.*

Since a Poisson bracket $\{\cdot,\cdot\}$ satisfies (i) and (ii) from Definition 6.1 it is also a Lie bracket and thus $(C^\infty(M),\{\cdot,\cdot\})$ is a Lie algebra.

For every $f \in C^\infty(M)$ the map $g \mapsto \{f,g\}$ is a derivation of the smooth functions on M and therefore defines a vector field, denoted by $X(f) \in \mathcal{X}_M$. In particular one has $dg(X(f)) = \{f,g\}$ and

$$X(\{f,g\}) = [X(f),X(g)] \text{ for all } f,g \in C^\infty(M),$$

where $[\cdot,\cdot]$ is the Lie bracket of vector fields. Thus $f \mapsto X(f)$ induces a Lie algebra homomorphism $\phi : (C^\infty(M),\{\cdot,\cdot\}) \to (\mathcal{X}_M,[\cdot,\cdot])$.

Definition 6.2. *Let $(M,\{\cdot,\cdot\})$ be a Poisson manifold. Two functions $f,g \in C^\infty(M)$ are said to be* ***in involution*** *if $\{f,g\} = 0$.*

Definition 6.3. *A* ***symplectic manifold*** *(M,Ω) is a differentiable manifold M with a non-degenerate closed 2-form Ω. The form Ω is called* ***symplectic form*** *of M.*

Definition 6.4. *A **Hamiltonian system** (M, Ω, H) is a symplectic manifold (M, Ω) with a smooth function $H : M \to \mathbb{R}$. The corresponding vector field $X(H)$ is called **Hamiltonian vector field**.*

Definition 6.5. *Let (M, Ω, H) be a Hamiltonian system. A function $f : M \to \mathbb{R}$ is called **integral of motion** for the Hamiltonian system (M, Ω, H) if f is preserved under the flow $\Phi_{X(H)}$ of the Hamiltonian vector field $X(H)$.*

From Definition 6.5 we immediately deduce that $f : M \to \mathbb{R}$ is an integral of motion if and only if $0 = \frac{d}{dt} f(\Phi_{X(H)})|_{t=0} = df(X(H)) = \{f, H\}$, i.e. f and H are in involution. From the formula

$$[X(f), X(H)] = X(\{f, H\}) \stackrel{!}{=} 0$$

we see that the corresponding Hamiltonian vector fields $X(f)$ and $X(H)$ commute. In particular the Hamiltonian H itself is an integral of motion.

Definition 6.6. *A Hamiltonian system (M, Ω, H) with $\dim(M) = 2n$ is called **completely integrable** if and only if the system has besides the Hamiltonian $H =: f_1$ additional $n - 1$ integrals of motion $f_2, \ldots, f_n$ such that the derivatives $df_1, \ldots, df_n$ are linear independent in T_p^*M for all $p \in M$.*

Definition 6.7. *A subspace of a vector space which is maximal isotropic with respect to a symplectic form Ω is called **Lagrangian**. A submanifold N of a symplectic manifold (M, Ω) is called Lagrangian if and only if T_pN is a Lagrangian subspace with respect to Ω for all $p \in N$.*

6.2 The phase space $(M_g^{\mathfrak{p}}, \Omega)$

In the following we will define the phase space of our integrable system. We need some preparation and first recall the generalized Weierstrass representation [17]. Set

$$\Lambda_{-1}^{\infty}\mathfrak{sl}_2(\mathbb{C}) = \{\xi_\lambda \in \mathcal{O}(\mathbb{C}^*, \mathfrak{sl}_2(\mathbb{C})) \mid (\lambda\xi_\lambda)_{\lambda=0} \in \mathbb{C}^*\epsilon_+\}.$$

A **potential** is a holomorphic 1-form $\xi_\lambda dz$ on $\mathbb{C}$ with $\xi_\lambda \in \Lambda_{-1}^{\infty}\mathfrak{sl}_2(\mathbb{C})$. Given such a potential one can solve the holomorphic ODE $d\phi_\lambda = \phi_\lambda\xi_\lambda$ to obtain a map $\phi_\lambda : \mathbb{C} \to \Lambda_r SL(2, \mathbb{C})$. Then Theorem 5.19 yields an extended frame $F_\lambda : \mathbb{C} \to \Lambda_r SU(2)$ via the r-Iwasawa decomposition

$$\phi_\lambda = F_\lambda B_\lambda.$$

It is proven in [17] that each extended frame can be obtained from a potential $\xi_\lambda dz$ by the Iwasawa decomposition. Note, that we have the inclusions

$$\mathcal{P}_g \subset \Lambda_{-1}^{\infty}\mathfrak{sl}_2(\mathbb{C}) \subset \Lambda_r\mathfrak{sl}_2(\mathbb{C}).$$

An extended frame $F_\lambda : \mathbb{C} \to \Lambda_r SU(2)$ is of **finite type**, if there exists $g \in \mathbb{N}$ such that the corresponding potential $\xi_\lambda dz$ satisfies $\xi_\lambda \in \mathcal{P}_g \subset \Lambda_{-1}^{\infty}\mathfrak{sl}_2(\mathbb{C})$. We say that a polynomial Killing field has minimal degree if and only if it has neither roots nor poles in $\lambda \in \mathbb{C}^*$. We will need the following proposition that summarizes two results by Burstall-Pedit [12, 13].

Proposition 6.8 ([27], Proposition 4.5)**.** *For an extended frame of finite type there exists a unique polynomial Killing field of minimal degree. There is a smooth 1:1 correspondence between the set of extended frames of finite type and the set of polynomial Killing fields without zeros.*

Consider the map $A : \xi_\lambda \mapsto A(\xi_\lambda) := -\lambda \det \xi_\lambda$ (see [27]) and set $\mathcal{P}_g^1(\mathbf{p}) := A^{-1}[\mathcal{M}_g^1(\mathbf{p})]$. Moreover, denote by $C_{\mathbf{p}}^\infty := C^\infty(\mathbb{R}/\mathbf{p})$ the Fréchet space of real infinitely differentiable functions of period $\mathbf{p} \in \mathbb{R}^+$. The above discussion and Remark 3.19 yield an injective map

$$\phi : \mathcal{P}_g^1(\mathbf{p}) \subset \Lambda_{-1}^\infty \mathfrak{sl}_2(\mathbb{C}) \to \phi[\mathcal{P}_g^1(\mathbf{p})] \subset C_{\mathbf{p}}^\infty \times C_{\mathbf{p}}^\infty,\ \xi_\lambda \mapsto (u(\xi_\lambda), u_y(\xi_\lambda)).$$

Definition 6.9. *Let $M_g^{\mathbf{p}}$ denote the space of $(u, u_y) \in C_{\mathbf{p}}^\infty \times C_{\mathbf{p}}^\infty$ (with fixed period $\mathbf{p}$) such that (u, u_y) is of finite type in the sense of Def. 3.13, where Φ_λ is of fixed degree $g \in \mathbb{N}_0$, and $\zeta_\lambda(0) \in \mathcal{P}_g^1(\mathbf{p})$ with $\zeta_\lambda = \Phi_\lambda - \lambda^{g-1}\overline{\Phi_{1/\bar{\lambda}}}^t$, i.e. $M_g^{\mathbf{p}} := \phi[\mathcal{P}_g^1(\mathbf{p})]$.*

Now we are able to prove the following

Lemma 6.10. *The map $\phi : \mathcal{P}_g^1(\mathbf{p}) \to M_g^{\mathbf{p}},\ \xi_\lambda \mapsto (u(\xi_\lambda), u_y(\xi_\lambda))$ is an embedding.*

Proof. From the previous discussion we know that $\phi : \mathcal{P}_g^1(\mathbf{p}) \to M_g^{\mathbf{p}}$ is bijective. We show that $\phi^{-1} : M_g^{\mathbf{p}} \to \mathcal{P}_g^1(\mathbf{p})$ is continuous. Assume that g is the minimal degree for $\xi_\lambda \in \mathcal{P}_g^1(\mathbf{p})$ (see Proposition 6.8). Then the Jacobi fields

$$(\omega_0, \partial_y\omega_0), \ldots, (\omega_{g-1}, \partial_y\omega_{g-1}) \in C^\infty(\mathbb{C}/\mathbf{p}) \times C^\infty(\mathbb{C}/\mathbf{p})$$

are linearly independent over $\mathbb{C}$ with all their derivatives up to order $2g+1$. We will now show that they stay linearly independent if we restrict them to $\mathbb{R}$. For this, suppose that they are linearly dependent on $\mathbb{R}$ with all their derivatives up to order $2g+1$. Since u solves the elliptic sinh-Gordon with analytic coefficients u is analytic on $\mathbb{C}$ [41]. Thus the $(\omega_i, \partial_y\omega_i)$ are analytic as well since they only depend on u and its k-th derivatives with $k \leq 2i+1 \leq 2g+1$ (see [45], Proposition 3.1). Thus they stay linearly dependent on an open neighborhood and the subset $M \subset \mathbb{C}$ of points such that these functions are linearly dependent is open and closed. Therefore $M = \mathbb{C}$, a contradiction!

