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für Kaçhatjan

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Chapter 1

General Introduction

"The macroeconomics of the future [...] will have to go beyond conventional late-twentieth-century methodology [...], by making the formation and revision of expectations an object of analysis in its own right [...]" - M. Woodford (2011).

The forward-looking behavior of economic agents is one of the salient features of modern macroeconomics. Agents understand that the effects of most of their economic decisions extend over several periods and, hence, that these effects can be improved by taking the future state of the world into account. However, future economic conditions are uncertain and need to be forecasted. And so, as agents' forecasts stand as crucial determinants of individual economic decisions and, consequently, of aggregate economic outcomes, the way in which the agents' *expectations formation mechanism* is modeled becomes a major factor driving our understanding of economics.

Since the fundamental works of Robert Lucas Jr. and Thomas Sargent in the early 1970s,[1] rational expectations has been the overwhelmingly predominant approach for imputing agents' expectations in macroeconomics. This hypothesis is typically defined as the mathematical conditional expectation of the relevant variables. It is an equilibrium concept that precludes systematic errors in agents' forecasts and assumes an efficient use of all of the information available to the agent, providing valuable inner-consistency.

Notwithstanding its omnipresence and success, rational expectations is not without shortcomings and limitations. In particular, one of the most prized properties of the Rational Expectations Hypothesis (REH) is, at the same time, one of its main drawbacks: the *discipline* that it imposes on economic models. This *discipline* is obtained as a consequence of the conformity that rational expectations imposes between agents beliefs about the future and the predictions of the model. In this way, agent's beliefs disappear as an extra component of the theory, subject to the arbitrariness and convenience of the economist, and are uniquely determined by the other components of the model. The cost of this *discipline* is, however, steep as agents' expectations cannot longer be an autonomous source of dynamics. A fact that dramatically limits their role.

As Sargent (2008) reminds us, one should be aware of the risks of deviating from the REH,

[1] Particularly, Lucas (1972) and Sargent (1973). These works, in turn, resumed the seminal work of Muth (1961).

since doing so "sends us into a *wilderness* because there is such a bewildering variety of ways to imagine discrepancies between objective and subjective distributions".[2] Nonetheless, understanding that the ultimate justification for adopting a specific economic model, besides its empirical performance, hinges on its ability to replicate our understanding and intuition of the economy, we should feel comfortable with judiciously relaxing this strong and restrictive assumption.

Moreover, as Milani (2012) points out, when bringing a model to the data one is, de facto, testing the economic theory and the assumed expectations formation mechanism jointly.[3] Therefore, when studying a particular economic problem, in order to avoid the risk of adopting misspecified theories or rejecting potentially valid ones, it is essential to consider different expectations formation mechanisms in the same way we consider alternative economic theories.

In this context, the study of adaptive learning has proven to bee one of the most extended approaches proposed to deviate from the REH.[4] A standard way of introducing adaptive learning, is by first noticing that rational expectations is typically based on a very strong informational assumption that presupposes agents to know the entire model including its parameters' values and the distributions of its exogenous shocks; a degree of knowledge that not even the theorist has.[5] If this amount of knowledge is reduced, agents are no longer able to derive the probability distributions that emerge in equilibrium and cannot use them to forecast relevant economic outcomes. At this point, adaptive learning assumes that agents form expectations using reduced-form models, which they estimate as new information becomes available. This is the usual starting point of the adaptive learning approach and, in particular, the common ground for the present work.

This dissertation is organized as three self-contained chapters which contribute to the literature of expectations formation in macroeconomics and of adaptive learning in particular.

In the second chapter, co-authored with Elena Rancoita, we study the empirical estimation of dynamic models under Adaptive Learning from a Bayesian perspective. We consider dif-

[2]Sargent is, in turn, paraphrasing Chris Sims. Subjective distributions refer, here, to the ones used by economic agents, while objective distributions refer to the ones that emerge from the model in equilibrium.

[3]Here, I am abusing the term theory by circumscribing it to the parts of the model different from the expectations formation mechanisms. Clearly, in a strict sense, the latter is also part of the theory.

[4]See, for example, Evans and Honkapohja (2001). An alternative important approach, though less extended, is the analysis of "eductive stability" proposed by Roger Guesnerie, see Guesnerie (2005). See also Frydman and Phelps (2013) for compilation of recent examples of non rational expectations' works.

[5]Even considering Muth's original ideas about rational expectations, in which in order to forecast economic outcomes, agents take into consideration the causal processes behind them and that their understanding of this casual processes is precisely the model itself, agents knowledge about the economy remains highly demanding.

ferent approaches to estimate macroeconomic models with adaptive learning and we examine their relative performances in terms of bias, accuracy and computational cost. Existing work estimating adaptive learning models use strong simplifying assumptions, such as deterministic learning rules and certainty of non-observable states, so as to circumvent the problem of dealing with large non-linear state space models. We compare the performance of the existing approach with two other ones: first, we propose a new approach based on the linearization of the learning rules, which allows for the use of linear-filters and can address a wider range of models - we show the conditions under which the linearized system converges to the same equilibrium as the original one- ; second, we consider a recently developed non-linear filter, the Smolyak Kalman Filter, which considerably reduces the problem of the curse of dimensionality affecting likelihood based non-linear estimators. Using the Cobweb model as a testing laboratory we find that the costs of linerization associated to our method are not significant, while the ones associated to the approach found in the literature can be substantial in terms of the estimates of agents' beliefs. In addition, we find that the computational costs associated with the newly devised approach are substantially lower than the ones associated with the Smolyak Kalman Filter.

In Chapter 3, I model sentiment as exogenous shocks to the beliefs agents have about the future and I study their role in the generation of business cycle fluctuations in the US economy. Considering a standard New Keynesian model of the business cycle, I introduce agents that update beliefs about the parameters of their forecasting models using newly observed data and exogenous sentiment shocks. The model is then estimated using the new estimation methodology proposed in Chapter 2, as the other methods cannot estimate this model. After accounting for the different degrees of freedom, I find that the resulting learning model fits the data significantly better than its non-sentiment version and than its rational expectations counterpart. Furthermore, I show how sentiment is an important driver of economic fluctuations, accounting for a substantial fraction of the variability of aggregate variables at business cycle frequencies, ranging from 20 to 50%. In particular, sentiment related to investment is very important to explain the variability of real output, consumption and investment. Finally, I explore the concrete role played by sentiment in the US over the last 50 years with a historic shock decomposition. The exercise suggests that sentiment is the main responsible for the slow recovery following a recession, as agents' pessimistic views take time to dissipate. In addition, it is interesting to observe that while sentiment moved in line with other shocks in the eve and during the great recession - pushing inflation below its steady state value - around the beginning of 2010 sentiment started to move in the opposite direction - creating inflationary pressure that partially off-set what otherwise would have been an even lower inflation level.

In Chapter 4, I study the effect of Internal Rationality on the stability of monetary policy. While maintaining the optimal behavior of economic agents, Internal Rationality relaxes the strong informational assumption imposed by Rational Expectations and provides well-defined microfoundations for models under Adaptive Learning. This concept, introduced in Adam and Marcet (2011) in the context of an asset pricing model, allows to overcome the arbitrariness and potential inconsistencies that usually afflict standard learning models. I extend this framework to a basic New Keynesian model, à la Clarida et al. (1999), and study its implications for monetary policy. The resulting model requires agents to make forecasts only of non-choice variables appearing in their maximization problems and for all infinite future periods. In particular, agents need to form expectations about their future real wages and profits. I find that Taylor-type rules that react on contemporaneous inflation are found to be desirable if and only if the so-called Taylor principle is satisfied, while for Taylor-type rules that react to current forecasts of one period ahead inflation the Taylor principle is found to be necessary but not sufficient, weakening the standard desideratum for monetary policy rules.

Finally, let me remark that all chapters of this Ph.D. thesis are written as independent essays. Each chapter contains its own introduction and appendices that provide supplementary materials such as additional graphs and tables as well as data sources. Hence, the essays can be read in any order. References from all three chapters can be found in one bibliography at the end of this dissertation.

Chapter 2

Estimating Dynamic Adaptive Learning Models: Comparing Existing and New Approaches[1]

[1]Joint with Elena Rancoita.

2.1 Introduction

While the theory of adaptive learning has made significant progress in the last 20 years, most of its findings have not yet been tested against the data. One of the main reasons for this is that dynamic models under learning constitute relatively large non-linear system of equations which are complex and computationally costly to estimate. In this context, we compare three different full information likelihood based methodologies to estimate macroeconomic models under adaptive learning and study their relative performance in terms of bias, accuracy and computational cost. First, we consider the prevailing approach in the literature, which abstracts from all uncertainty in the learning dynamics - limiting the range of models it can estimate. Second, we consider the Smolyak Kalman Filter, a non-linear filter which considerably reduces the, generally prohibitive, effects of the curse of dimensionality.[2] And, finally, we consider a new strategy that we devise and that is based on the linearization of the learning expectations formation mechanism, which, similarly to the strategy followed by the first approach, allows to circumvent the problems arising in the estimation of non-linear dynamic adaptive learning models but that is applicable to a wider range of models.

To better understand the problems related to the estimation of dynamic adaptive learning models and explain the scarcity of empirical literature on learning, let us briefly describe what the adaptive learning (AL) hypothesis entails. AL assumes that agents form expectations using *subjective* probability distributions that do not coincide with the ones that emerge in equilibrium. These subjective probability distributions are generally embodied in reduced form models that agents are assumed to estimate in order to make forecasts. Then, agents learn in the sense that they periodically update these estimates in an attempt to discover the "true" value of the parameters of their forecasting models (equivalently, in their attempt to learn the "true" probability distributions). The implied dynamics is self-referential, insofar as agents' subjective probability distributions affect, through agents' expectations, the economic outcomes that are later going to be used to update those same initial distributions. Moreover, since these distributions are adjusted (or adapted) gradually, these dynamics can generate significant amount of persistence in these models. In comparison, rational expectations can be understood as imposing the additional requirement that agents' subjective probability distributions coincide with the objective ones that emerge in equilibrium; thus solving for a

[2]The curse of dimensionality refers to the phenomenon in which the computational costs associated to the utilization or estimation of a model grow exponentially with the number of variables in the model.

fixed point and removing agents' incentives to revise their beliefs.[3]

The difficulties faced when estimating a learning model arise precisely from the non-linearities of the agents' reduced form forecasting models and of their evolution. Learning models, like RE models, are usually solved upon a first order (log-) linearization conditional on the expectations formation mechanism. Moreover, in line with the resulting form of the equilibrium of their rational expectations counterparts, learning models assume that agents' reduced form forecasting models are linear. However, as already mentioned, the coefficients of these forecasting models are estimated by the agents and, thus, are functions of previous estimates and other unobservable states in the model. Therefore, they become unobservable state variables themselves. This creates two potential sources of non-linearities in learning models. First, agents' estimates of these coefficients might multiply other unobservable state variables of the model.[4] Second, estimates usually are updated by means of a non-linear function of other states in the model, e.g. using an ordinary least squares estimator. Therefore, while under RE the whole model is linear and its likelihood can be easily computed with the Kalman Filter, under learning the model becomes non-linear and for the computation of the likelihood one generally has to resort to non-linear filters. The problem with non-linear filters is that they suffer from the curse of dimensionality. A problem that quickly turns to be computationally prohibitive and that is further exacerbated under learning as the state space expands to accommodate agents' beliefs.

We employ a simple learning version of the Cobweb model to evaluate the relative performance, in terms of bias, accuracy and computational cost, of existing and new estimation approaches.

The prevailing approach for the estimation of dynamic learning models can be found in Milani (2005, 2007) and Slobodyan and Wouters (2012, 2012). All these papers feature non-rational expectations formation mechanisms, which are modeled with non-linear learning updating rules. Nonetheless, they all compute the likelihood with the Kalman Filter. This is possible because of the strong implicit assumption that beliefs and their evolution is certain. In other words, all uncertainty in beliefs formation is neglected. This has two implications: first, it means that the modeler has a point prior on the postulated form of the beliefs updating equations; and second, in the case where beliefs are conditioned on uncertain states, that these

[3] In the learning literature, the term beliefs is often used interchangeably with the term estimates, though, to be precise, beliefs refer to agents' probability distributions from which estimates are constructed.

[4] An example when this does not occur is when agents are only estimating a constant.

can be approximated by their posterior mean. In this way, conditional on parameters, initial beliefs and the expectations formation mechanism, the model is linear and its likelihood can be computed with the Kalman Filter. Put in another way, by abstracting from all uncertainty in the expectations formation mechanism, beliefs evolve as time-varying parameters. Henceforth, we will refer to this approach as the Milani, Slobodyan and Wouters (MSW) method.

Neglecting the uncertainty of the expectation formation mechanism is a very strong assumption and although it facilitates the estimation of the model, it can imply very poor estimates and forecasts. Indeed, the estimation of unobservable states variables entails a large degree of uncertainty that needs to be modeled and estimated. On the one hand, one should consider the econometrician's uncertainty about the initial beliefs and the other unobservable states. On the other hand, one should also take into account the econometrician's uncertainty about the beliefs' evolution, by allowing for measurement errors or shocks in the learning rules.

The two new estimation approaches that we consider are able to overcome the limitations of the MSW method and are suitable to be applied to a wider range of models, though they rely on approximations of their own. The first method, which we devise, is based upon the linearization of the expectations formation mechanism. The justification for this approach is twofold. First, it allows us to have a linear model whose likelihood can be easily computed with the Kalman Filter, while still being able to accommodate uncertainty in the learning part of the model. And second, learning models already neglect non-linearities as they are generally solved upon a (log-) linearization. We argue that there is no clear criteria by which certain non-linearities should be considered while others discarded. The linearization of the learning dynamics affects agent beliefs' evolution and this might affect the dynamic of the model and eventually its convergence. We show that if the linearization is done around the associated Rational Expectations Equilibrium (REE), the resulting model converges to it under similar conditions as its original non-linear version.

The second method that we consider is the Smolyak Kalman Filter (Winschel and Krätzig (2010)), a developed filter based on the Quadrature Kalman Filter that instead of the tensor product builds upon the Smolyak operator (Smolyak (1963)). This approach is again non-optimal, as it is based on the approximation of the integrals required in the predictive and filtering steps of the filter and on the approximation of any non-Gaussian densities by Gaussian-sums. Yet the approach allows us to consider uncertainty in the learning updating equations while keeping non-linearities. The advantage of this filter with respect to other non-linear filters, such as Particle Filters and Quadrature Sum Filters, which are more frequently found in the literature, is that it is considerably less affected by the curse of dimensionality. This makes it an appealing non-linear filter for learning models.

We base our analysis on simulated data and on three main "exercises" aimed at understanding the relative performance of the three estimation methods in terms of bias, accuracy and computational cost. From our simulations, it turns out that the cost of linearizing the model around the REE is very small and that both alternative methods perform better than the MSW approach in terms of bias in most of the cases. Most importantly, when exogenous unobservable state variables are included in the model the MSW produces estimates of the beliefs which are negatively correlated with the true beliefs' process. In this case, the largest difference in the performance of the three methods can be observed and in particular between the MSW and the linearized approach. In addition, the linear approach is considerably faster than the Smolyak Kalman Filter and, therefore, more promising for the estimation of medium/large-scale DSGE models.

To fix ideas and present our estimation methods we will consider constant-gain learning, as it is one of the most popular ways of modeling adaptive learning. Nevertheless, the methods can also be applied to more general types of adaptive learning, such as Bayesian learning or Least Squares learning. The paper is organized as follows. In section 2.2 we review the most relevant theoretical and empirical results in the adaptive learning literature. In section 2.3, we introduce a simple version of the Cobweb model with constant gain learning that we will use throughout the paper. In section 2.4 we provide the detailed description of the three estimation methods that we want to compare. In section 2.5, we measure the relative advantages and disadvantages of all three approaches by means of three estimation exercises. In section 2.6 we conclude.

2.2 Literature Review

In this section, we review some of the most relevant results of the theoretical and empirical literature on adaptive learning. The significant difference in the development of the theoretical and empirical literature underlines the need of an efficient method to estimate these models.

Adaptive learning has been applied to many diverse problems including monetary policy design, hyperinflation and deflation dynamics, the study of asset pricing stylized facts and business cycle fluctuations. Orphanides and Williams (2005) find that the design of monetary policy should take into account its effect on agents' expectations formation. In particular, tight inflation control and the communication of the policy target might help prevent the costs of imperfect knowledge. Bullard and Mitra (2002) show how the effectiveness of monetary policy is sensitive to the manner in which agents form expectations, suggesting that monetary policy authorities should focus only on policies which induce a 'learnable' rational expectations equilibria. Evans and Honkapohja (2003, 2003) study optimal monetary policy rules under discretion and commitment in the context of adaptive learning and challenge the results found under RE.

Williams (2004), Eusepi and Preston (2008) and Huang, Liu and Zha (2009), among others, are examples of the implementation of adaptive learning in business cycle models. Williams (2004) considers the quantitative importance of different types of learning on the equilibrium volatility and persistence of economic variables in business cycle models, such as consumption, GDP and inflation. He finds that when agents learn on the structure of the economy the amplification and propagation of economic shocks become much larger than when they learn on the parameters of the reduced form solution. Eusepi and Preston (2008) show that business cycle fluctuations can become self-fulfilling in the presence of learning and that optimistic or pessimistic beliefs have an impact on the marginal rate of substitution between different variables, increasing the equilibrium volatility of macroeconomic variables. Huang, Liu and Zha (2009) find that introducing learning in a real business cycle model reduces the wealth effect of a neutral technology shock, and increases the substitution effect.

The introduction of learning in asset pricing models has also yielded promising results. Timmermann (1996) showed that learning could generate excess volatility in asset prices. While, Adam, Marcet and Nicolini (2008) showed how, in the context of a standard consumption based asset pricing model, learning can generate realistic amounts of stock price volatility and can quantitatively account for the observed volatility of returns, the volatility and persistence of the price-dividend ratio and the predictability of lon-horizon returns.

Despite all this important theoretical evidence, there are still only few examples actually estimating learning models. Among the most relevant there is Sargent, Williams and Zha (2006), Milani (2004, 2007) and Slobodyan and Wouters (2012, 2012). In particular, Milani (2004, 2007) estimates a small-scale monetary DSGE model under adaptive learning that features habit formation in consumption and inflation indexation. He finds that, differently than under rational expectations, the estimated degrees of habit formation and inflation indexation are reduce to almost zero, showing that learning might be an important factor behind data persistence. Slobodyan and Wouters (2012, 2012) construct and estimate learning versions of the Smets and Wouters (2007) model. They do not only find that these models overcome some of the shortcomings of their rational expectations counterparts as indicated by the DSGE-VAR methodology for identifying misspecifications (see Del Negro, Schorfheide, Smets and Wouters (2007)), but that they also significantly improve the model's fit to the data.

2.3 Model: a Simple Case of the Cobweb Model

In this section, we introduce a simple version of the Cobweb model through which we illustrate the different sources of non-linearities in learning models and that we use throughout the paper to study the different performance of the three estimation methods that we consider.[5]

As discussed in the introduction, learning models differ from RE ones in the way agents form their expectations. For this reason and to better disentangle the different sources of non-linearities, we first present the model using a generic expectations operator, that we indicate with $E^{\star}$.

The Cobweb model describes the equilibrium on a competitive goods market as the intersection between a demand and a supply, which we define as follows:

$$\begin{aligned} d_t &= np_t + v_t^d && (2.1) \\ s_t &= mE_{t-1}^{\star}[p_t] + rx_{t-1} + v_t^s && (2.2) \\ x_t &= \mu + \rho x_{t-1} + u_t && (2.3) \end{aligned}$$

where $r,\ \mu \in \mathbb{R}$, $n < 0$ and $m > 0$. The first equation defines the demand, d_t, as a negative function of current prices. Eq.(2.2), defines the supply, s_t, as a positive function of the expected current price, conditional on information up to the previous period, and on $x_{t-1} \in \mathbb{R}$, an exogenous random variable (e.g. input costs or some economic slack indicator). One can think of this set up as depicting a situation in which production materializes one period after firms make their production decisions. We further assume, that agents can observe x_{t-1} but the econometrician estimating the system may not. This assumption is quite reasonable in that firms deciding how much to produce, have probably better information on the factors

[5] We borrow this example from Evans and Honkapohja (2001)

affecting their production in their sector than an econometrician looking at the aggregate market. Additionally, this allows us to construct simple exercises for studying the properties of the estimation methods. The variable x_t follows a simple stationary AR(1) process, $\mid \rho \mid < 1$, with unconditional variance $\frac{\sigma_u^2}{1-\rho^2}$. And, finally, v_t^d and v_t^s are unobserved random shocks with means zero and variances σ_{vd}^2 and σ_{vs}^2, respectively.

The equilibrium of the model is given by the intersection of the demand and the supply and it summarizes the determinants of prices: firms' previous period expectations of current prices and the exogenous variable, x_{t-1}, i.e.

$$p_t = \alpha E_{t-1}^{\star}\left[p_t\right] + \beta x_{t-1} + w_t^p, \tag{2.4}$$

where $w_t^p = \frac{v_t^s - v_t^d}{n}$, $\beta = \frac{r}{n}$ and $\alpha = \frac{m}{n} < 0$.

Note that equation (2.4) is linear conditional on expectations. As we will see, this model is linear under rational expectations, but becomes non-linear under adaptive learning.Then, before deriving the equilibrium of the model under AL, let us solve the model under RE.

The assumption of rational expectations implies that agents form expectations using the equilibrium probability distributions. In this simple model, this entails that agents' subjective distributions, used in $E^{\star}$, coincide with the distribution defined by eq.(2.4). Applying the expectations operator to both sides of equation (2.4), one can easily solve for agents' price expectations under rational expectations, yielding

$$E_{t-1}^{RE}\left[p_t\right] = \frac{\beta}{1-\alpha} x_{t-1}. \tag{2.5}$$

Then, by substituting (2.5) in (2.4) we obtain the rational expectations equilibrium of the model as a function of the exogenous variable, x_{t-1}, and the shock, w_t^p:

$$p_t = \alpha \frac{\beta}{1-\alpha} x_{t-1} + \beta x_{t-1} + w_t^p = \frac{\beta}{1-\alpha} x_{t-1} + w_t^p \tag{2.6}$$

From eq. (2.6) we can observe that the distribution of the equilibrium price is the same that the agents used to form their expectations.

Under adaptive learning, agents are assumed to form expectations using reduced form models and to periodically estimate these models as new information becomes available. Furthermore, given that the learning literature is generally interested in small deviations from RE, agents are usually assumed to know the correct functional form of the associated rational expectations equilibrium and to estimate some of its parameters or coefficients. Even though this is not necessary, we will keep this assumption here.[6] In the case of our Cobweb model, this implies that agents do not know how prices are exactly formed in equilibrium, i.e. eq.(2.6), but that they do know that prices depend linearly on the exogenous variable x_{t-1}. In other words, agents in our model are assumed to form expectation using the following simple model,

$$p_t = a_{t-1} + b_{t-1}x_{t-1} + w_t^p, \tag{2.7}$$

where a_{t-1} and b_{t-1} are estimated from historical data and, thus, might not coincide with their REE values, i.e. 0 and $\frac{\beta}{1-\alpha}$, respectively.[7] In the adaptive learning literature, eq. (2.7) is often referred to as the *Perceived Law of Motion* (PLM), as it depicts agents' perception of the law of motion of the variables that agents forecast.

Then, if agents form expectations using (2.7), these will be given by

$$E_{t-1}^{AL}\left[p_t\right] = a_{t-1} + b_{t-1}x_{t-1} \tag{2.8}$$

and the implied actual price realization will be given by

$$p_t = \alpha a_{t-1} + \left(\alpha b_{t-1} + \beta\right) x_{t-1} + w_t^p. \tag{2.9}$$

Equation (2.9) is known as the *Actual Law of Motion* (ALM). Only when

$$\left(a_{t-1} = \alpha a_{t-1} \Leftrightarrow a_{t-1} = 0\right) \quad \wedge \quad \left(b_{t-1} = \alpha b_{t-1} + \beta \Leftrightarrow b_{t-1} = \frac{\beta}{1-\alpha}\right) \tag{2.10}$$

the *perceived* and the *actual* law of motions coincide, subjective and objective distributions equate and the REE realizes.[8]

From the results in (2.10), we can observe that under the assumption of AL the coefficients of the equilibrium price equation are unobservable time varying state variables, while under RE they are constant.

[6] One can also depart from under or overparametrizations of the equilibrium law of motion or even from non-nested forms. For a detail study of adaptive learning see Evans and Honkapohja 2001(22).

[7] Notice that we included a constant in the reduced form forecasting models of agents and that both parameters' estimates, a_{t-1} and b_{t-1}, are indexed by time. These indices indicate that the estimates in period t are condition on information up to period $t-1$.

[8] For completeness, let us define the function that maps the parameters of the PLM into the parameters of the ALM: $T(a,\ b) = (\alpha a, \alpha b + \beta)$. This mapping is called the T-map. The study of the stability properties of a learning model can, in many cases, be reduced to the study of the properties of its T-map.

To complete the description of the model under adaptive learning, we still need to define how agents periodically estimate the parameters of their forecasting models, a_{t-1} and b_{t-1}. Following much of the literature, we will assume that agents do this by means of constant-gain learning. This learning scheme is one of the most popular ways of modeling agents learning behavior and, for our Cobweb example, can be written in a recursive manner as,

$$\theta_{t-1} = \theta_{t-2} + \gamma R_{t-1}^{-1} X_{t-2}' \left(p_{t-1} - X_{t-2}\theta_{t-2}\right) \tag{2.11}$$

$$R_{t-1} = R_{t-2} + \gamma \left(X_{t-2}' X_{t-2} - R_{t-2} \right) \tag{2.12}$$

where $\theta_{t-1} = [a_{t-1},\ b_{t-1}]^T$, $X_{t-2} = [1,\ x_{t-2}]$, R_{t-1} is an estimate of the second moments of X_{t-2}, and γ is a small positive number. We will refer to θ_t, as well as to a_{t-1}, b_{t-1} and R_{t-1}, as agents' *beliefs*, and to equations (2.11) and (2.12) as the *learning rules*.

In each period, as new information becomes available, agents update their estimates of the coefficients of their forecasting models, θ_{t-1}, according to (2.11) and (2.12). In particular, current beliefs, θ_{t-1}, equal previous beliefs plus a correction term that depends on the last forecast error. These learning rules can be thought of as a deviation from Ordinary Least Squares. Equations (2.11) and (2.12) are nothing else than the recursive representation of the Ordinary Least Squares estimator where the forecast errors no longer have an equal weight, but an exponentially decreasing one as they become older.[9]

Then, the expectations formation mechanism under adaptive learning for the simple Cobweb model discussed in this paper is given by equations (2.8), (2.11) and (2.12). The Cobweb model, which now embeds also the learning rules, has now new unobservable states: a_{t-1}, b_{t-1} and R_{t-1}. Furthermore, as a consequence of these new states, the model has now become non-linear: first, expectations are non-linear, as they entail the product of two states, $b_{t-1}x_{t-1}$;[10] and second, these new states are non-linear functions of other states in the model.[11]

Finally, as mentioned before, we want to allow for a new source of uncertainty in the expectations formation mechanism. This uncertainty could alternatively be understood as modeling a measurement error, capturing the ignorance of the economist about how the model fits the actual behavior of agents or as a shock to beliefs, capturing other information used by agents to condition their beliefs, such as sentiment or other psychological factors.

[9]To retrieve the original Ordinary Least Squares estimator, one just needs to replace γ by t^{-1}.

[10]To be more precise, expectations are non-linear from the economist perspective, though they continue to be linear from the agent's perspective. But, since we are addressing the estimation of the model we are precisely interested in the economist's point of view.

[11]See equations (2.8), (2.11) and (2.12); non-linearities are marked in red.

The whole model including these latter uncertainty shocks, that we denote by w_t^{ab}, takes the following form:

$$
\begin{aligned}
p_t &= \alpha a_{t-1} + (\alpha b_{t-1} + \beta)\, x_{t-1} + w_t^p && (2.13)\\
x_{t-1} &= \mu + \rho x_{t-2} + u_{t-1} && (2.14)\\
\theta_{t-1} &= \theta_{t-2} + \gamma R_{t-1}^{-1} X_{t-2}' \left(p_{t-1} - X_{t-2}\theta_{t-2}\right) + w_t^{ab} && (2.15)\\
R_{t-1} &= R_{t-2} + \gamma \left(X_{t-2}' X_{t-2} - R_{t-2}\right) && (2.16)\\
& a_0,\ x_0,\ R_0 \ given && (2.17)
\end{aligned}
$$

where $w_t^{ab} = \begin{bmatrix} w_t^a \\ w_t^b \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_{w^a} & 0 \\ 0 & \sigma_{w^b} \end{bmatrix}\right)$.[12] The model defined by (2.13)-(2.17) constitutes a state space model, where, if we assume prices to be observable, eq.(2.13) is the so called *measurement or observable equation* and equations (2.14)-(2.17) are the *state equations*.

2.4 Estimation Methods

As shown in the previous section, learning models constitute non-linear state space models which means that their estimation with Bayesian methods is a complex task. The problem lies in the computation of the likelihood, which requires to keep track of the states' distributions in the presence of non-linearities. The complication is precisely that the non-linearities of the system make it virtually impossible to analytically derive the posterior distribution of the unobserved states. Therefore, one needs to resort to techniques based on approximations which are usually computationally costly.

In economics, one of the most popular of such methods is the Particle Filter (for a short review of the most important methods, see Arulampalam, Maskell, Gordon and Clapp (2002), Arasaratnam, Haykin and Elliot (2007), Gustafsson, Gunnarsson, Bergman, Forssell, Jansson and Karlsson (2002)). However, as most non-linear filters, it suffers form the curse of dimensionality, which means that its computational costs grow exponentially with the number of states that need to be estimated. Moreover, these costs become prohibitive for most DSGE

[12] Clearly we could introduce a measurement error in the equation describing the law of motion of R_t. However, we omit it for simplicity.

learning models: for example, learning versions of the Smets and Wouters (2003, 2007) model or the New-Area Wide Model of Christoffel, Coenen and Warne (2008).

