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On the Identification of Multivariate Correlated Unobserved Components Models

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On the Identification of Multivariate Correlated Unobserved Components Models [*]

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Abstract

This paper analyses identification for multivariate unobserved components models in which the innovations to trend and cycle are correlated. We address order and rank criteria as well as potential non-uniqueness of the reduced-form VARMA model. Identification is shown for lag lengths larger than one in case of a diagonal vector autoregressive cycle. We also discuss UC models with common features and with cycles that allow for dynamic spillovers.

Keywords: Unobserved components models, Identification, VARMA

JEL classification: C32, E32

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1 Introduction

Traditionally, in unobserved components (UC) models with stochastic trend and autoregressive (AR) cycle the innovations to the state variables were assumed to be uncorrelated, e.g. Harvey (1985), Clark (1987). Balke & Wohar (2002) and Morley, Nelson & Zivot (2003) allowed for correlation of the UC shocks. The latter authors state that identification in a univariate setting can be achieved if the lag polynomial in the cycle is at least of second order.

This paper demonstrates how identification can be derived for multivariate correlated UC models as the natural generalization of the univariate setting to higher dimensions, e.g. Sinclair (2009). The multivariate case allows analysing richer economic interactions and provides a larger information set for the purpose of decompositions and forecasts, e.g. Oh, Zivot & Creal (2008). Further papers in the literature using models treated here comprise Morley (2007), Startz & Tsang (2010), Weber (2011), Klinger & Weber (2014).

We clarify the role of uniqueness of the reduced-form vector autoregressive integrated moving average (VARIMA) model and present a rigorous treatment of the order and rank conditions. To the best of our knowledge identification of correlated UC models has only been discussed with respect to the order condition with the exception of the univariate UC model in Morley et al. (2003).

We find identification for given lag lengths larger than one. This criterion has to be fulfilled for all cyclical components. However, the lag structure does not have to be complete, i.e. the parameters associated with lower lags can be zero. Furthermore, we address UC models with common features and extend the usual UC specification with separate cyclical dynamics to the interesting case of a non-diagonal VAR cycle.

The paper is structured as follows. The subsequent section introduces the framework of correlated UC models. Section 3 discusses identification. This comprises our general approach, the order and rank criteria as well as the cases of common features and dynamically interacting cycles. The last section concludes.

2 Correlated Unobserved Component Model

Consider the following correlated UC model of for the $K \times 1$ random vector y_t, see Morley et al. (2003) and Sinclair (2009) for univariate and multivariate cases, respectively,

$$\begin{aligned} y_t &= \tau_t + c_t \\ \tau_t &= \mu + \tau_{t-1} + \eta_t \\ c_t &= B_1 c_{t-1} + \cdots + B_p c_{t-p} + \varepsilon_t, \end{aligned} \tag{1}$$

with

$$v_t = \begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \sim iidN(0, \Sigma_v), \tag{2}$$

where

$$\Sigma_v = \begin{pmatrix} \Sigma_\eta & \Sigma_{\eta\varepsilon} \\ \Sigma'_{\eta\varepsilon} & \Sigma_\varepsilon \end{pmatrix}.$$

Thus, the trend component τ_t follows a multivariate random walk, while the cyclical component c_t has a vector autoregressive (VAR) structure for which we make the following assumptions. The parameter matrices $B_1, \ldots, B_p$ are diagonal with typical diagonal elements $b_{kk,i}$, $i = 1, \ldots, p$, $k = 1, \ldots, K$, and $B_p \neq 0$ such that

$$|I_K - B_1 z - \cdots - B_p z^p| \neq 0 \quad \text{for } |z| \leq 1. \tag{3}$$

Consequently, the cyclical part of each component in y_t is characterized by a stable AR process of order at most p. This UC model framework is labelled UC-VAR(p) in the following. We will address the case of a general non-diagonal VAR(p) cycle for c_t in subsection 3.6.