By considering all derivatives of (u, u_y) up to order $2g+1$ we get a small open neighborhood U of $(u, u_y) \in C^\infty(\mathbb{R}/\mathbf{p}) \times C^\infty(\mathbb{R}/\mathbf{p})$ such that the functions

$$(\widetilde{\omega}_0, \partial_y\widetilde{\omega}_0), \ldots, (\widetilde{\omega}_{g-1}, \partial_y\widetilde{\omega}_{g-1}) \in C^\infty(\mathbb{R}/\mathbf{p}) \times C^\infty(\mathbb{R}/\mathbf{p})$$

stay linearly independent for $(\widetilde{u}, \widetilde{u}_y) \in U$. Given $(u, u_y) \in U$ there exist numbers $a_0, \ldots, a_{g-1}$ such that the g vectors

$$((\omega_0(a_j), \partial_y\omega_0(a_j)), \ldots, (\omega_{g-1}(a_j), \partial_y\omega_{g-1}(a_j)))^t$$

are linearly independent. Recall that $(\omega_g, \partial_y\omega_g) = \sum_{i=0}^{g-1} c_i(\omega_i, \partial_y\omega_i)$ in the finite type situation, which assures the existence of a polynomial Killing field. Inserting these $a_0, \ldots, a_{g-1}$

into the equation $(\omega_g, \partial_y\omega_g) = \sum_{i=0}^{g-1} c_i(\omega_i, \partial_y\omega_i)$ we obtain an invertible $g\times g$ matrix and can calculate the c_i. This shows that the coefficients c_i continuously depend on $(u,u_y)\in U$. Thus for $(u,u_y)\in M_g^{\mathbf{p}}$ and $\varepsilon>0$ there exists a $\delta_\varepsilon>0$ such that

$$\|\xi_\lambda(u,u_y)-\xi_\lambda(\widetilde{u},\widetilde{u}_y)\|<\varepsilon$$

holds for all Cauchy data $(\widetilde{u},\widetilde{u}_y)\in M_g^{\mathbf{p}}$ with $\|(u,u_y)-(\widetilde{u},\widetilde{u}_y)\|<\delta_\varepsilon$, where the norm is given by the supremum of the first $2g+1$ derivatives. $\square$

Let us study the map

$$Y : M_g^{\mathbf{p}} \to \Sigma_g^{\mathbf{p}} \simeq \mathcal{M}_g^1(\mathbf{p}), \;\; (u,u_y)\mapsto Y(u,u_y)$$

that appears in the diagram

$$\begin{tikzcd} \mathcal{P}_g^1(\mathbf{p}) \arrow[d, "\phi"'] \arrow[dr, "A"] & \\ M_g^{\mathbf{p}} \arrow[r, "Y"] & \Sigma_g^{\mathbf{p}} \end{tikzcd}$$

Proposition 6.11. *The map* $Y : M_g^{\mathbf{p}} \to \Sigma_g^{\mathbf{p}} \simeq \mathcal{M}_g^1(\mathbf{p}),\; (u,u_y)\mapsto Y(u,u_y)$ *is a principle bundle with fibre Iso*$(Y(u,u_y)) \simeq$ *Pic*$_{g+1}^{\mathbb{R}}(Y(u,u_y)) \simeq (\mathbb{S}^1)^g$. *In particular* $M_g^{\mathbf{p}}$ *is a manifold of dimension* $2g$.

Proof. Due to Theorem 5.11 the space $\mathcal{M}_g^1(\mathbf{p})$ is a smooth g-dimensional manifold. From Proposition 4.12 in [27] we know that the mapping

$$A : \mathcal{P}_g^1(\mathbf{p}) \to \mathcal{M}_g^1(\mathbf{p}),\; \xi_\lambda \mapsto -\lambda\det(\xi_\lambda)$$

is a principal fibre bundle with fibre $(\mathbb{S}^1)^g$ and thus $\mathcal{P}_g^1(\mathbf{p})$ is a manifold of dimension $2g$. Due to Lemma 6.10 the map $\phi : \mathcal{P}_g^1(\mathbf{p}) \to M_g^{\mathbf{p}}$ is an embedding and thus $M_g^{\mathbf{p}}$ is a manifold of dimension $2g$ as well. $\square$

Note, that the structure of such “finite-gap manifolds” is also investigated in [18] and [7, 39]. It will turn out that $M_g^{\mathbf{p}}$ can be considered as a symplectic manifold with a certain symplectic form Ω. To see this, we closely follow the exposition of [44] and consider the phase space of $(q,p)\in C_{\mathbf{p}}^\infty\times C_{\mathbf{p}}^\infty$ equipped with the symplectic form

$$\Omega = \int_0^{\mathbf{p}} dq\wedge dp$$

and the Poisson bracket

$$\{f,g\} = \int_0^{\mathbf{p}} \langle \nabla f, J\nabla g\rangle\, dx \;\;\text{with}\;\; J=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}.$$

Here f and g are functionals of the form $h : C_{\mathbf{p}}^\infty\times C_{\mathbf{p}}^\infty\to\mathbb{R},\; (q,p)\mapsto h(q,p)$ and ∇h denotes the corresponding gradient of h in the function space $C_{\mathbf{p}}^\infty\times C_{\mathbf{p}}^\infty$. Note that there holds $\{f, g\} = \Omega(\nabla f, \nabla g)$. If we consider functionals H, f on the function space $M = C^\infty_{\mathbf{p}} \times C^\infty_{\mathbf{p}}$ we have

$$df(X) = \int_0^{\mathbf{p}} \langle \nabla f, X\rangle \, dx$$

and

$$X(H) = J\nabla H.$$

Since $X(H)$ is a vector field it defines a flow $\Phi : O \subset M \times \mathbb{R} \to M$ such that $\Phi((q_0, p_0), t)$ solves

$$\frac{d}{dt}\Phi((q_0, p_0), t) = X(H)(\Phi((q_0, p_0), t)) \;\text{ with }\; \Phi((q_0, p_0), 0) = (q_0, p_0).$$

In the following we will write $(q(t), p(t))^t := \Phi((q_0, p_0), t)$ for integral curves of $X(H)$ that start at (q_0, p_0). A direct calculation shows

$$\frac{d}{dt} f(q(t), p(t))\bigg|_{t=0} = df(X(H)) = \int_0^{\mathbf{p}} \langle \nabla f, J\nabla H\rangle \, dx = \{f, H\}$$

and we see again that f is an integral of motion if and only if f and H are in involution. Set $(q, p) = (u, u_y)$, where u is a solution of the sinh-Gordon equation, i.e.

$$\Delta u + 2\sinh(2u) = u_{xx} + u_{yy} + 2\sinh(2u) = 0.$$

Setting $t = y$ we can investigate the so-called sinh-Gordon flow that is expressed by

$$\frac{d}{dy}\begin{pmatrix} u \\ u_y \end{pmatrix} = \begin{pmatrix} u_y \\ -u_{xx} - 2\sinh(2u) \end{pmatrix} = J\nabla H_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial H_2}{\partial q} \\ \frac{\partial H_2}{\partial p} \end{pmatrix}$$

with the Hamiltonian

$$H_2(q, p) = \int_0^{\mathbf{p}} \tfrac{1}{2}p^2 - \tfrac{1}{2}(q_x)^2 + \cosh(2q)\, dx = \int_0^{\mathbf{p}} \tfrac{1}{2}(u_y)^2 - \tfrac{1}{2}(u_x)^2 + \cosh(2u)\, dx$$

and corresponding gradient

$$\nabla H_2 = (q_{xx} + 2\sinh(2q), p)^t = (u_{xx} + 2\sinh(2u), u_y)^t\,.$$

Remark 6.12. *Since we have a loop group splitting (the r-Iwasawa decomposition) in the finite type situation, all corresponding flows can be integrated. Thus the flow $(q(y), p(y))^t = (u(x, y), u_y(x, y))^t$ that corresponds to the* sinh*-Gordon flow is defined for all $y \in \mathbb{R}$.*

Due to Remark 6.12 $q(y) = u(x, y)$ is a periodic solution of the sinh-Gordon equation with $u(x + \mathbf{p}, y) = u(x, y)$ for all $(x, y) \in \mathbb{R}^2$. The Hamiltonian H_2 is an integral of motion, another one is associated with the flow of translation (here we set $t = x$) induced by the functional

$$H_1(q, p) = \int_0^{\mathbf{p}} p q_x \, dx = \int_0^{\mathbf{p}} u_y u_x \, dx$$

with

$$\frac{d}{dx}\begin{pmatrix} u \\ u_y \end{pmatrix} = \begin{pmatrix} u_x \\ u_{yx} \end{pmatrix} = J\nabla H_1.$$

6.3 Polynomial Killing fields and integrals of motion

The Pinkall-Sterling iteration generates a sequence of solutions to the homogeneous Jacobi equation. The iteration requires for a given ω_n to find τ_n solving

$$\begin{aligned}\partial\tau_n &= \partial^2\omega_n - 2\partial\omega_n\partial u,\\ \bar\partial\tau_n &= -e^{-2u}\omega_n\end{aligned}$$

and then defining $\omega_{n+1} = \partial\tau_n + 2\tau_n\partial u$. Here we use a slightly different normalization according to the exposition given in [36]. In order to obtain τ_n it is useful to introduce auxiliary functions ϕ_n with

$$\phi_n = \partial\omega_n - \tau_n$$

that satisfy

$$\begin{aligned}\partial\phi_n &= 2\partial\omega_n\partial u,\\ \bar\partial\phi_n &= -\omega_n\sinh(2u).\end{aligned}$$

In order to supplement ω_n and τ_n at each step to a parametric Jacobi field one has to find a function σ_n that satisfies

$$\begin{aligned}\partial\sigma_n &= -e^{-2u}\omega_n,\\ \bar\partial\sigma_n &= \bar\partial^2\omega_n - 2\bar\partial\omega_n\bar\partial u.\end{aligned}$$

Finally one has the formula

$$\sigma_n = -e^{-2u}(\partial\omega_{n-1} + \phi_{n-1}).$$

We will now describe how the functions $\varphi((\lambda,\mu),z) := F_\lambda^{-1}(z)v(\lambda,\mu)$ and $\psi((\lambda,\mu),z) := \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)\varphi((\lambda,\mu),z)$ can be used to describe the functions ω,σ,τ from the Pinkall-Sterling iteration. Note that we swapped the roles of φ and ψ since we follow the exposition given in [36]. First we see that ψ is a solution of

$$\begin{aligned}\partial\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} &= \frac{1}{2}\begin{pmatrix}\partial u & ie^{-u}\\ i\lambda^{-1}e^{u} & -\partial u\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},\\ \bar\partial\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} &= \frac{1}{2}\begin{pmatrix}-\bar\partial u & i\lambda e^{u}\\ ie^{-u} & \bar\partial u\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}.\end{aligned}$$

Likewise φ solves

$$\begin{aligned}\partial\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} &= -\frac{1}{2}\begin{pmatrix}\partial u & ie^{-u}\\ i\lambda^{-1}e^{u} & -\partial u\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix},\\ \bar\partial\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} &= -\frac{1}{2}\begin{pmatrix}-\bar\partial u & i\lambda e^{u}\\ ie^{-u} & \bar\partial u\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}.\end{aligned}$$

Proposition 6.13 ([36], Proposition 3.1)**.** *Let ψ and φ be solutions of the above system and define $\omega := \psi_1\varphi_1 - \psi_2\varphi_2$. Then*

(i) The functon $h = \psi^t\varphi$ satisfies $dh = 0$.

