# A state-constrained stochastic optimal control problem arising in portfolio liquidation

Inauguraldissertation
zur Erlangung des akademischen Grades
eines Doktors der Naturwissenschaften
Universität Mannheim

vorgelegt von
Mourad Lazgham
aus Ajaccio, Frankreich

Mannheim, 2015

| | |
|---|---|
| Dekan: | Professor Dr. Heinz Jürgen Müller, Universität Mannheim |
| Referent: | Professor Dr. Alexander Schied, Universität Mannheim |
| Korreferent: | Professor Dr. Christoph Kühn, Goethe Universität Frankfurt am Main |

Tag der mündlichen Prüfung: 28. Juli 2015

## Abstract

We consider a stochastic optimal control problem originating from a classical portfolio liquidation problem, which is related to an *expected utility maximization* problem under finite fuel constraint. This has been extensively studied in the existing literature for the case where the utility function is assumed to be an exponential function (see, e.g., Schied et al. (2010)). The purpose of this work is to investigate more general utility functions with exponential growth.

As a first main result, we can establish the existence and uniqueness of optimal strategies, under rather mild model assumptions. This is a core result, which will then allow us to derive regularity properties of the corresponding value function. In particular, we will establish the continuity and partial differentiability of the value function for the underlying maximization problem.

Second, we prove a *Bellman principle* for the underlying optimization problem. The proof of this optimality principle is facilitated in our case by the continuity property of the value function. With this at hand, we will show that the control problem considered is closely related to the solution of a nonlinear parabolic degenerated *Hamilton-Jacobi-Bellman (HJB) equation with singularity.* In particular, a verification theorem is proved.

Third, the Bellman principle turns out to be a crucial tool for characterizing our value function as the unique *viscosity* solution of an HJB equation. A comparison principle is derived, where, contrarily to mainstream results, the proof does not involve the use of the (classical) Crandall-Ishii lemma.

Numerical results and simulations conclude our work. For instance, we show that our value function can be considered as the unique viscosity solution of an HJB equation with *removed singularity* in the initial condition. This enables us then to implement converging numerical schemes, based on the *monotone schemes method* of Barles and Souganidis (1991). Matlab visualizations are presented in the last section of this thesis.

**Zusammenfassung**

Wir betrachten ein stochastiches Kontrollproblem, das aus einem klassischen Portfolioliquidierungsproblem abgeleitet wird, und das mit einer *Erwartungsnutzenmaximierung* mit "fuel constraint" zusammenhängt. Das wurde ausführlich für den Fall, in dem die Nutzenfunktion als die Exponentialfunktion angenommen wird, in der bisherigen Literatur untersucht (siehe z.B. Schied et al. (2010)). Das Ziel dieser Arbeit besteht darin, diesen Zusammenhang für allgemeine Nutzenfunktionen mit exponentiellem Wachstum zu analysieren.

Als ein erstes Hauptresultat weisen wir die Existenz und Eindeutigkeit optimaler Strategien unter relativ milden Modellannahmen nach. Dies ist ein Kernresultat, das es uns ermöglichen wird, analytische Eigenschaften der Wertfunktion herzuleiten. Insbesondere wird die Stetigkeit und partielle Differenzierbarkeit der Wertfunktion hergeleitet.

Zweitens zeigen wir, dass für das zugrunde liegende Optimierungsproblem ein *Bellmannsches Prinzip* gilt. Der Beweis dieses Optimalitätsprinzips wird durch die Stetigkeit der Wertfunktion erleichtert. Somit können wir dann zeigen, dass ein enger Zusammenhang zwischen unserem Kontrollproblem und einer nichtlinearen parabolischen degenerierten *Hamilton-Jacobi-Bellmann Gleichung (HJB) mit Singularität* besteht. Insbesondere beweisen wir einen Verifikationssatz.

Drittens benutzen wir das Bellmannsche Prinzip, um die Wertfunktion unseres Kontrollproblems als die eindeutige *Viskösitätslösung* einer HJB-Gleichung mit Singularität zu charakterisieren. Wir leiten ein Vergleichsprinzip her, wobei das (klassische) Lemma von Crandall und Ishii nicht in dem Beweis verwendet wird.

Zum Abschluss stellen wir die numerischen Ergebnisse dar. Es wird unter anderem gezeigt, dass unsere Wertfunktion als die eindeutige Viskositätslösung einer HJB-Gleichung *ohne Singularität* in der Anfangswertbedingung dargestellt werden kann. Dies erlaubt uns dann, konvergente numerische Verfahren zu implementieren. Matlabmodellierungen runden die Arbeit ab.

# Contents

# Chapter 1

# Introduction

## 1.1 Stochastic control problem, exponential growth utility and Hamilton-Jacobi-Bellman equation with singularity

At the end of the year 2007, french trader Jérôme Kerviel from the Société Générale, having anticipated falling market prices, took a huge trade position of almost €50 billion, which generated €1.47 billion in hidden profits (see statement of the French Court of Cassation in Louvel (2014)). When his employers were aware of the existence of this huge portfolio and the high risk generated by it, they decided to close out this position over three days of trading beginning January 18, 2008 and ending five days later on January 23. Consequently, this trade led to a loss estimated at €4.9 billion (also known as the 2008 Société Générale trading loss). Summarizing, about €6.5 billion were lost during the complete liquidation of this position, which had been taken by a single trader. This case illustrates one of the main issues encountered in financial markets when the number of shares traded attains a certain level: the shortfall of the demand and supply usually required for price determination in the market (also called lack of liquidity). With the number of shares traded in the market growing large, the price determination has to take into account the so called execution costs. These are implicit costs the agent must face which are no longer deterministic and simply linear in the number of shares. Hence, linear trading strategies do not suffice any longer when we aim at liquidating a large portfolio within a given short time period, without incurring considerable losses.

Motivated by the expansion of equity trading, Bertsimas and Lo (1998) are the first to derive a dynamic execution strategy which minimizes expected cost. However, as illustrated by the 2008 Société Générale case, we have to add to execution costs the volatility risk incurred when trading, which does not allow to minimize losses by only selling as slow as possible. This is, for example, treated in Almgren and Chriss (2001), in a discrete-time framework. In this work, a mean-variance maximization problem is considered where the execution costs are assumed to be linear and are split into a temporary and a permanent price impact component. They argue that under some conditions optimal strategies may be chosen among deterministic ones. Nevertheless, linear execution costs may not seem to be a realistic assumption in practice, as argued in Almgren (2003). Therefore, it may be reasonable to consider a nonlinear temporary impact function, in a discrete- as well as continuous-time framework. As opposed to the temporary impact, the permanent impact has to be linear in order to avoid quasi-arbitrage opportunities, as shown in Huberman and Stanzl (2004). When the initial assumptions are slightly modified, the mean-variance optimization can be improved by allowing intertemporal modifications of trade (see Almgren and Lorenz (2007)).

The mean-variance approach can also be regarded as an expected-utility maximization problem for an investor with constant absolute risk aversion, which was in part solved by Schied et al. (2010): in their framework, they prove the existence and uniqueness of an optimal trading strategy, which is moreover deterministic. The latter one can be computed by solving a nonlinear Hamilton equation. Furthermore, the corresponding value function is the unique classical solution of a nonlinear degenerated Hamilton-Jacobi-Bellman equation with singular initial condition. We propose in this work to generalize this framework by considering utility functions that lie between two exponential utility functions (also called CARA utility functions) or, equivalently, utility functions with bounded Arrow-Pratt coefficient. Note that this case was already treated for infinite-time horizon in a one-dimensional framework with linear temporary impact without drift; see Schied and Schöneborn (2009), as well as Schöneborn (2008). Here, a characterization of the optimal trading strategy as the unique bounded solution of a classical fully nonlinear parabolic equation (after an adequate transform) is obtained. Lower and upper bounds are given through the lower and upper absolute risk aversion coefficients. Moreover, it is shown that the optimal liquidation strategy is Markovian, and a feedback form is given. The optimal strategy is deterministic if and only if the utility function is an exponential function (or, equivalently, for CARA investors). The principal ingredient that allows them to obtain the preceding results is the fact that when considering infinite time horizon, the transformed optimal strategy solves a *classical* parabolic PDE, due to the fact that the time parameter does not appear in the equation. Then, by verification arguments, the value function can be identified as a smooth solution of a parabolic PDE. By restricting ourselves in this work to the *finite-time* horizon, however, we shall see below that the preceding arguments *cannot* be applied any longer in this context. Thus, we have to think differently and, in particular, focus on solutions that are no longer classical.

Mathematically speaking, this work treats an expected-utility stochastic control problem and in this framework, we cannot expect the optimal strategy to be deterministic any longer. Indeed, most of the results proved for exponential utility functions make use of the Doléans-Dade exponential combined with the martingale property, as a common change of measure technique. However, these techniques cannot be applied in general to the case of utility functions with exponential growth. Moreover, we will have to face major integrability, measurability and regularity issues. Since the considered equation takes into account a time parameter, and no classical solutions are given in closed form so far (contrarily to the case of infinite-time horizon), we cannot expect to derive easily a classical solution to the corresponding HJB equation. Nevertheless, this will be overcome by referring to the notion of *viscosity solutions*, which corresponds to a weak local characterization of the value function. In order to establish this characterization, we will have to use a dynamic programming principle, also known as the *Bellmann principle*. In most of the literature, the proof of the Bellmann principle is omitted, due to its complexity. Although in some cases proving that the value function is the viscosity solution of a corresponding HJB equation can be carried out by using a weak version of the Bellman principle (see Bouchard and Touzi (2011) or Bouchard and Nutz (2012)), the assumptions made in such works are not suitable for our value function. Hence, the Bellman principle will have to be proved here. Fortunately, its demanding proof will be facilitated by the fact that in our case the value function is continuous. In the next step, the uniqueness of the viscosity solution of the corresponding HJB equation is proved by using a *comparison principle*. Such a comparison principle was first established in Crandall et al. (1992), for unbounded semi-continuous viscosity solutions which grow at most linearly. In Pham (2009), the comparison principle derived requires that the viscosity solution's growth is at most polynomial and is established by using the well-known *Crandall-Ishii lemma*. This could at first sight be applied to our problem by introducing an immediate change of variable transform. However, this would require a uniform Lipschitz condition of the *Hamiltonian operator*, and this assumption is not satisfied in our setting. Another version of the comparison principle for unbounded viscosity solutions can be found in Koike and Ley (2011), where the gradient term is supposed to grow superlinearly, and where the coefficients are locally Lipschitz-continuous. But this requires convexity of the gradient term or, equivalently, it requires us to impose a kind of degeneracy condition on the gradient. Thus, the comparison principles established so far require conditions that are not satisfied in our situation. This is due to the fact that a quotient term appears in our HJB equation and the gradient term does not satisfy any convexity property. Nevertheless, there is a way out: since the Hessian term in our HJB equation is one-dimensional and our viscosity solution will be shown to be continuous, we will be able to overcome the difficulties described above and derive a valid comparison principle, without the use of the classical Crandall-Ishii's Lemma: indeed, we will simply have to apply a Taylor formula instead.

If we wish to model numerically the value function and the corresponding optimal trading strategy, we have to deal with some technical issues. In a one-dimensional framework, under some conditions, the value function and its optimal strategy can be explicitly constructed, as shown in Schied and Schöneborn (2007). In the case where the utility function is supposed to be an exponential utility function, the integrated path of the optimal strategy can be obtained as the unique solution of a Hamilton equation, as proved in Schied et al. (2010). However, in general, the value function and the optimal strategy must be computed numerically. To implement a numerical scheme, we have to use some convergence result. The most popular one is presented in Barles and Souganidis (1991). It provides sufficient conditions such that a given numerical scheme converges to the viscosity solution of an HJB equation. Nevertheless, this requires a monotonicity property of the underlying scheme, and this might be difficult to establish, in general. The Monte Carlo method can also be used to numerically solve fully nonlinear parabolic PDEs, in the context of *second-order backward stochastic differential equations* as introduced in Cheridito et al. (2007). In this work, it is established that if a comparison principle holds, and if the related second-order backward stochastic differential equation satisfies some Lipschitz conditions, then a stochastic representation of the underlying PDE can be provided. This can then be used to compute the solution numerically, with the help of Monte Carlo simulation. The backward probabilistic scheme developed in their work can also be applied without referring to the notion of the backward stochastic differential equation, as argued in Touzi (2013), Chapter 12. In this latter work, approximation schemes are established that allow them to numerically compute viscosity solutions of some PDEs, even for the case where these solutions have exponential growth. Nevertheless, this method requires Lipschitz conditions and the non-degeneracy of the nonlinear parabolic PDE. These requirements cannot be fulfilled in our case. Thus, we will need to modify the Barles-Souganidis convergence result and to use the fact that the second-order term is one-dimensional in our case in order to be able to construct converging numerical schemes.

## 1.2 Summary of results

In a first part of this work, after setting up our framework and making clearer our definition of utility functions with exponential growth, given with the help of the Arrow-Pratt coefficient of risk aversion, we recall some useful properties of exponential value functions. We continue with proving the convexity property of the value function for a general class of utility functions with exponential growth. The next main result, perhaps the most important one, states the existence and uniqueness of an optimal strategy. Its proof is mainly an analytical one and does not require the previously established boundedness property of optimal strategies. As a direct consequence of this result, we can show that the associated value function is continuously differentiable in its revenues parameter (and even twice continuously differentiable, if the utility function is supposed to have a convex and decreasing derivative; this condition is fulfilled if, e.g., the utility function is a convex combination of exponential utility functions). The relatively long proof of the continuity of the value function concludes this chapter. We will prove it in two ways, the second way requiring some assumptions and being longer, but enabling us to derive an approximation sequence for the value function, from below.

In the third chapter, using the continuity property of the value function, we prove the underlying Bellman principle. In this proof we face measurability issues, and we have to restrict ourselves to considering the Wiener space to make matters clearer. This will be carried out without referring to the measurable selection arguments, typically used in proofs of the dynamic programming principle where no a priori regularity of the value function is known to hold; see, e.g., Meyer (1966) or Wagner (1980), Rieder (1978). Note that in most of the literature where the Bellman principle is related to stochastic control problems, its (rigorous) proof is simply omitted, or the reader is referred to the above literature. When the value function is supposed to be continuous, an easier version of its proof can be found in Krylov (2009) or Bertsekas and Shreve (1978). We use this principle to establish a tight connection between our expected utility maximization problem and an Hamilton-Jacobi-Bellman equation with a singularity in the initial condition. Furthermore, a certain quotient term will also be the source of numerical instabilities in our case, as seen in the last chapter.
More precisely, we show that our value function, satisfying an initial condition with singularity, has to be a classical solution of the associated HJB equation, if it is smooth enough. We prove in the next step a verification theorem which states that, under certain conditions, if this HJB equation has a classical solution, this is the unique solution and it is equal to the value function. A relation between the optimal strategy, the value function and the solution of an SDE will be also established, which can be useful in order to numerically compute the optimal liquidation strategy.

In the fourth chapter, we recall the notion of a viscosity solution. We show that our value function is not only a viscosity solution of the HJB equation, but also the *unique* solution, by using a comparison principle. This comparison principle is proved here without utilizing the Crandall-Ishii lemma. We only use a Taylor expansion on some test functions. The main issue encountered in this chapter is due to the fact that our value function has exponential growth in all parameters, and this does not allow us to apply the arguments usually used in this case, which consist in adding a penalizing supersolution of the HJB equation which grows larger than the value function itself and then working toward a contradiction (as, for instance, in Pham (2009)). Moreover, most of those arguments require a uniform Lipschitz condition of the Hamiltonian operator, which is not the case in our work. However, the *continuity* property of the value function will enable us to overcome these issues.

In the closing chapter we provide some numerical results related to the HJB equation. We first begin by relaxing the exponential growth requirement on the value function, using a classical change of variables formula. Applying again an affine transform (with an adequate function) to the previously transformed value function, we can remove the singularity in the initial condition. With this simplification at hand, we can prove a numerical convergence result similar to the one in Barles and Souganidis (1991). Though, since we are still facing an "instability" in the auxiliary HJB equation (due to a quotient in the first-order term) and we have no uniform Lipschitz property in its coefficients, no known results can be directly applied here. Hence, we have to modify the requirements of the convergence result in Barles and Souganidis (1991). Here, it is the fact that our second-order term is one-dimensional that will be crucial to establish converging numerical schemes. However, as it turns out, we cannot obtain the stability of the scheme without requiring some unfavorable Courant-Friedrichs-Lewy conditions. The figures provided at the end of the chapter illustrate these results.

## 1.3 Acknowledgement

I am deeply grateful to my supervisor Prof. Alexander Schied. Not only numerous and very precious ideas originated with him, but he also helped me to build up certain notions whose introduction enabled me to overcome issues and give rigorous proofs in cases where the gap between intuition and a formal proof turned out to be very large (see, for instance, the proof of the Bellman Principle). I am also deeply indebted to Prof. Schied for all I could learn from him during our cooperation in tutorials, and during other educational activities at the university of Mannheim.

Furthermore, I would like to thank Dr. Aurélien Alfonsi for inviting me to Paris to give a talk on one of the weekly scheduled seminars of the "Groupe de travail et modélisation stochastique et finance". During this visit, we had various valuable discussions and, in particular, he suggested to consider some special cases of utility functions. I would also like to thank all my (former) colleagues: Iryna Voloshchenko, Andrea Carlesso, Tao Zhang, as well as our visitor Jose Arturo Infante Acevedo for all the mathematical discussions we had and trading hints we discovered together, in Mannheim. A special thank goes to Florian Klöck for his extensive support during my dissertation project and to Alexander Kalinin for reviewing one of the last versions of my work. I would also like to thank Susanne Spether for all those pleasant and very useful conversations we had together and for her support during my stay at the "Lehrstuhl für Wirtschaftsmathematik I". I am likewise indebted to Prof. Simone Göttlich and to Prof. Andreas Neuenkirch for pointing out the difficulties involved in implementing numerical implicit schemes in the case considered here.

Last but not least, I gratefully acknowledge support by Deutsche Forschungsgemeinschaft through Research Grant SCHI 500/3-1.

# Chapter 2

# Optimal control problem and its value function

## 2.1 Model and preliminaries

### 2.1.1 Setup of the model

Taking $X_0 \in \mathbb{R}^d$, we consider a stochastic process $X_t = (X_t^1, \dots, X_t^d)$ starting in $X_0$ at time $t = 0$ which has to fulfill the boundary condition $X_T = 0$. For example, we can think of a basket of shares in $d$ risky assets an investor can choose to liquidate a large market order, where we describe by $X_t^i$ the number of shares of the $i$-th asset held at time $t$. The return process, or revenues process, is modeled through stochastic integration of $X_t$ with respect to the vector of prices $P_t = (P_t^1, \dots, P_t^d)$. As in the Almgren and Chriss (2001) model, we assume that the latter process can be split in two parts in order to model the influence of the variation of the process $X$ on it. To this end, we write

$$P_t = \widetilde{P}_t + I_t,$$

where $\widetilde{P}_t$ denotes the unperturbed price process, which models the prices that would have been available to a small investor if the large investor was not trading, while $I_t$ denotes the market impact which represents the discount that has to be borne by the investor in order to, e.g., liquidate his holdings. The corresponding revenues are then given by

$$\mathcal{R}_T^X = \int_0^T \dot{X}_s \cdot P_s \, ds = X_0 \widetilde{P}_0 + \int_0^T X_s \, d\widetilde{P}_s - \int_0^T \dot{X}_s \cdot I_s \, ds.$$

The economic interpretation is as follows. The first term is the face value of our portfolio. The stochastic integral term is associated to the volatility risk which has to be taken into account when buying/selling throughout the time interval $[0, T]$ rather than instantaneously. The last term is the "fee" or execution costs, which can arise, for instance, from the variations of $X$ performed in order to reach our boundary condition.

Modeling the price of an illiquid asset with price impact has been the source of extensive studies: see, e.g., Bank and Baum (2004), Jarrow (1994), Çetin et al. (2004), or Kraft and Kühn (2011), to mention only a few. In our setting, following the Almgren and Chris model, we split the price impact in two parts. More precisely, we split the price impact in a temporary price impact and a permanent price impact: $I_t = I_t^{temp} + I_t^{perm}$. This implies the following decomposition for the execution costs:

$$\int_0^T \dot{X}_t \cdot I_t \, dt = \int_0^T \dot{X}_t \cdot I_t^{perm} \, dt + \int_0^T \dot{X}_t \cdot I_t^{temp} \, dt = C^{perm} + C^{temp}.$$

Having in mind the basic demand and supply principle, we can give here again an economic interpretation of both price impacts. The permanent price impact can be regarded as the price an investor is willing to pay for an immediate sell/buy of a given number of given shares: this impact affects the price process from the beginning until the end of the trade and is unfavorable with respect to the market price. After being impacted by the introduction of a large amount of shares, which have to be sold/bought, the current price will be locally influenced (in time) by each fluctuation of $X$: this is the temporary impact (see also Schied et al. (2010) for more details on the economic interpretation).

We begin by modeling first the permanent impact. As assumed in Kyle (1985), one of the first models to consider price impact, we suppose the permanent impact to be linear in our shares process: this and also a symmetry property it exhibits exclude quasi arbitrage opportunities, as argued in Huberman and Stanzl (2004). Hence, the permanent price impact is represented in the form

$$I_t^{perm} = \Gamma(X_t - X_0).$$

Following Schied et al. (2010), we assume that the temporary price impact affects only the trading speed, i.e., $\dot{X}_t^i$. This can be written in the form

$$(I_t^{\text{temp}})^i = h^i(\dot{X}_t^i)$$

for a possibly nonlinear function $h : \mathbb{R}^d \longrightarrow \mathbb{R}^d$. This general formulation of the temporary impact can incorporate cross impacts generated by a large order of one or more assets in the portfolio.

**Assumption 2.1.1.** To model the temporary price impact, we assume that there exists a non-negative, strictly convex function $f$ with superlinear growth satisfying $\lim_{|x| \longrightarrow \infty} \frac{f(x)}{|x|} = \infty$, such that

$$f(x) = x \cdot h(x).$$

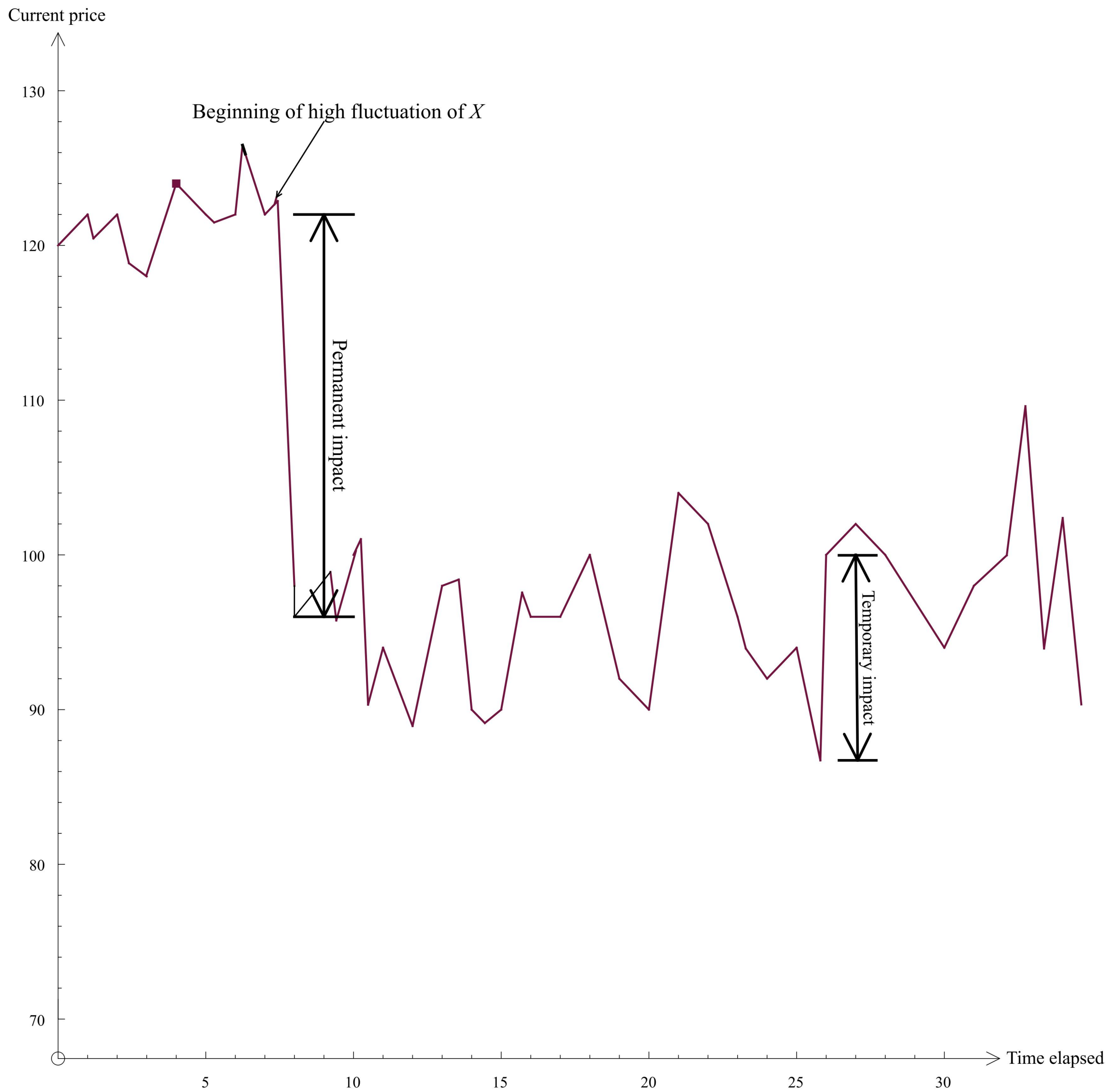



Figure 2.1: Example of price impact on a stock price

This gives us the following execution costs:

$$\mathcal{C}^{temp} = \int_0^T f(\dot{X}_t)\, dt. \tag{2.1}$$

*Remark* 2.1.2. The superlinear growth property can be interpreted as follows: strategies with high fluctuations (e.g., liquidation strategies which consist in liquidating a large amount of shares within a very short time period) generate high costs and therefore cannot be optimal. The convexity property illustrates, in particular, that the temporary impact generated by two fluctuations is larger than the one generated by only one fluctuation whose size is the sum of both the single fluctuations: splitting one big trade in two smaller trades can help to reduce the execution costs (without considering the volatility risk). Note that we have here $f(x) = 0$ if and only if $x = 0$. $\diamondsuit$

We use Almgren's model (developed in Almgren (2003)) to describe the unperturbed price process. This is done by using a Bachelier model in the following form:

$$\widetilde{P}_t^i = \widetilde{P}_0^i + \sum_{j=1}^m \sigma^{ij} B_t^j + b^i t, \qquad i = 1, \ldots, d,$$

for an initial price vector $\widetilde{P}_0 \in \mathbb{R}^d$ and a standard $m$-dimensional Brownian motion $B$ starting in 0 with drift $b \in \mathbb{R}^d$ and volatility matrix $\sigma = (\sigma^{ij}) \in \mathbb{R}^{d\times m}$. In order to avoid non-zero trading strategies with zero cost (i.e., quasi-arbitrage opportunities for the unperturbed price process, such as round trips for investors whose activities/trades do not influence the underlying price), we assume that the drift vector $b$ is orthogonal to the kernel of the covariance matrix $\Sigma = \sigma\sigma^\top$. From now on we can model our revenues over $[0, T]$ in the following way:

$$\mathcal{R}_T^X = R_0 + \int_0^T X_t^\top \sigma\, dB_t + \int_0^T b \cdot X_t\, dt - \int_0^T f(\dot{X}_t)\, dt. \tag{2.2}$$

Here again, there is an economic interpretation. The first term, $R_0 = X_0^\top \widetilde{P}_0 - \frac{1}{2} X_0^\top \Gamma X_0$, can be viewed as the face value of a portfolio including the discount associated to the introduction of this order in the market (which is represented by subtracting the term $\frac{1}{2} X_0^\top \Gamma X_0$). The stochastic integral term models the accumulated volatility risk, whereas the second integral term represents the linear drift applied to our state process. The last one stands for the cumulative cost of the temporary price impact.

In order for our revenues process to be well-defined, we have to make some assumptions in our framework. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_t)_{0\le t\le T}$ satisfying the usual conditions. In conjunction with the notation of Schied and Schöneborn (2008), we denote by

$$\mathcal{X}_{det}(T, X_0) = \left\{ X : [0, T] \to \mathbb{R}^d \ \text{ absolutely continuous, } \ \text{with given } X_0 \text{ and } X_T = 0 \right\},$$

the set of the deterministic processes whose speed liquidation processes $\dot{X}_t$ are defined $\lambda$-almost everywhere, where $\lambda$ stands for the Lebesgue-measure on $[0, T]$. Further, by

$$\mathcal{X}(T, X_0) := \\ \left\{ (X_t)_{t\in[0,T]} \text{ adapted, } t \to X_t \in \mathcal{X}_{det}(T, X_0) \text{ a.s., and } \sup_{0\le t\le T} |X_t| \in \mathrm{L}^\infty(\mathbb{P}) \right\},$$

we denote the set of the $\mathbb{P}\otimes\lambda$-a.e. bounded stochastic processes whose speed liquidation processes $\dot{X}_t$ can be defined $\mathbb{P}\otimes\lambda$-a.e., due to the absolute continuity assumption.

*Remark* 2.1.3. From a hedging point of view, the absolute continuity of $X$ seems to be very restrictive, since this does not englobe the Black-Scholes Delta hedging, for example. However, from a mathematical point of view, this serves as a reasonable starting point for developing a theory of optimal control problems for functions with bounded variation. $\diamondsuit$

In order to give a preference relation on the preceding set, we use here a utility function and consider the following expected-utility maximization problem:

$$\sup_{X\in\mathcal{X}(T,X_0)} \mathbb{E}\left[u\left(\mathcal{R}_T^X\right)\right], \tag{2.3}$$

which is viewed in, e.g., Bertsimas and Lo (1998) as the most natural approach to model execution costs. Here, $u : \mathbb{R} \longrightarrow \mathbb{R}$ is a strictly concave and increasing function. From an economic point of view, we say that our investor is a Von-Neumann-Morgenstern rational investor.

Further, it will be convenient to parametrize elements in $\mathcal{X}(T, X_0)$ (as done in Schied and Schöneborn (2008)). Toward this end, for $\xi$ progressively measurable and $\xi_t$ with values in $\mathbb{R}^d$, for $t \le T$, let us denote by

$$\dot{\mathcal{X}}_0(T, X_0) = \left\{ \xi \,|\, X_t = X_0 - \int_0^t \xi_s \, ds \text{ a.s., for } X \in \mathcal{X}(T, X_0) \right\}$$

the set of *control processes* or speed processes of $X$, when referring to a given process $X$. From now on, we denote our revenues process by $\mathcal{R}^\xi$, for a given $\xi \in \dot{\mathcal{X}}_0(T, X_0)$, to insist on the dependence on $\xi$. The pair $(X^\xi, \mathcal{R}^\xi)$ is then the solution of the following controlled stochastic differential equation:

$$\begin{cases} d\mathcal{R}_t^\xi = X_t^\top \sigma dB_t + b \cdot X_t \, dt - f(-\xi_t) \, dt, \\ dX_t = -\xi_t \, dt, \\ \mathcal{R}_{|t=0}^\xi = R_0 \text{ and } X_{|t=0} = X_0. \end{cases} \tag{2.4}$$

This can be rewritten in the following way:

$$\begin{cases} dY_t = d(\mathcal{R}_t^\xi, X_t) = \overline{\sigma}(t, Y_t, \xi_t) d\overline{B}_t + \overline{b}(t, Y_t, \xi_t) dt \\ Y_0 = (R_0, X_0), \end{cases} \tag{2.5}$$

where $\overline{\sigma}(t, y = (x, r), \xi) = (x^\top \sigma, 0)$, $\overline{B} = (B, W)$ with $W$ a one-dimensional Brownian motion, and $\overline{b}(t, (x, r), \xi) = (b \cdot x - f(\xi), -\xi)$. Note that even if $W$ does not play a role here, this can be useful to introduce a noise in our controlled process $X$ in order, for instance, to approximate our degenerate HJB equation established later on by a non-degenerate one (e.g., by setting, for all $\varepsilon > 0$, $\overline{\sigma}^\varepsilon(t, y, \xi) = (x^\top \sigma, \varepsilon)$). To this end, we can also suppose that $W$ is independent of $B^j, j = 1, \ldots, m$. We denote by $\dot{\mathcal{X}}(T, X_0)$ the subset of all control processes $\xi \in \dot{\mathcal{X}}_0(T, X_0)$ which satisfy the additional requirement

$$\mathbb{E}\left[\int_0^T |\overline{\sigma}(t, y, \xi_t)|^2 + |\overline{b}(t, y, \xi_t)| dt\right] = \mathbb{E}\left[\int_0^T \left(X_t^\xi\right)^\top \Sigma X_t^\xi + |b \cdot X_t^\xi - f(\xi_t)| + |\xi_t| \, dt\right] < \infty, \tag{2.6}$$

for $y \in \mathbb{R}^{d+1}$. With this assumption and with the above uniform Lipschitz condition in $y$, on $\overline{\sigma}$ and $\overline{b}$, it can be shown (see, e.g., Protter (2004), Chapter 5, Theorem 6) that the preceding SDE has a unique strong solution. Thus, our process $Y^\xi = (X^\xi, \mathcal{R}^\xi)$ is called the *controlled process*.

For the sake of convenience, we enlarge the preceding set $\dot{\mathcal{X}}(T, X_0)$ by introducing the following notation: we denote by $\dot{\mathcal{X}}^1(T, X_0)$ the set of the liquidation strategies whose paths satisfy (2.6), but are not necessarily uniformly bounded, i.e.,

$$\begin{aligned}
&\dot{\mathcal{X}}^1(T, X_0) \\
&:= \left\{\xi \,\middle|\, \left(X_t^\xi := X_0 - \int_0^t \xi_s \, ds\right)_{t \in [0,T]} \text{ adapted, } t \to X_t^\xi(\omega) \in \mathcal{X}_{det}(T, X_0) \, \mathbb{P}\text{-a.s.}\right\} \\
&\bigcap \left\{\xi \,\middle|\, \mathbb{E}\left[\int_0^T \left(X_t^\xi\right)^\top \sigma X_t^\xi + |b \cdot X_t^\xi - f(\xi_t)| + |\xi_t| \, dt\right] < \infty\right\} \\
&\supseteq \dot{\mathcal{X}}(T, X_0).
\end{aligned}$$

The maximization problem can therefore be reformulated in the following form:

$$\sup_{\xi \in \dot{\mathcal{X}}^1(T, X_0)} \mathbb{E}\left[u\left(\mathcal{R}_T^\xi\right)\right]. \tag{2.7}$$

In our work, we will consider a special class of utility functions. These functions will have a bounded Arrow-Pratt coefficient of absolute risk aversion, i.e., there shall exist two positive constants $A_i, i = 1, 2$, such that

$$0 < A_1 \leq -\frac{u''(x)}{u'(x)} \leq A_2 \quad \text{for all } x \in \mathbb{R}. \tag{2.8}$$

From the preceding inequality, we can suppose w.l.o.g. that $0 < A_1 < 1 < A_2$, which implies that

$$\exp(-A_1 x) \geq \frac{u'(x)}{u'(0)} \geq \exp(-A_2 x), \qquad \text{for } x \geq 0, \tag{2.9}$$

and

$$\exp(-A_1x) \leq \frac{u'(x)}{u'(0)} \leq \exp(-A_2x), \qquad \text{for } x < 0, \tag{2.10}$$

which gives us the following inequality for the utility function:

$$\frac{u'(0)}{A_1}(1-\exp(-A_1x)) \geq u(x)-u(0) \geq \frac{u'(0)}{A_2}(1-\exp(-A_2x)) = u_2(x).$$

Setting w.l.o.g. $u'(0)=1, u(0)=0$ (by translating $u$ vertically and/or multiplying it by a constant if necessary) and using then the inequalities

$$\frac{1}{A_1} - \exp(-A_1x) \geq \frac{1}{A_1}(1-\exp(-A_1x))$$

and

$$\frac{1}{A_2}(1-\exp(-A_2x)) \geq -\exp(-A_2x),$$

we finally have that the following inequality is fulfilled by $u$:

$$u_1(x) = \frac{1}{A_1} - \exp(-A_1x) \geq u(x) \geq -\exp(-A_2x) = u_2(x). \tag{2.11}$$

*Remark* 2.1.4. For some applications, it will be more convenient to combine both inequalities (2.9) and (2.10) to obtain the following one:

$$\exp(-A_1x) \leq u'(x) \leq \exp(-A_2x)+1, \qquad \text{for } x \in \mathbb{R}. \tag{2.12}$$

$\diamondsuit$

From Schied et al. (2010), it is known that for exponential utility functions, i.e., utility functions in the form $a - b\exp(-cx)$, where $a \in \mathbb{R}$ and $b, c > 0$, there exists a unique deterministic and continuous strategy solving the maximization problem (2.3). Moreover, the corresponding value function, i.e., the value function generated by the exponential expected-utility maximization problem, is the unique continuously differentiable solution of a Hamilton-Jacobi-Bellmann equation. We are going to use this strong result in order to derive the existence of an optimal control under the assumption (2.11). First note that from the preceding we get

$$\sup_{\xi \in \dot{\mathcal{X}}^1(T,X_0)} \mathbb{E}\left[u_1\left(\mathcal{R}_T^{\xi}\right)\right] \geq \sup_{\xi \in \dot{\mathcal{X}}^1(T,X_0)} \mathbb{E}\left[u\left(\mathcal{R}_T^{\xi}\right)\right] \geq \sup_{\xi \in \dot{\mathcal{X}}^1(T,X_0)} \mathbb{E}\left[u_2\left(\mathcal{R}_T^{\xi}\right)\right], \tag{2.13}$$

which can be rewritten as follows:

$$V_1(T,X_0,R_0) = \mathbb{E}\left[u_1\left(\mathcal{R}_T^{\xi_1^*}\right)\right] \geq \mathbb{E}\left[u\left(\mathcal{R}_T^{\xi}\right)\right] \geq \mathbb{E}\left[u_2\left(\mathcal{R}_T^{\xi_2^*}\right)\right] = V_2(T,X_0,R_0), \tag{2.14}$$

where, for $i=1,2$, $V_i,\ i=1,2$, denotes the corresponding exponential value function and $\xi_i^*$ is the corresponding optimal strategy. Another convenient formulation of an exponential value function $\overline{V}$, where the utility function is given by $\exp(-Ax)$ (we drop the minus sign in front of the exponential term and let $A \in [A_1, A_2]$), which can be found in Schied et al. (2010), is the following one:

$$\overline{V}(T, X_0, R_0) = \exp\left[ - AR_0 + A \inf_{\dot{\mathcal{X}}_{det}(T,X_0)} \int_0^T \mathcal{L}(X_t^\xi, \xi_t)\, dt \right], \tag{2.15}$$

where $\mathcal{L}$ denotes the Lagrangian operator defined by

$$\mathcal{L}(q,p) = \frac{A}{2} q^\top \Sigma q - b^\top q + f(-p).$$

In our work, we propose to study some important properties of the following value function:

$$V(T, X_0, R_0) = \sup_{\xi \in \dot{\mathcal{X}}^1(T,X_0)} \mathbb{E}\left[ u\left( \mathcal{R}_T^\xi \right) \right] \tag{2.16}$$

where the utility function $u$ satisfies (2.11). To this end, we first need to explore some properties of exponential value functions.