The set-up (1) - (3) results in a reduced-form VARIMA($p, 1, p$) representation. Its canonical form, compare e.g. Schleicher (2003), reads as

$$B(L)\Delta y_t = B(1)\mu + B(L)\eta_t + \Delta\varepsilon_t \tag{4}$$

$$= c + \Theta(L)u_t, \tag{5}$$

where $B(L) = B_0 - B_1 L - \cdots - B_p L^p$ and $\Theta(L) = \Theta_0 + \Theta_1 L + \cdots + \Theta_p L^p$ are K-dimensional lag-polynomials of order p with $B_0 = \Theta_0 = I_K$, and $c = B(1)\mu$. The polynomials in row i and column j of $B(L)$ and $\Theta(L)$ will be denoted by $b_{ij}(L)$ and $\theta_{ij}(L)$, respectively. Accordingly, the i-th row of $\Theta(L)$ is given by $\Theta_{i\bullet}(L) = [\theta_{i1}(L), \ldots, \theta_{iK}(L)]$. Moreover, we have $u_t \sim iidN(0, \Sigma_u)$. The representation (5) is due to a multivariate version of Granger's Lemma, compare e.g. Lütkepohl (1984, Lemma 1). The reduced-form autocovariance structure of the vector MA part $m_t = \Theta(L)u_t$ is described by the matrices $\Gamma_h = \mathrm{E}(m_t m'_{t-h}) = \sum_{i=0}^{p-h} \Theta_{i+h}\Sigma_u\Theta'_i$, $h = 1, 2, \ldots$, such that $\Gamma_h = 0$ for $h > p$.

3 Identification of Multivariate UC Models

3.1 Identification approach

We analyse whether the parameters of a given structural UC model with a diagonal cycle can be identified from its implied reduced-form VARIMA (5). The label 'given UC model' refers in particular to a given set of orders $\{p_1, \ldots, p_K\}$ of the individual AR cycles. This is a common assumption in the literature, see e.g. Hotta (1989), Morley et al. (2003), Sinclair (2009).

Identification of the UC model parameters is complicated by the fact that VARMA models are not generally identified, see e.g. Lütkepohl (2005, Ch. 12). It is not automatically guaranteed that the VAR and vector MA (VMA) components in (5) are uniquely separable. Hence, it does not seem reasonable to directly assume that the reduced-form VAR parameters are known (and equal to the structural VAR parameters in (4)).

However, VARMA models with a diagonal VAR component and instantaneous parameter matrices $B_0 = I_K$ and $\Theta_0 = I_K$ are identifiable if there are no common roots to $b_{kk}(z)$ and $\Theta_{k\bullet}(z)$ for none $k = 1, \ldots, K$, i.e. if there is no value z^* such that $b_{kk}(z^*) = 0$ and $\Theta_{k\bullet}(z^*) = 0$, see Dufour & Pelletier (2011, Assumption 3.13, Theorem 3.14). Nevertheless, it may be possible that such common roots are present if $\Sigma_{\eta\varepsilon} \neq 0$.[1] Accordingly, there could exist a reduced-form VARIMA representation with a set of lower AR orders. To simplify the discussion in the following we assume for a moment that the orders of all cyclical components in the given UC-

[1] If the structural error term vectors η_t and ε_t were indeed uncorrelated, then common roots in the VAR and MA lag operators are ruled out, compare Harvey (1989, Section 4.4) for the univariate case.

VAR(p) are equal to p and that the same common roots are present in all K components. Then, in case of common roots a reduced-form VARIMA($p^*, 1, p^*$) with $p^* < p$ exists.

This lower-order VARIMA would correspond to a (potentially infinite) set of observationally equivalent structural UC model parametrizations with a VARMA($p^*, p^* - 1$) cycle, UC-VARMA($p^*, p^* - 1$), in which the MA component may be absent.[2] Hence, the original UC-VAR(p) model is over-parametrized in the sense that alternative UC models with an implied lower-order VARIMA representation exist. Therefore, it is not reasonable to rely on the UC-VAR(p). If a UC-VAR(p^*) model exists, then this model is identified whenever $p^* \geq 2$ according to our results obtained in the following subsections. The treatment of identification of UC-VARMA models is beyond the scope of this note. However, a UC-VARMA($p^*, p^* - 1$) would not be identifiable due to the failure of the corresponding necessary order condition, see e.g. Morley et al. (2003), Oh et al. (2008) for the univariate case.

