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A Rationale for Non-Monotonic Group-Size Effect in Repeated Provision of Public Goods*

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Abstract

This paper analyzes the impact of a group-size change on contributing incentives in repeated provision of pure public goods. We develop a model in which the group members interact repeatedly and might be temporarily constrained to contribute to the public goods production. We show that an increase in the group size generates two opposite effects – the standard free-riding effect and the novel large-scale effect, which enhances cooperative incentives. Our results indicate that the former effect dominates in relatively large groups while the latter in relatively small groups. We provide therefore a rationale for non-monotonic group-size effect which is consistent with the previous empirical and experimental findings.

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Keywords: pure public goods, repeated game, non-monotonic group-size effect.

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1 Introduction

This paper addresses the question of private provision of public goods in a repeated setting. We consider an environment in which individuals interact repeatedly and might be temporarily constrained to contribute to the public goods production. The examples include review posting or file sharing in online communities, open-source software developing in programmer communities, problem solving in teams, fundraising in charitable organizations, information sharing in alliances, etc. In these examples, potential contributors might be temporarily unable to make contributions. Indeed, online community members might have no files to share, programmers or team members might lack expertise in solving particular tasks, donors might be financially constrained in some periods, while allies might have no relevant information to reveal. In this paper, we develop a repeated framework with temporarily constrained group members and provide the conditions under which cooperation can be sustained. Our main contribution is to analyze the effect of group size on the members' incentives to cooperate, which is called the *group-size effect*.

We consider an infinite-horizon model. A group of selfish infinitely-lived agents can provide a pure public good in each period. Public goods are assumed not to accumulate over time. The group members cannot directly communicate or make monetary transfers. Each group member might become temporarily unable to contribute to the public good production in some periods, which is his private information. Moreover, in the baseline model, individual contributions do not accumulate within one period. Indeed, think of a public good as a piece of information relevant in that period. If at least one group member reveals it then the public good is provided in that period. We relax this assumption in Section 5.2 in which a framework with cumulative contributions is analyzed.

We assume further that the cost of public good production exceeds a contributor's private benefit. Thus, cooperation cannot be sustained in a one-shot game. We consider next a repeated setting in which the group members observe whether the public goods have been provided or not in the previous periods, and condition their contributing decisions on this public history. In this setting, cooperative incentives arise from the repeated interactions. While it is not surprising that repeated interactions can incentivize cooperation, the mechanism at work is novel here. Each member who is able to make contribution faces the following trade-off. On the one hand, he has strong incentives to free-ride as contributing is costly and the public good might as well be provided by other able members. On the other hand, he might be constrained in some future periods and thus will have to rely on contributions by his non-constrained peers. However, those will contribute only if cooperation in the group has not broken down by then. We show that cooperation can be sustained in equilibrium if the group members are sufficiently patient. We provide therefore a rationale for public good contribution in a setting with selfish agents in which private benefits from making contributions are assumed away.

Our main finding is that of group-size effect. We show that for a range of parameter values, cooperative incentives increase with the group size in relatively small groups but decrease with the group size in relatively large groups. In short, the group-size effect is positive in small groups but negative in large groups. The intuition for this result is as follows. On the one hand, free-riding incentives strengthen with an increase in the group size. Each member realizes that in larger groups, it is more likely that other members are able to contribute to the public good provision and thus less likely that his own deviation triggers punishment. An increase in the group size thus has a negative impact on cooperative incentives. On the other hand, there is also a positive impact since continuation value of cooperation increases with the group size. Intuitively, the larger the group, the more members will be able to contribute to the public good production in the future periods. Each member thus expects the higher future payoffs in the periods in which he becomes constrained and so has extra incentives to sustain cooperation in the periods in which he faces no constraints. We call this effect a "large-scale effect."

There are therefore two opposite forces at work when the group size changes – the free-riding effect and the large-scale effect. We show that the aggregate impact depends on the initial group size. Indeed, the large-scale effect dominates in relatively small groups while the free-riding effect dominates in relatively large groups. The most intuitive way to see this result is to consider the limit cases of the group size. In a single-member group, there are no other members to rely on when a member faces constraints. As a result, contributing today implies no future benefits to the member and so he decides to shirk. In an infinitely-large group, no member will contribute either as his deviation is highly unlikely to trigger punishment. Then, cooperation has to be sustained in a medium-size group if it can be sustained at all.

Our result about non-monotonic group-size effect is closely related to the issue of community design. In online communities, for example, interactions and contributions fluctuate considerably over time. Many once active communities suddenly become quiet. Some of them revive later on while others never recover since then. This suggests that community members might tacitly apply collective punishment after observing a drop in contributions. Moreover, some organizations switch from open to restricted membership or start charging membership fees once they gain popularity. One possible explanation for this practice is that communities might want to limit the group size in order to mitigate the free-riding problem.[1] Indeed, the trade-off between large-scale effect and free-riding effect implies that as a group size grows, an increase in the probability of having more able members comes at a cost of lower individual incentives, which does not pay off in relatively large groups.

Our findings contribute to the literature on voluntary provision of pure public goods which goes back to the pioneering work of Samuelson (1954, 1955) followed by Olson (1965), Chamberlin (1974), Andreoni (1988), among many others. In these studies, an individual's marginal return from public good contribution decreases with the group size. Each group member therefore tends to shift a larger share of his resources to private consumption as the group expands. As a result, the free-riding effect arises such that individual contributions decrease with the group size.[2] Although theoretically sound, this reasoning fails to explain significant contribution rates observed in many real world communities. This is somewhat resolved in the literature on impure public good provision which assumes that agents gain certain private benefits or warm glow (such as moral satisfaction and joy of giving) from their own contribution (see Steinberg 1987, Andreoni 1989, 1990, among many others). Then, the impact of a group size on agents' private benefits is crucial in determining the group-size effect. For example, if private benefits from giving increase with the number of recipients then cooperative incentives might dominate free-riding incentives even in large groups (see Andreoni 2007).

There is also a growing empirical and experimental literature on the group-size effect. The evidence is mixed, however. Ledyard (1995) reports that free-riding behavior aggravates with the group size while Goeree et al. (2002) find no clear group-size effect. In turn, Zhang and Zhu (2011) show that an exogenous reduction in group size lowers average contribution levels of the remaining contributors at Chinese Wikipedia. The evidence of non-monotonic group-size effect is reported by Isaac and Walker (1988) and Isaac et al. (1994). Those are experimental studies analyzing free-riding and cooperative behaviors in groups of different sizes. The experiments were run for 10, 40 and 60 decision rounds and per-period endowment of each individual was his private information. The authors report a positive group-

[1] Obviously, there could be many other reasons for restricting membership and introducing membership fees (such as network effects, critical mass effects or congestion).

[2] Similar results are also reported in the mechanism design literature on public good provision (see Güth and Hellwig 1987, Mailath and Postlewaite 1990, Hellwig 2003).

size effect in relatively small groups (of size 4 and 10) but no group-size effect in relatively large groups (of size 40 and 100), which is somewhat in line with our theoretical findings.

This paper is also related to the literature on dynamic free-rider problems which focuses mainly on durable contribution settings.[3] To the best of our knowledge, the only paper that studies a setting with non-durable contributions is that by Pecorino (1999). He assumes perfect monitoring, non-constrained agents and large groups, and shows that cooperation can be sustained even in an infinitely-large group. In contrast, we consider imperfect monitoring, temporarily constrained individuals and finite groups. We can then compare cooperative incentives in finite groups of different sizes and explicitly identify the group-size effect.

Another strand of literature to which this paper is related is that on free-rider problem in team production. Holmstrom (1982) emphasizes that free-rider problem is generic in teams and shows that budget breaking might be needed in order to mitigate externalities from joint production. In contrast to Holmstrom (1982) and other follow-up studies, we don't build on the agency framework but rather study cooperative incentives induced by the repeated interactions. Kandel and Lazear (1992) develop a model in which free-rider problem can be somewhat alleviated through peer pressure. Since the payoffs depend on aggregate performance, the team members indeed have incentives to impose pressure on their shirking peers through physical or mental punishment. Kandel and Lazear (1992) conclude that peer pressure works well in homogeneous groups in which the members understand each other's tasks and so can effectively monitor each other. Our findings suggest however that skill heterogeneity might actually enhance cooperation in teams. Workers of different skills coming up with different ideas is the only source of cooperative incentives in our setting. A member works hard on a task that he is able to solve only if he expects his team mates to have necessary skills he lacks to solve some other tasks in the future.