In this section we present the three estimation approaches that we will examine in the rest of the paper and that address this computational problem. First, we consider the prevailing estimation approach in the literature, used by Milani (2004, 2007) and Slobodyan and Wouters (2012, 2012). Their approach circumvents the curse of dimensionality problem by neglecting all uncertainty in the non-linear parts of the model, rendering it the facto linear and so, circumventing the problem of having to resort to non-linear filters. Next, as a second method, we consider the Smolyak Kalman Filter (Winschel and Krätzig (2010)), a non-linear filter that significantly reduces the curse of dimensionality relative to the Particle Filter. Even though it is the most apt non-linear filter, also the Smolyak Kalman Filter becomes computationally too costly when dealing with medium and large size systems. Finally, we compare the above mentioned estimation techniques with a new approach that we devise and that is suitable for the estimation of medium and large scale DSGE models, without loosing in precision and speed. Our approach is based on the linearization of the expectations formation mechanism which transforms the model into a fully linear system.

In what follows, we briefly describe all three methods before discussing their empirical performance when applying them to the Cobweb model in the remaining sections.

2.4.1 The Literature Approach (MSW Approach)

The method used by Milani and Smets and Wouters, and to which we will refer to as the MSW approach, consists in abstracting from all uncertainty in the expectations formation mechanism. Which, as we have shown in the previous section, is the only source of non-linearities in the model. By doing so, the beliefs evolve deterministically and, in the estimation, they can be thought of as time-varying parameters.

To see this more clearly, let us briefly consider our simple Cobweb Model defined by equations (2.13)-(2.17). Assuming that we observe x_t and w_t^{ab} and that we know θ_0 and R_0 with certainty, each period we can use equations (2.15-2.16) to recursively compute agents' beliefs. Then, a_{t-1} and b_{t-1} are known variables, implying that the actual law of motion is linear. And, the likelihood can be computed with the standard Kalman Filter. The advantage of this method is that it is very fast and simple. Furthermore, under these demanding conditions, this approach is an optimal way of computing the likelihood.

However, in most interesting learning models these assumptions do not hold. Beliefs are usually conditioned on unobservable states and initial beliefs, as well as on shocks which are generally not known.[13] Under these conditions, the approach relies on strong approximations. First, unobservable state variables that enter the learning rules of the model are approximated by their means. Second, there is no room for unobservable shocks to beliefs that might capture important determinants of agents' behavior. And third, the approach does not allow the econometrician to use data in order to improve her inference about agents' beliefs. In particular, the last two points amount to having a mass one prior on the form of the learning rules.

This strategy delivers a very simple and practical method at the cost of not estimating agents' beliefs and their distributions, an important part of model. As we will show, the estimation method we devise, will also rely on the Kalman Filter to compute the likelihood. However, instead of abstracting from uncertainty it abstracts form non-linearities.

2.4.2 Smolyak Kalman Filter (SKF Approach)

Ideally, one would like to have a Bayesian estimation method which is fast and able to optimally estimate non-linear dynamic state space models. However, when it comes to these types of models we are forced to resort to sub-optimal non-linear filters which usually suffer from the curse of dimensionality. As mentioned above, the MSW approach circumvents the problem of depending on non-linear filters by abstracting from the uncertainty in the expectations formation mechanism, while the method we devise circumvents the problem by abstracting from non-linearities. Therefore, we would like to be able to compare both those methods with a third one that can take into account the non-linearities of the model and its uncertainty simultaneously. The most popular estimation method able to deal with non-linear state space models is the Particle Filter (for an overview see Arulampalam, Maskell, Gordon and Clapp (2002) and Gustafsson, Gunnarsson, Bergman, Forssell, Jansson and Karlsson (2002)). But this filter suffers significantly from the course of dimensionality. Therefore we consider a faster filter, the Smolyak Kalman Filter (SKF), which potentially could be also applied for the estimation of medium-scale DSGE models with adaptive learning.

The idea behind the SKF is similar to the more popular Quadrature Kalman Filter as they

[13]For example, in the Smets and Wouters' new Keynesian Model agents need to construct forecast of the future value of capital and of its return, which are unobservable states to the economist.

are both based on the evaluation of the joint (multidimensional) density of the state variables only on some grid points at each iteration of the filter. The main difference between the Smolyak and the Quadrature Kalman Filters is how they construct the multidimensional grid needed for the computation of the joint density and that starts from the one-dimensional sparse-grid defining the domain of each state variable. Instead of using the usual tensor product, the SKF is based on the Smolyak operator (Smolyak (1963)) which provides as good as an approximation with far fewer points. For a detail study of this filter see Winschel and Krätzig (2010), Kotecha and Djuric (2003) and Arasaratnam, Haykin and Elliot (2007) and the references therein.

A critical assumption of the SKF is that the states' posterior distribution is approximated with a Gaussian distribution. However, in most non-linear model, the states' posterior is not a Gaussian distribution. The Smolyak Sum Filter, which is based on the SKF, overcomes this problem by approximating all non-Gaussian states' posterior distributions with a Gaussian mixture. For this reason, the Smolyak Sum Filter is well suited for the estimations of non-linear state space models and we would like ideally to use this method. However, the Smolyak Sum Filter is much more computationally costly than the Smolyak Kalman Filter to the point that we decided not to use it.[14] In addition, in two of the three exercises described below we assume that the process x_t is observable also to the econometrician, implying that the state variables' posterior distributions are Gaussian. For this reason using the SKF instead of the Smolyak Sum Filter should not imply large estimation errors in the following simulation exercises.

Another setback of the SKF is that at each step the filter needs to factorize an estimated co-variance matrix which, due to computer accuracy, tends to loose its positive definite property. For a discussion on the problem see Arasaratnam, Haykin and Elliot (2007).

In theory, using a non-linear filter, one can treat the deep parameters of the model as state variables, without the need of a two steps estimation (i.e. a filter for the likelihood conditional on parameters, and a Metropolis-Hastings algorithm on top to estimate the parameters). This implies a fully Bayesian approach to the estimation of the whole model. Coming back to our simple Cobweb model, the model including the state equations for the deep parameters

[14] As we will explain later, we conducted three Monte Carlo exercises to compare the three methods and the Smolyak Sum Filter would have taken more than one day to do only one simulation of the Monte Carlo.

estimation can be written as

$$
\begin{aligned}
p_t &= \alpha a_{t-1} + (\alpha b_{t-1} + \beta)\, x_{t-1} + w_t^p && (2.18)\\
x_{t-1} &= \rho x_{t-2} + u_{t-1} && (2.19)\\
\theta_{t-1} &= \theta_{t-2} + \gamma R_{t-1}^{-1} X_{t-2}' \left(p_{t-1} - X_{t-2}\theta_{t-2}\right) + w_t^{ab} && (2.20)\\
R_{t-1} &= R_{t-2} + \gamma \left(X_{t-2}' X_{t-2} - R_{t-2} \right) && (2.21)\\
\alpha_t &= \alpha_{t-1} && (2.22)\\
\beta_t &= \beta_{t-1} && (2.23)\\
\rho_t &= \rho_{t-1} && (2.24)\\
\gamma_t &= \gamma_{t-1} && (2.25)\\
\sigma_{p,t} &= \sigma_{p,t-1} && (2.26)\\
\theta_0, \alpha_0, \beta_0, \rho_0, \gamma_0, \sigma_{p,0} & \quad given && (2.27)
\end{aligned}
$$

where parameters are modeled as dynamic constants. However, in the exercises below, we are interested in comparing the ability of the three different methods in dealing with non-linearities and not interested in comparing the results from using the Metropolis Hastings versus other Bayesian methods (like the SSF). Therefore, we will use the three approaches to estimate only equations (2.18)-(2.21).

2.4.3 Linearization of the Learning Model (LKF Approach)

The last approach that we consider is a new approach that we devise and that consists of a linearization of the whole learning model. There are at least two reasons why one may want to do this. First, this method is particularly simple and fast. The linearized model can be estimated by computing its likelihood with the Kalman Filter and then generating the posterior distribution of the parameters with a Metropolis-Hastings algorithm.[15] Second, as we discussed before, most economic models used in the applied literature relay on (log-) linearized structural equations. The beliefs updating rules and the way beliefs enter expectations are thus the only source of non-linearity in the model. Given that it is unclear why one should keep certain non-linearities while neglecting others, we propose to use a fully linearized

[15] We want to remain circumscribed to Bayesian estimation approaches, otherwise one could, for example, use Maximum Likelihood.

system.

The estimation approach is then based on a first order linearization of the learning model at hand. For the simple case of our Cobweb model (2.13)-(2.17) a first order linearization around a generic point, $\left\{\overline{a},\ \bar{b},\ \overline{R},\ \overline{x},\ \overline{p},\ \overline{w^p},\ \overline{w^{ab}}\right\}$, yields the following system of equations,

$$\begin{aligned} p_t &= \overline{p}+\alpha\left(a_{t-1}-\overline{a}\right)+\left(\alpha\overline{b}+\beta\right)\left(x_{t-1}-\overline{x}\right) \\ &\quad +\alpha\overline{x}\left(b_{t-1}-\overline{b}\right)+\left(w_t^p-\overline{w^p}\right) \end{aligned} \tag{2.28}$$

$$x_{t-1} = \rho x_{t-2}+u_{t-1} \tag{2.29}$$

$$\begin{aligned} \theta_{t-1} &= \overline{\theta}+\left(\theta_{t-2}-\overline{\theta}\right)+\gamma\overline{R}^{-1}\overline{X}'\left[p_{t-1}-\overline{p}-a_{t-2}+\overline{a}-\overline{x}\left(b_{t-2}-\overline{b}\right)\right. \\ &\quad \left.-\overline{b}\left(x_{t-2}-\overline{x}\right)\right]+\gamma\left(R_{t-1}^{-1}-\overline{R}^{-1}\right)\overline{X}'\left[\overline{p}-\overline{a}-\overline{b}\overline{x}\right] \\ &\quad +\gamma\overline{R}^{-1}\left(X_{t-2}-\overline{X}\right)'\left[\overline{p}-\overline{a}-\overline{b}\overline{x}\right]+\left(w_t^{ab}-\overline{w^{ab}}\right) \end{aligned} \tag{2.30}$$

$$R_{t-1} = \overline{R}+(1-\gamma)\left(R_{t-2}-\overline{R}\right)+2\gamma\bar{X}'\left(X_{t-2}-\overline{X}\right) \tag{2.31}$$

Looking at equation (2.30), we can observe that R_{t-1}^{-1} only appears multiplying the forecast error evaluated at the linearization point, i.e. $\overline{p}-\overline{a}-\overline{b}\overline{x}$. Therefore, by appropriately choosing a point around which to linearize we can significantly simplify the model. We consider then, the perfect foresight equilibrium associated to the model, i.e. $\left\{\overline{a},\ \bar{b},\ \overline{R},\ \overline{x},\ \overline{p},\ \overline{w^p},\ \overline{w^{ab}}\right\}$ $=\left\{0,\ \frac{\beta}{1-\alpha},\ \begin{pmatrix} 1 & 0 \\ 0 & \sigma_x^2 \end{pmatrix},\ 0,\ 0,\ 0,\ 0\right\}$.[16] This implies that the forecast error evaluated at the linearization point is zero in (2.30), which allows us to ignore the dynamics in (2.31). This is extremely convenient as R_{t-1} is generally a high-dimensional object that is hard to estimate. Also, variables in R_{t-1} have only second order effects on the economic outcomes. The system can then be re-written as:

$$p_t = \alpha a_{t-1}+\frac{\beta}{1-\alpha}x_{t-1}+w^p \tag{2.32}$$

$$x_t = \rho x_{t-1}+u_t \tag{2.33}$$

$$\theta_{t-1} = \theta_{t-2}+\gamma\overline{R}^{-1}\overline{X}'\left(p_{t-1}-a_{t-2}-\frac{\beta}{1-\alpha}x_{t-2}\right)+w_t^{ab}. \tag{2.34}$$

The resulting model (2.32)-(2.34) is linear conditional on the deep parameters of the model

[16] Notice that R_t is linearized around the theoretical second moments of X. If we were to linearize it around the perfect foresight equilibrium, it would not be invertible.

and its likelihood can be computed with the Kalman Filter.[17]

One question that remains open is the effect that the linearization has on the dynamics of the model. In particular, we would like to know whether this linearized version of the Cobweb model under AL (eq. (2.32)-(2.34)) has similar asymptotic dynamics to the ones of the non-linearized version (eq. (2.13)-(2.17)). More precisely, we would like to know what happens to θ_{t-1} and to the equilibrium price, p_t, as $t \to \infty$ in both versions of the model, and how they relate.

Assume that any REE of a model can be described as a reduced form model with parameter values θ^{ree}. Then, following Evans and Honkapohja (2001), we know that under constant gain learning θ_{t-1} can at most be expected to converge to a distribution around θ^{ree}. Moreover, they show that this convergence is mainly govern by the Expectational stability (E-stability) of the REE in question.[18] Therefore, we would like to know two things: first, the relation between the set of RE equilibria associated to each version of the model; and second, the relation between the respective conditions that make them E-stable.

We say that a REE is associated to a given model under AL if and only if it is an equilibrium of that model under RE. To determine the E-stability of a REE we need to determine the stability of the following differential equation,

$$\frac{d\theta}{d\tau} = T(\theta) - \theta \tag{2.35}$$

in a neighborhood of the associated θ^{ree}; where $T(\cdot)$ denotes the T-map of the model and τ denotes "notional" time.[19]

As we have already shown in the previous section, for the model (2.13)-(2.17), the T-map is defined by

$$T(a_{t-1},\ b_{t-1}) = (\alpha a_{t-1}, \alpha b_{t-1} + \beta) \tag{2.36}$$

and its unique associated REE is parametrized by $\theta^{ree} = (0, \frac{\beta}{1-\alpha})$.[20] This REE is E-stable if the eigenvalues of $D_\theta[T(\theta^{ree}) - \theta^{ree}]$ have real parts smaller than zero. This is satisfied if and

[17] Note that in this particular case, when linearizing $\overline{x} = 0$. For this reason the linearized learning rule for b_{t-1} becomes $b_{t-1} = b_{t-2}$ and therefore cannot be identified in the estimation. This will be mostly the case, as models are generally solved upon a log-linearization around the steady state and variables are defined as percentage deviations from it.

[18] Evans and Honkapohja (2001) find that the Expectational stability of a REE provides the main conditions required for the asymptotic stability (or "learnability") of that REE for a wide range of adaptive learning schemes.

[19] The T-map maps the parameters of the PLM to the parameters of the ALM in the model.

[20] Notice that REEs correspond to resting points of the differential equation (2.35) and, consequently, to fix points of the T-map.

only if $\alpha < 1$.

In turn, for the model (2.32)-(2.34), the T-map is defined as

$$T(a_{t-1},\ b_{t-1}) = (\alpha a_{t-1}, \frac{\beta}{1-\alpha}) \tag{2.37}$$

and its unique associated REE is parametrized by $\theta^{ree} = (0, \frac{\beta}{1-\alpha})$. Furthermore, this REE is E-stable if and only if $\alpha < 1$.

In this particular case, we have shown that both versions of the model have the same associated unique REE and that these equilibria are E-stable under the same condition ($\alpha < 1$). In what follows we present two propositions that generalize this result.

Consider the ALM of a generic constant-gain learning model:

$$y_t = T(\theta_{t-1})' \cdot z_t + e_t \tag{2.38}$$

where $y_t \in \mathbb{R}^{m\times 1}$ is a vector of endogenous variables, $z_t \in \mathbb{R}^{n\times 1}$ is a vector of exogenous variables and possibly the lags of some endogenous variables, $T(\cdot) \in \mathbb{R}^{n\times m} \to \mathbb{R}^{n\times m}$ is the T-map of the model, $\theta_{t-1} \in \mathbb{R}^{n\times m}$ is the vector of parameters of agents' PLM and e_t is white noise. Furthermore, let

$$\theta_t = \theta_{t-1} + \gamma R_t^{-1} z_t \left(y_t - z_t'\theta_{t-1}\right) \tag{2.39}$$

$$R_t = R_{t-1} + \gamma \left(z_t z_t' - R_{t-1}\right) \tag{2.40}$$

denote the associated learning updating equations. In addition, let $\mathcal{M}$ denote the model defined by equations (2.38)-(2.40) and let $\widetilde{\mathcal{M}}\left(\theta^{ree}\right)$ denote the linearization of $\mathcal{M}$ around the perfect foresight equilibrium associated to θ^{ree}.[21] Without loss of generality let us assume that $m = 1$.

Proposition 2.1. *If θ^{ree} is the vector of parameter values of a reduced form model describing a REE associated to $\mathcal{M}$, then θ^{ree} is the vector of parameter values of a reduced form model describing a REE associated to $\widetilde{\mathcal{M}}\left(\theta^{ree}\right)$. Furthermore, if $\frac{\partial T(\theta^{ree})'\cdot \bar{z}}{\partial \theta_{1,t-1}} \neq 1$ the inverse implication is also true.*[22]

[21] Again, R_t and R_{t-1} are linarized around the theoretical second moments of X.

[22] $\theta_{1,t-1}$ denotes here the first entry of the parameter vector θ_{t-1} and $\bar{z}$ the perfect foresight linearization point for z_t.

Proof. See the Appendix. $\square$

Proposition 2.2. *Let $\widetilde{T}(\cdot)$ denote the T-map of $\widetilde{\mathcal{M}}(\theta^{ree})$ and let θ^{ree} be the vector of parameter values of a reduced form model describing a REE associated to $\widetilde{\mathcal{M}}(\theta^{ree})$. Then, the REE associated to θ^{ree} is Exceptionally stable if and only if the real part of $\frac{\partial T(\theta^{ree})\bar{z}}{\partial \theta_{1,t-1}}$ is smaller than one.*

Proof. See the Appendix. $\square$

This result contrasts with the conditions required for a the E-stability of REE associated to (2.38)-(2.40). Namely, that the real parts of the eigenvalues of $D_\theta T(\theta^{ree})$ be smaller than one. In particular, when agents learn only about a constant, both conditions coincide.

In order to complete the proof for the convergence of θ_{t-1} to a distribution around a certain θ^{ree} see Theorem 7.9 in Evans and Honkapohja (2001). Given that the satisfaction of several conditions in the latter Theorem depend on the particular model considered we don't present results here. However, because of the linearization most conditions of the latter Theorem become significantly easier to check, and some can be proven to hold in general. We show this in the appendix.

2.5 Estimation Results

In this section we present three estimation exercises aimed at gaining insight on the relative performance of the different methods considered in the paper. As we discussed in the previous sections, all three methods are non-optimal, in the sense that they all rely on some type of approximation to compute the likelihood of the data conditional on the model and specific parameter values: the MSW estimation approach abstracts from all uncertainty in the expectations formation mechanism, the linearized approach is based on a first order approximation of the expectations formation mechanism and the SKF approximates the distributions of the states on some discrete "grid". In the following simulations, that use the Cobweb model (2.13)-(2.17) as a testing laboratory, we study the relative loss associated to each of the three methods.

Let us now describe how the exercises are constructed. In each exercise, a different specification of the Cobweb model (2.13)-(2.17) is assumed to be the true data generating process. Then, a Monte Carlo of 100 simulations is run to test the robustness of our results. The Monte Carlo is run over different combinations of the 'true' deep parameters used to generate the data and are randomly extracted from independent uniform distributions.[23] However, due to the computational costs, we limit our analysis to only a subset of the deep parameters (for example, in the first exercise α, β, and γ).[24] For each draw of the parameters, we use (2.13)-(2.17) to simulate data. Then, the likelihood of the generated data is computed using the three different estimation methods previously explained. Therefore, for each exercise we have three different estimations for each of the 100 initial Monte Carlo draws, corresponding to the three different methods. To make the exercises comparable, after computing the likelihood with the three different methods, we use in each case the Metropolis-Hastings algorithm to estimate the posterior distribution of the deep parameters of the model (see Chibb and Greenberg (1995)). The posterior distributions are then generated drawing at least 50000 times from a proposal density and until convergence is achieved.[25] Every time, at least the first 10% of the draws is discarded. The proposal density is set to a random walk with a variance proportional to the inverse of the Hessian of the posterior at the mode as in Geweke (1991, 1999). Furthermore, this variance is scaled to yield an acceptance ratio of about 0.34. The prior distributions of the deep parameters of the model are set equal across methods and Monte Carlo simulations. Even if a parameter is not included in the Monte Carlo simulation, if it is not assumed to be known to the econometrician, it is estimated, e.g. the standard deviation of the shock w_t^p. Finally, we work with a restricted sample of 200 observations in order to reproduce the estimates on a typical quarterly sample for US data of 50 years (e.g. Smets and Wouters (2007) or Slobodyan and Wouters (2012)).[26]

We compute several metrics in order to compare the three estimation techniques. First, we compare the methods on the basis of the Mean Squared Error (MSE) of both the deep parameters and the states' estimates and we decompose the MSE in bias and accuracy.[27]

[23]The supports of the uniform distribution cover a reasonable range of values for each parameter.

[24]The computations of the exercises as they are presented here took about one month using 4 computers.

[25]Convergence is checked using standard tests including the one proposed in Geweke (1999).

[26]We have further conducted the estimations using 4000 periods of simulated data, attempting to gain an idea of the asymptotic behavior of the estimators. Given the time costs of using such long data time series, we are not able to do the Monte Carlo exercise, and we restrict to a few parameter specifications for each exercise. The results do not significantly vary from the ones presented here.

[27]For any deep parameter of the model Y, we define the MSE as $\frac{1}{n}\sum_{i=1}^{n}\left(Y-\hat{Y}_i\right)^2$ where n is the number of Montecarlo repetitions and $\hat{Y}_i$ is the corresponding estimated value of the parameter. The bias is given

Then, we look at other measures like the Mean Absolute Percentage Error (MAPE) which rescales the error computed by the magnitude of the estimated variable.[28] We also look at the correlation between the actual realization of the states and their respective estimates, as in Geweke (1991, 1999), Fernandez-Villaverde and Rubio-Ramirez (2005), Milani (2004) among others. Finally, following Geweke's (1999), we compare the Marginal Likelihood of the different models. This concept allows us to compare two different models, even non-nested ones. In particular, we can use the marginal likelihood to compute the different models' posterior odds ratio, i.e.

$$\frac{p_{M_1,T}}{p_{M_2,T}} = \frac{p\left(Y^T \mid M_1\right)}{p\left(Y^T \mid M_2\right)} \cdot \frac{p_{M_1,0}}{p_{M_2,0}}$$

where M_1 and M_2 are two different models, $p_{i,0}$ stands for the models prior and $p_{i,1}$ for its posterior $(i = M_1,\ M_2)$. $p\left(Y^T \mid M_1\right)/p\left(Y^T \mid M_2\right)$ is the Bayes factor. Ratios larger than 1 would provide different degrees of evidence against model M_2. In all cases we assume that the prior distributions of the different models are the same.

2.5.1 Exercise I

The first exercise is constructed to provide some insight on the cost associated to the linearization of the LKF approach and to the approximations involved in the SKF. For this reason, we assume that no uncertainty enters the learning updating equation. The true data generating process is assumed to be given by,

$$\begin{aligned}
p_t &= \alpha a_{t-1} + (\alpha b_{t-1} + \beta) + w_t^p && (2.41)\\
x_{t-1} &= \rho x_{t-2} + u_{t-1} && (2.42)\\
\theta_{t-1} &= \theta_{t-2} + \gamma R_{t-1}^{-1} X_{t-2}\left(p_{t-1} - X_{t-2}\theta_{t-2}\right) && (2.43)\\
R_{t-1} &= R_{t-2} + \gamma\left(X_{t-2}^2 - R_{t-2}\right) && (2.44)\\
& a_0,\ x_0,\ R_0\ given
\end{aligned}$$

by $\frac{1}{n}\sum_{i=1}^{n}\left(Y - \hat{Y}_i\right)$ and the accuracy as $Var\left(\hat{Y}\right)$. For the unobserved states θ_t we employ a similar definition of MSE, bias and accuracy, with the only difference that n is given by the number of Monte Carlo replications times the sample time length.

[28] The MAPE of any estimated deep parameter is computed as $\frac{1}{n}\sum_{i=1}^{n}\left|\frac{Y-\hat{Y}_i}{Y}\right|$

where x_{t-1} is observable to the economist and to the agents and the true initial values of a_0, b_0 and R_0 are assumed to be known and set to their REE values. Under these assumptions, as mentioned before, the MSW approach is an optimal filter and a natural benchmark to study the losses associated with the other two methods. In the simulations we always set $\sigma_{w^p} = 0.1$ and we assume its prior distribution to have mean 0.1. Under these conditions, any difference in the estimations delivered by the other methods can be attributed to the approximations they respectively rest upon.

		MSW	LKF	SKF
α	MSE	0.0813	0.0797	0.0988
	Bias	0.1523	0.1490	0.2065
	Accuracy	0.0581	0.0575	0.0562
	MAPE	56.6%	55.4%	54.4%
β	MSE	0.0687	0.0680	0.0744
	Bias	-0.1349	-0.1341	-0.1605
	Accuracy	0.0505	0.0500	0.0486
	MAPE	18.3%	19.0%	18.0%
γ	MSE	0.0002	0.0002	0.0003
	Bias	-0.0041	-0.0046	-0.0104
	Accuracy	0.0002	0.0002	0.0002
	MAPE	40.5%	39.5%	48.6%

Table 2.1: Exercise I: Estimation of the deep parameters. MSE, bias, accuracy and MASE.

		MSW	LKF	SKF
	MSE	0.0000	0.0000	0.0000
a	Bias	0.0001	0.0001	0.0001
	Accuracy	0.0000	0.0000	0.0000
	$Corr(a,\ \hat{a})$	0.9698	0.9593	0.9610
	MSE	0.0006	-	0.0008
b	Bias	0.0016	-	-0.0015
	Accuracy	0.0005	-	0.0005
	$Corr(b,\ \hat{b})$	0.9630	-	0.9401

Table 2.2: Exercise I: Estimation of agents' beliefs. MSE, bias, accuracy and correlation with true beliefs.

	MSW	LKF	SKF
Average Time	1m 19s	3m 7s	26m 33s
Log-Marginal Likelihood	183.2034	182.9353	182.1206

Table 2.3: Exercise I: Average time and Log-Marginal Likelihood.

	LKF vs. MSW	SKF vs. MSW	LKF vs. SKF
Posterior Odds Ratios	0.7648	0.3386	2.2584

Table 2.4: Exercise I: Posterior Odds Ratios.

Table 2.1, 2.2, 2.3, and 2.4 show the estimation results of the first exercise. As we can see from Table 2.1, the MSE for all parameters and of all three methods are not very different although the LKF approach performs best. The LKF approach also delivers a lower bias for α and β, while the MSW does it for the gain parameters, though the differences are virtually negligible. The SKF is the most accurate method, follow by the LKF approach. The MSE of the estimates of the beliefs are very small (less than 10^{-4}) for all three methods as well. For this reason, it is difficult to say that one of the method performs best in terms of this metric (see Table 2.2). In Table 2.2, we can also observe the correlation between the true realization of agent's beliefs and the estimated ones. Both the estimated a_t and b_t are mostly correlated with the true beliefs when using the MSW method, even if the difference is very tiny between all three methods. The b_t coefficient cannot be estimated with the LKF as it cancels out in the linearization of the learning rules and it constitutes its main drawback. This occurs for the particular, though illustrative, setting we have chosen, as the mean of the process x_t equals zero.

The performance of all three methods does not give rise to large differences. In particular, the LKF approach does also not loose much with respect to the optimal MSW. One point to mention is the time required by each method. As expected, the SKF approach takes more time than the LKF one and, in turn, this one more than the MSW approach. This is partially explained by the fact that the SKF approach has to estimate three states, a_t, b_t and R_t, the LKF one, a_t, and the MSW none, as it computes them deterministically. Clearly, the magnitude of the loss in performance depends on the severity of the approximations and, for example, stronger non-linearities are to be expected to worsen the estimates of the LKF approach.

Table 2.4 shows the models' posterior odds ratios. According to the scale proposed by Jeffreys (1961)[29], even though the MSW approach dominates over the other ones, there is only a significant better performance of the LKF approach with respect to the SKF method. We conclude from this exercise that it does not seem to be a significant cost in the approximation of the LKF nor in the ones incurred by the SKF.

[29] Comparing two models, M_1 and M_2, following the suggestion of Jeffreys (1961) the interpretations of the Posterior odds are :

- $\frac{P_{M_1}}{P_{M_2}} < 1$ the null of M_2 is supported.
- $1 < \frac{P_{M_1}}{P_{M_2}} < 3.16$ some evidence against the null.
- $3.16 < \frac{P_{M_1}}{P_{M_2}} < 10$ substantial evidence against the null.
- $10 < \frac{P_{M_1}}{P_{M_2}} < 33.3$ strong evidence against the null.
- $33.3 < \frac{P_{M_1}}{P_{M_2}} < 100$ very strong evidence against the null.
- $100 < \frac{P_{M_1}}{P_{M_2}}$ decisive evidence against the null.

2.5.2 Exercise II

The second exercise is constructed to study the effect of ignoring the uncertainty in the learning updating rules rising from the x_t process when applying the MSW method. As discussed previously, any unobservable state (from the economist's perspective) entering the reduced form models agents use to construct forecasts, implies uncertainty in the knowledge that the economist has about the agents' beliefs. In the MSW, the learning rules are assumed to be deterministic functions and consequently these unobservable state variables are approximated with the mean of its last available probability distribution. This exercise aims at testing the cost of this assumption. For this reason, we consider the same model used in Exercise I as the true data generating process but we assume that the exogenous state x_t is now no-longer observable to the economist, i.e.

$$
\begin{aligned}
p_t &= \alpha a_{t-1} + (\alpha b_{t-1} + \beta)\, x_{t-1} + w^p && (2.45)\\
x_{t-1} &= \rho x_{t-2} + u_{t-1} && (2.46)\\
\theta_{t-1} &= \theta_{t-2} + \gamma R_{t-1}^{-1} X_{t-2} \left(p_{t-1} - X_{t-2}\theta_{t-2}\right) && (2.47)\\
R_{t-1} &= R_{t-2} + \gamma \left(X_{t-2}^2 - R_{t-2}\right) && (2.48)\\
& a_0,\ b_0,\ x_0,\ R_0,\ gven
\end{aligned}
$$

where the true initial values of $a_0,\ b_0$ and R_0 are again assumed to be known and set to their REE values. In the simulations we always set $\sigma_{wp} = \sigma_u = 0.1$ and we assume their prior distributions to have means 0.1. Under these conditions, we test the method only through the estimation of $\alpha,\ \beta,\ \gamma$ and ρ, as even though σ_{wp} and σ_u are also estimated they are not included in the Montecarlo exercise. In this case we expect the MSW to perform relatively worse than the other two methods, as it approximates x_{t-2} with its mean in the expectations formation mechanism.