In empirical applications the existence of common roots can be analysed by relying on the estimation results of the reduced-form VARIMA for a given set of AR lag orders and a VMA lag order equal to p. Therefore, we assume in line with the foregoing discussion that the lag orders of the individual AR cycles in the UC-VAR(p) model are minimal in the sense that y_t has no reduced form VARIMA representation with lag orders such that $p_k^* < p_k$ for at least one $k = 1, \ldots, K$. Then, the VAR component in (5) can be uniquely separated from the MA component. Therefore, we regard the reduced-form VAR parameters as given.

The latter result allows us to discuss identification within a system of linear equations that relates reduced-form and structural variance parameters, see subsections 3.2 to 3.4. To be precise, we only need to deal with the relevant necessary order and sufficient rank conditions. The subsections 3.2 to 3.4 also show that knowledge of the autocovariance structure of the VMA component of the reduced-form VARIMA (5) is sufficient to identify the structural (variance) parameters. Hence, there is no need to identify the MA parameter matrices $\Theta_1, \ldots \Theta_p$ by the

[2] There always exist at least one UC-VARMA($p, p-1$) representation for any given VARIMA($p, 1, p$) model, see Morley et al. (2003), Cochrane (1988) for the univariate case. However, there does not need to exist a UC-VAR(p) representation for a given VARIMA($p, 1, p$) since the UC-VAR(p) is a restricted version of a UC-VARMA($p, p-1$). We call two model parametrizations observationally equivalent if they give rise to the same joint density function of $\{y_t\}_{t=1}^T$. Obviously, the lower-order VARIMA($p^*, 1, p^*$) would be observationally equivalent to the original reduced-form VARIMA($p, 1, p$). Hence, the UC representations of the VARIMA($p^*, 1, p^*$) are observationally equivalent to the given UC-VAR(p).

commonly applied invertibility restriction on the VMA polynomial.

3.2 Order condition

The UC model (1) - (3) satisfies the order condition for identification as can be seen as follows. The Kp parameters in $B_1, \ldots, B_p$ are always identified since the VAR polynomial can be obtained from the reduced form (5). Therefore, the parameter vector μ is also identified. There remain $2K^2 + K$ structural variance parameters in Σ_v to be identified. Equating the autocovariance structures of $B(L)\eta_t + \Delta\varepsilon_t$ and $\Theta(L)u_t$ provides us with a link of these structural parameters to the reduced-form variance parameters, compare (4) and (5). Due to the symmetry of Γ_0, the reduced form contains $(K^2 + K)/2 + K^2p$ pieces of variance information in $\Gamma_0, \Gamma_1, \ldots, \Gamma_p$. Thus, the order condition is satisfied for $p \geq 2$ since $[(K^2 + K)/2 + K^2p] \geq [2K^2 + K]$ in this case. Except for $p = 2$ and $K = 1$, there are more reduced-form than structural-form variance parameters.

3.3 Rank condition: diagonal VAR(2) cycle

It remains to show that the structural variance parameters in Σ_v can indeed be uniquely recovered from the reduced-form VMA variance matrices $\Gamma_0, \Gamma_1, \ldots, \Gamma_p$. This requires to meet the relevant rank condition related to the system of equations linking the reduced-form and structural parameters. To simplify the understanding of the corresponding proof we start with the case $p = 2$. The proof relies on the construction of a particular submatrix that has full rank. This is easily achieved for a diagonal VAR(2) by assumption. The construction of the full rank submatrix is more involved for a VAR(p) cycle with $p > 2$ as as shown in subsection 3.4.

From the equivalence of (4) and (5) we first obtain

$$
\begin{aligned}
\operatorname{vec}(\Gamma_0) = \gamma_0 = {} & [I_{K^2} + (B_1 \otimes B_1) + (B_2 \otimes B_2)]\operatorname{vec}(\Sigma_\eta) + 2\operatorname{vec}(\Sigma_\varepsilon) \\
& + [(I_{K^2} + C_{KK}) + (I_K \otimes B_1) + C_{KK}(I_K \otimes B_1)]\operatorname{vec}(\Sigma_{\eta\varepsilon}) \\
\operatorname{vec}(\Gamma_1) = \gamma_1 = {} & [-(I_K \otimes B_1) + (B_1 \otimes B_2)]\operatorname{vec}(\Sigma_\eta) - \operatorname{vec}(\Sigma_\varepsilon) \\
& - [C_{KK} + (I_K \otimes B_1) - (I_K \otimes B_2)]\operatorname{vec}(\Sigma_{\eta\varepsilon}) \\
\operatorname{vec}(\Gamma_2) = \gamma_2 = {} & -(I_K \otimes B_2)\operatorname{vec}(\Sigma_\eta) - (I_K \otimes B_2)\operatorname{vec}(\Sigma_{\eta\varepsilon}),
\end{aligned}
\tag{6}
$$