### 2.1.2 Some properties of exponential value functions

In the sequel, $\overline{V}(T, X, R)$ will be taken as in (2.15) and we start by proving a boundedness result for the case of exponential value functions.

**Lemma 2.1.5.** *Let $T^n, X^n, R^n$ be a bounded sequence in $]0, T] \times \mathbb{R}^d \times \mathbb{R}$ such that $\inf_n T_n > 0$. Let us denote by $\xi^n \in \dot{\mathcal{X}}_{det}(T^n, X_0^n)$ the corresponding deterministic optimal strategy associated to $\overline{V}(T^n, X^n, R^n)$. Then $\int_0^{T^n} f(-\xi_t^n)\, dt$ is uniformly bounded in $n$.*

*Proof.* To show this, let us use (2.15) to write

$$\overline{V}(T^n, X^n, R^n) = \exp\left[ -AR^n + A \int_0^{T^n} \left( \frac{(X_t^{\xi^n})^\top \Sigma X_t^{\xi^n}}{2} - bX_t^{\xi^n} + f(-\xi_t^n) \right) dt \right]. \tag{2.17}$$

Because the left-hand side is bounded, due to the continuity of $\overline{V}$, the boundedness of the sequence $(T^n, X^n, R^n)$ and the fact that $\inf T^n > 0$ (recall that $\lim_{T\to 0} \overline{V}(T, X_0, R_0) = \infty$ when $X_0 \neq 0$, see Schied et al. (2010)), this implies that

$$\int_0^{T^n} \left( \frac{(X_t^{\xi^n})^\top \Sigma X_t^{\xi^n}}{2} - bX_t^{\xi^n} + f(-\xi_t^n) \right) dt$$

is also bounded in $n$. Because $\Sigma$ is positive-semidefinite, we have

$$\begin{aligned} \int_0^{T^n} &\left( \frac{(X_t^{\xi^n})^\top \Sigma X_t^{\xi^n}}{2} - bX_t^{\xi^n} + f(-\xi_t^n) \right) dt \\ &\geq \int_0^{T^n} \Big( - bX_t^{\xi^n} + f(-\xi_t^n) \Big)\, dt \end{aligned}$$

$$= \int_0^{T^n} \Big( b \cdot \xi_t^n t + f(-\xi_t^n) \Big)\, dt,$$

where the last equality is due to integration by parts. Because $\lim_{|x| \longrightarrow \infty} \frac{f(x)}{|x|} = \infty$, there exists a constant $C$ such that $\frac{|y|}{f(y)} \leq \eta$ for $|y| > C$, where $\eta$ is such that $|b|T\eta < 1$. Consider now the following set, $A_t^n = \{|\xi_t^n| \leq C\}$. We have by the integration by parts formula that

$$\begin{aligned}
&\int_0^{T^n} \Big( b \cdot \xi_t^n t + f(-\xi_t^n) \Big)\, dt \\
&\quad = \int_0^{T^n} \mathbb{1}_{A_t^n} \Big( b \cdot \xi_t^n t + f(-\xi_t^n) \Big)\, dt + \int_0^{T^n} \mathbb{1}_{(A_t^n)^c} \Big( b \cdot \xi_t^n t + f(-\xi_t^n) \Big)\, dt \\
&\quad \geq -M^n + \int_0^{T^n} \mathbb{1}_{A_t^n} f(-\xi_t^n)\, dt + \int_0^{T^n} \mathbb{1}_{(A_t^n)^c} f(-\xi_t^n) \Big( 1 + \frac{b \cdot \xi_t^n t}{f(-\xi_t^n)} \Big)\, dt \\
&\quad \geq -M^n + \int_0^{T^n} \mathbb{1}_{A_t^n} f(-\xi_t^n)\, dt + \int_0^{T^n} \mathbb{1}_{(A_t^n)^c} f(-\xi_t^n)(1 - |b|T^n\eta)\, dt \\
&\quad \geq \int_0^{T^n} f(-\xi_t^n)\, dt (1 - |b|T^n\eta) - M^n,
\end{aligned}$$

where $M^n = -|b|C(T^n)^2/2$. And because $T^n, M^n$ are bounded in $n$, it follows that $\sup_n \int_0^{T^n} f(-\xi_t^n)\, dt < \infty$. ∎

In the next lemma (see also Schied and Schöneborn (2008)), we give a useful upper bound for our value function, as previously defined in (2.16), with the help of the inequalities in (2.11).

**Lemma 2.1.6.** *Let $t \in ]0, T[$. Then we have*

$$|V(t, X_0, R_0)| \leq \frac{1}{A_1} + \exp\Big( - A_2 \Big( R_0 + A_2 |X_0|^2 |\Sigma| \frac{T}{6} + |b||X_0| \frac{T}{2} + t f\Big( \frac{-X_0}{t} \Big) \Big) \Big), \tag{2.18}$$

*where $|\Sigma|$ is the operator norm of $\Sigma$.*

*Proof.* First, note that we have by using (2.11), in conjunction with (2.14),

$$|V(T, X_0, R_0)| \leq \frac{1}{A_1} + |V_2(T, X_0, R_0)|.$$

Applying (2.15) and choosing the strategy $\widetilde{\xi}_s = \frac{X_0}{t}$, so that $X_s^{\widetilde{\xi}} = X_0 - \int_0^s \widetilde{\xi}_u\, du = X_0 \frac{t-s}{t}$, we have

$$\begin{aligned}
&|V_2(t, X_0, R_0)| \\
&\quad \leq \exp\Big( - A_2 R_0 + A_2 \inf_{\dot{\mathcal{X}}_{det}(t, X_0)} \int_0^t \mathcal{L}(X_s^{\xi}, \xi_s)\, ds \Big) \\
&\quad \leq \exp\Big( - A_2 R_0 + A_2 \int_0^t \mathcal{L}(X_s^{\widetilde{\xi}}, \widetilde{\xi}_s)\, ds \Big)
\end{aligned}$$

$$
\begin{aligned}
&= \exp\Big( - A_2 R_0 + A_2 \int_0^t A_2 \frac{X_0^\top \Sigma X_0}{2t^2}(t-s)^2 - b \cdot X_0 \frac{t-s}{t} + f\Big(-\frac{X_0}{t}\Big)\, ds \Big) \\
&\leq \exp\Big( - A_2 R_0 + A_2 \Big( \frac{X_0^\top \Sigma X_0}{6t^2} t^3 + |b||X_0| \frac{t}{2} + t f\Big(\frac{-X_0}{t}\Big)\Big)\Big) \\
&\leq \exp\Big( - A_2 \Big( R_0 + A_2 |X_0|^2 |\Sigma| \frac{T}{6} + |b||X_0| \frac{T}{2} + t f\Big(\frac{-X_0}{t}\Big)\Big)\Big),
\end{aligned}
$$

which gives us (2.18). ■

## 2.2 The optimization problem and its value function

### 2.2.1 Concavity property and initial condition satisfied by the value function

The aim of this subsection is to prove that the map

$$(X, R) \longmapsto V(T, X, R)$$

is concave, for fixed $T \in [0, \infty[$, and to derive the initial condition satisfied by $V$, where $V$ is the value function of the optimization problem as defined in (2.16). These are fundamental properties of the value function of the considered maximization problem. We start with the following lemma.

**Lemma 2.2.1.** *The map*

$$
\begin{aligned}
\textstyle\bigcup_{X \in \mathbb{R}^d} \dot{\mathcal{X}}^1(T, X) &\longrightarrow \mathbb{R} \\
\xi &\longmapsto \mathbb{E}[u(\mathcal{R}_T^\xi)]
\end{aligned}
\tag{2.19}
$$

*is concave.*

*Proof.* We first notice that

$$\xi \longmapsto \mathcal{R}_T^\xi$$

is concave $\mathbb{P}$-a.s. This is a direct consequence of the definition of $\mathcal{R}_T^\xi$: indeed, both maps $\xi \mapsto \int_0^T X_t^\xi \sigma \, dB_t$ as well as $\xi \mapsto \int_0^T X_t^\xi \cdot b \, dt$ are linear, $\mathbb{P}$-a.s., and $\xi \mapsto -\int_0^T f(-\xi_t)\, dt$ is concave, due to the convexity of $f$. Because $u$ is strictly increasing and concave, it follows that $\xi \longmapsto u(\mathcal{R}_T^\xi)$ is, $\mathbb{P}$-a.s., strictly concave. Taking now the expectation of the preceding map leads to the result. ■

The next proposition establishes the first regularity property of the value function: the concavity of the value function in the revenues parameter, with $T, X_0 \in ]0, \infty[ \times \mathbb{R}^d$ being fixed. This will enable us later to prove the differentiability of the value function in the revenues parameter, other parameters being fixed, with the help of the existence of an optimal strategy.

**Proposition 2.2.2.** *For fixed $T \in ]0,\infty[$,*

$$(X,R) \longmapsto V(T,X,R)$$

*is a concave function.*

*Proof.* Toward this end, let $X, \overline{X} \in \mathbb{R}^d, R, \overline{R} \in \mathbb{R}$ and $\lambda \in ]0,1[$. Further consider the strategies $\xi \in \dot{\mathcal{X}}^1(T,X)$ and $\overline{\xi} \in \dot{\mathcal{X}}^1(T,\overline{X})$. Note that $\lambda\xi + (1-\lambda)\overline{\xi} \in \dot{\mathcal{X}}(T, \lambda X + (1-\lambda)\overline{X})$. Let us denote

$$\mathcal{R}_T^{\lambda\xi+(1-\lambda)\overline{\xi}} := \int_0^T (X_t^{\lambda\xi+(1-\lambda)\overline{\xi}})^\top \sigma\, dB_t + \int_0^T b \cdot X_t^{\lambda\xi+(1-\lambda)\overline{\xi}}\, dt - \int_0^T f(-\lambda\xi + (1-\lambda)\overline{\xi}_t)\, dt.$$

We then have for fixed $\xi, \overline{\xi}$:

$$\begin{aligned}
&V(T, \lambda X + (1-\lambda)\overline{X}, \lambda R + (1-\lambda)\overline{R})) \\
&\geq \mathbb{E}\big[u\big(\lambda R + (1-\lambda)\overline{R} + \mathcal{R}_T^{\lambda\xi+(1-\lambda)\overline{\xi}}\big)\big] \\
&\geq \mathbb{E}\big[u\big(\lambda R + (1-\lambda)\overline{R}) + \lambda\mathcal{R}_T^{\xi} + (1-\lambda)\mathcal{R}_T^{\overline{\xi}}\big)\big] \\
&\geq \lambda\mathbb{E}\big[u\big(R + \mathcal{R}_T^{\xi}\big)\big] + (1-\lambda)\mathbb{E}\big[u\big(\overline{R} + \mathcal{R}_T^{\overline{\xi}}\big)\big],
\end{aligned}$$

where the first inequality is due to the definition of the value function $V$ at $(\lambda X + (1-\lambda)\overline{X}, \lambda R + (1-\lambda)\overline{R})$, and the second one follows from the fact that $\xi \mapsto \mathcal{R}_T^{\xi}$ is concave, as proved in Lemma 2.2.1, and $u$ is increasing. Finally, the third one is due the concavity of $u$. Taking now the supremum over $\xi$ ($\overline{\xi}$ being fixed), we obtain

$$V(T, \lambda X + (1-\lambda)\overline{X}, \lambda R + (1-\lambda)\overline{R})) \geq \lambda V(T,X,R) + (1-\lambda)\mathbb{E}\big[u\big(\overline{R} + \mathcal{R}_T^{\overline{\xi}}\big)\big].$$

Taking the supremum over $\overline{\xi}$ in the preceding equation, we obtain

$$V(T, \lambda X + (1-\lambda)\overline{X}, \lambda R + (1-\lambda)\overline{R})) \geq \lambda V(T,X,R) + (1-\lambda)V(T,X,\overline{R}),$$

which yields the assertion. ∎

Further, we establish the initial condition fulfilled by the value function.

**Proposition 2.2.3.** *Let $V$ be the value function of the maximization problem (2.16). Then $V$ fulfills the following initial condition.*

$$V(0,X,R) = \lim_{T\downarrow 0} V(T,X,R) = \begin{cases} u(R), & \text{if } X = 0, \\ -\infty, & \text{otherwise.} \end{cases} \tag{2.20}$$

*Proof.* We first note that if $X \neq 0$, then

$$\lim_{T\to 0} V(T,X,R) = -\infty,$$

because $V$ is supposed to lie between two CARA value functions which tend to $-\infty$ as $T$ goes to zero, if $X \neq 0$ (see Schied et al. (2010)). Suppose now that $X = 0$. We prove that

$$\lim_{T\to 0} V(T, 0, R) = u(R).$$

Observe first that

$$V(T, 0, R) \geq \mathbb{E}\Big[u\Big(\mathcal{R}_T^\xi\Big)\Big] = u(R),$$

by choosing the strategy $\xi_t = 0$ for all $t \in [0, T]$, $T > 0$. Since $V$ is increasing in $T$, for fixed $X, R$, the limit $\lim_{T\to 0}, V(T, X, R)$ exists, which implies that

$$\lim_{T\to 0} V(T, 0, R) \geq u(R).$$

We prove now the reverse inequality

$$\lim_{T\to 0} V(T, 0, R) \leq u(R). \tag{2.21}$$

Let $\xi$ be a round trip starting from 0 (i.e: $\xi \in \dot{\mathcal{X}}^1(T, 0)$). Applying Jensen's inequality to the concave utility function $u$, we get

$$\begin{aligned}
\mathbb{E}\big[u\big(\mathcal{R}_T^\xi\big)\big] &\leq u\big(\mathbb{E}\big[\mathcal{R}_T^\xi\big]\big) \\
&= u\bigg(R + \mathbb{E}\bigg[\int_0^T (X_t^\xi)^\top \sigma \, dBt + \int_0^T b \cdot X_t^\xi \, dt - \int_0^T f(-\xi_t) \, dt\bigg]\bigg) \\
&= u\bigg(R + \mathbb{E}\bigg[\int_0^T b \cdot X_t^\xi \, dt - \int_0^T f(-\xi_t) \, dt\bigg]\bigg),
\end{aligned}$$

where the final equality is due to the fact that the stochastic integral term is a martingale. We have to show now

$$\limsup_{T\downarrow 0} \mathbb{E}\bigg[\int_0^T b \cdot X_t^\xi \, dt - \int_0^T f(-\xi_t) \, dt\bigg] \leq 0. \tag{2.22}$$

To this end we use the integration by parts formula to infer

$$\int_0^T b \cdot X_t^\xi \, dt = \int_0^T tb \cdot \xi_t \, dt.$$

Hence, we have

$$\begin{aligned}
\mathbb{E}\bigg[\int_0^T b \cdot X_t^\xi \, dt - \int_0^T f(-\xi_t) \, dt\bigg] &= \mathbb{E}\bigg[\int_0^T tb \cdot \xi_t - f(-\xi_t) \, dt\bigg] \\
&\leq \int_0^T f^*(-bt) \, dt,
\end{aligned}$$

where $f^*$ designates the Fenchel-legendre transformation of the convex function $f$. As it will be seen in Remark 3.2.1, $f^*$ is convex, and in particular continuous, so that

$$\int_0^T f^*(-bt)\,dt \underset{T\downarrow 0}{\longrightarrow} 0,$$

which proves (2.22). Finally, using that $u$ is continuous and nondecreasing, we get

$$\begin{aligned}
\lim_{T\to 0} V(T,0,R) &= \liminf_{T\to 0} V(T,0,R) \\
&= \liminf_{T\to 0} \sup_{\xi\in\dot{\mathcal{X}}^1(T,0)} \mathbb{E}\big[u\big(\mathcal{R}_T^{\xi}\big)\big] \\
&\le \liminf_{T\to 0} \sup_{\xi\in\dot{\mathcal{X}}^1(T,0)} u\bigg(R+\mathbb{E}\bigg[\int_0^T b\cdot X_t^{\xi}\,dt - \int_0^T f(-\xi_t)\,dt\bigg]\bigg) \\
&\le u(R).
\end{aligned}$$

■

### 2.2.2 Existence and uniqueness of an optimal strategy

In this section we want to prove the existence and uniqueness of an optimal strategy for the considered maximization problem

$$\sup_{\xi\in\dot{\mathcal{X}}^1(T,X_0)} \mathbb{E}[u(\mathcal{R}_T^{\xi})],$$

where $u$ is a strictly concave and increasing function defined on $\mathbb{R}$ satisfying (2.11), and where $\mathcal{R}_T^{\xi}$ denotes the revenues associated with the liquidation strategy $\xi$ over the time interval $[0,T]$, as given in (2.2). The next theorem establishes the main result of the current section.

**Theorem 2.2.4.** *Let $T, X_0, R_0$ be in $]0,\infty[\times\mathbb{R}^d\times\mathbb{R}$, then there exists a unique $\xi^*\in\dot{\mathcal{X}}^1(T,X_0)$ such that*

$$V(T,X_0,R_0) = \sup_{\xi\in\dot{\mathcal{X}}^1(T,X_0)} \mathbb{E}[u(\mathcal{R}_T^{\xi})] = \mathbb{E}\Big[u\Big(\mathcal{R}_T^{\xi^*}\Big)\Big]. \tag{2.23}$$

The main idea of the proof is to show that a sequence of strategies $\xi^n$ such that

$$\mathbb{E}\big[u\big(\mathcal{R}_T^{\xi^n}\big)\big] \nearrow \sup_{\xi\in\dot{\mathcal{X}}^1(T,X_0)} \mathbb{E}\big[u\big(\mathcal{R}_T^{\xi}\big)\big]$$

lies in a weakly sequentially compact subset of the set $\dot{\mathcal{X}}^1(T,X_0)$, due to the fact that the function $u$ satisfies the inequalities (2.11), i.e., $u$ is between two exponential utility functions. We can therefore take a subsequence of the preceding sequence that converges weakly to a strategy $\xi^*$. The uniqueness of the optimal strategy will follow from the strict concavity of the map $\xi \longmapsto \mathbb{E}[u(\mathcal{R}_T^{\xi})]$. Due to inequality (2.14), we can w.l.o.g suppose that this sequence is such that

$$\mathbb{E}\left[\exp(-A_1\mathcal{R}_T^{\xi^n})\right] \leq 1 + 1/A_1 - V_2(T, X_0, R_0), \quad \text{for all } n \in \mathbb{N}, \tag{2.24}$$

where $V_2$ denotes the following CARA value function:

$$V_2(T, X_0, R_0) = \sup_{\xi \in \dot{\mathcal{X}}^1(T,X_0)} \mathbb{E}\big[-\exp\big(-A_2\mathcal{R}_T^{\xi}\big)\big].$$

We split the proof into several steps. We first prove a weak compactness property of certain subsets of $\dot{\mathcal{X}}^1(T, X_0)$. We start by recalling some fundamental functional analysis results. The first one is a classical characterization of convex closed sets (see e.g., Föllmer and Schied (2011), Theorem A.60 for a proof).

**Theorem 2.2.5.** *Suppose that $E$ is a locally convex space and that $\mathcal{C}$ is a convex subset of $E$. Then $\mathcal{C}$ is weakly closed if and only if $\mathcal{C}$ is closed with respect to the original topology of $E$.*

**Corollary 2.2.6.** *Let $\varphi : E \to ]-\infty; \infty]$ be a lower semi-continuous convex function with respect to the original topology of $E$. Then, $\varphi$ is lower semi-continuous with respect to the weak topology $\sigma(E', E)$, where $E'$ denotes the dual space of $E$. In particular, if $(x_n)$ converges weakly to $x$, then*

$$\varphi(x) \leq \liminf \varphi(x_n). \tag{2.25}$$

*Proof.* See e.g., Brezis (2011) (recall that if $\varphi$ is lower semi-continuous, then (2.25) holds, and conversely if $E$ is a metric space). ■

**Corollary 2.2.7.** *Let $(S, \mathcal{S}, \mu)$ be a measurable space, $F : \mathbb{R}^d \to \mathbb{R}$ a convex function bounded from below and $(x_n) \subset \mathrm{L}^1((S, \mathcal{S}, \mu); R^d)$. Suppose that $(x_n)$ converges to $x$, weakly. Then*

$$\int F(x)d\mu \leq \liminf \int F(x_n)d\mu.$$

*Further, if we suppose that $F : \mathbb{R}^d \to \mathbb{R}$ is concave and bounded from above, we have an analogous conclusion, i.e.,*

$$\int F(x)d\mu \geq \limsup \int F(x_n)d\mu.$$

*Proof.* We only show the first assertion, the second one being similar to prove. By using the preceding corollary, it is sufficient to prove that the convex map

$$\begin{aligned} L^1((S, \mathcal{S}, \mu); \mathbb{R}^d) &\longrightarrow [0, \infty] \\ \alpha &\longmapsto \int F(\alpha)\, d\mu \end{aligned}$$

is lower semi-continuous with respect to the strong topology of $\mathrm{L}^1((S,\mathcal{S},\mu);\mathbb{R}^d)$. To this end, let $c \in \mathbb{R}$ and $(x_n) \subset \mathrm{L}^1((S,\mathcal{S},\mu);\mathbb{R}^d)$ be a sequence that converges strongly to some $x \in \mathrm{L}^1((S,\mathcal{S},\mu);\mathbb{R}^d)$ and satisfies the condition $\int F(x_n)\,d\mu \le c$, for every $n$. We have to prove that

$$\int F(x)\,d\mu \le c.$$

By taking a subsequence if necessary, we can suppose that $(x_n)$ converges to $x$ $\mu$-a.e. Applying then Fatou's Lemma, we infer

$$\int F(x)\,d\mu = \int \liminf F(x_n)\,d\mu \le \liminf \int F(x_n)\,d\mu \le c,$$

which concludes the proof. ■

We can now establish the following lemma, which will be also useful for proving the continuity of the value function.

**Lemma 2.2.8.** *Let $(X_0^n, T^n) \subset \mathbb{R}^d \times \mathbb{R}$ be a sequence that converges to $(X_0, T)$ and set $\overline{T} = \sup_n T^n$. Further, consider a sequence $(\zeta^n)$ in $\dot{\mathcal{X}}^1(T^n, X_0^n)$ and $c > 0$ such that*

$$\mathbb{E}\left[\int_0^T f(-\zeta_t^n)\,dt\right] \le c. \tag{2.26}$$

*Suppose that $\zeta^n$ converges to $\zeta$ with respect to the weak topology on*

$$\mathrm{L}^1 := \mathrm{L}^1\Big(\Omega \times [0,T], \mathcal{F} \otimes \mathcal{B}([0,T]), (\mathbb{P} \otimes \lambda)\Big).$$

*Then $\zeta \in \dot{\mathcal{X}}^1(T, X_0)$ and, moreover,*

$$\mathbb{E}\left[\int_0^T f(-\zeta_t)\,dt\right] \le c. \tag{2.27}$$

*Proof.* First note that we have the canonical inclusion $\dot{\mathcal{X}}^1(T^n, X_0^n) \subseteq \dot{\mathcal{X}}^1(\overline{T}, X_0^n)$, by setting $\zeta^n = 0$ on $[T^n, \overline{T}]$. Suppose by way of contradiction that

$$\int_0^T \zeta\,dt \ne X_0.$$

Then, there exists a component $\zeta^i$, with $i \in \{1 \ldots d\}$, such that

$$\int_0^T \zeta_t^i\,dt \ne X_0^i.$$

When looking at that component, we can assume without loss of generality that $d = 1$, i.e., $\zeta = \zeta^i$, and work toward a contradiction. Under this assumption, there exists then a measurable set $\mathcal{A}$ with $\mathbb{P}(\mathcal{A}) > 0$, such that

$$\int_0^T \zeta_t\,dt > X_0 \quad \text{on } \mathcal{A} \qquad \text{or} \qquad \int_0^T \zeta_t\,dt < X_0 \quad \text{on } \mathcal{A}.$$

Without loss of generality, we can suppose that

$$\int_0^T \zeta_t \, dt > X_0 \quad \text{on } \mathcal{A}. \tag{2.28}$$

Because $\zeta^n \in \dot{\mathcal{X}}^1(T^n, X_0^n)$ converges to $\zeta$, weakly in $\mathrm{L}^1$, we have

$$0 = \mathbb{E}\left[\left(X_0^n - \int_0^{T^n} \zeta_t^n \, dt\right)\mathbb{1}_{\mathcal{A}}\right] = \mathbb{E}\left[\left(X_0^n - \int_0^{\overline{T}} \zeta_t^n \, dt\right)\mathbb{1}_{\mathcal{A}}\right]$$
$$\longrightarrow \mathbb{E}\left[\left(X_0 - \int_0^{\overline{T}} \zeta_t \, dt\right)\mathbb{1}_{\mathcal{A}}\right] = 0.$$

If $\overline{T} = T$, the result is proved, because the expectation on the right-hand side has to be negative, due to the assumption (2.28), which is a contradiction.

Suppose now that $\overline{T} > T$. It is sufficient to prove that $\zeta = 0$ on $[T, \overline{T}]$. To this end, set

$$\eta_t(\omega) := \mathbb{1}_{\{\zeta_t(\omega)>0\}}\mathbb{1}_{[T,\overline{T}]}(t).$$

Here again, due to the weak convergence of $\zeta^n$ to $\zeta$, the fact that

$$\eta \in \mathrm{L}^\infty\big((\Omega \times [0, \overline{T}], \mathcal{F} \otimes \mathcal{B}([0, \overline{T}]), (\mathbb{P} \otimes \lambda); \mathbb{R}^d)\big)$$

and $\zeta_n = 0$ on $[T^n, \overline{T}]$, we get

$$0 = \mathbb{E}\left[\int_{T^n}^{\overline{T}} \zeta_t^n \eta_t \, dt\right] \longrightarrow \mathbb{E}\left[\int_T^{\overline{T}} \zeta_t \eta_t \, dt\right] = 0.$$

Thus, $\{\zeta_t(\omega) > 0; t \in [T, \overline{T}]\}$ is a null set. By choosing $\eta_t(\omega) := \mathbb{1}_{\{\zeta_t(\omega)>0\}}\mathbb{1}_{[T,\overline{T}]}(t)$, we can prove in the same manner that $\{\zeta_t(\omega) < 0 \text{ on } [T, \overline{T}]\}$ is a null set. Hence, $\zeta = 0$ on $[T, \overline{T}]$. Therefore,

$$\int_0^T \zeta \, dt = X_0.$$

Using Corollary 2.2.7 we infer

$$\mathbb{E}\bigg[\int_0^T f(-\zeta_t) \, dt\bigg] \le \liminf_{n \longrightarrow \infty} \mathbb{E}\bigg[\int_0^T f(-\zeta_t^n) \, dt\bigg] \le c.$$

Now we want to show that $\zeta$ fulfills (2.6). Since $(\zeta^n) \subset \mathrm{L}^1$ and converges weakly to $\zeta$, we have that $\zeta \in \mathrm{L}^1$. Hence, we get

$$\mathbb{E}\bigg[\int_0^T \big(X_t^\xi\big)^\top \sigma X_t^\xi + |\xi_t| \, dt\bigg] \le \mathbb{E}\bigg[\int_0^T \|\zeta\|_{\mathrm{L}^1}^2 |\Sigma| + |\xi_t| \, dt\bigg] < \infty.$$

Further, we have

$$\mathbb{E}\bigg[\int_0^T |b \cdot X_t^\zeta - f(\zeta_t) \, dt\bigg] \le \mathbb{E}\bigg[\int_0^T |b| \|\zeta\|_{\mathrm{L}^1} + f(\zeta_t) dt\bigg] \le |b| \|\zeta\|_{\mathrm{L}^1} T + c < \infty,$$

and (2.6) is verified. Therefore $\zeta \in \dot{\mathcal{X}}^1(T, X_0)$, which concludes the proof. ∎

We can now prove a weak compactness property of a certain family of subsets, which lies in $\dot{\mathcal{X}}^1(T, X_0)$.

**Proposition 2.2.9.** *For $c > 0$, let*

$$\overline{K}_c = \Big\{\xi \in \dot{\mathcal{X}}^1(T, X_0) \big|\ \mathbb{E}\Big[\int_0^T f(-\xi_t)\, dt\Big] \le c\Big\}.$$

*Then $\overline{K}_c$ is a weakly sequentially compact subset of*

$$\mathrm{L}^1 := \mathrm{L}^1\big((\Omega \times [0,T], \mathcal{F} \otimes \mathcal{B}([0,T]), (\mathbb{P} \otimes \lambda)); \mathbb{R}^d\big).$$

*Proof.* We first prove that $\overline{K}_c$ is a closed convex set with respect to the strong topology of $\mathrm{L}^1$. The convexity of $\overline{K}_c$ is a direct consequence of the convexity of the map

$$\xi \longmapsto \mathbb{E}\Big[\int_0^T f(-\xi_t)\, dt\Big].$$

We show now the closedness of $\overline{K}_c$. To this end, let $\xi^n$ be a sequence in $\overline{K}_c$ that converges strongly to $\xi$. Especially, $\xi^n$ converges to $\xi$ in the weak sense and hence, we are in the setting of Lemma 2.2.8, which proves that $\xi \in \overline{K}_c$. Thus, $\overline{K}_c$ is convex and closed in $\mathrm{L}^1$. Hence, it is also closed with respect to the weak topology, as argued in Theorem 2.2.5. By the Dunford-Pettis theorem (Dunford and Schwartz (1988), Corollary IV.8.11), it remains to show that $\overline{K}_c$ is uniformly integrable to prove that $\overline{K}_c$ is weakly sequentially compact.

To this end, take $\varepsilon > 0$ and $\xi \in \overline{K}_c$. There exists $\alpha > 0$ such that $\frac{|\xi_t|}{f(-\xi_t)} \le \frac{\varepsilon}{c}$ for $|\xi_t| > \alpha$, due to the superlinear growth property of $f$. Due to the fact that $f(x) = 0$ if and only if $x = 0$, $1/f(-\xi_t)$ is well-defined on $\{|\xi_t| > \alpha\}$, and we hence get

$$\begin{aligned}
\mathbb{E}\Big[\int_0^T \mathbb{1}_{\{|\xi_t|>\alpha\}} |\xi_t|\, dt\Big] &= \mathbb{E}\Big[\int_0^T \mathbb{1}_{\{|\xi_t|>\alpha\}} \frac{|\xi_t|}{f(-\xi_t)} f(-\xi_t)\, dt\Big] \\
&\le \mathbb{E}\Big[\int_0^T \mathbb{1}_{\{|\xi_t|>\alpha\}} f(-\xi_t)\, dt\Big] \frac{\varepsilon}{c} \\
&\le \varepsilon,
\end{aligned}$$

which proves the uniform integrability of $\overline{K}_c$. ■

We show now a lemma which allows us to find lower and upper bound for the non-stochastic integral terms which appear in the revenue process.

**Lemma 2.2.10.** *Suppose that the drift $b \neq 0$. Let $\xi \in \dot{\mathcal{X}}^1(T, X_0)$ and $t^1, t^2 \in [0,T]$. There exists then a constant $C > 0$, depending on $f, b$ and $T$, such that*

$$\begin{aligned}
&\frac{5}{4}\int_{t_1}^{t_2} f(-\xi_t)\, dt + |b| C T^2/2 + b \cdot \big(t_1 X^\xi_{t_1} - t_2 X^\xi_{t_2}\big) \\
&\ge \int_{t_1}^{t_2} -b \cdot X^\xi_t + f(-\xi_t)\, dt \ge \frac{3}{4}\int_{t_1}^{t_2} f(-\xi_t)\, dt - |b| C T^2/2 + b \cdot \big(t_1 X^\xi_{t_1} - t_2 X^\xi_{t_2}\big).
\end{aligned}$$

*Proof.* Set $\gamma = \frac{1}{4|b|T}$. Because $\lim_{|x|\longrightarrow\infty} \frac{|x|}{f(x)} = 0$, there exists a constant $C_\gamma = C > 0$ such that $\frac{|y|}{f(y)} \leq \gamma$ for $|y| > C$. Consider now the set $A_t = \{|\xi_t| \leq C\}$ and set further $N = |b|CT^2$. Then:

$$
\begin{aligned}
&\int_{t_1}^{t_2} -b\cdot X_t^\xi + f(-\xi_t)\,dt \\
&= -tb\cdot X_t^\xi\Big|_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2} b\cdot\xi_t t + f(-\xi_t)\,dt \\
&= b\cdot\left(t_1X_{t_1}^\xi - t_2X_{t_2}^\xi\right) + \int_{t_1}^{t_2} \mathbb{1}_{A_t}\Big(b\cdot\xi_t t + f(-\xi_t)\Big)\,dt + \int_{t_1}^{t_2} \mathbb{1}_{A_t^c}\Big(b\cdot\xi_t t + f(-\xi_t)\Big)\,dt \\
&\geq b\cdot\left(t_1X_{t_1}^\xi - t_2X_{t_2}^\xi\right) - \int_{t_1}^{t_2} \mathbb{1}_{A_t}|b\cdot\xi_t|t\,dt + \int_{t_1}^{t_2} \mathbb{1}_{A_t}f(-\xi_t)\,dt \\
&\quad + \int_{t_1}^{t_2} \mathbb{1}_{A_t^c}f(-\xi_t)\Big(1 + \frac{b\cdot\xi_t t}{f(-\xi_t)}\Big)\,dt \\
&\geq b\cdot\left(t_1X_{t_1}^\xi - t_2X_{t_2}^\xi\right) - |b|C\int_{t_1}^{t_2} t\,dt + \int_{t_1}^{t_2} \mathbb{1}_{A_t}f(-\xi_t)\,dt \\
&\quad + \int_{t_1}^{t_2} \mathbb{1}_{A_t^c}f(-\xi_t)\Big(1 - \frac{|b||\xi_t|T}{f(-\xi_t)}\Big)\,dt \\
&\geq b\cdot\left(t_1X_{t_1}^\xi - t_2X_{t_2}^\xi\right) - |b|CT^2/2 + \int_{t_1}^{t_2} \mathbb{1}_{A_t}f(-\xi_t)\,dt \\
&\quad + \int_{t_1}^{t_2} \mathbb{1}_{A_t^c}f(-\xi_t)\Big(1 - \frac{|b||\xi_t|T}{f(-\xi_t)}\Big)\,dt \\
&\geq b\cdot\left(t_1X_{t_1}^\xi - t_2X_{t_2}^\xi\right) - |b|CT^2/2 + \int_{t_1}^{t_2} \mathbb{1}_{A_t}f(-\xi_t)\,dt + \frac{3}{4}\int_{t_1}^{t_2} \mathbb{1}_{A_t^c}f(-\xi_t)\,dt \\
&\geq b\cdot\left(t_1X_{t_1}^\xi - t_2X_{t_2}^\xi\right) + \frac{1}{4}\int_{t_1}^{t_2} \mathbb{1}_{A_t}f(-\xi_t)\,dt + \frac{3}{4}\int_{t_1}^{t_2} f(-\xi_t)\,dt - |b|CT^2/2 \\
&\geq b\cdot\left(t_1X_{t_1}^\xi - t_2X_{t_2}^\xi\right) + \frac{3}{4}\int_{t_1}^{t_2} f(-\xi_t)\,dt - |b|CT^2/2,
\end{aligned}
$$

which proves the lower inequality. To prove the upper inequality, it is sufficient to follow step by step the preceding arguments and to give an upper bound of the corresponding terms, instead of a lower bound. ■

In the next lemma, we show that a sequence of strategies in $\dot{\mathcal{X}}^1(T, X_0)$ such that the sequence of the corresponding expected utilities converges to the supremum in (2.23) can be chosen to lie in some $\overline{K}_m$, for $m$ large enough. This will be crucial for proving the existence of an optimal strategy. Here, we use the fundamental property (2.24) satisfied by the sequence $(\xi^n)$.

**Lemma 2.2.11.** *Let $(\xi^n)$ be a sequence of strategies such that*

$$\xi^n \in \dot{\mathcal{X}}^1(T, X_0) \text{ and } \mathbb{E}\left[u\left(\mathcal{R}_T^{\xi^n}\right)\right] \nearrow \sup_{\xi\in\dot{\mathcal{X}}^1(T,X_0)} \mathbb{E}\left[u\left(\mathcal{R}_T^{\xi}\right)\right]. \tag{2.29}$$

*Then, there exists a constant $m > 0$ such that*

$$\xi^n \in \overline{K}_m = \Big\{\xi \in \dot{\mathcal{X}}^1(T, X_0) \big|\, \mathbb{E}\Big[\int_0^T f(-\xi_t)\,dt\Big] \leq m\Big\},$$

*for every $n \in \mathbb{N}$.*

*Proof.* Set $\overline{M} := \overline{M}(T, X_0, R_0) = 1 + 1/A_1 - V_2(T, X_0, R_0)$. We first note that, due to (2.24), we have, for every $n$,

$$\mathbb{E}\Big[e^{-A_1\left(R_0+\int_0^T (X_t^{\xi^n})^\top \sigma\, dB_t + \int_0^T b\cdot X_t^{\xi^n}\,dt - \int_0^T f(-\xi^n{}_t)\,dt\right)}\Big] \leq 1/A_1 - V_2(T, X_0, R_0) = \overline{M}.$$

We prove now that, for every $n \in \mathbb{N}$,

$$\xi^n \in \widetilde{K}_\alpha = \Big\{\xi \in \dot{\mathcal{X}}^1(T, X_0) \big|\, \mathbb{E}\Big[\int_0^T -b\cdot X_t^\xi + f(-\xi_t)\,dt\Big] \leq \alpha\Big\}, \tag{2.30}$$

for $\alpha \geq \frac{\overline{M}-1}{A_1} + R_0$. To prove it, we use the fact that $e^x \geq 1 + x$, for all $x \in \mathbb{R}$, and we get

$$\begin{aligned}
\overline{M} &\geq \mathbb{E}\Big[e^{-A_1\left(R_0+\int_0^T (X_t^{\xi^n})^\top \sigma\, dB_t + \int_0^T b\cdot X_t^{\xi^n}\,dt - \int_0^T f(-\xi_t^n)\,dt\right)}\Big] \\
&\geq \mathbb{E}\Big[-A_1\Big(R_0 + \int_0^T (X_t^{\xi^n})^\top \sigma\, dB_t + \int_0^T b\cdot X_t^{\xi^n}\,dt - \int_0^T f(-\xi_t^n)\,dt\Big)\Big] + 1 \\
&= \mathbb{E}\Big[-A_1\Big(R_0 + \int_0^T b\cdot X_t^{\xi^n}\,dt - \int_0^T f(-\xi_t^n)\,dt\Big)\Big] + 1,
\end{aligned}$$

where the equality is due to the fact that $\mathbb{E}[\int_0^T (X_t^{\xi^n})^\top \sigma\, dB_t] = 0$. Indeed, due to (2.6), $Y_T := \int_0^T (X_t^{\xi^n})^\top \sigma\, dB_t$ is a true martingale, and hence $\mathbb{E}[Y_T] = \mathbb{E}[Y_0] = 0$. Then

$$\mathbb{E}\Big[\int_0^T -b\cdot X_t^{\xi^n} + f(-\xi_t^n)\,dt\Big] \leq \frac{\overline{M}-1}{A_1} + R_0,$$

and therefore (2.30) is true. Using now Lemma 2.2.10, we obtain

$$\alpha \geq \frac{\overline{M}-1}{A_1} + R_0 \geq E\Big[\int_0^T -b\cdot X_t^{\xi^n} + f(-\xi_t^n)\,dt\Big] \geq \frac{3}{4}\mathbb{E}\Big[\int_0^T f(-\xi_t^n)\,dt\Big] - N,$$

where $N$ has to be taken as in Lemma 2.2.10. Finally, for $m \geq \frac{4}{3}(\alpha + N)$, we arrive at

$$\mathbb{E}\Big[\int_0^T f(-\xi_t^n)\,dt\Big] \leq m,$$

which shows that $\xi^n \in \overline{K}_m$. ■

*Remark* 2.2.12. Due to the preceding lemma, we can w.l.o.g reformulate the maximization problem (2.23) in the following way:

$$V(T, X_0, R_0) = \sup_{\xi \in \dot{\mathcal{X}}^1(T,X_0)} \mathbb{E}\left[u\left(\mathcal{R}_T^\xi\right)\right] = \sup_{\xi \in \overline{K}_m} \mathbb{E}\left[u\left(\mathcal{R}_T^\xi\right)\right], \tag{2.31}$$

where $m$ has to be chosen such that

$$m \geq \frac{4}{3}\Big(\frac{-V_2(T, X_0, R_0)}{A_1} + R_0 + N\Big), \tag{2.32}$$

with $N$ being taken as in Lemma 2.2.10. ♢

In the following, we prove a fundamental property of the map $\xi \longmapsto \mathbb{E}\Big[u\big(\mathcal{R}_T^\xi\big)\Big]$, which we will also use to prove the continuity of the previously defined value function.