		MSW	LKF	SKF
α	MSE	0.1247	0.0965	0.1178
	Bias	0.0892	0.0541	0.0763
	Accuracy	0.1167	0.0936	0.1120
	MAPE	61.7%	58.2%	60.2%
β	MSE	0.1005	0.0913	0.0973
	Bias	-0.259	-0.206	-0.247
	Accuracy	0.0334	0.0489	0.0363
	MAPE	22.1%	20.6%	21.4%
γ	MSE	0.0010	0.0002	0.0004
	Bias	-0.0064	-0.0055	-0.0073
	Accuracy	0.0010	0.0002	0.0003
	MAPE	41.5%	40.3%	40.7%
ρ	MSE	0.3571	0.1273	0.5255
	Bias	0.1432	0.0878	0.2271
	Accuracy	0.3366	0.1196	0.04739
	MAPE	37.4%	21.7%	43.3%

Table 2.5: Exercise II: Estimation of the deep parameters. MSE, bias, accuracy and MASE.

		MSW	LKF	SKF
a	MSE	0.0441	0.0002	0.0010
	Bias	0.0439	0.0001	0.0006
	Accuracy	0.0000	0.0000	0.0000
	$Corr(a,\ \hat{a})$	-0.5692	0.9355	0.8902
b	MSE	0.0445	-	0.0012
	Bias	0.0402	-	-0.0011
	Accuracy	0.0012	-	0.0006
	$Corr(b,\ \hat{b})$	-0.4923	-	0.8801

Table 2.6: Exercise II: Estimation of agents' beliefs. MSE, bias, accuracy and correlation with true beliefs.

	MSW	LKF	SKF
Average Time	9m 2s	9m 43s	120m 34s
Log-Marginal Likelihood	266.3682	325.7631	255.7329

Table 2.7: Exercise II: Average time and Log-Marginal Likelihood.

	LKF vs. MSW	SKF vs. MSW	LKF vs. SKF
Posterior Odds Ratios -log points-	59.39	-10.64	70.03

Table 2.8: Exercise II: Posterior Odds Ratios.

The results of the second exercise are illustrated in Tables 2.5, 2.6, 2.7, and 2.8. We can observe that the differences between the three estimation methods are larger than in Exercise I, in particular, between the MSW and the other two other approaches. The LKF method again delivers the smallest MSE when estimating the deep parameters of the model, particularly for the gain parameter (see Table 2.5). The LKF approach also delivers the estimates with smallest bias and and better accuracy (with the exception of β). With respect to the mean absolute percentage errors, they remain in the same range as in the first exercise for α, β, and γ, though it is significantly smaller for the estimate of ρ delivered by the LKF approach.

Looking at the estimation of the beliefs, Table 2.6, one can observe how the introduction of uncertainty in the expectations formation mechanism creates a serious problem for the MSW approach. Most importantly, the MSW approach is not able to capture the correct correlation between its beliefs' estimates and the true ones. In addition, and as expected, having an unobservable state entering the learning dynamics reduces the estimation performance of both the LKF and SKF approaches, though, they continue to present very good results.

In terms of computational cost, the MSW approach now requires about the same time as the LKF one. This is mainly consequence of the need of the MSW approach to compute the inverse of a matrix (i.e. of R_{t-1}) that the LKF approach avoids. In addition, we already start to see the effects of the curse of dimensionality, as the SKF takes about two hours to estimate one Montecarlo simulation.

The posterior odds shown in table 2.8 indicate a decisive better performance of the LKF with respect to the other two methods. Posterior odds ratios very large, indicating decisive evidence against the null hypothesis that the two models are the same.

2.5.3 Exercise III

In the third and final exercise, we examine how the different approaches deal with a second source of uncertainty in the learning updating rules, the one coming directly from a shock to a_{t-1}.[30] As previously discussed, such a shock would allow to model, for example, factors that the agents use to condition their beliefs upon and that are orthogonal to the economic information included by the economist in the expectations formation mechanism. For instance, these factors may capture mood swings, psychological components of beliefs or other aspects that affect agents views about the economy and are important to explain economic dynamics. Alternatively, they could also be interpreted as a measurement error, that captures the economist's uncertainty about the unobserved beliefs.

For this exercise we assume the following model as the true data generating process :

$$\begin{aligned}
p_t &= \alpha a_{t-1} + (\alpha b_{t-1} + \beta)\, x_{t-1} + w_t^p && (2.49)\\
x_{t-1} &= \rho x_{t-2} + u_{t-1} && (2.50)\\
\theta_{t-1} &= \theta_{t-2} + \gamma R_{t-1}^{-1} X_{t-2}\,(p_{t-1} - X_{t-2}\theta_{t-2}) + w_t^{ab} && (2.51)\\
R_{t-1} &= R_{t-2} + \gamma\left(X_{t-2}^2 - R_{t-2}\right) && (2.52)\\
& a_0,\ b_0,\ x_0,\ R_0 \ given
\end{aligned}$$

It is the same model used in Exercise I, except that now we assume that there is a shock, w_t^{ab}, that hits a_{t-1}. This shock is modeled as a white noise process. To isolate this source of uncertainty we assume, as in the first exercise, that x_{t-1} is observable to the economist (remember that x_{t-1} is always assumed to be observable to the agents). Also as before, we maintain the assumption that the true initial values of θ_0 and R_0 are known and set to their REE values. In addition, we set $\sigma_{w^p} = \sigma_{w^a} = 0.1$ and $\sigma_{w^b} = 0$ and we assume their prior distributions to have means 0.1. Under these conditions, we test the method only through the estimation of α, β and γ, as even though σ_{w^p} and σ_{w^a} are also estimated they are not

[30]We have omitted the case in which the slope of agents' forecasting models are subject to a shock.

included in the Montecarlo exercise. As it was the case in Exercise II, we expect the MSW approach to perform worse than the other two methods, as it cannot take into account the uncertainty surrounding the learning rules.

		MSW	LKF	SKF
α	MSE	0.0604	0.0647	0.0658
	Bias	0.1376	0.1864	0.1921
	Accuracy	0.0415	0.0299	0.0109
	MAPE	50.7%	46.3%	49.1%
β	MSE	0.0739	0.0789	0.0744
	Bias	-0.1785	-0.1837	-0.1911
	Accuracy	0.0420	0.0452	0.0379
	MAPE	22.0%	18.9%	19.1%
γ	MSE	0.0004	0.0001	0.0004
	Bias	-0.0165	0.0017	-0.0112
	Accuracy	0.0001	0.0001	0.0002
	MAPE	62.8%	43.1%	49%

Table 2.9: Exercise III: Estimation of the deep parameters. MSE, bias, accuracy and MASE.

		MSW	LKF	SKF
a	MSE	0.0032	0.0024	0.0022
	Bias	0.0028	0.0023	0.0021
	Accuracy	0.0000	0.0000	0.0000
	$Corr(a,\ \hat{a})$	-0.2757	0.9254	0.7122
b	MSE	0.0011	-	0.0009
	Bias	0.0014	-	-0.0006
	Accuracy	0.0008	-	0.0008
	$Corr(b,\ \hat{b})$	0.9533	-	0.6542

Table 2.10: Exercise III: Estimation of agents' beliefs. MSE, bias, accuracy and correlation with true beliefs.

	MSW	LKF	SKF
Average Time	4m 4s	4m 54s	46m 33s
Log-Marginal Likelihood	273.6524	383.8036	301.1335

Table 2.11: Exercise III: Average time and Log-Marginal Likelihood.

	LKF vs. MSW	SKF vs. MSW	LKF vs. SKF
Posterior Odds Ratios -log points-	110	27	82

Table 2.12: Exercise III: Posterior Odds Ratios.

The results of the third exercise are summarized in Tables 2.9, 2.10, 2.11, and 2.12. As it can be observed, all three approaches perform better in terms of the estimation of the deep parameters in this case than in Exercise II. Notwithstanding the relatively equal performance of all approaches in terms of the estimation of the deep parameters, the gain parameter is better estimated by the LKF approach. And also, in terms of the mean average percentage error, the LKF appears to perform better than the two other methods. In particular, for the gain parameter, γ, the relative error of the LKF is 43% while the MAPE of the SKF and the MSW are respectively 49% and 62%.

The mean squared errors computed for the beliefs, as well as their bias and accuracy, are also similar among the three cases. As in the previous exercise, the correlation between the true belief process for the constant in agents' forecasting models, a_t, and its estimate is very large for the LKF method. The SKF approach again delivers high correlations (also for agents' estimates of the forecasting model's slope, b_t), though smaller than in Exercise II. However, the MSW method has problems matching the path of a_t as the correlation between the estimated a_t and the true process is negative. On the contrary the correlation between the estimated b_t and the true process is close to one. In terms of average time needed, the results show that the SKF approach is dominated by the other two methods. The time needed from the SKF is on average 46 minutes while the other two methods take about 5 minutes.

Finally, the posterior odds ratios shows a clear difference among the three methods. The LKF approach delivers the best fit of the model to the data. While the SKF approach dominates the MSW one.

2.6 Concluding Remarks

We have compared three different approaches suitable for the estimations of dynamic adaptive learning models with adaptive learning. These models are important as they present an alternative way of modeling expectations that has shown considerable potential to explain several economic puzzles and match economic data. We compare the method which is used in the few existing empirical works on learning with the Smolyak Kalman Filter and with a new approach we propose based on the the linearization of the expectations formation mechanism under adaptive learning. These latter two methods have not yet been applied to learning models and we find that they perform particularly well in our simulations.

We show in a series of exercises how the Bayesian estimation method prevailing in the literature, and that also relies on the Kalman Filter, cannot address the uncertainty in the learning updating equations properly. Furthermore, we find that our method provides as good an estimation in the cases in which no uncertainty in the learning updating equations is present. This suggests that there is no significant cost of approximating the non linear parts introduced by learning in a DSGE model. To get an idea of how much these last two methods loose or gain by not resorting to the more involved and time demanding non-linear filters, we compare our approach to the Smolyak Kalman Filter, an exponent of the large set of filters suited for the estimation of non-linear Dynamic State Space Models. We choose this filter because it is the least affected by the curse of dimensionality, a problem that turns non-linear filters prohibitive for most DSGE models under learning. We find that our method yields better estimates than the SKF, especially in terms of bias, when uncertainty in the learning updating equations is present. While the SKF approach, provides on average more accurate estimates of the deep parameters of the model. Additionally, while the LKF approach appears to dominate when it comes to the estimation of agents' beliefs about the constant of their forecasting models, by construction it cannot estimate the corresponding beliefs on the slope of those models. To estimate agents' beliefs about the slope of their forecasting models, the SKF approach appears to be the better option. However, the computational costs associated to the SKF approach are significantly larger than of the other methods, standing as a serious drawback of the method. Finally, using the marginal data of the density to compare the different approaches, we find that our method delivers a better fit to the data than the other two methods when uncertainty is present in the expectations formation mechanism. We argue that this is the most common and interesting case in macro learning models.

2.7 Appendix

In order to prove Proposition 2.1 and 2.2, we first need to linearize the generic learning model given by,

$$y_t = T(\theta_{t-1})' \cdot z_t + e_t \tag{2.53}$$

$$\theta_t = \theta_{t-1} + \gamma R_t^{-1} z_t \left(y_t - z_t' \theta_{t-1}\right) \tag{2.54}$$

$$R_t = R_{t-1} + \gamma \left(z_t z_t' - R_{t-1}\right) \tag{2.55}$$

around the perfect foresight equilibrium, $\left\{\bar{y}, \bar{z}, \bar{\theta}, \bar{R}\right\}$;[31] where $y_t \in \mathbb{R}^{m\times 1}$ is a vector of endogenous variables, $z_t \in \mathbb{R}^{n\times 1}$ is a vector of exogenous variables and possibly the lags of some endogenous ones. The operator $T(\cdot) \in \mathbb{R}^{n\times m} \to \mathbb{R}^{n\times m}$ is the T-map, which is the function that maps the parameters of the PLM to the parameters of the ALM. The vector $\theta_{t-1} \in \mathbb{R}^{n\times m}$ denotes agents' estimates of the coefficients of their reduced form forecasting models, called also beliefs.. The stochastic process e_t is a white noise.
Without loss of generality let us assume that $m = 1$. Then, (2.53) can be written as

$$\begin{aligned} y_t = {} & T_1\left(\theta_{1,t-1},\ \theta_{2,t-1}, ...,\ \theta_{n,t-1}\right) z_{1,t} + ... \\ & + T_n\left(\theta_{1,t-1},\ \theta_{2,t-1}, ...,\ \theta_{n,t-1}\right) z_{n,t} + e_t \end{aligned} \tag{2.56}$$

where, T_j indicates the $j - th$ row of the T-map vector and $\theta_{t-1} = \left(\theta_{1,t-1},\ \theta_{2,t-1}, ...,\ \theta_{n,t-1}\right)^t$. We further assume that $z_{1,t} = 1$, i.e. that y_t has an intercept.
We allow the ALM, equation (2.53) to be non-linear as T_j might be a non linear function of θ_{t-1} and $T_j\left(\theta_{t-1}\right)$ pre-multiplies $z_{j,t}$. The learning rules, equation (2.54) and (2.55), are also non-linear equations of the states. We are interested in the Rational Expectations Equilibria associated to the model, which, in turn,are parametrized by fixed points of the T-map and that we denote by $\theta^{ree} = (\theta_1^{ree},\ \theta_2^{ree}, ...,\ \theta_n^{ree})$. Then, the linearization of equation (2.53)

[31] R_t and R_{t-1} are linarized around the theoretical second moments of X.

around $\{\bar{y},\ \bar{z},\ \theta^{ree}\}$ yields,

$$\begin{aligned}
y_t &\approx T\left(\theta^{ree}\right)\overline{z} + \sum_{i=1}^{n} \frac{\partial T(\theta_{t-1}) \cdot z_t}{\partial \theta_{i,t-1}} \mid_{(\theta_t, z_t)=(\theta^{ree},\overline{z})} (\theta_{i,t-1} - \theta_i^{ree}) \\
&\quad + \sum_{i=2}^{n} \frac{\partial T(\theta_t) \cdot z_t}{\partial z_{i,t}} \mid_{(\theta_{t-1}, z_t)=(\theta^{ree},\overline{z})} (z_{i,t} - \overline{z}_i) + e_t \\
&= T\left(\theta^{ree}\right)\overline{z} + \sum_{i=1}^{n} \frac{\partial T(\theta_{t-1}) \cdot z_t}{\partial \theta_{i,t-1}} \mid_{(\theta_{t-1}, z_t)=(\theta^{ree},\overline{z})} (\theta_{i,t-1} - \theta_i^{ree}) \\
&\quad + T(\theta^{ree}) \cdot (z_t - \overline{z}) + e_t \\
&= \sum_{i=1}^{n} \frac{\partial T(\theta_{t-1}) \cdot z_t}{\partial \theta_{i,t-1}} \mid_{(\theta_{t-1}, z_t)=(\theta^{ree},\overline{z})} (\theta_{i,t-1} - \theta_i^{ree}) \\
&\quad + T(\theta^{ree}) \cdot z_t + e_t \\
&\equiv \widetilde{T}(\theta_{t-1}) \cdot z_t + e_t
\end{aligned} \tag{2.57}$$

where we define $\widetilde{T}(\cdot)$ as the linear T-map associated with $T(\cdot)$ and to θ^{ree}, which maps θ_{t-1} into $\widetilde{T}(\theta_{t-1}) \in \mathbb{R}^{n\times 1}$. $\widetilde{T}(\cdot)$ can be written as [32]

$$\widetilde{T}_1\left(\theta_{t-1}\right) = \sum_{i=1}^{n} \frac{\partial T(\theta_{t-1}) \cdot z_t}{\partial \theta_{i,t-1}} \mid_{(\theta_{t-1}, z_t)=(\theta^{ree},\overline{z})} (\theta_{i,t-1} - \theta_i^{ree}) + T_1(\theta^{ree}) \tag{2.58}$$

and

$$\widetilde{T}_j\left(\theta_{t-1}\right) = T_j(\theta^{ree}), \quad \forall j \in \{2, .., n\} \tag{2.59}$$

Next, we need to linearize the learning equations (2.54) and (2.55). Since the forecast error is zero at the point around which we linearize we do not need to keep track of R_{t-1} and we are only left we the linearized equation for θ_t, i.e.

$$\theta_t = \theta_{t-1} + \gamma \overline{R}^{-1} \overline{z} \left(y_t - \overline{y} - \overline{z}'(\theta_{t-1} - \theta^{ree}) - \left(\theta^{ree}\right)' (z_t - \overline{z}) \right)$$

which, after substituting y_t for the linearized T-map $\widetilde{T}(\cdot)$, can be re-written as,

$$\theta_t = \theta_{t-1} + \gamma \overline{R}^{-1} \overline{z} \left(z_t' \cdot \left(\widetilde{T}(\theta_{t-1}) - \theta_{t-1} \right) + e_t - \left(\theta^{\star}\right)' (z_t - \overline{z}) \right) \tag{2.60}$$

We further re-write the previous equation in the following succinct form, which defines the

[32]If we take $\overline{z}$ to be the log-linearized variables around the s.s., then $\overline{z} = (1,\ 0,\ \ldots,\ 0)$ and (2.58) is equal to

$$\widetilde{T}_1\left(\theta_{t-1}\right) = \frac{\partial T(\theta_{t-1})}{\partial \theta_{1,t-1}} \mid_{(\theta_{t-1}, z_t)=(\theta^{ree},\overline{z})} (\theta_{1,t-1} - \theta_i^{ree}) + T_1(\theta^{ree})$$

(2.59) remains clearly the same.

function $\widetilde{\mathcal{H}}(\cdot)$, and that will be used later on,

$$\theta_t = \theta_{t-1} + \gamma \widetilde{\mathcal{H}} \left(\theta_{t-1},\ z_t\right) \tag{2.61}$$

Let $\mathcal{M}$ denote the model defined by eq. (2.53)-(2.55) and let $\widetilde{\mathcal{M}}\left(\theta^{ree}\right)$ denote the linearization of $\mathcal{M}$ around the perfect foresight equilibrium associated to θ^{ree}, i.e. the model defined by eq. (2.57) and (2.60).

Proposition 2.1. *If θ^{ree} is the vector of parameter values of a reduced form model describing a REE associated to $\mathcal{M}$, then θ^{ree} is the vector of parameter values of a reduced form model describing a REE associated to $\widetilde{\mathcal{M}}\left(\theta^{ree}\right)$. Furthermore, if $\frac{\partial T(\theta^{ree})' \cdot \bar{z}}{\partial \theta_{1,t-1}} \neq 1$ the inverse implication is also true.*[33]

Proof of Proposition 2.1. Given that in the previous derivation of $\widetilde{\mathcal{M}}\left(\theta^{ree}\right)$, θ^{ree} denoted an arbitrary rational expectations equilibrium associated to $\mathcal{M}$, we just need to prove that θ^{ree} is also a rational expectations equilibrium associated to $\widetilde{\mathcal{M}}\left(\theta^{ree}\right)$. We will do this by showing that θ^{ree} is a fixed point of $\widetilde{T}(\cdot)$, i.e. that

$$\widetilde{T}\left(\theta^{ree}\right) = \theta^{ree}$$

Using the definition of the linearized T-map we have,

$$\begin{aligned}
\widetilde{T}_1\left(\theta^{ree}\right) &= \sum_{i=1}^{n} \frac{\partial T(\theta_{t-1}) \cdot z_t}{\partial \theta_{i,t-1}} |_{(\theta_{t-1}, z_t) = (\theta^{ree}, \overline{z})} \left(\theta_i^{ree} - \theta_i^{ree}\right) + T_1(\theta^{ree}) \\
&= T_1(\theta^{ree}) = \theta_1^{ree}
\end{aligned}$$

where the second equation follows from the definition of θ^{ree} fixed point of $T(\cdot)$.

And in addition, we have that for $j = 2, \ldots,\ n$,

$$\widetilde{T}_j\left(\theta^{ree}\right) = T_j(\theta^{ree}) = \theta_j^{ree}$$

where, again, the last equality holds by the fact that θ^{ree} is a fixed point of $T(\cdot)$. Thus θ^{ree} is a fixed point of $\widetilde{T}(\cdot)$ and hence a REE of it, which was what we wanted to prove. The reverse

[33] $\theta_{1,t-1}$ denotes here the first entry of the parameter vector θ_{t-1} and $\bar{z}$ the perfect foresight linearization point for z_t.

is not necessarily true.

Let $\theta^\star$ be a fixed point of $\widetilde{T}(\cdot)$. Then

$$\theta_j^\star = \widetilde{T}_j\left(\theta^\star\right) = T_j(\theta^{ree}) = \theta_j^{ree} \qquad \forall j \geq 2$$

thus $\theta_j^\star = \theta^{ree}$ for all $j \geq 2$. But for $j = 1$ we have that

$$\begin{aligned} \theta_1^\star = \widetilde{T}_1\left(\theta^\star\right) &= \sum_{i=1}^{n} \frac{\partial T(\theta_{t-1}) \cdot z_t}{\partial \theta_{i,t-1}} \,|_{(\theta_{t-1}, z_t) = (\theta^{ree}, \overline{z})}\, (\theta_i^\star - \theta_i^{ree}) + T_1(\theta^{ree}) \\ &= \frac{\partial T(\theta_{t-1}) \cdot z_t}{\partial \theta_{1,t-1}} \,|_{(\theta_{t-1}, z_t) = (\theta^{ree}, \overline{z})}\, (\theta_1^\star - \theta_1^{ree}) + \theta_1^{ree} \end{aligned}$$

or

$$0 = \left(\frac{\partial T(\theta_{t-1}) \cdot z_t}{\partial \theta_{1,t-1}} \,|_{(\theta_{t-1}, z_t) = (\theta^{ree}, \overline{z})} - 1 \right) (\theta_1^\star - \theta_1^{ree})$$

then, if the first factor of the above equation is zero we have infinitely many fixed points of $\widetilde{T}(\cdot)$ that are not of $T(\cdot)$.

Proposition 2.2. *Let $\widetilde{T}(\cdot)$ denote the T-map of $\widetilde{\mathcal{M}}\left(\theta^{ree}\right)$ and let θ^{ree} be the vector of parameter values of a reduced form model describing a REE associated to $\widetilde{\mathcal{M}}\left(\theta^{ree}\right)$. Then, the REE associated to θ^{ree} is Exceptionally stable if and only if the real part of $\frac{\partial T(\theta^{ree})\overline{z}}{\partial \theta_{1,t-1}}$ is smaller than one.*

Proof of Proposition 2.2. We will show that for the model $\widetilde{\mathcal{M}}\left(\theta^{ree}\right)$, the E-stability conditions for a REE parametrized by θ^{ree}, i.e. that the real part of the eigenvalues of $D_\theta \widetilde{T}(\theta^{ree})$ be smaller than one, are equivalent to having the real part of $\frac{\partial T(\theta^{ree}) \cdot \overline{z}}{\partial \theta_{1,t-1}}$ be smaller than one.

Then, let us first, write $D_\theta \widetilde{T}(\theta^{ree})$. Looking at eq. (2.58) and (2.59) we have that,

$$D_\theta \widetilde{T}(\theta^{ree}) = \begin{pmatrix} \frac{\partial T(\theta^{ree})\overline{z}}{\partial \theta_1} & \frac{\partial T(\theta^{ree})\overline{z}}{\partial \theta_2} & \cdots & \frac{\partial T(\theta^{ree})\overline{z}}{\partial \theta_n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{2.62}$$

Then the only non-zero eigenvalue of this operator is precisely $\frac{\partial T(\theta^{ree}) \cdot \overline{z}}{\partial \theta_{1,t-1}}$, which to guarantee that the REE associated to θ^{ree} is E-stable, needs to be smaller than one. Furthermore, if agents are only estimating a constant, this condition coincides with the E-stability condition of the REE associated to θ^{ree} for $\mathcal{M}$.

As already mentioned in the paper, the proof of convergence of θ_{t-1} to a distribution around θ^{ree} requires to check additional conditions, that generally depend on the particular model at hand and that are stated in Theorem 7.9 of Evans and Honkapohja (2001). However, because of the linearization, checking these conditions becomes significantly easier. Next we show how some of the conditions required for the convergence result in the latter Theorem always hold for $\widetilde{\mathcal{M}}\left(\theta^{ree}\right)$. We keep the same numeration as in Evans and Honkapohja (2001).

Assumption (A.2). *For any compact set $Q \subset D$, with D open set in $\mathbb{R}^n$, there exist K and q such that $\forall \theta \in Q$*

1. $\left|\widetilde{\mathcal{H}}\left(\theta,\ z\right)\right| \leq K\left(1+|z|^q\right)$

This holds since $\widetilde{\mathcal{H}}\left(\theta,\ z\right)$ is it self a polynomial in a compact set, thus it is bounded.

Assumption (A.3'). *For any compact set $Q \subset D$, with D open set in $\mathbb{R}^n$,$\widetilde{\mathcal{H}}\left(\theta,\ z\right)$ satisfies, $\forall \theta,\ \theta' \in Q$ and $z_1,\ z_2 \in \mathbb{R}^n$,*

1. $\mid \partial\widetilde{\mathcal{H}}\left(\theta,\ z_1\right)/\partial z - \widetilde{\mathcal{H}}\left(\theta,\ z_2\right)/\partial z \mid \leq L_1 \mid z_1 - z_2 \mid (1+ \mid z_1 \mid^{p_1} + \mid z_2 \mid^{p_1})$ for some $p_1 \geq 0$,
2. $\mid \widetilde{\mathcal{H}}\left(\theta,\ 0\right) - \widetilde{\mathcal{H}}\left(\theta',\ 0\right) \mid \leq L_2 \mid \theta - \theta' \mid$,
3. $\mid \partial\widetilde{\mathcal{H}}\left(\theta,\ z\right)/\partial z - \widetilde{\mathcal{H}}\left(\theta',\ z\right)/\partial z \mid \leq L_2 \mid \theta - \theta' \mid (1+ \mid z \mid^{p_2})$, for some $p_2 \geq 0$,

for some $L_1,\ L_2$.

For these assumption to hold it suffices for $\widetilde{\mathcal{H}}\left(\theta,\ z\right)$ to be twice continuously differentiable with bounded second derivatives on every Q.

Clearly, since $\widetilde{\mathcal{H}}\left(\theta,\ z\right)$ is a polynomial, it is twice continuously differentiable, and, furthermore, its second derivatives are continuous and thus bounded on every compact set. . Then $\widetilde{\mathcal{H}}\left(\theta,\ z\right) \in C^2(Q)$ for every Q.

Assumption (H.1). *$h(\theta)$ has continuous first and second derivative on D open, where*

$$h(\theta) = \lim_{t\to\infty} E\widetilde{\mathcal{H}}\left(\theta,\ z\right)$$

Then, we have that,

$$h(\theta) = \overline{R}^{-1}\overline{zz'}\left(\widetilde{T}(\theta) - \theta\right)$$

and if we set $\overline{R} = \overline{zz'}$, we have that

$$h(\theta) = \widetilde{T}(\theta) - \theta \tag{2.63}$$

This is a polynomial in θ and thus has continuous first and second derivatives on D.

Assumption (H.3). *$D_\theta h(\theta)$ is Lipschitz and all of the eigenvalues of $F = D_\theta h(\theta^{ree})$ have strictly negative real parts.*

Proposition 2.2 above, shows the conditions under which all of the eigenvalues of $D_\theta h(\theta^{ree})$ have strictly negative real parts, and they depend on the model at hand. However, since

$$D_\theta h(\theta) = \begin{pmatrix} \frac{\partial T(\theta^{ree})\overline{z}}{\partial \theta_1} - 1 & \frac{\partial T(\theta^{ree})\overline{z}}{\partial \theta_2} & \cdots & \frac{\partial T(\theta^{ree})\overline{z}}{\partial \theta_n} \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix}$$

is clearly independent of θ , then it satisfies Lipschitz conditions trivially. For completeness we write the Theorem 7.9 in Evans and Honkapohja (2001) below.

Theorem (see Theorem 7.9 Evans and Honkapohja (2001)): *Assume that Assumptions (A.2), (A.3'), (M.1)-(M.5), (H.1)-(H.3) and (N.1) hold. Consider the normalized random variable $U^{\gamma k}(t) = \gamma_k^{-1/2}\left[\underline{\theta}^{\gamma k} - \theta^\star\right]$. For any sequences $\tau_k \to \infty$, $\gamma_k \to 0$, the sequence of random variables $\left(U^{\gamma k}(\tau_k)\right)_{k \geq 0}$ converges in distribution to a normal random variable with zero mean and covariance matrix*

$$C = \int_0^\infty e^{sD_\theta h(\theta^\star)}\mathcal{R}(\theta^\star)e^{sD_\theta h(\theta^\star)'}ds \tag{2.64}$$

Then for small γ and large t, the distribution of θ_t is approximately given by

$$\theta_t \sim N(\theta^\star,\ \gamma C) \tag{2.65}$$

Chapter 3

Sentiment Shocks as Drivers of Business Cycles

'[...] a large proportion of our positive activities depend on spontaneous optimism rather than on a mathematical expectation, whether moral or hedonistic or economic. Most [...] of our decisions to do something positive [...] can only be taken as the result of animal spirits - a spontaneous urge to action rather than inaction, and not as the outcome of a weighted average of quantitative benefits multiplied by quantitative probabilities.

[...] often falling back [...] on whim or sentiment or chance.' Keynes, 1936[1]

3.1 Introduction

The rational expectations hypothesis (RE hypothesis) is the predominant approach for imputing expectations in macroeconomic applications. A number of recent studies show, however, that relatively small deviations from RE, such as the ones implied by adaptive learning (AL), can significantly improve the fit of business cycle models with the data (e.g. Slobodyan and Wouters (2012a, b), Milani (2007)). By endowing agents with subjective beliefs about their forecasting models, these studies arrive at alternative explanations for macroeconomic fluctuations, challenging the standard roles played by demand, technology and mark-up shocks and open a promising new field of research.

The standard approach in the AL literature consists of assuming that beliefs move only in response to economic outcomes or economic fundamentals, neglecting other important determinants of agents' expectations, such as purely subjective components of beliefs (see Akerlof and Shiller (2009)). This paper models these subjective views about the future as sentiment socks, defined as shocks to the beliefs agents entertain about their forecasting models, and explores their empirical importance for business cycle fluctuations in the context of an estimated New Keynesian model à la Smets and Wouters (see Smets and Wouters (2007)) with adaptive learning.