(ii) The function ω is in the kernel of the Jacobi operator and can be supplemented to a parametric Jacobi field with corresponding (up to complex constants)

$$\tau = \frac{i\psi_2\varphi_1}{e^u}, \quad \sigma = -\frac{i\psi_1\varphi_2}{e^u}.$$

Now we get the following

Proposition 6.14 ([36], Proposition 3.3)**.** *Let $h = \psi^t\varphi$. Then the entries of*

$$P(z) := \frac{\psi\varphi^t}{\psi^t\varphi} = \frac{1}{h}\begin{pmatrix}\psi_1\varphi_1 & \psi_1\varphi_2\\ \psi_2\varphi_1 & \psi_2\varphi_2\end{pmatrix}$$

have at $\lambda = 0$ the asymptotic expansions

$$\begin{aligned}
-\frac{i\omega}{2h} &= \frac{1}{\sqrt{\lambda}}\sum_{n=1}^{\infty}\omega_n(-\lambda)^n,\\
-\frac{i\tau}{h} &= \frac{\psi_2\varphi_1}{e^u h} = \frac{1}{\sqrt{\lambda}}\sum_{n=0}^{\infty}\tau_n(-\lambda)^n,\\
-\frac{i\sigma}{h} &= -\frac{\psi_1\varphi_2}{e^u h} = \frac{1}{\sqrt{\lambda}}\sum_{n=1}^{\infty}\sigma_n(-\lambda)^n.
\end{aligned}$$

By utilizing the involution ρ we can compute the asymptotic expansion of $P(z)$ at $\lambda = \infty$ from the expansion at $\lambda = 0$. Since we have $\bar{\tau} = \sigma$ we get $\rho^*(\frac{\omega}{h}) = \frac{\omega}{h}$ and $\rho^*(\frac{\bar{\tau}}{h}) = \frac{\sigma}{h}$. This yields

Corollary 6.15 ([36], Corollary 3.4)**.** *The entries of $P(z)$ have at $\lambda = \infty$ the asymptotic expansions*

$$\begin{aligned}
\frac{i\omega}{2h} &= \sqrt{\lambda}\sum_{n=1}^{\infty}(-1)^n\bar{\omega}_n\lambda^{-n},\\
\frac{i\tau}{h} &= \sqrt{\lambda}\sum_{n=1}^{\infty}(-1)^n\bar{\sigma}_n\lambda^{-n},\\
\frac{i\sigma}{h} &= \sqrt{\lambda}\sum_{n=0}^{\infty}(-1)^n\bar{\tau}_n\lambda^{-n}.
\end{aligned}$$

Definition 6.16. *Consider the asymptotic expansion*

$$\ln\mu = \frac{1}{\sqrt{\lambda}}\frac{i\mathbf{p}}{2} + \sqrt{\lambda}\sum_{n\geq 0}c_n\lambda^n \quad at \ \ \lambda = 0$$

and set $H_{2n+1} := (-1)^{n+1}\Re(c_n)$ and $H_{2n+2} := (-1)^{n+1}\Im(c_n)$ for $n \geq 0$.

Remark 6.17. *Since*

$$\ln\mu = \tfrac{1}{\sqrt{\lambda}}\tfrac{i\mathbf{p}}{2} + \sqrt{\lambda}\int_0^{\mathbf{p}} \left(-i(\partial u)^2 + \tfrac{i}{2}\cosh(2u)\right)\,dt + O(\lambda)$$

at $\lambda = 0$ we see that the functions H_1, H_2 are given by

$$\begin{aligned} H_1 &= \int_0^{\mathbf{p}} \tfrac{1}{2}u_y u_x\,dx, \\ H_2 &= -\int_0^{\mathbf{p}} \tfrac{1}{4}(u_y)^2 - \tfrac{1}{4}(u_x)^2 + \tfrac{1}{2}\cosh(2u)\,dx. \end{aligned}$$

These functions are proportional to the Hamiltonians that induce the flow of translation and the sinh-*Gordon flow respectively.*

We will now illustrate the link between the Pinkall-Sterling iteration from Proposition 3.17 and these functions H_n (which we call **Hamiltonians** from now on) and show that the functions H_n are pairwise in involution. Recall the formula

$$\frac{d}{dt}H((u,u_y)+t(\delta u,\delta u_y))|_{t=0} = dH_{(u,u_y)}(\delta u,\delta u_y) = \Omega(\nabla H(u,u_y),(\delta u,\delta u_y))$$

from the first section of this chapter. First we need the following lemma.

Lemma 6.18. *For the map $\ln\mu$ we have the variational formula*

$$\left.\frac{d}{dt}\ln\mu((u,u_y)+t(\delta u,\delta u_y))\right|_{t=0} = \int_0^{\mathbf{p}} \frac{1}{\varphi^t\psi}\psi^t\delta U_\lambda\varphi\,dx$$

with

$$\delta U_\lambda = \frac{1}{2}\begin{pmatrix} -i\delta u_y & i\lambda^{-1}e^{u}\delta u - ie^{-u}\delta u \\ i\lambda e^{u}\delta u - ie^{-u}\delta u & i\delta u_y \end{pmatrix}.$$

Proof. We follow the ansatz presented in [44], Section 6, and obtain for $F_\lambda(x)$ solving $\frac{d}{dx}F_\lambda = F_\lambda U_\lambda$ with $F_\lambda(0) = \mathbb{1}$ the variational equation

$$\left.\frac{d}{dx}\frac{d}{dt}F_\lambda(\delta u,\delta u_y)\right|_{t=0} = \left(\left.\frac{d}{dt}F_\lambda(\delta u,\delta u_y)\right|_{t=0}\right)U_\lambda + F_\lambda\delta U_\lambda$$

with

$$\left(\left.\frac{d}{dt}F_\lambda(\delta u,\delta u_y)\right|_{t=0}\right)(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\delta U_\lambda = \frac{1}{2}\begin{pmatrix} -i\delta u_y & i\lambda^{-1}e^{u}\delta u - ie^{-u}\delta u \\ i\lambda e^{u}\delta u - ie^{-u}\delta u & i\delta u_y \end{pmatrix}.$$

The solution of this differential equation is given by

$$\left(\left.\frac{d}{dt}F_\lambda(\delta u,\delta u_y)\right|_{t=0}\right)(x) = \left(\int_0^x F_\lambda(y)\delta U_\lambda(y)F_\lambda^{-1}(y)\,dy\right)F_\lambda(x)$$

and evaluating at $x = \mathbf{p}$ yields

$$\frac{d}{dt} M_\lambda(\delta u, \delta u_y)\Big|_{t=0} = \left(\int_0^{\mathbf{p}} F_\lambda(y) \delta U_\lambda(y) F_\lambda^{-1}(y)\, dy \right) M_\lambda.$$

Due to Lemma 5.25 there holds

$$\delta M_\lambda = \frac{d}{dt} M_\lambda(\delta u, \delta u_y)\Big|_{t=0} = \left[\sum_{i=1}^{2} \frac{(\delta v_i) w_i^t}{w_i^t v_i}, M_\lambda \right] + (P(\delta\mu) + \sigma^* P(\delta\mu)).$$

If we multiply the last equation with $\frac{w_1^t}{w_1^t v_1}$ from the left and v_1 from the right we get

$$\mu \frac{w_1^t \delta v_1}{w_1^t v_1} - \mu \frac{w_1^t \delta v_1}{w_1^t v_1} + \delta\mu \frac{w_1^t v_1}{w_1^t v_1} = \delta\mu = \mu \int_0^{\mathbf{p}} \frac{1}{\varphi^t \psi} \psi^t \delta U_\lambda \varphi \, dx$$

and therefore

$$\frac{d}{dt} \ln \mu(\delta u, \delta u_y)\Big|_{t=0} = \int_0^{\mathbf{p}} \frac{1}{\varphi^t \psi} \psi^t \delta U_\lambda \varphi \, dx.$$

This proves the claim. □

We now will apply Lemma 6.18 and Corollary 3.12 to establish a link between solutions ω_n of the homogeneous Jacobi equation from Proposition 3.17 and the Hamiltonians H_n.

Theorem 6.19. *For the series of Hamiltonians $(H_n)_{n\in\mathbb{N}_0}$ and solutions $(\omega_n)_{n\in\mathbb{N}_0}$ of the homogeneous Jacobi equation from the Pinkall-Sterling iteration there holds*

$$\nabla H_{2n+1} = (\Re(\omega_n(\cdot,0)), \Re(\partial_y \omega_n(\cdot,0))) \quad \text{and} \quad \nabla H_{2n+2} = (\Im(\omega_n(\cdot,0)), \Im(\partial_y \omega_n(\cdot,0))).$$

Proof. Considering the result of Lemma 6.18 a direct calculation gives

$$\begin{aligned}
\frac{d}{dt} \ln \mu(\delta u, \delta u_y)\Big|_{t=0} &= \int_0^{\mathbf{p}} \frac{1}{\varphi^t \psi} \psi^t \delta U_\lambda \varphi \, dx \\
&\overset{h=\varphi^t\psi}{=} \int_0^{\mathbf{p}} \frac{1}{2h} \psi^t \begin{pmatrix} -i\delta u_y & i\lambda^{-1} e^u \delta u - i e^{-u} \delta u \\ i\lambda e^u \delta u - i e^{-u} \delta u & i\delta u_y \end{pmatrix} \varphi \, dx \\
&= \int_0^{\mathbf{p}} \frac{1}{2h} ((\psi_2 \varphi_1 (i\lambda e^u - i e^{-u}) + \psi_1 \varphi_2 (i\lambda^{-1} e^u - i e^{-u}))\delta u \\
&\qquad - i(\psi_1 \varphi_1 - \psi_2 \varphi_2) \delta u_y)\, dx \\
&\overset{\text{Prop. 6.13}}{=} \int_0^{\mathbf{p}} \frac{1}{2h} \left((\lambda e^{2u} \tau - \tau - \lambda^{-1} e^{2u} \sigma + \sigma) \delta u - i\omega\, \delta u_y \right)\, dx \\
&\overset{\text{Prop. 3.17}}{=} \int_0^{\mathbf{p}} \frac{1}{2h} \left(-(\partial\omega - \bar\partial\omega) \delta u - i\omega\, \delta u_y \right)\, dx \\
&= \int_0^{\mathbf{p}} \frac{1}{2h} (i\omega_y\, \delta u - i\omega\, \delta u_y)\, dx = \Omega(-\tfrac{i}{2h}(\omega, \omega_y), (\delta u, \delta u_y)) \\
&= \Omega\left(\tfrac{-i}{2h}(\Re(\omega), \Re(\omega_y)), (\delta u, \delta u_y)\right) + i\Omega\left(\tfrac{-i}{2h}(\Im(\omega), \Im(\omega_y)), (\delta u, \delta u_y)\right) \\
&= \sqrt{\lambda} \sum_{n\geq 0} (-1)^{n+1} \Omega\left((\Re(\omega_n), \Re(\partial_y \omega_n)), (\delta u, \delta u_y)\right) \lambda^n \\
&\quad + i\sqrt{\lambda} \sum_{n\geq 0} (-1)^{n+1} \Omega\left((\Im(\omega_n), \Im(\partial_y \omega_n)), (\delta u, \delta u_y)\right) \lambda^n
\end{aligned}$$