where the vec-operator stacks the columns of a matrix below each other and C_{mn} is the $(mn \times mn)$ commutation matrix with $\operatorname{vec}(A') = C_{mn}\operatorname{vec}(A)$ for any $(m \times n)$ matrix A.

As Γ_0 is symmetric, γ_0 can just provide $\frac{1}{2}K(K+1)$ linearly independent equations. Therefore, we consider $\gamma_0^* = \operatorname{vech}(\Gamma_0)$ in the following, where the vech-operator is defined to stack columnwise the elements on and below the main diagonal of a square matrix below each other. Let D_K be the $(K^2 \times \frac{1}{2}K(K+1))$ duplication matrix such that $\operatorname{vec}(A) = D_K\operatorname{vech}(A)$ for any symmetric $(K \times K)$ matrix A and define $D_K^+ = (D_K' D_K)^{-1} D_K'$. Since $D_K^+\operatorname{vec}(A) = \operatorname{vech}(A)$ if A is symmetric, see Lütkepohl (1996, Section 9.5), we can re-write system (6) as

$$
\gamma^* = B^* \sigma^*, \tag{7}
$$

where $\gamma^* = [\gamma_0^{*\prime} : \gamma_1' : \gamma_2']'$, $\gamma_0^* = D_K^+ \gamma_0$, $\sigma^* = [\operatorname{vech}(\Sigma_\eta)' : \operatorname{vec}(\Sigma_\varepsilon)' : \operatorname{vec}(\Sigma_{\eta\varepsilon})']'$, and

$$
B^* = \begin{bmatrix} D_K^+(I_{K^2} + B_1 \otimes B_1 + B_2 \otimes B_2)D_K & 2D_K^+ & 2D_K^+(I_{K^2} + I_K \otimes B_1) \\ (-I_K \otimes B_1 + B_1 \otimes B_2)D_K & -I_{K^2} & -(C_{KK} + I_K \otimes B_1 - I_K \otimes B_2) \\ -(I_K \otimes B_2)D_K & 0_{(K^2 \times K^2)} & -I_K \otimes B_2 \end{bmatrix}. \tag{8}
$$

We will show below that the $(K^* \times K^*)$ matrix B^*, $K^* = 2.5K^2 + 0.5K$, has full rank. Hence, the structural variance parameters can be recovered from the reduced-form parameters by $\sigma^* = B^{*-1}\gamma^*$. Note that the off-diagonal elements of the symmetric variance matrix Σ_ε appear twice in σ^*. However, we analyse identification with respect to a given UC model. Thus, γ^* contains the implied, i.e. correct, reduced-form VMA variance parameters such that $B^{*-1}\gamma^*$

indeed returns two identical sets of off-diagonal elements of Σ_ε.

To show that B^* is of full rank we have to assume that $b_{kk,2} \neq 0$ for all $k = 1, \dots, K$, i.e. all K components of c_t have to follow AR(2) processes. Furthermore, we introduce the following notation for easier reference

$$B^* = \begin{bmatrix} B^*_{[1,1]} & B^*_{[1,2]} & B^*_{[1,3]} \\ B^*_{[2,1]} & B^*_{[2,2]} & B^*_{[2,3]} \\ B^*_{[3,1]} & B^*_{[3,2]} & B^*_{[3,3]} \end{bmatrix}, \tag{9}$$

where the structure of the submatrices $B^*_{[i,j]}$, $i, j = 1, 2, 3$, corresponds to (8). Corresponding to the univariate set-up of Morley et al. (2003) one can see that B^* has a rank deficit only if there exists a $(K_\delta \times 1)$ vector $\delta = [\delta_1' : \delta_2']'$ with $K_\delta = 1.5K^2 + 0.5K$ such that

$$\delta_1' \left(B^*_{[1,1]} + 2D_K^+ B^*_{[2,1]} \right) + \delta_2' B^*_{[3,1]} = 0_{(1 \times \frac{1}{2}K(K+1))} \quad \text{and} \tag{10}$$

$$\left(\delta_1' \left(B^*_{[1,3]} + 2D_K^+ B^*_{[2,3]} \right) + \delta_2' B^*_{[3,3]} \right) D_K = 0_{(1 \times \frac{1}{2}K(K+1))}. \tag{11}$$