The remainder of the paper is organized as follows. Section 2 describes the setup of our baseline model. Section 3 characterizes the equilibrium. Section 4 proceeds with our main result about group-size effect. Section 5 presents alternative model

[3]The classical references include Fershtman and Nitzan (1991) who study a dynamic public goods problem, Marx and Matthews (2000) who consider a dynamic fundraising environment, Lockwood and Thomas (2002) who analyze a repeated Prisoner's Dilemma with irreversible contributions, and many others. In all these studies, individual contributions are assumed to accumulate over time and thus are (partially) durable. In this paper, we abstract from this intertemporal link and focus on environments with non-durable contributions which are quite common in real-world practice.

specifications and examines robustness of our results. Finally, section 6 concludes.

2 Model

Consider an infinite-horizon model. A group of N risk-neutral infinitely-lived individuals can produce one unit of indivisible public good in each period. Public goods are assumed not to accumulate over time. The group members cannot directly communicate or make monetary transfers.

We assume that some of the group members might be incapable to contribute to the public good production in some periods. Formally, in each period, a group member receives a personal shock which is independent and identically distributed across periods and individuals. A member is able to contribute one unit of the public good with probability q, and is not able to contribute with probability $1 - q$. In what follows, we call the former an able member while the latter an unable member. N_t denotes the number of able members in period t where $N_t \in \{0, 1, ..., N\}$. An able member's strategy space in period t is whether to contribute one unit of the public good or not.

We study first a benchmark case in which individual contributions do not accumulate within one period. If at least one individual contributes in period t then the public good is produced in that period. Then an able member gets payoff $v \geq 0$ while an unable member gets $u > 0$. Additional contributions generate no extra payoffs to the group members. However, if no member contributes in period t then the public good is not provided in that period and an able member gets $w \geq 0$ while an unable member gets 0. We assume that $0 \leq w \leq v$ which means that an able member's utility from the public good consumption is weakly higher than his utility from no public good option. The member's individual one-period payoffs are summarized in Table 1.

Public good provision is costly. In particular, an able member incurs a fixed cost c of providing one unit of the public good. We assume that $c > v - w$ which implies that the cost of production exceeds the private benefits for a contributor. It follows that no contribution will be made in a stage game.

	public good provided	public good not provided
unable member	u	0
able member	v	w

Table 1: Individual one-period payoffs.

We focus on symmetric pure strategies. We relax this assumption in Section 5.1 in which we consider correlated equilibrium. Suppose now that all able members contribute to the public good provision. Then the member's individual one-period *ex-ante* payoff is given by $q(v-c)+(1-q)(1-\alpha)u$, where $\alpha \equiv (1-q)^{N-1}$ is the probability that there are no other able members in the group. Suppose next that no able member contributes to the public good provision. Then the member's individual one-period *ex ante* payoff is qw. The following assumption guarantees that it is *ex ante* socially beneficial that all able members contribute to the public good production.

Assumption 1 $q(v-c-w)+(1-q)(1-\alpha)u \geq 0.$

The timing of events within one period is as follows. First, able members are randomly drawn by nature. Each member's ability to contribute to the public good production is his private information. Second, able members simultaneously and independently decide whether to contribute or not. Finally, all members observe whether the public good is provided and get corresponding payoffs.

Our framework is quite general and can be used to analyze different applications. We provide now several examples and turn then to equilibrium characterization.

Peer-to-Peer Networks Consider a file-sharing network of internet users. In each period, some users (with probability q) get a rare file of value $w>0$ and choose whether to share it with the rest of the group. Sharing the file does not generate extra benefits for the users, i.e., $v=w$. The cost of sharing is c. If the file is distributed then the rest of the group gets benefit $u>0$.

Intelligence Agencies Think of a group of countries facing a common threat (e.g., terror threat). In each period, an intelligence service of each country might get a piece of relevant information which can be then revealed at cost c to intelligence services of other countries. This would obviously benefit uninformed countries ($u>0$) but would bring no extra benefit to informed ones ($v=w>0$).

Online Reviews In each period, a new restaurant is opened. Some consumers might learn by chance how good it is (with probability q). They then decide whether to post an online review about the restaurant quality. Posting a review is costly and implies no extra benefits for the informed consumers, i.e., $v=w$. Still, posting a review would benefit the uninformed consumers by generating net payoff $u>0$ for them.

Open-Source Software Consider an open-source software application. With probability q each programmer comes up with an idea of how to improve the application (e.g., fixing bugs, improving efficiency, etc.) and then decides whether to make this improvement. The cost of improving the application is c. Each programmer gets benefit $v = u > 0$ if the software is upgraded, and $w = 0$ otherwise.

Teamwork Think of a team which is assigned a new task every period. With probability q each member knows how to perform the task at cost c. If the team succeeds in solving the task then each team member gets benefit $v = u > 0$. However, if the team fails then each member gets $w = 0$.

3 Equilibrium Characterization

Consider first a single stage game. In this case, able members will not contribute to the public good production since the cost exceeds their private benefits from contributing. This is the unique equilibrium in the static setting.

We turn next to the repeated setting in which all group members observe whether the public goods have been provided or not in the previous periods. The equilibrium strategy we consider is similar to that in Green and Porter (1984) and includes a cooperation phase and a punishment phase. An able member cooperates (i.e., contributes to the public good provision) in the cooperation phase and punishes (i.e., does not contribute) in the punishment phase. Punishment lasts for T periods and afterwards cooperation is restored.[4]

We denote the state in period t by $w_t \in \{0, 1\}$. Then $w_t = 0$ if in period t the game is in the punishment phase while $w_t = 1$ if it is in the cooperation phase. At the end of each period, the group members observe whether a public good has been provided in this period or not, which serves as a public signal for them.[5] We define the following binary signal space $\{\underline{y}, \overline{y}\}$ such that $y_t = \overline{y}$ either if $w_{t-1} = 1$ and a period-t public good is provided, or if in period t a punishment phase has just ended. In turn, $y_t = \underline{y}$ either if $w_{t-1} = 1$ but a period-t public good is not provided, or if in period t a punishment phase has not been yet ended. The solution concept is symmetric Public Perfect Equilibrium (SPPE) in which the members condition their actions only on the public history.[6] Then all able members contribute to the

[4] No personal enforcement is allowed. We therefore don't consider the punishments based on the statistical evaluation of an individual member's previous deviations.

[5] More generally, one can allow the public signal to be the units of contributions in the previous period. However, this will not change our result regarding group-size effect and able members' trade-off is more transparent in the present setting. In fact, we allow the more general signal space in the discussion of cumulative contributions in Section 5.2.

[6] It is well known from the literature on the repeated games with imperfect public monitoring that using public strategy is a best response to all other members' using public strategies.

public good production if in the previous period the public signal is $\overline{y}$, and don't contribute if in the previous period the public signal is $\underline{y}$. The transition between the states can be summarized as follows:

- $w_t = 1$ for $t = 0$;
- if $w_{t-1} = 1$ and $y_t = \overline{y}$ then $w_t = 1$; if $w_{t-1} = 1$ and $y_t = \underline{y}$ then $w_t = 0$;
- if $w_{t-1} = 0$ and $y_t = \overline{y}$ then $w_t = 1$; if $w_{t-1} = 0$ and $y_t = \underline{y}$ then $w_t = 0$.

We analyze now the able members' incentives to follow the prescribed strategy. If an able member contributes to the public good production in a punishment phase then the subsequent game will not be affected, but he gets negative net payoff $v - c - w$ in the current period. It follows that an able member has no incentive to contribute in a punishment phase and therefore will follow the prescribed strategy.

If an able member contributes to the public good production in a cooperation phase then his expected payoff is

$$v - c + \delta V^{+},$$

where $\delta \in (0, 1)$ is a common discount factor and V^{+} is the member's value function defined at the beginning of any period in the cooperation phase and before each member learns whether he is able to contribute or not. By following the prescribed strategy in the cooperation phase, the able member also ensures cooperation in the next period.

Suppose that the able member chooses to deviate in the cooperation phase. Then punishment will be trigged only if there are no other able members in the group. Therefore, the expected payoff from deviating in the cooperation phase is given by

$$(1 - \alpha) \left(v + \delta V^{+}\right) + \alpha \left(w + \delta V^{-}\right),$$

where V^{-} is the member's value function defined at the beginning of a punishment phase and before each member learns whether he is able to contribute or not. As before, $\alpha \equiv (1 - q)^{N-1}$ denotes the probability that there are no other able members in the group.