around $\lambda = 0$ due to Proposition 6.14. On the other hand we know from Corollary 3.12 that we have the following asymptotic expansion of $\ln\mu$ around $\lambda = 0$

$$\begin{aligned}\ln\mu &= \tfrac{1}{\sqrt{\lambda}}\tfrac{i\mathbf{p}}{2} + \sqrt{\lambda}\int_0^{\mathbf{p}} \left(-i(\partial u)^2 + \tfrac{i}{2}\cosh(2u)\right)\, dt + O(\lambda)\\ &= \tfrac{1}{\sqrt{\lambda}}\tfrac{i\mathbf{p}}{2} + \sqrt{\lambda}\sum_{n\geq 0} c_n\lambda^n\\ &= \tfrac{1}{\sqrt{\lambda}}\tfrac{i\mathbf{p}}{2} + \sqrt{\lambda}\sum_{n\geq 0}(-1)^{n+1}H_{2n+1}\lambda^n + i\sqrt{\lambda}\sum_{n\geq 0}(-1)^{n+1}H_{2n+2}\lambda^n.\end{aligned}$$

Thus we get

$$\begin{aligned}\frac{d}{dt}\ln\mu(\delta u, \delta u_y)\bigg|_{t=0} &= \sqrt{\lambda}\sum_{n\geq 0}(-1)^{n+1}\Omega\left(\nabla H_{2n+1}, (\delta u, \delta u_y)\right)\lambda^n\\ &\quad + i\sqrt{\lambda}\sum_{n\geq 0}(-1)^{n+1}\Omega\left(\nabla H_{2n+2}, (\delta u, \delta u_y)\right)\lambda^n\end{aligned}$$

and a comparison of the coefficients of the two power series yields the claim. $\square$

6.4 An inner product on $\Lambda_r\mathfrak{sl}_2(\mathbb{C})$

We already introduced a differential operator $L_\lambda(x,y) := \frac{d}{dx} + U_\lambda(x,y)$ such that the sinh-Gordon flow can be expressed in commutator form, i.e.

$$\frac{d}{dy}L_\lambda(x,y) = \frac{d}{dy}U_\lambda(x,y) = [L_\lambda(y), V_\lambda(x,y)] = \frac{d}{dx}V_\lambda(x,y) + [U_\lambda(x,y), V_\lambda(x,y)].$$

In the following we will translate the symplectic form Ω with respect to the identification $(u, u_y) \simeq U_\lambda$. First recall that the span of $\{\epsilon_+, \epsilon_-, \epsilon\}$ is $\mathfrak{sl}_2(\mathbb{C})$ and that the inner product

$$\langle\cdot,\cdot\rangle : \mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C}) \to \mathbb{C},\ (\alpha, \beta) \mapsto \langle\alpha,\beta\rangle := \operatorname{tr}(\alpha\cdot\beta)$$

is non-degenerate. We will now extend the inner product $\langle\cdot,\cdot\rangle$ to a non-degenerate inner product $\langle\cdot,\cdot\rangle_\Lambda$ on $\Lambda_r\mathfrak{sl}_2(\mathbb{C}) = \Lambda_r\mathfrak{su}_2(\mathbb{C}) \oplus \Lambda_r^+\mathfrak{sl}_2(\mathbb{C})$ such that

$$\langle\cdot,\cdot\rangle_\Lambda|_{\Lambda_r\mathfrak{su}_2(\mathbb{C})\times\Lambda_r\mathfrak{su}_2(\mathbb{C})} \equiv 0 \ \text{ and } \ \langle\cdot,\cdot\rangle_\Lambda|_{\Lambda_r^+\mathfrak{sl}_2(\mathbb{C})\times\Lambda_r^+\mathfrak{sl}_2(\mathbb{C})} \equiv 0,$$

i.e. $\Lambda_r\mathfrak{su}_2(\mathbb{C})$ and $\Lambda_r^+\mathfrak{sl}_2(\mathbb{C})$ are isotropic subspaces of $\Lambda_r\mathfrak{sl}_2(\mathbb{C})$ with respect to $\langle\cdot,\cdot\rangle_\Lambda$.

Lemma 6.20. *The map $\langle\cdot,\cdot\rangle_\Lambda : \Lambda_r\mathfrak{sl}_2(\mathbb{C}) \times \Lambda_r\mathfrak{sl}_2(\mathbb{C}) \to \mathbb{R}$ given by*

$$(\alpha,\beta) \mapsto \langle\alpha,\beta\rangle_\Lambda := \Im\left(Res_{\lambda=0}\tfrac{d\lambda}{\lambda}tr(\alpha\cdot\beta)\right)$$

is bilinear and non-degenerate. Moreover, there holds

$$\langle\cdot,\cdot\rangle_\Lambda|_{\Lambda_r\mathfrak{su}_2(\mathbb{C})\times\Lambda_r\mathfrak{su}_2(\mathbb{C})} \equiv 0 \ \text{ and } \ \langle\cdot,\cdot\rangle_\Lambda|_{\Lambda_r^+\mathfrak{sl}_2(\mathbb{C})\times\Lambda_r^+\mathfrak{sl}_2(\mathbb{C})} \equiv 0,$$

i.e. $\Lambda_r\mathfrak{su}_2(\mathbb{C})$ and $\Lambda_r^+\mathfrak{sl}_2(\mathbb{C})$ are isotropic subspaces of $\Lambda_r\mathfrak{sl}_2(\mathbb{C})$ with respect to $\langle\cdot,\cdot\rangle_\Lambda$.

Proof. The bilinearity of $\langle\cdot,\cdot\rangle_\Lambda$ follows from the bilinearity of $\mathrm{tr}(\cdot)$. Now consider a non-zero element $\xi = \sum_{i\in\mathcal{I}}\lambda^i\xi_i \in \Lambda_r\mathfrak{sl}_2(\mathbb{C})$ and pick out an index $j\in\mathcal{I}$ such that $\xi_j \neq 0$. Setting $\widetilde{\xi} = i\lambda^{-j}\overline{\xi}_j^t$ we obtain

$$\langle\xi,\widetilde{\xi}\rangle_\Lambda = \Im\left(\mathrm{Res}_{\lambda=0}\tfrac{d\lambda}{\lambda}\mathrm{tr}(\xi\cdot\widetilde{\xi})\right) = \mathrm{tr}(\xi_j\cdot\overline{\xi}_j^t)\in\mathbb{R}^+$$

since $\xi_j\neq 0$. This shows that $\langle\cdot,\cdot\rangle_\Lambda$ is non-degenerate, i.e. the first part of the lemma.

We will now prove the second part of the lemma, namely that $\Lambda_r\mathfrak{su}_2(\mathbb{C})$ and $\Lambda_r^+\mathfrak{sl}_2(\mathbb{C})$ are isotropic subspaces of $\Lambda_r\mathfrak{sl}_2(\mathbb{C})$ with respect to $\langle\cdot,\cdot\rangle_\Lambda$.

First we consider $\alpha^+ = \alpha^+_\lambda = \sum_i\lambda^i\alpha_i^+$, $\widetilde{\alpha}^+ = \sum_i\lambda^i\widetilde{\alpha}_i^+ \in \Lambda_r\mathfrak{su}_2(\mathbb{C})$ with

$$\alpha^+_\lambda = -\overline{\alpha^+_{1/\bar\lambda}}^t \quad\text{and}\quad \widetilde{\alpha}^+_\lambda = -\overline{\widetilde{\alpha}^+_{1/\bar\lambda}}^t.$$

Then one obtains

$$\begin{aligned}\langle\alpha^+,\widetilde{\alpha}^+\rangle_\Lambda &= \Im\left(\mathrm{Res}_{\lambda=0}\tfrac{d\lambda}{\lambda}\mathrm{tr}(\alpha^+\cdot\widetilde{\alpha}^+)\right)\\ &= \Im(\mathrm{tr}(\alpha^+_{-1}\cdot\widetilde{\alpha}^+_1 + \alpha^+_0\cdot\widetilde{\alpha}^+_0 + \alpha^+_1\cdot\widetilde{\alpha}^+_{-1})).\end{aligned}$$

A direct calculation gives

$$\begin{aligned}\overline{\mathrm{tr}(\alpha^+_{-1}\widetilde{\alpha}^+_1 + \alpha^+_0\widetilde{\alpha}^+_0 + \alpha^+_1\widetilde{\alpha}^+_{-1})} &= \mathrm{tr}((-\overline{\alpha}^+_{-1})^t(-\overline{\widetilde{\alpha}}^+_1)^t + (-\overline{\alpha}^+_0)^t(-\overline{\widetilde{\alpha}}^+_0)^t + (-\overline{\alpha}^+_1)^t(-\overline{\widetilde{\alpha}}^+_{-1})^t)\\ &\overset{!}{=} \mathrm{tr}(\alpha^+_{-1}\widetilde{\alpha}^+_1 + \alpha^+_0\widetilde{\alpha}^+_0 + \alpha^+_1\widetilde{\alpha}^+_{-1})\end{aligned}$$

and thus $\mathrm{tr}(\alpha^+_{-1}\widetilde{\alpha}^+_1 + \alpha^+_0\widetilde{\alpha}^+_0 + \alpha^+_1\widetilde{\alpha}^+_{-1})\in\mathbb{R}$. This shows

$$\langle\alpha^+,\widetilde{\alpha}^+\rangle_\Lambda = \Im(\mathrm{tr}(\alpha^+_{-1}\cdot\widetilde{\alpha}^+_1 + \alpha^+_0\cdot\widetilde{\alpha}^+_0 + \alpha^+_1\cdot\widetilde{\alpha}^+_{-1})) = 0.$$