**Proposition 2.2.13.** *The map $\xi \longmapsto \mathbb{E}\Big[u\big(\mathcal{R}_T^\xi\big)\Big]$ is upper semi-continuous on $\dot{\mathcal{X}}^1(T, X_0)$ with respect to the weak topology in* $\mathrm{L}^1$.

*Proof.* Since

$$\xi \longmapsto \mathbb{E}\Big[u\big(\mathcal{R}_T^\xi\big)\Big]$$

is concave, due to Lemma 2.2.1, it is sufficient to show that the preceding map is upper semi-continuous with respect to the strong topology of $\mathrm{L}^1$, due to Corollary 2.2.6. Toward this end, let $(\widetilde{\xi}^n)$ be a sequence in $\dot{\mathcal{X}}^1(T, X_0)$ that converges to $\xi \in \dot{\mathcal{X}}^1(T, X_0)$, strongly in $\mathrm{L}^1$. Since we are dealing with a metric space, we can use the following characterization of upper semi-continuity at $\xi$:

$$\limsup_k \mathbb{E}\Big[u\big(\mathcal{R}_T^{\widetilde{\xi}^{s_k}}\big)\Big] \leq \mathbb{E}\Big[u\big(\mathcal{R}_T^\xi\big)\Big]. \tag{2.33}$$

But we also have that $\widetilde{\xi}^n$ converges weakly to $\xi$ and hence, we can directly apply Corollary 2.2.7 to obtain (2.33), and this concludes the proof. ■

Now we are ready for the proof of the existence and uniqueness of the optimal strategy for our optimization problem.

*Proof of Theorem 2.2.4.* Let $(\xi^n)_{n\in\mathbb{N}}$ be such that

$$\xi^n \in \dot{\mathcal{X}}^1(T, X_0, R_0) \;\text{ and }\; \mathbb{E}\left[u\left(\mathcal{R}_T^{\xi^n}\right)\right] \nearrow \sup_{\xi \in \dot{\mathcal{X}}^1(T,X_0)} \mathbb{E}\left[u\left(\mathcal{R}_T^\xi\right)\right].$$

Lemma 2.2.11 implies that there exists a subsequence $\xi^{n_k}$ of $\xi^n$ and a $\xi^* \in \dot{\mathcal{X}}^1(T, X_0)$ such that $\xi^{n_k} \longrightarrow \xi^*$, weakly in $\mathrm{L}^1$. Due to Proposition 2.2.13, we get

$$\begin{aligned} V(T, X_0, R_0) &= \limsup_k \mathbb{E}\Big[u\big(\mathcal{R}_T^{\xi^{n_k}}\big)\Big] \\ &\leq \mathbb{E}\Big[u\big(\mathcal{R}_T^{\xi^*}\big)\Big], \end{aligned}$$

which proves that $\xi^*$ is an optimal strategy for the maximization problem (2.16). The uniqueness of the optimal strategy follows from the fact that $\dot{\mathcal{X}}^1(T, X_0)$ is a convex set and that $\xi \longmapsto \mathbb{E}[u(\mathcal{R}_T^\xi)]$ is strictly concave. ■

It is established in Schied et al. (2010) that the optimal strategies for CARA value functions are such that the corresponding revenues have finite exponential moments, i.e.:

$$\mathbb{E}\Big[\exp\big(-\lambda\mathcal{R}_T^{\xi^{*,i}}\big)\Big] < \infty,$$

for all $\lambda > 0$, where $\xi^{*,i}$, $i = 1,2$, denotes the optimal strategy for the value function with respective CARA term equal to $A_i$, $i = 1,2$. This is due to the fact that the optimal strategies are deterministic, and hence $\int_0^T (X_t^{\xi^{*,i}})^\top \sigma\, dB_t$ has finite exponential moments. If $\lambda \leq A_1$, then

$$\mathbb{E}\Big[\exp\big(-\lambda\mathcal{R}_T^{\xi^{*}}\big)\Big] < \infty$$

holds, where $\xi^*$ is as in (2.23). Indeed, as for the inequalities in (2.11), we can show that there exists $C_\lambda > 0$, such that we have:

$$C_\lambda - \exp(-\lambda x) \geq \frac{1}{A_1} - \exp(-A_1 x) \geq u(x), \quad \text{for all } x \in \mathbb{R},$$

and this implies

$$\mathbb{E}\Big[\exp\big(-\lambda\mathcal{R}_T^{\xi^{*,\lambda}}\big)\Big] \leq \mathbb{E}\Big[\exp\big(-\lambda\mathcal{R}_T^{\xi^{*}}\big)\Big] \leq C_\lambda - \mathbb{E}\Big[u\big(\mathcal{R}_T^{\xi^{*}}\big)\Big] < \infty.$$

However, it is not clear whether or not the preceding holds for $\lambda > A_1$. In order to avoid integrability issues, we have to make in the sequel the following assumptions.

**Assumption 2.2.14.** In order to be able to rely on integrability properties of the strategies $\xi \in \dot{\mathcal{X}}^1(T, X_0)$, we suppose that the moment generating function of the revenues of the optimal strategy, denoted by $M_{\mathcal{R}_T^{\xi^*}}$, is defined for $2A_2$, where we set

$$M_{\mathcal{R}_T^{\xi^*}}(A) := \mathbb{E}\big[\exp(-A\mathcal{R}_T^{\xi^*})\big].$$

Thus, we will restrict ourselves to the following set of strategies:

$$\dot{\mathcal{X}}^1_{2A_2}(T, X_0) := \Big\{\xi \in \dot{\mathcal{X}}^1(T, X_0) \,|\, \mathbb{E}\big[\exp(-2A_2\mathcal{R}_T^{\xi})\big] \leq M_{\mathcal{R}_T^{\xi^*}}(2A_2) + 1\Big\} \tag{2.34}$$

In the sequel, we show that the preceding set is a closed convex set.

**Proposition 2.2.15.** *The set*

$$\dot{\mathcal{X}}^1_{2A_2}(T, X_0) := \Big\{\xi \in \dot{\mathcal{X}}^1(T, X_0) \,|\, \mathbb{E}\big[\exp(-2A_2\mathcal{R}_T^{\xi})\big] \leq M_{\mathcal{R}_T^{\xi^*}}(2A2) + 1\Big\}$$

*is a closed convex set with respect to the strong topology in* $\mathrm{L}^1$ *(and hence with respect to the weak topology).*

*Proof.* Due to the convexity of the map $\xi \mapsto \mathbb{E}[\exp(-A(\mathcal{R}_T^{\xi})]$ (see Lemma 2.2.1), the preceding set is convex. We show now that it is closed in $\mathrm{L}^1$. To this end, take a sequence $(\zeta^n)$ in $\dot{\mathcal{X}}^1_{2A_2}(T, X_0, R_0)$ that converges to $\zeta$ in $\mathrm{L}^1$. Since $\zeta^n$ converges also weakly to $\zeta$, we can use Corollary 2.2.7 to obtain

$$\mathbb{E}\big[\exp(-2A_2\mathcal{R}_T^{\zeta})\big] \le \liminf \mathbb{E}\big[\exp(-2A_2\mathcal{R}_T^{\zeta^n})\big] \le M_{\mathcal{R}_T^{\xi^*}}(2A_2) + 1,$$

and this completes our proof. ■

*Remark* 2.2.16. As argued before, if $M_{\mathcal{R}_T^{\xi^*}}(2A_2) < \infty$, then we also have

$$M_{\mathcal{R}_T^{\xi^*}}(A) < \infty, \qquad \text{for all } 0 < A < 2A_2.$$

If we suppose that $u$ is a convex combination of CARA utility functions, then we have that $M_{\mathcal{R}_T^{\xi^*}}$ is defined on $[A_1, A_2]$. However, we need to have that $M_{\mathcal{R}_T^{\xi^*}}(2A_2)$ is also defined, since we will have to apply the Cauchy-Schwarz inequality to prove the continuity of the value function (see the last section of this chapter). ♢

## 2.3 Regularity properties of the value function

### 2.3.1 Partial Differentiability of the value function

In this section we prove that the value function $V$ is differentiable with respect to the parameter $R \in \mathbb{R}$, for fixed $(T, X) \in ]0, \infty[\times\mathbb{R}^d$. Surprisingly, we just need the existence of the optimal strategy to prove it. Compared to the proof of the continuity of the value function in its parameters, this proof is essentially easier, due to fact that, for fixed $T$, the value function is concave. We begin by proving the following proposition.

**Proposition 2.3.1.** *Let $\xi \in \dot{\mathcal{X}}^1_{2A_2}(T, X_0)$. Then,*

$$R_0 \longmapsto \mathbb{E}\big[u\big(\mathcal{R}_T^{\xi} + R_0\big)\big]$$

*is twice differentiable on $\mathbb{R}$, with first and second derivative given by $\mathbb{E}\big[u'\big(\mathcal{R}_T^{\xi}\big)\big]$ and $\mathbb{E}\big[u''\big(\mathcal{R}_T^{\xi}\big)\big]$, respectively, where*

$$\mathcal{R}_T^{\xi} = R_0 + \int_0^T (X_t^{\xi})^{\top}\sigma\, dB_t + \int_0^T b \cdot X_t^{\xi}\, dt - \int_0^T f(-\xi_t)\, dt.$$

Before beginning with the proof, we need to prove the following lemma.

**Lemma 2.3.2.** *Let $g$ be a real-valued locally integrable function on $[0, \infty[$, such that*

$$\int_0^x g(t)\, dt \ge 0, \qquad \textit{for all } x > 0. \tag{2.35}$$

*Then*

$$\limsup_{x\to\infty} g(x) \ge 0.$$

*Proof.* We work toward a contradiction. Suppose that there exists $\varepsilon > 0$, such that

$$\limsup_{x\to\infty} g(x) < -2\varepsilon.$$

Then there exists $x_0 > 0$, such that

$$g(x) \leq -\varepsilon, \qquad \text{for all } x \geq x_0.$$

Thus, we get

$$\begin{aligned}
\int_0^x g(t)\,dt &= \int_0^{x_0} g(t)\,dt + \int_{x_0}^x g(t)\,dt \\
&\leq \int_0^{x_0} g(t)\,dt - \varepsilon(x - x_0) \\
&< 0, \qquad \text{for } x \text{ large enough,}
\end{aligned}$$

which is in contradiction with (2.35). ∎

We can now prove Proposition 2.3.1.

*Proof.* By translating $u$ horizontally if necessary, we can assume without loss of generality that $R_0 = 0$. Thus, we have to prove that

$$r \longmapsto \mathbb{E}\big[u\big(\mathcal{R}_T^\xi + r\big)\big]$$

is differentiable at $r = 0$, with derivative $\mathbb{E}\big[u'\big(\mathcal{R}_T^\xi\big)\big]$. Since $u$ is concave, increasing and lies in $C^1(\mathbb{R})$, $u'$ is decreasing and positive, and hence, it is sufficient to prove that

$$\mathbb{E}\big[u'\big(\mathcal{R}_T^\xi - 1\big)\big] < \infty \tag{2.36}$$

to be allowed to differentiate under the integral sign. Due to inequalities (2.11), we can write $\exp(A_2 x) \geq -u(-x)$, for all $x \in \mathbb{R}$. Therefore, we get

$$\exp(A_2x)+u(-x) = \int_0^x \Big(\frac{1}{A_2}\exp(A_2x) - u'(-x)\Big)\,dx + u(0) - \frac{1}{A_2} \geq 0, \qquad \text{for all } x \geq 0.$$

Hence, by translating $u$ vertically if necessary, the conditions of Lemma 2.3.2 apply, with $g(x) = \frac{1}{A_2}\exp(A_2x) - u'(-x)$ on $[0,\infty[$. Therefore, we can find a constant $C > 0$ such that

$$u'(-x) \leq C(\exp(A_2x) + 1), \qquad \text{for all } x \geq 0.$$

And finally we get

$$\begin{aligned}
\mathbb{E}\big[u'\big(\mathcal{R}_T^\xi - 1\big)\big] &= \mathbb{E}\big[u'\big(\mathcal{R}_T^\xi - 1\big)\mathbb{1}_{\{\mathcal{R}_T^\xi - 1 < 0\}}\big] + \mathbb{E}\big[u'\big(\mathcal{R}_T^\xi - 1\big)\mathbb{1}_{\{\mathcal{R}_T^\xi - 1 \geq 0\}}\big] \\
&\leq C(\mathbb{E}\big[\exp\big(-A_2\mathcal{R}_T^\xi\big)\big] + 1) + \mathbb{E}\big[u'\big(\mathcal{R}_T^\xi - 1\big)\mathbb{1}_{\{\mathcal{R}_T^\xi - 1 \geq 0\}}\big]
\end{aligned}$$

$$< \infty,$$

since $u'$ is bounded on $[0, \infty[$ and $\mathbb{E}\big[\exp\big(-A_2\mathcal{R}_T^\xi\big)\big] < \infty$, due to the assumption on $\xi$. Finally, (2.36) holds, and the proposition is proved for the case of the first derivative. For the second one, we take $0 < \eta < 1$ and $r \in\, ]-\eta, \eta[$. We wish to prove that

$$\sup_{r \in\, ]-\eta,\eta[} \mathbb{E}\big[\big|u''\big(\mathcal{R}_T^\xi + r\big)\big|\big] < \infty. \tag{2.37}$$

To this end, we use inequality (2.8) and the preceding arguments, to write

$$\begin{aligned}
\mathbb{E}\big[\big|u''\big(\mathcal{R}_T^\xi + r\big)\big|\big] &= \mathbb{E}\left[\frac{-u''\big(\mathcal{R}_T^\xi + r\big)}{u'\big(\mathcal{R}_T^\xi + r\big)} \cdot u'\big(\mathcal{R}_T^\xi + r\big)\right] \\
&\le \mathbb{E}\big[A_2 u'\big(\mathcal{R}_T^\xi + r\big)\big] \\
&\le \mathbb{E}\big[A_2 u'\big(\mathcal{R}_T^\xi - 1\big)\big] \\
&< \infty,
\end{aligned}$$

which proves the result for the case of the second derivative. This completes the proof. ■

In order to apply the preceding proposition to show the desired regularity properties of the value function, we need another result, since the optimal strategy depends on the parameter $R$ without, a priori, any known control of this dependence. Fortunately, the concavity of the value function in the revenues parameter permits us to give here a simple proof of the above statement (however, only for the first derivative). To this end, we consider now a family of concave $C^1$-functions $f_\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ and define

$$f(x) = \sup_\alpha f_\alpha(x).$$

Note that the supremum is not necessarily concave. However, if $f$ is concave in a neighborhood of a point $t$, then the following proposition gives us a sufficient condition under which $f$ is differentiable at this point.

**Lemma 2.3.3.** *Let $f_\alpha : \mathbb{R} \longrightarrow \mathbb{R}$, where $\alpha \in \Lambda$, be a family of concave $C^1$-functions which are uniformly bounded from above. Define*

$$f(x) = \sup_{\alpha \in \Lambda} f_\alpha(x).$$

*Suppose further that there exist $t \in \mathbb{R}$ and $\eta > 0$ such that $f$ is concave on $]t-\eta, t+\eta[$ and there exists $\alpha_t^* \in \Lambda$ such that $f(t) = f_{\alpha_t^*}(t)$. Then, $f$ is differentiable at $t$ with derivative*

$$f'(t) = f'_{\alpha_t^*}(t).$$

*If we suppose moreover that $\alpha_t^*$ is uniquely determined, then $f'$ is continuous at $t$.*

*Proof.* By translating the function $f$ if necessary, we can suppose without loss of generality that $t = 0$. Because $f$ is supposed to be concave in a neighborhood of $t = 0$, we only have to prove that $f'_+(0) \geq f'_-(0)$. To this end, let $\varepsilon > 0$ and $\alpha_0^* \in \Lambda$ be such that $f(0) = f_{\alpha_0^*}(0)$. Because $f_{\alpha_0^*}$ is concave and differentiable at 0, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $0 < h \leq \delta$ we have

$$\frac{f_{\alpha_0^*}(h) - f_{\alpha_0^*}(0)}{h} \geq \frac{f_{\alpha_0^*}(-h) - f_{\alpha_0^*}(0)}{-h} - \varepsilon.$$

Thus we get

$$\begin{aligned} \frac{f(h) - f(0)}{h} &\geq \frac{f_{\alpha_0^*}(h) - f_{\alpha_0^*}(0)}{h} \\ &\geq \frac{f_{\alpha_0^*}(-h) - f_{\alpha_0^*}(0)}{-h} - \varepsilon \\ &\geq \frac{f(-h) - f(0)}{-h} - \varepsilon, \end{aligned}$$

where the first and the last inequalities come from the definition of $f$. By sending $h$ to zero, we get

$$f'_+(0) \geq f'_{\alpha_0^*}(0) \geq f'_-(0) - \varepsilon,$$

for every $\varepsilon > 0$. Thus $f$ is differentiable.

Suppose now that $\alpha_t^*$ is uniquely determined, and suppose to the contrary that $f'$ is not continuous at $t$. Since $f$ is concave on $]t-\eta, t+\eta[$ and hence $f'$ is nonincreasing on $]t - \eta, t + \eta[$, the left- and right-hand limits at $t$ exist, and we would have that

$$f'(t^-) = f'_{\alpha_{t-}^*}(t^-) > f'(t^+) = f'_{\alpha_{t+}^*}(t^+),$$

where $\alpha_{t-}^*, \alpha_{t+}^* \in \Lambda$. Using the fact that $f'_{\alpha_{t-}^*}$ is continuous at $t$, due to the assumption on the family of functions $(f_\alpha)_{\alpha \in \Lambda}$, we must have on the one hand that $\alpha_{t-}^* \neq \alpha_{t+}^*$. However, as a direct consequence of the definition of $\alpha_t^*$ and the continuity of $f$, we must equally have, on the other hand,

$$f(t) = f_{\alpha_t^*}(t) = f(t^+) = f_{\alpha_{t+}^*}(t^+) = f_{\alpha_{t-}^*}(t^-).$$

Hence, the uniqueness of $\alpha_t^*$ would imply $\alpha_t^* = \alpha_{t-}^* = \alpha_{t+}^*$ in this case, which is a contradiction. Thus, the lemma is established. ∎

We can now prove the main theorem of this subsection

**Theorem 2.3.4.** *The value function is continuously differentiable in $R$, and we have the formula*

$$V_r(T, X, R) = \mathbb{E}\big[u'\big(\mathcal{R}_T^{\xi^*}\big)\big],$$

*where $\xi^*$ is the optimal strategy associated to $V(T, X, R)$.*

*Proof.* The proof is a direct consequence of Lemma 2.3.3, when applied to the family of concave functions $(R \mapsto \mathbb{E}[u(\mathcal{R}_T^\xi + R)])_{\xi \in \dot{\mathcal{X}}^1_{2A_2}(T,X_0)}$. Indeed, this is a family of concave $C^1$-functions (due to Proposition 2.3.1). The existence and uniqueness of an optimal strategy (Theorem 2.2.4) and the concavity of $R \mapsto V(T,X,R)$, for fixed $T,X$ (Proposition 2.2.2), yield that the remaining conditions of the preceding lemma are satifsfied. ■

**Corollary 2.3.5.** *Suppose that $u'$ is convex and decreasing. Then, the value function is twice differentiable, with second derivative*

$$V_{rr}(T,X,R) = \mathbb{E}\big[u''\big(\mathcal{R}_T^{\xi^*}\big)\big],$$

*where $\xi^*$ is the optimal strategy associated to $V(T,X,R)$.*

*Proof.* The proof is similar to the one of Theorem 2.3.4 and is obtained by applying Lemma 2.3.3 to $u'$, and Proposition 2.3.1. ■

*Remark* 2.3.6. We are in the setting of the preceding corollary if, e.g., $u$ is a convex combination of exponential utility functions or more generally, if $(-u)$ is a complete monotone function, i.e., if $\forall n \in \mathbb{N}^* : (-1)^n(-u)^{(n)} \geq 0$. According to the Hausdorff-Bernstein-Widder's theorem (cf. Widder (1941) or Donoghue (1974) Chapter 21), this is equivalent to the existence of a Borel measure $\mu$ on $[0,\infty[$, such that

$$-u(x) = \int_0^\infty e^{-xt}\, d\mu(t).$$

◇

### 2.3.2 Continuity of the value function

In general, the value function should not be expected to be smooth; see Tourin (2011) or Yong and Zhou (1999), where an example of a discontinuous value function is given. Under strong conditions, say, the boundedness of the set of our control process, it can be shown (see Touzi (2004)) that the value function is continuous. In this section we wish to prove in our framework that the value function $V$ defined in (2.16) is indeed continuous. Here again, we will use the fundamental assumption on our utility function, namely that $u$ lies between two exponential utility functions. With this assumption at hand, we can use two crucial properties of the maximization problem for exponential utility functions: the first one is that the associated exponential value function is continuous. The second one, which can be proved in a more general setting (say, for Levy processes), is that the optimal strategy is deterministic; see Schied et al. (2010).

The proof of the continuity of our value function will be split in two propositions. We first prove its upper semi-continuity and then its lower semi-continuity. To prove the upper semi-continuity we use the same techniques as are used to prove the existence of the optimal strategy for the maximization problem (2.16). The main idea to prove the lower semi-continuity is to use a convex combination of the optimal strategy for (2.16) and the optimal strategy of the corresponding exponential value function at a certain well-chosen point. Here, we have to distinguish between two cases; the case where the value function is approximated from above, and the case where the value function is approximated from below, in time. To pursue our work, some assumptions have to be made. In the sequel, for $\xi \in \dot{\mathcal{X}}^1(T, X_0)$ we will automatically set $\xi_t = 0$ for $t \geq T$.

**Proposition 2.3.7.** *The value function is upper semi-continuous on* $]0,\infty[\times\mathbb{R}^d\times\mathbb{R}$.

*Proof.* Take $\big(T, X_0, R_0\big) \in\, ]0,\infty[\times\mathbb{R}^d\times\mathbb{R}$ and let $\big(T^n, X_0^n, R_0^n\big)_n$ be a sequence that converges to $\big(T, X_0, R_0\big)$. We have to show that

$$\limsup_n V(T^n, X_0^n, R_0^n) \leq V(T, X_0, R_0). \tag{2.38}$$

Since $\big(T^n, X_0^n, R_0^n\big)_n$ is bounded and by using (2.14) and the fact that $V_i(T^n, X_0^n, R_0^n)$ are also bounded, it follows that $\limsup_n V(T^n, X_0^n, R_0^n) < \infty$. Hence, by taking a subsequence if necessary, we can suppose that $(V(T^n, X_0^n, R_0^n))$ converges to $\limsup_n V(T^n, X_0^n, R_0^n)$. Let $\xi^n$ be the optimal strategy associated to $V(T^n, X_0^n, R_0^n)$, which exists, for every $n \in \mathbb{N}$, due to Theorem 2.2.4. In the sequel we prove, as in Lemma 2.2.11, that the sequence $\xi^n$ lies in a weakly sequentially compact set. Note that this proposition can be proved without using Assumption 2.2.14.

First step: We set $\widetilde{T} := \sup_n T^n$. We show that, for every $n \in \mathbb{N}$, we have $\xi^n \in \overline{\mathcal{K}}_m$, provided that $m$ large enough, where

$$\overline{\mathcal{K}}_m = \Big\{\xi \in \overline{\mathcal{C}}\big(\dot{\mathcal{X}}^1(T^n, X_0^n)\big)_n \big|\, \mathbb{E}\Big[\int_0^{\widetilde{T}} f(-\xi_t)\,dt\Big] \leq m\Big\},$$

with $\overline{\mathcal{C}}(\dot{\mathcal{X}}^1(T^n, X_0^n))_n$ denoting the closed convex hull of the sequence of sets $(\dot{\mathcal{X}}^1(T^n, X_0^n))_n$. To this end, we use Remark 2.2.12, in which it is noted that we can choose $\xi^n \in \overline{K}_{m_n}$, where

$$\overline{K}_{m_n} = \Big\{\xi \in \dot{\mathcal{X}}^1(T^n, X_0^n)\big|\, \mathbb{E}\Big[\int_0^{\widetilde{T}} f(-\xi_t)\,dt\Big] \leq m_n\Big\},$$

$m_n$ having to be chosen such that

$$m_n \geq \frac{4}{3}\Big(\frac{-V_2(\widetilde{T}, X_0^n, R_0^n)}{A_1} + R_0^n + N\Big),$$

where $N$ depends only on $f, b$ and $\widetilde{T}$. Take now $m \in \mathbb{R}$ such that

$$m \geq \sup_n m_n.$$

Note that such $m$ exists because $(X_0^n, R_0^n)$ is bounded and $V_2$ is continuous. Then it follows that

$$\mathbb{E}\bigg[\int_0^{\widetilde{T}} f(-\xi_t^n)\,dt\bigg] \leq m, \qquad \text{for all } n \in \mathbb{N}.$$

Taking now the convex hull of the sequence of sets $(\dot{\mathcal{X}}^1(T^n, X_0^n))_n$, we conclude that $\xi^n \in \overline{\mathcal{K}}_m$ , for all $n \in \mathbb{N}$ .

<u>Second step:</u> We prove that $\overline{\mathcal{K}}_m$ is weakly sequentially compact. To this end, we first prove that it is a closed convex set in $\mathrm{L}^1$. First, note that $\overline{\mathcal{K}}_m$ is convex. This is because

$$\xi \longmapsto \mathbb{E}\bigg[\int_0^{\widetilde{T}} f(-\xi_t)\,dt\bigg]$$

is a convex map (due to the convexity of $f$) defined on the convex set $\overline{\mathcal{C}}\big(\dot{\mathcal{X}}^1(T^n, X_0^n)\big)_n$.

We show then that it is closed with respect to the $\mathrm{L}^1$-norm. Denote by $\overline{\mathcal{C}}(X_0^n)_n$ the closed convex hull of the sequence $(X_0^n)_n$, which is a bounded set in $\mathbb{R}^d$, because $(X_0^n)_n$ is bounded. We show that for $\xi \in \overline{\mathcal{K}}_m$ there exists $\widetilde{X}$ in $\overline{\mathcal{C}}(X_0^n)_n$ such that $\xi \in \dot{\mathcal{X}}^1(\widetilde{T}, \widetilde{X})$. To this end, we write $\xi$ as a convex combination of $\xi^{n_i} \in \dot{\mathcal{X}}^1(T^{n_i}, X_0^{n_i})$,

$$\xi = \lambda_1 \xi^{n_1} + \cdots + \lambda_s \xi^{n_s},$$

where $\sum_{i=1}^s \lambda_i = 1$, $\lambda_i \geq 0$. By expressing then the constraint on $\xi^{n_i}$, we get

$$\lambda_i \int_0^{T^i} \xi_t^{n_i}\,dt = \lambda_i X_0^{n_i},$$

which implies that

$$\int_0^{\widetilde{T}} \xi_t\,dt = \sum_{i=1}^s \lambda_i \int_0^{T^i} \xi_t^{n_i}\,dt = \sum_{i=1}^s \lambda_i X_0^{n_i} = \widetilde{X}.$$

Take now a sequence $(\widetilde{\xi}^q)_q$ of $\overline{\mathcal{K}}_m$ which converges in the $\mathrm{L}^1$-norm to a liquidation strategy $\widetilde{\xi}$. We prove that $\widetilde{\xi} \in \dot{\mathcal{X}}^1(\widetilde{T}, \widetilde{X})$ for $\widetilde{X}$ which lies in $\overline{\mathcal{C}}(X_0^n)_n$. As previously remarked, there exists a sequence $(\widetilde{X}^q)_q \in \overline{\mathcal{C}}(X_0^n)_n$ such that $\widetilde{\xi}^q \in \dot{\mathcal{X}}^1(\widetilde{T}, \widetilde{X}^q)$. Hence, we have

$$\int_0^{\widetilde{T}} \widetilde{\xi}^q\,dt = \widetilde{X}^q, \quad \mathbb{P}\text{-a.s.}$$

Replacing $(\widetilde{X}^q)_q$ by a subsequence if necessary, we can suppose that it converges to some $\widetilde{X}$, because this sequence is bounded. Moreover, $\widetilde{X}$ lies in $\overline{\mathcal{C}}(X_0^n)_n$. Because $(\widetilde{\xi}_q)_q$ converges weakly to $\widetilde{\xi}$, we are now in the setting of Lemma 2.2.8, which ensures that $\widetilde{\xi} \in \dot{\mathcal{X}}^1(\widetilde{T}, \widetilde{X})$, as well as

$$\mathbb{E}\bigg[\int_0^{\widetilde{T}} f(-\widetilde{\xi}_t)\,dt\bigg] \leq m.$$

Hence, this proves that $\overline{\mathcal{K}}_m$ is a closed subset of $\mathrm{L}^1$.

As $\overline{\mathcal{K}}_m$ is also convex, it is likewise closed with respect to the weak topology of $\mathrm{L}^1$. Thus, it is sufficient to prove that $\overline{\mathcal{K}}_m$ is uniformly integrable. To this end, take $\varepsilon > 0$ and $\xi \in \overline{\mathcal{K}}_m$. There exists $\alpha > 0$ such that $\frac{|\xi_t|}{f(-\xi_t)} \leq \frac{\varepsilon}{m}$, for $\left|\xi_t\right| > \alpha$, due to the superlinear growth property of $f$. Due to the fact that $f(x) = 0$ if and only if $x = 0$, $1/f(-\xi_t)$ is well-defined on $\{|\xi_t| > \alpha\}$, and hence

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \mathbb{1}_{\{|\xi_t|>\alpha\}}\left|\xi_t\right| dt\right] &= \mathbb{E}\left[\int_0^T \mathbb{1}_{\{|\xi_t|>\alpha\}}\frac{\left|\xi_t\right|}{f(-\xi_t)} f(-\xi_t)\, dt\right] \\
&\leq \mathbb{E}\left[\int_0^T \mathbb{1}_{\{|\xi_t|>\alpha\}} f(-\xi_t)\, dt\right]\frac{\varepsilon}{c} \\
&\leq \varepsilon,
\end{aligned}$$

which proves the uniform integrability of $\overline{\mathcal{K}}_m$. We thus have proved that $\overline{\mathcal{K}}_m$ is weakly sequentially compact.

Last step: We have proved that $(\xi^n)_n$ is a sequence in the weakly sequentially compact set $\overline{\mathcal{K}}_m$. Thus, there exist a subsequence $\xi^{n_k}$ of $\xi^n$ and some $\widetilde{\xi} \in \overline{\mathcal{K}}_m$ such that $\xi^{n_k}$ converges to $\widetilde{\xi}$, weakly in $\mathrm{L}^1$. We are here again in the settings of Lemma 2.2.8, which allows us us to deduce that $\widetilde{\xi} \in \dot{\mathcal{X}}^1(T, X_0)$. Finally, because $\xi \longmapsto \mathbb{E}[u(\mathcal{R}_T^{\xi})]$ is upper semi-continuous with respect to the weak topology of $\mathrm{L}^1$, due to Proposition 2.2.13, we get

$$\begin{aligned}
\limsup_n V(T^n, X_0^n, R_0^n) &= \limsup_k V(T^{n_k}, X_0^{n_k}, R_0^{n_k}) \\
&= \limsup_k \mathbb{E}\Big[u\big(\mathcal{R}_T^{\xi^{n_k}}\big)\Big] \\
&\leq \mathbb{E}\Big[u\big(\mathcal{R}_T^{\widetilde{\xi}}\big)\Big] \\
&\leq V(T, X_0, R_0),
\end{aligned}$$

where the last inequality is due to both the definition of $V$ at $(T, X_0, R_0)$ and the fact that $\widetilde{\xi} \in \dot{\mathcal{X}}^1(T, X_0)$. This concludes the proof of the upper semi-continuity of $V$. ■

In the following, we prove the lower semi-continuity of the value function $V$. Contrarily to the proof of the upper semi-continuity of $V$, we will have to consider two cases; when the sequence of time converges from above and from bellow to a fixed time $T$. For the latter case, we need first to prove the subsequent proposition, which derives a certain lower semi-continuity property of the value function within time, for fixed $X_0, R_0 \in \mathbb{R}^d \times \mathbb{R}$. The difficult part of this proof is due to the fact that accelerating the strategy when we approximate the time from below cannot be useful to prove the result, since we are then facing measurability/adaptivity issues.

Therefore we have to use other techniques, and this makes the proof essentially more involved than the previous one.

We first need to prove the following lemma, which gives a sufficient condition to assure the convergence of the sequence $\mathbb{E}[u(\mathcal{R}_T^{\eta^n})]$ to $\mathbb{E}[u(\mathcal{R}_T^{\eta})]$, when $\mathcal{R}_T^{\eta^n}$ converges to $\mathcal{R}_T^{\eta}$, in probability.

**Lemma 2.3.8.** *Let $\eta^n \in \dot{\mathcal{X}}^1(T, X_0)$ be a sequence such that $\mathcal{R}_T^{\eta^n}$ converges to $\mathcal{R}_T^{\eta}$, in probability, where $\eta \in \dot{\mathcal{X}}^1(T, X_0)$. Suppose moreover that $(\exp(-2A_2\mathcal{R}_{T^n}^{\eta^n}))_n$ is uniformly bounded in $\mathrm{L}^2$. Then we have*

$$\mathbb{E}\Big[u\Big(\mathcal{R}_T^{\eta^n}\Big)\Big] \underset{n\longrightarrow\infty}{\longrightarrow} \mathbb{E}\Big[u\Big(\mathcal{R}_T^{\eta}\Big)\Big]. \tag{2.39}$$

*Proof.* We need to prove that $(u(\mathcal{R}_T^{\eta^n}))_n$ is uniformly bounded in $\mathrm{L}^2$. But this is a direct consequence of the fact that $(\mathbb{E}[u^+(\mathcal{R}_T^{\eta^n})])_n$ is bounded and that, for all $n \in \mathbb{N}$, $\mathbb{E}[(u^-(\mathcal{R}_T^{\eta^n}))^2] \le \mathbb{E}[\exp(-2A_2\mathcal{R}_{T^n}^{\eta^n})]$, due to inequality (2.11) (where $u^+$ (resp., $u^-$) denotes the positive (resp., negative) part of $u$). Since $\mathbb{E}[\exp(-2A_2\mathcal{R}_{T^n}^{\eta^n})] < \infty$, applying Vitali's convergence theorem we conclude that

$$\mathbb{E}\Big[u\Big(\mathcal{R}_T^{\eta^n}\Big)\Big] \underset{n\longrightarrow\infty}{\longrightarrow} \mathbb{E}\Big[u\Big(\mathcal{R}_T^{\eta}\Big)\Big].$$

■

Now we are ready to prove the following proposition. Note that we will prove it in two ways, where the second one will need among others the additional assumption that the optimal strategy lies in the set $\dot{\mathcal{X}}_{8A_2}^1(T, X_0)$.

**Proposition 2.3.9.** *Let $(T, X_0, R_0) \in \,]0,\infty[\times\mathbb{R}^d \times \mathbb{R}$ and $T^n$ be a sequence of positive real numbers that converges from below to $T$, i.e., $T^n \uparrow T$. Then we have*

$$\liminf_n V(T^n, X_0, R_0) \ge V(T, X_0, R_0). \tag{2.40}$$

*Proof.* <u>FIRST VERSION:</u> For this proof, we will need Assumption 2.2.14. Let $(T, X_0, R_0) \in ]0,\infty[\times\mathbb{R}^d \times \mathbb{R}$ and $\xi \in \dot{\mathcal{X}}_{2A_2}^1(T, X_0)$. Define

$$\begin{aligned} \varphi^\xi : \,]0,\infty[ &\longrightarrow \mathbb{R} \\ \overline{T} &\longmapsto \mathbb{E}\big[u\big(\mathcal{R}_{\overline{T}}^\xi\big)\big]. \end{aligned}$$

Note that $\varphi^\xi$ is constant on $[T,\infty[$. We show that $\varphi^\xi$ is continuous at $T$. To this end, it is sufficient to take a sequence $(T^n)$ such that $T^n \uparrow T$ and to prove that

$$\varphi^\xi(T^n) \longrightarrow \varphi^\xi(T), \tag{2.41}$$

or, equivalently, that

$$\mathbb{E}\big[u\big(\mathcal{R}_{T^n}^\xi\big)\big] \longrightarrow \mathbb{E}\big[u\big(\mathcal{R}_T^\xi\big)\big].$$

Note that we easily have the $\mathbb{P}$-a.s. convergence of $\int_0^{T^n} b \cdot X_t^\xi \, dt$ to $\int_0^T b \cdot X_t^\xi \, dt$. We can also prove that

$$\int_0^{T^n} (X_t^\xi)^\top \sigma \, dB_t \underset{n\to\infty}{\longrightarrow} \int_0^T (X_t^\xi)^\top \sigma \, dB_t, \quad \mathbb{P}\text{-a.s.,}$$

and that

$$\int_0^{T^n} f\big( - \xi_t\big) \, dt \underset{n\to\infty}{\longrightarrow} \int_0^T f(-\xi_t) \, dt, \quad \mathbb{P}\text{-a.s.}$$

Therefore,

$$\mathcal{R}_{T^n}^\xi = \int_0^{T^n} (X_t^\xi)^\top \sigma \, dB_t + \int_0^{T^n} b \cdot X_t^\xi \, dt - \int_0^{T^n} f(-\xi_t) \, dt \underset{n\to\infty}{\longrightarrow} \mathcal{R}_T^\xi, \quad \mathbb{P}\text{-a.s.} \tag{2.42}$$

And because $u$ is continuous, we obtain

$$\lim_n u\big(\mathcal{R}_{T^n}^\xi\big) = u\big(\mathcal{R}_T^\xi\big), \quad \mathbb{P}\text{-a.s.} \tag{2.43}$$

Now, we have to prove the boundedness of the sequence $(\mathbb{E}[\exp(-2A\mathcal{R}_{T^n}^\xi)])_n$:

$$\begin{aligned}
&\mathbb{E}\big[\exp\big( - 2A\mathcal{R}_{T^n}^\xi\big)\big] \\
&= \mathbb{E}\Big[\exp\Big( - 2A\Big(\int_0^{T^n} (X_t^\xi)^\top \sigma \, dB_t + \int_0^{T^n} b \cdot X_t^\xi \, dt - \int_0^{T^n} f(-\xi_t) \, dt\Big)\Big)\Big] \\
&= \mathbb{E}\Big[\exp\Big( - 2A\Big(\mathbb{E}\Big[\int_0^{T^n} (X_t^\xi)^\top \sigma \, dB_t + \int_0^{T^n} b \cdot X_t^\xi \, dt - \int_0^{T^n} f(-\xi_t) \, dt\Big|\mathcal{F}_{T^n}\Big]\Big)\Big)\Big] \\
&\le \mathbb{E}\Big[\exp\Big( - 2A\Big(\mathbb{E}\Big[\int_0^{T} (X_t^\xi)^\top \sigma \, dB_t + \int_0^{T} b \cdot X_t^\xi \, dt - \int_{T^n}^{T} b \cdot X_t^\xi \, dt \\
&\qquad - \int_0^{T} f(-\xi_t) \, dt\Big|\mathcal{F}_{T^n}\Big]\Big)\Big)\Big] \\
&\le K\mathbb{E}\Big[\exp\Big( - 2A\Big(\mathbb{E}\Big[\int_0^{T} (X_t^\xi)^\top \sigma \, dB_t + \int_0^{T} b \cdot X_t^\xi \, dt - \int_0^{T} f(-\xi_t) \, dt\Big|\mathcal{F}_{T^n}\Big]\Big)\Big)\Big] \\
&\le K\mathbb{E}\Big[\mathbb{E}\Big[\exp\Big( - 2A\Big(\int_0^{T} (X_t^\xi)^\top \sigma \, dB_t + \int_0^{T} b \cdot X_t^\xi \, dt - \int_0^{T} f(-\xi_t) \, dt\Big)\Big)\Big|\mathcal{F}_{T^n}\Big]\Big] \\
&= K\mathbb{E}\Big[\exp\Big( - 2A\Big(\int_0^{T} (X_t^\xi)^\top \sigma \, dB_t + \int_0^{T} b \cdot X_t^\xi \, dt - \int_0^{T} f(-\xi_t) \, dt\Big)\Big)\Big] \\
&= K\mathbb{E}\big[\exp\big( - 2A\mathcal{R}_T^\xi\big)\big] \\
&< \infty,
\end{aligned}$$

where $K = \exp(T|b|\|X^\xi\|_{\mathrm{L}^2})$ is obtained by using Hölder's inequality, and where the finiteness of the last term is due to the fact that $\xi \in \dot{\mathcal{X}}^1_{2A_2}(T, X_0)$. Thus, the sequence $(u(\mathcal{R}_{T^n}^\xi)$ is uniformly bounded in $\mathrm{L}^2$, so using Vitali's convergence theorem we infer

$$\mathbb{E}\big[u\big(\mathcal{R}_{T^n}^\xi\big)\big] \underset{n\to\infty}{\longrightarrow} \mathbb{E}\big[u\big(\mathcal{R}_T^\xi\big)\big],$$

which proves (2.41). Hence, $\varphi^\xi$ is continuous at $T$. Therefore $\sup_{\xi\in\dot{\mathcal{X}}^1_{2A_2}(T,X_0)}\varphi^\xi$ is lower semi-continuous at $T$, because it is the supremum of a family of (lower semi-) continuous functions. Since

$$\sup_{\xi\in\dot{\mathcal{X}}^1_{2A_2}(T,X_0)}\varphi^\xi(T)=V(T,X_0,R_0),$$

this proves in particular that for every sequence of time $T^n$ that converges from below to $T$, we have

$$\liminf_n \sup_{\xi\in\dot{\mathcal{X}}^1_{2A_2}(T,X_0)}\varphi^\xi(T^n)\geq \sup_{\xi\in\dot{\mathcal{X}}^1_{2A_2}(T,X_0)}\varphi^\xi(T)=V(T,X_0,R_0), \tag{2.44}$$

which proves (2.40). ■

SECOND VERSION:

We present here an alternative proof of Proposition 2.3.9. Although this version of the proof is longer than the preceding one, it has the advantage that an approximation property/method for the value function in question, through value functions where the respective suprema are taken among strategies with "delayed information", is disclosed (Proposition 2.3.14). By "delayed information" we mean that, for a given time horizon $T$, strategies when considered at time $T$ are $\mathcal{F}_{\alpha T}$-measurable random variables, with $0<\alpha<1$. This can be also useful for numerical purposes. However, we have to make the following assumptions.