We can define the value functions V^{+} and V^{-} recursively. At the beginning of any period in a cooperation phase, a member anticipates that with probability q he will be able to contribute to the public good provision while with probability $1-q$ he will not be able to do so. Following the prescribed strategy, an able member contributes to the public good production and gets payoff $v - c + \delta V^+$. In turn, an unable member's payoff is determined by the rest of the group. If other group members are able to contribute then they follow the prescribed strategy and do contribute. This generates benefit u for the unable member and ensures cooperation in the next period. However, if there are no other able members in the group then the public good is not provided which triggers punishment. The unable member's expected payoff is δV^- in this case. It follows then that

$$V^+ = q\left(v - c + \delta V^+\right) + (1-q)\left[(1-\alpha)\left(u + \delta V^+\right) + \alpha\delta V^-\right]. \tag{1}$$

At the beginning of a punishment phase, all members realize that for T periods no public goods will be produced but afterwards cooperation will be restored. They therefore expect per-period payoff qw for T periods and $\delta^T V^+$ afterwards. It implies that

$$V^- = \sum_{\tau=0}^{T-1} \delta^\tau qw + \delta^T V^+ = \frac{1-\delta^T}{1-\delta} qw + \delta^T V^+. \tag{2}$$

Substituting (2) into (1) and rearranging yields the continuation value of cooperation V^+:

$$V^+ = \frac{(1-q)(1-\alpha)u + q(v-c) + (1-q)\alpha\delta\left(\frac{1-\delta^T}{1-\delta}\right)qw}{1 - q\delta - \delta(1-q)(1-\alpha+\alpha\delta^T)}. \tag{3}$$

An able member follows the prescribed strategy in the cooperation phase if and only if his payoff from cooperating exceeds that from deviating:

$$v - c + \delta V^+ \geq (1-\alpha)\left(v + \delta V^+\right) + \alpha\left(w + \delta V^-\right). \tag{4}$$

Substituting V^+ and V^- into (4) and simplifying yields the necessary and sufficient condition for sustaining cooperation:

$$\alpha\left(\frac{(1-\alpha)\left[(1-q)u + q(v-w)\right]}{c - \alpha(v-w)} - 1\right) \geq \frac{1-\delta}{\delta(1-\delta^T)}. \tag{5}$$

The right-hand side of (5) is a decreasing function of T. It goes to infinity when T approaches 0, which implies that without punishment, cooperation cannot be sustained. It converges to $\frac{1-\delta}{\delta}$ when a grim-trigger strategy (under which T goes to infinity) is applied. Therefore, a longer punishment phase makes cooperation easier to sustain, but it also leads to the value loss on the equilibrium path because punishment can be trigged even if no one has deviated. Since the right-hand side of (5) is always positive then a necessary condition for sustaining cooperation for given T is

$$(1-\alpha)(1-q)u+[(1-\alpha)q+\alpha](v-w)>c. \tag{6}$$

Note moreover that the right-hand side of (5) is a decreasing function of both δ and T. For any positive but finite T, it reaches its maximum of ∞ when δ approaches 0 and its minimum of $\frac{1}{T}$ when δ approaches 1. Thus for given T, as long as

$$\alpha\left(\frac{(1-\alpha)[(1-q)u+q(v-w)]}{c-\alpha(v-w)}-1\right)\geq\frac{1}{T}, \tag{7}$$

and the members are patient enough, cooperation can be sustained. When a grim-trigger strategy is used ($T=\infty$) so that the right-hand side of (7) becomes zero, cooperation essentially requires the left-hand side of (7) to be positive. The result is summarized in the following proposition. (Proofs of this and other propositions are given in the Appendix.)

Proposition 1 *As long as condition (6) holds, there exists a threshold discount factor* $\overline{\delta}\in(0,1)$ *associated with some punishment length* $T<\infty$ *such that cooperation is sustained for all* $\delta\geq\overline{\delta}$.

Comparing (6) with Assumption 1 makes it clear that for $c\in[(1-\alpha)(1-q)u+[(1-\alpha)q+\alpha](v-w), v-w+\frac{1-q}{q}(1-\alpha)u]$, cooperation is *ex ante* socially beneficial but it cannot be sustained in equilibrium. Note also that applying a grim-trigger strategy ($T=\infty$) is often suboptimal as it forgoes potential surplus from cooperation by punishing forever after one period of non-provision. As long as the able members' incentives are compatible with cooperation, the shorter the punishment phase the better. Therefore, the optimal punishment length, T_δ^*, is characterized by

$$\alpha\left(\frac{(1-\alpha)[(1-q)u+q(v-w)]}{c-\alpha(v-w)}-1\right)=\frac{1-\delta}{\delta\left(1-\delta^{T_\delta^*}\right)} \tag{8}$$

if the solution exists. Otherwise, we set $T_\delta^*=\infty$. This corresponds to the case in which the group never recovers to cooperation after one period of non-provision.

4 Group-Size Effect

We turn next to the group-size effect to study the impacts of a group-size change on the members' incentives to cooperate. Consider first the non-deviation condition

(4). Substituting (2) into (4) and rearranging yields

$$\underbrace{\alpha}_{\text{free-riding}} \left[\delta \left(1-\delta^{T}\right) \underbrace{V^{+}}_{\text{large scale}} + \left(v - w - \delta q w \frac{1-\delta^{T}}{1-\delta} \right) \right] \geq c. \tag{9}$$

Group size N affects only α and V^{+} which are both on the left-hand side of (9). α denotes the probability that there are no other able members in the group and so decreases with N. This effect makes the left-hand side of (9) smaller as N increases. V^{+} is the continuation value of cooperation and is also affected by a change in the group size N. The following lemma shows that an increase in N actually makes V^{+} (and therefore the left-hand side of (9)) larger.

Lemma 1 *V^{+} increases with N in equilibrium.*

It follows therefore that there are two opposite forces at work when the group size increases. The first force is the conventional free-riding effect reflected by α in (9). Intuitively, an able member has more incentives to deviate in a larger group. He realizes that the larger the group, the more likely there are other able members in the group and so the less likely his own deviation is to trigger punishment. The second force is what we call a large-scale effect, reflected by V^{+} in (9). An able member wants cooperation to be sustained in order to enjoy public good benefits even in the periods when he will be unable to contribute and so will depend on the other members' contributions. The more members there are in the group, the more likely there will be able members in the periods when he will be unable to contribute and so the larger the continuation value of cooperation.

We use the optimal punishment length, T_{δ}^{*}, to measure the group size effect. Suppose that a finite solution to (8) exists when the group size is N'. Now increase the group size to $N'' > N'$. If the optimal punishment length T_{δ}^{*} becomes larger or even infinite under N'' then we say that the group size effect is negative. If the optimal punishment length becomes shorter after the group size increases to N'', the group-size effect is positive.

Consider the left-hand side of equation (8). If $c \geq q(v-w) + (1-q)u$ then it is nonpositive for all $N \geq 2$ and thus cooperation cannot be sustained for any group size. If $c < q(v-w) + (1-q)u$ then the left-hand side of (8) is positive for some $N \geq 2$. We show that it has an inverted-U shape (i.e., first increases but then decreases in N) if $c > \widehat{c}$ where

$$\widehat{c} \equiv \frac{1}{2}\left(v - w - (1-q)(1-2q)(u - v + w)\right) + \frac{1}{2}\sqrt{\left((1-q)u + q(v-w)\right)\left(u - (u - v + w)\left(5q - 8q^2 + 4q^3\right)\right)}.$$

However, if $c \leq \widehat{c}$ then the left-hand side of (8) strictly decreases in N. The results are summarized in the following proposition.