This follows from the fact that $B^*_{[1,2]} = 2D_K^+$, $B^*_{[2,2]} = -I_{K^2}$, and $B^*_{[3,2]} = 0_{(K^2 \times K^2)}$. Note in this respect that the only way to reduce $B^*_{[1,2]}$ to zero by adding linear transformations of rows of the second block of columns of B^* is to add $2D_K^+ B^*_{[2,2]}$. Furthermore, any $(1 \times K^2)$ zero vector created by $\delta_1' \left(B^*_{[1,3]} + 2D_K^+ B^*_{[2,3]} \right) + \delta_2' B^*_{[3,3]}$ is transformed into a $(1 \times \frac{1}{2}K(K+1))$ zero vector when post-multiplied by D_K.

Since $\delta_1' \left(B^*_{[1,3]} + 2D_K^+ B^*_{[2,3]} \right) = 2\delta_1' D_K^+ (I_K \otimes B_2)$ and $(I_K \otimes B_2) = -B^*_{[3,3]}$ is of full rank by assumption, (11) implies $\delta_2' = 2\delta_1' D_K^+$. Plugging the latter identity into (10) yields

$$\begin{aligned} &\delta_1' \left(B^*_{[1,1]} + 2D_K^+ B^*_{[2,1]} \right) + \delta_2' B^*_{[3,1]} \\ &= \delta_1' \left(D_K^+ \left[I_{K^2} + B_1 \otimes B_1 + B_2 \otimes B_2 + 2(B_1 \otimes B_2) - 2(I_K \otimes B_1) - 2(I_K \otimes B_2) \right] D_K \right) \\ &= \delta_1' \left(D_K^+ \left[(I_K - B_1 - B_2) \otimes (I_K - B_1 - B_2) \right] D_K \right) \\ &= 0_{(1 \times \frac{1}{2}K(K+1))} \end{aligned} \tag{12}$$

as a requirement for a rank deficit. The second equality follows from Lütkepohl (1996, Section 9.5.4, Rule (3)). Since the VAR(2) process is stable, the matrix $I_K - B_1 - B_2$ has full rank. Due to Lütkepohl (1996, Section 9.5.4, Rule (2)) the same holds true for the matrix $D_K^+ [(I_K - B_1 - B_2) \otimes (I_K - B_1 - B_2)] D_K$. Therefore, no vector δ can exist that satisfies the conditions (10) and (11). This implies that B^* is of full rank K^*. Hence, the rank condition for identification of the structural parameters is satisfied if every cycle component has an AR(2) structure.

3.4 Rank condition: diagonal VAR(p) cycle

For a diagonal VAR(p) cycle we obtain the matrix representation as

$$\gamma^* = B^+ \sigma^*,$$

where $\gamma^* = [\gamma_0^{*\prime} : \gamma_1' : \cdots : \gamma_p^*]'$, $\gamma_i = \text{vec}(\Gamma_i)$, $i = 0, 1, \ldots, p$, $\gamma_0^* = D_K^+ \gamma_0$, $\sigma^* = [\text{vech}(\Sigma_\eta)' : \text{vec}(\Sigma_\varepsilon)' : \text{vec}(\Sigma_{\eta\varepsilon})']'$, and

$$B^+ = \begin{bmatrix} D_K^+(I_{K^2} + \sum_{i=1}^p B_i \otimes B_i) D_K & 2D_K^+ & 2D_K^+(I_{K^2} + I_K \otimes B_1) \\ (-I_K \otimes B_1 + \sum_{i=1}^{p-1} B_i \otimes B_{i+1}) D_K & -I_{K^2} & -C_{KK} - I_K \otimes B_1 + I_K \otimes B_2 \\ (-I_K \otimes B_2 + \sum_{i=1}^{p-2} B_i \otimes B_{i+2}) D_K & 0_{(K^2 \times K^2)} & -I_K \otimes B_2 + I_K \otimes B_3 \\ \vdots & \vdots & \vdots \\ (-I_K \otimes B_{p-1} + B_1 \otimes B_p) D_K & 0_{(K^2 \times K^2)} & -I_K \otimes B_{p-1} + I_K \otimes B_p \\ -(I_K \otimes B_p) D_K & 0_{(K^2 \times K^2)} & -I_K \otimes B_p \end{bmatrix}. \quad (13)$$