Now consider $\beta^- = \sum_{i\geq 0}\lambda^i\beta_i^-$, $\widetilde{\beta}^- = \sum_{i\geq 0}\lambda^i\widetilde{\beta}_i^- \in \Lambda_r^+\mathfrak{sl}_2(\mathbb{C})$ where $\beta_0^-,\widetilde{\beta}_0^-$ are of the form

$$\beta_0^- = \begin{pmatrix} h_0 & e_0\\ 0 & -h_0\end{pmatrix},\quad \widetilde{\beta}_0^- = \begin{pmatrix}\widetilde{h}_0 & \widetilde{e}_0\\ 0 & -\widetilde{h}_0\end{pmatrix}$$

with $h_0,\widetilde{h}_0\in\mathbb{R}$ and $e_0,\widetilde{e}_0\in\mathbb{C}$. Then one gets

$$\begin{aligned}\langle\beta^-,\widetilde{\beta}^-\rangle_\Lambda &= \Im\left(\mathrm{Res}_{\lambda=0}\tfrac{d\lambda}{\lambda}\mathrm{tr}(\beta^-\cdot\widetilde{\beta}^-)\right)\\ &= \Im(\mathrm{tr}(\beta_0^-\cdot\widetilde{\beta}_0^-))\\ &= \Im(2h_0\widetilde{h}_0)\\ &= 0.\end{aligned}$$

This yields the second claim and concludes the proof. $\square$

6.5 The symplectic form Ω and Serre duality

This section incorporates the results from Chapter 4, 5 and 6 and establishes a connection between the symplectic form Ω and Serre Duality 2.24. Moreover, we will show that $(M_g^{\mathbf{p}}, \Omega, H_2)$ is a completely integrable Hamiltonian system.

Definition 6.21. *Let* $H^0_{\mathbb{R}}(Y,\Omega) := \{\omega \in H^0(Y,\Omega) \,|\, \overline{\eta^*\omega} = -\omega\}$ *be the real part of* $H^0(Y,\Omega)$ *with respect to the involution* η.

The following observation is based on [24] and [47]. Let $R(\lambda,\mu) = 0$ be the equation that defines the Riemann surface $Y(u,u_y)$ corresponding to (u,u_y). Taking the total differential we get

$$\frac{\partial R}{\partial \lambda} d\lambda + \frac{\partial R}{\partial \mu} d\mu = 0 \tag{$*$}$$

and differentiating with respect to $(\delta u, \delta u_y)$ yields

$$\frac{\partial R}{\partial \lambda} \dot{\lambda}\, dt + \frac{\partial R}{\partial \mu} \dot{\mu}\, dt + \dot{R}\, dt = 0. \tag{$**$}$$

With the help of $(*)$ we get

$$d\mu = -\frac{\frac{\partial R}{\partial \mu}}{\frac{\partial R}{\partial \lambda}} d\lambda \quad \text{and} \quad d\lambda = -\frac{\frac{\partial R}{\partial \lambda}}{\frac{\partial R}{\partial \mu}} d\mu.$$

Now a direct calculation using $(*)$ and $(**)$ shows that for the form $\omega = \delta \ln \mu(\delta u, \delta u_y)\frac{d\lambda}{\lambda} - \delta \ln \lambda(\delta u, \delta u_y)\frac{d\mu}{\mu}$ there holds

$$\begin{aligned} \omega = \delta \ln \mu(\delta u, \delta u_y)\frac{d\lambda}{\lambda} - \delta \ln \lambda(\delta u, \delta u_y)\frac{d\mu}{\mu} &= -\frac{\delta R(\delta u, \delta u_y)}{\mu \frac{\partial R}{\partial \mu}} \frac{d\lambda}{\lambda} \\ &\overset{!}{=} \frac{\delta R(\delta u, \delta u_y)}{\lambda \frac{\partial R}{\partial \lambda}} \frac{d\mu}{\mu}. \end{aligned}$$

Thus we can choose that either $\delta \ln \lambda = 0$ or $\delta \ln \mu = 0$. In the following we will usually impose the first condition $\delta \ln \lambda = 0$. Since η is given by $(\lambda,\mu) \mapsto (1/\bar{\lambda}, \bar{\mu})$ we have

$$\eta^* \overline{\delta \ln \mu} = \delta \ln \mu \quad \text{and} \quad \eta^* \overline{\tfrac{d\lambda}{\lambda}} = -\tfrac{d\lambda}{\lambda}.$$

Thus we arrive at the map $\omega : T_{(u,u_y)} M_g^{\mathbf{p}} \to H^0_{\mathbb{R}}(Y(u,u_y),\Omega)$ given by

$$(\delta u, \delta u_y) \mapsto \omega(\delta u, \delta u_y) := \delta \ln \mu(\delta u, \delta u_y) \frac{d\lambda}{\lambda}.$$

Remark 6.22. *Due to Theorem 5.11 we can identify the space* $T_{Y(u,u_y)}\Sigma_g^{\mathbf{p}}$ *of infinitesimal non-isospectral (but iso-periodic) deformations of* $Y(u,u_y)$ *with the space* $H^0_{\mathbb{R}}(Y(u,u_y),\Omega)$ *via the map* $c \mapsto \omega(c) := \frac{c}{\nu}\frac{d\lambda}{\lambda} = \delta \ln \mu \frac{d\lambda}{\lambda}$. *Therefore* ω *can be identified with* dY, *the derivative of* $Y : M_g^{\mathbf{p}} \to \Sigma_g^{\mathbf{p}}$. *Due to Proposition 6.11, the map* $Y : M_g^{\mathbf{p}} \to \Sigma_g^{\mathbf{p}}$ *is a submersion. Thus the map* $\omega : T_{(u,u_y)} M_g^{\mathbf{p}} \to H^0_{\mathbb{R}}(Y(u,u_y),\Omega)$ *is surjective.*

Note that deformations which keep the period $\mathbf{p}$ fixed indeed correspond to a holomorphic 1-form ω since in that case we have

$$\omega = \delta \ln \mu \frac{d\lambda}{\lambda} = \frac{c(\lambda)}{\lambda}\frac{d\lambda}{\nu} = \frac{\sum_{i=1}^{g} c_i \lambda^i}{\lambda}\frac{d\lambda}{\nu} = \sum_{i=1}^{g} c_i \frac{\lambda^{i-1} d\lambda}{\nu}.$$

Definition 6.23. *Let $L_{(u,u_y)} \subset T_{(u,u_y)} M_g^{\mathbf{p}}$ be the kernel of the the map $\omega : T_{(u,u_y)} M_g^{\mathbf{p}} \to H^0_{\mathbb{R}}(Y(u,u_y),\Omega)$, i.e. $L_{(u,u_y)} = \ker(\omega)$.*

Now we are able to formulate and prove the main result of this work. The proof is based on the ideas and methods presented in the proof of [47], Theorem 7.5.

Theorem 6.24.

(i) There exists an isomorphism of vector spaces $d\Gamma_{(u,u_y)} : H^1_{\mathbb{R}}(Y(u,u_y),\mathcal{O}) \to L_{(u,u_y)}$.

(ii) For all $[f] \in H^1_{\mathbb{R}}(Y(u,u_y),\mathcal{O})$ and all $(\delta u, \delta u_y) \in T_{(u,u_y)} M_g^{\mathbf{p}}$ the equation

$$\Omega(d\Gamma_{(u,u_y)}([f]),(\delta u,\delta u_y)) = i\, Res([f]\,\omega(\delta u,\delta u_y)) \tag{6.5.1}$$

holds. Here the right hand side is defined as in the Serre Duality Theorem 2.24.

(iii) $(M_g^{\mathbf{p}},\Omega,H_2)$ is a Hamiltonian system. In particular, Ω is non-degenerate on $M_g^{\mathbf{p}}$.

Remark 6.25.

(i) From the Serre Duality Theorem 2.24 we know that the pairing $Res : H^1(Y(u,u_y),\mathcal{O}) \times H^0(Y(u,u_y),\Omega) \to \mathbb{C}$ is non-degenerate.

(ii) $L_{(u,u_y)} \subset T_{(u,u_y)} M_g^{\mathbf{p}}$ is a maximal isotropic subspace with respect to the symplectic form $\Omega : T_{(u,u_y)} M_g^{\mathbf{p}} \times T_{(u,u_y)} M_g^{\mathbf{p}} \to \mathbb{R}$, i.e. $L_{(u,u_y)}$ is Lagrangian.

Proof.

(i) Let $[f] = [(f_0, \eta^* \bar{f}_0)] \in H^1_{\mathbb{R}}(Y,\mathcal{O})$ be a cocycle with representative f_0 as defined in Lemma 4.15. Then we get

$$A_{f_0}(x) := P_x(f_0) + \sigma^* P_x(f_0) = \sum_{i=0}^{g-1} c_i \lambda^{-i}(P_x(\lambda^{-1}\nu) + \sigma^* P_x(\lambda^{-1}\nu)) = \sum_{i=0}^{g-1} c_i \lambda^{-i} \zeta_\lambda(x)$$

with

$$\delta U_\lambda(x) = [A^+_{f_0}(x), L_\lambda(x)] = [L_\lambda(x), A^-_{f_0}(x)].$$

due to Theorem 5.24 and moreover

$$\delta v(x) = -A^-_{f_0}(x) v(x)$$

holds due to Remark 5.23 with $A^-_{f_0}(x) = -\sum \frac{(\delta v_i(x))w_i^t(x)}{w_i^t(x)v_i(x)}$. From Lemma 5.27 we know that in general

$$\delta U_\lambda(x) = \left[L_\lambda(x), -\sum_{i=1}^{2} \frac{(\delta v_i(x))w_i^t(x)}{w_i^t(x)v_i(x)}\right] + (P_x(\tfrac{\delta \ln \mu}{\mathbf{p}}) + \sigma^* P_x(\tfrac{\delta \ln \mu}{\mathbf{p}})).$$

Since $\delta U_\lambda(x) = [L_\lambda(x), A^-_{f_0}(x)]$ we see that $\delta \ln \mu(\delta u^{f_0}, \delta u_y^{f_0}) = 0$ and consequently

$$(\delta u^{f_0}, \delta u_y^{f_0}) \in \ker(\omega).$$

Thus we have an injective map $d\Gamma_{(u,u_y)} : H^1_{\mathbb{R}}(Y(u,u_y), \mathcal{O}) \to L_{(u,u_y)}$. Due to Remark 6.22 we know that $\omega : T_{(u,u_y)} M_g^{\mathbf{p}} \to H^0_{\mathbb{R}}(Y(u,u_y), \Omega)$ is surjective. Since $\dim H^0_{\mathbb{R}}(Y(u,u_y), \Omega) = g$ there holds $\dim L_{(u,u_y)} = \dim \ker(\omega) = g$ and thus $d\Gamma_{(u,u_y)} : H^1_{\mathbb{R}}(Y(u,u_y), \mathcal{O}) \to L_{(u,u_y)}$ is an isomorphism of vector spaces.