Proposition 2 *If $\widehat{c} < c < q(v-w) + (1-q)u$ then the group-size effect is positive in small groups but negative in large groups. If $c \leq \widehat{c}$ then the group-size effect is always negative.*

According to Proposition 2, for $c \leq \widehat{c}$, free-riding effect prevails regardless of the group size. However, for $\widehat{c} < c < q(v-w) + (1-q)u$, the relationship between the group size and cooperative incentive is non-monotonic. While an increase in the group size intensifies both the free-riding incentives and the continuation value of cooperation, its aggregate impact depends on the current group size. We show that an increase in the group size enhances cooperation in relatively small groups but hinders cooperation in relatively large groups. Indeed, in a small group, an individual deviation is quite likely to trigger punishment and the continuation value of cooperation is low. As the group size increases, an individual deviation is somewhat less likely to trigger punishment while the continuation value of cooperation increases considerably. An increase in the group size then boosts the value of cooperation more than it boosts the free-riding incentives. In contrast, in large groups, a group-size increase just slightly affects the continuation value of cooperation and so enhances the free-riding incentives more than cooperative incentives. It follows therefore that large-scale effect dominates free-riding effect only when the group size is relatively small.

Proposition 2 therefore suggests that in case of $\widehat{c} < c < q(v-w) + (1-q)u$, for any patience level, cooperation is easier to be sustained in medium-size groups in which free-riding incentives and cooperative incentive are well balanced. Intuitively, consider now the limit cases in which N approaches 1 or converges to infinity. In a single-member group, the member has no incentive to contribute today as there is no future reward for doing so. Therefore, no contribution will be made in this case. In an infinitely-large group, there always exist unconstrained members who follow the cooperation strategy and so a deviation will not trigger punishment. As a result, no member will contribute either. Then, cooperation has to be sustained in a medium-size group if it can be sustained at all.

Even though the model is very stylized, it yields empirically testable predictions. Controlling for all other variables and changing N alone, one can test non-monotonic group-size effect predicted by our results. The testable hypotheses might be as follows. First, the group-size effect is initially increasing and then decreasing. Second, cooperation is more likely to be sustained in medium-size groups.

5 Discussion and Extensions

In this section, we relax some of the important assumptions of the model and discuss robustness of our results. We first consider the case in which the group members condition their contribution decisions on their observation of a signal. This allows them to somewhat coordinate their decisions and so makes cooperation easier to be sustained. We next relax the assumption of non-cumulative individual contributions and study the case of linear public good technology. We show that under some mild assumptions, our result about group-size effect holds in these extensions of the baseline model.

5.1 Correlated Equilibrium

In the baseline model, we assume that the group members cannot directly communicate with each other. Though realistic in some situations, this assumption might be too restrictive in some others. Moreover, one might underestimate the level of cooperation in a group if coordination is completely ruled out. Indeed, in our baseline model, every able member contributes on the equilibrium path while only one contribution is needed for public good provision. Therefore, all but one contributions are wasted which substantially reduces the value of cooperation.

In this section, we keep the assumption of no direct communication but suppose that the group members base their contribution decisions on their observation of a signal. Assume that the nature randomly determines an order of contribution making. For example, member 1 is assigned to make contribution first, member 2 is second, ..., member i is ith. Denote by $\mathcal{O}$ the finite set of all possible orders. We assume that all orders in $\mathcal{O}$ are realized with the equal probability. The group members don't know their position in the sequence. In each period, a member observes the signal only when it is his turn to make contribution and no contribution has been made so far in that period. The period ends either when one contribution has been made or when all group members have been called for contribution making. At the end of the period, all members know whether the contribution has been made or not. The prescribed equilibrium strategy requires that an able member makes contribution if he observes the signal. The punishment will be triggered if no contribution is made in the previous period. We characterize correlated equilibrium in which no group member wants to deviate from the prescribed strategy if the others don't deviate.[7]

Although the group members still cannot directly communicate with each other, the signal mechanism works as a coordinating device and guarantees that at most one contribution per period is made on the equilibrium path. Therefore, the contribution waste of the baseline model does not arise here, which implies higher value of cooperation.

Think of the following interpretation of this setting. In each period, there might be several members able to contribute to the public good provision. Since the probability of two members making contribution at exactly the same time is tiny, it is plausible to assume that the timing of contribution making is sequential. The explanation for this might be different time availability assigned randomly to the group members. Then an able member contributes to the public good production only if he observes that no contribution is made so far by his peers. However, if at the moment of contributing an able member observes that one contribution has been already made then he does not need to contribute again. For example, if an informed member reveals a piece of useful information to the rest of the group then other initially informed members will not do so.

We turn next to the analysis of the able members' incentives to follow the prescribed strategy. (Indeed, an unable member cannot make contribution even if he observes the signal.) Similarly to the baseline model studied in Section 3, an able member here has no incentive to contribute in a punishment phase and so will follow the prescribed strategy. Consider now his incentives in a cooperation phase. Suppose that an able member receives the signal and so knows that it is his turn to contribute. The expressions for the expected payoffs from cooperating and deviating are the same as in the baseline model except for the value functions which we denote here by $\tilde{V}^+$ and $\tilde{V}^-$. Obviously, the value function defined at the beginning of a punishment phase satisfies $\tilde{V}^- = V^-$. The value function defined at the beginning

[7]Alternatively, one could consider mixed-strategy equilibrium in which able members randomize between making and not making contribution.

of any period in a cooperation phase, $\tilde{V}^{+}$, is

$$\tilde{V}^{+} = q\left(v - \beta c + \delta V^{+}\right) + (1-q)\left[(1-\alpha)\left(u + \delta V^{+}\right) + \alpha\delta V^{-}\right],$$

where β is the probability that conditional on being able, a member is the first one among all able members to receive the signal. β is given by

$$\beta \equiv \sum_{k=0}^{N-1} \binom{N-1}{k} q^{k}(1-q)^{N-1-k}\frac{1}{k+1} = \frac{1-(1-q)\alpha}{Nq}.$$

Note that on the equilibrium path in our correlated equilibrium, only the able member who receives the signal first, actually incurs the cost c of public good production.

Following the similar steps as in Section 3 yields the non-deviation condition for an able member who receives the signal:

$$\alpha\left(\frac{(1-\alpha)\left[(1-q)\,u + q\,(v-w)\right] + (1-\beta)\,cq}{c - \alpha\,(v-w)} - 1\right) \geq \frac{1-\delta}{\delta(1-\delta^{T})}. \tag{10}$$

It is natural to expect that in correlated equilibrium, cooperation is easier to be sustained than in the baseline model. Indeed, the signal mechanism generates at most one contribution per period and so enhances the value of cooperation. As a result, the left-hand side of (10) is strictly greater than that of (5).

We turn now to our main research question, namely, the group-size effect. It is *a priori* not clear how an increase in the group size affects the members' incentives to cooperate. To answer this question, we study the impacts of a change in N on the left-hand side of non-deviation condition (10). Note that on the left-hand side of (10), both α and β depend on N, and α enters both in the numerator and denominator, which considerably complicates the analysis of the general case. Then, to obtain a clear-cut result, we restrict our attention to the case in which an able member gets the same level of utility from the public good consumption and from no public good option, $v = w$ (as in the examples of information sharing).

Consider the left-hand side of (10) for $v = w$. If $c \geq u$ then it is nonpositive for all $N \geq 2$ and therefore cooperation cannot be sustained for any group size. If $c < u$ then it is positive for some $N \geq 2$. We show that it has an inverted-U shape (i.e., first increases and then decreases in $N \geq 2$) either if $q \leq \frac{1}{2}$ or if $q > \frac{1}{2}$ and $c > \widehat{f}(q)\,u$ where

$$\widehat{f}(q) \equiv \frac{4\,(1-q)\,(1-2q)\ln(1-q)}{(2-q)\,q - 2\,(1+2q\,(1-q))\ln(1-q)}.$$

However, if $q > \frac{1}{2}$ and $c \leq \widehat{f}(q)\,u$ then the left-hand side of (10) strictly decreases in $N \geq 2$. We formalize the results in the following proposition.

Proposition 3 *If $q \leq \frac{1}{2}$ and $0 < c < u$, or if $q > \frac{1}{2}$ and $\widehat{f}(q)\,u < c < u$, then the group-size effect is positive in small groups and negative in large groups. If $q > \frac{1}{2}$ and $0 < c \leq \widehat{f}(q)\,u$ then the group-size effect is always negative.*

Proposition 3 suggests that for a relatively large range of parameter values, the main insights of our baseline model also hold in the case of correlated equilibrium. Here, because of implicit coordination generated by the signal mechanism, the probability of an able member contributing is strictly lower than 1. The more members there are in the group, the less likely an able member is to receive the signal to contribute. Therefore, the large-scale effect is amplified here relative to the baseline case. As a result, in relatively small groups, the large-scale effect again dominates the free-riding effect. Still, the important trade-off between the two effects highlighted in the previous section remains true in the correlated equilibrium. Indeed, in relatively large groups, the free-riding effect dominates the (even amplified) large-scale effect since deviation is quite unlikely to trigger punishment and thus incentives to free ride are stronger than those to cooperate.

