University of Mannheim / Department of Economics

Working Paper Series

Why prediction markets work: The role of information acquisition and endogenous weighting

Christoph Siemroth

Working Paper 14-29

December 2014

Why prediction markets work:
The role of information acquisition and endogenous weighting*

Christoph Siemroth†
University of Mannheim

First version: January 10, 2014
This version: December 3, 2014

Abstract

In prediction markets, investors trade assets whose values are contingent on the occurrence of future events, like election outcomes. Prediction market prices have been shown to be consistently accurate forecasts of these outcomes, but we don't know why. I formally illustrate an information acquisition explanation. Traders with more wealth to invest have stronger incentives to acquire information about the outcome, thus tend to have better forecasts. Moreover, their trades have larger weight in the market. The interaction implies that a few well-situated traders can move the asset price toward the true value. One implication for institutions aggregating information is to put more weight on votes of agents with larger stakes, which improves on equal weighting, unless prior distribution accuracy and stakes are negatively related.

Keywords: Information Acquisition, Information Aggregation, Forecasting, Futures Markets, Prediction Markets
JEL Classification: D83, D84, G13

*I am grateful to Pierre Boyer, Antonio Cabrales, Jean-Edouard Colliard, Martin Cripps, Hans Peter Grüner, Felix Jarman, Ernst Maug, Marco Ottaviani, Lionel Page, Peter Norman Sørensen, Konrad Stahl, Philipp Zahn, and seminar participants in Mannheim as well as participants at the 7th RGS conference Dortmund, the 17th SGF conference Zurich, and EEA-ESEM Toulouse 2014 for discussion and very helpful comments. This research was supported by the German Research Foundation (DFG) via SFB 884.

†Department of Economics, L7, 3-5, 68131 Mannheim, Germany. E-Mail: christoph.siemroth@gess.uni-mannheim.de.

1 Introduction

In a 2003 forecasting tournament, participants predicted outcomes of football games throughout a season to win prizes. Probability forecasts were rated with a quadratic scoring rule, so only participants with consistently accurate forecasts would be in the top ranks. Two mock entrants simply used the prices from two different prediction markets as their forecasts, and placed 6th and 8th out of almost 2,000 participants (Servan-Schreiber et al., 2004). More generally, prediction markets have been shown to provide better forecasts than polls in political elections (e.g., Forsythe et al., 1992; Berg et al., 2008), expert forecasts in sports (Spann and Skiera, 2009), or sales forecasts in business (Plott and Chen, 2002).

One kind of asset traded in these markets is called winner-take-all (WTA) contract. It pays out 1 if and only if a pre-specified condition is fulfilled, otherwise it pays out 0. For example, the IEM prediction market for the 2012 US Presidential election traded a Democratic and a Republican contract, which would pay out $1 if and only if the respective candidate obtained the majority of popular votes cast for the two major parties. Consequently, the price of a WTA contract may be interpreted as the market probability estimate that the respective candidate wins the election. With similar contracts, market-based predictions can be obtained for virtually all areas beyond politics.

Why do these markets predict so accurately? There are no satisfying explanations thus far. As Berg and Rietz (2006) state, "exactly how prediction markets become efficient is something of a mystery." The main goal of this paper is to provide and formally illustrate a theory. In what I shall call information acquisition explanation, traders have stronger incentives to acquire information about the unknown outcome the larger their endowment. Consequently, high endowment traders are better informed. Moreover, high endowment traders have larger impact on the market price, because they can buy more assets. This interaction implies that few, but well-situated traders can move the market price—interpreted as prediction—in the right direction, thereby explaining the observed accuracy. Unlike many financial market models, the explanation does not rely on the presence of insiders nor the ability of traders to infer information from asset prices. Even markets with traders who have systematically biased opinions about the outcomes can produce accurate forecasts, because of effective incentives for information acquisition and weighting by investment volume.

The model extends the competitive equilibrium notion from Manski (2006) with the possibility to acquire information. In the model, traders start out with heterogeneous beliefs about the outcome of the election. Based on this prior belief, endowment and asset prices, they decide whether to acquire information, whose accuracy depends on their information acquisition effort. Consequently, the share of informed traders and noise traders (driven by opinion) in the market is endogenously determined, which explains partially where beliefs originate and when beliefs (and market forecasts) tend to be accurate.

I establish that traders with prior beliefs close to the 'market estimate' (price), or with high endowment, have the strongest incentive to acquire information. The interpretation is that traders with extreme opinions about the outcome do not expect to be swayed by evidence, and hence do not acquire it, while traders with opinions close to the market price acquire information, because it might change their investment decision. Moreover, high endowment traders have stronger incentives to acquire information, because more is at stake for them. Comparative statics show that the forecast error of the market price is usually reduced in response to an endowment increase of all traders, because information acquisition is supported. It also shows that a shift of prior beliefs toward the true outcome usually improves forecasts, but may in rare cases *increase* the forecast error.

One lesson for institutional design is that giving more weight to votes or investments of high endowment agents might improve information aggregation. Unless accuracy of prior beliefs and endowment are negatively related, high endowment agents tend to have better information, which can be exploited via weighting. An empirically testable implication is that forecasts should be better with weighting by stake rather than equal weighting. Moreover, forecasts should be better if investment per trader is larger in the market.

WTA prediction markets are analytically identical to parimutuel betting without fees. Prediction market prices are translated into odds like probabilities. For example, if the price of the Democratic contract is p, and the complement is priced at $1 - p$, then the odds of a Democratic victory are $(1 - p)/p$ in the corresponding betting market. Thus, the results presented here also apply to betting markets.

While the idea that prediction (or stock) markets provide incentives to search for information is not new (e.g., Servan-Schreiber et al., 2004; Wolfers and Zitzewitz, 2004; Arrow et al., 2008), this is the first paper to formalize it in order to explain prediction market accuracy, and demonstrate its interaction with heterogeneous priors and the endogenous weighting implied by the market. Existing models are designed to address different questions (see next section). They typically assume an arbitrary belief distribution or give traders informative signals by default, so that forecast accuracy is a direct consequence of the accuracy of these primitives. In contrast, quality of information is endogenous in my model, so it is more suitable to explain the accuracy or inaccuracy of prediction markets. In a numerical example, the model with information acquisition has a 10 percentage points smaller forecast error on average than the model without information acquisition (Manski, 2006), holding all exogenous parameters constant. Moreover, the model motivates a different view on the interpretation of prediction market prices (e.g., Manski, 2006, Wolfers and Zitzewitz, 2006). Instead of comparing prices to statistics of the belief distribution, the key question is whether beliefs are driven by information rather than opinion (section 3).

1.1 Related literature

I confine attention to the relevant theoretical contributions in this section.[1] An informal explanation of prediction market accuracy was put forth by Forsythe et al. (1992). In what they dub marginal trader hypothesis (MTH), they argue that "prices are determined by the marginal trader" and that marginal traders are "free of judgment bias" (i.e., have accurate beliefs about the outcome). According to their definition (p. 1158 and fn. 21), a marginal trader is a trader who places limit orders within 2 cents of the current price. However, they are vague on how marginal traders "set prices." Why should they have the power to set prices, while biased traders—who may be as convinced of the correctness of their beliefs as unbiased traders—do not? Indeed, the existence of a group with perfect forecast is not sufficient for an accurate market forecast if wealth is bounded. A small subset of informed traders cannot always outbid hordes of biased traders to keep the price at the fair value. Serrano-Padial (2012), for example, demonstrates this point formally. Nor is their presence necessary—possible biases of Republicans and Democrats may cancel out to accurately predict the election vote share. Consequently, the MTH does not explain the consistently good performance of prediction markets.

My information acquisition explanation has similarities to the MTH, in that both imply that a small group of well informed traders can influence the market price to the better. However, the information acquisition explanation differs in that informed traders need not be free from judgment biases, need not have the power to set prices, and need not have beliefs equal to what is implied by the market price. Interestingly, Forsythe et al. (1992) find that marginal traders (with good forecasts about the outcome) have larger investments than the rest, which is captures the information acquisition explanation well.

Instead of explaining predictive accuracy, most of the theoretical literature on betting markets tries to explain the favorite long-shot bias, i.e., that odds often underestimate the favorite's chances, while the long-shot's chances are overestimated. Explanations are provided for example by Ali (1977), Ottaviani and Sørensen (2009), Page and Clemen (2013), and Ottaviani and Sørensen (forthcoming). For a more complete overview of explanations for the favorite long-shot bias, see the references in Ottaviani and Sørensen (2010). In all of these models, the origin of beliefs is either unmodeled or informative signals are obtained by default with exogenous precision.

To my knowledge, information acquisition has been incorporated only once in the context of prediction markets. Hanson and Oprea (2009) investigate whether a manipulator can drive the price away from the fundamental value of the asset. Since his presence as well as the strength of the manipulation preference is common knowledge, informed traders

[1] A good introduction to prediction markets with examples is provided by Wolfers and Zitzewitz (2004). An overview of the large betting literature in economics can be found in Sauer (1998), and Thaler and Ziemba (1988) provide an introduction to empirical anomalies in betting markets. Tziralis and Tatsiopoulos (2007) give an extensive overview of the prediction market literature with categorization into subfields.

may react to the manipulation attempt by acquiring more precise signals, thus raising prediction market accuracy on average. In contrast to my model, they use the quantal response equilibrium concept, all distributions are assumed to be common knowledge, all random variables are normally distributed, and there are no budget constraints.

Among the first to consider information acquisition in financial economics more generally, Grossman and Stiglitz (1980) show that a fully revealing rational expectations equilibrium (REE) with costly information acquisition does not exist. The following literature (e.g., Verrecchia, 1982; Barlevy and Veronesi, 2000; Peress, 2004) focuses on noisy rational expectations equilibria, where the price is affected by noise and only partially revealing to retain the incentive for information acquisition. Recently, Van Nieuwerburgh and Veldkamp (2010) investigated the information acquisition problem in a portfolio choice problem with information capacity constraints.

The main difference of my equilibrium concept (and much of the prediction/betting market literature, e.g., Ali, 1977; Manski, 2006) to REE is that traders do not infer information from prices (see definition 1 in the next section). There are two possible justifications. First, traders do not know "how equilibrium prices are related to initial information" (Radner, 1979) as in a REE. In actual prediction markets, traders do not know beliefs or endowments of other traders, and do not know if trades are motivated by hedging or manipulation motives (Rothschild and Sethi, 2013), i.e., they do not have the necessary 'models' to extract information from prices. Moreover, many types of prediction markets are seldomly run (e.g., elections) or even unique (e.g., Saddam Hussein being ousted by June 2003, Wolfers and Zitzewitz, 2004), so possibilities for learning rational expectations may be limited.