**Assumption 2.3.10.** We suppose that the optimal strategy lies in $\dot{\mathcal{X}}^1_{8A_2}(T,X_0)$. Further, we assume a consistency property of the strategies when multiplying them by a positive constant: if we take $\eta\in\dot{\mathcal{X}}^1_{8A_2}(T,X_0)$ and $\alpha>0$, then we assume to also have that $\alpha\eta$ lies in $\dot{\mathcal{X}}^1_{8A_2}(T,\alpha X_0)$. With this scaling property, the set $\dot{\mathcal{X}}^1_{8A_2}(T,X_0)$ is still a closed convex set, and this will avoid in the following an integrability issue, without affecting the existence of an optimal strategy $\xi^*$ as in (2.23). We will also use suppose that $\|X^{\xi^*}\|_{\mathrm{L}^\infty}<\infty$.

We first introduce notations. The control process $\xi\in\dot{\mathcal{X}}^1(T,X_0)$, will be considered here as an $\mathbb{R}^d$-valued random variable on $\Omega\times[0,T]$, with the following measure

$$\overline{P}:=\mathbb{P}\otimes\frac{1}{T}dt.$$

Let $\overline{\mathcal{F}}$ be the progressive $\sigma$- Algebra for $\Omega\times[0,T]$ and $\overline{\mathcal{F}}^\alpha$ the progressive $\sigma$- algebra with respect to $(\mathcal{F}_{\alpha t})$. Note that $\overline{\mathcal{F}}^\alpha\subset\overline{\mathcal{F}}$. For $\alpha\in]0,1[$, set

$$\dot{\mathcal{X}}^{1,\alpha}_{8A_2}(T,X_0,R_0):=\left\{\xi\in\dot{\mathcal{X}}^1_{8A_2}(T,X_0)\middle|\xi\text{ progressiv. meas. wrt }\overline{\mathcal{F}}^\alpha\right\}\subsetneq\dot{\mathcal{X}}^1_{8A_2}(T,X_0), \tag{2.45}$$

which denotes the set of anticipated strategies with anticipation parameter $\alpha$. We denote by

$$V_\alpha(T,X_0,R_0):=\sup_{\zeta\in\dot{\mathcal{X}}^{1,\alpha}_{8A_2}(T,X_0)}\mathbb{E}\left[u\left(\mathcal{R}^\zeta_T\right)\right] \tag{2.46}$$

the value function where the supremum is taken over the previously introduced set. Note that it can be shown in the same manner as in the proof of Theorem 2.2.4 that the preceding optimization problem has a unique solution (this is since we have that the corresponding set $\overline{K}_c^\alpha$, where

$$\overline{K}_c^\alpha = \Big\{ \xi \in \dot{\mathcal{X}}_{8A_2}^{1,\alpha}(T, X_0) \Big|\, \mathbb{E}\Big[\int_0^T f(-\xi_t)\, dt\Big] \leq c \Big\},$$

can be shown to be a weakly sequentially compact subset of $\mathrm{L}^1$, for all $c > 0$). We prove now the following two lemmas: the first one is a convergence property. Its simple proof is based on the integration by parts formula for the stochastic integral.

**Lemma 2.3.11.** *Let $\xi^n \in \dot{\mathcal{X}}^1(T, X_0)$ converge to some $\xi \in \dot{\mathcal{X}}^1(T, X_0)$ in the $\mathrm{L}^1[0,T]$-weak convergence sense, $\mathbb{P}$-a.s. Then*

$$\int_0^T (X_t^{\xi^n})^\top \sigma\, dB_t \underset{n\to\infty}{\longrightarrow} \int_0^T (X_t^{\xi})^\top \sigma\, dB_t, \quad \mathbb{P}\text{-}a.s.$$

*Proof.* By using the integration by parts formula, we get

$$\begin{aligned}\int_0^T (X_t^{\xi^n})^\top \sigma\, dB_t &= (X_T^{\xi^n})^\top \sigma B_T - (X_0^{\xi^n})^\top \sigma B_0 + \langle X^{\xi^n}, B\rangle_T + \int_0^T (\xi_t^n)^\top \sigma B_t\, dt \\ &= \int_0^T (\xi_t^n)^\top \sigma B_t\, dt, \quad \mathbb{P}\text{-a.s.},\end{aligned}$$

where we use for the first inequality the fact that $X_T^{\xi^n} = 0$, because $\xi \in \dot{\mathcal{X}}^1(T, X_0)$ and $B_0 = 0$. Therefore, since $t \mapsto B_t$ is a.s. continuous and, in particular, a.s. bounded on $[0,T]$, we infer

$$\int_0^T (X_t^{\xi^n})^\top \sigma\, dB_t = \int_0^T (\xi_t^n)^\top \sigma B_t\, dt \longrightarrow \int_0^T (\xi_t)^\top \sigma B_t\, dt = \int_0^T (X_t^{\xi})^\top \sigma\, dB_t, \quad \mathbb{P}\text{-a.s.},$$

where the last equality is obtained again by the integration by parts formula. ■

*Remark* 2.3.12. In the preceding lemma, if we require only the convergence in probability of the sequence $(\int_0^T (X_t^{\xi^n})^\top \sigma\, dB_t)$ to $\int_0^T (X_t^{\xi})^\top \sigma\, dB_t$, instead of an almost sure convergence, then the proof is a direct consequence of the counterpart of the Lebesgue dominated convergence theorem for the stochastic integral (see, e.g., Theorem 2.12 in Revuz and Yor (1999)). Most of the time in our work, this version will be sufficient. ♢

**Lemma 2.3.13.** *Let $\alpha \in\, ]0,1[$ and $(T^n)$ be a sequence of positive real numbers converging from below to $T$. Then*

$$\liminf_n V(T^n, X_0, R_0) \geq V_\alpha(T, X_0, R_0). \tag{2.47}$$

*Proof.* Set $\alpha_n = \frac{T^n}{T}$ and let $n_0 \in \mathbb{N}$ be such that $\alpha_n \geq \alpha$, for $n \geq n_0$. Take $\xi^\alpha \in \dot{\mathcal{X}}^{1,\alpha}_{8A_2}(T, X_0)$ and define, for $n \geq n_0$,

$$\xi^n_t := \frac{1}{\alpha_n} \xi^\alpha_{t/\alpha_n}.$$

We will prove that

$$\mathcal{R}^{\xi^n}_{T^n} = \int_0^{T^n} (X^{\xi^n}_t)^\top \sigma \, dB_t + \int_0^{T^n} b \cdot X^{\xi^n}_t \, dt - \int_0^{T^n} f(-\xi^n_t) \, dt \underset{n\to\infty}{\longrightarrow} \mathcal{R}^{\xi^\alpha}_T, \quad \mathbb{P}\text{-a.s.}, \tag{2.48}$$

by individually considering each term on the right-hand side of the preceding identity, starting from the left. From its definition, $\xi^n$ converges to $\xi$ with respect to the $\mathrm{L}^1[0,T]$-(weak) topology, $\mathbb{P}$-a.s., and thus we can use Lemma 2.3.11, whence

$$\int_0^{T^n} (X^{\xi^n}_t)^\top \sigma \, dB_t \underset{n\to\infty}{\longrightarrow} \int_0^T (X^{\xi^\alpha}_t)^\top \sigma \, dB_t, \quad \mathbb{P}\text{-a.s.}$$

Using the change of variables formula, we get

$$\begin{aligned}
\int_0^t \xi^n_s \, ds &= \frac{1}{\alpha_n} \int_0^t \xi_{s/\alpha_n} \, ds \\
&= \int_0^{t\alpha_n} \xi_s \, ds - X_0 + X_0 \\
&= X_0 - X^{\xi^\alpha}_{t/\alpha_n}, \quad \mathbb{P}\text{-a.s.}, \qquad \text{for all } t \in [0, T^n], \qquad (*)
\end{aligned}$$

and therefore $\int_0^{T^n} b \cdot X^{\xi^n}_t \, dt$ converges, $\mathbb{P}$-a.s., to $\int_0^T b \cdot X^{\xi^\alpha}_t \, dt$. Note that by replacing $t$ with $T^n$ in (*), we obtain furthermore that $X^{\xi^n}_{T^n} = 0$. Using Assumption 2.3.10, since $\xi^\alpha \in \dot{\mathcal{X}}^1_{8A_2}(T, X_0)$, we have that $2\xi^\alpha \in \dot{\mathcal{X}}^1_{8A_2}(T, 2X_0)$, and therefore we also have $2\xi^\alpha \in K_{m_2}$, as mentioned in Remark 2.2.12. This implies in particular that $\mathbb{E}[\int_0^T f\big(-2\xi^\alpha_t\big) \, dt] \leq m_2$. Hence, by taking $n$ large enough, we can write for the remaining integral in (2.48) :

$$\begin{aligned}
\int_0^{T^n} f(-\xi^n_t) \, dt &= \int_0^{T^n} f\big(-\xi^\alpha_{t/\alpha_n}/\alpha_n\big) \, dt \\
&= \alpha_n \int_0^T f\big(-\xi^\alpha_t/\alpha_n\big) \, dt \\
&\leq \alpha_n \int_0^T f\big(-2\xi^\alpha_t\big) \, dt \\
&< \infty, \quad \mathbb{P}\text{-a.s.},
\end{aligned}$$

where we use the convexity and positivity of $f$, as well as the fact that $f(0) = 0$. Indeed, for $0 \leq \alpha < \beta$ and for every $x \in \mathbb{R}^d$, there exists $\mu \in \,]0,1[$ such that $\alpha x = \mu\beta x + (1-\mu) \cdot 0$. And hence we obtain

$$f(\alpha x) \leq \mu f(\beta x) + (1-\mu) f(0) \leq f(\beta x).$$

Finally we get

$$\int_0^{T^n} f\big(-\xi_t^n\big)\,dt \underset{n\to\infty}{\longrightarrow} \int_0^T f(-\xi_t^\alpha)\,dt, \quad \mathbb{P}\text{-a.s.}$$

Therefore (2.48) is proved. Due to the continuity of $u$, we further have

$$\lim_n u\big(\mathcal{R}_{T^n}^{\xi^n}\big) = u\big(\mathcal{R}_T^{\xi^\alpha}\big), \quad \mathbb{P}\text{-a.s.} \tag{2.49}$$

We have to show now that the family $(\exp(-A_2\mathcal{R}_{T^n}^{\xi^n}))_n$ is bounded in $\mathrm{L}^2(\Omega,\mathcal{F},P)$, in order to apply Lemma 2.3.8. To this end, we use

$$X_t^{\xi^n} = X_{t/\alpha_n}^{\xi^\alpha}$$

to write

$$\begin{aligned}
&\mathbb{E}\big[\exp\big(-2A_2\mathcal{R}_{T^n}^{\xi^n}\big)\big]\\
&= \mathbb{E}\Big[\exp\Big(-2A_2\Big(\int_0^{T^n}(X_t^{\xi^n})^\top\sigma\,dB_t + \int_0^{T^n} b\cdot X_t^{\xi^n}\,dt - \int_0^{T^n} f(-\xi_t^n)\,dt\Big)\Big)\Big]\\
&= \mathbb{E}\Big[\exp\Big(-2A_2\int_0^{T^n}\Big(X_{t/\alpha_n}^{\xi^\alpha}\Big)^\top\sigma\,dB_t + \int_0^{T^n} b\cdot X_{t/\alpha_n}^{\xi^\alpha}\,dt - \int_0^{T^n} f(-\xi_{t/\alpha_n}^\alpha/\alpha_n)\,dt\Big)\Big]\\
&= \mathbb{E}\Big[\exp\Big(-2A_2\Big(\alpha_n\int_0^{T}(X_t^{\xi^\alpha})^\top\sigma\,dB_t + \alpha_n\int_0^{T} b\cdot X_t^{\xi^\alpha}\,dt - \alpha_n\int_0^{T} f(-\xi_t^\alpha/\alpha_n)\,dt\Big)\Big)\Big]\\
&\le \alpha_n\mathbb{E}\Big[\exp\Big(-2A_2\Big(\int_0^{T}(X_t^{\xi^\alpha})^\top\sigma\,dB_t + \int_0^{T} b\cdot X_t^{\xi^\alpha}\,dt - \int_0^{T} f(-2\xi_t^\alpha)\,dt\Big)\Big)\Big]\\
&\quad +1-\alpha_n\\
&\le \mathbb{E}\Big[\exp\Big(-A_2\Big(\int_0^{T}(X_t^{2\xi^\alpha})^\top\sigma\,dB_t + \int_0^{T} b\cdot X_t^{2\xi^\alpha}\,dt - \int_0^{T} f(-2\xi_t^\alpha)\,dt\Big)\Big)\Big] + 1\\
&= \mathbb{E}\big[\exp\big(-A_2R_T^{2\xi^\alpha}\big)\big] + 1\\
&< \infty,
\end{aligned}$$

where the finiteness of the last term is due to Assumption 2.3.10. Hence, we get

$$\mathbb{E}\big[u\big(\mathcal{R}_{T^n}^{\xi^n}\big)\big] \underset{n\longrightarrow\infty}{\longrightarrow} \mathbb{E}\big[u\big(\mathcal{R}_T^{\xi^\alpha}\big)\big].$$

Finally, we write

$$\begin{aligned}
\liminf_n V(T^n, X_0, R_0) &\ge \liminf_n \mathbb{E}\big[u\big(\mathcal{R}_{T^n}^{\xi^n}\big)\big]\\
&= \mathbb{E}\big[u\big(\mathcal{R}_T^{\xi^\alpha}\big)\big].
\end{aligned}$$

Since this holds for every $\xi^\alpha \in \dot{\mathcal{X}}_{8A_2}^{1,\alpha}(T, X_0)$, (2.47) follows. ∎

It remains to prove the following proposition.

**Proposition 2.3.14.** *We have*

$$V_\alpha(T, X_0, R_0) \underset{\alpha\uparrow 1}{\nearrow} V(T, X_0, R_0), \tag{2.50}$$

*where $V_\alpha$ is defined as in (2.46).*

*Proof.* Let $\xi^*$ be the optimal strategy associated to $V(T, X_0, R_0)$. Take a sequence $(\alpha_n)$ such that $\alpha_n \uparrow 1$ and set

$$\xi^n := \overline{\mathbb{E}}\big[\xi^* \big| \overline{\mathcal{F}}^{\alpha_n}\big],$$

where $\overline{\mathbb{E}}\big[\,\cdot\,|\overline{\mathcal{F}}^{\alpha_n}\big]$ denotes the conditional expectation with respect to the probability measure $\overline{P}$. We need to prove the following equality:

$$\int_0^t \xi_s^n \, ds = \mathbb{E}\bigg[\int_0^t \xi_s^* \, ds \bigg| \mathcal{F}_{\alpha_n t}\bigg], \quad \text{for a.e. } t. \tag{2.51}$$

To this end, take $A_{nt} \in \mathcal{F}_{\alpha_n t}$, so as to have $A_{nt} \times [0,t] \in \overline{\mathcal{F}}^{\alpha_n}$. Calculating

$$\begin{aligned}
\frac{1}{T}\mathbb{E}\bigg[\int_0^t \xi_s^n \, ds \mathbb{1}_{A_{nt}}\bigg] &= \frac{1}{T}\mathbb{E}\bigg[\int_0^T \xi_s^n \mathbb{1}_{A_{nt}\times[0,t](.,s)} \, ds\bigg] \\
&= \frac{1}{T}\mathbb{E}\bigg[\int_0^T \overline{\mathbb{E}}\big[\xi^* \big| \overline{\mathcal{F}}^{\alpha_n}\big]_s \mathbb{1}_{A_{nt}\times[0,t](.,s)} \, ds\bigg] \\
&= \frac{1}{T}\mathbb{E}\bigg[\int_0^T \xi_s^* \mathbb{1}_{A_{nt}\times[0,t](.,s)} \, ds\bigg] \\
&= \frac{1}{T}\mathbb{E}\bigg[\int_0^t \xi_s^* \, ds \mathbb{1}_{A_{nt}}\bigg] \\
&= \frac{1}{T}\mathbb{E}\bigg[\mathbb{E}\bigg[\int_0^t \xi_s^* \, ds \,\bigg|\, \mathcal{F}_{\alpha_{nt}}\bigg] \mathbb{1}_{A_{nt}}\bigg]
\end{aligned}$$

yields the result. In particular, we thus have $X_T^{\xi^n} = X_0$. Indeed, we can write

$$\int_0^T \xi_s^n \, ds = \mathbb{E}\bigg[\int_0^T \xi_s^* \, ds \bigg| \mathcal{F}_{\alpha_n T}\bigg] = \mathbb{E}\Big[X_0 \Big| \mathcal{F}_{\alpha_n T}\Big] = X_0.$$

Moreover, it is easy to see that

$$\big|X_t^{\xi^n}\big| = \bigg|\int_0^t \xi_s^n \, ds\bigg| = \bigg|\mathbb{E}\bigg[\int_0^t \xi_s^* \, ds \bigg| \mathcal{F}_{\alpha_n t}\bigg]\bigg| \leq \mathbb{E}\bigg[\bigg|\int_0^t \xi_s^* \, ds\bigg| \,\bigg|\mathcal{F}_{\alpha_n t}\bigg] \leq \|X^{\xi^*}\|_{\mathrm{L}^\infty}.$$

With this at hand,

$$\mathbb{E}\Big[u\Big(\mathcal{R}_T^{\xi^n}\Big)\Big] \leq V_{\alpha_n}(T, X_0, R_0) \leq V(T, X_0, R_0).$$

The martingale convergence theorem (or Lévy's zero-one law) further implies

$$\xi^n \longrightarrow \xi^*, \quad \text{both } \overline{P}\text{-a.s. and in } \mathrm{L}^1 := \mathrm{L}^1(\Omega \times [0,T], \overline{\mathcal{F}}, \overline{P}).$$

Now, we want to show

$$\mathbb{E}\big[u\big(\mathcal{R}_T^{\xi^n}\big)\big] \underset{n\longrightarrow\infty}{\longrightarrow} \mathbb{E}\big[u\big(\mathcal{R}_T^{\xi^*}\big)\big]. \tag{2.52}$$

To this end, we first prove that

$$\mathcal{R}^{\xi^n}_{T^n} = \int_0^{T^n} (X^{\xi^n}_t)^\top \sigma \, dB_t + \int_0^{T^n} b \cdot X^{\xi^n}_t \, dt - \int_0^{T^n} f(-\xi^n_t) \, dt \underset{n\to\infty}{\longrightarrow} \mathcal{R}^{\xi^*}_T, \quad \text{in probability}, \tag{2.53}$$

up to a subsequence, by individually considering each integral term, starting from the left. First, note that, due to the convergence of $\xi^n$ to $\xi$ in $\mathrm{L}^1$, we also have that

$$\int_0^T (\xi^n_t)^\top \sigma B_t \, dt \underset{n\to\infty}{\longrightarrow} \int_0^T (\xi^*_t)^\top \sigma B_t \, dt, \quad \text{in probability.}$$

Hence, by the integration by parts formula, we get

$$\int_0^T (X^{\xi^n}_t)^\top \sigma \, dB_t \underset{n\to\infty}{\longrightarrow} \int_0^T (X^{\xi^*}_t)^\top \sigma \, dB_t$$

and, in a similar vein, we have

$$\int_0^T b \cdot X^{\xi^n}_t \, dt \underset{n\to\infty}{\longrightarrow} \int_0^T b \cdot X^{\xi^*}_t \, dt,$$

both in probability. For the remaining integral, we use Jensen's inequality to obtain

$$\begin{aligned} f(-\xi^n_t) &= f\big(-\overline{\mathbb{E}}\big[\xi^* \big| \overline{\mathcal{F}}^{\alpha_n}\big]\big) \\ &\leq \overline{\mathbb{E}}\big[f(-\xi^*)\big|\overline{\mathcal{F}}^{\alpha_n}\big]. \end{aligned} \tag{2.54}$$

Moreover, we have

$$\begin{aligned} \overline{\mathbb{E}}\big[\big|\overline{\mathbb{E}}\big[f(-\xi^*)\big|\overline{\mathcal{F}}^{\alpha_n}\big]\big|\big] &= \overline{\mathbb{E}}\big[\overline{\mathbb{E}}\big[f(-\xi^*)\big|\overline{\mathcal{F}}^{\alpha_n}\big]\big] \\ &= \overline{\mathbb{E}}\big[f(-\xi^*)\big] \\ &= \frac{1}{T}\mathbb{E}\bigg[\int_0^T f(-\xi^*_t)\, dt\bigg] \\ &\leq m/T, \end{aligned}$$

since $\xi^* \in \overline{K}_m$, by using here again Remark 2.2.12. Combining the preceding with (2.54), we can apply the dominated convergence theorem of Lebesgue to infer

$$\frac{1}{T}\mathbb{E}\bigg[\int_0^T f(-\xi^n_t)\, dt\bigg] \underset{n\to\infty}{\longrightarrow} \frac{1}{T}\mathbb{E}\bigg[\int_0^T f(-\xi^*_t)\, dt\bigg].$$

By taking a subsequence if necessary, we finally get

$$\int_0^T f\big(-\xi^n_t\big)\, dt \underset{n\to\infty}{\longrightarrow} \int_0^T f(-\xi^*_t)\, dt \quad \mathbb{P}\text{-a.s.},$$

which proves (2.53).

Consider now the following filtration

$$\mathcal{G}_n := \mathcal{F}_{\alpha_n T},\ n \in \mathbb{N}.$$

Now, we show that

$$\int_0^T \overline{\mathbb{E}}\big[f(-\xi^*)\big|\overline{\mathcal{F}}^{\alpha_n}\big]_t\,dt$$

is a martingale with respect to the filtration $(\mathcal{G}_n)_n$. To this end, consider $n \geq m$ two integers and $A_m \in \mathcal{G}_m$. We have that $A_m \times [0,T] \in \overline{\mathcal{F}}^{\alpha_m} \subset \overline{\mathcal{F}}^{\alpha_n}$, which allows us to infer

$$\begin{aligned}
\mathbb{E}\bigg[\int_0^T \overline{\mathbb{E}}\big[f(-\xi^*)\big|\overline{\mathcal{F}}^{\alpha_n}\big]_t\,dt\mathbb{1}_{A_m}\bigg] &= \mathbb{E}\bigg[\int_0^T \overline{\mathbb{E}}\big[f(-\xi^*)\big|\overline{\mathcal{F}}^{\alpha_n}\big]_t\mathbb{1}_{A_m\times[0,T]}(.,t)\,dt\bigg]\\
&= T\overline{\mathbb{E}}\bigg[\int_0^T \overline{\mathbb{E}}\big[f(-\xi^*)\big|\overline{\mathcal{F}}^{\alpha_n}\big]_t\mathbb{1}_{A_m\times[0,T]}(.,t)\,dt\bigg]\\
&= T\overline{\mathbb{E}}\bigg[\int_0^T f(-\xi^*_t)\mathbb{1}_{A_m\times[0,T]}(.,t)\,dt\bigg]\\
&= T\overline{\mathbb{E}}\bigg[\int_0^T \overline{\mathbb{E}}\big[f(-\xi^*)\big|\overline{\mathcal{F}}^{\alpha_m}\big]_t\mathbb{1}_{A_m\times[0,T]}(.,t)\,dt\bigg]\\
&= \mathbb{E}\bigg[\int_0^T \overline{\mathbb{E}}\big[f(-\xi^*)\big|\overline{\mathcal{F}}^{\alpha_m}\big]_t\,dt\mathbb{1}_{A_m}\bigg],
\end{aligned}$$

and this proves the assertion. In the same manner, we can prove that

$$\int_0^T t\overline{\mathbb{E}}\big[b\cdot\xi^*\big|\overline{\mathcal{F}}^{\alpha_n}\big]_t\,dt$$

is also a martingale with respect to $(\mathcal{G}_n)_n$, and therefore we can write

$$\int_0^T t\overline{\mathbb{E}}\big[b\cdot\xi^*\big|\overline{\mathcal{F}}^{\alpha_n}\big]_t + \overline{\mathbb{E}}\big[f(-\xi^*)\big|\overline{\mathcal{F}}^{\alpha_n}\big]_t\,dt = \mathbb{E}\bigg[\int_0^T t\,b\cdot\xi^* + f(-\xi^*)dt\bigg|\mathcal{G}_n\bigg].$$

Note that the strategy $\xi^n$ is almost surely constant on $[T^n, T]$. Now, we show that $(\mathbb{E}[\exp(-A_2\mathcal{R}_T^{\xi^n})])$ is bounded in $\mathrm{L}^2$, by using the previously established martingale property as follows:

$$\begin{aligned}
&\mathbb{E}\big[\exp\big(-2A_2\mathcal{R}_T^{\xi^n}\big)\big]\\
&= \mathbb{E}\bigg[\exp\bigg(-2A_2\bigg(\int_0^T (X_t^{\xi^n})^\top\sigma\,dB_t + \int_0^T b\cdot X_t^{\xi^n}\,dt - \int_0^T f(-\xi_t^n)\,dt\bigg)\bigg)\bigg]\\
&= \mathbb{E}\bigg[\exp\bigg(-2A_2\bigg(\int_0^{\alpha_n T} (X_t^{\xi^n})^\top\sigma\,dB_t + \int_{\alpha_n T}^T (X_t^{\xi^n})^\top\sigma\,dB_t\\
&\quad - \int_0^T t\overline{\mathbb{E}}\big[b\cdot\xi^*\big|\overline{\mathcal{F}}^{\alpha_n}\big]_t\,dt - \int_0^T f\big(-\overline{\mathbb{E}}[\xi^*\big|\overline{\mathcal{F}}^{\alpha_n}]_t\big)\,dt\bigg)\bigg]\\
&= \mathbb{E}\bigg[\exp\bigg(-2A_2\bigg(\int_{\alpha_n T}^T (X_t^{\xi^n})^\top\sigma\,dB_t\bigg)\bigg)
\end{aligned}$$

$$
\begin{aligned}
&\cdot \exp\Big( - 2A_2\Big( \int_0^{\alpha_n T} (X_t^{\xi^n})^\top \sigma \, dB_t - \int_0^T t\overline{\mathbb{E}}\big[b\cdot\xi^* \big|\overline{\mathcal{F}}^{\alpha_n}\big]_t - \overline{\mathbb{E}}\big[f(-\xi^*)\big|\overline{\mathcal{F}}^{\alpha_n}\big]_t \, dt\Big)\Big)\Big] \\
&= \mathbb{E}\Big[ \exp\Big( - 2A_2\Big( \int_{\alpha_n T}^{T} (X_t^{\xi^n})^\top \sigma \, dB_t\Big)\Big) \\
&\quad \cdot \exp\Big( - 2A_2 \, \mathbb{E}\Big[ \int_0^{\alpha_n T} (X_t^{\xi^n})^\top \sigma \, dB_t - \int_0^T t\, b\cdot\xi^* + f(-\xi^*) dt \Big| \mathcal{G}_n \Big]\Big)\Big] \\
&\leq \mathbb{E}\Big[ \exp\Big( - 4A_2\Big( \int_{\alpha_n T}^{T} (X_t^{\xi^n})^\top \sigma \, dB_t\Big)\Big)\Big] \\
&\quad \cdot \mathbb{E}\Big[ \exp\Big( - 4A_2 \, \mathbb{E}\Big[ \int_0^{\alpha_n T} (X_t^{\xi^n})^\top \sigma \, dB_t + \int_0^T t\, b\cdot\xi^* - f(-\xi^*) dt \Big) \Big| \mathcal{G}_n \Big]\Big] \\
&\leq \mathbb{E}\Big[ \exp\Big( - 4A_2\Big( \int_{\alpha_n T}^{T} (X_t^{\xi^n})^\top \sigma \, dB_t\Big)\Big)\Big] \\
&\quad \cdot \mathbb{E}\Big[ \mathbb{E}\Big[ \exp\Big( - 4A_2 \int_0^{\alpha_n T} (X_t^{\xi^n})^\top \sigma \, dB_t + \int_0^T t\, b\cdot\xi^* - f(-\xi^*) dt \Big) \Big| \mathcal{G}_n \Big]\Big] \\
&= \mathbb{E}\Big[ \exp\Big( - 8A_2^2\Big( \int_{\alpha_n T}^{T} (X_t^{\xi^n})^\top \Sigma (X_t^{\xi^n}) \, dt\Big)\Big)\Big] \\
&\quad \cdot \mathbb{E}\Big[ \exp\Big( - 4A_2\Big( \int_0^{\alpha_n T} (X_t^{\xi^n})^\top \sigma \, dB_t + \int_0^T t\, b\cdot\xi^* - f(-\xi^*) dt \Big)\Big] \\
&\leq \exp\big( - 8A_2^2 \|X^{\xi^*}\|_\infty^2 |\Sigma| T\big) \cdot \mathbb{E}\Big[ \exp\big( - 4A_2\Big( \int_0^{\alpha_n T} (X_t^{\xi^n})^\top \sigma \, dB_t \\
&\quad + \int_0^T (X_t^{\xi^*} - X_t^{\xi^*})^\top \sigma \, dB_t + \int_0^T t\, b\cdot\xi^* - f(-\xi^*) dt \Big)\Big] \\
&\leq \exp\big( - 2A_2^2 \|X^{\xi^*}\|_\infty^2 |\Sigma| T\big) \cdot \mathbb{E}\Big[ \exp\Big( - 8A_2\Big( \int_0^{T} (X_t^{\xi^n} \mathbb{1}_{[0,\alpha_n T]} - X_t^{\xi^*})^\top \sigma \, dB_t\Big)\Big)\Big] \\
&\quad \cdot \mathbb{E}\Big[ \exp\Big( - 8A_2\Big( \int_0^{T} (X_t^{\xi^*})^\top \sigma \, dB_t + \int_0^T t\, b\cdot\xi^* - f(-\xi^*) dt \Big)\Big] \\
&\leq \mathbb{E}\Big[ \exp\Big( - 32A_2^2\Big( \int_0^{T} (X_t^{\xi^n} \mathbb{1}_{[0,\alpha_n T]} - X_t^{\xi^*})^\top \Sigma (X_t^{\xi^n} \mathbb{1}_{[0,\alpha_n T]} - X_t^{\xi^*}) \, dt\Big)\Big)\Big] \\
&\quad \cdot \exp\big( - 8A_2^2 \|X^{\xi^*}\|_\infty^2 |\Sigma| T\big) \cdot \mathbb{E}\big[ \exp\big( - 8A_2 \mathcal{R}_T^{\xi^*}\big)\big] \\
&\leq \exp\big( - 140A_2^2 \|X^{\xi^*}\|_\infty^2 |\Sigma| T\big) \cdot \mathbb{E}\big[ \exp\big( - 8A_2 \mathcal{R}_T^{\xi^*}\big)\big] \\
&< \infty.
\end{aligned}
$$

This proves (2.52). Hence, (2.50) follows. ∎

We can now prove the lower semi-continuity of the value function $V$.

**Proposition 2.3.15.** *The value function is lower semi-continuous on* $]0,\infty[\times\mathbb{R}^d\times\mathbb{R}$.

*Proof.* Let $(T, X_0, R_0) \in ]0,\infty[\times\mathbb{R}^d\times\mathbb{R}$ and $(T^n, X_0^n, R_0^n)_n$ be a sequence that converges to $(T, X_0, R_0)$. We have to show that

$$\liminf_n V(T^n, X_0^n, R_0^n) \geq V(T, X_0, R_0). \tag{2.55}$$

We split the proof of (2.55) in two parts, in which we suppose first that $T^n \downarrow T$ and then $T^n \uparrow T$ (for this latter case, we will use Proposition 2.3.9).

<u>First case:</u> Suppose that $T^n \downarrow T$. We set

$$\lambda_n = \begin{cases} |X_0^n - X_0|, & \text{if } |X_0^n - X_0| \neq 0, \\ \frac{1}{n}, & \text{otherwise,} \end{cases} \tag{2.56}$$

which belongs to $]0,1[$, for $n$ large enough. Let $\widehat{X}_0^n \in \mathbb{R}^d$ be such that

$$X_0^n = (1-\lambda_n)X_0 + \lambda_n \widehat{X}_0^n.$$

Consider the following sequence of strategies

$$\xi_t^n := (1-\lambda_n)\xi_t^* + \lambda_n \widehat{\xi}_t^n,$$

where $\xi^*$ is the optimal strategy associated to $V(T, X_0, R_0)$, and $\widehat{\xi}^n$ is the optimal strategy associated to $V_2(T^n, \widehat{X}_0^n, R_0^n)$ (and hence deterministic).
Note that, due to the choice of $\lambda_n$, $\widehat{X}_0^n$ is bounded: indeed, we have

$$\widehat{X}_0^n = \frac{X_0^n - X_0}{\lambda_n} + \lambda_n + X_0,$$

which term is bounded, due to the boundedness of $X_0^n$ and the definition of $\lambda_n$. Hence, $V_2(T^n, \widehat{X}_0^n, R_0^n)$ is bounded in $n$, which implies that $\int_0^{T^n} f(-\widehat{\xi}_t^n)\,dt$ (which is deterministic) is also bounded in $n$, as proved in Lemma 2.1.5. Since $f$ has superlinear growth and is positive, this implies that $\int_0^{T^n} |-\widehat{\xi}_t^n|\,dt$ is also bounded in $n$.

Observe now that we have

$$\int_0^{T^n} \xi_t^n\,dt = (1-\lambda_n)\int_0^{T^n} \xi_t^*\,dt + \lambda_n \int_0^{T^n} \widehat{\xi}_t^n\,dt = (1-\lambda_n)X_0 + \lambda_n \widehat{X}_0^n = X_0^n,$$

where the last equality follows from the fact that $T^n \geq T$ and $\xi_t^* = 0$, for $t \geq T$. Moreover, $\xi^n$ is such that it fulfills (2.6), due to the convexity of $f$ and the boundedness of $\widehat{\xi}^n$. Thus, $\xi^n \in \dot{\mathcal{X}}^1_{2A_2}(T^n, X_0^n)$.

We now show that

$$\mathcal{R}_{T^n}^{\xi^n} = \int_0^{T^n} (X_t^{\xi^n})^\top \sigma\,dB_t + \int_0^{T^n} b \cdot X_t^{\xi^n}\,dt - \int_0^{T^n} f(-\xi_t^n)\,dt \underset{n\to\infty}{\longrightarrow} \mathcal{R}_T^{\xi^*}, \quad \mathbb{P}\text{-a.s.}, \tag{2.57}$$

by individually consedering each term, starting from the left.

Because $\int_0^{T^n} |\widehat{\xi}_t^n|\,dt$ is uniformly bounded, $\xi^n$ converges to $\xi^*$ in $\mathrm{L}^1[0,T]$, $\mathbb{P}$-a.s. Indeed, we write

$$\mathbb{E}\Big[\int_0^{T^n} |\xi_t^n - \xi_t^*|\,dt\Big] = \lambda_n \mathbb{E}\Big[\int_0^{T^n} |\widehat{\xi}_t^n - \xi_t^*|\,dt\Big]$$

$$= \lambda_n\Big(\mathbb{E}\Big[\int_0^{T^n} |\widehat{\xi}_t^n|\,dt\Big] + \mathbb{E}\Big[\int_{T^n}^T |\xi_t^*|\,dt\Big]\Big)$$
$$\underset{n\to\infty}{\longrightarrow} 0.$$

Therefore, we can use Lemma 2.3.11 to infer

$$\int_0^{T^n} (X_t^{\xi^n})^\top \sigma\, dB_t \underset{n\to\infty}{\longrightarrow} \int_0^T (X_t^{\xi^*})^\top \sigma\, dB_t.$$

Due to the formula $X_t^{\xi^n} = (1-\lambda_n)X_t^{\xi^*} + \lambda_n X_t^{\widehat{\xi}^n}$, $\mathbb{P}$-a.s., for all $t \in [0, T^n]$, we can rewrite the second integral term of (2.57) in the following way:

$$\begin{aligned}\int_0^{T^n} b \cdot X_t^{\xi^n}\, dt &= (1-\lambda_n)\int_0^{T^n} b \cdot X_t^{\xi^*}\, dt + \lambda_n \int_0^{T^n} b \cdot X_t^{\widehat{\xi}^n}\, dt \\ &= (1-\lambda_n)\int_0^{T} b \cdot X_t^{\xi^*}\, dt + \lambda_n \int_0^{T^n} b \cdot X_t^{\widehat{\xi}^n}\, dt,\end{aligned}$$

which converges $\mathbb{P}$-a.s. to $\int_0^T b \cdot X_t^{\xi^*}\, dt$, because $\int_0^{T^n} b \cdot X_t^{\widehat{\xi}^n}\, dt$ is uniformly bounded and $\lambda_n$ is a null sequence.