5.2 Cumulative Contributions

In the baseline model, we assume that individual contributions do not accumulate within one period. We relax this assumption here to consider a setting in which public good consumption strictly increases in the number of individual contributions, as in the cases of dispersed information, ideas and labor input. So, if each able member cooperates, an increase in the group size affects not only the probability of a single unit being provided but the expected total number of contributions. Our final goal is then to study the group-size effect in this environment.

Consider the following production technology. If the total number of contributions is $k = 0, 1, ..., N$ then the public good of size $P(k)$ is provided and each group member gets utility $P(k)$ from consuming it. So the public good production is deterministic. Alternatively, $P(k)$ might be interpreted as the probability of the public good being successfully provided. We assume that $P(k)$ is strictly increasing in k and that $P(0) = 0$. Able members' utility is equal to 0 in case the public good is not provided.

In each period, an able member decides whether to contribute one unit to the public good production or not. The cost of contributing is $c > 0$. To be consistent with our baseline model, we assume that $P(k) - P(k-1) < c$ for $k = 1, ..., N$. So, an able member has no incentive to make contribution in a stage game. Then, the unique equilibrium in the static setting entails no public good contribution.

We study next the repeated setting in which all group members observe the total number of contributions made in the previous periods. As in the baseline model, the solution concept here is SPPE. To characterize SPPE, we follow the approach developed by Abreu et al. (1986, 1990). According to this approach, a SPPE value can be decomposed into a member's current payoff and the continuation value, denoted by $v(k)$, which is a mapping from the set of all public outcomes, $k = 0, 1, ..., N$, into the set of SPPE values, $V^*(N, \delta)$. Note that each able member choosing not to contribute constitutes a SPPE and so $V^*(N, \delta)$ is non-empty: $0 \in V^*(N, \delta)$. It is also assumed that the group members have access to a public randomization device at the end of each period. Then $v(k)$ can be a probability distribution over $V^*(N, \delta)$. As a consequence, $V^*(N, \delta)$ is a convex set. Then, applying Abreu et al.'s approach, we can show that $V^*(N, \delta)$ is a closed interval of the form $[0, \overline{v}(N, \delta)]$.

We focus the analysis on the upper bound of $V^*(N, \delta)$, $\overline{v}(N, \delta)$, which is the highest value a SPPE can achieve. We believe it is reasonable to do so since the SPPE with the highest value will be a natural candidate in case the group members have ever a chance to coordinate on equilibrium selection. To construct a SPPE, we also have to choose a continuation value from $V^*(N, \delta)$. Abreu et al. (1986, 1990) show that any value in $V^*(N, \delta)$ can be achieved by the following trigger strategy with the *bang-bang* property. At the beginning of the game, every able member contributes to the public good provision. Then, after observing the public outcome k, with probability $1 - \eta(k)$ all group members continue to play the same cooperative strategy in the next period, and with probability $\eta(k)$ they switch to static Nash forever. We can then find the highest value, $\overline{v}(N, \delta)$, among those trigger strategy SPPE values.

We assume without loss of generality that $P(k) = k$. The following problem then characterizes the highest SPPE value, $\overline{v}(N,\delta)$:

$$\begin{aligned}
&\max_{\eta(k)} v = qv^a + (1-q)\,v^u \\
\text{s.t.}\quad & v^a = (N-1)\,q + 1 - c + \delta v \sum_{k=0}^{N-1} (1-\eta(k+1)) \binom{N-1}{k} q^k (1-q)^{N-k-1}, \\
& v^u = (N-1)\,q + \delta v \sum_{k=0}^{N-1} (1-\eta(k)) \binom{N-1}{k} q^k (1-q)^{N-k-1}, \\
& v^a \geq v^u.
\end{aligned}$$

Here, v^a is an able member's discounted payoff if he follows the prescribed strategy while v^u is an unable member's discounted payoff. In each period in which cooperation is sustained, every member expects other members to provide $(N-1)\,q$ units of public good. Moreover, an able member also contributes one unit at cost c. These are the members' current payoffs. The discounted continuation value of cooperation is δv multiplied by the probability of cooperation being sustained. The *ex ante* expected value v is thus $qv^a + (1-q)\,v^u$. In the equilibrium, the incentive compatibility (IC) constraint has to hold so that an able member has no incentive to mimic an unable member: $v^a \geq v^u$.

Substituting the expressions for v^a and v^u into v and the IC constraint and rearranging yields

$$\begin{aligned}
v &= q(1-c) + (N-1)\,q + \delta v \sum_{k=0}^{N} \binom{N}{k} q^k (1-q)^{N-k} (1-\eta(k)), \quad (11) \\
c - 1 &\leq \delta v \sum_{k=0}^{N-1} \binom{N-1}{k} q^k (1-q)^{N-k-1} (\eta(k) - \eta(k+1)).
\end{aligned}$$

In the equilibrium, the IC constraint has to hold with equality. We prove it by contradiction. For the IC constraint to be satisfied, some $\eta(k)$ have to be strictly greater than zero. Suppose that the IC constraint does not bind in the equilibrium. Then, there exists at least one positive $\eta(k)$ which can be decreased by a tiny amount $\varepsilon > 0$ while the IC constraint is still satisfied. However, decreasing this $\eta(k)$ will increase v (since v is a decreasing function of all $\eta(k)$). Thus, a non-binding IC constraint is not optimal, which leads to a contradiction.

In what follows, we use the method developed by Abreu et al. (1991). According to this method, solving for $\overline{v}(N,\delta)$ is equivalent to finding the maximum likelihood test to detect deviation. We next show this formally. Define

$$\mathcal{L} \equiv \frac{q\sum_{k=0}^{N-1}\binom{N-1}{k}q^k(1-q)^{N-1-k}\eta(k)+(1-q)\sum_{k=0}^{N-1}\binom{N-1}{k}q^k(1-q)^{N-1-k}\eta(k)}{q\sum_{k=0}^{N-1}\binom{N-1}{k}q^k(1-q)^{N-1-k}\eta(k+1)+(1-q)\sum_{k=0}^{N-1}\binom{N-1}{k}q^k(1-q)^{N-1-k}\eta(k)}.$$

The numerator is the *ex ante* probability of triggering punishment if one able member deviates and does not make contribution. This probability is then the measure of punishment off the equilibrium path. The denominator is the *ex ante* probability of triggering punishment if no member deviates. This probability is therefore the measure of punishment on the equilibrium path and represents the equilibrium value loss. Then the likelihood ratio $\mathcal{L}$ measures how effective the punishment is in deterring deviation per unit of the value loss on the equilibrium path. The larger $\mathcal{L}$ is, the heavier punishment is imposed on the deviator per unit of the equilibrium value loss.

Since

$$\mathcal{L}-1=\frac{q\sum_{k=0}^{N-1}\binom{N-1}{k}q^k(1-q)^{N-1-k}(\eta(k)-\eta(k+1))}{\sum_{k=0}^{N}\binom{N}{k}q^k(1-q)^{N-k}\eta(k)}, \tag{12}$$

then the binding IC constraint can be written as a function of $\mathcal{L}-1$ in the following way:

$$c-1=\frac{\delta v}{q}(\mathcal{L}-1)\sum_{k=0}^{N}\binom{N}{k}q^k(1-q)^{N-k}\eta(k).$$

Clearly, $\mathcal{L}$ has to be strictly greater than 1 for the IC constraint to hold. Substituting the IC constraint into the value function (11) yields

$$v=\frac{1}{1-\delta}\left[q(N-1)-q(c-1)\left(1+\frac{1}{\mathcal{L}-1}\right)\right], \tag{13}$$

which after rearranging becomes

$$v=\frac{1}{1-\delta}\left[(1-q)q(N-1)+q(q(N-1)+1-c)\right]-\frac{1}{1-\delta}\frac{q(c-1)}{\mathcal{L}-1}.$$

The first term above is the expected value in case cooperation can be sustained forever. The second term is the expected loss resulting from future punishments on the equilibrium path. (13) implies that v is a strictly increasing function of $\mathcal{L}$. Therefore, the optimal punishment scheme maximizes $\mathcal{L}$ subject to the binding IC constraint, which is equivalent to finding the maximum likelihood test for detecting deviation.