Second, the behavioral economics literature has demonstrated repeatedly that subjects substantially overweight their own opinions and signals. For example, Weizsäcker (2010) shows in a meta study of information cascade experiments that subjects stick to their own information more than half of the time even though it would be optimal to follow the contradicting collective information of others. Also demonstrating the overconfidence in their own beliefs/information, 89% of traders responded they believed to be better informed than others in a survey accompanying the IEM 2004 presidential election market (Berg and Rietz, 2006). Hence, even if traders had the necessary knowledge to infer information from prices, they would not fully act on it. My model takes this view to the extreme in that no inference from prices occurs. Thus, traders can be viewed as myopic in the sense that they ignore the informativeness of prices, but otherwise act rationally given this ignorance. The models of Hong and Stein (1999) and Eyster et al. (2013), for example, also include traders who ignore the informational content of prices for behavioral reasons.

2 A model of costly information acquisition

An unknown state of the world (briefly 'outcome') from the set $\{A, B\}$, $B = A^c$, is exogenously given, and will be publicly revealed in the future. For example, suppose a presidential candidate (incumbent) faces a challenger in an upcoming election. Then A represents victory by the incumbent in the election, whereas B means the challenger is victorious. Formally, $\theta = 1$ iff A and $\theta = 0$ iff B, which is the parameter to be predicted.

The market is populated by a continuum of risk-neutral traders. Traders may be heterogeneous in their endowment $\omega_i \in (0, \overline{\omega}]$, $\overline{\omega} < \infty$, which is distributed with cdf $W(\omega_i)$. Moreover, each trader i is characterized by prior belief $q_i \in [0, 1]$, drawn from a continuous and strictly increasing cdf $Q(q_i|\theta, \omega_i)$, which is i's subjective estimate of $\Pr(\theta = 1)$. Since I am interested in the information content in the market, I assume prior heterogeneity is due to differences in opinion, not due to prior differences in information. This is a deviation from the common prior assumption imposed in the majority of the literature, and implies that traders may disagree about prospects. The specification of prior belief distribution Q allows for a dependence with endowment ω_i.

Traders do not receive an informative signal about θ by default. Instead, they may acquire a private binary signal, which is costly in terms of effort. The precision of the signal is an increasing function of effort $e_i \geq 0$, $\nu(e_i) = \Pr(s_i = 1|\theta = 1, e_i) = \Pr(s_i = 0|\theta = 0, e_i)$. The interpretation is that a trader can run Internet searches, read magazines, or talk to experts (signal), which influences his beliefs (posterior). But the effort cost may be too large, so a trader may rather rely on his opinion (prior) to make the investment decision. Effort costs are *not* paid out of the endowment, and enter linearly in the utility function.

Two state-contingent futures contracts—also called winner-take-all contracts—are traded in the prediction market. One A-contract pays 1 iff $\theta = 1$ to the holder, and 0 otherwise. Conversely, one B-contract pays 1 iff $\theta = 0$. The contracts are issued by the market maker. The prediction market is thus a complete one-period Arrow-Debreu security market, like the IEM prediction market described in the introduction. Let the price of the A-contract be p, and the price of the B-contract $1 - p$, to rule out arbitrage opportunities. The profit per A-contract held if A occurs is the difference of value 1 and the price p; the loss per A-contract if B occurs is p, the price paid.

The timing of trader decisions is shown in Figure 1. Each trader first decides on information acquisition effort e_i, or equivalently signal precision $\nu_i := \nu(e_i)$, for a given price p ($t = 0$). Then he receives a signal with precision ν_i, and computes posterior $\pi_i := \Pr(\theta = 1|s_i)$ using Bayes' rule. The posterior is equal to the prior if zero effort, i.e., precision $\nu_i = 1/2$, is chosen. Based on posterior π_i, endowment ω_i, and price p, the trader decides which option to invest in by specifying investment volume $a_i(\pi_i, \omega_i, p), b_i(\pi_i, \omega_i, p)$ ($t = 1$). Finally, the outcome θ is revealed and the assets pay out ($t = 2$).

The equilibrium price p^* can be viewed as the market's probability estimate that A

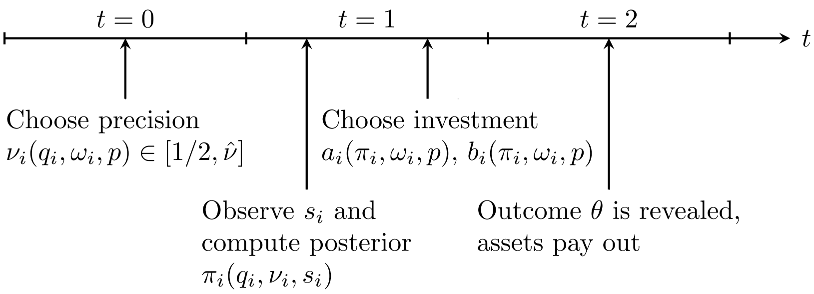

Figure 1: Timing of decisions for trader i and payout.

occurs. The more traders believe that A is going to occur, the more invest in A, thus raising the price (forecast) p^*. The forecast error of the market is $|p^* - \theta|$, so the best forecast is for the market price to equal the fundamental value of the asset (i.e., 1 or 0). In competitive equilibrium, the price equates the aggregate demand for A and B-contracts, so that money is redistributed from losers to winners, and the market operates at zero profit.

The competitive equilibrium concept used here requires that price p^* induces information acquisition, which leads to beliefs and asset demands clearing the market for that same price.

Definition 1. *A competitive equilibrium with endogenous information acquisition requires*

1. *a precision level function $\nu_i(q_i, \omega_i, p)$ for all i, which maximizes expected utility at $t = 0$ anticipating optimal behavior at $t = 1$,*

2. *posterior beliefs $\pi_i(q_i, \nu_i, s_i)$ for all i, computed via Bayes' rule,*

3. *investment functions $a_i(\pi_i, \omega_i, p)$, $b_i(\pi_i, \omega_i, p)$ for all i, which maximize expected utility subject to $a_i + b_i \leq \omega_i$ at $t = 1$, and*

4. *an equilibrium price p^*, which induces information acquisition $\nu_i(q_i, \omega_i, p^*)$ at $t = 0$, leading to beliefs $\pi_i(q_i, \nu_i, s_i)$ and investments $a_i(\pi_i, \omega_i, p^*), b_i(\pi_i, \omega_i, p^*)$ clearing the asset market at $t = 1$, i.e., for $p^* \in (0, 1)$,*

$$\int_0^{\overline{\omega}} \int_0^1 \frac{a_i(\pi_i, \omega_i, p^*)}{p^*} dQ(q_i|\theta, \omega_i) dW(\omega_i) = \int_0^{\overline{\omega}} \int_0^1 \frac{b_i(\pi_i, \omega_i, p^*)}{1 - p^*} dQ(q_i|\theta, \omega_i) dW(\omega_i).$$

The difference to an Arrow-Debreu equilibrium is that the equilibrium price must simultaneously induce information acquisition levels ν_i and clear the market for the resulting investment functions a_i, b_i. Consequently, the equilibrium concept yields a single equilibrium price, even though in principle the sequential decisions of the traders (Figure 1) allow for different prices at $t = 0$ and $t = 1$. To motivate this equilibrium notion, suppose a Walrasian auctioneer announces an initial price p_0 at $t = 0$. Traders make their information acquisition decision, update their beliefs and trade at $t = 1$, which leads to market clearing price $p_1 \neq p_0$ if p_0 does not fulfill the above definition. Thus, if we were to repeat the procedure with p_1 as initial price, traders might make different information acquisition decisions and ultimately different investment decisions, possibly leading to yet another price. If this tâtonnement procedure stops for some $p_0 = p_1$, then $p_0 = p^*$ as defined above.

Trader expectations about the eventual market price p^* at $t = 1$ are a necessary consequence of the static trading model, because traders need a price to evaluate the value of information when deciding whether to acquire information, but a price only forms later ($t = 2$). According to the equilibrium definition, traders are able to predict the eventual market clearing price p^* at $t = 1$, but infer no information from it as they would in a rational expectations equilibrium (REE).[2] I show in the appendix that a simple sequential trading process—where traders invest sequentially (without price expectations) on the basis of posted prices, which adjust in response to investments—converges to the equilibrium price of the static model. Thus, the assumption that traders can accurately forecast the market clearing price is not crucial for the results—it is an artifact of the static model.

2.1 Investment and information acquisition decision

In this section, I determine the optimal individual information acquisition and investment decisions for a given asset price p. In the next section, individual decisions will be aggregated to determine the equilibrium price p^*. Going backwards on the time line, taking posterior π_i and price p as given, the investment problem of the risk neutral trader at $t = 1$ is

$$\max_{a_i, b_i \geq 0} \pi_i[(1-p)a_i/p - b_i] + (1-\pi_i)[pb_i/(1-p) - a_i] \text{ s.t. } a_i + b_i \leq \omega_i.$$

That is, the trader believes A occurs with posterior probability π_i, yielding a profit of $(1-p)a_i/p$ on the a_i investment, and a loss of all b_i investment. Payoffs for B follow similarly. The linear utility function yields a corner solution, which is $a_i = \omega_i,\ b_i = 0$ if $\pi_i > p$ and $a_i = 0,\ b_i = \omega_i$ if $\pi_i < p$. Henceforth, I will use the short-hand $\alpha_i := \omega_i/p$ and $\beta_i := \omega_i/(1-p)$ to denote the amount of A or B-contracts bought, respectively.

Anticipating these investment decisions, the trader decides how much costly effort to spend, which determines the precision of the signal. Note that the effort choice at $t = 0$ has to induce investment behavior which depends on the signal at $t = 1$ ('discriminating signal precision'), otherwise the effort cost is incurred for no benefit. For example, if the resulting posterior is $\pi_i(s_i = 0, q_i, e_i) < \pi_i(s_i = 1, q_i, e_i) < p$, then exerting effort $e_i > 0$ would not make a difference in the investment decision—i invests in B for either realization of the signal—and therefore cannot be optimal. The minimum discriminating effort level is

$$\tilde{e}_i = \min_e \{e : \pi_i(s_i = 1, q_i, e) \geq p \geq \pi_i(s_i = 0, q_i, e)\}.$$

I omit explicit expressions for the minimum discriminating precision level $\tilde{\nu}_i$, because Proposition 1 shows that the 'discrimination constraint' is never binding.

[2] For a justification of this ignorance, see the related literature section.

The expected utility from choosing a discriminating signal precision *before* observing the signal ($t = 0$) is

$$EU(e_i \geq \tilde{e}_i) = q_i\nu(e_i)(1-p)\alpha_i - q_i(1-\nu(e_i))\omega_i + (1-q_i)\nu(e_i)p\beta_i - (1-q_i)(1-\nu(e_i))\omega_i - e_i.$$

That is, from his prior perspective, trader i anticipates that he will invest in A iff $s_i = 1$ and in B iff $s_i = 0$, that the signal will be correct with probability $\nu(e_i)$, and wrong with probability $1 - \nu(e_i)$, and he weighs each case according to his prior belief q_i.