After accounting for the different degrees of freedom, I show that the model with sentiment shocks fits the data significantly better than the model without sentiment shocks. In particular, the model with AL and sentiment shocks has an improved ability to match the observed data covariances. Both results also hold when compared to the model under RE. A forecast error variance decomposition exercise shows the large importance of sentiment shocks as drivers of economic fluctuations in the United States: they account for up to 50% of all

[1]Keynes (1936), Chapter 12, The State of Long-term Expectation.

variability in aggregate variables at business cycle frequencies. The concrete role played by sentiment is explored in a simple shock decomposition exercise in which retrieved historical shocks are fed back into the estimated model. This exercise suggests that sentiment shocks display a common pattern for real variables, amplifying their fluctuation over the cycle, acting as alternating waves of optimism and pessimism. In particular, this role appears to be stronger during recessions when agents persistent pessimistic views take time to revert, thereby slowing down the subsequent recovery. Furthermore, the decomposition exercise suggests that sentiment shocks also played an important role in the historic evolution of price inflation. While these shocks accounted, on average, for about a third of inflation deviations from steady state over the pre-Volcker period, they appear to have remained largely at bay during the 'Great Moderation', reinforcing the idea that inflation is largely expectations-driven.[2] In addition, sentiment shocks are found to generate inflationary preassure after the 'Great Recession', which partially off-set what otherwise would have been and even lower inflation level. This is the only significant episode in the price inflation series in which sentiment shocks display an opposite effect to the one of the other shocks in the model; possibly picking up the effects of the Quantitive Eassing programs.

There is a vast universe of potential forces that may drive expectations away from what past data or *market fundamentals* suggest. Keynes, for example, emphasised the role of speculation and of what he famously termed as 'animal spirits'. More recently, Akerlof and Shiller (2009) addressed the effects that emotional and psychological factors have on economic decision making. They revisit Keynes' idea of animal spirits and identify several different categories, including the state of confidence, money illusion and the role of stories for shaping behaviour, among others. Moreover, there are other factors that, perhaps today more than ever, reinforce or exacerbate these sentiments, such as media and their ability to shape and coordinate public opinion. As mentioned above, in this paper, and borrowing the term proposed by Milani (2013), these emotional, psychological or social mood drivers are broadly defined as sentiment and they are modelled as shocks shifting the beliefs agents entertain about the reduced form models they use to form expectations; thus, affecting agents' forecasts about future economic variables.

The model is estimated using Bayesian techniques and U.S. canonical data. However, given the nature of the model at hand standard Bayesian estimation approaches cannot be

[2]This result should, however, bee taken with caution since this may also respond to a policy change which the monetary policy rule in the model does not allow for.

employed directly and, therefore, the paper employs a new estimation strategy proposed in Arias and Rancoita (2013). The standard approach to estimate models with learning consists of abstracting from all uncertainty in the expectation formation mechanism model, including sentiment shocks, thereby rendering the model de facto linear and, thus, allowing the likelihood to be computed with the standard Kalman Filter.[3] Arias and Rancoita (2013) propose an alternative estimation approach that relies on linearizing the belief updating equations.[4] This allows making use of the Kalman Filter and accommodating sentiment shocks at the same time. This paper is the first to introduce such an approach in a medium-size learning model. For completeness, I also compare the standard estimation approach with the new one devised in Arias and Rancoita (2013).

This paper is closely related to Milani (2013) and Slobodyan and Wouters (2012). Milani (2013) is concurrent work, attempting to answer a similar question as the one studied here. It proposes a two step estimation procedure, which consists of first identifying sentiment shocks in the U.S. business cycle, using the difference between survey data and what the model under AL suggests expectations should have been. It then incorporates these shocks as exogenous sentiment shocks into the estimation of the model. It finds evidence suggesting that these type of shocks play a significant role in the U.S. economy, in particular sentiment shocks related to investment decisions.

The present paper addresses the question in a different way. First, it relies on a single step estimation approach that estimates sentiment shocks and other model parameters jointly. In addition, the set of sentiment shocks considered here is significantly larger and includes sentiment related to all forward-looking variables that need to be forecasted in the model instead of restricting it to the ones related to investment, consumption and inflation.[5] Another important difference is that, the expectation formation mechanism follows Slobodyan and Wouters (2012), where agents are assumed to use small forecasting models and update them using Bayes rule, instead of the reduced form models of the same form as the minimum state variable (MSV) solution and constant-gain learning algorithm adopted in Milani (2013). The reason for this choice is twofold: first the more simple small forecasting models simplify the

[3]By approximating all uncertain states in the non linear parts of the model by 'certain' estimates, beliefs behave as time-varying parameters.

[4]The validity of such an approximation rest largely on the validity of the original log-linearization around the model's steady state and some stability conditions for the learning dynamics, analogous to the ones found in the learning literature; see Arias and Rancoita (2013).

[5]The reason behind this restriction in Milani's paper is that the estimation makes use of data on expectations from the Survey of Professional Forecasters which is limited to those variables.

estimation considerably; and second, they have a better empirical performance. As Slobodyan and Wouters (2012) show, in the context of a standard new Keynesian model, this learning scheme considerably improves the model's fit to the data relative to other more elaborate forms, including the MSV solution.

Another strand of related literature is represented by Cogley and Sargent (2008) and Suda (2013). Cogley and Sargent (2008) studies the role of particularly pessimistic initial priors in a Bayesian learning version of a simple asset pricing model. It suggests that an event like the Great Depression may alter beliefs, generating pessimistic initial conditions, in a way that can explain part of the equity premium puzzle in the postwar U.S. data. Suda (2013) builds on the same idea and studies the effects of one time 'shattered' beliefs, consequence of one time events such as the Great Depression. He does this in the context of a standard equilibrium business cycle model where agents learn about the probabilities characterizing productivity in the economy, via Bayesian methods. He finds that sufficiently large shocks to beliefs of agents can have a quantitatively important and persistent impact in the macroeconomy. Both works provide evidence of the importance of non-rational beliefs as drivers of macroeconomic dynamics and in particular of subjective interpretations of events as 'shifters' of agents beliefs. The paper at hand tries to build a framework where these shocks can be appropriately studied.

Beaudry et al. (2011) constitutes another recent effort that supports the importance of sentiment or psychological reasons behind economic dynamics. Using sign-restriction based identification schemes, it tries to isolate macroeconomic fluctuations that appear most likely driven by mood swings. Its findings suggest that this may be the main force driving business cycles.

From a more general perspective, this paper is part of an ample literature that attempts to explain busines cycle fluctuation as the result of sources other than standard demand, technology and mark-up shocks. This includes, in particular, the 'News Shocks' literature that explains economic fluctuations, partially, as a consequence of the anticipatory behavior of agents to information about future shock realizations. Beaudry and Portier (2006) is here the classic example, where agents recieve news about future technoloy shocks, while Schimitt-Grohe and Uribe (2012) brings the idea further, allowing for news about diverse shocks and incorporating them into a DSGE framework. In turn, Blanchard et al. (2013) assume news to be noisy, adding a signal extraction problem to the standard set up. This noise and the exisitng posibility that news about future shocks may not realize, could be associated to subjective views on future events which may be driven by sentiment, psychological factors or mood swings, though their original interpretation is different. Similar considerations could be extended to the literature on sunspots in RE and to the literature on noise shocks. See Benhabib and Farmer (1994) and Angeletos (2008) for the former and Lorenzoni (2010) for the latter.

The rest of the paper is structured as follows. Section 3.2 describes the core of the model and the expectation formation mechanism. Section 3.3 discusses the estimation methodology and the data. Section 3.4 presents and discusses the results. Section 3.5 concludes.

3.2 Model

The paper explores the role of shocks to beliefs in business cycle fluctuations in the context of a standard New Keynesian model (see Christiano, Eichenbaum and Evans (2005), Smets and Wouters (2007)), featuring both nominal and real rigidities. The version adopted here follows Slobodyan and Wouters (2012), who replace the traditional rational expectations hypothesis with adaptive learning. This not only makes the information assumption on agents more realistic but, as they show, it improves the fit of the model to the data, as measured by the marginal data density and the implied models' posterior odds - indicators of the relative likelihood of the models to generate the observed data and used to compare models in a Bayesian framework. In addition, learning helps reducing the degree of dependency on highly autocorrelated exogenous stochastic structures and mechanical sources of endogenous persistence, which are generally accepted as drawbacks of the rational expectations literature (Milani (2012)). This is consequence of the endogenous propagation introduced by learning and that manifests in the gradual update of agents beliefs.

The model consists of three main sectors: Households, Firms and a Government. In the first sector, representative households seek to maximize their lifetime utility subject to a budget constraint, where the former may rise either from consumption (relative to a habit component) or from leisure (agents choose how much to work). Each period, they also decide how much to invest in capital, taking into account its adjustment costs, and bonds and choose the utilization rate of capital, depending on its return. Households provide their labor to a union which, then, sets wages subject to a Calvo pricing scheme with indexation. This is consequence of the monopolistic power the union creates by differentiating the labor provided by the households.

In the second sector, firms are further divided into two sub-sectors: an intermediate goods sub-sector, where firms, choosing capital and labor, produce differentiated goods, creating a monopolistic-competition market where prices are set subject, also, to a Calvo scheme with indexation; and a final goods sub-sector, in which firms combine the intermediate goods into a final good that, in a perfectly competitive market, sell to consumers, investors and the government. Finally, the third sector is given by the monetary authority which, by means of a Taylor type rule, determines the short-term nominal interest rate as a function of inflation and output deviations from their respective targets.

The model is completed by the expectation formation mechanism, which will be introduced next, and the stochastic structure of the model. In particular, the latter can be divided in two to parts: first, the standard structural shocks used to match the data in these type of models and which remains the same as in Smets and Wouters (2007) and Slobodyan and Wouters (2012); and second, the shocks given by agents' sentiment about the future. In the paper the former are referred as standard shocks, while the latter as sentiment shocks. The model, then, comprises 7 different standard shocks use to match the 7 U.S. data series used in the estimation: a total factor productivity shock (TFP), a risk-premium shock, a government expenditure shock, an investment-specific technical change shock (IST), a monetary policy shock, a price mark-up shock and a wage mark-up shock. The first five shocks are modeled as AR(1) processes while the last two as i.i.d. shocks.[6] In addition, government expenditure is further affected by the innovation of TFP, since in the estimation government expenditure also includes net exports, which can be affected by productivity movements. The model is briefly presented in its log-linearized form in Appendix I.

3.2.1 Expectations Formation

According to the model, agents need to forecast the future value of seven endogenous variables to take decisions: consumption, investment, hours worked, inflation, the price and the return of capital and real wages.[7] To construct these expectations, they are assumed to use small reduced-form AR(2)-forecasting models. However, the parameters of these processes are unknown to them and need to be estimated. The assumption here is that agents entertain beliefs about those parameters in the form of some distribution and, as new information

[6] Under RE both mark-up shocks are usually modeled as a persistent process, e.g. as ARMA(1,1). However, adaptive learning generates sufficient endogenous persistence to abstract from such a structure.

[7] See Appendix I.

becomes available, they update this distribution using Bayes rule. Beliefs, then, encompass the parameters' distribution, how this distribution is updated each period, and how it relates to the data agents observe (i.e. through the specific AR(2) models. See equations (3.1) and (3.2))

This type of adaptive learning deviates from the traditional rational expectations assumption in three ways. First, the resulting probability measure used to forecast future variables does not need to coincide with the one implied by the model. Second, this probability measure, reflected in the parameters of the forecasting models, evolves over time inducing further dynamics. Third, the information set that agents use to form expectations is smaller than under RE. In particular, it is also smaller than the standard way adaptive learning is modelled. Under adaptive learning, agents are generally endowed with knowledge about the correct form of the rational expectations equilibrium of interest. A common rationale for this is given by the assumption that agents know the model in the same way as the researcher but do not know the values the parameters take, which prevents them from deriving the RE equilibrium.[8] The reason for using small reduced-form forecasting models is twofold. On the one hand it considerably simplifies the estimation costs and on the other, it can be justified from an empirical perspective. Slobodyan and Wouters (2012), show that these type of small forecasting models significantly improve the fit of the model to the data and help produce impulse response functions in line with DSGE-VAR models.

The following state space model describes how agents perceive the law of motion of forward-looking variables and the way parameters evolve,

$$y_t^f = X_t \beta_t + u_t \tag{3.1}$$

$$\beta_t = (1-\rho)\bar{\beta} + \rho\beta_{t-1} + v_t \ , \tag{3.2}$$

where y_t^f, is the vector containing the seven forward variables agents need to forecast each period and $X_t \in \mathbb{R}^{7\times 21}$ is a matrix that contains for each forward looking variable its first two lags and a constant, i.e. the AR(2) process. Agents believe the parameters characterizing that model, $\beta_t \in \mathbb{R}^{21\times 1}$, follow an autoregressive process around $\bar{\beta}$, where $\rho \leq 1$. Then, departing from Gaussian distributions for the initial states and assuming Gaussian errors, u_t and v_t, agents use observations of y_t^f to construct the distribution of β_t. Since the model is linear,

[8] Following this logic, RE would further assume that agents also know the value of the parameters of the model. Therefore, they would know more than the researcher.

this can be optimally done by means of the familiar Kalman Filter and belief evolution is fully characterized by the dynamics of their first two moments,

$$\begin{aligned} \beta_{t|t} &= \beta_{t|t-1} + \overbrace{P_{t|t-1} X_{t-1}^T \left[\Sigma + X_{t-1} P_{t|t-1} X_{t-1}^T\right]^{-1}}^{:=K_t} \left(y_t^f - X_{t-1}\beta_{t|t-1}\right) &\quad (3.3) \\ P_{t|t} &= P_{t|t-1} - K_t X_{t-1} P_{t|t-1} &\quad (3.4) \end{aligned}$$

where $\beta_{t|t-1} = (1-\rho)\bar{\beta} + \rho\beta_{t-1|t-1}$ is the predicted mean and $P_{t|t-1} = \rho^2 P_{t-1|t-1} + V$ the predicted covariance matrix of the states. V is the covariance matrix of v_t and Σ the covariance matrix of u_t. Furthermore,K_t is the Kalman gain, which optimally determines how much past beliefs need to be adjusted in the direction of the forecast error, $y_t^f - X_{t-1}\beta_{t|t-1}$, by considering the uncertainty of the latter relative to the uncertainty of the prior. Equations (3.3) and (3.4) are then referred to as the beliefs updating equations.

Intuitively, each period t agents need to construct expectations about next periods forward looking variables, i.e. $E_t\left(y_{t+1}^f\right)$. For this, and even though y_t^f is assumed to be known at time t,[9] agents use information up to period $t-1$ to update their beliefs about the distribution of the parameters of their forecasting models. Then, agents use the updated distributions, summarized by $\beta_{t|t-1}$, to generate the necessary expectations to take their economic decisions, $E_t\left(y_{t+1}^f\right) = X_t\beta_{t|t-1}$.[10] This closes the model. Plugged into equations (3.8)-(3.21) the model becomes backward looking and it can be estimated.

The main innovative point of this paper is that agents may deviate from the way the researcher models how beliefs are updated, i.e. deviate from (3.3) and (3.4). At any given period, agents may condition their beliefs on subjective information that pushes them away from what the data suggests. In the spirit of Milani (2013), these drivers are defined as sentiment and encompass a wide range of factors, from psychological and social ones to plain speculation. In the benchmark model these deviations take a simple form, identically and independently distributed shocks to the constant of the respective reduced form models. Equation (3.3) is then extended and written as,

$$\beta_{t|t} = \beta_{t|t-1} + P_{t|t-1} X_{t-1}^T \left[\Sigma + X_{t-1} P_{t|t-1} X_{t-1}^T\right]^{-1} \left(y_t^f - X_{t-1}\beta_{t|t-1}\right) + \xi_{t+1}\ , \quad (3.5)$$

where ξ_{t+1} is vector of appropriate size containing the sentiment shocks of agents.

[9]This is the standard assumption of 'time-t' dating, which avoids simultaneity between y_t^f and $\beta_{t|t}$. It is done for technical simplicity, see Evans and Honkapohja (2001).

[10]Remember, X_t contains information up to $t-1$.

3.3 Estimation Methodology

Adaptive learning introduces non-linearities in an otherwise linear model. Not only the beliefs updating equations are non-linear (see (3.4) and (3.5)), but the function used by agents to form expectations becomes non-linear as well, $E_t\left(y^f_{t+1}\right) = X_t\beta_{t|t-1}$. This is because the parameters of the reduced-form models used to forecast forward variables, and that under RE were constants, are now dynamic states, and the researcher needs to keep track of their evolution. In principle, Bayesian estimation of such a model would have to relay on some non-linear filter, such as Particle Filters or Quadrature Filters (Arulampalam et al. (2002)). However, these types of filters suffer from the so-called *curse of dimensionality*. That means that the computational costs increase exponentially with the dimension of the model. With learning the computational problem is further exacerbated since the state space is augmented to include agents' beliefs. In the model at hand the computational cost of such an estimation approach becomes prohibitive.

The literature proposes a simple way of circumventing the problem. By abstracting from all uncertainty in the beliefs updating equations, the evolution of beliefs can be computed deterministically such that beliefs behave as time-varying parameters. This makes the model effectively linear in the states and its marginal likelihood can be computed with the simple Kalman Filter.[11] To do this, the method relies on approximating the distribution of the unobserved forward looking variables that enter the beliefs updating equations, y^f_t, X_{t-1}, by a unit mass distribution at their last estimated mean (which in turn is affected by this approximation).

Since the objective of this paper is to study sentiment shocks and these are a source of uncertainty in beliefs, abstracting from uncertainty in the latter is not a feasible strategy here. Hence, this paper adopts the method introduced in Arias and Raincoita (2013) for the estimation of DSGE models under AL. The method is based on the linearization around the steady state equilibrium under rational expectations of what, in any case, is a largely linear model. Then, the marginal likelihood of the resulting linear model can be easily computed by means of the Kalman Filter. The validity of such a linearization is a direct consequence of the accuracy of the log linearization done on the optimality conditions of the model in the first place (equations (3.8)-(3.21)).

[11]Examples of this approach can be found in Milani (2005, 2007, 2013) and Slobodyan and Wouters (2007, 2012).

The particular linearization point has the advantage that it renders the forecast error, $y_t^f - X_{t-1}\beta_{t|t-1}$, equal to zero. This implies that one can neglect the dynamics of the matrix estimating the second moments of the beliefs, P. Then, after linearization, the beliefs updating equation can be written as,

$$\beta_t = (1-\rho)\bar{\beta} + \rho\left\{\beta_{t-1} + M\left(y_{t-1}^f - \bar{X}\beta_{t-1} - X_{t-2}\bar{\beta}\right)\right\} + \xi_t \ , \tag{3.6}$$

where $M = P^* X^{T*}\left[\Sigma + X^* P^* X^{T*}\right]^{-1}$[12]. Equation (3.6) is now the single equation describing the evolution of beliefs.

To estimate the model, seven US time series over a period ranging from the first quarter of 1965 to the fourth quarter of 2013 are used: real GDP, short-term nominal interest rate (Federal Funds rate), real consumption, real investment, hours worked, inflation and real wages. This gives rise to the following measurement equation,

$$\begin{pmatrix} dlGDP_t \\ FEDFUNDS_t \\ dlCons_t \\ dlINV_t \\ lHours_t \\ dlP_t \\ dlWage_t \end{pmatrix} \equiv O_t = \underbrace{\begin{pmatrix} \gamma \\ r \\ \gamma \\ \gamma \\ l \\ \pi \\ \gamma \end{pmatrix}}_{:=constant} + \begin{pmatrix} \Delta y_t \\ r_t \\ \Delta c_t \\ \Delta i_t \\ l_t \\ \pi_t \\ \Delta w_t \end{pmatrix}$$

which together with equations (3.8)-(3.21) and (3.6) complete the model.[13]

The whole model can be brought to state space form and succinctly written as,

$$\begin{cases} O_t = & Z_t^{obs} \\ Z_t = \mu + & G \cdot Z_{t-1} + V \cdot \epsilon_t \ , \end{cases} \tag{3.7}$$

where $Z_t = \left[Y_t', \omega_t', Y_{t-1}^{f'}, Y_{t-2}^{f'}, dobs_t', \beta_t'\right]$ is an appropriately stacked vector of $Y_t = [k_t, y_t, r_t, c_t, i_t, l_t, \pi_t, q_t, r_t^k, w_t]'$, $\omega_t = \left[\varepsilon_t^a, \varepsilon_t^b, \varepsilon_t^g, \varepsilon_t^q, \varepsilon_t^r, \varepsilon_t^p, \varepsilon_t^w\right]$, observables and beliefs[14].

The likelihood of the model is computed with the Kalman Filter and the posterior distri-

[12] $\beta_{t|t-1} = (1-\rho)\bar{\beta} + \rho\beta_{t-1|t-1}$ has been used and the subindices have been simplified, $t-1 \mid t-1 \equiv t-1$.

[13] γ denotes the period trend growth rate of real GDP, consumption, investment and wages; π denotes the periods steady state inflation rate as l and r do the same for hours worked and the nominal interest rate respectively. *dl* stands for log first difference and *l* for log.

[14] The state space form of the model is derived in the appendix.

butions of the parameters are generated by means of a Metropolis-Hastings algorithm. The priors for the parameters of the model are taken from Smets and Wouters (2007) and additionally include the priors for the standard deviations of the sentiment shocks (see Tables 3.3, 3.4 and 3.5 in Appendix II). Following Slobodyan and Wouters (2012), σ_0, the parameter setting the proportion to $\left(X\ \Sigma^{-1}X\right)^{-1}$ of the initial covariance matrix of the belief coefficients around which the linearization is done it is set to 0.03. The remaining parameters are estimated. Initial beliefs for each parameter draw are set to the implied rational expectations value. The estimation starts with the search for the mode of the log-posterior distribution of the parameters, which combines the log-likelihood of the data conditional on the model and the parameters with the log-prior knowledge about the parameters.

Four different expectation formation mechanisms are considered throughout the paper. First, the model derived under rational expectations, RE, the predominant assumption in macroeconomics and hence a natural benchmark. This specification corresponds to the model in Smets and Wouters (2007) and is the only one that requires more persistent processes to model the mark-up shocks. As shown in Slobodyan and Wouters (2012), models under adaptive learning generate enough endogenous persistence so that mark-up shocks are correctly captured by i.i.d. processes. As previously discussed, the model with sentiment shocks is based on the one with small-forecasting reduced form models introduced in Slobodyan and Wouters (2012). To incorporate these type of shocks, the model's expectation formation mechanism is linearized. Therefore, to better understand the contribution and role of sentiment and disentangle them from the effects of the linearization relative to the non-linear adaptive learning scheme, two other specifications are estimated: a model with non-linear adaptive learning, AL, which coincides with the model in Slobodyan and Wouters (2012) and the linearized version of it, without sentiment shocks, LAL. Finally, the core of this paper, the model with sentiment shocks, which, is also linearized, LALwS. The only difference of the model under RE and under AL from their respective original versions is that they are estimated using larger and updated samples, extending until the last quarter of 2013.

3.4 Results

Sentiment shocks were introduced, in subsection 3.2.1, as i.i.d., however, after a first estimation, the resulting historical innovations presented some significant correlations, both between sentiment shocks and also between sentiment shocks and some *standard* shocks. To cope with this issue, and guided by the results in the first estimation, the stochastic structure is adjusted (see Appendix II). The new specification allows the sentiment shocks on consumption and hours worked to depend on the risk premium innovation in the same way as the exogenous spending process is allowed to depend on TFP. Similarly, sentiment on investment, price and return on capital are allowed to depend on the innovation of the IST shock and sentiment about wage inflation may depend on the innovation of the wage mark-up shock. Finally, the sentiment shock for price inflation is allowed to depend on the innovations of both the price and the wage mark-up shocks. Furthermore, to account for the correlation observed among some sentiment shocks, a feature that is in fact desired, since optimism and pessimism are likely to be contagious, a series of dependencies between the sentiment shocks are estimated. For all of these parameters, their priors are set to be a beta distribution, adjusted to the interval $[-1,\ 1]$ with mean 0 and standard deviation 0.45; for the parameters accounting for the dependency of price inflation sentiment on price mark-up and of wage inflation sentiment on wage mark-up, these priors are further restricted to the interval $[0,\ 1]$. This final version of the stochastic structure is set to be the baseline and the one used in the rest of the paper.

Tables 3.3, 3.4 and 3.5 in Appendix II report the posterior distributions statistics for all model specifications. Standard test were used to determine the identification of the different parameters estimates in the model, including Geweke (1992) convergence tests, the plotting of the Metropolis-Hastings draws, the testing for different means in sub-samples of the draws yielded by the MH algorithm, and the plotting of the likelihood as a function of each parameter. Results indicate that all parameters are identified using 400,000 draws and burning the first 10%.

Structural parameters remain mostly unchanged and are robust to the expectations formation mechanism assumed. Although, the mean of the posterior distributions do change for some parameters, there is a strong overlapping between their respective 5th - 95th quantiles intervals.

There are, however, some noteworthy exceptions. First, the inverse of the intertemporal elasticity of substitution (IES) for consumption, σ_c, when sentiment shocks are present, it is estimated to be much lower (0.43). As the willingness to shift consumption across time increases with the higher IES, so does the effect of the interest rate on consumption and in turn on the whole economy. In addition, increases in the hours worked now lead to positive and larger increases in current consumption, opposite to the small and negative effect that it has in the models without sentiment (see equation (3.9)).

Second, estimating the model with sentiment shocks delivers an IST shock which is less volatile and persistent (see σ_q and ρ_q in Table 1). The smaller size of the IST shocks seem to be compensated by two effects of the higher IES that help match the data: First, the overall impact of the IST shock on investment becomes larger. Second, the lower σ_c, increases the impact of the sentiment shock about investment on the economy (in particular on real output, consumption and investment).

All together, sentiment shocks are estimated to be small relative to the other standard shocks present. Notwithstanding, they enter through a quite persistent belief process with a mean autoregression coefficient estimate of 0.97 (ρ in last row of Table 3). Their relevance will become apparent once the forecast error variance decomposition is considered later on.

A last point to notice regards the posterior estimates of the non-linear and linear versions of the adaptive learning model without sentiment shocks, they present two main differences. First, under linear adaptive learning, the risk-premium shock is estimated to be less volatile and considerably less persistent. This goes in line with the featured stronger habits in consumption that make the latter more persistent and less responsive to changes in the interest rate. In addition, the value of the capital stock becomes more sensitive to the risk-premium shock, partially compensating for its smaller variability. Second, the elasticity of the capital adjustment cost function, φ, returns to the levels of rational expectations when the adaptive learning scheme is linearized, reducing the effect of the current value of the capital stock on current investment and increasing the sensitivity of the capital accumulation dynamics to how efficient investment is (captured by the IST shock).

3.4.1 Marginal Data Density

One important dimension to evaluate when comparing models is the degree in which they are able to explain the data. Comparing the marginal data density of different models is the most accepted way of comparing two different models as shown by the ample literature on the matter (e.g. Fernandez-Villaverde and Rubio-Ramirez (2004), Milani (2007), Slobodyan and Wouters (2012)). This indicator is an average over the parameter space of the value of the likelihood of the data conditional on the model, and that accounts for the different degrees of freedom of the different models. In particular, it can be used to compute two different models' posterior odds, which indicate how more likely a model is to generate the observed data, relative to another. Alternatively, the marginal data density can be interpreted as reflecting the model's out of sample prediction performance. Table 3.1 shows the marginal data density, expressed in log-points, estimated for the four types of expectation formation mechanisms that are considered in the paper.[15]

Table 3.1: Marginal Data Density

RE	AL	LAL	LALwS
−1134.9	−1106.7	−1117.4	−1104.5

As the fourth column of the table shows, the model with adaptive learning and sentiments shocks fits the data better than all other specifications. Furthermore, confirming the results in Slobodyan and Wouters (2012) and Milani (2007), adaptive learning, both in its linear and non-linear form, improves the model's fit to the data with respect to its rational expectations counterpart. In terms of the models' posterior odds, the linearized adaptive learning model, LAL, is 17.5 log-points more likely to produce the observed data than the RE model; which in posterior probabilities terms means that the RE model is assigned zero probability relative to any of the learning models (since the other learning specifications present even larger marginal data densities).[16] Moreover, the model with sentiment shocks, LALwS, presents a significant improvement relative to the non-linear and linear adaptive learning versions without sentiment shocks. The evidence is again substantial in favour of sentiment shocks, as even when compared to the non-linear version of adaptive learning without sentiment, AL, the model LALwS is about 9 times more likely to produce the observed data. This is an important result as it provides strong evidence suggesting that sentiment shocks are an important feature of economic expectations. Interestingly, comparing the AL and the LAL models shows how the linearizing strategy seems to reduce the ability of the model to fit the data - relative to the non linear adaptive learning strategy.

[15]The marginal likelihood of the data, for this and most of interesting models, is approximated by the weighted harmonic mean of the posterior likelihoods generated with the Metropolis-Hastings algorithm. Weights are given by a truncated multivariate normal evaluated at the corresponding de-meaned parameter draw following Geweke (1998). The marginal data density is defined as

$$\int_{\Theta} \mathcal{L}\left(y^T|\theta,\ M\right)\pi(\theta)\ d\theta$$

, where M denotes a particular model, $\theta \in \Theta$ the parameters of the model, y^T the data, $\pi(\theta)$ the prior of the parameters and $\mathcal{L}\left(y^T|\theta,\ M\right)$ the likelihood of the data conditional on the parameters and the model.

[16]This is true if one departs, as it is done here, from an agnostic point of view in which both models are given the same prior probabilities of being the true one, i.e. 0.5.

In spite of the apparent cost of linearizing the expectation formation mechanism relative to the costs of abstracting form uncertainty in beliefs when sentiment is not present, the linearization strategy significantly outperforms the non-linear adaptive learning model in matching the variance in the observed variables - it also outperforms the RE model in this respect.[17] The non-linear version tends to generate too much variance, in particular, for hours worked, real investment growth and the interest rate. Furthermore, the fact that it is a linear model allows for a correct (conditional on the linearization) variance decomposition, an exercise that is presented next.