(ii) For an isospectral variation of $U_\lambda(x)$ we have

$$\delta U_\lambda(x) = [L_\lambda, B^-(x)] = \tfrac{d}{dx} B^-(x) - [U_\lambda(x), B^-(x)]$$

with a map $B^-(x) : \mathbb{R} \to \Lambda_r^+ \mathfrak{sl}_2(\mathbb{C})$. In the following $\oplus$ will denote the $\Lambda_r \mathfrak{su}_2(\mathbb{C})$-part of $\Lambda_r \mathfrak{sl}_2(\mathbb{C}) = \Lambda_r \mathfrak{su}_2(\mathbb{C}) \oplus \Lambda_r^+ \mathfrak{sl}_2(\mathbb{C})$ and $\ominus$ will correspond to the second summand $\Lambda_r^+ \mathfrak{sl}_2(\mathbb{C})$. Since $\delta U_\lambda(x)$ lies in the $\oplus$-part and $\frac{d}{dx} B^-(x)$ lies in the $\ominus$-part, we see that $\delta U_\lambda(x)$ is equal to the $\oplus$-part of the commutator expression $-[U_\lambda(x), B^-(x)]$. Writing U for $U_\lambda(x)$ and B^- for $B^-(x)$ we get

$$\begin{aligned} \delta U &= \lambda^{-1} \delta U_{-1} + \lambda^0 \delta U_0 + \lambda \delta U_1 \\ &\overset{!}{=} \big(\lambda^{-1}[B_0^-, U_{-1}] + \lambda^0([B_1^-, U_{-1}] + [B_0^-, U_0]) \\ &\quad + \lambda([B_2^-, U_{-1}] + [B_1^-, U_0] + [B_0^-, U_1])\big)_\oplus. \end{aligned}$$

Thus we arrive at three equations

$$\delta U_{-1} = [B_0^-, U_{-1}], \quad \delta U_0 = \big([B_1^-, U_{-1}] + [B_0^-, U_0]\big)_\oplus,$$
$$\delta U_1 = \big([B_2^-, U_{-1}] + [B_1^-, U_0] + [B_0^-, U_1]\big)_\oplus.$$

Recall that U_λ is given by

$$U_\lambda = \frac{1}{2} \begin{pmatrix} -iu_y & i\lambda^{-1} e^u + ie^{-u} \\ i\lambda e^u + ie^{-u} & iu_y \end{pmatrix}$$

and consequently

$$\delta U_\lambda = \frac{1}{2} \begin{pmatrix} -i\delta u_y & i\lambda^{-1} e^u \delta u - ie^{-u}\delta u \\ i\lambda e^u \delta u - ie^{-u} \delta u & i\delta u_y \end{pmatrix}.$$

We can now use the above equations in order to obtain relations on the coefficients B_0^- and B_1^- of $B^- = \sum_{i\geq 0} \lambda^i B_i^-$ where B_0^- is of the form

$$B_0^- = \begin{pmatrix} h_0 & e_0 \\ 0 & -h_0 \end{pmatrix} \quad \text{with} \quad h_0 \in \mathbb{R},\, e_0 \in \mathbb{C},$$

and B_1^- is of the form $B_1^- = \left(\begin{smallmatrix} h_1 & e_1 \\ f_1 & -h_1 \end{smallmatrix}\right)$. Since $\delta U_{-1} = [B_0^-, U_{-1}]$ a direct calculation yields

$$B_0^- = \begin{pmatrix} \frac{1}{2}\delta u & e_0 \\ 0 & -\frac{1}{2}\delta u \end{pmatrix}.$$

Moreover, the sum $[B_1^-, U_{-1}] + [B_0^-, U_0]$ is given by

$$\begin{pmatrix} \frac{i}{2}e_0 e^{-u} - \frac{i}{2}f_1 e^u & ih_1 e^u + ie_0 u_y + \frac{i}{2}e^{-u}\delta u \\ -\frac{i}{2}e^{-u}\delta u & -\frac{i}{2}e_0 e^{-u} + \frac{i}{2}f_1 e^u \end{pmatrix}.$$

For the diagonal entry of $[B_1^-, U_{-1}] + [B_0^-, U_0]$ the $\oplus$-part is given by the imaginary part and therefore

$$-\tfrac{i}{2}\delta u_y = \tfrac{i}{2}\Re(e_0)e^{-u} - \tfrac{i}{2}\Re(f_1)e^u.$$

Thus we get

$$\delta u_y = \Re(f_1)e^u - \Re(e_0)e^{-u}.$$

For $B^-(x) := A_{f_0}^-(x)$ we obtain

$$\begin{aligned}
\Im\left(\operatorname{tr}(\delta U_0 A_{f_0,0}^-) + \operatorname{tr}(\delta U_{-1} A_{f_0,1}^-)\right) &= \Im\left(-i\delta u_y h_0 - \tfrac{i}{2}e_0 e^{-u}\delta u + \tfrac{i}{2}f_1 e^u \delta u\right) \\
&= -\delta u_y \Re(h_0) + \tfrac{1}{2}\delta u \left(\Re(f_1)e^u - \Re(e_0)e^{-u}\right) \\
&= \frac{1}{2}\left(\delta u \delta u_y^{f_0} - \delta u^{f_0}\,\delta u_y\right).
\end{aligned}$$

Now a direct calculation gives

$$\begin{aligned}
\frac{1}{2}\Omega(d\Gamma_{(u,u_y)}([f]), (\delta u, \delta u_y)) &= \frac{1}{2}\int_0^{\mathbf{p}} (\delta u^{f_0}\delta u_y - \delta u\, \delta u_y^{f_0})dx \\
&= -\int_0^{\mathbf{p}} \Im\left(\operatorname{tr}(\delta U_0 A_{f_0,0}^-) + \operatorname{tr}(\delta U_{-1} A_{f_0,1}^-)\right) dx \\
&= -\int_0^{\mathbf{p}} \langle \delta U_\lambda(x), A_{f_0}^-(x)\rangle_\Lambda dx \\
&\overset{\delta U_\lambda \in \oplus}{=} -\int_0^{\mathbf{p}} \langle \delta U_\lambda(x), A_{f_0}(x)\rangle_\Lambda dx.
\end{aligned}$$

Setting $\widehat{P}_x(\frac{\delta \ln \mu}{\mathbf{p}}) := P_x(\frac{\delta \ln \mu}{\mathbf{p}}) + \sigma^* P_x(\frac{\delta \ln \mu}{\mathbf{p}})$, we further obtain

$$\begin{aligned}
\frac{1}{2}\Omega(d\Gamma_{(u,u_y)}([f]),(\delta u,\delta u_y)) &\overset{\text{Lem. 5.27}}{=} -\int_0^{\mathbf{p}} \langle [L_\lambda(x), B^-(x)], A_{f_0}(x)\rangle_\Lambda dx \\
&\quad - \int_0^{\mathbf{p}} \langle \widehat{P}_x(\tfrac{\delta \ln \mu}{\mathbf{p}}), A_{f_0}(x)\rangle_\Lambda dx \\
&= \int_0^{\mathbf{p}} \langle [B^-(x), L_\lambda(x)], A_{f_0}(x)\rangle_\Lambda dx \\
&\quad - \int_0^{\mathbf{p}} \langle \widehat{P}_x(\tfrac{\delta \ln \mu}{\mathbf{p}}), A_{f_0}(x)\rangle_\Lambda dx.
\end{aligned}$$

Recall, that $\mathrm{tr}([B^-(x), L_\lambda(x)] \cdot A_{f_0}(x)) = \mathrm{tr}(B^-(x) \cdot [L_\lambda(x), A_{f_0}(x)])$. Moreover, there holds $[L_\lambda(x), A_{f_0}(x)] = 0$ and we get

$$\begin{aligned}
\frac{1}{2}\Omega(d\Gamma_{(u,u_y)}([f]),(\delta u,\delta u_y)) &= \int_0^{\mathbf{p}} \langle B^-(x), [L_\lambda(x), A_{f_0}(x)]\rangle_\Lambda dx \\
&\quad - \int_0^{\mathbf{p}} \langle \widehat{P}_x(\tfrac{\delta \ln \mu}{\mathbf{p}}), A_{f_0}(x)\rangle_\Lambda dx \\
&= - \int_0^{\mathbf{p}} \langle \widehat{P}_x(\tfrac{\delta \ln \mu}{\mathbf{p}}), A_{f_0}(x)\rangle_\Lambda dx \\
&= - \int_0^{\mathbf{p}} \langle A_{f_0}(x), \widehat{P}_x(\tfrac{\delta \ln \mu}{\mathbf{p}})\rangle_\Lambda dx.
\end{aligned}$$