The expected utility of not acquiring information is

$$EU(e_i = 0) = \begin{cases} q_i(1-p)\alpha_i - (1-q_i)p\alpha_i = (q_i - p)\alpha_i & \text{if } q_i > p, \\ (1-q_i)p\beta_i - q_i(1-p)\beta_i = (p - q_i)\beta_i & \text{if } q_i < p. \end{cases} \tag{1}$$

A trader prefers information acquisition, i.e., a positive effort level, if and only if the benefits from the more informed investment decision at least equal the effort costs. Hence, positive effort, assuming $\nu_i := \nu(e_i) \geq \tilde{\nu}_i$, is incentive compatible iff for $q_i > p$

$$q_i\nu_i(1-p)\alpha_i - q_i(1-\nu_i)p\alpha_i + (1-q_i)\nu_i p\beta_i - (1-q_i)(1-p)(1-\nu_i)\beta_i - e_i \geq (q_i - p)\alpha_i. \tag{2}$$

Solving the first order condition ($\nu_{ee} < 0$) for the LHS, the unconstrained optimal positive effort level is

$$e_i^* = \nu_e^{-1}\left(\frac{1}{q_i\alpha_i + (1-q_i)\beta_i}\right) > 0,$$

where ν_e is the partial derivative with respect to effort. The optimal level exists under the Inada conditions $\nu_e(e) \to 0$ as $e \to \infty$ and $\nu_e(e) \to \infty$ as $e \to 0$.

For explicit solutions, I assume a specific form for the effort-precision function,

$$\nu(e) = \min\left\{\frac{1}{2}(\sqrt{e} + 1), \hat{\nu}\right\}, \ \hat{\nu} < 1,$$

so that $\nu(e = 0)$ is normalized to 1/2 (an uninformative signal). Because the square root function is unbounded, I impose an upper bound $\hat{\nu} < 1$, so that traders cannot learn the true state of the world perfectly by investing a lot of effort. From the first order condition, the optimal unconstrained precision is

$$\nu_i^* := \nu(e_i^*) = \frac{q_i\alpha_i + (1-q_i)\beta_i + 4}{8}.$$

The upper and lower bound of incentive compatible precisions $\underline{\nu}_i := \underline{\nu}(q_i, \omega_i, p)$ and $\overline{\nu}_i := \overline{\nu}(q_i, \omega_i, p)$ if $q_i > p$ and $\nu_i \geq \tilde{\nu}_i$ are the solutions to the quadratic equation (2),

$$(\underline{\nu}_i, \overline{\nu}_i) = \frac{q_i\alpha_i + (1-q_i)\beta_i + 4}{8} \pm \sqrt{\frac{(q_i\alpha_i + (1-q_i)\beta_i + 4)^2}{64} - 1/4(1 + q_i\alpha_i)}. \tag{3}$$

For $q_i < p$, the RHS (expected utility without information acquisition) of (2) changes, resulting in solutions

$$(\underline{\nu}_i, \overline{\nu}_i) = \frac{q_i\alpha_i + (1-q_i)\beta_i + 4}{8} \pm \sqrt{\frac{(q_i\alpha_i + (1-q_i)\beta_i + 4)^2}{64} - 1/4(1 + (1-q_i)\beta_i)}. \quad (4)$$

In short, i acquires information iff $[\underline{\nu}_i, \overline{\nu}_i] \cap [\tilde{\nu}_i, \hat{\nu}] \neq \varnothing$, and chooses ν_i^* if it is in the intersection. The following proposition characterizes the information acquisition decision.

Proposition 1.

i. Incentive compatible precision levels always discriminate, i.e., $\underline{\nu}(q_i, \omega_i, p) \in \mathbb{R} \implies \underline{\nu}(q_i, \omega_i, p) \geq \tilde{\nu}(q_i, p)$.

ii. The incentive compatibility constraint to acquire information becomes less stringent as $|q_i - p|$ *decreases. If* $q_i \leq p \leq 1/2$ *or* $1/2 \leq p \leq q_i$*, then* $\frac{\partial \nu_i}{\partial |q_i - p|} \leq 0$*. Moreover, there exists a positive probability mass of traders with prior* q_i *around* p *who acquire information.*

iii. For sufficiently large ω_i*, the incentive compatibility constraint becomes less stringent as* ω_i *increases. If, for some* $\omega_i > 0$*, informative signals are acquired, then increasing* ω_i *strictly increases signal precision until* $\nu_i = \hat{\nu}$*.*

Proof. See Appendix. □

Traders who have a prior belief closer to the price p are more willing to acquire information (*ii.*). To understand the intuition, note that traders compute the expected utility based on their prior belief when deciding about acquiring information. Whenever the prior deviates considerably from the price, the trader expects large gains $|q_i - p|$ per contract without information acquisition (see (1)). This can be interpreted as a trader having a strong opinion about the outcome, who does not expect to be swayed by evidence and hence does not acquire it. Conversely, the trader expects only small gains based on his prior if q_i is close to p, so his opinion about the outcome is not as strong, and he is willing to acquire information and possibly revise his beliefs if there is evidence contradicting his prior.

All else equal, larger endowment makes information acquisition more likely, or increases information seeking effort, if endowment is sufficiently large (*iii.*). The intuition is that while endowment (and thus potential gains and losses) scale up, the cost of acquiring information remains the same. Thus, due to higher stakes, traders want to be 'more certain' that their investment decision will be the right one. Peress (2004) obtains a similar comparative static for the same reason in his noisy rational expectations model.

These results are illustrated in Figure 2, in which the optimal precision level ν^*, the range of incentive compatible precision levels $[\underline{\nu}, \overline{\nu}]$ (conditional on being discriminating) and the minimum discriminating precision level $\tilde{\nu}$ for different values of ω, p, q_i are plotted. It illustrates how a larger endowment increases the range of priors where information acquisition

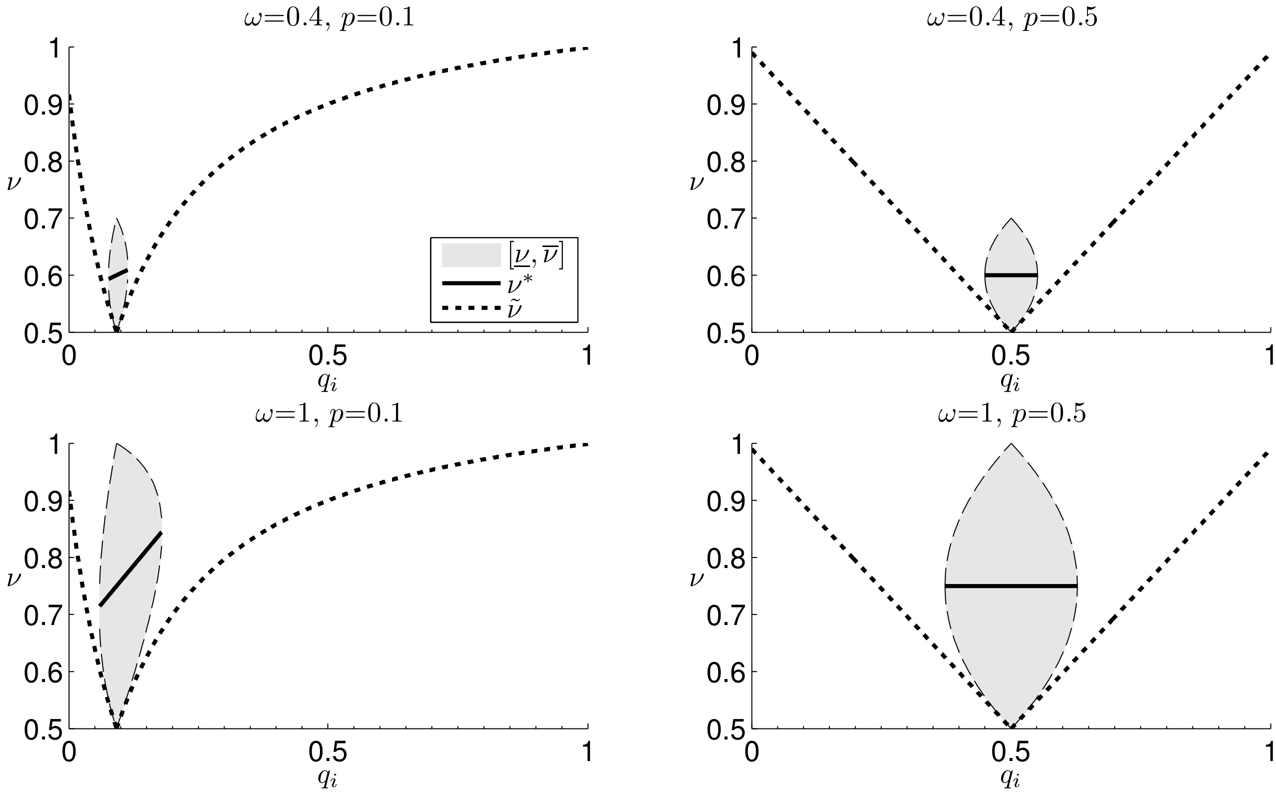

Figure 2: Information acquisition decision: optimal (solid line), incentive compatible (shaded area) and minimum discriminating (dotted line) precision levels, with varying endowment ω and price p, depending on prior belief q_i.

is incentive compatible, and increases the chosen precision, until ν^* reaches upper bound $\hat{\nu}$ (which is not included in the figure for visibility). The optimal precision ν^* is increasing in prior q_i for small p and decreasing for large p. The intuition is that A-contracts are cheap for small p, so investing in A if the state of the world is indeed A gives large profits (this is the equivalent of winning a long-shot bet). This is when the trader wants to make sure to make the right investment. Thus, if q_i increases, the trader believes this case to be more likely and increases the probability to make the right investment, i.e., increases ν^*.

Note that the motivation of the traders is not to make the best prediction possible. If that were the case, traders would also want to obtain information if the prior is far from the price. Instead, the motivation is to maximize utility, and that might favor opinion over independent information to save effort costs. Note also that traders tend to acquire less information the larger the coefficient of absolute risk aversion (Cabrales et al., 2014).

2.2 Competitive equilibrium

How does the possibility of information acquisition affect the forecast of the prediction market? The following example illustrates how information acquisition can improve the forecast compared to a model without, as for example used in Gjerstad (2005), Wolfers and Zitzewitz (2006), or Manski (2006).

Example. Suppose $\theta = 1$, each trader has endowment $\omega = 1$ and priors q_i are uniformly distributed, i.e., $q_i \sim U(0,1)$. Without information acquisition, the equilibrium price is $p_0^* = 1/2$, because at that price half of the traders are willing to invest in A and half in B, thus clearing the market. The equilibrium price divides traders in those who always (i.e., for each state of the world) invest in A ($q_i > p_0^*$), and in traders who always invest in B ($q_i < p_0^*$). I shall call these 'uninformed' traders, because their decision is based solely on their prior, and therefore independent of the true state of the world.