We prove in the Appendix that the optimal punishment scheme is characterized by the public outcome $\tilde{k} \in \{0, 1, ..., N\}$ such that $\eta(k) = 1$ for $k < \tilde{k}$, $0 < \eta(\tilde{k}) < 1$, $\eta(k) = 0$ for $k > \tilde{k}$. $\tilde{k}$ and $\eta(\tilde{k})$ are found through the following procedure:
- set $\tilde{k} = 0$ and check the IC constraint for $0 < \eta(0) < 1$, $\eta(k) = 0$, $k > 0$; if the IC constraint holds then this is the optimal punishment scheme and $\eta(0)$ is characterized by the binding IC constraint; if the IC constraint does not hold then
- set $\tilde{k} = 1$ and check the IC constraint for $\eta(0) = 1$, $0 < \eta(1) < 1$, $\eta(k) = 0$, $k > 1$; if the IC constraint holds then this is the optimal punishment scheme and $\eta(1)$ is pinned down by the binding IC constraint; if the IC constraint does not hold then
- move to $\tilde{k} = 2$, check the IC constraint, and continue this process until the IC constraint is satisfied for some $\tilde{k}$.

The following proposition summarizes the result.

Proposition 4 *The optimal SPPE value, $\overline{v}(N, \delta)$, is supported by the following cut-off strategy: $\eta(k) = 1$ for $k < \tilde{k}$, $0 < \eta(\tilde{k}) < 1$, $\eta(k) = 0$ for $k > \tilde{k}$. $\tilde{k}$ and $\eta(\tilde{k})$ are found through the procedure described above.*

According to Proposition 4, the optimal punishment scheme is characterized by a public outcome $\tilde{k}$ such that $\eta(k) = 1$ for $k < \tilde{k}$, $0 < \eta(\tilde{k}) < 1$, $\eta(k) = 0$ for $k > \tilde{k}$. Substituting the optimal punishment scheme into (12) yields

$$\mathcal{L} - 1 = \frac{q\left[\binom{N-1}{\tilde{k}-1} q^{\tilde{k}-1}(1-q)^{N-\tilde{k}}\left(1-\eta(\tilde{k})\right) + \binom{N-1}{\tilde{k}} q^{\tilde{k}}(1-q)^{N-\tilde{k}-1}\eta(\tilde{k})\right]}{\sum_{k=0}^{\tilde{k}-1}\binom{N}{k} q^{k}(1-q)^{N-k} + \binom{N}{\tilde{k}} q^{\tilde{k}}(1-q)^{N-\tilde{k}}\eta(\tilde{k})}. \tag{14}$$

We next study the group-size effect to check robustness of the baseline model results. In this setting with cumulative contributions, we measure the group-size effect by the impact of a group-size change on the optimal cut-off strategy. The group-size effect is negative if an increase in N makes $\tilde{k}$ larger or (in case $\tilde{k}$ remains unchanged) makes $\eta(\tilde{k})$ larger. Indeed, in this case, a more likely punishment is needed to sustain cooperation. The group-size effect is positive if an increase in N makes $\tilde{k}$ smaller or (in case $\tilde{k}$ remains unchanged) makes $\eta(\tilde{k})$ smaller. Then, cooperation can be sustained with a less likely punishment.

Ideally, we would like to characterize the group-size effect for the whole range of parameters N, c, q and δ. Although Proposition 4 provides an algorithm for finding the upper bound of $V^*(N,\delta)$, we have no explicit expressions for $\tilde{k}$ and $\eta(\tilde{k})$ in terms of N and other parameters. This considerably complicates our task. In what follows, we restrict our attention to the case of sufficiently large δ, i.e., sufficiently patient group members:

$$\delta > \left(1 + \left(\frac{q(N-1)}{c-1} - 1\right)(1-q)^{N-1}\right)^{-1} \text{ for } N > 1 + \frac{c-1}{q}. \tag{15}$$

In this case, $\tilde{k} = 0$ characterizes the optimal punishment scheme. To see this, substitute $\tilde{k} = 0$ into (14) which yields

$$\mathcal{L} - 1 = \frac{q(1-q)^{N-1}\eta(0)}{(1-q)^N \eta(0)} = \frac{q}{1-q}.$$

Then substituting $\mathcal{L} - 1$ into the value function (13) yields

$$v = \frac{1}{1-\delta}\left[q(N-1) - (c-1)\right],$$

which is positive for $N > 1 + \frac{c-1}{q}$. Finally, substituting $\mathcal{L} - 1$ and v into the binding IC constraint and rearranging yields

$$\eta(0) = \frac{(1-\delta)(c-1)}{\delta\left[q(N-1) - (c-1)\right](1-q)^{N-1}},$$

which satisfies $\eta(0) \in (0,1)$ for δ specified in (15). Therefore, $\tilde{k} = 0$ indeed characterizes the optimal punishment scheme in this case.

We consider now an increase in the group size from N to $N+1$. The group-size effect is positive if $\tilde{k} = 0$ still characterizes the optimal punishment scheme for $N+1$ and $\eta(0)$ decreases with an increase in the group size. The group-size effect is negative otherwise. Intuitively, if an increase in the group size favors cooperation then the new punishment should be less likely than the original one. We show in the Appendix that for relatively large groups, the group size effect is negative, i.e., an increase in the group size leads to a more likely punishment. For relatively small groups, the group-size effect is shown to be positive under some assumptions. Proposition 5 formalizes the result.

Proposition 5 *Suppose that in a group of size N, the members are patient enough* *such that (15) holds. Then cooperation is sustained with* $\tilde{k} = 0$*. Consider an increase in the group size to* $N + 1$*. Then the group-size effect is positive for relatively small* N *and negative for relatively large* N *if and only if* $0 < q < \frac{1}{2}$ *or* $\frac{1}{2} \leq q < 1$ *and* $2q < c < 1 + q$*. Otherwise, the group-size effect is always negative.*

Proposition 5 indicates that for some range of parameter values, our insights about non-monotonic group-size effect hold in this environment with cumulative contributions. The forces at work here are similar to those of the baseline model. On the one hand, an increase in the group size boosts the continuation value of cooperation, which advances cooperative incentives of the members. On the other hand, it also enhances free-riding incentives, which hinders cooperation. Obviously, the latter effect prevails in large groups in which an individual deviation is not likely to trigger punishment and therefore free-riding incentives dominate cooperative incentives. In small groups, however, the former effect might prevail as an individual deviation is quite likely to trigger punishment (and so free-riding incentives are not that high) while the continuation value of cooperation grows swiftly with the group-size (and so do cooperative incentives).

6 Conclusion

This paper studies the group-size effect in public goods provision. We develop a model in which the group members interact repeatedly and might be temporarily constrained to contribute to the public goods production. In our setting, an increase in the group size enhances both the future value of cooperation and the temptation to free ride. We show that the former effect dominates in relatively small groups while the latter in relatively large groups. As a result, the group-size effect is positive in small communities but negative in large communities.

Our findings provide a novel rationale for public goods contribution and non-monotonic group-size effect. The underlying mechanism is fairly simple and easy to extend. For example, one can consider heterogeneous individuals and study optimal group composition. According to our analysis (available upon request), optimal group composition often requires a mixture of individuals who are more likely to be unconstrained with those who are more likely to get temporarily constrained as the former have higher incentives to shirk. One can also think about introducing a more elaborated production technology into the model. Indeed, in the case of cumulative contributions, public good production can be interpreted as information aggregation process similar to that in voting models. Then, with our model in hand, one can analyze the optimal size of deliberating committees and efficient aggregation of information (for a related work, see Koriyama and Szentes, 2009). Finally, we assume in this paper that every group member can become temporarily constrained and so the probability of all members being constrained is strictly positive in each period. Our setting can be easily extended to the case with a fixed number (but random identity) of unconstrained individuals in each period. We leave these and other potential extensions for future research.

Appendix

Proof of Proposition 1.