3.4.2 Forecast Error Variance Decomposition

To better understand the main drivers of economic fluctuations, Figures 3.1 and 3.2 present the forecast error variance decomposition for real GDP and investment growth; and for the price inflation and the Federal Funds rate respectively (the decomposition of the other observed variables is not shown, but briefly described). Several horizons are considered, ranging from 1 quarter to 25 years, which is taken as the unconditional forecast error and denoted with ∞. It is worth mentioning, that this variance decomposition cannot be defined for the non-linear adaptive learning model and is only possible because of the linerization strategy used to estimate the learning models with sentiment. Again, results show the importance of sentiment shocks.

In the shorter horizons, the three model specifications present a roughly similar role for the different standard shocks; while, sentiment shocks, taken together, account for about 10 percent of the variability of all observables. In particular, risk premium, government expenditure and TFP shocks still explain the major part of the variability behind real variables in the short run. Adaptive learning does not seem to introduce important changes relative to rational expectations in the short run either. The two main differences between the rational expectations and the adaptive learning specifications, in their two variants, are given by the change in the persistence's perception of the mark-up shocks, that results in the price mark-up shock becoming the only driver of inflation variations in the short run and a general relative larger role of the price mark-up with respect to the wage mark-up shocks and the expanded role of the risk-premium shock relative to the IST shock, that becomes even more important for real GDP, consumption and investment. The latter, when sentiment shocks are added,

[17]Not reported.

is further explained by the significant lower estimates for the persistence and variance of the IST shock and by the larger estimated intertemporal elasticity of substitution that augments the effect of risk premium shocks

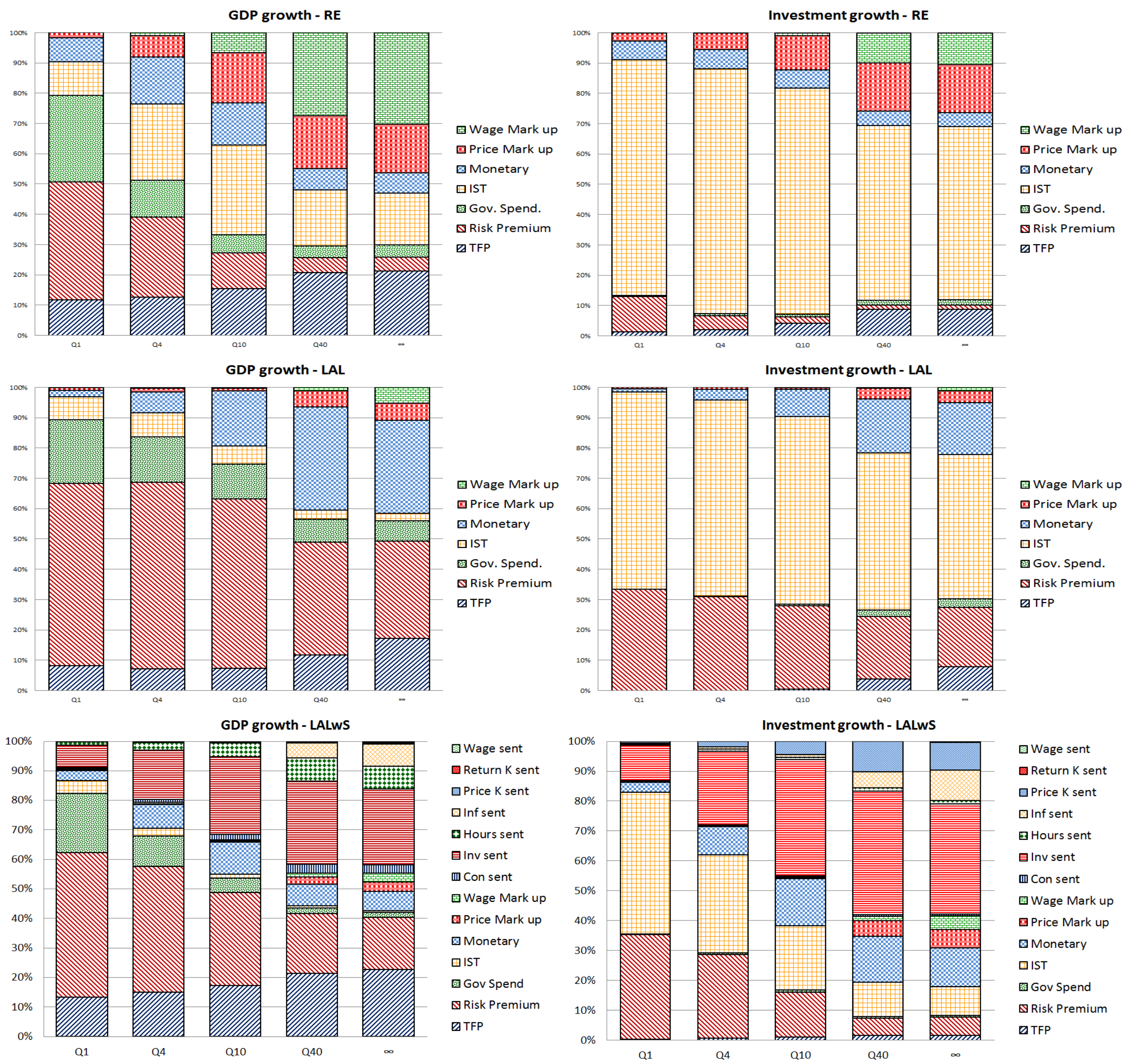


Figure 3.1: Forecast Error Variance Decomposition : Output and Investment.

At business cycle frequencies, illustrated here with the 10 and 40 quarters decompositions, the picture changes. The contribution of sentiment shocks considerably expands, showing their importance as an explanation for economic dynamics. In the longer run, except for wage inflation, where they account for about a fifth of its total variation (still a significant fraction) the role of sentiment shocks becomes largest and accounts for about half of all variations. This responds, in particular, to the long-lived effect that sentiment shocks have through the beliefs updating mechanism, which is a highly persistent process. In line with the findings of Milani (2013), sentiment shocks associated to investment appear to be particularly important for real variables. Specially, this shock stands as the main source of variation of real investment in the medium and long run, explaining up to 40 percent of it. It also accounts for approximately a quarter of the variation in GDP, where the risk-premium and TFP shocks still play a relevant role. In addition, the existing result that wage mark-up shocks dramatically reduce their role when adaptive learning is in place, is also found.

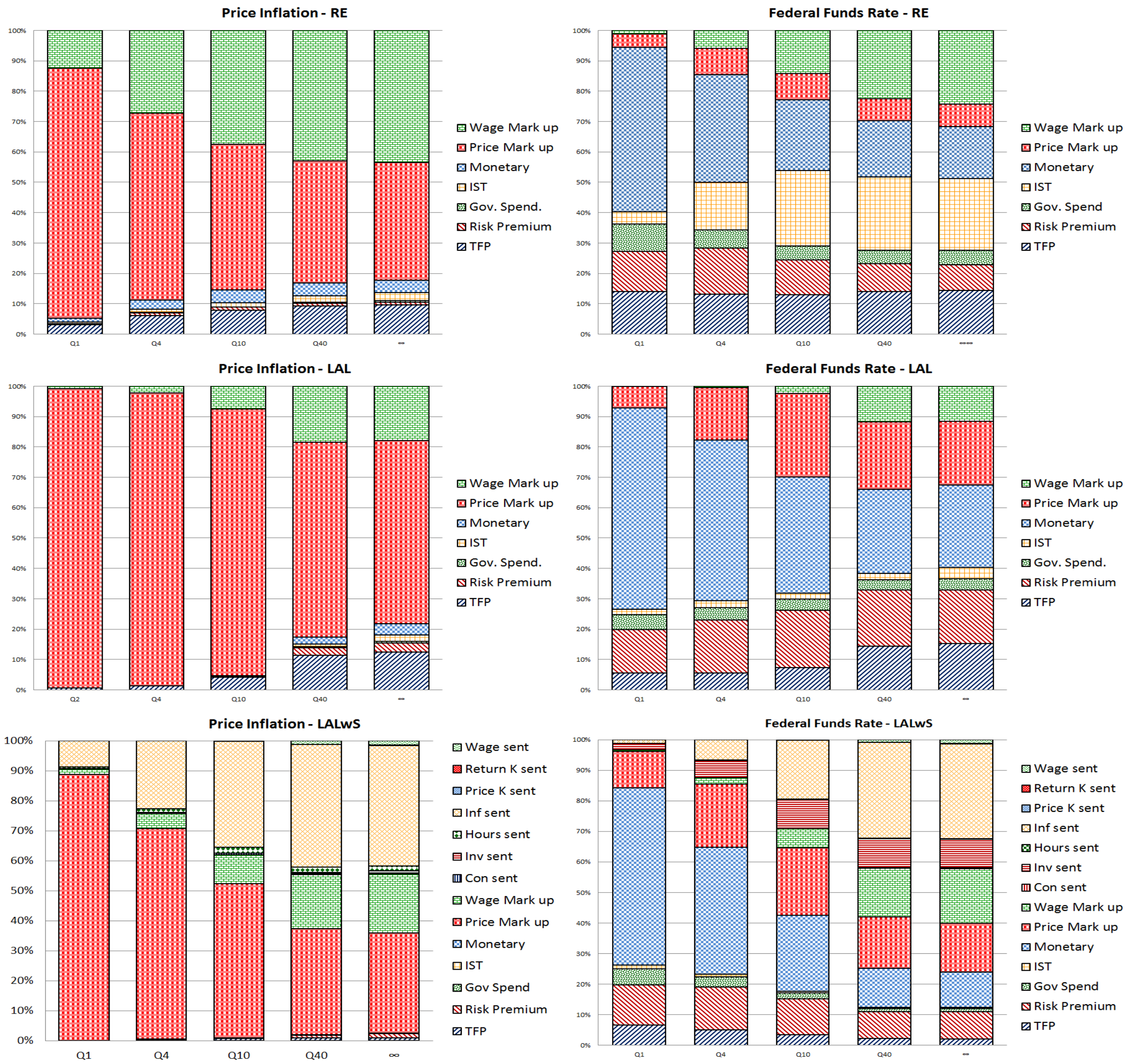


Figure 3.2: Forecast Error Variance Decomposition : Price Inflation and Federal Funds Rate.

In turn, consumption, which in the long run under rational expectations was largely explained by wage mark-up shocks, under learning is mostly explained by the risk-premium and monetary shocks. Once sentiment is introduced, this decomposition changes further, giving room to the above mentioned investment sentiment shock and to hours sentiment shock, two shocks that directly affect the intertemporal consumption Euler equation. Finally, the sentiment shock related to price inflation turns into the major source of variation of inflation in the long run, followed by the standard price and wage mark-up shocks, which continue to play an important role for this variable; wage inflation remains largely determined by the wage mark-up shock, as in the rational expectations case, but under adaptive learning to a larger extent; lastly, variations in the Federal Funds Rate at business cycle frequencies, are mainly driven by the sentiment shock associated to inflation, while the role of TFP is significantly decreased and the IST shock, as in the case of adaptive learning with no sentiment shocks, plays virtually no role.

3.4.3 Historical Variance Decomposition

Table 3.2 presents the historical variance decomposition obtained from the model with sentiment shocks (Tables 3.6 and 3.7 present the corresponding results for the RE and LAL versions respectively - AL is a non-linear model and thus cannot be used to linearly decompose the variance in a comparable fashion).

It shows that sentiment accounted for a substantial fraction of the overall observed variance in GDP, Consumption and Investment growth, explaining from 13.5 to 24.3 percent, and in the Fed Funds rate, hours worked and price inflation, where the contribution is even higher, ranging from 38.5 up to 54.6 percent. Agents' sentiment about investment and inflation seems to play the larger roles. Specifically, agents' sentiment about price inflation accounts for about a third of the variance in inflation and about a quarter of the variance in the interest rate, while sentiment about investment is responsible for approximately a fifth of the variance in real investment growth and more than a quarter of the one in hours worked. On the other hand, sentiment related to consumption and wage inflation play a smaller role, while sentiment about the price and the return of capital are almost negligible. When sentiment is included in the model, the contribution of the risk-premium shock becomes lower in line with its overall estimated lower impact on the economy. A similar case is presented for the IST and monetary shocks, which are estimated to have smaller innovations and persistence. Most of this lower variability is then taken over by sentiment shocks.

Table 3.2: LAL with Sent. Shocks - variance decomposition in %

	Δy	r	Δc	Δi	l	π_p	π_w
Structural Shocks:	86.5	61.5	83.4	75.7	45.4	58.7	96.1
TFP	12.0	2.1	6.2	0.5	4.1	1.0	0.3
Risk Premium	40.8	8.1	66.6	27.4	24.2	1.2	1.7
Gov. Exp.	24.7	1.1	5.3	0.4	3.4	0.1	0.1
IST	4.3	0.3	1.4	38.1	1.1	0.1	0.0
Monetary	4.7	12.7	5.0	7.6	8.8	0.1	0.2
Price markup	0.9	15.4	1.0	1.7	2.4	33.8	6.6
Wage markup	0.6	18.2	0.3	0.7	4.5	20.4	89.2
Sentiment Shocks:	13.5	38.5	16.6	24.3	54.6	41.3	3.9
Consumption	0.7	0.2	3.0	0.2	5.4	0.4	0.2
Investment	9.7	9.4	7.4	19.3	28.5	0.6	0.6
Hours Worked	1.7	0.1	3.7	0.4	12.4	1.8	0.2
Inflation	0.9	28.1	0.9	2.1	7.3	37.2	0.1
Price of Capital	0.1	0.1	0.4	2.1	0.5	0.1	0.0
Return of Capital	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Wage	0.0	1.2	0.1	0.0	0.8	1.5	1.9

The remaining differences of the learning model with sentiment from its RE counterpart are largely inherited from the learning dynamics. Therefore, they are studied in the context of the learning model without sentiment.

First, a similar result as in Slobodyan and Wouters (2012) is found when comparing the contributions of the price and wage markup shocks to inflation, between the RE and the LAL versions. Under learning, both shocks are estimated as iid, consequently, agents do not distinguish between their persistence, yielding a relatively larger role for the price mark-up shock, which under RE was the least persistent one. The adaptive learning dynamics further affect two other shocks and their contributions in terms of the variance generated. The risk-premium shock and the IST shock are estimated to be less volatile and less persistent respectively. However, a higher estimate for the habits component increases the effect of the risk-premium shock on real investment and on the price of the capital stock compensating for the smaller shocks, which explains why the risk-premium shock becomes more important under learning. Finally, the smaller contribution of the monetary shock also reflects the larger role played by habits under learning, as they make consumption less sensitive to changes in the interest rate in the corresponding Euler equation.

3.4.4 Shock Decomposition Exercise

After having established the large importance that sentiment shocks play at business cycle frequencies, this section presents a simple shock decomposition exercise aiming at studying the concrete role of sentiment shocks over the cycle. Figures 3.3 and 3.4 plot the evolution of real output, consumption and investment and of price inflation respectively, together with the underlying contribution that can be attributed to standard and sentiment shocks respectively. The red bars depict the evolution of the variables when sentiment shocks are turned off; the blue bars show the sentiment shocks' quarterly contribution, that is linearly added to yield the black line, which depicts the actually observed path for the variables.

This exercise shows the important role that sentiment shocks play in the determination of economic fluctuations, which, if not present, would have yielded quite different economic dynamics. Moreover, sentiment shocks display a distinctive and common pattern for the three observed real variables in the model that translates into a reinforcing behavior, amplifying the economic cycles. The computed correlation between output growth and the contribution of sentiment shocks to output growth is of about 0.71. The analogue correlation for consumption and investment growth are also high, 0.63 and 0.65 respectively.

More interestingly, during recessions, this reinforcing effect of sentiment shocks, which acts as a pessimistic wave, is not necessarily immediately reverted and can endure for several years slowing down the recovery - even though economic fundamentals may have already recovered. Except for the recessions experienced during the 80's, this seems to have been the case for all recessions since the seventies, where the pessimistic views of agents played an important part in avoiding a quick exit.

This observation is in line with the persistent effect of sentiment on beliefs, suggesting that it takes time before agents perceptions about the economy are reverted into a neutral or optimistic state. During the last two recessions these pessimistic views have been particularly strong. Sentiment remained pessimistic long after the recession was over, particularly for consumption. This contrasts with the situation experienced during the 90's and the 80's, when sentiment bounced back relatively quickly and became positive until the next recession.

The results are also worthwhile analyzing in the case of the evolution of price inflation. A significantly large fraction of the high inflation experienced during the seventies and beginning of the eighties appears to be due to agents' sentiment. Pessimistic views, which this time translate into positive contributions, account on average for 37% of the quarterly price inflation deviations from steady state over that period. A point that underscores the idea that inflation is mainly driven by expectations.

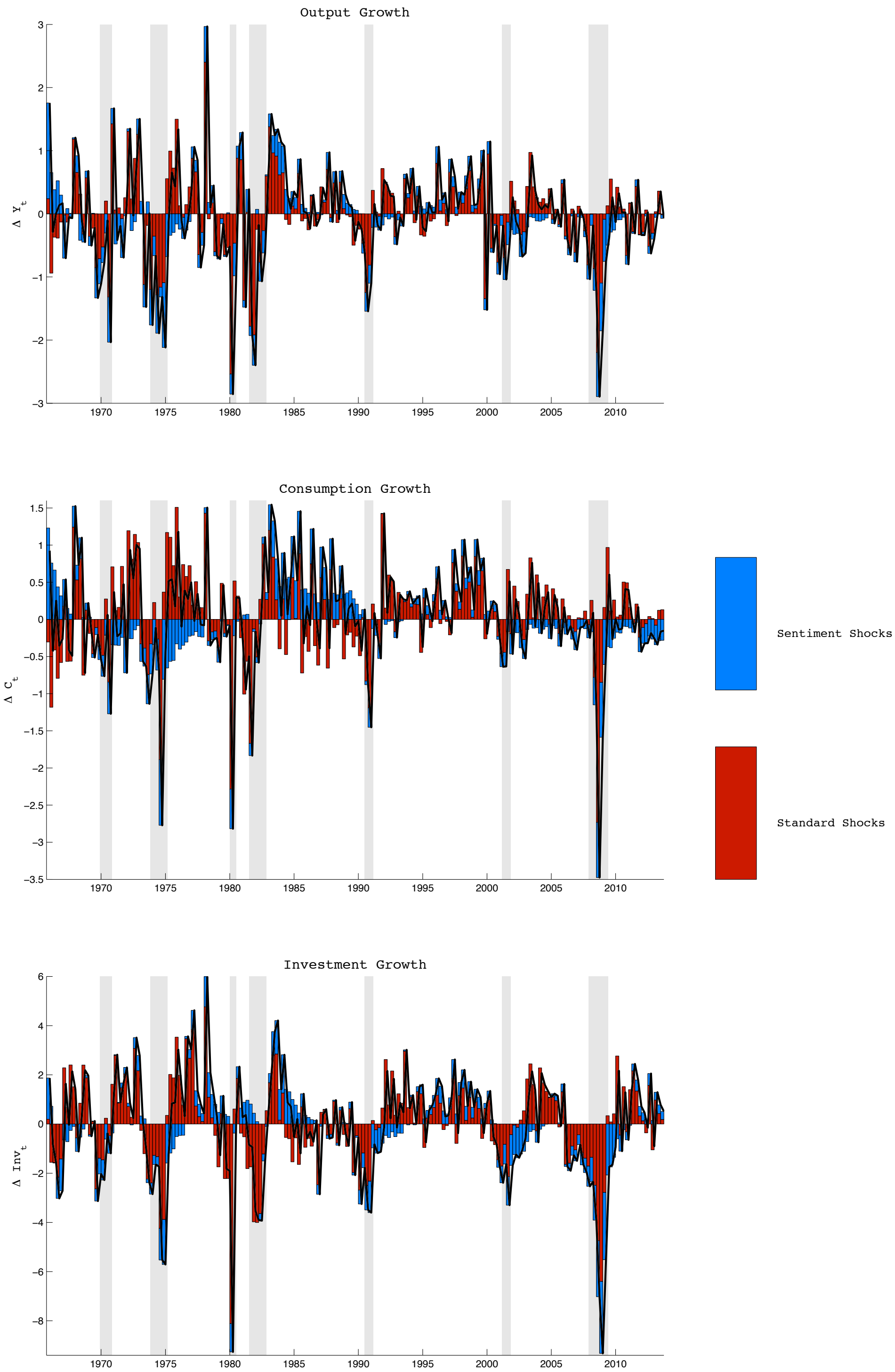


Figure 3.3: Shock decomposition. Real variables: Output, Consumption and Investment.

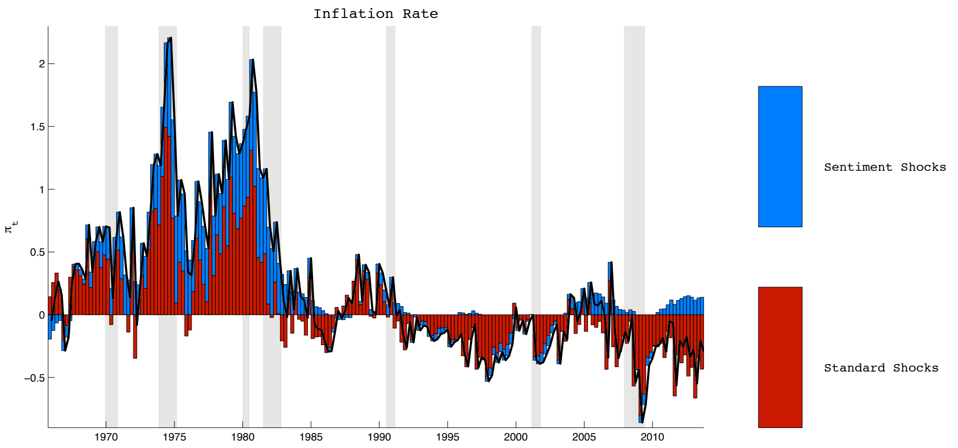


Figure 3.4: Shock decomposition: Inflation

This period, which commenced its decline with the appointment of Paul Volcker as chairman of the Federal Reserve Board, was then followed by a period of relatively low and stable inflation and a very much limited role of sentiment. However, this point should be taken with caution, as sentiment may be picking up a change in the policy regime that the Taylor rule in the model does not allow for. Perhaps more interesting, is the observation that after the 'Great Recession', around the beginning of 2010, sentiment started to display the opposite effect to the one of produced by standard shocks.[18] This inflationary pressure that sentiment generated partially off-set what otherwise would have been an even lower inflation level. Possibly picking up some of the effects of the Quantitative Easing programs.

[18]Correlations between price inflation and the contribution of sentiment shocks to price inflation show this change, as while the correlation after the 'Great Inflation' and til 2010 was of 0.6, it becomes 0.16 during the subsequent years.

3.5 Concluding Remarks

This paper studies the role of sentiment shocks as a source of business cycle fluctuations. In the context of a standard New Keynesian Model with adaptive learning and applying a novel estimation methodology it presents strong evidence suggesting that sentiment shocks are important drivers of economic fluctuations and therefore, that they need to be considered. The relevance of this result is strengthened by the ability of the learning model to fit the data significantly better when sentiment is present than otherwise - and, in turn, better than when rational expectations are assumed. Agents' subjective views used to form beliefs challenge the standard sources of macroeconomic dynamics. In particular, sentiment is responsible for a substantial portion of the medium and long run variability in real GDP, investment and consumption growth as well as in inflation, hours worked and the nominal interest rate, accounting for up to about half of it. Coinciding with the results in Milani (2013), investment is the most important sentiment shock driving the real variables in the model, which brings to mind Keynes' idea of animal spirits. In a straightforward shock decomposition exercise, sentiment shocks are found to display a pro-cyclical reinforcing effect, while a particular role over the cycle is identified for real GDP, investment and consumption growth. Evidence suggests that agents tend to become pessimistic during and well after a recession, slowing the subsequent recovery down. Also, sentiment shocks are found to play an important role in the historic evolution of price inflation: a role that seems to have changed after the high inflation period of the 70's and beginning of the 80's. Finally, sentiment shocks appears to be catching some of the effects of the Quantitative Easing programs, as they are found responsible for positive inflationary pressure that partially off-set what otherwise would have been an even lower inflation level.

3.6 Appendix I

3.6.1 Model

The model is briefly presented in its log-linearized form around the stationary steady state and consists of 14 endogenous variables in the same number of equations and 7 exogenous shocks.[19]

The economy's aggregate resource constraint captures how output (y_t) is allocated either to consumption (c_t), investment (i_t), the cost of adjusting the utilization level of capital (u_t) or to the exogenous government spending (ε_t^g):[20][21]

$$y_t = \frac{c_*}{y_*}c_t + \frac{i_*}{y_*}i_t + \frac{r_*^k k_*}{y_*}u_t + \varepsilon_t^g \ , \tag{3.8}$$

where $\frac{c_*}{y_*}$, $\frac{i_*}{y_*}$, $\frac{r_*^k k_*}{y_*}$ denote the steady state shares of consumption, investment, and cost of changing capital utilization relative to output, respectively.[22]

A typical consumption Euler equation,

$$c_t = (1-c_1)c_{t-1} + c_1 E_t c_{t+1} + c_2(l_t - E_t l_{t+1}) - c_3(r_t - E_t \pi_{t+1}) + \varepsilon_t^b \ , \tag{3.9}$$

describes current consumption's (c_t) dependence on past and expected future consumption, on expected hours worked growth (l_t), and on the real interest rate. ε_t^b stands for the exogenous process followed by the risk premium. The parameters are given by $c_1 = \frac{1}{1+\eta/\gamma}$, $c_2 = \frac{c_1(\sigma_c - 1)w_* L_*/C_*}{\sigma_c}$, and $c_3 = \frac{1-\eta/\gamma}{\sigma_c}$, where η is the habit formation parameter and σ_c denotes the inverse of the intertemporal elasticity of substitution.

The investment Euler equation,

$$i_t = i_1 i_{t-1} + (1-i_1)E_t i_{t+1} + i_2 q_t + \varepsilon_t^q \ , \tag{3.10}$$

with parameters $i_1 = (1+\beta\gamma^{1-\sigma_c})^{-1}$ and $i_2 = \frac{i_1}{\gamma^2\varphi}$ characterizes the dependence of current investment (i_t) on past and next periods expected investment and on the real value of the

[19]Later on, the exogenous stochastic structure will be augmented to include sentiment shocks.

[20]For a detailed step by step derivation see the appendix contained in Smets and Wouters (2007). The description here largely follows Smets and Wouters (2007) and Milani (2013).

[21]Includes net exports.

[22]All lower case variables represent log-deviations from their respective steady state value unless stated otherwise.

capital stock, (q_t). It also depends on an investment-specific technological (IST) change shock, ε^q. As usual, β denotes the discount factor and φ represents the adjustment costs in investment.

The evolution of the value of capital is given by,

$$q_t = (1 - q_1)E_t q_{t+1} + q_1 E_t r^k_{t+1} - (r_t - E_t \pi_{t+1}) + q_2 \varepsilon^b_t \ , \tag{3.11}$$

where $q_1 = \frac{r^k_*}{r^k_* + (1-\delta)}$, $q_2 = \frac{\sigma_c(1+\eta/\gamma)}{1-\eta/\gamma}$, δ is the depreciation rate, and r^k_* the steady-state rental rate of capital. It is a function of its expected future value, of the expected rental rate of capital, and of the return on assets held by households.

Aggregate supply is given by a Cobb-Douglas production function,

$$y_t = \phi_p(\alpha k^s_t + (1-\alpha)l_t + \varepsilon^a_t) \ . \tag{3.12}$$

Output is produced using capital services k^s_t and labour l_t as inputs, with shares determined by α. ε^a is total factor productivity (TFP), while ϕ_p reflects the existence of fixed costs in production and corresponds to the price mark up in steady state.

Capital services, in turn, are a fraction of the capital stock in the previous period (capital is assumed to need one quarter to become operational), determined by the degree of capital utilization u_t.

$$k^s_t = k_{t-1} + u_t \ . \tag{3.13}$$

Moreover, the degree of capital utilization is a positive function of the rental rate of capital,

$$u_t = \frac{1-\psi}{\psi} r^k_t \ , \tag{3.14}$$

where ψ is the elasticity of the capital utilization cost function.

Capital accumulation dynamics are given by,

$$k_t = k_1 k_{t-1} + (1-k_1)i_t + k_2 \varepsilon^q_t \ , \tag{3.15}$$

where $k_1 = \frac{1-\delta}{\gamma}$ and $k_2 = (1-k_1)(1+\beta\gamma^{1-\sigma_c})\gamma^2\varphi$. The capital stock net of depreciation increases with investment but depends on how efficient those investments are, ε^q.

Price mark up is defined as the difference between the marginal product of labour and wages (w),

$$\mu_t^p = \alpha(k_t^s - l_t) + \varepsilon_t^a - w_t \tag{3.16}$$

While inflation is determined by the following New Keynesian Phillips curve,

$$\pi_t = \pi_1 \pi_{t-1} + \pi_2 E_t \pi_{t+1} - \pi_3 \mu_t^p + \varepsilon_t^p \ , \tag{3.17}$$

where $\pi_1 = \frac{\iota_p}{1+\beta\gamma^{1-\sigma_c}\iota_p}$, $\pi_2 = \frac{\beta\gamma^{1-\sigma_c}}{1+\beta\gamma^{1-\sigma_c}\iota_p}$ and $\pi_3 = \frac{(1-\beta\gamma^{1-\sigma_c}\xi_p)(1-\xi_p)}{\xi_p((\phi_p-1)\varepsilon_p+1)}\frac{1}{1+\beta\gamma^{1-\sigma_c}\iota_p}$, with ι_p denoting the indexation to past inflation of those prices that where not re-optimized, ξ_p the Calvo parameter regulating the price stickiness and ε_p the curvature of the Kimball goods market aggregator. Current inflation depends on both lagged and expected future inflation and also on the price mark-up and a price mark-up disturbance, ε^p.

The rental rate of capital depends negatively on the capital to labor ratio and positively on the real wage,

$$r_t^k = -(k_t^s - l_t) + w_t \ . \tag{3.18}$$

In the labor market the wage markup is characterized by,

$$\mu_t^w = w_t - \left(\sigma_l l_t + \frac{1}{1-\eta/\gamma}\left(c_t - \frac{\eta}{\gamma}c_{t-1}\right)\right) \ , \tag{3.19}$$

i.e. the difference between the real wage and the marginal rate of substitution between consuming and working. σ_l denotes the inverse of the Frisch elasticity of labor supply.