The previous section showed that information acquisition is incentive compatible only for traders with prior close to the price. When allowing information acquisition (keeping the price constant), traders with $q_i < p_0^* = 1/2$ close to $1/2$ turn from uninformed B-traders into informed traders acquiring information, and the share of those investing in the correct asset A improves from 0 to $\nu_i(q_i, \omega, p_0^*) > 1/2$. Conversely, traders with $q_i > p_0^*$ close to $1/2$ turn from uninformed A-traders into informed traders, and the share of those investing in A decreases from 1 to $\nu_i(q_i, \omega, p_0^*) > 1/2$. The mass of traders turned into informed traders is equal for both sides about $1/2$ if $p = 1/2$ (see Figure 2), so the mass of traders in the population willing to invest in the correct asset A at price $p = 1/2$ increases. Consequently, $p = 1/2$ is not an equilibrium price. Indeed, if aggregate demand is weakly decreasing in p, the equilibrium price p^* must be closer to the true value $\theta = 1$ to clear the market. $\square$

As the example illustrates, the possibility of information acquisition can sway traders with incorrect initial opinion to acquire information, revise their beliefs and invest in the correct outcome instead, thus improving the market forecast.

More formally, let $\nu(q_i, \omega_i, p)$ denote the precision level resulting from the endogenously chosen information acquisition effort. For readability, I will omit the conditioning set of cdf $Q(q_i|\theta, \omega_i)$ in the following. From Definition 1, the equilibrium price p^* is implicitly defined as the fixed point of ($\mathbf{1}\{.\}$ is the indicator function)

$$\int_0^{\overline{\omega}} \int_0^1 a_i(\pi_i, \omega_i, p^*)/p^* dQ(q_i) dW(\omega_i) = \int_0^{\overline{\omega}} \int_0^1 b_i(\pi_i, \omega_i, p^*)/(1-p^*) dQ(q_i) dW(\omega_i)$$

$$\iff \int_0^{\overline{\omega}} \omega_i \Bigg(\int_0^1 \mathbf{1}\{\nu(q_i, \omega_i, p^*) = 1/2\} \mathbf{1}\{q_i \geq p^*\} + \mathbf{1}\{\nu(q_i, \omega_i, p^*) > 1/2\}$$

$$[\theta \nu(q_i, \omega_i, p^*) + (1-\theta)(1 - \nu(q_i, \omega_i, p^*))] dQ(q_i) - p^* \Bigg) dW(\omega_i) = 0,$$

where $\mathbf{1}\{\nu(q_i, \omega_i, p^*) = 1/2\}\mathbf{1}\{q_i \geq p^*\}$ indicates uninformed A-traders, who do not acquire information and always invest in A, and the remaining term is the contribution of informed traders to A-investment, who invest according to their information. They invest in A only in $\nu(q_i, \omega_i, p^*)$ of the cases if the true state of the world is A, otherwise in $1 - \nu(q_i, \omega_i, p^*)$ of the cases. The equivalence is shown in the proof of Proposition 2 (*ii.*). Rewriting the share of uninformed A-investors in the population at p^*,

$$\int_0^1 \mathbf{1}\{\nu(q_i,\omega_i,p^*)=1/2\}\mathbf{1}\{q_i \geq p^*\}dQ(q_i) = \int_{p^*}^1 \mathbf{1}\{\nu(q_i,\omega_i,p^*)=1/2\}dQ(q_i) = 1-Q(t_u(p^*)),$$

where t_u (t_l) is the upper (lower) threshold for incentive compatible priors. Explicit expressions for the thresholds, which depend on p and ω_i, are derived in the appendix. Since the precision is bounded by $\hat{\nu}$, for $\theta = 1$ the share of informed investors can be written as

$$\int_0^1 \mathbf{1}\{\nu(q_i,\omega_i,p^*) > 1/2\}\nu(q_i,\omega_i,p^*)dQ(q_i) = \int_{t_l}^{t_u} \min\left\{\nu^*(q_i,\omega_i,p^*),\hat{\nu}\right\}dQ(q_i).$$

Hence, the equilibrium price p^* if $\theta = 1$ fulfills

$$\int_0^{\overline{\omega}} \left(1 - Q(t_u) + \int_{t_l}^{t_u} \min\left\{\frac{q_i\alpha_i + (1-q_i)\beta_i + 4}{8}, \hat{\nu}\right\} dQ(q_i) - p^*\right)\omega_i dW(\omega_i) = 0. \quad (5)$$

Proposition 2.

i. There exists an equilibrium price p^* fulfilling definition 1,

ii. which is implicitly defined as the fixed point of (5)*.*

iii. Assuming homogeneous endowment ($\omega_i = \omega$), the equilibrium is unique for ω such that $\hat{\nu}$ is binding,

iv. and unique if aggregate investment is non-increasing in the price.

Proof. See Appendix. □

As usual, non-increasing aggregate investment is crucial for uniqueness. Typically, A-investment decreases in response to a price increase, because uninformed A-traders with q_i close to t_u turn into informed traders, who only invest in A in $\nu_i < 1$ of the cases, and informed traders with q_i close to t_l turn into uninformed B-traders, who never invest in A. Let us call this the price effect. Still, the investment for the A-contract need not necessarily be decreasing in the price, because of an interaction with information acquisition. Since the precision chosen by informed traders $\nu^*(q_i,\omega_i,p) = (q_i\omega_i/p + (1-q_i)\omega_i/(1-p)+4)/8$ increases for large p, a price increase may trigger more information acquisition, which results in more investments in A for some informed traders if $\theta = 1$. ν^* is increasing in p for large p (holding q_i constant), because outcome B becomes more of a long-shot bet with large potential gains, so making the correct investment is more valuable. However, this effect is strong enough to reverse the price effect only if p is large and the distribution of priors is very unsmooth. Since this is a very specific situation, I assume prior distributions to be such that aggregate investment is weakly decreasing in the price in the following. The validity of this assumption is confirmed in all numerical examples.

The following proposition gives comparative statics regarding the market price if (without loss of generality) $\theta = 1$. The results are sufficient conditions for an endowment increase to reduce forecast error (increase the equilibrium price), and for a shift of priors toward the true value to reduce forecast error.

Proposition 3. *Comparative statics if investment is non-increasing in the price, ω is homogeneous and $\theta = 1$.*

i. Larger endowment ω increases information in the market as measured by

$$I_A = \int_{t_l}^{t_u} \min\{\nu^*(q_i), \hat{\nu}\} dQ(q_i),$$

i.e., more traders obtain information of higher precision, unless $\hat{\nu}$ is binding.

ii. Larger endowment increases the equilibrium price p^* if and only if

$$\int_{t_l}^{t_u} \mathbf{1}\{\nu^*(q_i) < \hat{\nu}\} \frac{\partial \nu^*}{\partial \omega} dQ(q_i) - \min\{\nu^*(t_l), \hat{\nu}\} q(t_l) \frac{\partial t_l}{\partial \omega} > (1 - \min\{\nu^*(t_u), \hat{\nu}\}) q(t_u) \frac{\partial t_u}{\partial \omega}.$$

iii. Any change in the prior distribution from Q to R such that $Q(q_i) > R(q_i)\ \forall q_i \in (0,1)$ increases the market price p^* if $\hat{\nu}$ is binding for all informed traders, unless all traders in the Q-market already acquire information.

iv. A larger upper bound on the signal precision $\hat{\nu}$ increases the equilibrium price p^* if $\hat{\nu}$ is binding for a positive probability mass of traders, otherwise it has no effect.

Proof. See Appendix. □

While larger endowment always increases information in the market as measured by investment from informed traders in the correct asset A (*i.*), it does not imply that the forecast error always decreases in response to an endowment increase. This result is stated in (*ii.*) and illustrated in Figure 3 for a discrete endowment increase, which increases the range of priors who acquire information (i.e., t_l decreases and t_u increases) and increases the chosen precision ν^*. Some uninformed A-traders, who always invest in the correct asset A, become informed traders, and only a share $\nu^* < 1$ of those invests in A, because signals are imperfect. Hence, investment in A (and consequently the equilibrium price) can decrease with an endowment increase if and only if the density on the upper threshold $q(t_u)$ is very large compared to the density at the lower threshold $q(t_l)$ and the ν^* increase for all informed traders, i.e., if and only if the density of traders who reduce investment in the correct asset is large compared to the density of traders who increase investment. The RHS of the condition in (*ii.*) is the investment loss represented by the white bordered area in the figure, and the LHS is represented by the dotted area. It can be shown that an endowment increase for uniformly distributed priors always increases the equilibrium price. Moreover,

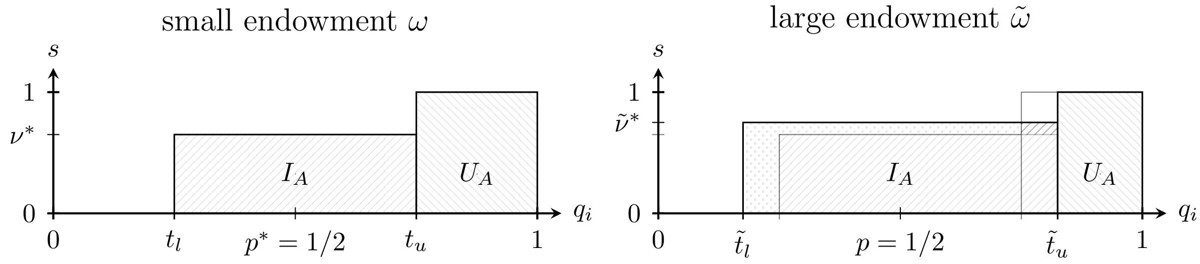

Figure 3: The effect of an endowment increase for all traders from ω to $\tilde{\omega}$ on the share of investors s who invest in asset A if $\theta = 1$.

The figure shows traders investing in A for both endowment levels (shaded area), investing in A only for larger endowment (dotted area), and investing in A only for smaller endowment (white bordered area). Hence, aggregate investment and equilibrium price increase only if the density weighted dotted area is larger than the weighted white bordered area. I_A is the aggregate investment in A by informed traders, while U_A is investment in A by uninformed traders.

this comparative static holds in all numerical cases with normal distribution in section 2.3. As with uniqueness, it may fail to hold for unsmooth prior distributions.

The price increases if probability mass is shifted towards the true outcome θ and if the precision cap is binding (*iii.*). However, if the precision cap is not binding, this need not necessarily hold, again because of an interaction with information acquisition. If $p > 1/2$, then the optimal precision $\nu^*(q_i, \omega_i, p)$ is decreasing in q_i, because the profitable state B (price only $1 - p < 1/2$) is considered less likely. Thus, if the mass of traders with high q_i increases, then it can decrease the mass of agents investing in A. It is therefore possible that traders with *more accurate* opinions produce *worse* forecasts, because they invest less in information acquisition. The price always increases in response to increases of ω or μ in the numerical example using discrete changes (section 2.3).