The $(K^+ \times K^*)$ matrix B^+, $K^+ = (p+0.5)K^2 + 0.5K$, can be re-written using the submatrix-block structure

$$B^+ = \begin{bmatrix} B^+_{[1,1]} & B^+_{[1,2]} & B^+_{[1,3]} \\ B^+_{[2,1]} & B^+_{[2,2]} & B^+_{[2,3]} \\ \vdots & \vdots & \vdots \\ B^+_{[p+1,1]} & B^+_{[p+1,2]} & B^+_{[p+1,3]} \end{bmatrix}.$$

To show that the matrix B^+ has full column rank K^* we first obtain a quadratic $(K^* \times K^*)$ matrix B^* corresponding to (9) by elementary row operations applied to B^+. To be precise, a three-by-three submatrix structure is generated with $B^*_{[3,3]}$ being of full rank K^2. To this end, we have to assume that $b_{kk,i} \neq 0$ for each component k, $k = 1, \ldots, K$, for at least one lag i with $2 \leq i \leq p$. Then, there will always exist an index sequence $q_1, q_2, \ldots, q_n$ with $2 \leq q_1 < q_2 < \cdots < q_n \leq p-1$ and a scalar $c \in \mathbb{N}_0$ such that $B^a = \sum_{s=1}^{n} B_{q_s} + cB_p$ is of full rank. Next, define the coefficients c_i, $i = q_1+1, q_1+2, \ldots, p+1$, by $c_i = s$ for $q_s + 1 \leq i \leq q_{s+1}$ with $s = 1, 2, \ldots, n$ and $q_{n+1} = p+1$. Then, we have $\sum_{i=q_1+1}^{p+1} c_i B^+_{[i,3]} + cB^+_{[p+1,3]} = -(I_K \otimes B^a)$. Accordingly, the third block of rows of B^* is generated by $B^*_{[3,j]} = \sum_{i=q_1+1}^{p+1} c_i B^+_{[i,j]} + cB^+_{[p+1,j]}$ for $j = 1, 2, 3$. We set $B^*_{[2,j]} = \sum_{i=2}^{q_1} B^+_{[i,j]} - \sum_{i=q_1+1}^{p+1}(c_i - 1)B^+_{[i,j]} - cB^+_{[p+1,j]}$, $j = 1, 2, 3$, in order to generate the second block of rows. Finally, the first block of rows is given by $B^*_{[1,j]} = B^+_{[1,j]}$, $j = 1, 2, 3$.

Equipped with these definitions we have $\delta_1' \left(B^*_{[1,3]} + 2D_K^+ B^*_{[2,3]} \right) = -2\delta_1' D_K^+ B^*_{[3,3]}$ with $B^*_{[3,3]} = -(I_K \otimes B^a)$ being of full rank. Plugging this result into (11) yields again $\delta_2' = 2\delta_1' D_K^+$. Then, from (10), we obtain $\delta_1' \left(D_K^+ \left[(I_K - B_1 - \cdots - B_p) \otimes (I_K - B_1 - \cdots - B_p) \right] D_K \right) = 0_{(1 \times \frac{1}{2}K(K+1))}$ as a requirement for a rank deficit which cannot be satisfied since the VAR(p) cycle in (1) is stable. It follows that $\mathrm{rk}(B^+) = K^*$. Hence, in line with the arguments of the previous subsection, a correlated UC model with a stable diagonal VAR(p) cycle is overidentified if each individual cyclical component has at least an AR(2) structure and $K > 1$.