Wage dynamics are determined by,

$$w_t = w_1 w_{t-1} + (1-w_1)E_t(w_{t+1} + \pi_{t+1}) - w_2\pi_t + w_3\pi_{t-1} - w_4\mu_t^w + \varepsilon_t^w \tag{3.20}$$

where $w_1 = (1+\beta\gamma^{1-\sigma_c})^{-1}$, $w_2 = \frac{1+\beta\gamma^{1-\sigma_c}\iota_w}{1+\beta\gamma^{1-\sigma_c}}$, $w_3 = \frac{\iota_w}{1+\beta\gamma^{1-\sigma_c}\iota_w}$ and $w_4 = \frac{1}{1+\beta\gamma^{1-\sigma_c}\iota_p}$ $\frac{(1-\beta\gamma^{1-\sigma_c}\xi_w)(1-\xi_w)}{\xi_w((\phi_w-1)\varepsilon_w+1)}$. As for prices, ι_w is the degree of wage indexation to past inflation, ξ_w is the Calvo parameter regulating wage stickiness and ε_w is the curvature of the Kimball labor market aggregator. The current real wage depends on its past and expected future values; on the past, current and expected future value of inflation; and on the wage mark-up and the wage mark-up disturbance ε^w.

Finally, monetary policy is assumed to follow a Taylor type rule,

$$r_t = \rho_r r_{t-1} + (1-\rho_r)\left[\chi_\pi \pi_t + \chi_y (y_t - y_t^\star)\right] + \chi_{\Delta y}(\Delta y_t - \Delta y_t^\star) + \varepsilon_t^r \ , \tag{3.21}$$

where the short-term nominal interest rate is gradually adjusted to changes in inflation, the output gap and a monetary policy shock, ε^r.[23]

The stochastic sub-structure given by the seven shocks introduced above, TFP, risk-premium, government expenditure, investment-specific technical change, monetary, price mark-up and wage mark-up are set in the following way: the first five shocks are modeled as AR(1) processes while the last two as i.i.d. shocks.[24] In addition, government expenditure is further affected by the innovation of TFP, since in the estimation government expenditure also includes net exports, which can be affected by productivity movements.

[23]The output gap is defined as the difference between actual output and potential output, $y_t - y_t^\star$. Where the latter, $y_t^\star$, in turn, is defined as the output that would prevail in the economy under flexible prices.

[24]Under RE both mark-up shocks are usually modeled as a persistent process, e.g. as ARMA(1,1). However, adaptive learning generates sufficient endogenous persistence to abstract from such a structure.

3.7 Appendix II

3.7.1 Estimations

This section presents the posterior estimation of parameters.

Table 3.3: Posterior Estimates: stochastic structure.

- mean and 5% ÷ 95% quantiles reported -

	Prior(*mean, std*)	RE	AL	LAL	LALwS
σ_a	$\Gamma^{-1}(0.1, 2)$	0.44 0.40÷0.48	0.44 0.41÷0.49	0.44 0.41÷0.49	0.47 0.42÷0.52
σ_b	$\Gamma^{-1}(0.1, 2)$	0.22 0.17÷0.26	0.15 0.12÷0.17	0.10 0.07÷0.15	0.16 0.13÷0.21
σ_g	$\Gamma^{-1}(0.1, 2)$	0.50 0.46÷0.55	0.48 0.44÷0.53	0.48 0.44÷0.52	0.50 0.46÷0.55
σ_q	$\Gamma^{-1}(0.1, 2)$	0.36 0.31÷0.42	0.42 0.37÷0.47	0.35 0.28÷0.42	0.17 0.12÷0.24
σ_r	$\Gamma^{-1}(0.1, 2)$	0.22 0.20÷0.25	0.21 0.20÷0.23	0.21 0.20÷0.23	0.20 0.19÷0.22
σ_p	$\Gamma^{-1}(0.1, 2)$	0.13 0.10÷0.15	0.15 0.13÷0.16	0.12 0.10÷0.15	0.14 0.12÷0.16
σ_w	$\Gamma^{-1}(0.1, 2)$	0.35 0.32÷0.39	0.34 0.31÷0.37	0.36 0.32÷0.40	0.34 0.30÷0.38
σ_c^s	$\Gamma^{-1}(0.1, 2)$	–	–	–	0.05 0.03÷0.10
σ_i^s	$\Gamma^{-1}(0.1, 2)$	–	–	–	0.14 0.05÷0.25
σ_l^s	$\Gamma^{-1}(0.1, 2)$	–	–	–	0.06 0.03÷0.10
σ_π^s	$\Gamma^{-1}(0.1, 2)$	–	–	–	0.04 0.03÷0.06
$\sigma_{p_k}^s$	$\Gamma^{-1}(0.1, 2)$	–	–	–	0.08 0.03÷0.16
$\sigma_{r_k}^s$	$\Gamma^{-1}(0.1, 2)$	–	–	–	0.07 0.03÷0.13
$\sigma_{w_\pi}^s$	$\Gamma^{-1}(0.1, 2)$	–	–	–	0.05 0.03÷0.08
ρ_a	$Beta(0.5, 0.2)$	0.96 0.94÷0.98	0.99 0.99÷0.99	0.97 0.96÷0.99	0.97 0.95÷0.99
ρ_b	$Beta(0.5, 0.2)$	0.36 0.21÷0.55	0.64 0.48÷0.76	0.29 0.14÷0.49	0.13 0.04÷0.27
ρ_g	$Beta(0.5, 0.2)$	0.98 0.96÷0.99	0.97 0.96÷0.99	0.97 0.96÷0.99	0.96 0.95÷0.98
ρ_q	$Beta(0.5, 0.2)$	0.79 0.72÷0.86	0.38 0.28÷0.49	0.43 0.28÷0.58	0.15 0.05÷0.28
ρ_r	$Beta(0.5, 0.2)$	0.13 0.05÷0.22	0.15 0.06÷0.25	0.14 0.06÷0.24	0.08 0.03÷0.15
ρ_p	$Beta(0.5, 0.2)$	0.90 0.82÷0.96	–	–	–
ρ_w	$Beta(0.5, 0.2)$	0.96 0.94÷0.98	–	–	–
θ_p	$Beta(0.5, 0.2)$	0.77 0.63÷0.87	–	–	–
θ_w	$Beta(0.5, 0.2)$	0.93 0.88÷0.97	–	–	–
a_{g^b}	$N(0.5, 0.25)$	0.51 0.37÷0.64	0.55 0.42÷0.67	0.55 0.42÷0.68	0.46 0.33÷0.59

The models under adaptive learning, following Slobodyan and Wouters (2012), feature a simpler process for both mark-up shocks with respect to the model under RE. While under RE these shocks followed an ARMA (1,1) process, under learning they are can be modeled as

Table 3.4: Posterior Estimates: stochastic structure cross-correlations.

- mean and 5% ÷ 95% quantiles reported -

	Prior(*mean, std*)	RE	AL	LAL	LALwS
θ_{b-c}	$Beta\,(0.5, 0.2)$	–	–	–	$\underset{-0.53 \div 0.74}{0.15}$
θ_{b-l}	$Beta\,(0.5, 0.2)$	–	–	–	$\underset{-0.51 \div 0.73}{0.19}$
θ_{q-i}	$Beta\,(0.5, 0.2)$	–	–	–	$\underset{-0.47 \div 0.65}{0.10}$
θ_{q-p_k}	$Beta\,(0.5, 0.2)$	–	–	–	$\underset{-0.68 \div 0.73}{0.03}$
θ_{q-r_k}	$Beta\,(0.5, 0.2)$	–	–	–	$\underset{-0.72 \div 0.73}{0.00}$
$\theta_{p-\pi_p}$	$Beta\,(0.5, 0.2)$	–	–	–	$\underset{0.02 \div -0.25}{0.11}$
$\theta_{w-\pi_w}$	$Beta\,(0.5, 0.2)$	–	–	–	$\underset{0.03 \div 0.43}{0.19}$

white noise ($\rho_p,\ \rho_w,\ \theta_p,\ \theta_w$ are the persistence and moving average parameters respectively).[25] Furthermore, when sentiment shocks are included a series of shocks cross effects are estimated, namely:

$$\begin{cases} s_t^c &= e_t^c &+ \theta_{b-c} e_t^b \\ s_t^i &= e_t^i &+ \theta_{q-i} e_t^q \\ s_t^l &= e_t^l &+ \theta_{b-l} e_t^b \\ s_t^{\pi_p} &= e_t^{\pi_p} &+ \theta_{p-\pi_p} e_t^p \\ s_t^{p_k} &= e_t^{p_k} &+ \theta_{q-p_k} e_t^q \\ s_t^{r_k} &= e_t^{r_k} &+ \theta_{q-r_k} e_t^q \\ s_t^{\pi_w} &= e_t^{\pi_w} &+ \theta_{w-l} e_t^w \end{cases} \tag{3.22}$$

The first column of the right hand side is given by the sentiment shocks, the second column is given by the effect of the innovations of the standard shocks on the sentiment shocks.

[25] σ: standard deviation of shocks.ρ: $AR(1)$ coefficient.θ: $MA(1)$ coefficient.a_{g^b}: the effect of TFP innovations on exogenous demand.

Table 3.5: Posterior Estimates: structural parameters.

- mean and 5% ÷ 95% quantiles reported -

	Prior(*mean, std*)	RE	AL	LAL	LALwS
φ	$N\,(4, 1.5)$	5.23 (3.73÷6.89)	2.92 (2.09÷4.52)	5.59 (4.05÷7.30)	6.77 (5.11÷8.60)
σ_c	$N\,(1.5, 0.37)$	1.29 (1.11÷1.51)	1.65 (1.40÷1.91)	1.38 (1.16÷1.61)	0.43 (0.28÷0.65)
η	$Beta\,(0.7, 0.1)$	0.74 (0.65÷0.81)	0.66 (0.58÷0.75)	0.84 (0.78÷0.88)	0.87 (0.81÷0.91)
σ_l	$N\,(2.0, 0.5)$	1.40 (0.56÷2.28)	1.56 (0.83÷2.36)	1.77 (0.94÷2.56)	1.28 (0.54÷2.03)
ξ_p	$Beta\,(0.5, 0.1)$	0.76 (0.68÷0.83)	0.74 (0.69÷0.79)	0.75 (0.68÷0.81)	0.75 (0.61÷0.87)
ξ_w	$Beta\,(0.5, 0.1)$	0.78 (0.68÷0.86)	0.82 (0.77÷0.87)	0.79 (0.74÷0.85)	0.71 (0.64÷0.78)
ι_p	$Beta\,(0.5, 0.15)$	0.24 (0.11÷0.38)	0.32 (0.18÷0.49)	0.40 (0.20÷0.64)	0.26 (0.12÷0.42)
ι_w	$Beta\,(0.5, 0.15)$	0.64 (0.42÷0.83)	0.33 (0.17÷0.53)	0.41 (0.22÷0.63)	0.39 (0.19÷0.61)
ψ	$Beta\,(0.5, 0.15)$	0.72 (0.57÷0.86)	0.70 (0.51÷0.86)	0.65 (0.45÷0.82)	0.66 (0.47÷0.83)
ϕ_p	$N\,(1.25, 0.12)$	1.72 (1.57÷1.90)	1.63 (1.47÷1.80)	1.63 (1.45÷1.83)	1.59 (1.39÷1.82)
ρ_R	$Beta\,(0.75, 0.1)$	0.82 (0.78÷0.86)	0.90 (0.87÷0.93)	0.91 (0.88÷0.93)	0.86 (0.83÷0.89)
r_π	$N\,(1.5, 0.25)$	1.55 (1.34÷1.80)	1.63 (1.33÷1.94)	1.61 (1.34÷1.92)	1.67 (1.44÷1.91)
r_y	$N\,(0.12, 0.05)$	0.04 (0.02÷0.07)	0.11 (0.06÷0.16)	0.08 (0.03÷0.13)	0.07 (0.03÷0.11)
$r_{\Delta y}$	$N\,(0.12, 0.05)$	0.16 (0.12÷0.19)	0.12 (0.09÷0.15)	0.12 (0.09÷0.15)	0.13 (0.10÷0.16)
$\bar{\pi}$	$\Gamma\,(0.62, 0.1)$	0.81 (0.64÷0.97)	0.62 (0.51÷0.73)	0.68 (0.53÷0.84)	0.69 (0.53÷0.87)
$100(\beta^{-1}-1)$	$\Gamma\,(0.25, 0.1)$	0.15 (0.07÷0.24)	0.14 (0.06÷0.24)	0.16 (0.08÷0.29)	0.25 (0.13÷0.39)
$\bar{l}$	$N\,(5.0, 2.0)$	6.62 (4.56÷8.63)	7.58 (6.19÷8.85)	6.78 (4.53÷8.96)	6.91 (5.20÷8.84)
γ	$N\,(0.4, 0.1)$	0.41 (0.38÷0.43)	0.43 (0.39÷0.46)	0.41 (0.37÷0.44)	0.42 (0.38÷0.45)
α	$N\,(0.3, 0.05)$	0.20 (0.18÷0.23)	0.18 (0.15÷0.21)	0.18 (0.15÷0.21)	0.16 (0.13÷0.19)
ρ	$Beta\,(0.5, 0.29)$	–	0.97 (0.96÷0.98)	0.96 (0.90÷0.98)	0.97 (0.94÷0.99)

3.7.2 Historic Variance Decomposition

Table 3.6: LAL no shocks - variance decomposition in %

Structural Shocks:	Δy	r	Δc	Δi	l	π_p	π_w
TFP	8.4	13.4	0.8	0.3	9.0	11.1	0.0
Risk Premium	52.8	17.7	85.6	32.3	33.6	2.9	0.8
Gov. Exp.	24.1	4.0	0.8	0.2	11.3	0.7	0.0
IST	6.9	4.2	0.4	62.3	6.4	2.5	0.0
Monetary	5.3	26.9	8.7	3.4	27.9	3.4	0.4
Price markup	1.9	21.7	3.3	1.4	5.7	62.0	5.5
Wage markup	0.6	11.6	0.6	0.2	6.0	17.5	93.3

Table 3.7: RE - var decomposition in %

Structural Shocks:	Δy	r	Δc	Δi	l	π_p	π_w
TFP	9.7	16.1	1.7	2.2	3.8	11.2	0.8
Risk Premium	32.0	8.5	66.9	6.6	7.4	1.0	0.8
Gov. Exp.	21.1	4.9	2.5	0.6	10.1	0.6	0.0
IST	16.8	24.9	3.3	76.4	16.7	2.7	2.1
Monetary	10.2	17.4	13.4	6.0	9.4	4.4	1.5
Price markup	6.4	6.7	4.8	6.7	14.7	38.5	17.0
Wage markup	3.8	21.5	7.4	1.6	37.2	41.7	77.7

3.8 Appendix III

3.8.1 State Space Form

This section briefly introduces the model in its state space form which is the basis for the likelihood computation and, in turn, for the Bayesian estimation. It consists of an observation and a process equation. The process equation describes the law of motion of the states, that is the economic model, while the observation equation maps them into the data.

The *structural* model given by equations (3.8)-(3.21) constitutes the main building block for our process equation. The model can first be written in its matrix form as

$$Y_t = BY_{t-1} + CY_t + D\hat{E}_t Y_{t+1} + E\omega_t + F\omega_{t-1} \tag{3.23}$$

$$\omega_t = \rho_\omega \omega_{t-1} + S\epsilon_t \tag{3.24}$$

where $Y_t = \left[k_t, y_t, r_t, c_t, i_t, l_t, \pi_t, q_t, r_t^k, w_t\right]'$, is a vector of endogenous states, $\omega_t = \left[\varepsilon_t^a, \varepsilon_t^b, \varepsilon_t^g, \varepsilon_t^q, \varepsilon_t^r, \varepsilon_t^p, \varepsilon_t^w\right]$ the vector containing all seven standard shocks and ϵ_t is a 14×1 random vector of innovations to ω, that also includes the 7 innovations to sentiment shocks. The matrices B, C, D, E, F, ρ_ω and S are then functions of the parameters of the model, θ, of the appropriate size. Notice that the model still includes the subjective expectations operator $\hat{E}_t$.

Expectations are formed by means of reduced-form models whose coefficients are updated using Bayes rules. In particular, following Slobodyan and Wouters (2012) these models are simple $AR(2)$, i.e. linear on the first two lags of the relevant variable that needs to be forecasted and include a constant. From the perspective of the researcher the coefficients estimated by the agents each period become states, rendering the model non-linear. After linearizing, expectations are constructed as

$$\hat{E}_t Y_{t+1} = \bar{\beta}_1 \, Y_t + \bar{\beta}_2 \, Y_{t-1} + \beta_t$$

and the model can be solved for the current states,

$$\begin{pmatrix} Y_t \\ \omega_t \end{pmatrix} = N\beta_t + T \begin{pmatrix} Y_{t-1} \\ \omega_{t-1} \end{pmatrix} + R\epsilon_t \tag{3.25}$$

where [26]

$$N = \begin{pmatrix} \left(Id - C - D\bar{\beta}_1\right)^{-1} D \\ 0_{(7\times 21)} \end{pmatrix}$$

$$T = \begin{pmatrix} \left[\left(Id - C - D\bar{\beta}_1\right)^{-1}\left(B + D\bar{\beta}_2\right)\right] & \left[\left(Id - C - D\bar{\beta}_1\right)^{-1}\left(E\rho_\omega + F\right)\right] \\ 0_{(7\times 10)} & \rho \end{pmatrix}$$

$$R = \begin{pmatrix} \left(Id - C - D\bar{\beta}_1\right)^{-1} ES \\ S \end{pmatrix}$$

The linearized version of the optimal Bayesian updating rules for beliefs can be written as

$$\beta_t = \rho\beta_{t-1} + \rho M \left\{ Y^f_{t-1} - \bar{\beta}_1 \, Y^f_{t-2} - \bar{\beta}_2 \, Y^f_{t-3} - \beta_{t-1} \right\} + \Omega\epsilon_t \tag{3.26}$$

where the exponent f denotes that only the rows corresponding to the forward variables are selected and the matrix M is a composite that includes the Kalman gain evaluated at the REE. Variables appearing with a bar on top denote the corresponding point around which the equation was linearized.

Considering that the mapping of the data to the model is given by,

[26] In our exercise $\mu = 0$.

$$\begin{pmatrix} dlGDP_t \\ FEDFUNDS_t \\ dlCons_t \\ dlINV_t \\ lHours_t \\ dlP_t \\ dlWage_t \end{pmatrix} \equiv O_t = \underbrace{\begin{pmatrix} \gamma \\ r \\ \gamma \\ \gamma \\ l \\ \pi \\ \gamma \end{pmatrix}}_{:=constant} + \begin{pmatrix} \Delta y_t \\ r_t \\ \Delta c_t \\ \Delta i_t \\ l_t \\ \pi_t \\ \Delta w_t \end{pmatrix}$$

Finally, the state vector is defined as $Z_t \equiv \left[Y_t', \omega_t', Y_{t-1}^{f'}, Y_{t-2}^{f'}, dobs_t', \beta_t'\right]'$ and write the model in its state space form

$$\begin{cases} O_t = & Z_t^{obs} \\ Z_t = \mu + & G \cdot Z_{t-1} + V \cdot \epsilon_t \end{cases} \tag{3.27}$$

where obs denotes that only $dobs_t$ is selected from Z_t, $\mu = (0_{1\times 31}, trend', 0_{1\times 7})'$ and

$$G = \begin{pmatrix} T + N\rho M Sel^f & -N\rho M\bar{\beta}_1 & -N\rho M\bar{\beta}_2 & 0 & N\rho\left(Id_7 - M\right) \\ Sel^f & 0 & 0 & 0 & 0 \\ 0 & Id_7 & 0 & 0 & 0 \\ \left(T + N\rho M Sel^f\right)^{obs} - Sel^{obs} & \left(-N\rho M\bar{\beta}_1\right)^{obs} & \left(-N\rho M\bar{\beta}_2\right)^{obs} & 0 & \left(N\rho\left(Id_7 - M\right)\right)^{obs} \\ \rho M Sel^f & -\rho M\bar{\beta}_1 & -\rho M\bar{\beta}_2 & 0 & \rho\left(Id - M\right) \end{pmatrix} \tag{3.28}$$

$$V = \begin{pmatrix} R + N\Omega \\ 0 \\ 0 \\ (R + N\Omega)^{obs} \\ \Omega \end{pmatrix} \tag{3.29}$$

where Sel^f is a matrix selecting the forward variables in $\left(Y'_{t-1}, \omega'_{t-1}\right)$ and *obs*, again, denotes that only the rows corresponding to the observable variables are selected. The model given by (3.27) is used to compute the likelihood of the data for each parameter draw generated by the MH algorithm.

3.8.2 Data

The model is estimated using data on seven US macroeconomic variables: real GDP, short term interest rate (Federal Funds rate), real consumption, real investment, hours worked, Inflation and real wages. The data is constructed as in Slobodyan and Wouters (2012) and updated to include the latest releases. For example, real GDP is expressed in billions of chained 2009 dollars. It covers the period ranging from the first quarter of 1966 till the fourth of 2013 and includes a pre sample of 4 quarters starting in 1965. Aggregate real variables are expressed in per capita terms and all series have been adjusted for seasonality. Furthermore, all variables are expressed in percentage points.

Chapter 4

Monetary Policy under Internal Rationality

4.1 Introduction

The forward-looking behavior of economic agents is one of the salient features of modern macroeconomics and brings about the critical issue of how expectations formation should be modeled. Since the work of Muth (1961), and specially after the contributions of Lucas (1972) and Sargent (1973), Rational Expectations (RE) has been the main paradigm under which expectations are formed. Notwithstanding its success, the RE hypothesis suffers from several shortcomings which led to the proposal of alternative approaches. Among them, Adaptive Learning (AL) has become one of the most fruitful ones. However, the AL literature rests upon a series of ad-hoc assumptions and degrees of freedom that render the modeling scheme in part arbitrary. To cope with these issues Adam and Marcet (2011) formalize the concept of Internal Rationality (IR), a decision theoretic framework that serves as well-defined microfoundations for Adaptive Learning models. This paper applies the internal rationality framework to a simple New Keynesian model and uses it to study the stability properties of different monetary policy rules.

From a formal point of view, rational expectations can be defined as the mathematical conditional expectation generated by the probability distributions that emerge from the model in equilibrium. Thus, by construction, expectations are pinned down by the rest of the model.[1] This disciplines them, as it does not leave any degrees of freedom for how expectations are to be determined. Moreover, it strongly simplifies models, as one does not need to keep track of the beliefs of agents anymore. However, the tight link between fundamentals and expectations that RE impose prevents the latter from being an autonomous source of dynamics, creating several drawbacks. For example, to match the data DSGE models under RE generally need to resort to a series of mechanical sources of persistence (e.g. habit formation in consumption, inflation indexation or adjustment costs of investment), as well as highly auto correlated exogenous shocks that have been criticized as not sufficiently supported by actual evidence.[2] Moreover, RE rest upon a strong informational assumption that requires agents to know the entire model to a degree that not even the theorist has (e.g. agents are assumed to know the value of all the parameters in the model).

Adaptive learning departs from a less demanding and, thus, more realistic informational

[1]Multiple equilibrium considerations are at this point omitted.

[2]For an overview of the shortcomings faced by Rational Expectations in Macroeconomics, see the survey of Milani (2012) and the references therein.

assumption, which, necessarily, prevents agents from deriving the objective probability distributions that would prevail under RE. Then, adaptive learning assumes that agents form expectations using reduced fom models. However, the parameters of those models are unknown to the agents and each period they need to estimate them using the latest available information.[3] This relaxes the relation between fundamentals and expectations creating further dynamics and improving the model's performance. Cogely and Sargent (2008) and Adam et al. (2015), show how models under AL are able to explain several asset pricing puzzles that remained elusive under RE. While Milani (2007) and Slobodyan and Wouters (2012) show how AL creates further endogenous persistence in DSGE models reducing their dependency on both mechanical sources of persistence and highly autocorrelated exogenous processes. Moreover, recent empirical evidence as the one presented in Slobodyan and Wouters (2012) among others, shows how adaptive learning can considerably improve a model's fit to the data relative to their RE counterparts.

In addition, there is an inherent value to having different expectations formation mechanisms available, since most models actually represent a simultaneous test of both the underlying theory and the expectations formations mechanism itself. Consequently, as stressed by MacCallum (1999), Taylor (1999) and Milani (2012) among others, a misspecification of the latter may lead to the rejection of a possibly valid theory or to the adoption of models that would be discarded under different expectations assumptions. Therefore, and given the degree of uncertainty in economics, the wider the range of expectations formation mechanisms under which a policy achieves its goals the more desirable it becomes.

Notwithstanding, as pointed out by Adam and Marcet (2011), adaptive learning fails to provide a clear and consistent framework where deviations from RE can be properly studied. In particular it is not clear whether agents remain rational under AL. There are two main reasons for this. The first one, is due to the manner in which AL is introduced. Instead of setting the subjective expectations operator already from the outset, in the definition of agents' problems, AL departs from the optimality conditions of the model under RE and only then replaces the rational expectations operator by the subjective one assumed under AL. This is an important source of arbitrariness as optimality conditions under RE can be expressed in different equivalent ways, which under learning may lead to different results. For example, Adam, Marcet and Nicolini (2015) and Timmermann (1996) both study a simple asset pricing model under learning. However, while the first one departs form a one period ahead optimal

[3]See Evans & Honkapohja (2001) for a detail treatise of Adaptive Learning.

pricing condition and finds a significant effect of learning, the second one departs from the classical equation that sets prices equal to the expected discounted sum of future dividends and finds a rather modest effect. Something similar can be observed in the context of a simple monetary model. While Bullard and Mitra (2002) depart form the usual consumption Euler equation, Preston (2005) uses the inter-temporal budget constraint to derive decision rules that involve agents expectations of the whole future, leading to different results. Both approaches are known as the Euler Equation (EE) approach and the Infinite Horizon (IH) approach respectively. In principle, all these different setups could be justified provided an ad-hoc information set for agents (see for example, Evans, Honkapojha & Mitra (2012) for a discussion on how the EE and IH approach can be reconciled). However, and even if those ad-hoc assumptions were accepted, it is not clear whether agents behavior remain optimal under those particular conditions. Moreover, a theory whose results depend on the manner in which it is stated is not an appealing one.

The second reason is given by the manner in which agents are assumed to estimate the parameters of the reduced form models they use to form expectations. For most stochastic approximation algorithms assumed by the learning literature, such as ordinary least squares learning or constant gain learning, it is not clear whether they arise form agents using information in an optimal manner (optimal in a Bayesian sense).

Internal Rationality seeks to solve these issues by recognizing two complementary conditions comprised within the REH. The first one, *internal rationality*, stipulates that agents make fully optimal decisions given a well-defined system of beliefs about those variables they need to learn about (non-choice variables) and the information they have about the economy. The second one, *external rationality*, states that agents subjective probability beliefs coincide with the objective ones as they emerge in equilibrium. The proposed approach consists of relaxing the external rationality assumption while keeping the internal rationality one. Thus, creating a more natural and sensible starting point for economic models.

This paper derives a basic New Keynesian model under Internal Rationality and uses it to study the stability properties of different monetary policy rules. The resulting model yields decision rules that, similar to the IH approach of Preston (2005), require agents to forecast variables for the whole infinite future. However, differently form the IH and EE approaches agents under IR do not form expectations about own choice variables, such as own consumption or labor choices. Instead, agents under IR are required to forecast their future real wages and profits. Two Taylor-type rules are then considered. First a monetary policy rule in which the nominal interest rate is set proportional to the current period's inflation level and a second one that reacts to expectations of inflation one period ahead. Both rules are representative of an extended class of feedback rules suggested by various authors. The two monetary rules are then studied in order to determined whether agents that at some point and for some reason are away of a particular REE could coordinate on it and *learn* it.

The first rule is found to be learnable if the so-called Taylor principle holds. A principle that states that relative aggressive responses to inflation generally lead to unique rational expectations equilibria and that therefore are desirable. This result coincides with the one found under RE and for AL under both IH and EE approaches. For the second rule, however, the Taylor-principle is found to be necessary but not sufficient. A result that contrasts with the conclusions reached under RE and AL in its EE approach, and coincides with the one found under the IH approach. This is an important result, since it is generally argued that a judicious monetary policy is one that satisfies the Taylor principle.

The rest of the paper is structured as follows. Section 4.2 derives the core of the model. Section 4.3 describes the expectation formation mechanism. Section 4.4 briefly discusses the theory used to analyse the stability of monetary policy. Section 4.5 presents and discusses the results. Section 4.6 concludes.

4.2 Model

This section derives the model under Internal Rationality which is later used to study monetary policy. It is based on the basic New Keynesian Model under Rational Expectations found in Gali (2008) and belongs to the family of models in the literature which, under different expectations formation mechanisms, analyze simple Taylor-type monetary policy rules.[4]

4.2.1 Households' Problem

The economy is populated by a continuum of infinitely-lived households facing the same utility maximization problem. Each household chooses consumption, C_T^j, hours worked, N_T^j,

[4] Clarida et al (1999) is a classical example for this framework under RE while Bullard and Mitra (2002) and Preston (2005) are examples for when different Adaptive Learning expectation formation mechanisms are assumed.

and how much to invest in one period bonds, B_T^j, subject to a simple budget constraint requiring the periods' outgoings to be smaller than the periods' total wealth, i.e.

$$\max_{\{C_t^j, B_t^j, N_{tj}\}_{T=t}^{\infty}} \hat{E}_t^j \sum_{T=t}^{\infty} \beta^{T-t} U\left(C_T^j, N_T^j; \xi_T\right) \tag{4.1}$$

$$s.t. \;\; P_T C_T^j + B_T^j \;\; \leq \;\; W_T N_T^j + (1 + i_{T-1}) B_{T-1}^j + D_T \tag{4.2}$$

where β stands for the one period discount factor, P_T is the price of one consumption unit in period T, W_T is the nominal wage payed for one unit of labor and i_{t-1} is the nominal interest rate valid from period $t-1$ to period t. Furthermore, households are payed dividends, D_T, as they are assumed to own the firms in the economy. The period utility $U\left(C_T^j, N_T^j; \xi_T\right)$ is assumed to be continuous, twice differentiable, with $U_c > 0$, $U_{cc} \leq 0$, $U_n \leq 0$, $U_{nn} \leq 0$ and $U_{nc} = 0$ and to depend on a preference shock, ξ_T. Finally, $\hat{E}_t^j$ denotes the subjective expectations operator used by each household to construct forecasts of future variables. It is defined on an underlying subjective probability space, $\left(\Omega, \mathcal{F}, P^j\right)$, and even though it is assumed to be homogenous among agents, they do not know this to be true. A particularly sensible departure point, since endowing agents with knowledge about the beliefs of all other agents in the economy, and further imposing its common knowledge, is an extremely strong assumption.[5] Choice variables, as well as the subjective expectation operator corresponding to each agent in the economy are indexed with the letter j and helps distinguish between what agents know and what the aggregate economy ultimately yields.