An increase of the precision cap $\hat{\nu}$ can be interpreted as the forecasting problem becoming easier, or better information becoming available. Unsurprisingly, part (*iv.*) shows that larger $\hat{\nu}$ never increases the forecast error, and decreases it if $\hat{\nu}$ is binding.

2.3 Numerical example

2.3.1 Preliminaries

The purpose of this section is threefold. First, it demonstrates that this model produces better predictions than the model without information acquisition (Manski, 2006). Second, it quantifies the impact of exogenous variables on p^*. And third, it shows the comparative statics are less ambiguous for discrete changes than the theoretical results may let on.

I calculate the equilibrium price p^* numerically for various parameter values. Priors are assumed to be normally distributed, but any probability mass $\int_1^\infty q(q_i)dq_i$ is bunched at 1 (similarly all mass below 0 is bunched at 0). This is preferable to truncation, as the truncated mass depends on parameters μ, σ, which hinders identification of comparative static effects.[3] Consequently, $\mu \neq 1/2$ is not actually the mean of the prior distribution, but merely a position parameter.

The equilibrium price with information acquisition is defined in (5), where (Φ denotes the cdf of the standard normal distribution, ϕ its density function)

$$Q(q_i) = \Phi\left(\frac{q_i - \mu}{\sigma}\right), \; q(q_i) = \frac{1}{\sigma}\phi\left(\frac{q_i - \mu}{\sigma}\right) \;\; \forall q_i \in (0,1).$$

The equilibrium price without information acquisition, where traders rely solely on their priors, is given by (e.g., Manski, 2006)

$$\int_0^{\overline{\omega}} (1 - Q(p_0^*) - p_0^*)\omega_i dW(\omega_i) = 0.$$

Endowment is homogeneous, with ω as a parameter to be varied. Moreover, I confirmed numerically that the equilibrium is unique.

Table 1 displays the equilibrium price with information acquisition (p^*), and without (p_0^*), as well as the corresponding parameter values $\mu, \sigma, \omega, \theta$. The forecast error of the market prediction $p^* - \theta$ can be easily computed. It also shows I_j, $j = A$ iff $\theta = 1$, $j = B$ iff $\theta = 0$, which is the probability mass of informed traders investing in the correct outcome in equilibrium. This is a measure of information in the market. A high value of I_j either means many traders acquire information, or that information is good (i.e., signals with high precision are acquired). For $\theta = 1$, $I_A = \int_{t_l}^{t_u} \min\{\nu^*(q_i, \omega, p^*), \hat{\nu}\} dQ(q_i)$. U_j is the probability mass of uninformed traders investing in the correct outcome, i.e., the mass of traders that does not acquire information in equilibrium, but has prior q_i that makes them invest in the correct option. For $\theta = 1$, $U_A = 1 - Q(t_u)$. From the market clearing condition, $U_A + I_A = p^*$.

2.3.2 Numerical results

A higher endowment ω leads to smaller forecast error $|p^* - \theta|$. The reason is that, first, more endowment leads to a higher mass of traders acquiring information. And second, traders who do acquire information choose more precise signals. This is reflected by I_j, the probability mass of informed traders investing in the correct outcome, which is increasing in endowment. As endowment rises, the market price is more and more driven by information rather than prior beliefs. For example, at $\omega = 5$, $\theta = 1$ and $\sigma = 0.1$, the probability mass

[3] The cdf of the truncated normal distribution can be increasing in μ for fixed thresholds 0 and 1 at some $x \in (0,1)$ because of truncation. Thus, an increase of μ does not make the prior distribution unambiguously more accurate for $\theta = 1$, since there may be points x where the mass of traders with prior $q_i \leq x$ increases in μ. Hence, μ is not a suitable (unambiguous) position parameter—a negative price response to a μ-increase could be due to changes in truncation, which we are not interested in, or due to reduced information acquisition. To solve this problem, I assume any truncated mass is located at the corners, so that the prior belief distribution in the interior is just the cdf of a normal distribution, which is always decreasing in response to an increase of μ.

Table 1: Equilibrium price with information acquisition p^* for various parameter values.

ω	μ	σ	θ	p^*	I_B	U_B	p_0^*
0.5	0.1	0.1	0	0.18	0.13	0.69	0.19
0.5	0.1	0.3	0	0.27	0.07	0.66	0.28
0.5	0.5	0.1	0	0.49	0.29	0.22	0.5
0.5	0.5	0.3	0	0.49	0.1	0.41	0.5
0.5	0.7	0.1	0	0.65	0.23	0.12	0.66
0.5	0.7	0.3	0	0.61	0.09	0.3	0.61
1	0.1	0.1	0	0.15	0.32	0.52	0.19
1	0.1	0.3	0	0.24	0.17	0.59	0.28
1	0.5	0.1	0	0.43	0.53	0.03	0.5
1	0.5	0.3	0	0.46	0.24	0.3	0.5
1	0.7	0.1	0	0.6	0.39	0.01	0.66
1	0.7	0.3	0	0.58	0.22	0.2	0.61
5	0.1	0.1	0	0.04	0.77	0.18	0.19
5	0.1	0.3	0	0.09	0.51	0.4	0.28
5	0.5	0.1	0	0.13	0.87	0	0.5
5	0.5	0.3	0	0.24	0.69	0.07	0.5
5	0.7	0.1	0	0.28	0.72	0	0.66
5	0.7	0.3	0	0.37	0.6	0.02	0.61

ω	μ	σ	θ	p^*	I_A	U_A	p_0^*
0.5	0.1	0.1	1	0.2	0.12	0.08	0.19
0.5	0.1	0.3	1	0.28	0.07	0.21	0.28
0.5	0.5	0.1	1	0.51	0.29	0.22	0.5
0.5	0.5	0.3	1	0.51	0.1	0.41	0.5
0.5	0.7	0.1	1	0.67	0.25	0.42	0.66
0.5	0.7	0.3	1	0.62	0.09	0.53	0.61
1	0.1	0.1	1	0.22	0.22	0.01	0.19
1	0.1	0.3	1	0.3	0.17	0.13	0.28
1	0.5	0.1	1	0.57	0.53	0.03	0.5
1	0.5	0.3	1	0.54	0.24	0.3	0.5
1	0.7	0.1	1	0.71	0.54	0.17	0.66
1	0.7	0.3	1	0.65	0.22	0.43	0.61
5	0.1	0.1	1	0.44	0.44	0	0.19
5	0.1	0.3	1	0.45	0.45	0.01	0.28
5	0.5	0.1	1	0.87	0.87	0	0.5
5	0.5	0.3	1	0.76	0.69	0.07	0.5
5	0.7	0.1	1	0.93	0.93	0	0.66
5	0.7	0.3	1	0.85	0.66	0.19	0.61

Description: The table displays the equilibrium price with information acquisition p^* and without (p_0^*). Priors are drawn from distribution $\mathcal{N}(\mu, \sigma^2)$, and endowment is ω. The true state of the world is θ, so the forecast error is $p^* - \theta$. I_A is the probability mass of informed traders investing in A in equilibrium, while U_A is the probability mass of uninformed traders investing in A. The upper bound of the signal precision is $\hat{\nu} = 0.95$.

of uninformed A-traders U_A is 0 (rounded), so it is almost exclusively informed traders investing in A and supporting the equilibrium, while U_A is positive for lower endowment.

Further, μ closer to the true state θ decreases the absolute value of the forecast error. This not surprising, since it means there is a larger probability mass of uninformed traders for each given price p willing to invest in the right asset, all else equal.

The effect of σ is ambiguous. In general, the parameter shifts probability mass closer to or farther away from μ. If μ is small, then a larger σ means there is a larger probability mass at high values of q_i. Consequently, a larger σ increases forecast error for low μ and $\theta = 0$, or for large μ and $\theta = 1$. That is, there are more wrongly investing uninformed traders, which drive the market price in the wrong direction. Similarly, a larger σ improves the prediction if μ is far from θ, because it increases the mass of correctly investing uninformed traders. The effect is more pronounced with smaller ω, where the price is driven more by opinion.

The prediction in the model without information acquisition p_0^* is unaffected by endowment, because it does not influence beliefs. Moreover, the prediction is the same independent of the state of the world, because investment decisions are not correlated with θ. The forecasts in the model with information acquisition are never worse, and usually strictly better than in the model without. Averaging over the 36 cases in Table 1, the forecast error in the Manski model is larger by about 10.5 percentage points, given the same endowment and prior distribution. Therefore, allowing for information acquisition goes a long way in explaining forecast accuracy.

2.4 Endogenous weighting: endowment heterogeneity

In elections or votes, each "voice" has equal weight. For purposes of preference aggregation, this may be fair, but if voting is meant to determine an objectively correct state, then equal weighting need not yield the best outcome (e.g., Ashton and Ashton, 1985). Indeed, if information among traders cannot be shared, the optimal weighting is to give full weight to the group with the best information. However, it is usually difficult to identify this group ex ante. In the asset market, this dilemma is solved endogenously by weighting each bet with its wager, or each trade with its volume ("weighting effect"). Combining with the earlier result that larger endowment induces more information acquisition ("incentive effect"), the market endogenously gives higher weight to traders with better forecasts, all else equal.

In the following, I assume for simplicity there are two endowment groups (ω_H, Q_H) and (ω_L, Q_L) with $\omega_H > \omega_L$, represented by endowment and prior distribution. The share of traders of group H in the market is $0 < \gamma < 1$. The equilibrium price in a hypothetical market populated only by group H is p_H^*, and the price in the hypothetical low endowment market is p_L^*, which is a way of expressing the combination of the group prior distribution and its endowment in terms of a forecast. Lemma 7 in the appendix shows that if aggregate investment is non-increasing in the price, then the market price in the market with both groups fulfills $\min\{p_H^*, p_L^*\} < p^* < \max\{p_H^*, p_L^*\}$ for $p_H^* \neq p_L^*$. An immediate consequence of this result is the following.

Corollary 4. *If the market consists of two endowment groups $\omega_H > \omega_L$, and aggregate investment is non-increasing in the price for both groups, then the equilibrium price p^* has a smaller forecast error than the price in a market consisting only of the low endowment group p_L^* if and only if the forecast error in the high endowment market is smaller, i.e.,*

$$|\theta - p^*| < |\theta - p_L^*| \iff |\theta - p_H^*| < |\theta - p_L^*|.$$

This corollary implies that adding a group of high endowment traders or traders with better information to the market would improve the market prediction. To analyze the weighting effect, consider a uniform (equal) weight price u^*, defined in the general case as

$$\int_0^1 \mathbf{1}\{a_i(q_i, \omega_i, u^*) > b_i(q_i, \omega_i, u^*)\} dQ(q_i) = u^*.$$

The price is determined by equating the share of the number of investments in A (instead of the share of dollars in A) with the price u^* that induces these investments. This method gives equal weight to every trader independent of his investment volume, as in polling, but it is not a market clearing price whenever it diverges from p^*. Thus, the market maker might have to use his own funds to pay the winners, or he might make a profit. If endowment is homogeneous or if $p^*_H = p^*_L$, then $p^* = u^*$.