We prove now that

$$\int_0^T f\big(-(1-\lambda_n)\xi_t^* - \lambda_n\widehat{\xi}_t^n\big)\, dt \underset{n\to\infty}{\longrightarrow} \int_0^T f(-\xi_t^*)\, dt, \quad \mathbb{P}\text{-a.s.} \tag{2.58}$$

Due to the continuity of $f$, we have that

$$f\big(-(1-\lambda_n)\xi_t^* - \lambda_n\widehat{\xi}_t^n\big) \longrightarrow f\big(-\xi_t^*\big), \quad \mathbb{P}\text{-a.s.}$$

Using the convexity of $f$, we further get

$$0 \le f\big(-(1-\lambda_n)\xi_t^* - \lambda_n\widehat{\xi}_t^n\big) \le (1-\lambda_n) f\big(-\xi_t^*\big)\, dt + \lambda_n f\big(-\widehat{\xi}_t^n\big).$$

As $\int_0^T f(-\widehat{\xi}_t^n)\, dt$ is uniformly bounded in $n$, (2.58) is proved by using the dominated convergence theorem of Lebesgue. Therefore, (2.57) follows. Hence, by using the continuity of $u$, we have

$$\lim_n u\big(\mathcal{R}_{T^n}^{\xi^n}\big) = u\big(\mathcal{R}_T^{\xi^*}\big), \quad \mathbb{P}\text{-a.s.} \tag{2.59}$$

Further, setting $L := \sup_n V_2(T^n, \widehat{X}_0^n, R_0^n)$, we write

$$\begin{aligned}\exp(-2A_2\mathcal{R}_{T^n}^{\xi^n}) &\le \big((1-\lambda_n)\exp(-2A_2\mathcal{R}_{T^n}^{\xi^*}) + \lambda_n \exp(-2A_2\mathcal{R}_{T^n}^{\widehat{\xi}^n})\big) \\ &= \big((1-\lambda_n)\exp(-2A_2\mathcal{R}_{T}^{\xi^*}) + \lambda_n \exp(-2A_2\mathcal{R}_{T^n}^{\widehat{\xi}^n})\big) \\ &\le \big((1-\lambda_n)M_{\mathcal{R}_T^{\xi^*}}(2A2) + \lambda_n L\big) \\ &< \infty,\end{aligned}$$

where the first inequality follows from the convexity of $\xi \longmapsto \exp(-2A\mathcal{R}^{\xi}_{T^n})$, the following equality from the fact that $T^n \geq T$, and the last inequality from Assumption 2.2.14. Therefore, by applying Lemma 2.3.8, we have

$$\mathbb{E}\big[u\big(\mathcal{R}^{\xi^n}_{T^n}\big)\big] \underset{n\longrightarrow\infty}{\longrightarrow} \mathbb{E}\big[u\big(\mathcal{R}^{\xi^*}_{T}\big)\big].$$

Finally we can write

$$\begin{aligned}\liminf_n V(T^n, X_0^n, R_0^n) &\geq \liminf_n \mathbb{E}\left[u\left(\mathcal{R}^{\xi^n}_{T^n}\right)\right]\\ &= \mathbb{E}\left[u\left(\mathcal{R}^{\xi^*}_{T}\right)\right]\\ &= V(T, X_0, R_0),\end{aligned}$$

which proves (2.55) when $T^n \downarrow T$.
<u>Second case:</u> Suppose now that $T^n \uparrow T$. We set $\lambda_n$ and $\widehat{X}_0^n \in \mathbb{R}^d$ as in (2.56). Let us consider the following sequence of strategies

$$\xi^n_t := (1-\lambda_n)\xi^{*,n}_t + \lambda_n \widehat{\xi}^n_t,$$

where $\xi^{*,n}$ is the optimal strategy associated to $V(T^n, X_0, R_0)$ and $\widehat{\xi}^n$ for $V_2(T^n, \widehat{X}_0^n, R_0^n)$.

Here again, it can be shown as above that $\xi^n \in \dot{\mathcal{X}}^1_{2A_2}(T^n, X_0^n)$. We can therefore write

$$\begin{aligned}\liminf_n V(T^n, X_0^n, R_0^n) &\geq \liminf_n \mathbb{E}\big[u\big(\mathcal{R}^{\xi^n}_{T^n}\big)\big]\\ &= \liminf_n \mathbb{E}\big[u\big(\mathcal{R}^{(1-\lambda_n)\xi^{*,n}+\lambda_n\widehat{\xi}^n}_{T^n}\big)\big]\\ &\geq \liminf_n \big((1-\lambda_n)\mathbb{E}\big[u\big(\mathcal{R}^{\xi^{*,n}}_{T^n}\big)\big] + \lambda_n\mathbb{E}\big[u\big(\mathcal{R}^{\widehat{\xi}^n}_{T^n}\big)\big]\big)\\ &\geq \liminf_n (1-\lambda_n)V(T^n, X_0, R_0) + \liminf_n \lambda_n V_2(T^n, X_0^n, R_0^n)\\ &\geq V(T, X_0, R_0).\end{aligned}$$

Here, we have used Lemma 2.2.1 for the second inequality, inequality (2.14) for the third one, and Proposition 2.3.9 in conjunction with the fact that $V_2(T^n, X_0^n, R_0^n)$ is bounded and $\lambda_n$ is a null sequence, for the last one. This proves (2.55) when $T^n \uparrow T$, which concludes the proof. ■

As a consequence of Proposition 2.3.7 and Proposition 2.3.15, we obtain the following main result of this section:

**Theorem 2.3.16.** *The value function $V$ is continuous on $]0,\infty[\times\mathbb{R}^d\times\mathbb{R}$.*

# Chapter 3

# Bellman principle and Hamilton-Jacobi-Bellman equation

## 3.1 The Bellman principle and the construction of $\varepsilon$-maximizers.

In this section we prove the Bellman principle of optimality underlying our maximization problem (2.16). To this end, we use $\varepsilon$-maximizers constructed on a bounded region. Their existence is proved by using an approximating sequence of strategies. Thus, we avoid here the use of a measurable selection theorem, which appears typically in optimal control theory. The dynamic programming principle will be a key result to prove both a verification theorem and a theorem stating that our value function is a solution, in the viscosity sense (see Chapter 4), of a Hamilton-Jacobi-Bellman equation. From now on, for a fixed time $T \in ]0, \infty[$, we will consider the time-reversed value function: $t \mapsto V(T-t, X_0, R_0)$. This will enable us, in the next section, to set an initial condition that reflects the global *fuel constraint* imposed on strategies. For fixed time $T \in ]0, \infty[$, we assume now that $(\Omega, \mathcal{F}, \mathbb{P})$ is the *canonical Wiener Space.*

**Theorem 3.1.1.** (Bellman Principle) *Let* $(T, X_0, R_0) \in ]0, \infty[\times \mathbb{R}^d \times \mathbb{R}$. *Then we have*

$$V(T, X_0, R_0) = \sup_{\xi \in \dot{\mathcal{X}}^1(T, X_0)} \mathbb{E}\big[V\big(T-\tau, X^{\xi}_{\tau}, \mathcal{R}^{\xi}_{\tau}\big)\big], \tag{3.1}$$

*for every stopping time* $\tau$ *taking values in* $[0, T[$.

*Remark* 3.1.2. Note that Bouchard and Touzi (2011) developed a weak formulation of the dynamic principle, which can be used to derive the viscosity property of the corresponding value function, in some optimal control problems. However, this requires the following concatenation property (*Assumption A*) of the strategies: for $\xi, \eta \in \dot{\mathcal{X}}^1(T, X_0)$ and a stopping time $\tau \in [0, T[$, we must have that

$\xi \mathbb{1}_{[0,\tau]} + \eta \mathbb{1}_{]\tau,T]} \in \dot{\mathcal{X}}^1(T, X_0)$, which is however not the case in general, and therefore is not usable in our work. In Bouchard and Nutz (2012), another weak formulation of the dynamic principle with generalized state constraints is formulated. Here again, a concatenation property (*Assumption B*) in the following form is required: for $\xi, \eta \in \dot{\mathcal{X}}^1(T, X_0)$ and a time $s \in [0, T]$, it must hold that $X^\xi_t = X^\xi_s - \int_s^t \eta_u \, du$, for $t \le s$, which is again not the case in general, and thus cannot be directly applied here. $\diamondsuit$

We will have to split the proof of Theorem 3.1.1 in two parts. First, let us make the following assumption on $f$.

**Assumption 3.1.3.** From now on, we suppose that $f$ has at most a polynomial growth of degree $p$, i.e., there exists $C > 0$ such that

$$f(x) \le C(1 + |x|^p), \qquad \text{for all } x \in \mathbb{R}^d.$$

Further, in order to avoid measurability issues, we need to suppose that, for $T \in ]0, \infty[$, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ is the canonical Wiener space. Taking this perspective, let us start with proving some measurability results. Here also, we will restrict our attention to strategies that lie in $\dot{\mathcal{X}}^1_{2A_2}(T, X_0, R_0)$, as mentioned in Assumption 2.2.14. We will need the following fundamental lemma.

**Lemma 3.1.4.** *For $\omega \in \Omega$, define the map $\phi_\omega : \Omega \to \Omega$ by*

$$\phi_\omega(\widetilde{\omega}) = \begin{cases} \omega(s), & \text{for } s \in [0, \tau(\omega)], \\ \omega(\tau(\omega)) + \widetilde{\omega}(s) - \widetilde{\omega}(\tau(\omega)), & \text{for } s \in ]\tau(\omega), T], \end{cases}$$

*where $\tau$ is a stopping time with value in $[0, T]$. Moreover, for $\xi \in \dot{\mathcal{X}}^1(T, X_0)$ we define*

$$\xi^\omega_t(\widetilde{\omega}) := \xi_t \circ \phi_\omega(\widetilde{\omega}).$$

*Then, for $\mathbb{P}$-a.e. $\omega$,*

$$\mathbb{E}\left[u\big(\mathcal{R}^\xi_T\big)\big|\mathcal{F}_\tau\right](\omega) = \mathbb{E}\left[u\big(\mathcal{R}^\xi_\tau + \mathcal{R}^{\xi^\omega}_{\tau,T}\big)\big|\mathcal{F}_\tau\right](\omega) = \mathbb{E}\Big[u\big(\mathcal{R}^\xi_\tau(\omega) + \mathcal{R}^{\xi^\omega}_{\tau(\omega),T}\big)\Big], \tag{3.2}$$

*where $R^{\widetilde{\xi}}_{t,T}$ denotes the revenues generated by the strategy $\xi^\omega$ during the time period $[t, T]$, i.e:*

$$R^{\widetilde{\xi}}_{t,T} = \int_t^T (X^{\widetilde{\xi}}_s)^\top \sigma \, dB_s + \int_t^T b \cdot X^{\widetilde{\xi}}_s \, ds - \int_t^T f(-\widetilde{\xi}_s) \, ds.$$

To prove the preceding Lemma, we have to use the three following lemmas. The proof of the first one can be found in, e.g., Revuz and Yor (1999) (as a consequence of Levy's characterization of Brownian motion) or Hunt and Kennedy (2004).

**Lemma 3.1.5.** *Let $\tau$ be a bounded stopping time and $(B_t)_{t\in[0,\infty[}$ a Brownian motion. Then $\widetilde{B}_t := B_{t+\tau} - B_\tau$ is a Brownian motion independent of $\mathcal{F}_\tau$.*

The next lemma uses the Dynkin's $\pi$-$\lambda$ theorem. See, e.g., Williams (1991) for more details.

**Lemma 3.1.6.** *Let $F : \mathbb{R}^2 \longrightarrow [0, \infty[$ be a measurable function, $X$ independent of a sigma-algebra $\mathcal{A}$ and $Y$ $\mathcal{A}$-measurable. Then,*

$$\mathbb{E}[F(X,Y)\big|\mathcal{A}](\omega) = \mathbb{E}[F(X,Y(\omega))] \quad \mathbb{P}\text{-}a.s. \tag{3.3}$$

*Proof.* Let us first consider $A = (A_1 \times A_2), A_i \in \mathcal{B}(\mathbb{R}), i = 1, 2$, and set

$$F(x,y) := \mathbb{1}_{A_1 \times A_2}(x,y) = \mathbb{1}_{A_1}(x)\mathbb{1}_{A_2}(y).$$

We write then

$$\begin{aligned}
\mathbb{E}[F(X,Y)](\omega) &= \mathbb{E}[\mathbb{1}_{A_1}(X)\mathbb{1}_{A_2}(Y)\big|\mathcal{A}](\omega) \\
&= \mathbb{1}_{A_2}(Y(\omega))\mathbb{E}[\mathbb{1}_{A_1}(X)\big|\mathcal{A}](\omega) \\
&= \mathbb{1}_{A_2}(Y(\omega))\mathbb{E}[\mathbb{1}_{A_1}(X)] \\
&= \mathbb{E}[\mathbb{1}_{A_1}(X)\mathbb{1}_{A_2}Y(\omega)],
\end{aligned}$$

where we use the fact that $Y$ is $\mathcal{A}$-measurable for the second equality and the independence of $X$ for the third one. Consider now

$$\mathcal{D} := \{A \in \mathcal{B}(\mathbb{R}^2) \,\big|\, (3.3) \text{ holds for } F = \mathbb{1}_A\}.$$

Then $\mathcal{D}$ is a Dynkin system which contains $\mathcal{C} := \{A_1 \times A_2 \big| A_i \in \mathcal{B}(\mathbb{R})\}$. Due to the stability of the set $\mathcal{C}$ under intersection, it follows that $\mathcal{D} \supset \sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R}^2)$. Using the monotone convergence theorem, (3.3) follows for an arbitrary $F$. ■

The next lemma is a consequence of both preceding results.

**Lemma 3.1.7.** *Let $H : \Omega \longrightarrow [0, \infty[$ be a measurable function, $\tau$ a stopping time with values in $[0, T[$ and $\phi_w$ be defined as in Lemma 3.1.4, for $\omega \in \Omega$. Then we have*

$$\mathbb{E}[H\big|\mathcal{F}_\tau](\omega) = \mathbb{E}[H \circ \phi_\omega] \quad \mathbb{P}\text{-}a.s.$$

We can now prove Lemma 3.1.4

*Proof of Lemma 3.1.4.* First, note that

$$\begin{aligned}
\mathcal{R}^\xi_T \circ \phi_\omega(\widetilde{\omega}) &= \mathcal{R}^\xi_\tau \circ \phi_\omega(\widetilde{\omega}) + \mathcal{R}^\xi_{\tau,T} \circ \phi_\omega(\widetilde{\omega}) \\
&= \mathcal{R}^\xi_\tau(\omega) + \mathcal{R}^{\xi^\omega}_{\tau(\omega),T}(\widetilde{\omega}),
\end{aligned}$$

for $\mathbb{P}$ -a.e. $\widetilde{\omega} \in \Omega$. Due to the fact that $u$ is bounded from above, we can apply the preceding Lemma to $H := -u(\mathcal{R}^\xi_T)$ (by translating $u$ vertically if necessary), and we finally get (when dropping the minus sign in front of $u$)

$$\begin{aligned}
\mathbb{E}\big[u\big(\mathcal{R}^\xi_T\big)\big|\mathcal{F}_\tau\big](\omega) &= \mathbb{E}\big[u\big(\mathcal{R}^\xi_T \circ \phi_\omega\big)\big] \\
&= \mathbb{E}\big[u\big(\mathcal{R}^\xi_\tau(\omega) + \mathcal{R}^{\xi^\omega}_{\tau(\omega),T}\big)\big],
\end{aligned}$$

which proves the lemma. ■

The following lemma yields an upper bound for an exponential value function at some stopping time with values in $[0,T[$. It uses the notations of Lemma 3.1.4. For $d=1$, an analogous result can be found in Schied and Schöneborn (2008).

**Lemma 3.1.8.** *Let $\overline{V}(T,X_0,R_0)=\inf_{\xi\in\dot{\mathcal{X}}_{det}(T,X_0)}\mathbb{E}\big[\exp(-A\mathcal{R}_T^\xi)\big]$ and $\tau$ be a stopping time with values in $[0,T[$. We then have*

$$\overline{V}(T-\tau,X_\tau^\zeta,\mathcal{R}_\tau^\zeta)\le\mathbb{E}\big[\exp(-A\mathcal{R}_T^\zeta)|\mathcal{F}_\tau\big],\quad \mathbb{P}\text{-}a.s., \tag{3.4}$$

*for every $\zeta\in\dot{\mathcal{X}}^1(T,X_0)$*

*Proof.* Let $\tau\le T$ be a stopping time and $\zeta\in\dot{\mathcal{X}}^1(T,X_0)$. By

$$\mathcal{R}_{s,T}^\zeta=\int_s^T(X_t^\zeta)^\top\sigma\,dB_t+\int_s^T b\cdot X_t^\zeta\,dt-\int_s^T f(-\zeta_t)\,dt \tag{3.5}$$

we denote the revenues generated by $\zeta$ over the time interval $[s,T]$ . Using (2.15), we can express $\overline{V}$ in the following way, for every $\omega\in\Omega$,

$$\overline{V}(T-\tau(\omega),X_\tau^\zeta(\omega),\mathcal{R}_\tau^\zeta(\omega))=\exp\Big(-A\mathcal{R}_\tau^\zeta(\omega)+A\inf_{\widetilde{\zeta}\in\dot{\mathcal{X}}_{det}(T-\tau(\omega),X_\tau^\zeta(\omega))}\int_\tau^T\mathcal{L}(X_t^{\widetilde{\zeta}},\widetilde{\zeta}_t)\,dt\Big).$$

Let us next set

$$Y^\zeta=e^{-A\int_\tau^T(X_t^\zeta)^\top\sigma\,dB_t-\frac{1}{2}\int_\tau^T A^2(X_t^\zeta)^\top\Sigma X_t^\zeta\,dt}.$$

We then have for every $\zeta\in\dot{\mathcal{X}}^1(T,X_0)$ and almost every $\omega\in\Omega$:

$$\begin{aligned}
&\mathbb{E}\Big[\exp(-A\mathcal{R}_{\tau,T}^\zeta)|\mathcal{F}_\tau\Big](\omega)\\
&=\mathbb{E}\Big[\exp\Big(-A\Big(\int_\tau^T(X_t^\zeta)^\top\sigma\,dB_t+\int_\tau^T b\cdot X_t^\zeta\,dt-\int_\tau^T f(-\zeta_t)\,dt\Big)\Big)\Big|\mathcal{F}_\tau\Big](\omega)\\
&=\mathbb{E}\Big[Y^\zeta\exp\Big(A\int_\tau^T\mathcal{L}(X_t^\zeta,\zeta_t)\,dt\Big)\Big|\mathcal{F}_\tau\Big](\omega)\\
&\ge\mathbb{E}\Big[Y^\zeta\exp\Big(A\inf_{\widetilde{\zeta}\in\dot{\mathcal{X}}_{det}(T-\tau(\omega),X_\tau^\zeta(\omega))}\int_\tau^T\mathcal{L}(X_t^{\widetilde{\zeta}},\widetilde{\zeta}_t)\,dt\Big)\Big|\mathcal{F}_\tau\Big](\omega)\\
&=\mathbb{E}\Big[Y^\zeta e^{A\mathcal{R}_\tau^\zeta(\omega)}\overline{V}(T-\tau(\omega),X_\tau^\zeta(\omega),\mathcal{R}_\tau^\zeta(\omega)|\mathcal{F}_\tau\Big](\omega)\\
&=\exp\big(A\mathcal{R}_\tau^\zeta(\omega)\big)\overline{V}(T-\tau(\omega),X_\tau^\zeta(\omega),\mathcal{R}_\tau^\zeta(\omega))\mathbb{E}\big[Y^\zeta|\mathcal{F}_\tau\big](\omega).
\end{aligned}$$

Here, we have used (3.5) for the first equality and the monotonicity property of the conditional expectation for the inequality.

It remains to show that

$$\mathbb{E}\big[Y^\zeta|\mathcal{F}_\tau\big]=1,\quad \mathbb{P}\text{-a.s.}. \tag{3.6}$$

Indeed, this will prove the result, because we also have that

$$\mathbb{E}\big[\exp(-A\mathcal{R}_T^\zeta)|\mathcal{F}_\tau\big](\omega)=\mathbb{E}\big[\exp\big(-A\big(\mathcal{R}_{\tau,T}^\zeta+\mathcal{R}_\tau^\zeta(\omega)\big)\big)|\mathcal{F}_\tau\big](\omega)$$

$$= \exp\big(-A\mathcal{R}_\tau^\zeta(\omega)\big)\mathbb{E}\big[\exp\big(-A\mathcal{R}_{\tau,T}^\zeta(\omega)\big)\big|\mathcal{F}_\tau\big](\omega),$$

by using (3.2). To prove (3.6), let us define the following process

$$Z_t^\zeta = e^{-A\int_0^t (X_u^\zeta)^\top \sigma\, dB_u - \frac{1}{2}\int_0^t A^2 (X_u^\zeta)^\top \Sigma X_u^\zeta\, du},$$

which is a true martingale, due to Girsanov's theorem, since $X^\zeta$ has to fulfill (2.6), due to the assumption on $\zeta$. Therefore, we have

$$\begin{aligned}\mathbb{E}\big[Z_T^\zeta|\mathcal{F}_\tau\big] &= \mathbb{E}\big[Y^\zeta Z_\tau^\zeta|\mathcal{F}_\tau\big]\\ &= Z_\tau^\zeta \mathbb{E}\big[Y^\zeta|\mathcal{F}_\tau\big]\\ &= Z_\tau^\zeta,\end{aligned}$$

which proves (3.6) and hence also our lemma. ■

We wish now to prove the following fundamental proposition:

**Proposition 3.1.9.** *Let $\xi \in \dot{\mathcal{X}}^1_{2A_2}(T, X_0)$ and $\tau$ be a stopping time with values in $[0, T[$. Then we have*

$$V(T, X_0, R_0) \geq \mathbb{E}\big[V\big(T-\tau, X_\tau^\xi, \mathcal{R}_\tau^\xi\big)\big]. \tag{3.7}$$

This proposition will follow from the subsequent lemma and the theorem on the existence of $\varepsilon$-maximizers on a bounded region. The latter one will be proved without the use of a measurable selection argument, by simply using the continuity of the value function and the existence of an optimal strategy for the maximization problem (2.16).

The next lemma allows us to restrict our problem to a region where the parameters $T, X_0$ and $R_0$ are bounded. Indeed, outside this region (with the bound of the parameters having to be taken large enough), the following result proves that the right-hand side term of (3.7) can be chosen smaller than $\varepsilon$.

**Lemma 3.1.10.** *Let $\xi \in \dot{\mathcal{X}}^1_{2A_2}(T, X_0)$. Under the assumptions and notations of Proposition 3.1.9 there exists $N = N_\varepsilon \in \mathbb{N}$, such that*

$$\mathbb{E}\Big[\big|V(T-\tau, X_\tau^\xi, \mathcal{R}_\tau^\xi)\big|\mathbb{1}_{\{|X_\tau^\xi|\vee|\mathcal{R}_\tau^\xi|>N\}}\Big] \leq \varepsilon. \tag{3.8}$$

*Proof.* We first prove that

$$\mathbb{E}\big[|V_2(T-\tau, X_\tau^\xi, \mathcal{R}_\tau^\xi)|\big] < \infty, \tag{3.9}$$

where we have $|V_2(T, X_0, R_0)| = \inf_{\zeta\in\dot{\mathcal{X}}(T,X_0)} \mathbb{E}\big[\exp(-A_2\mathcal{R}_T^\zeta)\big]$. This is a direct consequence of Lemma 3.1.8. Indeed, we can write

$$\mathbb{E}\big[|V_2(T-\tau, X_\tau^\xi, \mathcal{R}_\tau^\xi)|\big] \leq \mathbb{E}\big[\mathbb{E}\big[\exp(-A_2\mathcal{R}_T^\xi)|\mathcal{F}_\tau\big]\big]$$

$$= \mathbb{E}\big[\exp(-A_2 \mathcal{R}_T^{\xi})\big] < \infty.$$

Here, the first inequality is due to (3.4), and the last one follows from the fact that $\xi \in \dot{\mathcal{X}}^1_{2A_2}(T, X_0)$. Thus (3.9) follows, and hence, there exists $N \in \mathbb{N}$ such that

$$\mathbb{E}\Big[\big(|V_2(T-\tau, X_\tau^{\xi}, \mathcal{R}_\tau^{\xi})| + 1/A_1\big)\mathbb{1}_{\{|X_\tau^{\xi}| \vee |\mathcal{R}_\tau^{\xi}| > N\}}\Big] \le \varepsilon.$$

Using

$$|V(T, X_0, R_0)| \le |V_2(T, X_0, R_0)| + 1/A_1, \quad \text{for all } (T, X_0, R_0) \in ]0, \infty[\times \mathbb{R}^d \times \mathbb{R},$$

which is due to (2.14), we infer (3.8). ∎

We can now state and prove the following fundamental theorem of this section.

**Theorem 3.1.11** (Existence of the $\varepsilon$-maximizers on a bounded region)**.** *With the notations of Proposition 3.1.9, Lemma 3.1.4 and Lemma 3.1.10, there exists a progressively measurable process* $\widetilde{\xi}^{\cdot} = \widetilde{\xi}^{\cdot,\tau,\varepsilon} \in \dot{\mathcal{X}}^1_{2A_2}(T - \tau(.), X_\tau^{\xi}(.))$ *such that for* P*-a.e.* $\omega \in \big\{|X_\tau^{\xi}| \wedge |\mathcal{R}_\tau^{\xi}| \le N\big\}$,

$$V\big(T - \tau(\omega), X_\tau^{\xi}(\omega), \mathcal{R}_\tau^{\xi}(\omega)\big) \le \mathbb{E}\Big[u\Big(\mathcal{R}_\tau^{\xi}(\omega) + \mathcal{R}^{\widetilde{\xi}^{\omega,\tau,\varepsilon}}_{\tau(\omega),T}\Big)\Big] + \varepsilon. \tag{3.10}$$

*Proof.* The proof of this result is split in several steps. Let us first consider a simple process $\xi$ which is allowed to take only countably many values and a discrete stopping time $\tau$. The existence of the $\varepsilon$-maximizers is easier to prove in this case, because we are not facing any measurability problems.

In a second step, we consider an arbitrary process $\xi \in \dot{\mathcal{X}}^1_{2A_2}(T, X_0)$ and a stopping time $\tau$ taking values in $[0, T[$. The process $\xi$ can then be approximated by simple processes as in the first step, with respect to the topology of the $\mathrm{L}^p$-norm, where $p$ has to be chosen such that $f(x) \le C(1 + |x|^p)$ (see Assumption 3.1.3).

In a third step, we show by compactness arguments that the corresponding sequence of $\varepsilon$-maximizers (as obtained in the first step) converges weakly to a process $\xi^{\tau,\varepsilon}$.

In a last step, we show that $\xi^{\tau,\varepsilon}$ is the $\varepsilon$-maximizer we were looking for.
As seen in Remark 2.2.12, we will use the fact that a process $\xi \in \dot{\mathcal{X}}^1_{2A_2}(T, X_0)$ lies, in particular, in the set $\overline{K}_m(T, X_0)$, for a constant $m > 0$, where

$$\overline{K}_m(T, X_0) = \Big\{\xi \in \dot{\mathcal{X}}^1(T, X_0) \,\Big|\, \mathbb{E}\Big[\int_0^T f(-\xi_t)\, dt\Big] \le m\Big\}.$$

<u>First step:</u> Let $\varepsilon > 0$. For $L \in \mathbb{N}$ and $i \in \{0, \dots, 2^L\}$ define

$$t_i = i\frac{T}{2^L}.$$

Let $\xi \in \dot{\mathcal{X}}^1_{2A_2}(T, X_0)$ be defined as follows:

$$\xi_t(\omega) = \sum_{i=1}^{2^L} \xi_i(\omega)\mathbb{1}_{[t_i,t_{i+1}[}(t), \tag{3.11}$$

where $\xi_i$ takes values in the set $\{z_{i,p} \,|\, p \in \mathbb{N}, z_{i,p} \in \mathbb{R}^d\}$ . Let $\tau$ be a stopping time taking values in the set $\{t_0, t_1, ..., t_{2^L}\}$. Define $\Omega_{i,p_i} := \{\xi_i = z_{i,p_i}\}$ and $\Gamma_j := \{\tau = t_j\}$. Note that $\Gamma_j$ and $\Omega_{i,p_i}$ can be empty. For every $t \in [0, T]$, we have

$$X^\xi_t = X_0 - \sum_{i=1}^{k-1} \xi_i(t_{i+1} - t_i) - \xi_k(t - t_k), \tag{3.12}$$

where $k$ is such that $t \in [t_k, t_{k+1}[$. We can therefore write

$$X^\xi_\tau(\omega) = X_0 - \sum_{i=1}^{q-1} z_{i,p_i}(t_{i+1} - t_i), \tag{3.13}$$

for every $\omega \in \bigcap_{i=1}^q \Omega_{i,p_i} \cap \Gamma_q$. Because $V$ and $u$ are continuous (see Theorem 2.3.16), $V$ is uniformly continuous on $C_N := [t_1, T] \times \overline{B}(0, N) \times [-N, N]$ (where $\overline{B}(0, N)$ denotes the $d$-dimensional euclidian closed ball with radius $N$), and $u$ is uniformly continuous on $[-N, N]$. Therefore, we can find $\delta_N$ such that, for every $t^i, x^i, r^i, i = 1, 2$, we have

$$|(t^1 - t^2, x^1 - x^2, r^1 - r^2)| < \delta_N \Rightarrow |V(t^1, x^1, r^1) - V(t^2, x^2, r^2)| \vee |u(r^1) - u(r^2)| < \varepsilon.$$

Further, take $L \in \mathbb{N}$ such that

$$\frac{N}{2^L} < \delta_N,$$

and introduce

$$\mathbb{G} := \big\{((1, p_1), \dots, (q, p_q))|q \in \{0, \dots, 2^L\}, p_1, \dots, p_q \in \mathbb{N}\big\}.$$

Setting

$$\begin{aligned} &r_j := -N + \frac{jN}{2^L}, \quad x_g := X_0 - \sum_{i=1}^{q-1} z_{i,p_i}(t_{i+1} - t_i), \\ &j \in \{1, ..., 2^{L+1}\}, \quad g \in \mathbb{G}, \text{ with } g = ((1, p_1), \dots, (q, p_q)), \end{aligned}$$

we can now define the following grid:

$$\Gamma_N = \Big\{(t_i, x_g, r_l)|i \in \{0, ..., 2^L\}, j \in \{0, ..., 2^{L+1}\}, g \in \mathbb{G}\Big\} \cap C_N.$$

When

$$\Big(\tau(\omega), X^\xi_\tau(\omega), \mathcal{R}^\xi_\tau(\omega)\Big) \in \{t_i\} \times \{x_g\} \times [r_l, r_{l+1}[ \cap C_N,$$

we set

$$\gamma_N(\omega) := (T - t_i, x_g, r_l).$$

Note that $\gamma_N$ is $\mathcal{F}_\tau$-measurable. Let us denote by $\xi^{*,\gamma_N(\omega)}$ the optimal strategy associated to $V(\gamma_N(\omega))$ (which exists due to Theorem (2.2.4)). Then, the process $\xi^{*,\gamma_N(\omega)}$ is well-defined for every $\omega \in \{|X^\xi_\tau| \wedge |\mathcal{R}^\xi_\tau| \leq N\}$. Moreover, it belongs to the set $\dot{\mathcal{X}}^1_{2A_2}(T-t_i, x_g) = \dot{\mathcal{X}}^1_{2A_2}(T-\tau(\omega), X^\xi_\tau(\omega))$ (note that if $\tau(\omega) = T$ and $x_g = 0$, then $\gamma_N(\omega) = (0, 0, r_l)$, for some $r_l$, which implies that $V(\gamma_N(\omega)) = u(r_l)$, and therefore $\xi^{*,\gamma_N(\omega)} = 0$ is well-defined in this case, too). Furthermore, we have by construction

$$V(T - t_i, x_g, r_l) = \mathbb{E}\Big[u\Big(r_l + \mathcal{R}^{\xi^{*,\gamma_N(\omega)}}_{\tau(\omega),T}\Big)\Big]. \tag{3.14}$$

So, we obtain on $\{|X^\xi_\tau| \wedge |\mathcal{R}^\xi_\tau| \leq N\}$:

$$\begin{aligned}
&\Big|V(T - \tau(\omega), X^\xi_\tau(\omega), \mathcal{R}^\xi_\tau(\omega)) - \mathbb{E}\Big[u\Big(\mathcal{R}^\xi_\tau(\omega) + \mathcal{R}^{\xi^{*,\gamma_N(\omega)}}_{\tau,T}(\omega)\Big)\Big]\Big| \\
&\quad \leq \Big|V(T - \tau(\omega), X^\xi_\tau(\omega), \mathcal{R}^\xi_\tau(\omega)) - V(\gamma_N(\omega))\Big| \\
&\quad + \Big|V(\gamma_N(\omega)) - \mathbb{E}\Big[u\Big(\mathcal{R}^\xi_\tau(\omega) + \mathcal{R}^{\xi^{*,\gamma_N(\omega)}}_{\tau(\omega),T}\Big)\Big]\Big| \\
&\quad = \Big|V(T - t_i, x_g, \mathcal{R}^\xi_\tau(\omega)) - V(T - t_i, x_g, r_l)\Big| \\
&\quad + \Big|\mathbb{E}\Big[u\Big(r_l + \mathcal{R}^{\xi^{*,\gamma_N(\omega)}}_{\tau(\omega),T}\Big)\Big] - u\Big(\mathcal{R}^\xi_\tau(\omega) + \mathcal{R}^{\xi^{*,\gamma_N(\omega)}}_{\tau(\omega),T}\Big)\Big]\Big| \\
&\quad \leq \varepsilon + \varepsilon \\
&\quad = 2\varepsilon,
\end{aligned}$$

where the last inequality is due to the uniform continuity of $V$ and of $u$. Thus, we have found a process $\xi^{*,\gamma_N(.)} = \widetilde{\xi}^{.,\tau,\varepsilon} \in \dot{\mathcal{X}}^1_{2A_2}(T - \tau(.), X^\xi_\tau(.))$ such that (3.10) holds for every $\omega \in \{|X^\xi_\tau| \wedge |\mathcal{R}^\xi_\tau| \leq N\}$. Moreover,

$$\widetilde{\xi}^{.,\tau,\varepsilon} \in \overline{K}_{m^\varepsilon}(T - \tau(.), X^\xi_\tau(.)),$$

where $m^\varepsilon$ has to be chosen as in (2.32).

<u>Second step:</u> Now, let $\xi$ and $\tau$ be arbitrary. We can find a sequence of processes $\xi^k$ as in the first step, such that $\xi^k$ converges to $\xi$ in $\mathrm{L}^p$, i.e.,

$$\mathbb{E}\Big[\int_0^T |\xi^k_t - \xi_t|^p \, dt\Big] \longrightarrow 0,$$

where $p$ is chosen according to Assumption 3.1.3. Moreover, this sequence of processes may be chosen to lie in $\dot{\mathcal{X}}^1_{2A_2}(T, X_0)$, as argued in Assumption 2.2.14. We will prove that

$$\mathcal{R}^{\xi^k}_T \underset{k\to\infty}{\longrightarrow} \mathcal{R}^\xi_T, \quad \text{in probability.} \tag{3.15}$$

Due to Lemma 2.3.11, we have that

$$\int_t^T (X_s^{\xi^k})^\top \sigma\, dB_s \underset{k\to\infty}{\longrightarrow} \int_t^T (X_s^{\xi})^\top \sigma\, dB_s, \quad \mathbb{P}\text{-a.s.}$$

We have moreover, as a direct consequence of the $\mathrm{L}^p$ convergence of $\xi^k$ to $\xi$,

$$\int_t^T b\cdot X_s^{\xi^k}\, ds \underset{k\to\infty}{\longrightarrow} \int_t^T b\cdot X_s^{\xi}\, ds, \quad \mathbb{P}\text{-a.s.},$$

and

$$\int_t^T f(-\xi_s^k)\, ds \underset{k\to\infty}{\longrightarrow} \int_t^T f(-\xi_s)\, ds, \quad \text{in } \mathrm{L}^1,$$

(due to the growth condition imposed on $f$ in Assumption 3.1.3), and hence in probability. This establishes (3.15).

Third step: We can find a sequence of stopping times $(\tau_k)$ (with values in $[0,T[$) as in the first step, such that $\tau_k \downarrow \tau \quad \mathbb{P}$-a.s. As seen in the first step above, for each $k\in\mathbb{N}$, we can find $\widetilde{\xi}^{\cdot,\tau_k,\varepsilon} \in \overline{K}_{m^\varepsilon}(T-\tau_k(.), X_{\tau_k}^{\xi^k}(.))$, such that

$$V\big(T-\tau_k(\omega), X_{\tau_k}^{\xi^k}(\omega), \mathcal{R}_{\tau_k}^{\xi^k}(\omega)\big) \le \mathbb{E}\Big[u\Big(\mathcal{R}_{\tau_k}^{\xi^k}(\omega) + \mathcal{R}_{\tau_k(\omega),T}^{\widetilde{\xi}^{\omega,\tau_k,\varepsilon}}\Big)\Big] + \varepsilon \tag{3.16}$$

for P-a.e $\omega \in \big\{\big|X_{\tau_k}^{\xi^k}\big| \wedge \big|\mathcal{R}_{\tau_k}^{\xi^k}\big| \le N\big\}$. Moreover, we have that $\widetilde{\xi}^{\cdot,\tau_k,\varepsilon} \in \overline{\mathcal{K}}_{m^\varepsilon}$ with

$$\overline{\mathcal{K}}_{m^\varepsilon} = \Big\{\xi \in \overline{\mathcal{C}}\big(\dot{\mathcal{X}}^1_{2A_2}(T-\tau_k(.), X_{\tau_k}^{\xi}(.))\big)_k \Big|\ \mathbb{E}\Big[\int_{\tau(.)}^T f(-\xi_t)\, dt\Big] \le m^\varepsilon\Big\},$$

where $\overline{\mathcal{C}}(\dot{\mathcal{X}}^1_{2A_2}(T-\tau_k(.), X_{\tau_k}^{\xi}(.)))_k$ denotes the closed convex hull of the sequence of sets $\big(\dot{\mathcal{X}}^1_{2A_2}(T-\tau_k(.), X_{\tau_k}^{\xi}(.))\big)_k$. Recall that we set here

$$\zeta_t = 0, \text{ for } \quad t\in[\tau(.), \tau_k(.)], \text{ when } \zeta \in \dot{\mathcal{X}}^1_{2A_2}(T-\tau_k(.), X_{\tau_k}^{\xi}(.)),$$

since $\tau(.) \le \tau_k(.), \quad \mathbb{P}$-a.s.
Because $\overline{\mathcal{K}}_{m^\varepsilon}$ is weakly sequentially compact, as proved in Proposition 2.3.7, there exists $\widetilde{\xi}^{\tau,\varepsilon} \in \overline{\mathcal{K}}_{m^\varepsilon}$ such that, by passing to a subsequence if necessary, $\widetilde{\xi}^{k,\tau_k,\varepsilon}$ converges to $\widetilde{\xi}^{\tau,\varepsilon}$, weakly in $\mathrm{L}^1$. Using now Lemma 2.2.8, we have that $\widetilde{\xi}^{\tau,\varepsilon} \in \overline{\mathcal{K}}_{m^\varepsilon}, \quad \mathbb{P}$-a.s. on $\{|X_{\tau_k}^{\xi^k}| \wedge |\mathcal{R}_{\tau_k}^{\xi^k}| \le N\}$.