Under internal rationality agents are required to entertain beliefs about all external variables (i.e. not depending on their own decisions) entering the problems they face. That, together with the agents assumed optimizing behavior, defines functional relations between the agents' own choice variables and external variables. As a result, rational agents cannot entertain beliefs about own choice variables that violate those optimal functional relations and consequently, to arrive to their optimal decision rules we must substitute them out. For this purpose the budget constraint (4.2) needs to be substituted forward which together with the appropriate non-Ponzi constraint, $\lim_{T\to\infty} \prod_{s=0}^{T} \frac{1}{1+i_{t-1+s}} B_{t+T} = 0$, yields an inter temporal

[5] Introducing heterogeneous beliefs in this setup becomes straight forward.

budget constraint of the following form,

$$B^j_{t-1} = \sum_{k=0}^{\infty} \left(\prod_{s=0}^{k} \frac{1}{1 + i_{t-1+s}} \right) \left(P_{t+k} C^j_{t+k} - W_{t+k} N^j_{t+k} - D^j_{t+k} \right) \tag{4.3}$$

This equation has a clear intuitive interpretation. The agent's life time expenditures net of their life time income can only be financed by the agent's initial wealth.

In order to arrive to a linear consumption decision rule, the inter temporal budget constraint, (4.3), together with the first order conditions of the household's maximization problem are, as usual, log-linearized. The resulting optimal allocation relations are given by the following system of equations, valid for all $T \geq t$,

$$\hat{E}^j_t \sum_{k=0}^{\infty} \beta^{k+1} \left\{ \overline{PC} \hat{c}^j_{t+k} - \overline{WN} \left(\hat{\omega}_{t+k} + \hat{n}^j_{t+k} \right) - \overline{D} \hat{\delta}^j_{t+k} \right\} = 0 \tag{4.4}$$

$$\hat{E}^j_t \left\{ \hat{c}^j_T + \sigma^{-1} \left(\hat{i}_T - \hat{\pi}_{T+1} + g_T \right) \right\} = \hat{E}^j_t \left(\hat{c}^j_{T+1} \right) \tag{4.5}$$

$$\hat{E}^j_t \left\{ \varphi^{-1} \left(\hat{\omega_T} - \sigma \hat{c}^j_T \right) \right\} = \hat{E}^j_t \left(\hat{n}^j_T \right) \tag{4.6}$$

Equation (4.4) is the log-linear counterpart of equation (4.3) where variables with a bar on top denote steady state values and ' ˆ ' denotes percentage deviations of a variable from its corresponding steady state. In particular, $\hat{\omega}_{t+k}$ denotes real wages and $\hat{\delta}_{t+k}$ real dividends.[6] Note that bond holdings are no longer present. This is because bonds are assumed to be in zero net supply. Equation (4.5) describes the agents usual consumption Euler equation, where $\sigma^{-1} \equiv -\frac{U_c}{U_{cc} C}$ denotes the inter temporal elasticity of substitution for consumption, $\hat{\pi}_{T+1}$ is price inflation and $g_t \equiv \frac{U_{c\xi} \bar{\xi}}{U_c} \Delta \hat{\xi}_{t+1}$ a redefined preference shock. Finally, equation (4.6) describes the optimal labor allocation for households, where $\varphi^{-1} \equiv \frac{U_n}{U_{nn} \bar{N}}$.

This system can then be easily solved for $\hat{c}^j_t$: First, the labor supply equation can be used to get rid of all future labor choices in the inter temporal budget constraint; Second, the Euler equation can be solved backwards to express every future consumption choice, $\hat{c}^j_T$, as a function of the previous periods real interest rates, preference shocks and the current household's consumption choice, $\hat{c}^j_t$. The resulting condition can then be used to get rid of future consumption choices in the inter temporal budget constraint. Thus, one is left with an equation describing the household's optimal behavior as a function of their subjective beliefs

[6] Except for the nominal interest rate which is defined as $\hat{i} = ln\,(1 + i_t)$.

and their implied forecasts of external economic conditions for their whole future,

$$\begin{aligned}\hat{c}_t^j &= -\sigma^{-1}\sum_{k=0}^{\infty}\beta^{k+1}\hat{E}_t^j\left(\hat{i}_{t+k}-\hat{\pi}_{t+1+k}\right)+\frac{(1-\beta)\left(1+\varphi^{-1}\right)}{\beta\left(\frac{\overline{PC}}{\overline{WN}}+\sigma\varphi^{-1}\right)}\sum_{k=0}^{\infty}\beta^{k+1}\hat{E}_t^j\left(\hat{\omega}_{t+k}\right) \\ &+\frac{(1-\beta)\overline{D}}{\beta\left(\overline{PC}+\sigma\varphi^{-1}\overline{WN}\right)}\sum_{k=0}^{\infty}\beta^{k+1}\hat{E}_t^j\left(\hat{\delta}_{t+k}\right)-\sigma^{-1}\sum_{k=0}^{\infty}\beta^{k+1}\hat{E}_t^j\left(\hat{g}_{t+k}\right)\end{aligned} \tag{4.7}$$

This contrasts with the one period ahead decision rule under rational expectations given by

$$\hat{c}_t = E_t\left(\hat{c}_{t+1}\right)+\sigma^{-1}\left(\hat{i}_t - E_t\left(\hat{\pi}_{t+1}\right)+g_t\right) \tag{4.8}$$

and which is many times used as a departing point for AL models.[7] The reason for this difference is very simple. Consider equation (4.7) again. Shifted one period forward and applying the subjective expectations operator, $\hat{E}_t^j$, on both sides of the equality one gets that,

$$\begin{aligned}\hat{E}_t^j\left(\hat{c}_{t+1}^j\right) &= -\sigma^{-1}\sum_{k=0}^{\infty}\beta^{k+1}\hat{E}_t^j\left(\hat{i}_{t+1+k}-\hat{\pi}_{t+2+k}\right) \\ &+\frac{(1-\beta)\left(1+\varphi^{-1}\right)\overline{WN}}{\beta\left(\overline{PC}+\sigma\varphi^{-1}\overline{WN}\right)}\sum_{k=0}^{\infty}\beta^{k+1}\hat{E}_t^j\left(\hat{\omega}_{t+1+k}\right) \\ &+\frac{(1-\beta)\overline{D}}{\beta\left(\overline{PC}+\sigma\varphi^{-1}\overline{WN}\right)}\sum_{k=0}^{\infty}\beta^{k+1}\hat{E}_t^j\left(\hat{\delta}_{t+1+k}\right) \\ &-\sigma^{-1}\sum_{k=0}^{\infty}\beta^{k+1}\hat{E}_t^j\left(\hat{g}_{t+1+k}\right)\end{aligned}$$

Then, it can be easily observed that $\hat{E}_t^j\left(\hat{c}_{t+1}^j\right)$ could be used to substitute for the infinite sums on the RHS of equation (4.7) collapsing it to a one step ahead consumption decision rule (as in (4.8)). However, while under rational expectations that step is valid[8] under AL it is clearly not. Forming expectations about own future consumption by means of an estimated reduce form model that describes its law of motion is not only non rational (since agents do not forecast their own decisions) but, moreover, it does not coincide with the right hand side of the equation which depends on expectations of external variables - based on estimated reduce form models themselves. Is in this way that equation (4.8) is not valid under internal rationality and instead agents' behavior is determined according to (4.7).

[7] Bullard & Mitra (2002), Evans & Honkapojha (2008) among others.

[8] Considering the steps made to derive equation (4.7), it is clear that the same equation holds under rational expectations (i.e. when the subjective expectations operator, $\hat{E}_t^j$, is substituted for the objective one, E_t).

4.2.2 Firms' Problem

The production in the economy encompasses two sectors. An *intermediate goods* sector formed by a continuum of firms producing differentiated goods and a *final goods* sector consisting of a single firm using the intermediate goods as inputs to produce the unique consumption good in the economy.

The intermediate goods are transformed into the final good by means of a standard CES production technology

$$Y_t = \left(\int_0^1 Y_t^{i\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}} \tag{4.9}$$

where $\epsilon > 1$ stands for the elasticity of substitution between inputs, Y_t denotes the final good production level and Y_t^i the intermediate good produced by firm i. Each period, the final goods firm minimizes the cost of producing any given production level yielding a set of demand schedules for each intermediate good Y_t^i as a function of their price p^i, the aggregate price level in the economy, P_t, and the final good's production level, Y_t, and which is assumed to be known by the intermediate firms,

$$Y_t^i = \left(\frac{p^i}{P_t} \right)^{-\epsilon} Y_t \tag{4.10}$$

where the price level is further defined as $P_t \equiv \left(\int_0^1 P_t^{i1-\epsilon} di \right)^{\frac{1}{1-\epsilon}}$. Finally, the final goods firm sells its output in a competitive market.

As mentioned above, the intermediate goods sector is populated by a continuum of firms, of mass one, each producing a differentiated intermediate good by means of a simple Cobb-Douglas production technology, $Y_t^i = A_t N_t^i$. The differentiated products provide firms with monopolistic power which is used to set prices. However, they do so according to a standard Calvo pricing scheme (see Calvo (1983)).[9] Thus, every period only a fraction $1-\alpha \in (0,1)$ of firms is allowed to re-set their prices, meaning that on average prices have a duration equal to $\frac{1}{1-\alpha}$ periods.

Therefore, each period, the firms allowed to set prices do it by maximizing the discounted sum of expected future profits subject to the demand they face and taking into account the

[9]Equivalently a quadratic cost function for nominal price adjustments as proposed in Rotemberg (1982) could be assumed.

possibility that they will not be able to change their prices in the future, i.e.

$$\max_{p^i} \hat{E}_t^i \sum_{k=0}^{\infty} \alpha^k Q_{t+k} \left[p^i Y_{t+k}^i - W_{t+k} N_{t+k}^i \right] \tag{4.11}$$

$$s.t. \;\; Y_{t+k}^i = \left(\frac{p^i}{P_{t+k}} \right)^{-\epsilon} Y_{t+k} \tag{4.12}$$

where $Q_{t+k} \equiv \prod_{s=1}^{k} (1 + i_{t+s-1})^{-1}$, $Q_t \equiv 1$ and p^i denotes the price set by firms. Again, as it was the case for households, firms are homogenous, implying that each period the firms that are allowed to change their prices choose the same price. Nevertheless, this is not know to firms.

The first order condition for this problem can be written as,

$$\hat{E}_t^i \sum_{k=0}^{\infty} \alpha^k Q_{t+k} Y_{t+k} P_{t+k}^{\epsilon} \left[p^i - \mu \frac{W_{t+k}}{A_{t+k}} \right] = 0 \tag{4.13}$$

and after an appropriate log-linearization the price setting decision can be expressed as,

$$\hat{p}^i = (1 - \alpha\beta) \hat{E}_t^i \sum_{k=0}^{\infty} (\alpha\beta)^k \{ \hat{\omega}_{t+k} - \hat{a}_{t+k} + \hat{p}_{t+k} \} \tag{4.14}$$

Firms, then, set prices considering the probability they will not be able to do it again in the future and thus, taking into account changes in their expected future real costs and on the price level of the economy.

4.2.3 Aggregate Equilibrium

A series of steady state relations, complementary equations and equilibrium conditions provide the remaining structure to derive the economy's aggregate equilibrium. Market clearing in the goods and labor market together with the production technology yield the following steady state identity,

$$\bar{C} = \bar{Y} = \bar{N} \tag{4.15}$$

while the first order condition for firms sets the real wage in steady state equal to the inverse of the mark-up,

$$\frac{\bar{W}}{\bar{P}} = \frac{1}{\mu} \tag{4.16}$$

The definition of households dividends implies that in steady state,

$$\bar{D} = \bar{Y}\bar{P} - \bar{W}\bar{N} = \bar{W}\bar{Y}(\mu - 1) \tag{4.17}$$

Also, since firms are homogenous and each period a fraction $1-\alpha$ is able to re-set prices, the economy's price level evolves as follows

$$P_t = \left(\alpha P_{t-1}^{1-\epsilon} + (1-\alpha)\left(p^i\right)^{1-\epsilon}\right)^{\frac{1}{1-\epsilon}}$$

which after linearization can be written as,

$$\hat{p}^i = \frac{\hat{\pi}_t}{1-\alpha} + \hat{p}_{t-1} \tag{4.18}$$

Finally, the following conditions implied by market clearing conditions, the labor optimality condition, the production technology and the dividends definition hold,[10]

$$\begin{aligned} \hat{c}_t^j &= \hat{c}_t = \hat{y}_t = \hat{y}_t^i && (4.19)\\ \hat{\omega}_t &= (\varphi + \sigma)\,\hat{y}_t - \varphi \hat{a}_t && (4.20)\\ \hat{\delta}_t &= \frac{\mu - 1 - \varphi - \sigma}{\mu - 1}\hat{y}_t + \frac{\varphi + 1}{\mu - 1}\hat{a}_t && (4.21) \end{aligned}$$

Note that these conditions only hold in period t, stressing that they arise from market equilibrium conditions and that they are not known to the agents.

Then, using equations (4.15)-(4.21) and integrating over all firms and households we can write the economy's Philips curve as,

$$\hat{\pi}_t = \kappa \hat{y}_t + \frac{(1-\alpha)}{\alpha}\hat{E}_t \sum_{k=1}^{\infty} (\alpha\beta)^k \left\{(1-\alpha\beta)\,\hat{w}_{t+k} - (1-\alpha\beta)\,\hat{a}_{t+k} + \hat{\pi}_{t+k}\right\} - \Omega \hat{a}_t \tag{4.22}$$

where $\kappa = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha}(\varphi + \sigma)$ and $\Omega = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha}(\varphi + 1)$ and the economy's IS curve as,

$$\begin{aligned} \hat{y}_t &= -\sigma^{-1}\hat{i}_t - \sigma^{-1}\sum_{k=1}^{\infty}\beta^k \hat{E}_t\left(\hat{i}_{t+k} - \beta^{-1}\hat{\pi}_{t+k}\right) + \frac{(1-\beta)\left(1+\varphi^{-1}\right)}{\frac{\mu}{1-\gamma} + \sigma\varphi^{-1}}\sum_{k=1}^{\infty}\beta^{k-1}\hat{E}_t\left(\hat{\omega}_{t+k}\right) \\ &\quad + \frac{(1-\beta)(\mu - 1 + \gamma)}{\mu + \sigma\varphi^{-1}(1-\gamma)}\sum_{k=1}^{\infty}\beta^{k-1}\hat{E}_t\left(\hat{\delta}_{t+k}\right) - \sigma^{-1}\sum_{k=0}^{\infty}\beta^k \hat{E}_t\left(\hat{g}_{t+k}\right) \qquad (4.23) \end{aligned}$$

[10]They are expressed in their log-linearized form.

where $\hat{E}_t = \int_{[0,1]} \hat{E}_t^j dj$. Since agents where assume to be homogenous we also have that $\hat{E}_t = \hat{E}_t^j = \hat{E}_t^i$, but in general the operator $\hat{E}_t$ does not need to satisfy standard probability properties and is only the average of expectations prevailing in the economy.

Equations (4.22)-(4.23) determine the evolution of inflation and output as a function of agents expectations of economic conditions for the whole infinite future and constitute the core of the New Keynesian model under Internal Rationality. This setup is similar to the one proposed by Preston (2005), however, here agents do not entertain beliefs about future own choice variables.

The exogenous processes for the technology shock and the preference shock are assumed to follow simple AR(1) process, i.e.

$$\hat{a}_t = \rho_a \hat{a}_{t-1} + \varepsilon_t^a \tag{4.24}$$

$$\hat{g}_t = \rho_g \hat{g}_{t-1} + \varepsilon_t^g \tag{4.25}$$

where $\rho_a,\ \rho_g \in (0,1)$. To close the model two more things remain. The way in which the monetary authority conducts policy, i.e. sets the short term nominal interest rate, $\hat{i}_t$. And the way in which agents form expectations.

4.3 Agents' Beliefs

One particularly important building block of a learning model is the beliefs agents are assumed to entertain each period and how they are assumed to evolve over time. Internal rationality stresses the need of properly defining this system of beliefs and of making it consistent with the optimal behavior of agents. In other words, internal rationality requires a well defined probability space and an optimal and consistent updating of the distributions there defined.

Agents beliefs encompass three components, a model that describes their perception about the law of motion of the variables they are required to make forecast of; a prior distribution of the parameters that characterize that reduced form model; and a strategy to update that prior as new data becomes available in time.

Regarding their perceived law of motion, and following the literature, agents use a reduced form model of the same form as the minimum state variable rational expectations equilibrium

(MSV-REE). In the context of our simple New Keynesian model, and given the interest rate policy rules later considered, the rational expectation equilibrium in question is always linear on a constant and the two exogenous processes, $\hat{a}_t$ and $\hat{g}_t$. That means that agents construct expectations by means of the following model

$$y_t = x_t'\theta + \eta_t \tag{4.26}$$

where $y_t = \left(\hat{i}_t, \hat{\pi}_t, \hat{\omega}_t, \hat{\delta}_t\right)'$, $x_t = (1, \hat{a}_t, \hat{g}_t)'$, $\theta \in \mathbb{R}^{3\times 1}$ and $\eta_t \in \mathbb{R}^{4\times 1}$ is the standard error term, $\eta_t \sim \mathcal{N}(0, \Sigma)$. Learning is, then, nothing else than the search for the true value of θ by using new information to update the beliefs agents have about it, i.e. its distribution. Bayes rule provides the optimal way to do that. And in particular, by assuming the appropriate prior and likelihood distribution forms, the Bayesian updating rule gives rise to the familiar Least Squares Learning scheme and allows us to make use of the whole machinery of the standard learning theory to study the model dynamics.

With that purpose in mind, the likelihood function is chosen to be of the multivariate-normal form, which given (4.26) translates into

$$L(y_t \mid x_t, \theta, \Sigma) = (2\pi)^{-\frac{5}{2}} \mid \Sigma \mid^{-\frac{1}{2}} e^{\frac{1}{2}(y_t - x_t'\theta)'\Sigma^{-1}(y_t - x_t'\theta)} \tag{4.27}$$

where $\Sigma = \sigma^2 I_5$, $\sigma^2 = \phi^{-1}$ is a random variable and I_5 is the identity matrix of order 5. Then, it needs to be further assumed that the prior agents have for θ and ϕ[11] is given by a Normal-Gamma distribution, a conjugate prior of the normal distribution, i.e.

$$\pi\left(\theta, \phi; \theta_0, N_0^{-1}, a_0, b_0\right) \propto \phi^{a-1} e^{-b\phi} e^{-\frac{\phi}{2}(\theta-\theta_0)' N_0 (\theta-\theta_0)} \tag{4.28}$$

Under these conditions Bayes rule specifies how the posterior, i.e. the updated beliefs distributions using the last data point, looks like,

$$\begin{aligned}
\pi\left(\theta, \phi \mid y_t; \theta_1, N_1^{-1}, a_1, b_1\right) &\propto L(y_t \mid x_t, \theta, \phi I_5)\, \pi\left(\theta, \phi; \theta_0, N_0^{-1}, a_0, b_0\right) \\
&\propto \phi^{\frac{1}{2}} e^{\frac{\phi}{2}(y_t - x_t'\theta)'(y_t - x_t'\theta)} \phi^{a_0 - 1} e^{-b_0\phi} \\
&\quad e^{-\frac{\phi}{2}(\theta-\theta_0)' N_0 (\theta-\theta_0)}
\end{aligned}$$

[11] It is more convenient to work with the precision instead of the variance.

After completing the square and some minor algebra the posterior can be re-written as

$$\pi\left(\theta,\phi \mid y_t;\theta_1,N_1^{-1},a_1,b_1\right) \propto \phi^{a_0+\frac{1}{2}-1}e^{-\phi\left[b_0+\frac{1}{2}\left(y_t'y_t+\theta_0'N_0\theta_0-\theta_1'N_1\theta_1\right)\right]} e^{-\frac{\phi}{2}(\theta-\theta_1)'N_1(\theta-\theta_1)}$$

which is nothing else than the kernel of a Normal-Gamma distribution with updated parameters given by,

$$\theta_1 = \left(x_tx_t'+N_0\right)^{-1}\left(x_ty_t+N_0\theta_0\right) \tag{4.29}$$

$$N_1 = x_tx_t'+N_0 \tag{4.30}$$

$$b_1 = b_0+\frac{1}{2}\left(y_t'y_t+\theta_0'N_0\theta_0-\theta_1'N_1\theta_1\right)$$

$$a_1 = a_0+\frac{1}{2}$$

Moreover, since agents use the mean of their distributions to take decisions, one only need to keep track of θ_t and, consequently, of N_t. Using the matrix inversion lemmas one can further re-write equations (4.29) and (4.30) as,

$$\begin{aligned} \theta_t &= \theta_{t-1}+\left(x_tx_t'+N_{t-1}\right)^{-1}x_t(y_t-x_t'\theta_{t-1}) \\ N_t &= x_tx_t'+N_{t-1} \end{aligned}$$

And finally, by setting $R_t = tN_t$ one gets

$$\begin{aligned} \theta_t &= \theta_{t-1}+\frac{1}{t}R_t^{-1}x_t(y_t-x_t'\theta_{t-1}) \\ R_t &= R_{t-1}+\frac{1}{t}\left(x_tx_t'+R_{t-1}\right) \end{aligned} \tag{4.31}$$

which are non other than the recursive least square formulas describing Ordinary Least Square Learning.

4.4 Learning Dynamics

In the previous section it has been shown how ordinary least squares learning can arise from a well-defined and dynamically consistent beliefs structure. Thus, the whole machinery of adaptive learning, as presented in Evans & Honkapohja (2001), can be put to work in order to study the dynamic properties of different monetary policies under internal rationality.

The interest is then to derive the conditions under which a particular rational expectations equilibria is learnable; whether departing from an initial temporary equilibrium different form the REE of interest, and given enough data, agents are able to *coordinate* and eventually converge to it. For this purpose, Evans & Honkapohja (2001) introduced the concept of expectational stability which shows how to derive conditions that guarantee that a rational equilibrium is indeed learnable or stable. Here, the idea behind this concept is briefly depicted.

Learning is characterized by a sequence of temporary equilibrium. Each period agents use their reduced form models together with their latest estimated parameters, $\beta_t \equiv (b_t,\ c_t,\ d_t)'$, to form the expectations needed to take decisions. In the model at hand, those forecasts about future variables take a simple form

$$\hat{E}_t^j y_T = b_t + c_t \rho_a^{T-t} \hat{a}_t + d_t \rho_g^{T-t} \hat{g}_t \quad \forall T > t$$

To take their decisions agents substitute these expectations in (4.22) and (4.23)[12], which together with market clearing conditions the monetary policy rule and so forth, imply the periods' realized inflation and output. This temporary equilibrium can be succinctly written as

$$\begin{pmatrix} \hat{\pi}_t \\ \hat{y}_t \end{pmatrix} = Ab_t + (B + Cc_t)\,\hat{a}_t + (D + Ed_t)\,\hat{g}_t$$

where $A,\ B,\ C,\ D$ and E are real matrices collecting appropriate terms.[13] Therefore, each period there is an implicit function mapping the estimated parameters of the reduced form models used by agents to form expectations into the parameters characterizing the law of motion that actually determines the period's equilibrium,

$$T\,(b_t,\ c_t,\ d_t) = (Ab_t,\ B + Cc_t,\ D + Ed_t)$$

Evans & Honkapohja (2001) refer to this mapping as the T-map and by means of stochastic approximation methods they show that the convergence under ordinary least square learning to a rational expectations equilibrium, β^{ree}, is governed by the stability properties of the

[12]Actually, they substitute them in (4.7) and (4.14), which after aggregation deliver the equivalent result as directly substituting in the aggregate equilibrium equations (4.22) and (4.23).

[13]Real wages, dividends and the nominal interest rate in equilibrium can be easely determined as a function of the periods inflation, output and shock realizations.

associated ordinary differential equation (o.d.e.)

$$\frac{d}{d\tau}(\beta) = T(\beta) - \beta$$

where τ represents "notional" time (an infinitesimal change in time).

It can be easily seen that the resting points of this o.d.e. are fixed points of T. Which in turn are nothing else than the REE of the model, i.e.$T(\beta^{ree}) = \beta^{ree}$. Therefore, a rational expectations equilibrium, β^{ree}, is expectationally stable, or learnable, if its associated differential equation is stable in a neighborhood of β^{ree}.[14] Furthermore, standard properties for ordinary differential equations state that a fixed point of $T(\cdot)$ is stable if and only if the eigenvalues of $J\left[T(\beta^{ree})\right]$ have real parts smaller than 1 (where J denotes the Jacobin operator). Therefore, when studying the stability properties of monetary policy rules it suffices to look for the conditions that guarantee that the real part of the eigenvalues associated to the corresponding T-map remain smaller than 1.

4.5 Monetary Stability

Here, the model under internal rationality, given by equations (4.22) - (4.25) together with (4.26) and (4.31), is used to study the stability properties of two representative monetary policy rules. Both rules set the short term nominal interest rate according to simple feedback rules that respond to inflation alone. The first one, is a contemporaneous policy rule given by

$$\hat{i}_t = \phi_\pi \hat{\pi}_t \tag{4.32}$$

where ϕ_π is assumed to be non-negative. This interest rate rule is generally used as a baseline specification in the literature and also here. However, it has been criticized as unrealistic, since the actual monetary authority does not have complete information on variables such as inflation in the same quarter as they need to take the decision about the nominal interest rate (see McCallum (1999) among others). Hence, a second interest rate rule is considered where this shortcoming is overcomed by specifying the rule as a forward looking one, i.e.

$$\hat{i}_t = \phi_\pi \hat{E}_t \hat{\pi}_{t+1} \tag{4.33}$$

[14] Alternatively, one can think that the rational expectations equilibrium of the model are given by the resting points of the differential equation.

Thus, the monetary authority is assumed to set the nominal interest rate in response to their forecasts of future inflation, for which they need to learn in the same way as the rest of the agents in the economy. It can be easily shown, that these expectations can be formed conditional on information up to period t or $t-1$ without affecting the stability conditions of the policy rule.

Proof. Using the theory briefly depict in section (4) the following two propositions, summarizing the learnability results for both interest rules, are proven.

Proposition 4.1 *Consider the internal rationality model given by (4.22) - (4.25), (4.26), (4.31), and the contemporaneous interest rate rule (4.32). Then, the Taylor principle,*

$$\phi_\pi > 1 \tag{4.34}$$

is necessary and sufficient for expectational stability.

Proof. See the Appendix.

This simple result coincides with the conclusions found under standard adaptive learning approaches (see Bullard & Mitra (2002) and Preston (2005)) and stresses the relevance of the Taylor principle. Namely, that relative aggressive responses to inflation generally lead to learnable rational expectations equilibria and that therefore are desirable as they keep the economy stable. However, as shown by the next proposition this is not always the case.

Proposition 4.2 *Consider the internal rationality model given by (4.22) - (4.25), (4.26), (4.31), and the forward looking interest rate rule (4.32). Then, the Taylor principle is necessary but not sufficient for expectational stability. In particular, the monetary policy should statisfy the following condition,*

$$\frac{1}{1-\beta} + \frac{\sigma(\beta+\beta\alpha-2)}{\kappa(1-\alpha\beta)} < \phi_\pi$$

Proof. See the Appendix. □

That is, even though the strong reactions to variations in the inflation level are still required, these reactions may not suffice to prevent the economy from drifting away. This is a very important result, as it is commonly argued that a judicious monetary policy is one that satisfies the Taylor principle. It can be observed, that the more patient the agents in the economy are, i.e. the higher the discount factor,β, is, the stronger the reaction of the monetary policy to changes in inflation should be.

The result differs from the one found by Bullard & Mitra (2002) under adaptive learning and the Euler Equation approach, where the Taylor principle continues a sufficient condition for learnability, stressing the importance of studying monetary policy under different expectations formation mechanisms. Preston (2005) studies a similar set of interest rules in a framework where agents also form expectations long into the future and arrives to the same stability conditions as under internal rationality. However, as it was already stressed, both setups differ in a number of ways, including their implied dynamics, rendering this coincidence not to be expected.

4.6 Concluding Remarks

This paper derives a basic New Keynesian model under Internal Rationality, a decision theoretic framework that serves as well-defined microfoundations for Adaptive Learning models. The resulting model yields decision rules which require agents to forecast variables for the whole infinite future including their future real wages and profits. The model is then used to study the expectational stability properties of two standard Taylor type interest rate rules. In the first one the monetary authority sets the nominal interest rate as a response to the current period's inflation and in the second one the interest rate reacts to expectations of inflation one period ahead. The first rule is found to be expectationally stable under internal rationality, i.e. learnable, if the so-called Taylor principle holds. A principle that states that relative aggressive responses to inflation generally lead to unique rational expectations equilibria and that therefore are desirable. For the second rule, however, the Taylor-principle is found to be necessary but not sufficient. A finding that differs from standard results under more classical expectations formation mechanisms. This is an important result, since it is generally argued that a judicious monetary policy is one that satisfies the Taylor principle. Moreover, it underscores the importance of considering monetary policies under diverse expectations formation mechanisms, as their performance hinge on the expectations specification in the model. Furthermore, ultimately, these alternative expectations formation mechanisms need to be empirically validated.