Note that the right-hand side of (5) is a continuous function, strictly decreasing in both δ and T. If $T=\infty$ then the right-hand side of (5) becomes $\frac{1-\delta}{\delta}$, which strictly decreases from ∞ to 0 when δ increases from 0 to 1. Then, from the Intermediate Value Theorem, there exists a $\overline{\delta}\in(0,1)$ such that

$$\alpha\left(\frac{(1-\alpha)\left[(1-q)\,u+q\,(v-w)\right]}{c-\alpha\,(v-w)}-1\right)=\frac{1-\overline{\delta}}{\overline{\delta}}.$$

For any $\delta>\overline{\delta}$, we have

$$\frac{1-\delta}{\delta}<\frac{1-\overline{\delta}}{\overline{\delta}}.$$

Since the right-hand side of (5) is continuous in T, there always exists a $T<\infty$ such that

$$\alpha\left(\frac{(1-\alpha)\left[(1-q)\,u+q\,(v-w)\right]}{c-\alpha\,(v-w)}-1\right)\geq\frac{1-\delta}{\delta\,(1-\delta^{T})}.$$

The proposition then follows. ∎

Proof of lemma 1.

Differentiating (3) with respect to α and simplifying yields

$$\operatorname{sign}\left(\frac{dV^{+}}{d\alpha}\right)=-\operatorname{sign}\left(q\delta\left(1-\delta^{T}\right)(-c-u-w+v)+\left(1-\delta^{T+1}\right)u\right).$$

Consider the argument of the sign function, $q\delta\left(1-\delta^{T}\right)(-c-u-w+v)+\left(1-\delta^{T+1}\right)u$, as a function of T. It is easy to check that its first-order derivative is proportional to $\delta^{T+1}\ln\delta$, which implies that the argument itself is monotonic in T. When $T=0$, it equals to $(1-\delta)\,u \geq 0$. When $T=\infty$, it equals to $q\delta(-c-u-w+v)+u$ which is strictly positive because

$$\begin{aligned} q\delta(-c-u-w+v)+u &= q\delta\,(v-w-c)+(1-q\delta)\,u \\ &\geq q\delta\,(v-w-c)+(1-\alpha)\,(1-q\delta)\,u > 0. \end{aligned}$$

The last inequality follows from Assumption 1. Therefore, the argument of the sign function is always positive for $T \in [0,\infty]$ and

$$\frac{dV^+}{d\alpha} < 0.$$

Since α is a decreasing function of N then

$$\frac{dV^+}{dN} > 0.$$

■

Proof of Proposition 2.

We use the first-order condition with respect to α to find critical points of the left-hand side of (5). There are two critical points:

$$\alpha_1 = \frac{1}{v-w}\left[c - \frac{\sqrt{c\,(1-q)\,(c-v+w)\,(u-v+w)\,((1-q)\,u+q\,(v-w))}}{(1-q)\,(u-v+w)}\right]$$

and

$$\alpha_2 = \frac{1}{v-w}\left[c + \frac{\sqrt{c\,(1-q)\,(c-v+w)\,(u-v+w)\,((1-q)\,u+q\,(v-w))}}{(1-q)\,(u-v+w)}\right].$$

α_2 is strictly larger than 1 and is therefore ruled out since $\alpha \in [0,1]$. α_1 is always smaller than 1. The second-order condition with respect to α is

$$\frac{-2c\,(c-v+w)\,[(1-q)\,u+q\,(v-w)]}{(c-\alpha\,(v-w))^3} < 0.$$

Therefore, the left-hand side of (5) is a strictly concave function of $\alpha \in [0,1]$. α_1 is smaller than or equal to 0 if and only if $q\,(v-w)+(1-q)\,u \leq c$. In this case, the left-hand side of (5) is nonpositive for $\alpha \in [0,1]$ and therefore cooperation cannot be sustained for any group size N. α_1 is larger than 0 if and only if $q\,(v-w)+$

$(1-q)\,u > c$. Then the left-hand side of (5) has a single peak at $(0,1)$ and is therefore first increasing and then decreasing in α. Since α is a strictly decreasing function of N then the left-hand side of (5) is also first increasing and then decreasing in N.

However, strictly speaking, α cannot exceed $1-q$ since at least two members are required to form a group. Thus, in order to prove that the left-hand side of (5) has an inverted-U shape one has to show that $\alpha_1 \in (0, 1-q)$. Then condition

$$2c > v - w - (1-q)\,(1-2q)\,(u-v+w) + \sqrt{((1-q)\,u + q\,(v-w))\,(u - (u-v+w)\,(5q-8q^2+4q^3))}$$

ensures that $\alpha_1 < 1-q$. However, if the above condition does not hold then $\alpha_1 \geq 1-q$ and thus the left-hand side of (5) is an increasing function of $\alpha \in (0, 1-q)$ and a decreasing function of $N \geq 2$. ■

Proof of Proposition 3.

Substituting $v = w$, α and β into the non-deviation condition (10) yields

$$(1-q)^{N-1}\,\frac{Nu(1-q)\big(1-(1-q)^{N-1}\big)-c\big(1-(1-q)^{N}+N(1-q)\big)}{cN} \geq \frac{1-\delta}{\delta(1-\delta^{T})}. \tag{16}$$

Consider the left-hand side of (16). Its first-order derivative is given by

$$\frac{(1-q)^{N-1}\Big(c\big(1-(1-q)^{N}+N\ln(1-q)\big[-N(1-q)+2(1-q)^{N}-1\big]\big)+uN^{2}\ln(1-q)\big[1-q-2(1-q)^{N}\big]\Big)}{cN^{2}}. \tag{17}$$

It is easy to check that the sign of (17) is equal to the sign of

$$\frac{c}{u} + \widehat{f}\,(N,q)\,,$$

where

$$\widehat{f}\,(N,q) \equiv \frac{N^{2}\ln(1-q)\big[1-q-2(1-q)^{N}\big]}{1-(1-q)^{N}+N\ln(1-q)\big[-N(1-q)+2(1-q)^{N}-1\big]}.$$

We next compare $\frac{c}{u}$ with $-\widehat{f}\,(N,q)$ to determine the sign of (17). For any $q \in (0,1)$, $-\widehat{f}\,(N,q)$ is an increasing function of $N \geq 2$. It takes its minimum of $\widehat{f}\,(q) \equiv \widehat{f}\,(2,q) = \frac{4(1-q)(1-2q)\ln(1-q)}{(2-q)q-2(1+2q(1-q))\ln(1-q)}$ when $N=2$ and approaches 1 from below when N goes to infinity. Consider the following three cases.

1. $\frac{c}{u} \geq 1$.

In this case, $\frac{c}{u}+\widehat{f}(N,q)>0$ and so the sign of (17) is strictly positive for all $N\geq 2$. So, the left-hand side of (16) is an increasing function of $N\geq 2$ bounded between $-\frac{1}{2}(1-q)\left(2-q^2-2(1-q)q\frac{u}{c}\right)<0$ and 0, and so nonpositive for all $N\geq 2$. Thus, cooperation cannot be sustained for any group size.

2. $\max\left[\widehat{f}(q),0\right]<\frac{c}{u}<1$, which amounts to $q\leq\frac{1}{2}$ and $c<u$, or $q>\frac{1}{2}$ and $\widehat{f}(q)\,u<c<u$.

   In this case, there exists a unique $\overline{N}>2$ such that $\frac{c}{u}+\widehat{f}\left(\overline{N},q\right)=0$ and $\frac{c}{u}+\widehat{f}(N,q)\gtreqless 0$ for $N\lesseqgtr\overline{N}$. Thus, the left-hand side of (16) has a single peak at $\overline{N}>2$ and approaches zero from above when N goes to infinity. It has therefore an inverted-U shape at $N\geq 2$.

3. $0<\frac{c}{u}\leq\max\left[\widehat{f}(q),0\right]$, which amounts to $q>\frac{1}{2}$ and $c\leq\widehat{f}(q)\,u$.

   In this case, $\frac{c}{u}+\widehat{f}(N,q)<0$ and so the sign of (17) is strictly negative for all $N\geq 2$. The left-hand side of (16) is a decreasing function of $N\geq 2$ bounded between $-\frac{1}{2}(1-q)\left(2-q^2-2(1-q)q\frac{u}{c}\right)>0$ and 0, and so positive for all $N\geq 2$.

To sum up, the left-hand side of (16) has an inverted-U shape at $N\geq 2$ either when $q\leq\frac{1}{2}$ and $c<u$ or when $q>\frac{1}{2}$ and $\widehat{f}(q)\,u<c<u$. It decreases in $N\geq 2$ when $q>\frac{1}{2}$ and $c\leq\widehat{f}(q)\,u$. ■

Proof of Proposition 4.