Proposition 5. *Suppose the market consists of two endowment groups $\omega_H > \omega_L$, and aggregate investment is non-increasing in the price for both groups. If $p^*_H > p^*_L$, then $p^* > u^*$. Similarly, $p^*_H < p^*_L$ implies $p^* < u^*$.*

Proof. See Appendix. □

Corollary 6. *If $|p^*_H - \theta| < |p^*_L - \theta|$, then endogenous weighting compared to equal weighting unambiguously reduces the forecast error, i.e., $|p^* - \theta| < |u^* - \theta|$, but the improvement is smaller than for optimal group weighting, i.e., $|p^*_H - \theta| < |p^* - \theta|$.*

As shown in the previous sections, a larger endowment for traders typically leads to smaller forecast error. Thus, the endogenous weighting implied by market clearing reduces forecast error if prior beliefs and endowment are independent, or if higher endowment groups tend to have more accurate priors. An evolutionary argument makes the latter condition plausible: traders with better priors have better forecasts, so they increase their endowment by making the right investments. Hence, over time high endowment traders are the ones with better priors (e.g., Servan-Schreiber et al., 2004). Endogenous weighting increases the forecast error only if the low endowment group has considerably better priors to compensate for the inferior information due to the incentive effect, so that p^*_L is closer to θ than p^*_H.

In summary, the endogenous weighting of the market improves on the effect of information acquisition by giving more weight to richer traders, which usually have better forecasts.

3 Discussion: Interpreting prediction market prices

Several authors have asked how prediction market prices are to be interpreted. Contrary to conventional wisdom, Manski (2006) shows that an equilibrium price with risk neutral traders, interpreted as probability forecast, may be far from the mean belief in the market. Gjerstad (2005) and Wolfers and Zitzewitz (2006) demonstrate that this disparity is smaller for risk averse agents. They use a set-up without information and with arbitrary distribution of beliefs, which may or may not be close to the actual outcome. Consequently, the mean belief is a convenient summary statistic, but it may not be a good predictor of the outcome. What we are instead interested in is how price p^* relates to the outcome θ. To answer this, the key question is whether the price is driven by opinion, which may be close or far from the truth, or by information and evidence, which is correlated with the outcome and therefore more reliable. As this model shows, the price can be anything from an aggregation of wrong opinions to a very accurate and informed forecast. And even if initial beliefs about the outcome are off, the prediction market may generate a good forecast if agents have sufficient incentives to seek out information and revise their beliefs and investments.

Of course, it is possible to interpret the belief distribution of Manski et al. as posterior distribution after information acquisition. Any information acquisition equilibrium based on prior distribution Q can be reached in the model without information acquisition with belief distribution R such that (assuming homogeneous endowment)

$$1 - R(p^*) = 1 - Q(t_u) + \int_{t_l(p^*)}^{t_u(p^*)} \min\left\{\frac{q_i\alpha_i(p^*) + (1-q_i)\beta_i(p^*) + 4}{8}, \hat{\nu}\right\} dQ(q_i) = p^*.$$

Yet, merely assuming a belief distribution without modeling its generation does not give any insights into its informational content. Moreover, meaningful comparative static analysis is not possible, as distribution R is fixed for primitives ω and Q, hence changes in information acquisition (e.g., incentive compatibility) would not be captured.

4 Conclusion

Costly information acquisition explains the existence of uninformed traders, who do not acquire information and rely only on their opinion when investing, and of informed traders. In my model, noise traders and informed traders evolve endogenously from an initial distribution of opinions and endowment. Thus, good forecasts are not explained merely by the existence of insiders in the market, and bad ones by their absence. Rather, the information acquisition explanation implies that accurate initial beliefs as well as large endowment—improving incentives for information acquisition—can lead to good forecasts, but low stakes, high information costs, or inaccurate beliefs may also lead to bad ones.

Overall, a few high endowment agents, who choose to be informed because of large stakes, can be sufficient to drive the market price in the right direction due to their large weight in the market. In conventional financial markets, these high endowment agents may be hedge funds and investment banking divisions, who can move millions of dollars and have access to research into potential investment opportunities that smaller investors cannot afford. The explanation also implies that larger weight on bets or votes of high endowment agents may improve information aggregation in other contexts, at least as long as pooling of information is difficult and other motives such as manipulation (Rothschild and Sethi, 2013) play a minor role. Endogenous weighting has a bias reducing effect, unless endowments and priors are sufficiently negatively related.

There appears to be only one empirical paper that attempts to shed light on the role of money as incentive in futures markets. Servan-Schreiber et al. (2004) compare predictions of a play money prediction market and a real stakes prediction market. They find no significant difference in forecast accuracy, but stress that the play money market has stronger selection of good forecasters, because large 'play endowment' can only be obtained with a record of correct predictions. Moreover, the top traders were allowed to redeem play money for prizes, thus providing different (rank order) incentives for information acquisition. Hence, a direct test of the information acquisition explanation, e.g., by endowing traders with money randomly to rule out selection effects, is yet to be done.

Appendix A: Proofs

Proof of Proposition 1. *i.* Setting $q_i = p$, both $\underline{\nu}$ and $\tilde{\nu}$ take value 1/2, the global minimum. The former increases faster as q_i deviates, hence the result follows.

ii. From *i.*, $\underline{\nu}_i \geq \tilde{\nu}$ for q_i close to p for any $\omega_i > 0$. Since $\underline{\nu}_i(p = 1/2) = 1/2$ and $\underline{\nu}_i(p)$ is continuous with finite slope, there exists a neighborhood around $q_i = p$ such that $\underline{\nu}_i < \hat{\nu}$ whenever $\hat{\nu} > 1/2$.

Setting $q_i = p$, $\overline{\nu}_i - \underline{\nu}_i = \sqrt{\omega_i}/2 > 0$. Since $\overline{\nu}_i, \underline{\nu}_i$ are continuous in q_i, there exists a neighborhood around $q_i = p$ such that $[\underline{\nu}_i, \overline{\nu}_i] \cap [\tilde{\nu}_i, \hat{\nu}] \neq \varnothing$. Because the density of q_i is strictly positive everywhere, there is a positive probability mass of traders with q_i such that information acquisition is incentive compatible.

The term within the square root of (3) or (4) is strictly decreasing in q_i if $q_i > p$ and strictly increasing if $q_i < p$. If the term is negative, then information acquisition is not incentive compatible. Thus, the incentive compatibility constraint becomes less stringent as q_i approaches p.

Because ν_i^* is increasing in q_i whenever $p < 1/2$ and decreasing whenever $p > 1/2$, the chosen precision is increasing in q_i for $q_i \leq p \leq 1/2$ and decreasing for $1/2 \leq p \leq q_i$. Together with the effect on the stringency of the IC, $\frac{\partial \nu_i}{\partial |q_i - p|} \leq 0$ if $1/2 \leq p \leq q_i$ or $q_i \leq p \leq 1/2$.

iii. I first show that the range of incentive compatible effort levels is nondecreasing for sufficiently large $\omega_i > 0$. Incentive compatible positive effort levels exist for a given q_i iff the term within the square root in (3) or (4) is nonnegative. This term becomes positive for sufficiently large ω_i, because all squared ω_i terms have positive sign, while one (linear) ω_i term has a negative sign.

To be shown: choice set $[\underline{\nu}_i, \overline{\nu}_i] \cap [\tilde{\nu}_i, \hat{\nu}]$ is nondecreasing and non-empty as $\omega_i \to \infty$. Since *i.* states $\tilde{\nu}_i \leq \underline{\nu}$, it remains to be shown that sufficiently large ω_i implies $\underline{\nu}_i < \hat{\nu}$.

Claim: $\lim_{\omega \to \infty} \underline{\nu}_i \to \tilde{\nu}$. To see this,[4] apply a Taylor approximation for $\underline{\nu}_i$ (see (3), (4)), which is of the form $y - \sqrt{y^2 - z} \approx z/(2y)$ and holds for $y >> z$. Then, as $\omega_i \to \infty$,

$$\frac{z}{2y} = \frac{1 + q_i \omega_i / p}{\omega_i [q_i/p + (1 - q_i)/(1 - p)] + 4} \to \frac{q_i/p}{q_i/p + (1 - q_i)/(1 - p)} = \tilde{\nu}_i.$$

[4] I am grateful to Ron Gordon for this idea.

The above approximation is obtained by Taylor expanding $\sqrt{1-x}$ at 1, $\sqrt{1-x} \approx 1 - 1/2x$ (omitting higher order terms, as these vanish asymptotically for x small). Then, using $x = z/y^2$,

$$y - \sqrt{y^2 - z} = y - y\sqrt{1 - z/y^2} \approx y - y\left(1 - \frac{z}{2y^2}\right) = \frac{z}{2y}.$$

The Taylor approximation becomes arbitrarily accurate as $\omega_i \to \infty$, since $1 - z/y^2 \to 1$. It is easy to verify that $\underline{\nu}_i$ converges from above. Therefore, the range of incentive compatible effort levels is nondecreasing for sufficiently large ω_i. Consequently, ν_i increases as ω_i increases until $\nu_i = \hat{\nu}$, as ν_i^* is strictly increasing in ω_i. □

Proof of Proposition 2. *i.* The optimal precision $\nu_i(q_i, \omega_i, p)$ is upper hemi-continuous (uhc) in the parameters and non-empty by Berge's maximum theorem, and since the maximizer is unique by concavity, uhc coincides with continuity. Consequently, we can write investment as function of the primitives, i.e., $a_i(\pi_i(q_i, \nu_i, s_i), \omega_i, p)$ as $a_i(q_i, \omega_i, p)$, same for $b_i(q_i, \omega_i, p)$.