3.5 Common Features

Morley (2007) considers common trends (cointegration) and Schleicher (2003) also allows for common cycles in UC models. In case of common trends we define the trend component by $\tau_t = \alpha \tau_t^*$, where α is a $K \times r$ matrix of full column rank and τ_t^* is a r-dimensional random walk with drift: $\tau_t^* = \mu + \tau_{t-1}^* + \eta_t$. Hence, there are r common trends, i.e. there exist $K - r$ linearly independent cointegration relations among the components in y_t. Similarly, the set-up of l common cycles is introduced by $c_t = \beta c_t^*$. Here, β is a $K \times l$ matrix of full column rank and c_t^* is a l-dimensional VAR(p) process: $c_t^* = B_1 c_{t-1}^* + \cdots + B_p c_{t-p}^* + \varepsilon_t$.

The common feature models are special cases of the set-up (1) - (3) and are thus identified whenever the above conditions are met. However, common features reduce the number of structural-form parameters such that the order condition can be already fulfilled for a lag length of one in the cycle. We illustrate this for the case of common trends.

First, we normalize α according to $\alpha = (I_r : \alpha^{*\prime})'$, where α^* is a $(K-r) \times r$ matrix. Hence, there are $(K-r)r$ structural parameters in α. Additionally, Σ_v contains $0.5K^2 + (0.5+r)K + 0.5r^2 + 0.5r$ structural variance parameters since η_t is now a r-dimensional vector. The VAR(1) cycle provides $1.5K^2 + 0.5K$ reduced-form variance parameters. Hence, the order condition requires $K^2 \geq 2Kr - 0.5r^2 + 0.5r$. E.g. in case of a single common trend ($r = 1$), the order condition holds for $K \geq 2$.

Nevertheless, we cannot verify that the rank condition is satisfied: setting up a system like (6), which links the reduced- and structural-form parameters, and re-writing it in the form of (7) produces a matrix B^* which is not of full rank. The same results are obtained for the case of common cycles.

3.6 Identification of UC models with a general VAR(p) cycle

Let B_i, $i = 1, \ldots, p$, be $K \times K$ parameter matrices such that the stability condition (3) is satisfied. Thus, dynamic spillovers in the cycles are introduced. We further assume that the lag orders of the VAR cycle of the given UC model are minimal. Then, one needs to additionally impose some structure on the VAR cycle to ensure that the VAR and MA components of the implied reduced form VARIMA can be uniquely separated. One potential approach in this respect is to assume that each of the K rows in the VAR cycle contains at least one term of order p. Then, the reduced form VARIMA would be in Echelon form with all Kronecker indices being equal to p, see e.g. Lütkepohl (2005, Section 12.1.2). Therefore, this VARIMA is identified in the sense that its VAR component can be uniquely separated.

Accordingly, the VAR parameters are regarded as given which allows us to proceed as in the case of a diagonal cycle. Obviously, the order condition still holds for the current set-up. Moreover, we can still use the structure of B^+ as in (13) and of B^* as in (9). The rank condition is met if at least one linear combination of the parameter matrices $B_1, \ldots, B_p$ has full rank. This follows from adopting the above approach regarding the diagonal VAR(p). The problem, however, is to provide primitive conditions on the VAR parameters that ensure the appropriate full-rank structure. E.g., it is not sufficient to assume that each cycle component has at least one AR term of order two or higher.

Nevertheless, in many applications one can expect that the rank condition is satisfied. Consider e.g. a VAR(2) cycle with $b_{kk,2} \neq 0$ for all $k = 1, \ldots, K$. It seems reasonable to assume that the spillovers at lag two, which are captured by the off-diagonal elements of B_2, are smaller than the diagonal elements $b_{11,2}, \ldots, b_{KK,2}$ such that B_2 is of full rank K. Then, the rank condition would be satisfied following the line of arguments in subsection 3.4.

4 Conclusion

We discuss identification in multivariate correlated UC models. In this context, we address uniqueness of the reduced-form VARIMA model as well as rigorously treat both the order and rank condition. This adds important theoretical results to the literature.

We extend the analysis to UC models with common features and non-diagonal VAR cycles, cases especially relevant to macroeconomic applications. In general, our results can provide a useful basis for growing strands of literature that apply and develop correlated UC models, e.g. Morley (2007), Sinclair (2009), Startz & Tsang (2010), Weber (2011), Klinger & Weber (2014).

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