Writing out the last equation yields

$$\begin{aligned}
\Omega(d\Gamma_{(u,u_y)}([f]),(\delta u,\delta u_y)) &= -\frac{2}{\mathbf{p}}\int_0^{\mathbf{p}} \Im\left(\mathrm{Res}_{y_0}\tfrac{d\lambda}{\lambda}\mathrm{tr}(A_{f_0}(x)\widehat{P}_x(\delta\ln\mu))\right) dx \\
&= -2\Im\left(\mathrm{Res}_{y_0}(f_0\cdot\delta\ln\mu\tfrac{d\lambda}{\lambda})\right) \\
&= i\left(\mathrm{Res}_{y_0}(f_0\cdot\delta\ln\mu\tfrac{d\lambda}{\lambda}) - \mathrm{Res}_{y_0}\overline{(f_0\cdot\delta\ln\mu\tfrac{d\lambda}{\lambda})}\right) \\
&= i\left(\mathrm{Res}_{y_0}(f_0\cdot\delta\ln\mu\tfrac{d\lambda}{\lambda}) - \mathrm{Res}_{y_\infty}\eta^*\overline{(f_0\cdot\delta\ln\mu\tfrac{d\lambda}{\lambda})}\right) \\
&= i\left(\mathrm{Res}_{y_0}(f_0\cdot\delta\ln\mu\tfrac{d\lambda}{\lambda}) + \mathrm{Res}_{y_\infty}(f_\infty\cdot\delta\ln\mu\tfrac{d\lambda}{\lambda})\right)
\end{aligned}$$

and thus

$$\Omega(d\Gamma_{(u,u_y)}([f]),(\delta u,\delta u_y)) = i\,\mathrm{Res}([f]\,\omega(\delta u,\delta u_y)).$$

(iii) In order to prove (iii) we have to show that Ω is non-degenerate on $M_g^{\mathbf{p}}$. From equation (6.5.1) we know that

$$\Omega((\delta u,\delta u_y),(\delta\widetilde{u},\delta\widetilde{u}_y)) = 0 \ \text{ for } \ (\delta u,\delta u_y),(\delta\widetilde{u},\delta\widetilde{u}_y) \in L_{(u,u_y)} = \ker(\omega).$$

Moreover, $\omega : T_{(u,u_y)}M_g^{\mathbf{p}} \to H^0_{\mathbb{R}}(Y(u,u_y),\Omega)$ is surjective since $\dim T_{(u,u_y)}M_g^{\mathbf{p}} = 2g$ and $\dim\ker(\omega) = g = \dim H^0_{\mathbb{R}}(Y(u,u_y),\Omega)$. Thus we have

$$T_{(u,u_y)}M_g^{\mathbf{p}}/\ker(\omega) \simeq H^0_{\mathbb{R}}(Y(u,u_y),\Omega)$$

and there exists a basis $\{\delta a_1, \ldots, \delta a_g, \delta b_1, \ldots, \delta b_g\}$ of $T_{(u,u_y)}M_g^{\mathbf{p}}$ such that

$$\operatorname{span}\{\delta a_1, \ldots, \delta a_g\} = \ker(\omega) \quad \text{and} \quad \omega[\operatorname{span}\{\delta b_1, \ldots, \delta b_g\}] = H^0_{\mathbb{R}}(Y(u, u_y), \Omega).$$

Now $L_{(u,u_y)} = \ker(\omega) \simeq H^1_{\mathbb{R}}(Y(u, u_y), \mathcal{O})$ and since the pairing from Serre duality is non-degenerate we obtain with equation (6.5.1) (after choosing the appropriate basis)

$$\Omega(\delta a_i, \delta b_j) = \delta_{ij} \quad \text{and} \quad \Omega(\delta b_i, \delta a_j) = -\delta_{ij}.$$

Summing up the matrix representation B_Ω of Ω on $T_{(u,u_y)}M_g^{\mathbf{p}}$ has the form

$$B_\Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & (*) \end{pmatrix}$$

and thus Ω is of full rank. This shows (iii) and concludes the proof of Theorem 6.24.

□

Corollary 6.26. *The map $d\Gamma_{(u,u_y)} : H^1_{\mathbb{R}}(Y(u, u_y), \mathcal{O}) \to L_{(u,u_y)}$ is given by*

$$(f_0, \eta^* \bar{f}_0) \mapsto (\delta u^{f_0}, \delta u_y^{f_0}) = Res_{\lambda=0} \tfrac{f_0}{2h}(\omega, \omega_y)\tfrac{d\lambda}{\lambda} + Res_{\lambda=\infty} \tfrac{\eta^* \bar{f}_0}{2h}(\omega, \omega_y)\tfrac{d\lambda}{\lambda}.$$

Proof. From Theorem 6.24 and the proof of Theorem 6.19 we can extract the formula

$$\begin{aligned}
\Omega\Big((\delta u^{f_0}, \delta u_y^{f_0}), (\delta u, \delta u_y)\Big) &= i\left(\operatorname{Res}_{\lambda=0}(f_0 \cdot \delta \ln \mu \tfrac{d\lambda}{\lambda}) + \operatorname{Res}_{\lambda=\infty}(\eta^* \bar{f}_0 \cdot \delta \ln \mu \tfrac{d\lambda}{\lambda})\right) \\
&= \operatorname{Res}_{\lambda=0}(f_0 \cdot \Omega(\tfrac{1}{2h}(\omega, \omega_y), (\delta u, \delta u_y))\tfrac{d\lambda}{\lambda}) \\
&\quad + \operatorname{Res}_{\lambda=\infty}(\eta^* \bar{f}_0 \cdot \Omega(\tfrac{1}{2h}(\omega, \omega_y), (\delta u, \delta u_y))\tfrac{d\lambda}{\lambda}) \\
&= \Omega\Big(\operatorname{Res}_{\lambda=0} \tfrac{f_0}{2h}(\omega, \omega_y)\tfrac{d\lambda}{\lambda} + \operatorname{Res}_{\lambda=\infty} \tfrac{\eta^* \bar{f}_0}{2h}(\omega, \omega_y)\tfrac{d\lambda}{\lambda}, (\delta u, \delta u_y)\Big).
\end{aligned}$$

Since Ω is non-degenerate due to Theorem 6.24, the claim follows immediately. □

Corollary 6.27. *The Hamiltonians $H_n : M_g^{\mathbf{p}} \to \mathbb{R}$ are in involution, i.e. $\{H_n, H_m\} = 0$ for $n, m = 1, \ldots, g$ and the Hamiltonian system $(M_g^{\mathbf{p}}, \Omega, H_2)$ is completely integrable.*

Proof. From Theorem 6.19 we know that

$$\nabla H_{2n+1} = (\Re(\omega_n(\cdot, 0)), \Re(\partial_y \omega_n(\cdot, 0))) \quad \text{and} \quad \nabla H_{2n+2} = (\Im(\omega_n(\cdot, 0)), \Im(\partial_y \omega_n(\cdot, 0))).$$

By choosing the appropriate f_0 we can deduce from Corollary 6.26 that the elements $(\Re(\omega_n(\cdot, 0)), \Re(\partial_y \omega_n(\cdot, 0)))$ and $(\Im(\omega_n(\cdot, 0)), \Im(\partial_y \omega_n(\cdot, 0)))$ lie in $L_{(u,u_y)} = \ker(\omega)$. Moreover, we get from Theorem 6.24 that there exist some $[f_n] \in H^1_{\mathbb{R}}(Y, \mathcal{O})$ such that

$$\{H_n, H_m\} = \Omega(\nabla H_n, \nabla H_m) = \Omega(d\Gamma_{(u,u_y)}([f_n]), \nabla H_m) = i \operatorname{Res}([f_n]\, \omega(\nabla H_m)).$$

This gives $\{H_n, H_m\} = i \operatorname{Res}([f_n], \omega(\nabla H_m)) = 0$ for $n, m = 1, \ldots, g$. □

Remark 6.28. *For the non-linear Schrödinger operator with a potential $q(x)$ with period $\mathbf{p} = 1$ the symplectic form is given by*

$$\Omega(\delta q, \delta\widetilde{q}) = \sum_{i\neq j} \int_0^1 \frac{\delta q_{ij}(x)\delta\widetilde{q}_{ji}(x)}{p_i - p_j} dx,$$

where the distinct p_i are the entries of the diagonal matrix $p = diag(p_1, \ldots, p_n)$ appearing in the corresponding Lax operator

$$L = \frac{d}{dx} + q(x) + \lambda p.$$

A direct calculation shows

$$\begin{aligned}
\Omega(\delta q, \delta\widetilde{q}) &= \sum_{i\neq j} \int_0^1 \frac{\delta q_{ij}(x)\delta\widetilde{q}_{ji}(x)}{p_i - p_j} dx \\
&= \int_0^1 tr(\delta q(x)\, ad^{-1}(p)\delta\widetilde{q}(x) - \delta\widetilde{q}(x)\, ad^{-1}(p)\delta q(x))dx \\
&= Res_{\lambda=\infty} \frac{d\lambda}{\lambda} \int_0^1 tr(\delta q(x)\, ad^{-1}(p)\delta\widetilde{q}(x) - \delta\widetilde{q}(x)\, ad^{-1}(p)\delta q(x))dx \\
&= Res_{\lambda=\infty} \frac{d\lambda}{\lambda} \int_0^1 tr([L, a^-(x)]\, ad^{-1}(p)[L, b^-(x)] \\
&\qquad - [L, b^-(x)]\, ad^{-1}(p)[L, a^-(x)])dx \\
&= Res_{\lambda=\infty} d\lambda \int_0^1 tr([L, a^-(x)]\, ad^{-1}(p)[\tfrac{1}{\lambda}L, b^-(x)] \\
&\qquad - [L, b^-(x)]\, ad^{-1}(p)[\tfrac{1}{\lambda}L, a^-(x)])dx \\
&= Res_{\lambda=\infty} d\lambda \int_0^1 tr([L, a^-(x)]b^-(x) - [L, b^-(x)]a^-(x))dx.
\end{aligned}$$

The techniques from the proof of [47], Theorem 7.5, lead to the reproduction of the symplectic form by Serre duality .