The investment maximization problem at $t = 1$ for each i is continuous, with maximizer a_i, b_i from the compact set $\omega_i \geq a_i + b_i \geq 0$. Applying Berge's maximum theorem, the investment correspondence is uhc in the parameters, non-empty and compact-valued. This implies that aggregate investment is uhc in the parameters (Aumann, 1976). Given risk-neutrality, investment is multi-valued if and only if $\pi_i = p$. In this case, investment is the budget line, which is a convex set. Writing (5) as mapping $[0, 1] \to [0, 1]$,

$$\int_0^{\overline{\omega}} \left(1 - Q(t_u) + \int_{t_l}^{t_u} \min\{\nu^*(q_i, p), \hat{\nu}\} dQ(q_i)\right) \omega_i dW(\omega_i) / \int_0^{\overline{\omega}} \omega_i dW(\omega_i) = p,$$

Kakutani's fixed point theorem guarantees the existence of p fulfilling definition 1.

ii. In equilibrium, the number of contracts demanded for either outcome must be identical, so that investments from the losers are identical to earnings of the winners. Without loss of generality, suppose $\theta = 1$. Then, denoting the mass of uninformed traders buying A-contracts at price p^* by U_A and the mass of informed A-traders by I_A (similarly for B), the equilibrium price fulfills

$$\int_0^{\overline{\omega}} \omega_i \frac{U_A + I_A}{p^*} dW(\omega_i) = \int_0^{\overline{\omega}} \omega_i \frac{U_B + I_B}{1 - p^*} dW(\omega_i)$$

$$\iff \int_0^{\overline{\omega}} \omega_i (U_A + I_A)(1 - p^*) - \omega_i (U_B + I_B) p^* dW(\omega_i) = 0$$

$$\iff \int_0^{\overline{\omega}} \omega_i (U_A + I_A - p^*) dW(\omega_i) = 0,$$

because $U_A + U_B + I_A + I_B = 1$, and because risk neutral agents invest the entire endowment ω_i. Replacing the U_A, I_A terms, this is the desired expression.

iii. Rewriting, $\int_{t_l(p)}^{t_u(p)} \min\{\nu^*, \hat{\nu}\} dQ = \hat{\nu}[Q(t_u) - Q(t_l)]$, since $\hat{\nu}$ is binding. A change in p changes this term by $\hat{\nu}[t'_u q(t_u) - t'_l q(t_l)]$, and changes $1 - Q(t_u)$ by $-t'_u q(t_u)$. Hence, $\hat{\nu}[t'_u q(t_u) - t'_l q(t_l)] - t'_u q(t_u) < 0$, since $\hat{\nu} \le 1$ and $t'_u, t'_l > 0$. Thus, the probability mass investing in A is strictly decreasing in p, so there is a unique p^* fulfilling equality (5).

iv. As before, writing (5) as

$$\int_0^{\overline{\omega}} \left(1 - Q(t_u) + \int_{t_l}^{t_u} \min\{\nu^*(q_i, p), \hat{\nu}\} dQ(q_i)\right) \omega_i dW(\omega_i) / \int_0^{\overline{\omega}} \omega_i dW(\omega_i) = p,$$

where the LHS is aggregate A-investment and weakly decreasing in p by assumption, immediately shows that the unique equilibrium is unique.

□

Proof of Proposition 3. *i.* Using Leibniz' integral rule,

$$\frac{\partial I_A}{\partial \omega} = \int_{t_l}^{t_u} \left(\frac{q_i}{8p^*} + \frac{(1-q_i)}{8(1-p^*)}\right) q(q_i) dq_i$$
$$+ \frac{t_u \alpha + (1-t_u)\beta + 4}{8} q(t_u) t'_u - \frac{t_l \alpha + (1-t_l)\beta + 4}{8} q(t_l) t'_l > 0,$$

because $\partial t_l / \partial \omega < 0$ and $\partial t_u / \partial \omega > 0$.

ii. Using Leibniz' integral rule to compute $\frac{\partial(5)}{\partial \omega}$ and rearranging gives the expression.

iii. Not all agents acquire information, which implies $t_l \in (0,1)$ or $t_u \in (0,1)$. The share of traders investing in A is $1 - Q(t_u) + \int_{t_l}^{t_u} \hat{\nu} q(q_i) dq_i = 1 - Q(t_u) + \hat{\nu}[Q(t_u) - Q(t_l)]$. For any change $Q(q_i) > R(q_i)\ \forall q_i \in (0,1)$, the share changes by $Q(t_u) - R(t_u) + \hat{\nu}[R(t_u) - Q(t_u) - R(t_l) + Q(t_l)] > 0$, since $[1-\hat{\nu}]Q(t_u) \ge [1-\hat{\nu}]R(t_u)$ and $Q(t_l) \ge R(t_l)$ with at least one strict inequality, as t_l or t_u is in the interior. The increase of the equilibrium price is implied by the implicit function theorem and non-increasing investment.

iv. If $\hat{\nu}$ is binding, then

$$\frac{\partial(5)}{\partial \hat{\nu}} = \int 1 dQ + \hat{\nu}[q(t_u)t'_u - q(t_l)t'_l] > 0,$$

since $t'_u > 0$ and $t'_l < 0$. Similarly, if it is not binding, the derivative is 0. □

Proof of Proposition 5. For $\theta = 1$, u^* is defined as

$$\gamma \left[1 - Q_H(t_u(u^*, \omega_H)) + \int_{t_l(u^*,\omega_H)}^{t_u(u^*,\omega_H)} \min\{\nu^*(u^*, \omega_H, q_i), \hat{\nu}\} dQ_H(q_i) - u^* \right]$$
$$+(1-\gamma) \left[1 - Q_L(t_u(u^*, \omega_L)) + \int_{t_l(u^*,\omega_L)}^{t_u(u^*,\omega_L)} \min\{\nu^*(u^*, \omega_L, q_i), \hat{\nu}\} dQ_L(q_i) - u^* \right] = 0.$$

Setting the price equal to p^*, which fulfills $p_H^* > p^* > p_L^*$ (Lemma 7), the first term within brackets is positive, while the second is negative. Since the relative weight of the first term is reduced compared to (6), because $\omega_H > \omega_L$, while the relative weight of the second is larger, the LHS is negative. Consequently, the unique solution u^* must be smaller than p^*. □

Lemma 7. *If the market consists of two groups with $p_H^* \neq p_L^*$, and aggregate investment is non-increasing in the price for both groups, then the unique market price p^* fulfills $\min\{p_H^*, p_L^*\} < p^* < \max\{p_H^*, p_L^*\}$.*

Proof. If $\theta = 1$, the equilibrium price is implicitly defined as

$$\gamma\omega_H \left[1 - Q_H(t_u(p^*, \omega_H)) + \int_{t_l(\omega_H)}^{t_u(\omega_H)} \min\{\nu^*(p^*, \omega_H, q_i), \hat{\nu}\} dQ_H(q_i) - p^* \right]$$
$$+(1-\gamma)\omega_L \left[1 - Q_L(t_u(p^*, \omega_L)) + \int_{t_l(\omega_L)}^{t_u(\omega_L)} \min\{\nu^*(p^*, \omega_L, q_i), \hat{\nu}\} dQ_L(q_i) - p^* \right] = 0. \tag{6}$$

Suppose without loss of generality $p_H^* > p_L^*$. By monotonicity of aggregate investment, the LHS when evaluated at $p = p_L^*$ is positive and therefore cannot be a market clearing price. Since investment in the A-contract is weakly decreasing in p in both groups, the LHS is strictly decreasing in p. Thus, the unique market clearing price must fulfill $p^* > p_L^*$. Similarly, the LHS evaluated at $p = p_H^*$ is positive, which implies $p^* < p_H^*$. □

Appendix B: Upper and lower threshold of incentive compatible priors

The goal is to find t_u, the upper threshold of prior belief q_i, where information acquisition is just incentive compatible. Note that, for large ω_i, precision levels ν_i^* may be greater 1 and therefore violate the axioms of probability. Moreover, for large ω_i, there is no type $q_i \in [0, 1]$ for which no information acquisition is preferable (i.e., there is no real solution for q_l, q_u below). This requires case distinctions. Ignoring constraints, the type that is indifferent between information acquisition and relying on his prior knowledge is found by setting the square root term in (3) equal to zero, which gives q_u, and including constraints we obtain

$$t_u = \begin{cases} q_u(p) = \frac{-y-\sqrt{y^2-4xz}}{2x} & \text{if } \nu^*(q_u) \leq \hat{\nu},\ q_u \leq 1 \text{ and } q_u \in \mathbb{R}, \\ q : \underline{\nu}(q) = \hat{\nu} \wedge q \geq p & \text{otherwise,} \end{cases}$$

where $x = (\alpha - \beta)^2$, $y = -2\beta^2 - 8\alpha - 8\beta + 2\alpha\beta$, $z = \beta^2 + 8\beta$. q_u is the smaller of the two solutions of the quadratic equation. Only the case $q_i \geq p$ has to be considered here, because at $q_i = p$ information acquisition is always incentive compatible (Proposition 1), so the upper threshold must be larger than the price. For the lower threshold type q_l, the square root term in (4) is similarly set to zero, resulting in

$$t_l = \begin{cases} q_l(p) = \frac{-y+\sqrt{y^2-4xz}}{2x} = 1 - q_u(1-p) & \text{if } \nu^*(q_l) \leq \hat{\nu},\ q_l \geq 0 \text{ and } q_l \in \mathbb{R}, \\ q : \underline{\nu}(q) = \hat{\nu} \wedge q \leq p & \text{otherwise,} \end{cases}$$

where $x = (\alpha - \beta)^2$, $y = -2\beta^2 + 8\alpha + 8\beta + 2\alpha\beta$, $z = \beta^2 - 8\beta$. This is the larger of the two solutions to the quadratic equation. The following properties of the threshold functions are used in the proofs. With respect to the price,

$$\frac{\partial q_u}{\partial p} > 0,\ \frac{\partial q_l}{\partial p} > 0, \text{ and } \frac{\partial q_u}{\partial p} > \frac{\partial q_l}{\partial p} \iff p < 1/2,\ \frac{\partial q_u}{\partial p} < \frac{\partial q_l}{\partial p} \iff p > 1/2.$$

Using symmetry,

$$\frac{\partial q_u}{\partial p} = (1 - q_l(1-p))' = q_l'(1-p) = q_u'(p),\ q_l'(1/2) = q_u'(1/2).$$

Moreover,

$$\frac{\partial q_u}{\partial \omega} > 0,\ \frac{\partial q_l}{\partial \omega} < 0, \text{ and } \frac{\partial q_u}{\partial \omega} > -\frac{\partial q_l}{\partial \omega} \iff p < 1/2,\ \frac{\partial q_u}{\partial \omega} < -\frac{\partial q_l}{\partial \omega} \iff p > 1/2.$$

Appendix C: Sequential trading

This section shows that the equilibrium price p^* of the static trading model (Definition 1) can also be reached under an additional assumption if traders invest given posted prices sequentially, and a market maker uses the investments to adjust the posted prices. More specifically, each trader $n = 1, 2, \ldots$ is an i.i.d. draw from the population, faces a price $p_{n-1}(\boldsymbol{a}, \boldsymbol{b})$, which is a function of previous investments $\boldsymbol{a} = (a_1, a_2, \ldots, a_{n-1})$ in A and $\boldsymbol{b} = (b_1, b_2, \ldots, b_{n-1})$ in B, acquires information (or not) and makes his investment choice given this price. The market maker posts prices $(p_1, p_2, \ldots)$, sells at these prices, and updates the posted prices depending on the sequentially arriving investments. The market maker uses the following pricing rule—which is identical to the parimutuel rule, except here the price applies only to the next trader—as a function of past investments, with arbitrary starting price:

$$p_{n-1} = \frac{\sum_{i=1}^{n-1} a_i}{\sum_{i=1}^{n-1}(a_i + b_i)}, \; p_0 \in [0,1], \; \text{for } n = 1, 2, \ldots .$$

To show convergence of this price process, I am going to use the following strong law of large numbers:

Theorem (Etemadi (1983))**.** *Let $\{a_i : i > 0\}$ be a sequence of non-negative random variables with finite second moments such that:*

a. $\sup_{i>0} \mathbb{E}a_i < \infty$*,*

b. $\mathbb{E}[a_i a_j] \leq \mathbb{E}a_i \mathbb{E}a_j, j > i$*, and*

c. $\sum_{i=1}^{\infty} \text{Var } a_i/i^2 < \infty$*.*

Then, as $n \to \infty$,

$$\left(\sum_{i=1}^{n} a_i - \mathbb{E}\left[\sum_{i=1}^{n} a_i \right] \right) /n \xrightarrow{\text{a.s.}} 0.$$

The crucial condition due to the non-independence of subsequent investments induced by price changes is non-positive auto-covariance (b). A larger investment in A increases the price of asset A, and so future investments in A are expected to be smaller. As the following proposition shows, this condition is fulfilled if aggregate investment is non-increasing in the price, which also implies that the equilibrium is unique (Proposition 2). Thus, the equilibrium of the static model can be reached in such a simple sequential trading process and traders do not need any price expectations as in the static model of definition 1; they just take the price they are offered.