4.7 Appendix

4.7.1 Proof of Proposition 4.1

In order to determine the conditions under which the contemporaneous interest rate rule (4.32) renders the model at hand expectationally stable one needs to consider its associated T-map. This function maps the estimated parameters of the reduced form models agents use to form expectations into the actual parameters of the laws of motion that determine endogenous variables in equilibrum. Let $\theta_t = (b_t, c_t, d_t) = \left(b_t^\pi, b_t^i, b_t^\omega, b_t^\delta, c_t^\pi, c_t^i, c_t^\omega, c_t^\delta, d_t^\pi, d_t^i, d_t^\omega, d_t^\delta\right)'$ denote the vector of estimated parameters in period t and let $T: \mathbb{R}^{12} \to \mathbb{R}^{12}$ denote the T-map. Let us further express the T-map in the following way,

$$\begin{aligned} T(\theta_t) = (&T_{b^\pi}(\theta_t), T_{b^i}(\theta_t), T_{b^\omega}(\theta_t), T_{b^\delta}(\theta_t), \\ &T_{c^\pi}(\theta_t), T_{c^i}(\theta_t), T_{c^\omega}(\theta_t), T_{c^\delta}(\theta_t), \\ &T_{d^\pi}(\theta_t), T_{d^i}(\theta_t), T_{d^\omega}(\theta_t), T_{d^\delta}(\theta_t)) \end{aligned}$$

where $T_{h^k}(\theta_t)$ denotes the component function of T which in turn defines the coefficient h of the law of motion for the endogenous variable k. Using the conditions (4.20) and (4.21) that hold in period t, one can simplify the T map and re-write it as a function of the coefficients of the law of motions for $\hat{\pi}_t$ and $\hat{y}_t$ alone, namely

$$\begin{aligned} T(\theta_t) = (&T_{b^\pi}(\theta_t), \phi_\pi T_{b^\pi}(\theta_t) + \phi_y T_{b^y}(\theta_t), \Lambda T_{b^y}(\theta_t) + t.i.\theta, \Gamma T_{b^y}(\theta_t) + t.i.\theta, \\ &T_{c^\pi}(\theta_t), \phi_\pi T_{c^\pi}(\theta_t) + \phi_y T_{c^y}(\theta_t), \Lambda T_{c^y}(\theta_t) + t.i.\theta, \Gamma T_{c^y}(\theta_t) + t.i.\theta, \\ &T_{d^\pi}(\theta_t), \phi_\pi T_{d^\pi}(\theta_t) + \phi_y T_{d^y}(\theta_t), \Lambda T_{d^y}(\theta_t) + t.i.\theta, \Gamma T_{d^y}(\theta_t) + t.i.\theta,) \end{aligned}$$

where $\Lambda = \varphi + \sigma$, $\Gamma = \frac{\mu - 1 - \varphi - \sigma}{\mu - 1}$ and $t.i.\theta$ stands for terms independent of θ (since we need to take derivatives with respect to θ those terms can be neglected). Then, Evans & Honkapohja (2001) show that in order to determine the learnability of a REE we only need to look at the the Jacobian of T. This is a real matrix of order 12 and can be written as

$$JT(\theta^{ree}) = \begin{pmatrix} A(b^{ree}) & 0 & 0 \\ 0 & A(c^{ree}) & 0 \\ 0 & 0 & A(d^{ree}) \end{pmatrix} \tag{4.35}$$

where $A(\cdot) \in \mathbb{R}^{4\times 4}$ and 0 are zero matrices of order 4, which result from the fact that the constant terms of the actual law of motion of the economy are independent of the estimated coefficients multiplying the exogenous shocks in the perceived law of motion entertained by agents, and that the coefficients multiplying any shock in the actual law of motion are independent of the estimated coefficients multiplying the remaining shock and the estimated constant terms in the reduced form models used by agents. Then, the eigenvalues of $JT(\theta^{ree})$ are given by the eigenvalues of $A(b^{ree})$, $A(c^{ree})$ and $A(d^{ree})$ which in turn are given by the roots of the following characteristic polynomial,

$$\det(A(x) - Id\lambda) =$$

$$\begin{vmatrix} \frac{\partial T_{x^\pi}(\theta_t)}{\partial x^\pi} - \lambda & \frac{\partial T_{x^\pi}(\theta_t)}{\partial x^i} & \frac{\partial T_{x^\pi}(\theta_t)}{\partial x^\omega} & \frac{\partial T_{x^\pi}(\theta_t)}{\partial x^\delta} \\ \phi_\pi \frac{\partial T_{x^\pi}(\theta_t)}{\partial x^\pi} + \phi_y \frac{\partial T_{x^y}(\theta_t)}{\partial x^\pi} & \phi_\pi \frac{\partial T_{x^\pi}(\theta_t)}{\partial x^i} + \phi_y \frac{\partial T_{x^y}(\theta_t)}{\partial x^i} - \lambda & \phi_\pi \frac{\partial T_{x^\pi}(\theta_t)}{\partial x^\omega} + \phi_y \frac{\partial T_{x^y}(\theta_t)}{\partial x^\omega} & \phi_\pi \frac{\partial T_{x^\pi}(\theta_t)}{\partial x^\delta} + \phi_y \frac{\partial T_{x^y}(\theta_t)}{\partial x^\delta} \\ \Lambda \frac{\partial T_{x^y}(\theta_t)}{\partial x^\pi} & \Lambda \frac{\partial T_{x^y}(\theta_t)}{\partial x^i} & \Lambda \frac{\partial T_{x^y}(\theta_t)}{\partial x^\omega} - \lambda & \Lambda \frac{\partial T_{x^y}(\theta_t)}{\partial x^\delta} \\ \Gamma \frac{\partial T_{x^y}(\theta_t)}{\partial x^\pi} & \Gamma \frac{\partial T_{x^y}(\theta_t)}{\partial x^i} & \Gamma \frac{\partial T_{x^y}(\theta_t)}{\partial x^\omega} & \Gamma \frac{\partial T_{x^y}(\theta_t)}{\partial x^\delta} - \lambda \end{vmatrix}$$

After a series of elementary operations, the above expression can be simplified and re-written as

$$\det(A(x) - Id\lambda) = \lambda^2 \begin{vmatrix} \frac{\partial T_{x^\pi}(\theta_t)}{\partial x^\pi} + \phi_\pi \frac{\partial T_{x^\pi}(\theta_t)}{\partial x^i} - \lambda & \frac{\partial T_{x^\pi}(\theta_t)}{\partial x^\omega} + \frac{\Gamma}{\Lambda} \frac{\partial T_{x^\pi}(\theta_t)}{\partial x^\delta} + \frac{\phi_y}{\Lambda} \frac{\partial T_{x^\pi}(\theta_t)}{\partial x^i} \\ \Lambda \frac{\partial T_{x^y}(\theta_t)}{\partial x^\pi} + \phi_\pi \Lambda \frac{\partial T_{x^y}(\theta_t)}{\partial x^i} & \Lambda \frac{\partial T_{x^y}(\theta_t)}{\partial x^\omega} + \Gamma \frac{\partial T_{x^y}(\theta_t)}{\partial x^\delta} + \phi_y \frac{\partial T_{x^y}(\theta_t)}{\partial x^i} - \lambda \end{vmatrix} \tag{4.36}$$

Thus, two eigenvalues are 0 and the other two are given by the roots of the following polynomial,

$$\begin{aligned} \lambda^2 - \lambda \left(\frac{\partial T_{x^\pi}(\theta_t)}{\partial x^\pi} + \phi_\pi \frac{\partial T_{x^\pi}(\theta_t)}{\partial x^i} + \Lambda \frac{\partial T_{x^y}(\theta_t)}{\partial x^\omega} + \Gamma \frac{\partial T_{x^y}(\theta_t)}{\partial x^\delta} + \phi_y \frac{\partial T_{x^y}(\theta_t)}{\partial x^i} \right) & \\ + \left(\frac{\partial T_{x^\pi}(\theta_t)}{\partial x^\pi} + \phi_\pi \frac{\partial T_{x^\pi}(\theta_t)}{\partial x^i} \right) \left(\Lambda \frac{\partial T_{x^y}(\theta_t)}{\partial x^\omega} + \Gamma \frac{\partial T_{x^y}(\theta_t)}{\partial x^\delta} + \phi_y \frac{\partial T_{x^y}(\theta_t)}{\partial x^i} \right) & \\ - \left(\frac{\partial T_{x^y}(\theta_t)}{\partial x^\pi} + \phi_\pi \frac{\partial T_{x^y}(\theta_t)}{\partial x^i} \right) \left(\Lambda \frac{\partial T_{x^\pi}(\theta_t)}{\partial x^\omega} + \Gamma \frac{\partial T_{x^\pi}(\theta_t)}{\partial x^\delta} + \phi_y \frac{\partial T_{x^\pi}(\theta_t)}{\partial x^i} \right) & = 0 \end{aligned}$$

where

$$\begin{aligned}
\frac{\partial T_{x^{\pi}}\left(\theta_t\right)}{\partial x^{\pi}} &= \chi\left\{\frac{\left(1-\alpha\right)\beta\rho_x}{1-\alpha\beta\rho_x}+\kappa\sigma^{-1}\frac{\rho_x}{1-\beta\rho_x}\right\}\\
\frac{\partial T_{x^{\pi}}\left(\theta_t\right)}{\partial x^{i}} &= -\chi\kappa\sigma^{-1}\frac{\beta\rho_x}{1-\beta\rho_x}\\
\frac{\partial T_{x^{\pi}}\left(\theta_t\right)}{\partial x^{\omega}} &= \chi\left\{\frac{\left(1-\alpha\right)\left(1-\alpha\beta\right)\beta\rho_x}{1-\alpha\beta\rho_x}+\kappa\frac{\left(1-\beta\right)\left(1+\varphi^{-1}\right)}{\mu+\sigma\varphi^{-1}}\frac{\rho_x}{1-\beta\rho_x}\right\}\\
\frac{\partial T_{x^{\pi}}\left(\theta_t\right)}{\partial x^{\delta}} &= \chi\kappa\frac{\left(1-\beta\right)\left(\mu-1\right)}{\mu+\sigma\varphi^{-1}}\frac{\rho_x}{1-\beta\rho_x}\\
\frac{\partial T_{x^{y}}\left(\theta_t\right)}{\partial x^{\pi}} &= \chi\left\{-\sigma^{-1}\phi_{\pi}\frac{\left(1-\alpha\right)\beta\rho_x}{1-\alpha\beta\rho_x}+\frac{\sigma^{-1}\rho_x}{1-\beta\rho_x}\right\}\\
\frac{\partial T_{x^{y}}\left(\theta_t\right)}{\partial x^{i}} &= -\chi\sigma^{-1}\frac{\beta\rho_x}{1-\beta\rho_x}\\
\frac{\partial T_{x^{y}}\left(\theta_t\right)}{\partial x^{\omega}} &= \chi\left\{-\sigma^{-1}\phi_{\pi}\frac{\left(1-\alpha\right)\left(1-\alpha\beta\right)\beta\rho_x}{1-\alpha\beta\rho_x}+\frac{\left(1-\beta\right)\left(1+\varphi^{-1}\right)}{\mu+\sigma\varphi^{-1}}\frac{\rho_x}{1-\beta\rho_x}\right\}\\
\frac{\partial T_{x^{y}}\left(\theta_t\right)}{\partial x^{\delta}} &= \chi\frac{\left(1-\beta\right)\left(\mu-1\right)}{\mu+\sigma\varphi^{-1}}\frac{\rho_x}{1-\beta\rho_x}
\end{aligned}$$

and $\chi = 1+\sigma^{-1}\phi_{\pi}\kappa$.

Given a real polynomial of degree two, $\lambda^2-\lambda B+C$, the real parts of its roots are smaller than one if and only if the following two conditions are satisfied

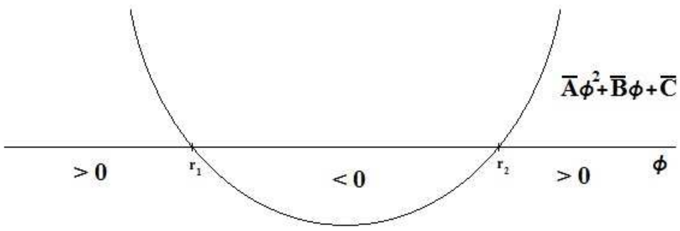
$$1)\quad if\quad B^2-4C\leq 0 \;\Rightarrow\; 0<2-B$$

$$2)\quad if\quad B^2-4C>0 \;\Rightarrow\; 0<2-B \;\wedge\; 0<1-B+C$$

Noticing that the discriminant can be written as a polynomial of order 2 in ϕ_{π}, i.e. $B^2-4C=\bar{A}\phi_{\pi}^2+\bar{B}\phi_{\pi}+\bar{C}$, we can look for the conditions on ϕ_{π} that make the discriminant larger or smaller than zero. After some simple algebra we observe that,

$$\bar{A}=\chi^2\beta^2\left(1-\alpha\beta\right)^2\left(\kappa-\left(1-\alpha\right)\left(1-\beta\right)\left(\varphi+\sigma\right)\right)^2>0$$

thus we face a parabola that has a minimum. Letting $r_1=\frac{-\bar{B}-\sqrt{\bar{B}^2-4\bar{A}\bar{C}}}{2\bar{A}}$ and $r_2=\frac{-\bar{B}-\sqrt{\bar{B}^2-4\bar{A}\bar{C}}}{2\bar{A}}$ denote its roots, we have



Furthermore, noticing that $\bar{B}^2 - 4\bar{A}\bar{C} > 0$, conditions 1) and 2) can then be re-written as

$$1) \quad if \qquad \phi_\pi \in [r_1, r_2] \cap \mathbb{R}^+ \qquad \Rightarrow \quad 0 < 2 - B$$

$$2) \quad if \quad \phi_\pi \in (0, \max\{0, r_1\}) \cup (\max\{0, r_2\}, \infty) \quad \Rightarrow \quad 0 < 2 - B \quad \wedge \quad 0 < 1 - B + C$$

Then, some algebra delivers the following equivalences

$$\begin{aligned} 0 &< 1 - B + C \iff \phi_\pi > Q \\ 0 &< 2 - B \qquad \iff \phi_\pi > R \end{aligned}$$

with

$$\begin{aligned} Q &= \frac{\rho_x(\sigma+\varphi) + \alpha^2\beta\rho_x(\sigma+\varphi) - \alpha\left(\sigma + \beta\rho_x^2\sigma + (1+\beta)\rho_x\varphi\right)}{(1-\alpha)(1-\alpha\beta)(\sigma+\varphi)} \\ R &= \frac{\alpha\left(\left(2 - \beta\rho_x(1-\rho_x) + \beta^2\rho_x^2\right)\sigma + \rho_x(1+\beta+\beta\rho_x)\varphi\right)}{(1-\alpha)(1-\alpha\beta)((1+\alpha)\beta\rho_x - 2)(\sigma+\varphi)} \\ &\quad + \frac{\left(\alpha^3\beta^2\rho_x^2 - \rho_x\right)(\sigma+\varphi) - \alpha^2\beta\rho_x\left((2+\beta\rho_x)\sigma + (1+\rho_x+\beta\rho_x)\varphi\right)}{(1-\alpha)(1-\alpha\beta)((1+\alpha)\beta\rho_x - 2)(\sigma+\varphi)} \end{aligned}$$

Which after some more calculations deliver the following inequalities,

$$R < Q < r_1 < r_2$$

Finally, since these conditions have to hold for $\rho_x \in (0, 1]$ and Q is increasing in ρ_x and since $\rho_b = 1$ implies $Q = 1$ we only have to look to the case in which $Q = 1$. Then, putting everything together we have the result we were looking for, namely, that a REE is expectationally stable if and only if

$$\phi_\pi \in (1, \infty)$$

4.7.2 Proof of Proposition 4.2

Following the same strategy as in the previous proof one can arrive to a characteristic polynomial of the form

$$\lambda^3 + \lambda^2 b + \lambda c + d = 0 \tag{4.37}$$

where

$$\begin{aligned}
b &= -\left(\frac{\partial T_{x^\pi}(\theta_t)}{\partial x^\pi} + \Lambda \frac{\partial T_{x^y}(\theta_t)}{\partial x^\omega} + \Gamma \frac{\partial T_{x^y}(\theta_t)}{\partial x^\delta} \right) \\
c &= -\phi_\pi \rho_x \frac{\partial T_{x^\pi}(\theta_t)}{\partial x^i} + \frac{\partial T_{x^\pi}(\theta_t)}{\partial x^\pi} \left(\Lambda \frac{\partial T_{x^y}(\theta_t)}{\partial x^\omega} + \Gamma \frac{\partial T_{x^y}(\theta_t)}{\partial x^\delta} \right) \\
&\quad - \frac{\partial T_{x^y}(\theta_t)}{\partial x^\pi} \left(\Lambda \frac{\partial T_{x^\pi}(\theta_t)}{\partial x^\omega} + \Gamma \frac{\partial T_{x^\pi}(\theta_t)}{\partial x^\delta} \right) \\
d &= \phi_\pi \rho_x \left\{ \frac{\partial T_{x^\pi}(\theta_t)}{\partial x^i} \left(\Lambda \frac{\partial T_{x^y}(\theta_t)}{\partial x^\omega} + \Gamma \frac{\partial T_{x^y}(\theta_t)}{\partial x^\delta} \right) - \frac{\partial T_{x^y}(\theta_t)}{\partial x^i} \left(\Lambda \frac{\partial T_{x^\pi}(\theta_t)}{\partial x^\omega} + \Gamma \frac{\partial T_{x^\pi}(\theta_t)}{\partial x^\delta} \right) \right\}
\end{aligned}$$

Using the Tartaglia formulas to derive conditions for the real parts of the polynomial to be smaller than 1, one arrives to the following necessary and sufficient conditions

$$\begin{aligned}
1) \qquad b &> 0 \\
2) \qquad d &> 0 \\
3) \qquad -d + bc &> 0
\end{aligned}$$

Given the complexity of the polynomial at hand, condition 3) does not provide a clear nor useful condition. However, after some simple algebra, condition 1) and 2) deliver the following restrictions,

$$1 < \phi_\pi \tag{4.38}$$

$$\frac{1}{1-\beta} + \frac{\sigma(\beta + \beta\alpha - 2)}{\kappa(1 - \alpha\beta)} < \phi_\pi \tag{4.39}$$

The first condition is nothing else than the Taylor principle, which is then necessary for achieving stability. However, the second condition implies that it is not sufficient. This can be easily seen by noticing that as $\beta \to 1$ the left hand side of inequality (4.39) tends to infinity.

Chapter 5

Bibliography

[1] K. Adam and A. Marcet. Internal Rationality, Imperfect Market Knowledge and Asset Prices. *Journal of Economic Theory*, 146:1224–1252, 2011.

[2] K. Adam, A. Marcet, and J.P. Nicolini. Stock Market Volatility and Learning. *Journal of Finance (forhtcoming)*, 2015.

[3] G. Akerlof and R. Shiller. *Animal Spirits: How Human Psychology Drives the Economy, and Why It Matters for Global Capitalism*. Princeton University Press, 2009.

[4] I. Arasaratnam, S. Haykin, and R. Elliott. Discrete-Time Nonlinear Filtering Algorithms Using Gauss–Hermite Quadrature. *Proceedings of the IEEE*, 95(5):953–977, 2007.

[5] A. Arias and E. Rancoita. Estimation of DSGE Models with Adaptive Learning. *mimeo*, 2013.

[6] M. Arulampalam, S. Maskell, N. Gordon, and T. Clapp. A Tutorial on Particle Filters for Online Nonlinear/Non-Gaussian Bayesian Tracking. *IEEE Transactions on Signal Processing*, 50(2):174–188, 2002.

[7] P. Beaudry, D. Nam, and J. Wang. Do Mood Swings Drive Business Cycles and Is It Rational? *NBER Working Paper Series*, 2011.

[8] P. Beaudry and P. Portier. Stock Price, News and Economic Fluctuations. *American Economic Review*, 96(4):1293–1307, 2006.

[9] J. Benhabib and R. Farmer. Indeterminancy and Increasing Returns. *Journal of Economic Theory*, 63:19–41, 1994.

[10] B. Bernanke and M. Woodford. *The Inflation-Targeting Debate*. University of Chicago Press, 2007.

[11] J. Boivin and M. Giannoni. Has Monetary Policy Become More Effective? *The Review of Economics and Statistics*, 88(3):445–462, 2006.

[12] W. Branch. Sticky Information and Model Uncertainty in Survey Data on Inflation Expectations. *Journal of Economic Dynamics and Control*, 31(1):245–276, 2007.

[13] J. Bullard and M. Mitra. Learning about Monetary Policy Rules. *Journal of Monetary Economics*, 49:1105–1129, 2002.

[14] G. Calvo. Staggered Prices in a Utility Maximizing Framework. *Journal of Monetary Economics*, 12:383– 398, 1983.

[15] S. Chib and E. Greenberg. Understanding the Metropolis-Hastings Algorithm. *The American Statistician*, 49(4):327–335, 1995.

[16] L. Christiano, M. Eichenbaum, and C. Evans. Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy. *Journal of Political Economy*, 113(1):1–46, 2005.

[17] K. Christoffel, G. Coenen, and A. Warne. The New Area-Wide Model of the Euro Area: a Micro-Founded Open-Economy Model for Forecasting and Policy Analysis. *ECB working paper*, 2008.

[18] R. Clarida, J. Gali, and M. Gertler. The Science of Monetary Policy: A New Keynesian Perspective. *Journal of Economic Literature*, 37:1661–1707, 1999.

[19] T. Cogley and T. Sargent. The Market Price of Risk and the Equity Premium: A Legacy of the Great Depression? *Journal of Monetary Economics*, 55:454–476, 2008.

[20] M. Del Negro, F. Schorfheide, F. Smets, and R. Wouters. On the Fit of New Keynesian Models. *Journal of Business & Economic Statistics*, 25(2):123–143, 2007.

[21] S. Eusepi and B. Preston. Expectations, Learning, and Business Cycle Fluctuations. *American Economic Review*, 101(6):2844–72, 2011.

[22] G. Evans and S. Honkapohja. *Learning and Expectations in Macroeconomics*. Princeton University Press, 2001.

[23] G. Evans and S. Honkapohja. Adaptive Learning and Monetary Policy Design. *Journal of Money, Credit and Banking*, pages 1045–1072, 2003.

[24] G. Evans and S. Honkapohja. Expectations and the Stability Problem for Optimal Monetary Policies. *The Review of Economic Studies*, 70(4):807–824, 2003.

[25] G. Evans and S. Honkapohja. Robust Learning Stability with Operational Monetary Policy Rules. *Working Papers Central Bank of Chile*, 2008.

[26] G. Evans, S. Honkapohja, and K. Mitra. Notes on Agents' Behavioral Rules Under Adaptive Learning and Recent Studies of Monetary Policy. *Center For Dynamic Macroeconomic Analysis Working Paper Series*, 2012.

[27] J. Fernández-Villaverde. The Econometrics of DSGE Models. *SERIES: Journal of the Spanish Economic Association*, 1(1):3–49, 2010.

[28] J. Fernández-Villaverde and J. Rubio-Ramírez. Comparing Dynamic Equilibrium Models to Data: a Bayesian Approach. *Journal of Econometrics*, 123:153–187, 2004.

[29] J. Fernández-Villaverde and J. Rubio-Ramírez. Estimating Dynamic Equilibrium Economies: Linear Versus Nonlinear Likelihood. *Journal of Applied Econometrics*, 20(7):891–910, 2005.

[30] J. Fernández-Villaverde and J. Rubio-Ramírez. Solving DSGE Models with Perturbation Methods and a Change of Variables. *Journal of Economic Dynamics and Control*, 30(12):2509–2531, 2006.

[31] R. Frydman. Individual Rationality, Decentraization and the Rational Expectations Hypothesis. In R. Frydman and E. Phelps, editors, *Individual Forecasting and Aggregate Outcomes: Rational Expectations Examined*, pages 97–122. Cambridge University Press.

[32] R. Frydman. Towards an Understanding of Market Processes: Individual Expectations, Learning and Convergence to Rational Expectations Equilibrium. *American Economic Review*, 72:652–668, 1982.

[33] R. Frydman and E. Phelps. *Rethinking Expectations - The Way Forward For Macroeconomic.* Prin, 2013.

[34] J. Galí. *Monetary Policy, Inflation, and the Business Cycle*. Princeton University Press, Princeton, New Jersey, 2008.

[35] J. Geweke. *Evaluating the Accuracy of Sampling-Based Approaches to the Calculation of Posterior Moments*, volume 196. Federal Reserve Bank of Minneapolis, Research Department, 1991.

[36] J. Geweke. Using Simulation Methods for Bayesian Econometric Models: Inference, Development and Communication. *Econometric Reviews*, 18(1):73, 1999.

[37] M. Giannoni and M. Woodford. How Forward-Looking is Optimal Monetary Policy? *Journal of Money, Credit and Banking*, pages 1425–1469, 2003.

[38] M. Giannoni and M. Woodford. Optimal Inflation-Targeting Rules. In *The Inflation-Targeting Debate*, pages 93–172. University of Chicago Press, 2004.

[39] R. Guesnerie. *Assesing Rational Expectations 2: "Eductive" Stability in Economics.* Cambridge, 2005.

[40] F. Gustafsson, F. Gunnarsson, N. Bergman, U. Forssell, J. Jansson, R. Karlsson, and P. Nordlund. Particle Filters for Positioning, Navigation, and Tracking. *Signal Processing, IEEE Transactions on*, 50(2):425–437, 2002.

[41] S. Honkapohja and K. Mitra. Learning Stability in Economies with Heterogeneous Agents. *Review of Economic Dynamics*, 9(2):284–309, 2006.

[42] K. Huang, Z. Liu, and T. Zha. Learning, Adaptive Expectations and Technology Shocks. *The Economic Journal*, 119(536):377–405, 2009.

[43] J. M. Keynes. *The General Theory of Employment, Interest and Money.* 2007 edition, 1936.

[44] J. Kotecha and P. Djuric. Gaussian Particle Filtering. *Signal Processing, IEEE Transactions on*, 51(10):2592–2601, 2003.

[45] R. Lucas Jr. Expectations and the Neutrality of Money. *Journal of Economic Theory*, 63:326–334, 1972.

[46] R. Lucas Jr. Econometric Policy Evaluation: a Critique. In *Carnegie-Rochester conference series on public policy*, volume 1, pages 19–46. Elsevier, 1976.

[47] G. Mankiw, R. Reis, and J. Wolfers. Disagreement about Inflation Expectations. *NBER Chapters*, 18:209–270, 2004.

[48] B. McCallum. Issues in the Design of Monetary Policy Rules. *Taylor, J., Woodford, M. (Eds.), Handbook of Macroeconomics*, 1999.

[49] R. Mehra and E. Prescott. The Equity Premium: a Puzzle. *Journal of monetary Economics*, 15(2):145–161, 1985.

[50] F. Milani. Persistence in monetary policy models: Indexation, habits and learning with long-horizon expectations. In *Econometric Society 2004 North American Summer Meetings*, number 172. Econometric Society, 2004.

[51] F. Milani. Expectations, Learning and Macroeconomic Persistence. *Journal of monetary Economics*, 54(7):2065–2082, 2007.

[52] F. Milani. The Modeling of Expectations in Empirical DSGE Models: a Survey. *Working Papers 121301, University of California-Irvine, Department of Economics*, 2012.

[53] F. Milani. Sentiment and the U.S. Business Cycle. *mimeo*, 2013.

[54] J. Muth. Rational Expectations and the Theory of Price Movements. *Econometrica*, 29:315–335, 1961.

[55] A. Orphanides and J. Williams. Imperfect Knowledge, Inflation Expectations, and Monetary Policy. *Bernanke and Woodford*, pages 201–234, 2005.

[56] A. Orphanides and J. Williams. Robust Monetary Policy with Imperfect Knowledge. *Journal of Monetary Economics*, 54(5):1406–1435, 2007.

[57] E. Phelps. The Trouble with Rational Expectations and the Problem of Inflation Stabilization. In R. Frydman and E. Phelps, editors, *Individual Forecasting and Aggregate Outcomes: Rational Expectations Examined*. Cambridge University Press, New York.

[58] B. Preston. Learning about Monetary Policy Rules when Long-Horizon Forecasts Matter. *International Journal of Central Banking*, 1 (2):81–126, 2005.

[59] J. Rotemberg. Sticky Prices in the United States. *Journal of Political Economy*, 90:1187–1211, 1982.

[60] T. Sargent. Rational Expectations, the Real Rate of Interest and the Natural Rate of Unemployment. *Brookings Papers on Economic Activity*, 2:429–472, 1973.

[61] T. Sargent. *The Conquest of American Inflation.* Princeton University Press, 2002.

[62] T. Sargent, N. Williams, and T. Zha. Shocks and Government Beliefs: the Rise and Fall of American Inflation. *National Bureau of Economic Research*, 2004.

[63] S. Schmitt-Grohe and M. Uribe. Solving Dynamic General Equilibrium Models Using a Second-Order Approximation to the Policy Function. *Journal of economic dynamics and control*, 28(4):755–775, 2004.

[64] S. Schmitt-Grohe and M. Uribe. What's News in Business Cycles. *Econometrica*, 80(6):2733–2764, 2012.

[65] S. Slobodyan and R. Wouters. Learning in a Medium-Scale DSGE Model with Expectations Based on Small Forecasting Models. *American Economic Journal: Macroeconomics*, 4(2):65–101, 2012.

[66] S. Slobodyan and R. Wouters. Learning in an Estimated Medium-Scale DSGE Model. *Journal of Economic Dynamics and control*, 36(1):26–46, 2012.

[67] F. Smets and R. Wouters. An Estimated Dynamic Stochastic General Equilibrium Model of the Euro Area. *Journal of the European economic association*, 1(5):1123–1175, 2003.

[68] F. Smets and R. Wouters. Comparing shocks and frictions in us and euro area business cycles: A Bayesian DSGE Approach. *Journal of Applied Econometrics*, 20(2):161–183, 2005.

[69] F. Smets and R. Wouters. Shocks and Frictions in U.S. Business Cycles: a Bayesian DSGE Approach. *American Economic Review*, 97(3):586–606, 2007.

[70] S. Smolyak. Quadrature and Interpolation Formulas for Tensor Products of Certain Classes of Functions. In *Dokl. Akad. Nauk SSSR*, volume 4, page 123, 1963.

[71] J. Suda. Belief Shocks and the Macroeconomy. *working papers series*, page 38, 2013.

[72] J. Taylor. *Monetary Policy Rules. Introduction.* University of Chicago Press, 1999.

[73] A. Timmermann. Excess Volatility and Predictability of Stock Prices in Autoregressive Dividend Models with Learning. *The Review of Economic Studies*, 63(4):523–557, 1996.

[74] N. Williams. Adaptive Learning and Business Cycles. *Work. Pap., Dep. Econ., Princeton Univ*, 2004.

[75] V. Winschel and M. Kraetzig. Solving, Estimating, and Selecting Nonlinear Dynamic Models without the Curse of Dimensionality. *Econometrica*, 78(2):803–821, 2010.

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