Last step: Notice first that we have

$$\limsup_k \mathbb{E}\Big[u\Big(\mathcal{R}_{\tau_k}^{\xi^k}(\omega) + \mathcal{R}_{\tau_k(\omega),T}^{\widetilde{\xi}^{\omega,\tau_k,\varepsilon}}\Big)\Big] \le \mathbb{E}\Big[u\Big(\mathcal{R}_{\tau}^{\xi}(\omega) + \mathcal{R}_{\tau(\omega),T}^{\widetilde{\xi}^{\omega,\tau,\varepsilon}}\Big)\Big] \tag{3.17}$$

for P-a.e $\omega \in \big\{\big|X_{\tau_k}^{\xi^k}\big| \wedge \big|\mathcal{R}_{\tau_k}^{\xi^k}\big| \le N\big\}$. Indeed, similarly to how it was established for $\xi \longmapsto \mathbb{E}\big[u\big(\mathcal{R}_T^{\xi}\big)\big]$, we can prove that $(r,\eta)\mapsto \mathbb{E}\big[u\big(r+\mathcal{R}_{t,T}^{\eta}\big)\big]$ is concave and thus we can apply Corollary 2.2.7 which proves (3.17). (Note that we cannot simply apply Fatou's lemma to prove (3.17), since it is not known whether or not

$$\limsup_k u\Big(\mathcal{R}^{\xi^k}_{\tau_k}(\omega) + \mathcal{R}^{\widetilde{\xi}^{\omega,\tau_k,\varepsilon}}_{\tau_k(\omega),T}\Big) \leq u\Big(\mathcal{R}^{\xi}_{\tau}(\omega) + \mathcal{R}^{\widetilde{\xi}^{\omega,\tau,\varepsilon}}_{\tau(\omega),T}\Big),$$

because we only have a weak convergence of $\widetilde{\xi}^{\omega,\tau_k,\varepsilon}$ to $\widetilde{\xi}^{\tau,\varepsilon}$.) Going back to (3.16) and passing to the limit superior on both sides of the inequality, we finally get, for P-a.e. $\omega \in \{|X^\xi_\tau| \wedge |\mathcal{R}^\xi_\tau| \leq N\}$,

$$\begin{aligned} V\big(T-\tau(\omega), X^\xi_\tau(\omega), \mathcal{R}^\xi_\tau(\omega)\big) &= \limsup_k V\big(T-\tau_k(\omega), X^{\xi^k}_{\tau_k}(\omega), \mathcal{R}^{\xi^k}_{\tau_k}(\omega)\big) \\ &\leq \limsup_k \mathbb{E}\Big[u\Big(\mathcal{R}^{\xi^k}_{\tau_k}(\omega) + \mathcal{R}^{\widetilde{\xi}^{\omega,\tau_k,\varepsilon}}_{\tau_k(\omega),T}\Big)\Big] + \varepsilon \\ &\leq \mathbb{E}\Big[u\Big(\mathcal{R}^{\xi}_{\tau}(\omega) + \mathcal{R}^{\widetilde{\xi}^{\omega,\tau,\varepsilon}}_{\tau(\omega),T}\Big)\Big] + \varepsilon, \end{aligned}$$

where the first equality is due to the continuity of $V$ in its arguments. And this proves (3.10).

■

We can now prove Proposition 3.1.9

*Proof of Proposition 3.1.9.* Take $\xi \in \dot{\mathcal{X}}^1_{2A_2}(T, X_0)$ and write:

$$\begin{aligned} &\mathbb{E}[V(T-\tau, X^\xi_\tau, \mathcal{R}^\xi_\tau)] \\ &\quad= \mathbb{E}\Big[V(T-\tau, X^\xi_\tau, \mathcal{R}^\xi_\tau)\mathbb{1}_{\{|X^\xi_\tau|\vee|\mathcal{R}^\xi_\tau|>N\}}\Big] + \mathbb{E}\Big[V(T-\tau, X^\xi_\tau, \mathcal{R}^\xi_\tau)\mathbb{1}_{\{|X^\xi_\tau|\wedge|\mathcal{R}^\xi_\tau|\leq N\}}\Big] \\ &\quad\leq \varepsilon + \int_\Omega \mathbb{E}\Big[u\Big(\mathcal{R}^\xi_\tau + \mathcal{R}^{\widetilde{\xi}^{\omega,\tau,\varepsilon}}_{\tau,T}\Big)\Big|\mathcal{F}_\tau\Big](\omega)\mathbb{P}(d\omega) + \varepsilon \\ &\quad= 2\varepsilon + \int_\Omega \mathbb{E}\Big[u\big(\mathcal{R}^\xi_\tau(\omega) + \mathcal{R}^{\widetilde{\xi}^{\omega,\tau,\varepsilon}}_{\tau(\omega),T}\big)\Big]\mathbb{P}(d\omega) \\ &\quad= 2\varepsilon + \mathbb{E}\Big[u\Big(\mathcal{R}^{\xi^{\tau,\varepsilon}}_T\Big)\Big] \\ &\quad\leq 2\varepsilon + V(T, X_0, R_0). \end{aligned}$$

Here, the first inequality is due to Lemma 3.1.10 and Theorem 3.1.11. The following equalities are due to Lemma 3.1.4, whereby the process $\xi^{\tau,\varepsilon}$ is defined as

$$\xi^{\tau,\varepsilon}_t(\omega) = \begin{cases} \xi_t(\omega) & \text{for } t \in [0, \tau(\omega)] \\ \widetilde{\xi}^{\omega,\tau,\varepsilon}_t(\omega) & \text{for } t \in [\tau(\omega), T]. \end{cases}$$

The last inequality follows from the definition of $V(T, X_0, R_0)$. ■

In Proposition 3.1.9, we have proved the inequality " $\geq$ " of equation (3.1). We have now to prove the reverse inequality. To this end, we need the following proposition which uses the notion of the essential supremum of a set $\Phi$ of random variables, denoted by $\operatorname{ess\,sup}_\Phi$.

**Proposition 3.1.12.** *With the notations of Lemma 3.1.4, we have*

$$V\Big(T-\tau(\omega), X_\tau^\xi(\omega), \mathcal{R}_\tau^\xi(\omega)\Big) = \operatorname*{ess\,sup}_{\xi^\omega \in \dot{\mathcal{X}}^1_{2A_2}(T-\tau(\omega), X_\tau^\xi(\omega))} \mathbb{E}\left[u(\mathcal{R}_\tau^\xi + \mathcal{R}_{\tau,T}^{\xi^\omega})|\mathcal{F}_\tau\right](\omega) \quad (3.18)$$

*for $\mathbb{P}$-a.e. $\omega$ on $\{|X_\tau^\xi| \wedge |\mathcal{R}_\tau^\xi| \leq N\}$.*

*Proof.* We recall the $\mathbb{P}$-a.s. equality fulfilled by $V(T-\tau, X_\tau^\xi, \mathcal{R}_\tau^\xi)$,

$$V\left(T-\tau(\omega), X_\tau^\xi(\omega), \mathcal{R}_\tau^\xi(\omega)\right) = \sup_{\xi^\omega \in \dot{\mathcal{X}}^1_{2A_2}(T-\tau(\omega), X_\tau^\xi(\omega))} \mathbb{E}\left[u\left(\mathcal{R}_\tau^\xi(\omega) + \mathcal{R}_{\tau,T}^{\xi^\omega}(\omega)\right)\right], \quad \mathbb{P}\text{-a.s.},$$

where $\xi^\omega$ is defined as in Lemma 3.1.4. Hence, this permits us to write

$$V(T-\tau(\omega), X_\tau^\xi(\omega), \mathcal{R}_\tau^\xi(\omega)) \geq \mathbb{E}\left[u\left(\mathcal{R}_\tau^\xi + \mathcal{R}_{\tau,T}^{\xi^\omega}\right)\Big|\mathcal{F}_\tau\right](\omega), \quad \text{P-a.s. },$$

for all $\xi^\omega \in \dot{\mathcal{X}}^1_{2A_2}(T-\tau(\omega), X_\tau^\xi(\omega))$. Using the definition of the essential supremum (see, e.g., Föllmer and Schied (2011), Definition A.34), it follows then

$$V(T-\tau(\omega), X_\tau(\omega), R_\tau^X(\omega)) \geq \operatorname*{ess\,sup}_{\xi^\omega \in \dot{\mathcal{X}}^1 2A_2(T-\tau(\omega), X_\tau^\xi(\omega))} \mathbb{E}\left[u\left(\mathcal{R}_\tau^\xi + \mathcal{R}_{\tau,T}^{\xi^\omega}\right)|\mathcal{F}_\tau\right](\omega), \tag{3.19}$$

which proves the inequality " $\geq$ " of (3.18). For the converse inequality, let $\widetilde{\xi}^{\omega,\tau,\varepsilon}$ be as in Theorem 3.1.11. We have on $\{|X_\tau^\xi| \wedge |\mathcal{R}_\tau^\xi| \leq N\}$ :

$$\mathbb{E}\left[u(\mathcal{R}_\tau^\xi + \mathcal{R}_{\tau,T}^{\widetilde{\xi}^{\omega,\tau,\varepsilon}})|\mathcal{F}_\tau\right](\omega) \geq V(T-\tau(\omega), X_\tau^\xi(\omega), \mathcal{R}_\tau^\xi(\omega)) - \varepsilon, \quad \mathbb{P}\text{-a.s.}$$

And therefore:

$$\operatorname*{ess\,sup}_{\xi^\omega \in \dot{\mathcal{X}}^1_{2A_2}(T-\tau(\omega), X_\tau^\xi(\omega))} \mathbb{E}\left[u(\mathcal{R}_\tau^\xi + \mathcal{R}_{\tau,T}^{\xi^\omega})|\mathcal{F}_\tau\right](\omega) \geq V(T-\tau(\omega), X_\tau^\xi(\omega), \mathcal{R}_\tau^\xi(\omega)) - \varepsilon, \quad \mathbb{P}\text{-a.s.}$$

Letting $\varepsilon$ go to 0 gives us the required inequality. ■

We can now prove Theorem 3.1.1.

*Proof of Theorem 3.1.1.* Thanks to Proposition 3.1.9, it remains to show only the inequality " $\leq$ " in (3.1). Let $\xi \in \dot{\mathcal{X}}^1_{2A_2}(T, X_0)$ and set $\widetilde{\xi}_s = \xi_{\tau+t} \in \dot{\mathcal{X}}^1_{2A_2}(T-\tau, X_\tau^\xi)$ for $s \geq \tau$ and $t \geq 0$. We get

$$\begin{aligned}
\mathbb{E}\Big[u\big(\mathcal{R}_T^\xi\big)\Big] &= \mathbb{E}\Big[u\Big(\mathcal{R}_\tau^\xi + \mathcal{R}_{\tau,T}^{\widetilde{\xi}}\Big)\Big] \\
&= \mathbb{E}\left[\mathbb{E}\left[u(\mathcal{R}_\tau^\xi + \mathcal{R}_{\tau,T}^{\widetilde{\xi}})|\mathcal{F}_\tau\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[u(\mathcal{R}_\tau^\xi + \mathcal{R}_{\tau,T}^{\widetilde{\xi}})|\mathcal{F}_\tau\right]\left(\mathbb{1}_{\{|X_\tau^\xi| \vee |\mathcal{R}_\tau^\xi| > N\}} + \mathbb{1}_{\{|X_\tau^\xi| \wedge |\mathcal{R}_\tau^\xi| \leq N\}}\right)\right] \\
&\leq \varepsilon + \mathbb{E}\Big[V(T-\tau, X_\tau^\xi, \mathcal{R}_\tau^\xi)\mathbb{1}_{\{|X_\tau^\xi| \wedge |\mathcal{R}_\tau^\xi| \leq N\}}\Big].
\end{aligned}$$

The last inequality is due to the definition of the essential supremum and Proposition 3.1.12 combined with Lemma 3.1.10. Taking the supremum over $\xi$, and then sending $\varepsilon$ to zero (which implies sending $N$ to infinity), we get the inequality " $\leq$ ". This yields the assertion. ■

## 3.2 The Hamilton-Jacobi-Bellman equation

We want to prove that $V$ fulfills a Hamilton-Jacobi-Bellman (HJB) equation, which is obtained via a classical heuristic derivation. To this end, let us first suppose that $V \in \mathcal{C}^{1,1,2}(]0,T] \times \mathbb{R}^d \times \mathbb{R})$. Let $\xi \in \dot{\mathcal{X}}^1_{2A_2}(T, X_0)$ and $t \in [0,T[$. By using Itô's formula and (2.4) we have:

$$\begin{aligned} dV(T-t, X_t^\xi, \mathcal{R}_t^\xi) &= -V_t(T-t, X_t^\xi, \mathcal{R}_t^\xi)dt - \xi_t \nabla_x V(T-t, X_t^\xi, \mathcal{R}_t^\xi)dt \\ &\quad + V_r(T-t, X_t^\xi, \mathcal{R}_t^\xi)(\sigma X_t^\xi dB_t + (b \cdot X_t^\xi - f(-\xi_t))dt) \\ &\quad + \frac{\left(X_t^\xi\right)^\top \Sigma X_t^\xi}{2} V_{rr}(T-t, X_t^\xi, \mathcal{R}_t^\xi)dt \\ &= \Big( -V_t - \xi_t \nabla_x V + V_r(b \cdot X_t^\xi - f(-\xi_t)) \\ &\quad + \frac{\left(X_t^\xi\right)^\top \Sigma X_t^\xi}{2} V_{rr}) \Big)(T-t, X_t^\xi, \mathcal{R}_t^\xi)\, dt + \left(X_t^\xi\right)^\top \sigma dB_t. \end{aligned}$$

In order to simplify the computation, let us introduce the following linear second-order operator $\mathcal{L}^\eta$, where for $\eta \in \mathbb{R}^d$,

$$\mathcal{L}^\eta v(T, X, R) := \left( \frac{X^\top \Sigma X}{2} v_{rr} + b \cdot X v_r - \left( \eta^\top \nabla_x v + f(-\eta) v_r \right) \right)(T, X, R). \quad (3.20)$$

Note that this operator is continuous in $\eta$, due to the continuity of $f$. Since we expect $V(T-t, X_t^\xi, \mathcal{R}_t^\xi)$ to be a local supermartingale, we should have, $\mathbb{P} \otimes \lambda$-a.e.,

$$\left( -V_t + \mathcal{L}^{\xi_t} V \right)(T-t, X_t^\xi, \mathcal{R}_t^\xi) \leq 0. \quad (3.21)$$

Since this must hold for every $\xi \in \dot{\mathcal{X}}^1_{2A_2}(T, X_0)$, and once the corresponding derivatives are continuous, this would lead to

$$\Big( -V_t + \sup_{\xi \in \mathbb{R}^d} \mathcal{L}^\xi V \Big)(T-t, X, R) \leq 0,$$

for all $(t, X, R) \in [0, T[ \times \mathbb{R}^d \times \mathbb{R}$. In order for $V(T-t, X_t^{\xi^*}, \mathcal{R}_t^{\xi^*})$ to be a martingale, where $\xi^*$ is optimal, we should then have that

$$\Big( -V_t + \mathcal{L}^{\xi_t^*} V \Big)\left(T-t, X_t^{\xi_t^*}, \mathcal{R}_t^{\xi^*}\right) = 0, \quad (3.22)$$

which would lead to

$$\Big( -V_t + \mathcal{L}^{\xi_t^*} V \Big)(T-t, X, R) = 0,$$

for all $(t, X, R) \in [0, T[ \times \mathbb{R}^d \times \mathbb{R}$. Moreover, the global liquidation constraint on $\xi$ is given by the following asymptotic limit:

$$V(0, X, R) = \lim_{T \downarrow 0} V(T, X, R) = \begin{cases} u(R), & \text{if } X = 0, \\ -\infty, & \text{otherwise.} \end{cases}$$

Thus, the preceding Hamiltion-Jacobi-Bellman equation can be rewritten as follows:

$$-V_t + \sup_{\xi \in \mathbb{R}^d} \mathcal{L}^{\xi} V = 0, \tag{3.23}$$

$$V(0, X, R) = \lim_{T \downarrow 0} V(T, X, R) = \begin{cases} u(R), & \text{if } X = 0, \\ -\infty, & \text{otherwise.} \end{cases} \tag{3.24}$$

Note that the validity of (3.24) has already been established in Proposition 2.2.3. The intuitive interpretation of the singularity in the initial condition is as follows: a strategy which will not lead to a complete liquidation of the portfolio within a given time period is highly penalized.

*Remark* 3.2.1. Since $f$ is positive and $\lim_{|x| \to \infty} f(x) = \infty$, equation (3.23) makes sense only when $V_r(t, x, r) > 0$, for every $(t, x, r) \in \,]0, T] \times \mathbb{R}^d \times \mathbb{R}$. However, this is in concordance with Theorem 2.3.4, since it has been proved there that the value function has, in particular, a strictly positive partial derivative in its third argument. Next, let us denote by

$$f^*(z) = \sup_x (x \cdot z - f(x))$$

the Fenchel-Legendre transformation of $f$. Note that $f^*$ is a finite convex function, due to the assumptions on $f$ (see Theorem 12.2 in Rockafellar (1997)). We show now that we can rewrite (3.23) as follows:

$$- V_t + b \cdot X_t \, V_r + \frac{X^\top \Sigma X}{2} V_{rr} + V_r f^* \Big( \frac{\nabla_x V}{V_r} \Big) = 0. \tag{3.25}$$

For that matter, we compute

$$\begin{aligned} 0 &= -V_t + \sup_{\xi \in \mathbb{R}^d} \mathcal{L}^{\xi} V \\ &= -V_t + b \cdot X \, V_r + \frac{X^\top \Sigma X}{2} V_{rr} + \sup_{\xi \in \mathbb{R}^d} -\big( \xi \cdot \nabla_x V + f(-\xi) V_r \big) \\ &= -V_t + b \cdot X \, V_r + \frac{X^\top \Sigma X}{2} V_{rr} + \sup_{\xi \in \mathbb{R}^d} \big( \xi \cdot \nabla_x V - f(\xi) V_r \big) \\ &= -V_t + b \cdot X \, V_r + \frac{X^\top \Sigma X}{2} V_{rr} + V_r \sup_{\xi \in \mathbb{R}^d} \big( \xi \cdot \frac{\nabla_x V}{V_r} - f(\xi) \big) \\ &= -V_t + b \cdot X \, V_r + \frac{X^\top \Sigma X}{2} V_{rr} + V_r f^* \Big( \frac{\nabla_x V}{V_r} \Big), \end{aligned}$$

which completes the argument. ♢

In order to make the relations between the Hamilton-Jacobi-Bellman equation and our value function clearer, we have to use the dynamic programming principle (Theorem 3.1.1). We first suppose that $V \in \mathcal{C}^{1,1,2}(]0, T] \times \mathbb{R}^d \times \mathbb{R})$ and show that $V$ is a classical solution of (3.23). We split the proof in two propositions.

**Proposition 3.2.2.** *Let $V$ be the value function of the maximization problem* (2.16)*. Suppose that $V \in \mathcal{C}^{1,1,2}(]0,T] \times \mathbb{R}^d \times \mathbb{R})$. Then, $V$ is a supersolution of* (3.23)*, i.e., $V$ fulfills the inequality*

$$\Big( - V_t + \sup_{\xi \in \mathbb{R}^d} \mathcal{L}^\xi V \Big)(t,x,r) \leq 0, \qquad \textit{for all } (t,x,r) \in\, ]0,T] \times \mathbb{R}^d \times \mathbb{R}. \tag{3.26}$$

Before beginning with the proof of the preceding proposition, let us briefly describe an easy way how to construct supersolutions of (3.23): the following lemma shows that the set of viscosity supersolutions is stable under the operation of linear combination with positive coefficients.

**Lemma 3.2.3.** *Let $V, \widetilde{V}$ be two supersolutions of* (3.23) *and $\varepsilon \geq 0$. Then $V + \varepsilon \widetilde{V}$ is a again supersolution of* (3.23)*.*

*Proof.* We write

$$\begin{aligned}
&-(V+\varepsilon\widetilde{V})_t + \frac{X^\top \Sigma X}{2}(V+\varepsilon\widetilde{V})_{rr} + b\cdot X\,(V+\varepsilon\widetilde{V})_r \\
&\quad + \sup_{\xi\in\mathbb{R}^d} \big(\xi\cdot\nabla_x(V+\varepsilon\widetilde{V}) - f(\xi)(V_t+\varepsilon\widetilde{V})_r\big) \\
&= -V_t - \varepsilon\widetilde{V}_t + \frac{X^\top \Sigma X}{2}(V_{rr}+\varepsilon\widetilde{V}_{rr}) + b\cdot X\,(V_r+\varepsilon\widetilde{V}_r) \\
&\quad + \sup_{\xi\in\mathbb{R}^d} \big(\xi\cdot(\nabla_x V + \varepsilon\nabla_x\widetilde{V}) - f(\xi)(V_r+\varepsilon\widetilde{V}_r)\big) \\
&\leq -V_t + \frac{X^\top \Sigma X}{2} V_{rr} + b\cdot X\,V_r + \sup_{\xi\in\mathbb{R}^d}(\xi\cdot\nabla_x V - f(\xi)V_r) \\
&\quad + \varepsilon\Big(\widetilde{V}_t + \frac{X^\top \Sigma X}{2}\widetilde{V}_{rr} + b\cdot X\,\widetilde{V}_r + \sup_{\xi\in\mathbb{R}^d}\big(\xi\cdot\nabla_x\widetilde{V} - f(\xi)\widetilde{V}_r\big)\Big) \\
&\leq 0,
\end{aligned}$$

where the first inequality follows by taking the supremum of a sum, and the second one is in conjunction with the fact that both $V$ and $\widetilde{V}$ are supersolutions. Thus, $V + \varepsilon\widetilde{V}$ is again a supersolution. ■

*Proof of Proposition 3.2.2.* Here, we use classical argumentations, as it can be found in, e.g., Crandall et al. (1992). Let $(t,x,r) \in [0,T[\, \times \mathbb{R}^d \times \mathbb{R}^d, \eta \in \mathbb{R}^d$ and $\varepsilon > 0$ such that $t + \varepsilon < T$. Define $\xi \in \dot{\mathcal{X}}^1_{A_2}([t,T],x)$ in the following way

$$\xi_s := \begin{cases} \eta, & \text{if } s \in [t, t+\varepsilon[, \\ -\frac{x-\varepsilon\eta}{T-(t+\varepsilon)}, & \text{if } s \in [t+\varepsilon, T], \end{cases}$$

and consider the corresponding processes $(X^\xi_s, \mathcal{R}^\xi_s)$ that verify $X^\xi_t = x, \mathcal{R}^\xi_t = r$. For all $k \in \mathbb{N}$ large enough, consider the following stopping time

$$\tau_k := \inf\big\{s > t | (s-t, X^\xi_s - x, \mathcal{R}^\xi_s - r) \notin [0, 1/k[\times B(0,\alpha)\times] - \alpha; \alpha[\big\},$$

where $B(0,\alpha)$ denotes the ball of radius $\alpha > 0$ centered at the origin in $\mathbb{R}^d$. By applying Itô's formula (which is possible here, due to the smoothness assumptions) together with Theorem 3.1.1, we have

$$\begin{aligned}
0 &\geq E\big[V(T-\tau_k, X^\xi_{\tau_k}, \mathcal{R}^\xi_{\tau_k}) - V(T-t,x,r)\big] \\
&= \mathbb{E}\bigg[ - \int_t^{\tau_k} V_t(T-s, X^\xi_s, \mathcal{R}^\xi_s)\, ds + \int_t^{\tau_k} V_r(T-s, X^\xi_s, \mathcal{R}^\xi_s)\, d\mathcal{R}^\xi_s \\
&\quad + \int_t^{\tau_k} \nabla_x V(T-s, X^\xi_s, \mathcal{R}^\xi_s)\, dX^\xi_s + \frac{1}{2}\int_t^{\tau_k} V_{rr}(T-s, X^\xi_s, \mathcal{R}^\xi_s)\, d\langle \mathcal{R}^\xi\rangle_s \bigg] \\
&= \mathbb{E}\bigg[ \int_t^{\tau_k} \Big( - V_t(T-s, X^\xi_s, \mathcal{R}^\xi_s) + \mathcal{L}^\xi V(T-s, X^\xi_s, \mathcal{R}^\xi_s) \Big)\, ds \bigg] \\
&\quad + \mathbb{E}\bigg[ \int_t^{\tau_k} (X^\xi_s)^\top \sigma V_r(T-s, X^\xi_s, \mathcal{R}^\xi_s)\, dB_s \bigg].
\end{aligned}$$

Due to the definition of $\tau_k$, the last expectation vanishes thanks to Doob's optional sampling theorem. Hence,

$$\mathbb{E}\bigg[ \int_t^{\tau_k} \Big( - V_t(T-s, X^\xi_s, \mathcal{R}^\xi_s) + \mathcal{L}^\xi V(T-s, X^\xi_s, \mathcal{R}^\xi_s) \Big)\, ds \bigg] \leq 0. \tag{3.27}$$

Because of the a.s. continuity in $s$ of the integrands, we have $\tau_k = t + 1/k$, for $k$ large enough. Thus, using the mean value theorem, we get that

$$k \int_t^{\tau_k} \Big( - V_t(T-s, X^\xi_s, \mathcal{R}^\xi_s) + \mathcal{L}^\xi V(T-s, X^\xi_s, \mathcal{R}^\xi_s) \Big)\, ds \tag{3.28}$$

converges a.s. to

$$- V_t(T-t,x,r) + \mathcal{L}^\eta V(T-t,x,r), \tag{3.29}$$

when $k$ goes to infinity. In addition, (3.28) is a.s. uniformly bounded in $k$. Indeed, due to the definition of $\tau_k$, the processes $X^\xi_t$ and $\mathcal{R}^\xi_t$ are bounded, and so are the terms $V_t, V_r$ and $\mathcal{L}^\xi V$ in the related integral, since they are continuous in both preceding quantities (and since we can find $\delta > 0$ such that for $k$ small enough we have $\tau_k < T - \delta$) . Thus, we can use the dominated convergence theorem to achieve

$$\begin{aligned}
&\mathbb{E}\bigg[ k\int_t^{\tau_k} \Big( - V_t(T-s, X^\xi_s, \mathcal{R}^\xi_s) + \mathcal{L}^\xi V(T-s, X^\xi_s, \mathcal{R}^\xi_s) \Big)\, ds \bigg] \\
&\qquad \underset{k\to\infty}{\longrightarrow} -V_t(T-t,x,r) + \mathcal{L}^\eta V(T-t,x,r).
\end{aligned}$$

Combining this with inequality (3.27), we finally obtain

$$- V_t(T-t,x,r) + \mathcal{L}^\eta V(T-t,x,r) \leq 0. \tag{3.30}$$

Since we chose $\eta$ arbitrarily, and due to the continuity of $\eta \longrightarrow \mathcal{L}^\eta V$, we can now take the supremum on the left-hand side of the last inequality, which gives

$$\Big( - V_t + \sup_{\eta\in\mathbb{R}^d} \mathcal{L}^\eta V \Big)(T-t,x,r) \leq 0.$$

■

**Proposition 3.2.4.** *Let $V$ be the value function of the maximization problem (2.16). Suppose that $V \in \mathcal{C}^{1,1,2}(]0,T] \times \mathbb{R}^d \times \mathbb{R})$. Then $V$ is a subsolution of (3.23), i.e., $V$ fulfills the inequality*

$$\Big( - V_t + \sup_{\xi \in \mathbb{R}^d} \mathcal{L}^{\xi} V \Big)(t,x,r) \geq 0, \qquad \text{for all } (t,x,r) \in ]0,T] \times \mathbb{R}^d \times \mathbb{R}. \tag{3.31}$$

*Proof.* We first note that a straightforward use of the same ideas as in the proof of Proposition 3.2.2 cannot work: there would appear an $\varepsilon$-term on the right-hand side of (3.27), with the help of the existence of $\varepsilon$-maximizers, so multiplying as before (3.27) by $k$ on both sides and sending $k$ to infinity would not permit us to conclude. Therefore, we will follow the proof of Touzi (2013), Proposition 3.5.

We assume that there exists $(t_0, x_0, r_0)$ such that

$$-V_t(T-t_0, x_0, r_0) + \sup_{\eta \in \mathbb{R}^d} \mathcal{L}^{\eta} V(T-t_0, x_0, r_0) < 0,$$

and work torward a contradiction, using an $\varepsilon$-maximizer. First, set

$$\varphi(T-t,x,r) = V(T-t,x,r) + \frac{\delta}{2} |(x,r) - (x_0, r_0)|^2.$$

Since we have

$$\begin{aligned} (V-\varphi)(T-t_0,x_0,r_0) &= 0, \quad \nabla_x (V-\varphi)(T-t_0,x_0,r_0) = 0, \\ (V-\varphi)_r(T-t_0,x_0,r_0) &= 0, \quad (V-\varphi)_t(T-t_0,x_0,r_0) = 0, \\ (V-\varphi)_{rr}(T-t_0,x_0,r_0) &= -\delta, \end{aligned}$$

and

$$(x,r) \longrightarrow - \inf_{\xi \in \mathbb{R}^d} (x \cdot \xi - f(-\xi) r) = \frac{1}{r} f^* \Big( \frac{x}{r} \Big)$$

is continuous on $\mathbb{R}^d \times ]0, \infty[$, it follows that

$$h(t_0,x_0,r_0) := -\varphi_t(T-t_0,x_0,r_0) + \sup_{\xi \in \mathbb{R}^d} \mathcal{L}^{\xi} \varphi(T-t_0,x_0,r_0) < 0,$$

for $\delta$ small enough.
We define now, for $\eta > 0$ small enough, the following neighborhood of $(T-t_0, x_0, r_0)$:

$$\mathcal{N}_\eta = \big\{ (t,x,r) | (t-t_0, x-x_0, r-r_0) \in ]-\eta, \eta[ \times B(0,\eta) \times ]-\eta,\eta[ \text{ and } h(t,x,r) < 0 \big\}.$$

Further, we set

$$\varepsilon = \min_{(T-t,x,r) \in \partial \mathcal{N}_\eta} (\varphi - V) = \frac{\delta}{2} \min_{\partial \mathcal{N}_\eta} |(T-t,x,r) - (T-t_0,x_0,r_0)|^2 > 0. \tag{3.32}$$

Take $\xi \in \dot{\mathcal{X}}^1_{2A_2}([t_0,T], X_0)$ and let us introduce the following stopping time

$$\tau := \inf \{ s > t_0 \mid (s, X^{\xi}_s, \mathcal{R}^{\xi}_s) \notin \mathcal{N}_\eta \}.$$

Due to the pathwise continuity of the corresponding state process, we have $(T-\tau, X^{\xi}_{\tau}, \mathcal{R}^{\xi}_{\tau}) \in \partial\mathcal{N}_{\eta}$, so that

$$(\varphi - V)(T-\tau, X^{\xi}_{\tau}, \mathcal{R}^{\xi}_{\tau}) \geq \varepsilon, \quad \mathbb{P}\text{-a.s.},$$

by using (3.32). Hence, applying Itô's formula we get

$$\begin{aligned}
&\mathbb{E}\Big[V\big(T-\tau, X^{\xi}_{\tau}, \mathcal{R}^{\xi}_{\tau}\big) - V(T-t_0, x_0, r_0)\Big] \\
&= \mathbb{E}\Big[V\big(T-\tau, X^{\xi}_{\tau}, \mathcal{R}^{\xi}_{\tau}\big) - \varphi\big(T-\tau, X^{\xi}_{\tau}, \mathcal{R}^{\xi}_{\tau}\big) + \varphi\big(T-\tau, X^{\xi}_{\tau}, \mathcal{R}^{\xi}_{\tau}\big) \\
&\quad - \varphi(T-t_0, x_0, r_0) + \varphi(T-t_0, x_0, r_0) - V(T-t_0, x_0, r_0)\Big] \\
&= \mathbb{E}\Big[V\big(T-\tau, X^{\xi}_{\tau}, \mathcal{R}^{\xi}_{\tau}\big) - \varphi\big(T-\tau, X^{\xi}_{\tau}, \mathcal{R}^{\xi}_{\tau}\big) + \varphi\big(T-\tau, X^{\xi}_{\tau}, \mathcal{R}^{\xi}_{\tau}\big) \\
&\quad - \varphi(T-t_0, x_0, r_0)\Big] \\
&\leq -\varepsilon + \mathbb{E}\Big[\varphi\big(T-\tau, X^{\xi}_{\tau}, \mathcal{R}^{\xi}_{\tau}\big) - \varphi(T-t_0, x_0, r_0)\Big] \\
&= -\varepsilon + \mathbb{E}\bigg[\int_{t_0}^{\tau} \Big( - \varphi_t(T-s, X^{\xi}_{s}, \mathcal{R}^{\xi}_{s}) + \mathcal{L}^{\xi}\varphi(T-s, X^{\xi}_{s}, \mathcal{R}^{\xi}_{s})\Big)\, ds\bigg] \\
&\quad + \mathbb{E}\bigg[\int_{t_0}^{\tau} (X^{\xi}_{s})^{\top}\sigma\varphi_r(T-s, X^{\xi}_{s}, \mathcal{R}^{\xi}_{s})\, dB_s\bigg].
\end{aligned}$$

The last expectation vanishes, due to the boundedness of the integrands on the stochastic interval $[t_0, \tau]$. Since $\big(-\varphi_t + \mathcal{L}^{\xi}\varphi\big)(s, X^{\xi}_{s}, \mathcal{R}^{\xi}_{s}) \leq 0$, on $[t_0, \tau]$, we have by using the above inequalities:

$$\begin{aligned}
V(T-t_0, x_0, r_0) &\geq \varepsilon + \mathbb{E}\bigg[V\big(T-\tau, X^{\xi}_{\tau}, \mathcal{R}^{\xi}_{\tau}\big) - \int_{t_0}^{\tau} \Big( - \varphi_t(T-s, X^{\xi}_{s}, \mathcal{R}^{\xi}_{s}) \\
&\qquad + \mathcal{L}^{\xi}\varphi\big(T-s, X^{\xi}_{s}, \mathcal{R}^{\xi}_{s}\big)\Big)\, ds\bigg] \\
&\geq \varepsilon + \mathbb{E}\big[V\big(T-\tau, X^{\xi}_{\tau}, \mathcal{R}^{\xi}_{\tau}\big)\big].
\end{aligned}$$

By taking the supremum over $\xi$ on the right-hand side and using Theorem 3.1.1, we infer (since $\varepsilon$ does not depend on $\xi$)

$$V(T-t_0, x_0, r_0) \geq \varepsilon + \sup_{\xi \in \dot{\mathcal{X}}^1_{2A_2}([t_0,T],X_0)} \mathbb{E}\big[V\big(T-\tau, X^{\xi}_{\tau}, \mathcal{R}^{\xi}_{\tau}\big)\big] = \varepsilon + V(T-t_0, x_0, r_0),$$

which is a contradiction with $\varepsilon > 0$. Therefore, we have

$$\Big( - V_t + \sup_{\eta \in \mathbb{R}^d} \mathcal{L}^{\eta} V\Big)(t, x, r) \geq 0, \qquad \text{for all } (t, x, r) \in \,]0, T] \times \mathbb{R}^d \times \mathbb{R}.$$

$\blacksquare$

**Theorem 3.2.5.** *Let $V$ be the value function of the maximization problem (2.16). Suppose that $V \in \mathcal{C}^{1,1,2}(]0,T] \times \mathbb{R}^d \times \mathbb{R})$. Then, $V$ is a classical solution of (3.23) with initial condition (3.24).*

*Proof.* Combining Propositions 3.2.2 and 3.2.4, we obtain that the value function $V$ is a classical solution to the equation

$$-v_t + \frac{X^\top \Sigma X}{2} v_{rr} + b \cdot X v_r + \sup_{\xi \in \mathbb{R}^d} \left( \xi^\top \nabla_x v - v_r f(\xi) \right) = 0. \tag{3.33}$$

Moreover, in Proposition 2.2.3 we already showed that $V$ fulfills the initial condition (3.24). ■

## 3.3 Verification theorem

In this section we give sufficient conditions which allow us to conclude that a smooth function $w$ satisfying (3.23) with initial condition (3.24) coincides with our value function $V$. This so-called *verification argument* relies essentially on Itô's lemma. See for example Touzi (2013) or Pham (2009) for further details. Note that here, due to existence and uniqueness of the optimal control for the value function $V$, we will only need the existence of a strong solution to an associated SDE, to ensure that $w = V$. Here again, we take as growth condition that $w$ lies between two CARA value functions.

**Theorem 3.3.1.** *Let $T > 0$ and suppose that $w \in \mathcal{C}^{1,1,2}(]0,T[\times \mathbb{R}^d \times \mathbb{R}) \cap \mathcal{C}(]0,T] \times \mathbb{R}^d \times \mathbb{R})$ fulfills the following inequality*

$$V_2(t,x,r) \le w(t,x,r) \le V_1(t,x,r), \tag{3.34}$$

*where $V_i$ is the CARA value function as defined in (2.14), for $i = 1,2$.*

1. *Assume further that $w$ satisfies the following conditions:*

$$0 \ge -w_t(T-t,x,r) + \sup_{\xi \in \mathbb{R}^d} \mathcal{L}^\xi w(T-t,x,r), \tag{3.35}$$

*for all $(t,x,r) \in [0,T[\times \mathbb{R}^d \times \mathbb{R}$, and*

$$\lim_{t \downarrow 0} w(t,x,r) = \begin{cases} w(0,0,r) \ge u(r), & \text{if } X = 0, \\ -\infty, & \text{otherwise}, \end{cases} \tag{3.36}$$

*on $]0,T] \times \mathbb{R}^d \times \mathbb{R}$. Then $w \ge V$ on $]0,T] \times \mathbb{R}^d \times \mathbb{R}$.*

2. *Suppose further that*

$$0 = -w_t(T-t,x,r) + \sup_{\xi \in \mathbb{R}^d} \mathcal{L}^\xi w(T-t,x,r), \tag{3.37}$$

*for all $(t,x,r) \in [0,T[\times \mathbb{R}^d \times \mathbb{R}$, and*

$$\lim_{T \downarrow 0} w(t,x,r) = \begin{cases} u(r), & \text{if } X = 0, \\ -\infty, & \text{otherwise}. \end{cases} \tag{3.38}$$

*Moreover, assume*

$$w_r(T-t,x,r) > 0, \text{ for all } t,x,r \text{ on } [0,T[\times\mathbb{R}^d\times\mathbb{R}. \tag{3.39}$$

*i. Then, the continuous function* $\widehat{\xi}:]0,T]\times\mathbb{R}^d\times\mathbb{R}\longmapsto\mathbb{R}^d$ *defined by*

$$\widehat{\xi}(t,x,r) := \nabla f^*\Big(\frac{\nabla_x w(t,x,r)}{w_r(t,x,r)}\Big) \tag{3.40}$$

*(where $f^*$ denotes the Fenchel-Legendre transform of $f$) is such that*

$$\begin{aligned}&-w_t(T-t,x,r)+\sup_{\xi\in\mathbb{R}^d}\mathcal{L}^{\xi}w(T-t,x,r)\\&=-w_t(T-t,x,r)+\mathcal{L}^{\widehat{\xi}(T-t,x,r)}w(T-t,x,r)\\&=0,\end{aligned} \tag{3.41}$$

*for every $(t,x,r)$ on $]0,T]\times\mathbb{R}^d\times\mathbb{R}$.*

*ii. If we suppose furthermore that there exists a strong solution $(X,\mathcal{R})$ to the following SDE:*

$$\begin{cases} d\mathcal{R}_t = (X_t)^{\top}\sigma dB_t + b\cdot X_t\,dt - f(-\widehat{\xi}(t,X_t,\mathcal{R}_t))\,dt,\\ dX_t = -\widehat{\xi}(t,X_t,\mathcal{R}_t)\,dt,\\ \mathcal{R}_{|t=0}=R_0 \text{ and } X_{|t=0}=X_0,\end{cases} \tag{3.42}$$

*such that $\widehat{\xi}(.,X,\mathcal{R})\in\dot{\mathcal{X}}^1_{2A_2}(T,X_0)$, then we have $w=V$ on $]0,T]\times\mathbb{R}^d\times\mathbb{R}$. The solution of the preceding SDE is unique and given by $(X_t^{\xi^*},R_t^{\xi^*})$, where $\xi^*$ denotes the optimal liquidation strategy for the value function $V(T,X_0,R_0)$. Moreover, the optimal control is given in feedback form, by*

$$\xi_t^* = \widehat{\xi}(T-t,X_t^{\xi^*},R_t^{\xi^*}), \quad (\mathbb{P}\otimes\lambda)\text{-a.s.}$$

*Remark* 3.3.2. Before we proceed to the proof, let us first make a few remarks.