Some $\eta(k)$ have to be positive for the IC constraint to hold. Otherwise, v^u will be strictly higher than v^a and therefore the IC constraint will be violated. Since some $\eta(k)$ are positive, $\mathcal{L}$ is well defined.

To find the highest $\mathcal{L}$, we need to find a public outcome k which is most likely to occur off the equilibrium path relative to its likelihood on the equilibrium path. For each public outcome, the power of test, $\mathcal{L}(k)$, is defined as the likelihood ratio of triggering punishment after public outcome k has been observed. Denote by $\theta^d(k)$ the probability of triggering punishment after observing k off the equilibrium path and by $\theta^c(k)$ the probability of triggering punishment after observing k on the equilibrium path. Then

$$\mathcal{L}(k) = \frac{\theta^d(k)}{\theta^c(k)} = \frac{q\cdot\binom{N-1}{k}q^k(1-q)^{N-1-k}+(1-q)\cdot\binom{N-1}{k}q^k(1-q)^{N-1-k}}{q\cdot\binom{N-1}{k-1}q^{k-1}(1-q)^{N-k}+(1-q)\cdot\binom{N-1}{k}q^k(1-q)^{N-1-k}} = \frac{N-k}{N}\frac{1}{1-q}.$$

It is easy to check that $\{\mathcal{L}(k)\}_{k=0}^{N}$ is a sequence strictly decreasing in k. Note that

$$\mathcal{L} = \frac{\sum_{k=0}^{N}\theta^d(k)\,\eta(k)}{\sum_{k=0}^{N}\theta^c(k)\,\eta(k)}.$$

Moreover,

$$\frac{\sum_{j=0}^{k-1}\theta^d(j)+\theta^d(k)\,\eta(k)}{\sum_{j=0}^{k-1}\theta^c(j)+\theta^c(k)\,\eta(k)} > \frac{\sum_{j=0}^{k}\theta^d(j)+\theta^d(k+1)\,\eta(k+1)}{\sum_{j=0}^{k}\theta^c(j)+\theta^c(k+1)\,\eta(k+1)} \tag{18}$$

for $k=0,...,N-1$. To see this formally, compare the left-hand side of the above inequality with the right-hand side. This is equivalent to comparing

$$\begin{aligned}&(1-\eta(k))\sum_{j=0}^{k-1}\left(\theta^c(k)\,\theta^d(j)-\theta^d(k)\,\theta^c(j)\right)+\\&\eta(k+1)\sum_{j=0}^{k-1}\left(\theta^c(k+1)\,\theta^d(j)-\theta^d(k+1)\,\theta^c(j)\right)+\\&\eta(k)\,\eta(k+1)\left[\theta^d(k)\,\theta^c(k+1)-\theta^c(k)\,\theta^d(k+1)\right]\end{aligned} \tag{19}$$

with zero. The first term equals zero for $k=0$ and is strictly positive for $k=1,...,N-1$ since $\frac{\theta^d(j)}{\theta^c(j)} > \frac{\theta^d(k)}{\theta^c(k)}$ with $j=0,...,k-1$. The second term is equal to zero for $k=0$ and is strictly positive for $k=1,...,N-1$ since $\frac{\theta^d(j)}{\theta^c(j)} > \frac{\theta^d(k+1)}{\theta^c(k+1)}$ with $j=0,...,k-1$. Finally, the last term is always positive since $\frac{\theta^d(k)}{\theta^c(k)} > \frac{\theta^d(k+1)}{\theta^c(k+1)}$. It follows that (19) is strictly positive and therefore (18) holds. (18) implies that

$$\frac{\theta^d(0)}{\theta^c(0)} > \frac{\theta^d(0)+\theta^d(1)\eta(1)}{\theta^c(0)+\theta^c(1)\eta(1)} > \frac{\theta^d(0)+\theta^d(1)+\theta^d(2)\eta(2)}{\theta^c(0)+\theta^c(1)+\theta^c(2)\eta(2)} > ... > \frac{\theta^d(0)+\theta^d(1)+...+\theta^d(N)\eta(N)}{\theta^c(0)+\theta^c(1)+...+\theta^c(N)\eta(N)}. \tag{20}$$

Thus, $\mathcal{L}$ reaches its maximum value of $\frac{\theta^d(0)}{\theta^c(0)}$ when $\eta(0) > 0$ and $\eta(1) = ... = \eta(N) = 0$. The next step is to check the IC constraint for this punishment scheme. If the IC constraint is satisfied then this is the optimal punishment scheme and $\eta(0)$ is characterized by the binding IC constraint. If the IC constraint does not hold then one has to consider the second largest value of $\mathcal{L}$ from ranking (20), $\frac{\theta^d(0)+\theta^d(1)\eta(1)}{\theta^c(0)+\theta^c(1)\eta(1)}$, which is reached when $\eta(0) = 1$, $\eta(1) > 0$ and $\eta(2) = ... = \eta(N) = 0$. If the IC constraint is satisfied for $\eta(0) = 1$, $\eta(1) > 0$ and $\eta(2) = ... = \eta(N) = 0$, then this is the optimal punishment scheme and $\eta(1)$ is characterized by the binding IC constraint. Otherwise, one has to consider the third largest value of $\mathcal{L}$ from ranking (20), check the IC constraint for the punishment scheme corresponding to this value of $\mathcal{L}$, and continue this process until the IC constraint is satisfied. ■

Proof of Proposition 5.

Denote $R(N,q) \equiv \left(1 + \left(\frac{q(N-1)}{c-1} - 1\right)(1-q)^{N-1}\right)^{-1}$. Suppose that $\delta > R(N_1, q)$. Then $\tilde{k} = 0$ characterizes the optimal punishment scheme for some $N_1 > 1 + \frac{c-1}{q}$. Consider an increase in the group size from N_1 to $N_1 + 1$. $\tilde{k} = 0$ will still characterize the optimal punishment scheme for $N_1 + 1$ if $R(N_1 + 1, q) < R(N_1, q)$, which amounts to

$$N_1 < \frac{c}{q}.$$

Assume that there is N_1 such that $1 + \frac{c-1}{q} < N_1 < \frac{c}{q}$ and consider the impact of a group-size increase from N_1 to $N_1 + 1$ on $\eta(0)$. It is easy to show that for $1 + \frac{c-1}{q} < N_1 < \frac{c}{q}$, $\eta(0)$ decreases with a group-size increase from N_1 to $N_1 + 1$, which means that the group-size effect is positive in this case. Otherwise, the group-size effect is negative. We next check when there exists $N_1 \geq 2$ such that $1 + \frac{c-1}{q} < N_1 < \frac{c}{q}$. Consider the following cases.

1. If $\frac{c}{q} \leq 2$ then there is no $N_1 \geq 2$ such that $1 + \frac{c-1}{q} < N_1 < \frac{c}{q}$ and therefore the group-size effect is always negative.

2. If $1 + \frac{c-1}{q} < 2 < \frac{c}{q}$ then there is $N_1 \in \left[2, \frac{c}{q}\right)$ such that $1 + \frac{c-1}{q} < N_1 < \frac{c}{q}$ and therefore the group-size effect is positive. Still, for $N_1 \geq \frac{c}{q}$ the group-size effect is negative.

3. If $2 \leq 1 + \frac{c-1}{q}$ and there is at least one integer between $1 + \frac{c-1}{q}$ and $\frac{c}{q}$ then the group-size effect is positive for this integer and negative for all $N_1 \geq \frac{c}{q}$.

4. If $2 \leq 1 + \frac{c-1}{q}$ but there is no integer between $1 + \frac{c-1}{q}$ and $\frac{c}{q}$ then the group-size effect is always negative.

If the distance between $1+\frac{c-1}{q}$ and $\frac{c}{q}$ is greater than 1 then there will be always at least one integer between them. It follows therefore that the group-size effect is positive for relatively small N_1 and negative for relatively large N_1 if and only if either $1+\frac{c-1}{q} < 2 < \frac{c}{q}$ or $2 \leq 1+\frac{c-1}{q}$ and $\frac{c}{q} - \left(1+\frac{c-1}{q}\right) > 1$. These conditions amount to $0 < q < \frac{1}{2}$ and $c > 1$ or $\frac{1}{2} \leq q < 1$ and $2q < c < 1+q$. ■

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