Proposition 8. *If aggregate investment is non-increasing in the price, then*

$$p_{n-1} \xrightarrow{\text{a.s.}} p^*,$$

with p^* as defined in Definition 1.

Proof. Clearly, a_i and b_i are nonnegative, and $\sup_{i>0} \mathbb{E}a_i < \infty$ due to $\omega_i \leq \bar{\omega} < \infty$. Moreover, since any a_i is finite, there is an upper bound c such that Var $a_i \leq c < \infty$, hence

$$\sum_{i=1}^{\infty} \text{Var } a_i/i^2 \leq \sum_{i=1}^{\infty} c/i^2 < \infty,$$

since the over-harmonic series is convergent. It remains to be shown that (b) above holds. Investment a_n by trader n depends on the offered price p_{n-1}. Hence, although trader characteristics are i.i.d. draws, the investment is not; it is influenced by previous investments causing price changes. Thus, for $\theta = 1$,

$$\begin{aligned}\mathbb{E}[a_n|p_{n-1}] = \mathbb{E}[a_n|a_1,\ldots,a_{n-1}] &= \int_0^1\int_0^{\bar{\omega}} a_n(q_n,\omega_n,p_{n-1})dQ(q_n)dW(\omega_n)\\ &= \int_0^{\overline{\omega}}\left(1-Q(t_u)+\int_{t_l}^{t_u}\min\{\nu^*(q_n,p_{n-1}),\hat{\nu}\}dQ(q_n)\right)\omega_n dW(\omega_n),\end{aligned}$$

which is just the aggregate investment in A in the static model at price p_{n-1}. Thus, the assumption of non-increasing aggregate investment implies $\mathbb{E}[a_i a_j] \le 0, j > i$, i.e., a large investment in A tends to trigger lower subsequent investments in A. In equilibrium (5), the share of aggregate investment in A equals the price of asset A,

$$\int_0^{\overline{\omega}}\left(1-Q(t_u)+\int_{t_l}^{t_u}\min\{\nu^*(q_i,p^*),\hat{\nu}\}dQ(q_i)\right)\omega_i dW(\omega_i)\Big/\int_0^{\overline{\omega}}\omega_i dW(\omega_i) = p^*.$$

Since aggregate investment is decreasing in the price, if $p_{n-1} < p^*$, then $\mathbb{E}[a_n|p_{n-1}] \ge \mathbb{E}[a_n|p^*]$ and $\mathbb{E}[b_n|p_{n-1}] \le \mathbb{E}[b_n|p^*]$, hence

$$\mathbb{E}\left[\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n (a_i+b_i)}\middle| p_{n-1}\right] \ge p_{n-1}.$$

Similarly, if $p_{n-1} > p^*$, then $\mathbb{E}[a_n|p_{n-1}] \le \mathbb{E}[a_n|p^*]$ and $\mathbb{E}[b_n|p_{n-1}] \ge \mathbb{E}[b_n|p^*]$. Therefore,

$$\mathbb{E}\left[\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n (a_i+b_i)}\right] \to p^*,$$

and

$$\mathbb{E}\left[\sum_{i=1}^n a_n\right]/n \to \mathbb{E}\left[\sum_{i=1}^n a_n\middle| p^*\right]/n \text{ and } \mathbb{E}\left[\sum_{i=1}^n b_n\right]/n \to \mathbb{E}\left[\sum_{i=1}^n b_n\middle| p^*\right]/n$$

for all p_0. Since $\mathbb{E}a_i\mathbb{E}a_j \ge 0$, Etemadi (1983)'s theorem implies

$$\left(\sum_{i=1}^n a_i - \mathbb{E}\left[\sum_{i=1}^n a_i\right]\right)/n \xrightarrow{\text{a.s.}} 0 \text{ and } \left(\sum_{i=1}^n (a_i+b_i) - \mathbb{E}\left[\sum_{i=1}^n (a_i+b_i)\right]\right)/n \xrightarrow{\text{a.s.}} 0.$$

Together,

$$p_{n-1} = \frac{\sum_{i=1}^{n-1} a_i}{\sum_{i=1}^{n-1}(a_i+b_i)} \xrightarrow{\text{a.s.}} p^*. \qquad \square$$

References

Ali, M. (1977): "Probability and utility estimates for racetrack bettors," *Journal of Political Economy*, 85, 803–815.

Arrow, K., R. Forsythe, M. Gorham, R. Hahn, R. Hanson, J. Ledyard, S. Levmore,

R. Litan, P. Milgrom, F. Nelson, et al. (2008): "The promise of prediction markets," *Science*, 320, 877–878.

Ashton, A. H. and R. H. Ashton (1985): "Aggregating subjective forecasts: Some empirical results," *Management Science*, 31, 1499–1508.

Aumann, R. J. (1976): "An elementary proof that integration preserves uppersemicontinuity," *Journal of Mathematical Economics*, 3, 15–18.

Barlevy, G. and P. Veronesi (2000): "Information acquisition in financial markets," *Review of Economic Studies*, 67, 79–90.

Berg, J., F. Nelson, and T. Rietz (2008): "Prediction market accuracy in the long run," *International Journal of Forecasting*, 24, 285–300.

Berg, J. E. and T. A. Rietz (2006): "The Iowa Electronic Markets: Stylized facts and open issues," in *Information Markets: A New Way of Making Decisions*, ed. by R. W. Hahn and P. C. Tetlock, AEI-Brookings Joint Center for Regulatory Studies, 142–169.

Cabrales, A., O. Gossner, and R. Serrano (2014): "The Inverse Demand for Information and the Appeal of Information Transactions," Working paper.

Etemadi, N. (1983): "On the laws of large numbers for nonnegative random variables," *Journal of Multivariate Analysis*, 13, 187–193.

Eyster, E., M. Rabin, and D. Vayanos (2013): "Financial Markets where Traders Neglect the Informational Content of Prices," Working paper.

Forsythe, R., F. Nelson, G. R. Neumann, and J. Wright (1992): "Anatomy of an experimental political stock market," *American Economic Review*, 82, 1142–1161.

Gjerstad, S. (2005): "Risk aversion, beliefs, and prediction market equilibrium," Working Paper, University of Arizona.

Grossman, S. J. and J. E. Stiglitz (1980): "On the impossibility of informationally efficient markets," *American Economic Review*, 70, 393–408.

Hanson, R. and R. Oprea (2009): "A manipulator can aid prediction market accuracy," *Economica*, 76, 304–314.

Hong, H. and J. C. Stein (1999): "A unified theory of underreaction, momentum trading, and overreaction in asset markets," *Journal of Finance*, 54, 2143–2184.

Manski, C. (2006): "Interpreting the predictions of prediction markets," *Economics Letters*, 91, 425–429.

Ottaviani, M. and P. N. Sørensen (2009): "Surprised by the parimutuel odds?" *American Economic Review*, 99, 2129–2134.

——— (2010): "Noise, information, and the favorite-longshot bias in parimutuel predictions," *American Economic Journal: Microeconomics*, 2, 58–85.

——— (forthcoming): "Price Reaction to Information with Heterogeneous Beliefs and Wealth Effects: Underreaction, Momentum, and Reversal," *American Economic Review.*

Page, L. and R. Clemen (2013): "Do prediction markets produce well calibrated probability forecasts?" *Economic Journal*, 123, 491–513.

Peress, J. (2004): "Wealth, information acquisition, and portfolio choice," *Review of Financial Studies*, 17, 879–914.

Plott, C. R. and K.-Y. Chen (2002): "Information aggregation mechanisms: Concept, design and implementation for a sales forecasting problem," Working Paper, California Institute of Technology.

Radner, R. (1979): "Rational expectations equilibrium: Generic existence and the information revealed by prices," *Econometrica*, 655–678.

Rothschild, D. M. and R. Sethi (2013): "Trading Strategies and Market Microstructure: Evidence from a Prediction Market," Working paper.

Sauer, R. (1998): "The economics of wagering markets," *Journal of Economic Literature*, 36, 2021–2064.

Serrano-Padial, R. (2012): "Naive traders and mispricing in prediction markets," *Journal of Economic Theory*, 147, 1882–1912.

Servan-Schreiber, E., J. Wolfers, D. M. Pennock, and B. Galebach (2004): "Prediction markets: Does money matter?" *Electronic Markets*, 14, 243–251.

Spann, M. and B. Skiera (2009): "Sports forecasting: a comparison of the forecast accuracy of prediction markets, betting odds and tipsters," *Journal of Forecasting*, 28, 55–72.

Thaler, R. and W. Ziemba (1988): "Parimutuel betting markets: Racetracks and lotteries," *Journal of Economic Perspectives*, 2, 161–174.

Tziralis, G. and I. Tatsiopoulos (2007): "Prediction markets: An extended literature review," *Journal of Prediction Markets*, 1, 75–91.

Van Nieuwerburgh, S. and L. Veldkamp (2010): "Information acquisition and under-diversification," *Review of Economic Studies*, 77, 779–805.

Verrecchia, R. E. (1982): "Information acquisition in a noisy rational expectations economy," *Econometrica*, 50, 1415–1430.

Weizsäcker, G. (2010): "Do we follow others when we should? A simple test of rational expectations," *American Economic Review*, 100, 2340–2360.

Wolfers, J. and E. Zitzewitz (2004): "Prediction Markets," *Journal of Economic Perspectives*, 18, 107–126.

——— (2006): "Interpreting prediction market prices as probabilities," *NBER Working Paper*, 12200.