1. In the special case where the utility function $u$ is a convex combination of exponential utility functions, i.e: $u(x)=\lambda u_1(x)+(1-\lambda)u_2(x)$, with $\lambda\in]0,1[$ and $u_i$ an exponential utility function, for $i=1,2$, we can easily prove the existence of $w$ satisfying (3.35) and the following boundary condition

$$\lim_{t\downarrow 0} w(t,x,r) = \begin{cases} w(0,0,r)=\lambda u_1(r)+(1-\lambda)u_2(r), & \text{if } X=0,\\ -\infty, & \text{otherwise,}\end{cases}$$

on $]0,T]\times\mathbb{R}^d\times\mathbb{R}$. Indeed, by setting $w=\lambda V_1+(1-\lambda)V_2$, where $V_i$ are the corresponding exponential value functions, we can show by using Lemma 3.2.3 that $w$ satisfies (3.35). Since $w$ satisfies also (3.36), we have thus that $w\geq V$. However, since the first inequality invoked in Lemma 3.2.3 is strict in general, this makes (3.35) strict in general, too. And hence (see the following proof of part 1.) $w>V$ in general.

2. Proving the existence (and uniqueness) of a strong solution of (3.42) can be very challenging, since
$$\nabla f^*\Big(\frac{\nabla_x w(t,x,r)}{w_r(t,x,r)}\Big)$$
is at most supposed to be continuous but has *no Lipschitz-continuity* property, due to the quotient term and the fact that $\nabla f^*$ can be superlinear.

3. With formula (3.40), we have a way to compute numerically the optimal liquidation strategy. However, this would require to first compute the gradient of the value function, which is not an easy task, since even computing the value function itself can present some issues, as seen in Chapter 5. Moreover, as mentioned above, the coefficients in the SDE do not satisfy a (locally) Lipschitz condition, and (up to our knowledge) no known converging method can be applied to solve the SDE (3.42).

$\diamondsuit$

*Proof.* To prove part 1, let $\xi \in \dot{\mathcal{X}}^1_{2A_2}(T, X_0)$, $t \in ]0, T[$ and $\tau_k$ be the following stopping time
$$\tau_k := \inf\Big\{ s > 0,\ |w_r\big(T-s, X_s^\xi, \mathcal{R}_s^\xi\big)| > k \Big\} \wedge t.$$
Note that $\tau_k \longrightarrow t$, a.s., when $k \longrightarrow \infty$. By Itô's formula we write:
$$\begin{aligned}
&w(T-\tau_k, X^\xi_{\tau_k}, \mathcal{R}^\xi_{\tau_k}) - w(T, X_0, R_0)\\
&= -\int_0^{\tau_k} w_t(T-s, X_s^\xi, \mathcal{R}_s^\xi)\,ds\\
&\quad + \int_0^{\tau_k} w_r(T-s, X_s^\xi, \mathcal{R}_s^\xi)\,d\mathcal{R}_s^\xi + \int_0^{\tau_k} \nabla_x w(T-s, X_s^\xi, \mathcal{R}_s^\xi)\,dX_s^\xi\\
&\quad + \frac{1}{2}\int_0^{\tau_k} w_{rr}(T-s, X_s^\xi, \mathcal{R}_s^\xi)\,d\langle \mathcal{R}^\xi\rangle_s\\
&= \int_0^{\tau_k} \Big( -w_t(T-s, X_s^\xi, \mathcal{R}_s^\xi) + \mathcal{L}^\xi w(T-s, X_s^\xi, \mathcal{R}_s^\xi)\Big)\,ds\\
&\quad + \int_0^{\tau_k} (X_s^\xi)^\top \sigma w_r(T-s, X_s^\xi, \mathcal{R}_s^\xi)\,dB_s
\end{aligned}$$
The definition of $\tau_k$, in conjunction with the integrability property of $X^\xi$, implies that the stochastic integral $\int_0^{\tau_k} (X_s^\xi)^\top \sigma w_r(T-s, X_s^\xi, \mathcal{R}_s^\xi)\,dB_s$ is a true martingale. Hence, its expectation vanishes, and by taking expectations on both sides we obtain:
$$\begin{aligned}
&\mathbb{E}\Big[w(T-\tau_k, X^\xi_{\tau_k}, \mathcal{R}^\xi_{\tau_k})\Big] - w(T, X_0, R_0)\\
&= \mathbb{E}\Big[\int_0^{\tau_k} \Big( -w_t(T-s, X_s^\xi, \mathcal{R}_s^\xi) + \mathcal{L}^\xi w(T-s, X_s^\xi, \mathcal{R}_s^\xi)\Big)\,ds\Big].
\end{aligned}$$

It follows then with (3.35) that

$$\mathbb{E}\Big[w(T-\tau_k, X^{\xi}_{\tau_k}, \mathcal{R}^{\xi}_{\tau_k})\Big] \le w(T, X_0, R_0). \tag{3.43}$$

In order to take the limit, in $k$, on the left-hand side of the preceding inequality, we need to prove a uniform integrability property of the sequence of random variables $(w(T-\tau_k, X^{\xi}_{\tau_k}, \mathcal{R}^{\xi}_{\tau_k}))$. Since $w$ is bounded from above, it is sufficient to prove a boundedness property of the sequence $(w^{-}(T-\tau_k, X^{\xi}_{\tau_k}, \mathcal{R}^{\xi}_{\tau_k}))$, in $\mathrm{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. To this end, let us write

$$\begin{aligned}
\big(w^{-}(T-\tau_k, X^{\xi}_{\tau_k}, \mathcal{R}^{\xi}_{\tau_k})\big)^2 &\le \big(V_2(T-\tau_k, X^{\xi}_{\tau_k}, \mathcal{R}^{\xi}_{\tau_k})\big)^2 \\
&\le \mathbb{E}\big[\exp(-A_2\mathcal{R}^{\xi}_{T})|\mathcal{F}_{\tau_k}\big]^2 \\
&\le \mathbb{E}\big[\exp(-2A_2\mathcal{R}^{\xi}_{T})|\mathcal{F}_{\tau_k}\big].
\end{aligned}$$

Here, the first inequality is due to (3.34), the second one follows from Lemma 3.1.8, and the last one is obtained with Jensen's inequality. Using the fact that $\xi \in \dot{\mathcal{X}}^1_{2A_2}(T, X_0)$, we have

$$\mathbb{E}\big[\mathbb{E}\big[\exp(-2A\mathcal{R}^{\xi}_{T})|\mathcal{F}_{\tau_k}\big]\big] = \mathbb{E}\big[\exp(-2A\mathcal{R}^{\xi}_{T})\big] \le M_{\mathcal{R}^{\xi*}_{T}}(2A2) + 1$$

and hence, $((w^{-}(T-\tau_k, X^{\xi}_{\tau_k}, \mathcal{R}^{\xi}_{\tau_k}))$ is bounded, in $\mathrm{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. The sequence $(w(T-\tau_k, X^{\xi}_{\tau_k}, \mathcal{R}^{\xi}_{\tau_k}))$ is thus uniformly integrable, and by using Vitali's convergence theorem, we finally get

$$\lim_{k\to\infty} \mathbb{E}\Big[w(T-\tau_k, X^{\xi}_{\tau_k}, \mathcal{R}^{\xi}_{\tau_k})\Big] = \mathbb{E}\Big[w(T-t, X^{\xi}_{t}, \mathcal{R}^{\xi}_{t})\Big] \le w(T, X_0, R_0). \tag{3.44}$$

We wish now to pass to the limit $t \uparrow T$ in the preceding equation. To this end, consider the following sequence of stopping times

$$\sigma_k := \inf\left\{t \ge 0 \Big| (T-t) f\left(\frac{X^{\xi}_t}{T-t}\right) \ge k\right\} \wedge T.$$

Note that $\sigma_k \longrightarrow T$, a.s., when $k$ goes to infinity. We wish to show that

$$\mathbb{E}\left[w(T-\sigma_k, X^{\xi}_{\sigma_k}, \mathcal{R}^{\xi}_{\sigma_k})\mathbb{1}_{\{\sigma_k<T\}}\right] \underset{k\to\infty}{\longrightarrow} 0. \tag{3.45}$$

From (3.34) we have

$$\begin{aligned}
\mathbb{E}\left[V_1(T-\sigma_k, X^{\xi}_{\sigma_k}, \mathcal{R}^{\xi}_{\sigma_k})\mathbb{1}_{\{\sigma_k<T\}}\right] &\ge \mathbb{E}\left[w(T-\sigma_k, X^{\xi}_{\sigma_k}, \mathcal{R}^{\xi}_{\sigma_k})\mathbb{1}_{\{\sigma_k<T\}}\right] \\
&\ge \mathbb{E}\left[V_2(T-\sigma_k, X^{\xi}_{\sigma_k}, \mathcal{R}^{\xi}_{\sigma_k})\mathbb{1}_{\{\sigma_k<T\}}\right].
\end{aligned}$$

It is hence sufficient to show that

$$\mathbb{E}\left[V_i(T-\sigma_k, X^{\xi}_{\sigma_k}, \mathcal{R}^{\xi}_{\sigma_k})\mathbb{1}_{\{\sigma_k<T\}}\right] \longrightarrow 0. \tag{3.46}$$

Now, Lemma 3.1.8 implies

$$\begin{aligned}\mathbb{E}\left[V_i(T-\sigma_k, X^{\xi}_{\sigma_k}, \mathcal{R}^{\xi}_{\sigma_k})\mathbb{1}_{\{\sigma_k<T\}}\right] &\le \mathbb{E}[\mathbb{E}\big[\exp(-A_i\mathcal{R}^{\xi}_T)|\mathcal{F}_{\sigma_k}\big]\mathbb{1}_{\{\sigma_k<T\}}\big]\\ &= \mathbb{E}\big[\exp(-A_i\mathcal{R}^{\xi}_T)\mathbb{1}_{\{\sigma_k<T\}}\big].\end{aligned}$$

By using the Lebesgue dominated convergence theorem, we then get

$$\mathbb{E}\big[\exp(-A_i\mathcal{R}^{\xi}_T)\mathbb{1}_{\{\sigma_k<T\}}\big] \underset{k\to\infty}{\longrightarrow} 0,$$

which proves (3.46). We have on the other hand:

$$\begin{aligned}\mathbb{E}\left[w(T-\sigma_k, X^{\xi}_{\sigma_k}, \mathcal{R}^{\xi}_{\sigma_k})\mathbb{1}_{\{\sigma_k=T\}}\right] &= \mathbb{E}\left[w(0,0,\mathcal{R}^{\xi}_{\sigma_k})\mathbb{1}_{\{\sigma_k=T\}}\right]\\ &\ge \mathbb{E}\left[u(\mathcal{R}^{\xi}_{\sigma_k})\mathbb{1}_{\{\sigma_k=T\}}\right], \qquad \text{due to (3.36)},\\ &= \mathbb{E}\big[u\big(\mathcal{R}^{\xi}_T\big)\big].\end{aligned}$$

Hence, by using (3.44), it follows that

$$\mathbb{E}\left[u(\mathcal{R}^{\xi}_T)\mathbb{1}_{\{\sigma_k=T\}}\right] + \mathbb{E}\left[w(T-\sigma_k, X^{\xi}_{\sigma_k}, \mathcal{R}^{\xi}_{\sigma_k})\mathbb{1}_{\{\sigma_k<T\}}\right] \le w(T, X_0, R_0).$$

Sending now $k$ to infinity, we finally get

$$\mathbb{E}\big[u(\mathcal{R}^{\xi}_T)\big] \le w(T, X_0, R_0).$$

Now, by taking the supremum over $\xi \in \dot{\mathcal{X}}^1_{2A_2}(T, X_0)$ on the left-hand side of the preceding inequality, we obtain

$$V(T, X_0, R_0) \le w(T, X_0, R_0),$$

which proves part 1.

We prove now part 2. Using Remark 3.2.1 together with assumption (3.39), we can rewrite (3.37) in the following way:

$$0 = \Big(-w_t + \frac{x^\top \Sigma x}{2} w_{rr} + b\cdot x\, w_r + \frac{1}{w_r} f^*\Big(\frac{\nabla_x w}{w_r}\Big)\Big)(T-t, x, r).$$

Using Theorem 26.5 in Rockafellar (1997) ($f$ has a superlinear growth, is strictly convex and continuously differentiable on $\mathbb{R}^d$), we have that $(\nabla f)^{-1} = \nabla f^*$ is well-defined and continuous. Hence, when setting

$$\widehat{\xi}(t, x, r) := \nabla f^*\Big(\frac{\nabla_x w(t, x, r)}{w_r(t, x, r)}\Big),$$

we obtain that $\widehat{\xi}$ is also continuous in $t, x$ and $r$ and fulfills (3.41), which proves part 2.i.

To prove part 2. ii, we suppose that there exists a strong solution $(X, \mathcal{R})$ to the following SDE:

$$\begin{cases} d\mathcal{R}_t = (X_t)^\top \sigma dB_t + b \cdot X_t \, dt - f(-\widehat{\xi}(t, X_t, \mathcal{R}_t)) \, dt, \\ dX_t = -\widehat{\xi}(t, X_t, \mathcal{R}_t) \, dt, \\ \mathcal{R}_{|t=0} = R_0 \text{ and } X_{|t=0} = X_0. \end{cases} \tag{3.47}$$

Consider now the following sequence of stopping times

$$\tau_k := \inf \Big\{ s > 0, \ |w_r\big(T - s, X_s, \mathcal{R}_s\big)| > k \Big\} \wedge t.$$

As before, $\tau_k \longrightarrow t$, a.s., when $k \longrightarrow \infty$. By Itô's formula we infer

$$\begin{aligned}
& w(T - \tau_k, X_{\tau_k}, \mathcal{R}_{\tau_k}) - w(T, X_0, R_0) \\
&= - \int_0^{\tau_k} w_t(T - s, X_s, \mathcal{R}_s) \, ds \\
&\quad + \int_0^{\tau_k} w_r(T - s, X_s, \mathcal{R}_s) \, d\mathcal{R}_s + \int_0^{\tau_k} \nabla_x w(T - s, X_s, \mathcal{R}_s) \, dX_s \\
&\quad + \frac{1}{2} \int_0^{\tau_k} w_{rr}(T - s, X_s, \mathcal{R}_s) \, d\langle \mathcal{R} \rangle_s \\
&= \int_0^{\tau_k} \Big( - w_t(T - s, X_s, \mathcal{R}_s) + \mathcal{L}^{\widehat{\xi}} w(T - s, X_s, \mathcal{R}_s) \Big) \, ds \\
&\quad + \int_0^{\tau_k} (X_s)^\top \sigma w_r(T - s, X_s, \mathcal{R}_s) \, dB_s.
\end{aligned}$$

The definition of $\tau_k$ and the integrability condition on $X$ imply that $\int_0^{\tau_k} (X_s)^\top \sigma w_r(T - s, X_s, \mathcal{R}_s) \, dB_s$ is a true martingale. Taking now the expectation on both sides of the preceding equation array yields

$$\begin{aligned}
& \mathbb{E}\big[w(T - \tau_k, X_{\tau_k}, \mathcal{R}_{\tau_k})\big] - w(T, X_0, R_0) \\
&= \mathbb{E}\Big[ \int_0^{\tau_k} \Big( - w_t(T - s, X_s, \mathcal{R}_s) + \mathcal{L}^{\widehat{\xi}} w(T - s, X_s, \mathcal{R}_s) \Big) \, ds \Big],
\end{aligned}$$

and by using (3.41), this gives us

$$\mathbb{E}\big[w(T - \tau_k, X_{\tau_k}, \mathcal{R}_{\tau_k})\big] = w(T, X_0, R_0). \tag{3.48}$$

The same arguments as are used in part 1 (namely, uniform integrability of the sequence of integrands $(w(T - \tau_k, X_{\tau_k}, \mathcal{R}_{\tau_k}))_k$) permit us to send $k$ to infinity, and so we get

$$\mathbb{E}\big[w(T - t, X_t, \mathcal{R}_t)\big] = w(T, X_0, R_0). \tag{3.49}$$

Here again, the same arguments as are used in part 1, permit us to set $t = T$ in the preceding equation. From (3.38), it follows that we must have $X_T = 0$ in order to retrieve

$$V(T, X_0, R_0) \geq \mathbb{E}\Big[ u\big(\mathcal{R}_T\big) \Big]$$

$$= \mathbb{E}\Big[ w(0, 0, \mathcal{R}_T) \Big]$$
$$= w(T, X_0, R_0),$$

where the first equality follows from (3.38), too. Hence, we thus have shown that $w \leq V$. Using the reverse inequality from part 1, we finally get $w = V$. Therefore it follows $(X, \mathcal{R}) = (X^{\xi^*}, \mathcal{R}^{\xi^*})$, due to the uniqueness of the optimal strategy (Theorem 2.2.4). Moreover,

$$\xi_t^* = \widehat{\xi}(T - t, X_t^{\xi^*}, \mathcal{R}_t^{\xi^*}), \quad (\mathbb{P} \otimes \lambda)\text{-a.s.},$$

which concludes the proof. ■

# Chapter 4

# Viscosity solutions of the HJB-equation.

In the previous chapter, we have used the dynamic programming principle to derive some connections between our maximization problem (2.16) and classical solutions of the Hamilton-Jacobi-Bellman equation (3.23). Unfortunately, the preceding method works out only if our value function $V$ is known to be smooth enough, or if there exists a classical solution to (3.23). It is shown in Yong and Zhou (1999), Chapter 4, Example 2.3, that even in the deterministic case, the value function may not be smooth. To overcome this difficulty and in order to include non-smooth functions, we will use now the notion of viscosity solutions. Since our value function is continuous, we will restrict our framework to the class of viscosity solutions that are continuous. A more general definition (i.e., among the class of locally bounded functions) of viscosity solution can be found, for instance, in Fleming and Soner (2006). With this definition, however, a strong comparison principle would imply that $V$ is again continuous.

## 4.1 The value function as viscosity solution of the Hamilton-Jacobi-Bellman equation.

We start by first considering an abstract definition of the notion of viscosity solution, as it can be found, for instance, in Touzi (2013) or Fleming and Soner (2006). We consider a nonlinear second-order degenerate partial differential equation

$$F(T-t, x, r, v(T-t, x, r), v_t(t, x, r), \nabla_x v(t, x, r), v_r(t, x, r), v_{rr}(t, x, r)) = 0, \quad (4.1)$$

where $F$ is a continuous function on $]0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$ taking values in $\mathbb{R}$ with a fixed $T > 0$, for $(t, x, r) \in ]0, T] \times \mathbb{R}^d \times \mathbb{R}$. We need to make the following crucial assumption on $F$.

*Viscosity solutions of the HJB-equation.*

**Assumption 4.1.1** (Ellipticity). For all $(t, x, r, q, p, s, m) \in ]0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ and $a, b \in \mathbb{R}$, it should hold

$$F(T-t, x, r, q, p, s, m, a) \leq F(T-t, x, r, q, p, s, m, b) \quad \text{whenever } a \geq b. \tag{4.2}$$

**Definition 4.1.2.** Let $v : ]0, T] \times \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function.

1. $v$ is called a *viscosity subsolution* of (4.1) if, for every $\varphi \in \mathcal{C}^{1,1,2}(]0, T] \times \mathbb{R}^d \times \mathbb{R})$ and every $(t^*, x^*, r^*) \in [0, T[ \times \mathbb{R}^d \times \mathbb{R}$, whenever $v - \varphi$ attains a local maximum at $(T - t^*, x^*, r^*) \in ]0, T] \times \mathbb{R}^d \times \mathbb{R}$, we have

$$F(., v, \varphi_t, \nabla_x \varphi, \varphi_r, \varphi_{rr})(T - t^*, x^*, r^*) \leq 0. \tag{4.3}$$

2. $v$ is a *viscosity supersolution* of (4.1) if, for every $\varphi \in \mathcal{C}^{1,1,2}(]0, T] \times \mathbb{R}^d \times \mathbb{R})$ and every $(t^*, x^*, r^*) \in [0, T[ \times \mathbb{R}^d \times \mathbb{R}$, whenever $v - \varphi$ attains a local minimum at $(T - t^*, x^*, r^*) \in ]0, T] \times \mathbb{R}^d \times \mathbb{R}$, we have

$$F(., v, \varphi_t, \nabla_x \varphi, \varphi_r, \varphi_{rr})(T - t^*, x^*, r^*) \geq 0. \tag{4.4}$$

3. $v$ is a *viscosity solution* of the equation (4.1) if $v$ is a viscosity subsolution and supersolution of the same equation.

*Remark* 4.1.3. We note the following.

1. The above definition is unchanged if the maximum or minimum point $(T - t^*, x^*, r^*)$ is global and/or strict. See Barles (2013) for further details.

2. We can suppose w.l.o.g. that $v(T - t^*, x^*, r^*) = \varphi(T - t^*, x^*, r^*)$. Otherwise, we can use the function $\psi$ defined as $\psi(T - t, x, r) := \varphi(T - t, x, r) + v(T - t^*, x^*, r^*) - \varphi(T - t^*, x^*, r^*)$.

3. The function $\varphi$ will be called a test function for $v$.

$\diamondsuit$

The following result justifies the introduction of this notion.

**Theorem 4.1.4.** *The value function $V$ is a viscosity solution of the Hamilton-Jacobi-Bellman equation (3.23) with initial condition (3.24).*

The proof is split in two propositions. We first prove that the value function is a viscosity supersolution, then we prove that it is a viscosity subsolution.

**Proposition 4.1.5.** *The value function $V$ is a viscosity supersolution of the Hamilton-Jacobi-Bellman equation (3.23) with initial condition (3.24).*

*Proof.* Using Definition 4.1.2, we wish to show that, for every $\varphi \in \mathcal{C}^{1,1,2}(]0,T] \times \mathbb{R}^d \times \mathbb{R})$ and every $(t^*, x^*, r^*) \in [0, T[\times \mathbb{R}^d \times \mathbb{R}$, whenever $V - \varphi$ attains a local minimum at $(T - t^*, x^*, r^*) \in ]0, T] \times \mathbb{R}^d \times \mathbb{R}$, we have

$$\begin{aligned}
0 &\geq -\varphi_t(T-t^*, x^*, r^*) + \frac{{x^*}^\top \Sigma x^*}{2} \varphi_{rr}(T-t^*, x^*, r^*) + b \cdot x^* \varphi_r(T-t^*, x^*, r^*) \\
&\quad + \sup_{\xi \in \mathbb{R}^d} \big(\xi^\top \nabla_x \varphi(T-t^*, x^*, r^*) - \varphi_r(T-t^*, x^*, r^*) f(\xi)\big) \\
&= \big(-\varphi_t + \sup_{\xi \in \mathbb{R}^d} \mathcal{L}^\xi \varphi\big)(T-t^*, x^*, r^*).
\end{aligned} \tag{4.5}$$

The idea of the proof is almost the same as in the proof of Proposition 3.2.2, but as $V$ is not necessarily smooth, we cannot apply Itô's formula to it now. However, due to the definition of the viscosity supersolution, we can use a test function $\varphi$ instead of $V$ to derive the inequality. We proceed as follows: let $(t^*, x^*, r^*) \in ]0, T] \times \mathbb{R}^d \times \mathbb{R}$ be such that $V - \varphi$ attains a local minimum at $(T - t^*, x^*, r^*)$. Let $\eta \in \mathbb{R}^d$ and $\varepsilon > 0$ such that $t^* + \varepsilon < T$ and define $\xi \in \dot{\mathcal{X}}^1_{A_2}([t^*, T], x)$ in the following way

$$\xi_s := \begin{cases} \eta, & \text{if } s \in [t^*, t^* + \varepsilon[, \\ -\frac{x - \varepsilon\eta}{T - (t^* + \varepsilon)}, & \text{if } s \in [t^* + \varepsilon, T], \end{cases}$$

and consider the corresponding processes $(X^\xi_s, \mathcal{R}^\xi_s)$ which satisfy $X^\xi_{t^*} = x^*, \mathcal{R}^\xi_{t^*} = r^*$. Choose $\alpha > 0$ such that the maximum is global on the region $]T - t^* - \alpha, T - t^* + \alpha] \times B(x^*, \alpha) \times [r^* - \alpha, r^* + \alpha]$, and consider the following sequence of stopping times

$$\tau_k := \inf\left\{ s > t^* \;\middle|\; (s - t^*, X^\xi_s, \mathcal{R}^\xi_s) \notin [0, \frac{1}{k}[\, \times B(x^*, \alpha) \; \times [r^* - \alpha, r^* + \alpha] \right\}.$$

By using Theorem 3.1.1 and taking $k$ large enough, we have

$$\begin{aligned}
0 &\geq \mathbb{E}[V(T-\tau_k, X^\xi_{\tau_k}, \mathcal{R}^\xi_{\tau_k}) - V(T-t^*, x^*, r^*)] \\
&= \mathbb{E}[V(T-\tau_k, X^\xi_{\tau_k}, \mathcal{R}^\xi_{\tau_k}) - \varphi(T-\tau_k, X^\xi_{\tau_k}, \mathcal{R}^\xi_{\tau_k}) \\
&\quad + \varphi(T-\tau_k, X^\xi_{\tau_k}, \mathcal{R}^\xi_{\tau_k}) - \varphi(T-t^*, x^*, r^*) \\
&\quad - \big(V(T-t^*, x^*, r^*) - \varphi(T-t^*, x^*, r^*)\big)] \\
&\geq \mathbb{E}[\varphi(T-\tau_k, X^\xi_{\tau_k}, \mathcal{R}^\xi_{\tau_k}) - \varphi(T-t^*, x^*, r^*)] \\
&= \mathbb{E}\bigg[-\int_{t^*}^{\tau_k} \varphi_t(T-s, X^\xi_s, \mathcal{R}^\xi_s)\, ds + \int_{t^*}^{\tau_k} \varphi_r(T-s, X^\xi_s, \mathcal{R}^\xi_s)\, d\mathcal{R}^\xi_s \\
&\quad + \int_{t^*}^{\tau_k} \nabla_x \varphi(T-s, X^\xi_s, \mathcal{R}^\xi_s)\, dX^\xi_s + \frac{1}{2} \int_{t^*}^{\tau_k} \varphi_{rr}(T-s, X^\xi_s, \mathcal{R}^\xi_s)\, d\langle \mathcal{R}^\xi \rangle_s \bigg] \\
&= \mathbb{E}\bigg[\int_{t^*}^{\tau_k} \Big(-\varphi_t(T-s, X^\xi_s, \mathcal{R}^\xi_s) + \mathcal{L}^\xi \varphi(T-s, X^\xi_s, \mathcal{R}^\xi_s)\Big)\, ds\bigg] \\
&\quad + \mathbb{E}\bigg[\int_{t^*}^{\tau_k} (X^\xi_s)^\top \sigma \varphi_r(T-s, X^\xi_s, \mathcal{R}^\xi_s)\, dB_s\bigg],
\end{aligned}$$

where the first inequality is due to the dynamic programming principle (3.1), and the second one follows from the minimum property of $V-\varphi$ at $(T-t^*,x^*,r^*)$. The second equality is due to Itô's lemma, applied to $\varphi$.

The second expectation in the last identity of the preceding array vanishes, since $\tau_k$ is defined as above and the term inside the corresponding expectation is a true martingale. Hence,

$$\mathbb{E}\Big[\int_{t^*}^{\tau_k}\Big(-\varphi_t(T-s,X_s^\xi,\mathcal{R}_s^\xi)+\mathcal{L}^\xi\varphi(T-s,X_s^\xi,\mathcal{R}_s^\xi)\Big)\,ds\Big]\le 0. \tag{4.6}$$

Moreover, due to the a.s. continuity (in $s$) of the integrands, we have $\tau_k=t+1/k$, for $k$ large enough. Thus, we can use the same arguments as in Proposition 3.2.2, and we get

$$\begin{aligned}\mathbb{E}\Big[k\int_{t^*}^{\tau_k}\Big(-\varphi_t(T-s,X_s^\xi,\mathcal{R}_s^\xi)+\mathcal{L}^\xi\varphi(T-s,X_s^\xi,\mathcal{R}_s^\xi)\Big)\,ds\Big]&\\ \underset{k\to\infty}{\longrightarrow}-\varphi_t(T-t^*,x^*,r^*)+\mathcal{L}^\eta\varphi(T-t^*,x^*,r^*).&\end{aligned}$$

Combining this with the inequality (4.6), we finally have

$$-\varphi_t(T-t^*,x,r)+\mathcal{L}^\eta\varphi(T-t^*,x^*,r^*)\le 0. \tag{4.7}$$

Since we chose $\eta$ arbitrarily, we can now take the supremum over $\eta\in\mathbb{R}^d$, due to the continuity of $\eta\longrightarrow\mathcal{L}^\eta\varphi$, for $\eta\in\mathbb{R}^d$, which gives us

$$\Big(-\varphi_t+\sup_{\eta\in\mathbb{R}^d}\mathcal{L}^\eta\varphi\Big)(T-t^*,x^*,r^*)\le 0.$$

■

**Proposition 4.1.6.** *The value function $V$ is a viscosity subsolution of the Hamilton-Jacobi-Bellman equation* (3.23) *with initial condition* (3.24)*.*

*Proof.* We wish to show that, for every $\varphi\in\mathcal{C}^{1,1,2}(]0,T]\times\mathbb{R}^d\times\mathbb{R})$ and every $(t^*,x^*,r^*)\in[0,T[\,\times\mathbb{R}^d\times\mathbb{R}$, whenever $V-\varphi$ attains a local maximum at $(T-t^*,x^*,r^*)\in\,]0,T]\times\mathbb{R}^d\times\mathbb{R}$, we have

$$\begin{aligned}0\le&-\varphi_t(T-t^*,x^*,r^*)+\frac{{x^*}^\top\Sigma x^*}{2}\varphi_{rr}(T-t^*,x^*,r^*)+b\cdot x^*\varphi_r(T-t^*,x^*,r^*)\\&+\sup_{\xi\in\mathbb{R}^d}\big(\xi^\top\nabla_x\varphi(T-t^*,x^*,r^*)-\varphi_r(T-t^*,x^*,r^*)f(\xi)\big)\\=&\big(-\varphi_t+\sup_{\xi\in\mathbb{R}^d}\mathcal{L}^\xi\varphi\big)(T-t^*,x^*,r^*).\end{aligned}\tag{4.8}$$

We follow here the idea of the proof of Proposition 3.2.4. However, as in the previous proposition, we will apply Itô's formula to the test function $\varphi$ instead of $V$. Let $\varphi\in\mathcal{C}^{1,2,2}(]0,T]\times\mathbb{R}^d\times\mathbb{R})$ and $(T-t^*,x^*,r^*)$ be such that

$$V(T-t^*,x^*,r^*)-\varphi(T-t^*,x^*,r^*)<V(T-t,x,r)-\varphi(T-t,x,r), \tag{4.9}$$

for $(T-t,x,r)$ in a neighborhood of $(T-t^*,x^*,r^*)$, and suppose by way of contradiction to (4.8) that

$$h(t,x,r) := \big(-\varphi_t + \sup_{\xi\in\mathbb{R}^d}\mathcal{L}^\xi\varphi\big)(T-t^*,x^*,r^*) < 0.$$

We will work toward a contradiction. Suppose without loss of generality that the left-hand side of (4.9) is equal to zero, as argued in Remark 4.1.3.

We define the following neighborhood of $(T-t^*,x^*,r^*)$:

$$\mathcal{N}_\eta = \big\{(t,x,r)|(t-t^*,x-x^*,r-r^*) \in ]-\eta,\eta[ \times B(0,\eta)\times ]-\eta,\eta[ \text{ and } h(t,x,r)<0\big\},$$

which is a non-empty set, for $\eta>0$ small enough, because $h$ is continuous. We set

$$2\varepsilon = \max_{\partial\mathcal{N}_\eta}(V-\varphi). \tag{4.10}$$

Note that $\varepsilon>0$ due to (4.9). Because of continuity of $V-\varphi$ and the fact that $V(T-t^*,x^*,r^*)-\varphi(T-t^*,x^*,r^*)=0$, there exists $(T-t_0,x_0,r_0)\in\mathcal{N}_\eta$ such that

$$(\varphi-V)(T-t_0,x_0,r_0)\le -\varepsilon.$$

Take $\xi\in\dot{\mathcal{X}}^1_{2A_2}([t_0,T],X_0)$ and let us introduce the following stopping time

$$\tau := \inf\{s>t_0 \mid (s,X_s^{\xi^\varepsilon},\mathcal{R}_s^\xi)\notin\mathcal{N}_\eta\}.$$

Due to the continuity of the state process, we have $(T-\tau,X_\tau^\xi,\mathcal{R}_\tau^\xi)\in\partial\mathcal{N}_\eta$, which implies that

$$(V-\varphi)(T-\tau,X_\tau^\xi,\mathcal{R}_\tau^\xi)\le 2\varepsilon,$$

due to (4.10). Hence, by using Itô's Lemma,

$$\begin{aligned}
&\mathbb{E}\Big[V\big(T-\tau,X_\tau^\xi,\mathcal{R}_\tau^\xi\big)\Big]-V(T-t_0,x_0,r_0)\\
&= \mathbb{E}\Big[V\big(T-\tau,X_\tau^\xi,\mathcal{R}_\tau^\xi\big)-\varphi\big(T-\tau,X_\tau^\xi,\mathcal{R}_\tau^\xi\big)+\varphi\big(T-\tau,X_\tau^\xi,\mathcal{R}_\tau^\xi\big)\\
&\qquad -\varphi(T-t_0,x_0,r_0)+\varphi(T-t_0,x_0,r_0)-V(T-t_0,x_0,r_0)\Big]\\
&\le 2\varepsilon+\mathbb{E}\Big[\varphi\big(T-\tau,X_\tau^\xi,\mathcal{R}_\tau^\xi\big)-\varphi(T-t_0,x_0,r_0)\Big]-\varepsilon\\
&\le \varepsilon+\mathbb{E}\Big[\int_{t_0}^\tau\Big(-\varphi_t(T-s,X_s^\xi,\mathcal{R}_s^\xi)+\mathcal{L}^\xi\varphi(T-s,X_s^\xi,\mathcal{R}_s^\xi)\Big)\,ds\Big]\\
&\qquad +\mathbb{E}\Big[\int_{t_0}^\tau (X_s^\xi)^\top\sigma\varphi_r(T-s,X_s^\xi,\mathcal{R}_s^\xi)\,dB_s\Big].
\end{aligned}$$

The latter term vanishes, due to the boundedness of the integrand on the stochastic interval $[t_0,\tau]$. Because moreover $\big(-\varphi_t+\mathcal{L}^\xi\varphi\big)(s,X_s^\xi,\mathcal{R}_s^\xi)\le 0$ on $[t_0,\tau]$, we have:

$$V(T-t_0,x_0,r_0)\ge -\varepsilon+\mathbb{E}\Big[V\big(T-\tau,X_\tau^\xi,\mathcal{R}_\tau^\xi\big)-\int_{t_0}^\tau\Big(-\varphi_t(T-s,X_s^\xi,\mathcal{R}_s^\xi)$$

$$+\mathcal{L}^{\xi}\varphi(T-s,X_s^{\xi},\mathcal{R}_s^{\xi})\Big)\,ds\Big]$$
$$\geq -\varepsilon+\mathbb{E}\big[V\big(T-\tau,X_{\tau}^{\xi},\mathcal{R}_{\tau}^{\xi}\big)\big].$$

By taking the supremum over $\xi$ on the right-hand side and using Theorem 3.1.1, we infer (since $\varepsilon$ does not depend on $\xi$)

$$V(T-t_0,x_0,r_0)\geq -\varepsilon+\sup_{\xi\in\dot{\mathcal{X}}^1_{2A_2}(T,X_0)}\mathbb{E}\big[V\big(T-\tau,X_{\tau}^{\xi},\mathcal{R}_{\tau}^{\xi}\big)\big]=-\varepsilon+V(T-t_0,x_0,r_0),$$

which is in contradiction to $\varepsilon>0$. Therefore we have the following inequality:

$$\Big(-\varphi_t+\sup_{\eta\in\mathbb{R}^d}\mathcal{L}^{\eta}\varphi\Big)(T-t^*,x^*,r^*)\geq 0,$$

which proves that $V$ is a viscosity subsolution of (3.23). ■

*Proof of Theorem 4.1.4.* Proposition 4.1.5 and 4.1.6 show that $V$ is a viscosity solution of (3.23). We already have shown in Proposition 2.2.3 that the value function $V$ fulfills the initial condition (3.24). Thus, Theorem 4.1.4 is proved. ■

## 4.2 Comparison principles and uniqueness results

In order to prove that our value function is the *unique* viscosity solution of (3.23) with initial condition (3.24), it will be convenient to add a term linear in $V$, in our initial HJB equation. We begin by defining classical solutions to this transformed equation and, in a second step, we will show that one may consider w.l.o.g. the HJB equation in this useful form:

$$\Big(-V_t+\beta V+\sup_{\xi\in\mathbb{R}^d}\mathcal{L}^{\xi}V\Big)(T-t,x,r)=0, \tag{4.11}$$

where $\beta<0$ and $(T-t,x,r)\in\,]0,T]\times\mathbb{R}^d\times\mathbb{R}$.

**Definition 4.2.1.** A function $U$ (resp., $V$) $\in\mathcal{C}^{1,1,2}(]0,T]\times\mathbb{R}^d\times\mathbb{R})$ is called a subsolution (resp., supersolution) of (4.11) if $U$ (resp., $V$) fulfills the following inequality:

$$0\leq\Big(-U_t+\beta U+\sup_{\xi\in\mathbb{R}^d}\mathcal{L}^{\xi}U\Big)(T-t,x,r),$$
$$\Big(\text{resp., } 0\geq\Big(-V_t+\beta V+\sup_{\xi\in\mathbb{R}^d}\mathcal{L}^{\xi}V\Big)(T-t,x,r),\Big)$$

for all $(t,x,r)\in[0,T[\times\mathbb{R}^d\times\mathbb{R}$.