We want to illustrate Theorem 6.24 and consider the first non-trivial case $g = 1$. It corresponds to a solution of the sinh-Gordon equation where the "higher flow" is given by the flow of translations and therefore the corresponding polynomial Killing field $\zeta_\lambda(x)$ solves

$$\tfrac{d}{dx}\zeta_\lambda = [\zeta_\lambda, U_\lambda] \text{ with } \zeta_\lambda(0) \in \mathcal{P}_1.$$

From Lemma 5.27 we know that a general variation of U_λ is given by

$$\begin{aligned}
\delta U_\lambda &= \left[L_\lambda, -\sum_{i=1}^{2} \frac{(\delta v_i(x))w_i^t(x)}{w_i^t(x)v_i(x)}\right] + \left(P_x(\tfrac{\delta \ln \mu}{\mathbf{p}}) + \sigma^* P_x(\tfrac{\delta \ln \mu}{\mathbf{p}})\right) \\
&= [L_\lambda, B^-(x)] + \left(P_x(\tfrac{\delta \ln \mu}{\mathbf{p}}) + \sigma^* P_x(\tfrac{\delta \ln \mu}{\mathbf{p}})\right) \\
&= \delta U_\lambda^{\mathrm{I}} + \delta U_\lambda^{\mathrm{N}}
\end{aligned}$$

and that δU_λ is a sum of an isospectral part $\delta U^{\mathrm{I}}_\lambda$ and a non-isospectral (but iso-periodic) part $\delta U^{\mathrm{N}}_\lambda$. Moreover, let

$$\delta U^{f_0}_\lambda = [L_\lambda, A^-_{f_0}(x)]$$

be a variation that corresponds to an element $(\delta u^{f_0}, \delta u^{f_0}_y) \in L_{(u,u_y)}$. In the present situation f_0 and $\delta \ln \mu$ are given by

$$f_0 = c_0 \frac{\nu}{\lambda} \text{ with } c_0 \in i\mathbb{R} \text{ and } \delta \ln \mu = c_1 \frac{\lambda}{\nu} \text{ with } c_1 \in \mathbb{R}.$$

Note that $c(\lambda) = c_1 \lambda$ is a polynomial of degree $g = 1$ since we are interested in iso-periodic deformations $\delta U^{\mathrm{N}}_\lambda$ that leave the period $\mathbf{p}$ fixed. Now $A_{f_0}(x)$ is given by

$$\begin{aligned} A_{f_0}(x) &= P_x(f_0) + \sigma^* P_x(f_0) = c_0(P_x(\lambda^{-1}\nu) + \sigma^* P_x(\lambda^{-1}\nu)) \\ &= c_0 \zeta_\lambda(x). \end{aligned}$$

Inserting this into equation (6.5.1) yields

$$\begin{aligned} \Omega((\delta u^{f_0}, \delta u^{f_0}_y), (\delta u, \delta u_y)) &= i \operatorname{Res}([f]\, \omega(\delta u, \delta u_y)) \\ &= -2\Im\left(\operatorname{Res}_{y_0}(f_0 \cdot \delta \ln \mu \tfrac{d\lambda}{\lambda})\right) \\ &= 2ic_0c_1 \in \mathbb{R}. \end{aligned}$$

The last equation equals zero if and only if $c_0 = 0$ or $c_1 = 0$.

7 Summary and outlook

In this chapter we summarize the results of this work, especially those which are new. We also give some remarks on other interesting questions that are beyond the scope of this thesis.

7.1 Summary

In this thesis we studied the sinh-Gordon equation and worked out the Hamiltonian framework for periodic finite type solutions, i.e. we identified the space of such solutions as a completely integrable Hamiltonian system $(M_g^{\mathbf{p}}, \Omega, H_2)$. Moreover, we were able to prove the classical features of integrable systems for that particular system. We now give an overview for the results of the various chapters.

In the *second chapter* we introduced the λ-dependent $\mathfrak{sl}_2(\mathbb{C})$-valued one-form α_λ with $\lambda \in \mathbb{C}^*$ following the exposition of Hitchin in [30] to obtain a $\mathbb{C}^*$-family of flat connections. The Maurer-Cartan equation for α_λ is the sinh-Gordon equation

$$\Delta u + 2\sinh(2u) = 0,$$

which arises as the integrability condition for the λ-dependent extended frame F_λ that solves the equation

$$dF_\lambda = F_\lambda \alpha_\lambda \quad \text{with} \quad F_\lambda(z_0) = \mathbb{1}.$$

Moreover, it was possible to describe the transformation of F_λ and α_λ with respect to certain parameter transformations.

In the *third chapter* we introduced spectral data (Y, D) for periodic finite type solutions of the sinh-Gordon equation consisting of a spectral curve Y and a divisor D. We defined the monodromy M_λ of the λ-dependent frame F_λ and considered its asymptotic expansion around the points $\lambda = 0$ and $\lambda = \infty$. At these points M_λ has essential singularities. We were able to prove a formal diagonalization of the form α_λ around $\lambda = 0$ and also obtained a formal diagonalization of the monodromy M_λ around $\lambda = 0$.
Instead of taking a periodic u defined on $\mathbb{R}^2$ we studied a pair $(u, u_y) \in C^\infty(\mathbb{R}/\mathbf{p}) \times C^\infty(\mathbb{R}/\mathbf{p})$ with fixed period $\mathbf{p} \in \mathbb{R}$ that corresponds to u if one considers the coordinate y as a flow parameter. By introducing polynomial Killing fields $\zeta_\lambda(x)$ and the appropriate space of potentials $\mathcal{P}_g$ we parameterized the space of Cauchy data (u, u_y) of finite type and gave definitions for their spectral data $(Y(u, u_y), D(u, u_y))$ consisting of a spectral

curve $Y(u, u_y)$ and a divisor $D(u, u_y)$ on $Y(u, u_y)$.

In the *fourth chapter* we showed that the map $(u, u_y) \mapsto (Y(u, u_y), D(u, u_y))$ is a bijection and introduced a basis of $H^1(Y, \mathcal{O})$. Moreover, we translated the reality condition on M_λ and ξ_λ to this setting. We also investigated the Baker-Akhiezer function and its analytic properties in order to reconstruct the x-dependent eigenvectors of $M_\lambda(x) = F_\lambda^{-1}(x) M_\lambda F_\lambda(x)$ and $\zeta_\lambda(x)$.

The *fifth chapter* dealt with isospectral and non-isospectral deformations of the spectral data (Y, D). On the one hand we studied non-isospectral (but isoperiodc) deformations of spectral curves Y of genus g and showed that the space of such curves is a smooth g-dimensional manifold with the help of the Whitham deformations. This lead to the conclusion that the space of Cauchy data (u, u_y) that corresponds to such smooth spectral curves Y is a smooth $2g$-dimensional manifold.
We also introduced an isospectral group action on $\mathrm{Pic}^{\mathbb{R}}_{g+1}(Y)$ by means of Krichever's construction procedure for linear flows on $\mathrm{Pic}^{\mathbb{R}}_0(Y)$ and showed that

$$\mathrm{Iso}(Y) = \{(u, u_y) \mid Y(u, u_y) = Y\}$$

is parameterized by a g-dimensional torus $(\mathbb{S}^1)^g$. This degree of freedom corresponds to the degree of freedom for the movement of the divisor D in the Jacobian $\mathrm{Jac}(Y)$. Moreover, we calculated the infinitesimal deformations of ξ_λ and U_λ that result from that isospectral group action.

The *sixth chapter* combined the third, fourth and fifth chapter and dealt with the symplectic form Ω on the $2g$-dimensional phase space $M_g^{\mathbf{P}}$ as well as the Hamiltonian formalism for the sinh-Gordon hierarchy.
Due to the asymptotic expansion of the monodromy M_λ we were able to define a series of Hamiltonians $(H_n)_{n\in\mathbb{N}_0}$ on the phase space and showed that the series $(\omega_n)_{n\in\mathbb{N}_0}$ of solutions of the linearized sinh-Gordon equation

$$\bar{\partial}\partial\omega + \cosh(2u)\omega = \left(\tfrac{1}{4}\Delta + \cosh(2u)\right)\omega = 0,$$

that is obtained via the Pinkall-Sterling iteration, corresponds to the gradients of the Hamiltonians $(H_n)_{n\in\mathbb{N}_0}$ in the following way:

$$\nabla H_{2n+1} = (\Re(\omega_n(\cdot,0)), \Re(\partial_y\omega_n(\cdot,0))) \;\text{ and }\; \nabla H_{2n+2} = (\Im(\omega_n(\cdot,0)), \Im(\partial_y\omega_n(\cdot,0))).$$

We also showed that $(H_n)_{n\in\mathbb{N}_0}$ are involutive integrals of motion for the Hamiltonian system $(M_g^{\mathbf{P}}, \Omega, H_2)$. Moreover, we introduced an inner product on the loop Lie algebra $\Lambda_r\mathfrak{sl}_2(\mathbb{C})$ and used this inner product to establish the formula

$$\Omega(d\Gamma_{(u,u_y)}([f]), (\delta u, \delta u_y)) = i\,\mathrm{Res}([f]\,\omega(\delta u, \delta u_y))$$

that relates the symplectic form Ω to Serre duality as it was done in [47] for the non-linear Schrödinger operator. This is the main result of the thesis.

7.2 Outlook

The Bäcklund transformation provides a tool for isospectral transformations of given Cauchy data (u, u_y). A good exposition for the case of complex Fermi curves of finite genus corresponding to Dirac operators with periodic finite type potentials can be found in [48]. For a quaternionic divisor D with $\deg(D) = g + 1$ we see that the transformation

$$D \mapsto \widetilde{D} := D + y_0 + y_\infty - y - \eta(y)$$

preserves this property. By calculating the corresponding Baker-Akhiezer function $\widetilde{\psi}$, it should be possible to describe the transformation $(u, u_y) \mapsto (\widetilde{u}, \widetilde{u}_y)$ with respect to the transformation $D \mapsto \widetilde{D}$.

Another interesting question arises from a result in [47], where the non-linear Schrödinger operator with periodic potential $q(x)$ was investigated. It was shown that the points $(\lambda_i, \mu_i)_{i \in \mathcal{I}}$ of the corresponding divisor $D(q)$ are almost Darboux coordinates in the sense that

$$\Omega(\delta q, \delta\widetilde{q}) = \sum_i (\tfrac{d}{dt}\lambda_i(\delta q)|_{t=0})(\tfrac{d}{dt}\ln \mu_i(\delta\widetilde{q})|_{t=0}) - (\tfrac{d}{dt}\lambda_i(\delta\widetilde{q})|_{t=0})(\tfrac{d}{dt}\ln \mu_i(\delta q)|_{t=0}),$$

or in short form

$$\Omega = \sum_i d\lambda_i \wedge d\ln \mu_i.$$

An analogous result was proven in [1] for the finite-dimensional case. This result should carry over to the present situation if we replace λ_i by $\ln \lambda_i$ in the last equation. It would be interesting to relate our results to the existence of such Darboux coordinates in the present setting.

Finally, it should be possible to extend the present results to periodic solutions of the sinh-Gordon equation of infinite type to obtain a similar description as in [47].

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