**Lemma 4.2.2.** *Assume that $U$ (resp., $V$) $\in\mathcal{C}^{1,1,2}(]0,T]\times\mathbb{R}^d\times\mathbb{R})$ is a subsolution (resp., supersolution) of (3.23). Then, $\overline{U}(T-t,x,r):=\exp(\beta(T-t))U(T-t,x,r)$ (resp., $\overline{V}(T-t,x,r):=\exp(\beta(T-t)V(T-t,x,r))$ is a subsolution (resp., supersolution) of (4.11).*

*Proof.* Consider the case where $U$ is a subsolution of (3.23). A straightforward calculation yields then:

$$
\begin{aligned}
&\Big(-\overline{U}_t + \beta\overline{U} + \sup_{\xi\in\mathbb{R}^d}\mathcal{L}^\xi\overline{U}\Big)(T-t,x,r)\\
&= -\beta\overline{U} - \exp(\beta(T-t)U_t + \beta\overline{U}\\
&\quad + \exp(\beta(T-t))\Big(\frac{X^\top\Sigma X}{2}U_{rr} + b\cdot XU_r + \sup_{\xi\in\mathbb{R}^d}(\xi\cdot\nabla_x U - f(\xi)U_r)\Big)\\
&= \exp(\beta(T-t))\Big(-U_t + \frac{X^\top\Sigma X}{2}U_{rr} + b\cdot XU_r + \sup_{\xi\in\mathbb{R}^d}(\xi\cdot\nabla_x U - f(\xi)U_r)\Big)\\
&\geq 0,
\end{aligned}
$$

and $\overline{U}$ is thus a subsolution of (4.11). In the same way, we can show that if $V$ is a supersolution of (3.23), $\overline{V}$ is a supersolution of (4.11). ∎

### 4.2.1 Classical comparison principle

In the classical case, we show that the comparison principle is, fortunately, a straightforward application of the verification theorem.

**Theorem 4.2.3** (Comparison principle)**.** *Let $U, V \in \mathcal{C}^{1,1,2}(]0,T]\times\mathbb{R}^d\times\mathbb{R})$ be such that $U$ is a subsolution to (4.11) and $V$ is a supersolution to (4.11), both satisfying the growth condition*

$$V_2(T-t,x,r) \leq w(T-t,x,r) \leq V_1(T-t,x,r), \tag{4.12}$$

*where $w$ can be chosen to be either $U$ or $V$. We suppose that $U$ and $V$ satisfy the boundary condition*

$$\limsup_{t\to 0} U(t,x,r) - V(t,x,r) \leq 0, \qquad \textit{for fixed } x,r\in\mathbb{R}^d\times\mathbb{R}. \tag{4.13}$$

*Then $U \leq V$ on $]0,T]\times\mathbb{R}^d\times\mathbb{R}$.*

*Proof.* Denote $\tilde{u}(r) := V(0,0,r)$. Since $V$ has a strictly positive partial derivative in $r$ (see Remark (3.2.1)), using (4.12) at $(0,0,r)$ it follows that $\tilde{u}$ is a utility function which lies between two exponential utility functions. Denote by $\widetilde{V}$ the corresponding value function of the maximization problem (2.16) generated by this utility function. We are now in the setting of the verification theorem 3.3.1, which gives us that $\widetilde{V} \leq V$. Since $\tilde{u}(r) \geq U(0,0,r)$, using (4.13), we can apply here again the verification theorem 3.3.1 (part 1. can also be proved with a reverse inequality in (3.35)). Therefore, it follows that $\widetilde{V} \geq U$. Hence, our theorem is proved. ∎

*Remark* 4.2.4. In the classical case, the common argument which consists in penalizing the supersolution and then working toward a contradiction, as it can be found in, e.g., Pham (2009) for the polynomial case, does not seem to work here, even after several attempts. Indeed, if we followed the idea of the previously mentioned work, we would be looking for a function $\varphi$ such that for every $\varepsilon > 0$, $U$ subsolution and $V$ supersolution, we have

$$\lim_{|x|,|r|\to\infty} \sup_{[0,T[}(U - V_\varepsilon)(T-t,x,r) \leq 0, \qquad \text{for all } \varepsilon > 0, \tag{4.14}$$

where $V_\varepsilon = \varepsilon\varphi + V$ has to be a supersolution. However, $(V_\varepsilon)_r$ has to be strictly positive, since otherwise it cannot be a supersolution, because then $\sup_{\xi\in\mathbb{R}^d}(\xi\cdot\nabla_x\varphi - f(\xi)\varphi_r) = \infty$. But this seems to be difficult (even impossible) to obtain, due to (4.14) (e.g., for fixed $x$ and $r$ converging to both $-\infty$ and $\infty$) and the growth condition satisfied by $U$ and $V$ (recalling that $V_i$ verify inequality (2.18)). $\diamondsuit$

### 4.2.2 Strong comparison principle for viscosity solutions

We wish to prove now a strong comparison principle for viscosity solutions. Since our value function is known to be continuous, we can restrict this comparison principle to functions which are continuous (i.e., we do not involve here definitions of lower or upper semi-continuous functions). There are several comparison principles for unbounded viscosity solutions; let us mention the one of Koike and Ley (2011), which states a comparison principle for nonlinear degenerate parabolic equations. Nevertheless, this cannot be applied here, since the requirements (13), (14) and (15) in Koike and Ley (2011) cannot be fulfilled in our case, again due to the term $\sup_{\xi\in\mathbb{R}^d}(\xi\cdot\nabla_x V - f(\xi)V_r) = V_r f^*(\nabla_x V/V_r)$.

In order to prove the strong comparison principle theorem, we need to introduce an equivalent definition of a viscosity solution with the help of the notion of *subjet* and *superjet*, as it can be found in, e.g., Pham (2009).

**Definition 4.2.5.** Let $U$ be a continuous function on $]0,T]\times\mathbb{R}^d\times\mathbb{R}$. The *second-order superjet* of a function $U$ at a point $(t^*,x^*,r^*) \in [0,T[\times\mathbb{R}^d\times\mathbb{R}$ is the set $\mathcal{J}^{2,+}U(T-t^*,x^*,r^*)$ of elements $(\bar{q},\bar{p},\bar{s},\bar{m}) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}$ satisfying

$$\begin{aligned} U(T-t,x,r) \leq{}& U(T-t^*,x^*,r^*) + \bar{q}(t-t^*) + \bar{p}\cdot(x-x^*) + \bar{s}(r-r^*) \\ &+ \frac{1}{2}\bar{m}(r-r^*)^2 + o(|t-t^*| + |x-x^*| + |r-r^*|^2). \end{aligned} \tag{4.15}$$

We define similarly the *second-order subjet* of a continuous function $V$, defined on $]0,T]\times\mathbb{R}^d\times\mathbb{R}$, at a point $(t^*,x^*,r^*) \in [0,T[\times\mathbb{R}^d\times\mathbb{R}$ : this is the set of elements $(\bar{q},\bar{p},\bar{s},\bar{m}) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}$ satisfying

$$\begin{aligned} V(T-t,x,r) \geq{}& V(T-t^*,x^*,r^*) + \bar{q}(t-t^*) + \bar{p}\cdot(x-x^*) + \bar{s}(r-r^*) \\ &+ \frac{1}{2}\bar{m}(r-r^*)^2 + o(|t-t^*| + |x-x^*| + |r-r^*|^2). \end{aligned} \tag{4.16}$$

We denote this set by $\mathcal{J}^{2,-}V(T-t^*,x^*,r^*)$.

*Remark* 4.2.6. Let $(t^*, x^*, r^*) \in [0, T[\times\mathbb{R}^d \times \mathbb{R}$ be a local minimum point of $(V - \varphi)(T - t, x, r)$, where $\varphi \in \mathcal{C}^{1,1,2}(]0, T] \times \mathbb{R}^d \times \mathbb{R})$. Then, a second-order Taylor expansion of $\varphi$ yields:

$$\begin{aligned}
&V(T-t,x,r) \geq V(T-t^*,x^*,r^*) - \varphi(T-t^*,x^*,r^*) + \varphi(T-t,x,r)\\
&= V(T-t^*,x^*,r^*) - \varphi_t(T-t^*,x^*,r^*)(t-t^*) + \nabla_x\varphi(T-t^*,x^*,r^*)(x-x^*)\\
&\quad +\varphi_r(T-t^*,x^*,r^*)(r-r^*) + \frac{1}{2}\varphi_{rr}(T-t^*,x^*,r^*)(r-r^*)^2\\
&\quad +o(|t-t^*| + |x-x^*| + |r-r^*|^2),
\end{aligned} \tag{4.17}$$

which implies that

$$(-\varphi_t, \nabla_x\varphi_x, \varphi_r, \varphi_{rr})(T-t^*,x^*,r^*) \in \mathcal{J}^{2,-}V(T-t^*,x^*,r^*). \tag{4.18}$$

In the same manner for $U$, we consider $(t^*, x^*, r^*) \in [0, T[\times\mathbb{R}^d \times \mathbb{R}$ to be a local maximum point of $(U - \varphi)(T - t, x, r)$, where $\varphi \in \mathcal{C}^{1,1,2}(]0, T] \times \mathbb{R}^d \times \mathbb{R})$. Then,

$$\begin{aligned}
&U(T-t,x,r) \leq U(T-t^*,x^*,r^*) + \varphi(T-t,x,r) - \varphi(T-t^*,x^*,r^*)\\
&= U(T-t^*,x^*,r^*) - \varphi_t(T-t^*,x^*,r^*)(t-t^*) + \nabla_x\varphi(T-t^*,x^*,r^*)(x-x^*)\\
&\quad +\varphi_r(T-t^*,x^*,r^*)(r-r^*) + \frac{1}{2}\varphi_{rr}(T-t^*,x^*,r^*)(r-r^*)^2\\
&\quad +o(|t-t^*| + |x-x^*| + |r-r^*|^2),
\end{aligned} \tag{4.19}$$

implying

$$(-\varphi_t, \nabla_x\varphi_x, \varphi_r, \varphi_{rr})(T-t^*,x^*,r^*) \in \mathcal{J}^{2,+}U(T-t^*,x^*,r^*). \tag{4.20}$$

Actually, the converse property also holds: for any $(q, p, s, m) \in \mathcal{J}^{2,+}U(T-t^*, x^*, r^*)$, there exists $\varphi \in \mathcal{C}^{1,1,2}(]0, T] \times \mathbb{R}^d \times \mathbb{R})$ such that

$$(-\varphi_t, \nabla_x\varphi_x, \varphi_r, \varphi_{rr})(T-t^*,x^*,r^*) = (q,p,s,m).$$

See Lemma 4.1 in Fleming and Soner (2006) for a construction of such a $\varphi$. ◇

We can now state the alternative definition of a viscosity solution of equation (4.11).

**Lemma 4.2.7.** *Let $v$ be a continuous function on $]0, T] \times \mathbb{R}^d \times \mathbb{R}$.*

1. *$v$ is a viscosity subsolution of (4.11) on $]0, T] \times \mathbb{R}^d \times \mathbb{R}$ if and only if for all $(t, x, r) \in [0, T[\times\mathbb{R}^d \times \mathbb{R}$ and all $(q, p, s, m) \in \mathcal{J}^{2,+}v(T - t, x, r)$ we have*

$$0 \leq q + \beta v(T-t,x,r) + \frac{x^\top \Sigma x}{2} m + b \cdot x\, s + \sup_{\xi\in\mathbb{R}^d} \left(\xi^\top p - s f(\xi)\right). \tag{4.21}$$

2. *$v$ is a viscosity supersolution of (4.11) on $]0, T] \times \mathbb{R}^d \times \mathbb{R}$ if and only if for all $(t, x, r) \in [0, T[\times\mathbb{R}^d \times \mathbb{R}$ and all $(q, p, s, m) \in \mathcal{J}^{2,-}v(T - t, x, r)$ we have*

$$0 \geq q + \beta v(T-t,x,r) + \frac{x^\top \Sigma x}{2} m + b \cdot x\, s + \sup_{\xi\in\mathbb{R}^d} \left(\xi^\top p - s f(\xi)\right). \tag{4.22}$$

*Proof.* We prove only *1*, noting that *2* can be proved similarly. Suppose that $v$ fulfills the inequality (4.21) for all $(t,x,r) \in [0,T[\times\mathbb{R}^d\times\mathbb{R}$ and all $(q,p,s,m) \in \mathcal{J}^{2,+}v(T-t,x,r)$. Take now $\varphi \in \mathcal{C}^{1,1,2}(]0,T]\times\mathbb{R}^d\times\mathbb{R})$ and consider $(t^*,x^*,r^*) \in [0,T[\times\mathbb{R}^d\times\mathbb{R}$ a local maximum point of $(V-\varphi)(T-t,x,r)$. Due to Remark 4.2.6 and relation (4.17), $\varphi$ fulfills (4.3), which implies that $v$ is a viscosity subsolution.

Suppose now that $v$ is a viscosity subsolution and let $(q,p,s,m) \in \mathcal{J}^{2,+}v(T-t^*,x^*,r^*)$. As mentioned above, at the end of Remark 4.2.6, there exists $\varphi \in \mathcal{C}^{1,1,2}(]0,T]\times\mathbb{R}^d\times\mathbb{R})$ such that

$$(-\varphi_t, \nabla_x\varphi_x, \varphi_r, \varphi_{rr})(T-t^*,x^*,r^*) = (q,p,s,m).$$

By using (4.16) together with (4.17), we obtain that $(T-t^*,x^*,r^*)$ is a local maximum of $v-\varphi$. Thus $\varphi$ fulfills (4.3), which proves that $(q,p,s,m)$ fulfills (4.21). ■

We can now state and prove a strong comparison principle. The first part of the following proof is similar to what can be found in Pham (2009). However, some adaptations have to be made because of growth and boundary conditions: in fact, since we can use the local definition of a viscosity solution, and since the functions in question are continuous, we do not need to penalize the supersolution. In particular, we do not need to use the Crandall-Ishii lemma in the last part of our proof. Indeed, in our HJB equation the term with the second derivative is only one-dimensional. Therefore, matters simplify and we just need to apply the Taylor formula to find adequate elements of the sub- and superjet of $U$ and $V$, respectively, to work toward a contradiction.

**Theorem 4.2.8.** *Let $U$ (resp., $V$) be a continuous viscosity subsolution (resp., continuous viscosity supersolution) of (4.11), defined on $]0,T]\times\mathbb{R}^d\times\mathbb{R}$, satisfying the growth condition*

$$V_2(t,x,r) \le v(t,x,r) \le V_1(t,x,r), \quad \textit{for all } (t,x,r) \in ]0,T]\times\mathbb{R}^d\times\mathbb{R}, \tag{4.23}$$

*where $v$ can be chosen to be $U$ or $V$. We suppose that $U$ and $V$ satisfy the boundary condition*

$$\limsup_{t\to 0}\big(U(t,x,r) - V(t,x,r)\big) \le 0, \quad \textit{for fixed } x,r \in \mathbb{R}^d\times\mathbb{R}, \tag{4.24}$$

*Then $U \le V$ on $]0,T]\times\mathbb{R}^d\times\mathbb{R}$.*

*Proof.* Suppose that (4.24) is true and assume by way of contradiction that there exists $(t^*,x^*,r^*) \in [0,T[\,\times\mathbb{R}^d\times\mathbb{R}$ such that $(U-V)(T-t^*,x^*,r^*) > 0$. Since $U-V$ is continuous on $]0,T]\times\mathbb{R}^d\times\mathbb{R}$, we can suppose w.l.o.g that the supremum of $U-V$ on a compact subset is attained at some $(T-t^*,x^*,r^*)$, i.e.,

$$\bar{m} = \sup_{K\subset[0,T[\times\mathbb{R}^d\times\mathbb{R}} (U-V)(T-t,x,r) = (U-V)(T-t^*,x^*,r^*) > 0, \tag{4.25}$$

where $K$ is compact with non-empty interior. We use now the doubling of variables technique, developed first by Kružkov (1970), and consider for any $\varepsilon > 0$ the functions

$$\Phi_\varepsilon(t,t',x,x',r,r') := U(t,x,r) - V(t',x',r') - \varphi_\varepsilon(t,t',x,x',r,r'), \tag{4.26}$$

$$\varphi_\varepsilon(t,t',x,x',r,r') := \frac{1}{\varepsilon}\big(|t-t'|^2 + |x-x'|^2 + |r-r'|^2\big). \tag{4.27}$$

Let $[0,\eta]\times\overline{B}(0,r)\times[r^*-\alpha,r^*+\alpha]\subset K$ be a compact neighborhood of $(t^*,x^*,r^*)$, where $0<\eta<T$, $0<\alpha<r^*$ and $r>0$. Then, on the compact neighborhood $[0,\eta]^2\times\overline{B}(0,r)^2\times[r^*-\alpha,r^*+\alpha]^2$, the continuous function $\Phi_\varepsilon$ attains its maximum, denoted by $m_\varepsilon$, at some $(T-t_\varepsilon,T-t'_\varepsilon,x_\varepsilon,x'_\varepsilon,r_\varepsilon,r'_\varepsilon)$. We show that

$$m_{\varepsilon_n}\to\bar{m} \quad\text{and}\quad \varphi(T-t_{\varepsilon_n},T-t'_{\varepsilon_n},x_{\varepsilon_n},x'_{\varepsilon_n},r_{\varepsilon_n},r'_{\varepsilon_n})\to 0, \tag{4.28}$$

for some sequence $(\varepsilon_n)$ with $\varepsilon_n\to 0$. First note that

$$\begin{aligned}
\bar{m} &= \Phi_\varepsilon(T-t^*,T-t^*,x^*,x^*,r^*,r^*)\\
&= (U-V)(T-t^*,x^*,r^*) - \varphi_\varepsilon(T-t^*,T-t^*,x^*,x^*,r^*,r^*)\\
&\le U(T-t_\varepsilon,x_\varepsilon,r_\varepsilon) - V(T-t'_\varepsilon,x'_\varepsilon,r'_\varepsilon) - \varphi_\varepsilon(T-t_\varepsilon,T-t'_\varepsilon,x_\varepsilon,x'_\varepsilon,r_\varepsilon,r'_\varepsilon) \qquad (4.29)\\
&= m_\varepsilon\\
&\le U(T-t_\varepsilon,x_\varepsilon,r_\varepsilon) - V(T-t'_\varepsilon,x'_\varepsilon,r'_\varepsilon). \qquad (4.30)
\end{aligned}$$

Since $((T-t_\varepsilon,T-t'_\varepsilon,x_\varepsilon,x'_\varepsilon,r_\varepsilon,r'_\varepsilon))_{\varepsilon>0}$ belongs to the compact set $[0,\eta]^2\times\overline{B}(0,r)^2\times[r^*-\alpha,r^*+\alpha]^2$, we can find a sequence $(T-t_{\varepsilon_n},T-t'_{\varepsilon_n},x_{\varepsilon_n},x'_{\varepsilon_n},r_{\varepsilon_n},r'_{\varepsilon_n})$, where $\varepsilon_n\downarrow 0$, which converges to some $(T-\tilde{t},T-\tilde{t}',\tilde{x},\tilde{x}',\tilde{r},\tilde{r}')$, as $n\to\infty$.

The boundedness of the sequence $(U(T-t_{\varepsilon_n},x_{\varepsilon_n},r_{\varepsilon_n})-V(T-t'_{\varepsilon_n},x'_{\varepsilon_n},r'_{\varepsilon_n}))_n$ implies that $\big(\varphi_{\varepsilon_n}(T-t_{\varepsilon_n},T-t'_{\varepsilon_n},x_{\varepsilon_n},x'_{\varepsilon_n},r_{\varepsilon_n},r'_{\varepsilon_n})\big)_n$ is also bounded (from above), due to inequality (4.29). As $n$ goes to $\infty$, $\varepsilon_n$ tends to 0. Therefore, by using (4.27), we must have that

$$T-\tilde{t}=T-\tilde{t}',\ \tilde{x}=\tilde{x}',\ \tilde{r}=\tilde{r}',$$

and

$$\bar{m}=U(T-\tilde{t},\tilde{x},\tilde{r})-V(T-\tilde{t},\tilde{x},\tilde{r}),$$

using inequality (4.30) and the definition of $\bar{m}$. We can therefore suppose w.l.o.g. that $\tilde{t}=t^*,\tilde{x}=x^*,\tilde{r}=r^*$. Letting $\varepsilon_n$ go to 0 in (4.30), we get

$$\bar{m}\le\lim_{n\to\infty}m_{\varepsilon_n}\le(U-V)(T-t^*,x^*,r^*)=\bar{m},$$

and thus (4.28) is proved.
Further, we have that $\varphi_\varepsilon\in\mathcal{C}^{1,1,2}([0,T]\times\mathbb{R}^d\times\mathbb{R})$ and

$(T-t_\varepsilon,x_\varepsilon,r_\varepsilon)$ is a local maximum of

$$(t,x,r) \to U(T-t,x,r) - \varphi_\varepsilon(T-t,T-t'_\varepsilon,x,x'_\varepsilon,r,r'_\varepsilon), \qquad (4.31)$$

resp.,

$$\begin{aligned}&(T-t'_\varepsilon,x'_\varepsilon,r'_\varepsilon) \text{ is a local minimum of}\\ &\qquad (t',x',r') \to V(T-t',x',r') + \varphi_\varepsilon(T-t_\varepsilon,T-t',x_\varepsilon,x',r_\varepsilon,r'). \quad (4.32)\end{aligned}$$

Indeed, we write on the neighborhood $[0,\eta] \times \overline{B}(0,r) \times [r^*-\alpha,r^*+\alpha]$ of $(T-t^*,x^*,r^*)$:

$$\begin{aligned}
&U(T-t_\varepsilon,x_\varepsilon,r_\varepsilon) - \varphi_\varepsilon(T-t_\varepsilon,T-t'_\varepsilon,x_\varepsilon,x'_\varepsilon,r_\varepsilon,r'_\varepsilon) - (U(T-t,x,r)\\
&\quad -\varphi_\varepsilon(T-t,T-t'_\varepsilon,x,x'_\varepsilon,r,r'_\varepsilon))\\
&= U(T-t_\varepsilon,x_\varepsilon,r_\varepsilon) - V(T-t'_\varepsilon,x'_\varepsilon,r'_\varepsilon) - \varphi_\varepsilon(T-t_\varepsilon,T-t'_\varepsilon,x_\varepsilon,x'_\varepsilon,r_\varepsilon,r'_\varepsilon)\\
&\quad +V(T-t'_\varepsilon,x'_\varepsilon,r'_\varepsilon) - (U(T-t,x,r) - \varphi_\varepsilon(T-t,T-t'_\varepsilon,x,x'_\varepsilon,r,r'_\varepsilon))\\
&= m_\varepsilon - (U(T-t,x,r) - V(T-t'_\varepsilon,x'_\varepsilon,r'_\varepsilon) - \varphi_\varepsilon(T-t,T-t'_\varepsilon,x,x'_\varepsilon,r,r'_\varepsilon))\\
&\geq 0,
\end{aligned}$$

due to the definition of $m_\varepsilon$. Thus, (4.31) follows. In the same manner we can prove (4.32). Our purpose now is to use formulas (4.19) and (4.17) in order to find an adequate element of $\mathcal{J}^{2,+}U(T-t_\varepsilon,x_\varepsilon,r_\varepsilon)$ and of $\mathcal{J}^{2,-}V(T-t'_\varepsilon,x'_\varepsilon,r'_\varepsilon)$ to conclude. To this end, we compute the following derivatives of $(t,x,r) \mapsto \varphi_\varepsilon(T-t,T-t'_\varepsilon,x,x'_\varepsilon,r,r'_\varepsilon)$ at $(T-t_\varepsilon,x_\varepsilon,r_\varepsilon)$:

$$\begin{aligned}
(\varphi_\varepsilon)_t(T-t_\varepsilon,T-t'_\varepsilon,x_\varepsilon,x'_\varepsilon,r_\varepsilon,r'_\varepsilon) &= \frac{2}{\varepsilon}(t_\varepsilon - t'_\varepsilon),\\
(\varphi_\varepsilon)_r(T-t_\varepsilon,T-t'_\varepsilon,x_\varepsilon,x'_\varepsilon,r_\varepsilon,r'_\varepsilon) &= \frac{2}{\varepsilon}(r_\varepsilon - r'_\varepsilon),\\
\nabla_x(\varphi_\varepsilon)(T-t_\varepsilon,T-t'_\varepsilon,x_\varepsilon,x'_\varepsilon,r_\varepsilon,r'_\varepsilon) &= \frac{2}{\varepsilon}(x_\varepsilon - x'_\varepsilon)\\
(\varphi_\varepsilon)_{rr}(T-t_\varepsilon,T-t'_\varepsilon,x_\varepsilon,x'_\varepsilon,r_\varepsilon,r'_\varepsilon) &= \frac{2}{\varepsilon}.
\end{aligned}$$

Because $\frac{r_\varepsilon - r^*}{\varepsilon}, \frac{r'_\varepsilon - r^*}{\varepsilon} \to 0$, as $\varepsilon$ goes to 0, due to (4.28), we can choose a neighborhood $[0,\eta]\times\overline{B}(0,r)\times[r^*-\alpha_\varepsilon,r^*+\alpha_\varepsilon]$ of $(t^*,x^*,r^*)$, such that $\frac{\alpha_\varepsilon}{\varepsilon} \to 0$, as $\varepsilon$ goes to 0. Using this and (4.31), and inserting the derivatives of $(t,x,r) \mapsto \varphi_\varepsilon(T-t,T-t'_\varepsilon,x,x'_\varepsilon,r,r'_\varepsilon)$ at $(T-t_\varepsilon,x_\varepsilon,r_\varepsilon)$ in (4.19), we can compute:

$$\begin{aligned}
&U(T-t,x,r) - U(T-t_\varepsilon,x_\varepsilon,r_\varepsilon)\\
&\leq -\varphi_\varepsilon(T-t_\varepsilon,T-t'_\varepsilon,x_\varepsilon,x'_\varepsilon,r_\varepsilon,r'_\varepsilon) + \varphi_\varepsilon(T-t,T-t'_\varepsilon,x,x'_\varepsilon,r,r'_\varepsilon)\\
&= -(\varphi_\varepsilon)_t(T-t_\varepsilon,T-t'_\varepsilon,x_\varepsilon,x'_\varepsilon,r_\varepsilon,r'_\varepsilon)(t-t_\varepsilon)\\
&\quad + \nabla_x(\varphi_\varepsilon)(T-t_\varepsilon,T-t'_\varepsilon,x_\varepsilon,x'_\varepsilon,r_\varepsilon,r'_\varepsilon)(x-x_\varepsilon)\\
&\quad + (\varphi_\varepsilon)_r(T-t_\varepsilon,T-t'_\varepsilon,x_\varepsilon,x'_\varepsilon,r_\varepsilon,r'_\varepsilon)(r-r_\varepsilon)\\
&\quad + \int_{r_\varepsilon}^{r} \frac{(r-s)^2}{2}(\varphi_\varepsilon)_{rr}(T-t_\varepsilon,T-t'_\varepsilon,x_\varepsilon,x'_\varepsilon,s,r'_\varepsilon)\,ds
\end{aligned}$$

$$
\begin{aligned}
&+o(|t-t_\varepsilon|+|x-x_\varepsilon|)\\
&=-\frac{2}{\varepsilon}(t_\varepsilon-t'_\varepsilon)(t-t_\varepsilon)+\frac{2}{\varepsilon}(x_\varepsilon-x'_\varepsilon)(x-x_\varepsilon)+\frac{2}{\varepsilon}(r_\varepsilon-r'_\varepsilon)(r-r_\varepsilon)\\
&\quad+\int_{r_\varepsilon}^{r}\frac{(r-s)^2}{2}\frac{2}{\varepsilon}\,ds+o(|t-t_\varepsilon|+|x-x_\varepsilon|)\\
&=-\frac{2}{\varepsilon}(t_\varepsilon-t'_\varepsilon)(t-t_\varepsilon)+\frac{2}{\varepsilon}(x_\varepsilon-x'_\varepsilon)(x-x_\varepsilon)+\frac{2}{\varepsilon}(r_\varepsilon-r'_\varepsilon)(r-r_\varepsilon)\\
&\quad-\frac{1}{3\varepsilon}(r-r_\varepsilon)(r-r_\varepsilon)^2+o(|t-t_\varepsilon|+|x-x_\varepsilon|)\\
&\le-\frac{2}{\varepsilon}(t_\varepsilon-t'_\varepsilon)(t-t_\varepsilon)+\frac{2}{\varepsilon}(x_\varepsilon-x'_\varepsilon)(x-x_\varepsilon)+\frac{2}{\varepsilon}(r_\varepsilon-r'_\varepsilon)(r-r_\varepsilon)+\frac{2\alpha_\varepsilon}{3\varepsilon}(r-r_\varepsilon)^2\\
&\quad+o(|t-t_\varepsilon|+|x-x_\varepsilon|+|r-r_\varepsilon|^2).
\end{aligned}
$$

Hence, using Remark 4.2.6 we have proved that

$$
\Big(-\frac{2}{\varepsilon}(t_\varepsilon-t'_\varepsilon),\frac{2}{\varepsilon}(x_\varepsilon-x'_\varepsilon),\frac{2}{\varepsilon}(r_\varepsilon-r'_\varepsilon),\frac{2\alpha_\varepsilon}{3\varepsilon}\Big)\in\mathcal{J}^{2,+}U(T-t_\varepsilon,x_\varepsilon,r_\varepsilon). \tag{4.33}
$$

Further, we look for an adequate element of $\mathcal{J}^{2,-}V(T-t'_\varepsilon,x'_\varepsilon,r'_\varepsilon)$. To this end, as before, we compute the following derivatives of $(t',x',r')\mapsto\varphi_\varepsilon(T-t_\varepsilon,T-t',x_\varepsilon,x',r_\varepsilon,r')$ at $(T-t'_\varepsilon,x'_\varepsilon,r'_\varepsilon)$:

$$
\begin{aligned}
(\varphi_\varepsilon)_{t'}(T-t_\varepsilon,T-t'_\varepsilon,x_\varepsilon,x'_\varepsilon,r_\varepsilon,r'_\varepsilon)&=\frac{2}{\varepsilon}(t'_\varepsilon-t_\varepsilon),\\
(\varphi_\varepsilon)_{r'}(T-t_\varepsilon,T-t'_\varepsilon,x_\varepsilon,x'_\varepsilon,r_\varepsilon,r'_\varepsilon)&=\frac{2}{\varepsilon}(r'_\varepsilon-r_\varepsilon),\\
\nabla_{x'}(\varphi_\varepsilon)(T-t_\varepsilon,T-t'_\varepsilon,x_\varepsilon,x'_\varepsilon,r_\varepsilon,r'_\varepsilon)&=\frac{2}{\varepsilon}(x'_\varepsilon-x_\varepsilon),\\
(\varphi_\varepsilon)_{r'r'}(T-t_\varepsilon,T-t'_\varepsilon,x_\varepsilon,x'_\varepsilon,r_\varepsilon,r'_\varepsilon)&=\frac{2}{\varepsilon}.
\end{aligned}
$$

As before for $U$, inserting the derivatives of $(t',x',r')\to\varphi_\varepsilon(T-t_\varepsilon,T-t',x_\varepsilon,x',r_\varepsilon,r')$ at $(T-t'_\varepsilon,x'_\varepsilon,r'_\varepsilon)$ into (4.17), we have in conjunction with (4.32):

$$
\begin{aligned}
&V(T-t,x,r)-V(T-t'_\varepsilon,x'_\varepsilon,r'_\varepsilon)\\
&\ge\varphi_\varepsilon(T-t_\varepsilon,T-t'_\varepsilon,x_\varepsilon,x'_\varepsilon,r_\varepsilon,r'_\varepsilon)-\varphi_\varepsilon(T-t_\varepsilon,T-t',x_\varepsilon,x',r_\varepsilon,r')\\
&=(\varphi_\varepsilon)_{t'}(T-t_\varepsilon,T-t'_\varepsilon,x_\varepsilon,x'_\varepsilon,r_\varepsilon,r'_\varepsilon)(t'-t'_\varepsilon)\\
&\quad-\nabla_{x'}(\varphi_\varepsilon)(T-t_\varepsilon,T-t'_\varepsilon,x_\varepsilon,x'_\varepsilon,r_\varepsilon,r'_\varepsilon)(x'-x'_\varepsilon)\\
&\quad-(\varphi_\varepsilon)_{r'}(T-t_\varepsilon,T-t'_\varepsilon,x_\varepsilon,x'_\varepsilon,r_\varepsilon,r'_\varepsilon)(r'-r'_\varepsilon)\\
&\quad-\int_{r'_\varepsilon}^{r'}\frac{(r'-s)^2}{2}(\varphi_\varepsilon)_{r'r'}(T-t_\varepsilon,T-t'_\varepsilon,x_\varepsilon,x'_\varepsilon,r_\varepsilon,s)\,ds+o(|t'-t'_\varepsilon|+|x'-x'_\varepsilon|)\\
&=\frac{2}{\varepsilon}(t'_\varepsilon-t_\varepsilon)(t-t'_\varepsilon)-\frac{2}{\varepsilon}(x'_\varepsilon-x_\varepsilon)(x-x'_\varepsilon)-\frac{2}{\varepsilon}(r'_\varepsilon-r_\varepsilon)(r-r'_\varepsilon)-\frac{1}{\varepsilon}\int_{r'_\varepsilon}^{r'}(r'-s)^2\,ds\\
&\quad+o(|t'-t'_\varepsilon|+|x'-x'_\varepsilon|)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2}{\varepsilon}(t'_\varepsilon - t_\varepsilon))(t - t'_\varepsilon) - \frac{2}{\varepsilon}(x'_\varepsilon - x_\varepsilon)(x - x'_\varepsilon) - \frac{2}{\varepsilon}(r'_\varepsilon - r_\varepsilon)(r - r'_\varepsilon) \\
&\quad + \frac{1}{3\varepsilon}(r' - r'_\varepsilon)(r' - r'_\varepsilon)^2 + o(|t' - t'_\varepsilon| + |x' - x'_\varepsilon| + |r' - r'_\varepsilon|^2). \\
&\geq \frac{2}{\varepsilon}(t'_\varepsilon - t_\varepsilon)(t - t'_\varepsilon) - \frac{2}{\varepsilon}(x'_\varepsilon - x_\varepsilon)(x - x'_\varepsilon) - \frac{2}{\varepsilon}(r'_\varepsilon - r_\varepsilon)(r - r'_\varepsilon) - \frac{2\alpha_\varepsilon}{3\varepsilon}(r' - r'_\varepsilon)^2. \\
&\quad + o(|t' - t'_\varepsilon| + |x' - x'_\varepsilon| + |r' - r'_\varepsilon|^2),
\end{aligned}
$$

This shows, thanks to Remark 4.2.6, that

$$
(-\frac{2}{\varepsilon}(t_\varepsilon - t'_\varepsilon), \frac{2}{\varepsilon}(x_\varepsilon - x'_\varepsilon), \frac{2}{\varepsilon}(r_\varepsilon - r'_\varepsilon), -\frac{2\alpha_\varepsilon}{3\varepsilon}) \in \mathcal{J}^{2,-}V(T - t'_\varepsilon, x'_\varepsilon, r'_\varepsilon). \tag{4.34}
$$

Applying Lemma 4.2.7 to the viscosity subsolution $U$ and (4.33), we finally obtain

$$
\begin{aligned}
0 \leq & -\frac{2}{\varepsilon}(t_\varepsilon - t'_\varepsilon) + \beta U(T - t_\varepsilon, x_\varepsilon, r_\varepsilon) + \frac{2}{\varepsilon}(r_\varepsilon - r'_\varepsilon) b \cdot x_\varepsilon + \frac{\alpha_\varepsilon {x_\varepsilon}^\top \Sigma x_\varepsilon}{3\varepsilon} \\
& + \frac{2}{\varepsilon} \sup_{\xi \in \mathbb{R}^d} \left( \xi^\top (x_\varepsilon - x'_\varepsilon) - (r_\varepsilon - r'_\varepsilon) f(\xi) \right).
\end{aligned} \tag{4.35}
$$

Proceeding in the same manner for $V$ and using the viscosity supersolution property of Lemma 4.2.7, as well as (4.34), we also get

$$
\begin{aligned}
0 \geq & -\frac{2}{\varepsilon}(t_\varepsilon - t'_\varepsilon) + \beta V(T - t'_\varepsilon, x'_\varepsilon, r'_\varepsilon) + \frac{2}{\varepsilon}(r_\varepsilon - r'_\varepsilon) b \cdot x'_\varepsilon - \frac{\alpha_\varepsilon {x'_\varepsilon}^\top \Sigma x'_\varepsilon}{3\varepsilon} \\
& + \frac{2}{\varepsilon} \sup_{\xi \in \mathbb{R}^d} \left( \xi^\top (x_\varepsilon - x'_\varepsilon) - (r_\varepsilon - r'_\varepsilon) f(\xi) \right).
\end{aligned} \tag{4.36}
$$

By subtracting (4.35) from (4.36), we then get:

$$
\begin{aligned}
0 \leq & \beta(U(T - t_\varepsilon, x_\varepsilon, r_\varepsilon) - V(T - t'_\varepsilon, x'_\varepsilon, r'_\varepsilon)) + \frac{2}{\varepsilon}(r_\varepsilon - r'_\varepsilon) b \cdot (x_\varepsilon - x'_\varepsilon) \\
& + \frac{\alpha_\varepsilon}{3\varepsilon}\left( {x_\varepsilon}^\top \Sigma x_\varepsilon + {x'_\varepsilon}^\top \Sigma x'_\varepsilon \right)
\end{aligned}
$$

Sending now $\varepsilon$ to 0 and using the fact that $\frac{\alpha_\varepsilon}{\varepsilon}, r_\varepsilon - r'_\varepsilon, |x_\varepsilon - x'_\varepsilon| \to 0$, when $\varepsilon \to 0$, we get

$$
0 \leq \beta(U - V)(T - t^*, x^*, r^*). \tag{4.37}
$$

Because $\beta < 0$, (4.37) is in contradiction with (4.25). Thus, we have shown that $U \leq V$ on $]0, T] \times \mathbb{R}^d \times \mathbb{R}$. ■

The following uniqueness result directly follows from the above theorem.

**Corollary 4.2.9.** *The value function defined in* (2.16) *is the* unique *viscosity solution of* (3.23) *with initial condition* (3.24).

*Proof.* Let $U$ be another solution of (3.23) with initial condition (3.24) satisfying the growth condition

$$
V_2(t, x, r) \leq U(t, x, r) \leq V_1(t, x, r), \quad \text{for all } (t, x, r) \in ]0, T] \times \mathbb{R}^d \times \mathbb{R}.
$$

Then we have

$$\lim_{t\to 0}\big(U(t,x,r)-V(t,x,r)\big)=0, \qquad \text{for fixed } x,r\in\mathbb{R}^d\backslash\{0_{\mathbb{R}^d}\}\times\mathbb{R},$$

which can be extended to $\mathbb{R}^d\times\mathbb{R}$. Hence, by using Theorem 4.2.8 we deduce that $U\leq V$. Since both $U$ and $V$ are viscosity sub- and supersolution, respectively, we conclude by reversing the preceding inequality. ∎

*Remark* 4.2.10. In the one-dimensional framework, adding a term of the form $\varepsilon V_{xx}$ in equation (4.11), for $\varepsilon>0$, does not change the conclusion of the preceding theorem: indeed, we could apply step by step the same arguments as above to obtain the analogous conclusion for the strong comparison result. This allows us to approximate our degenerate parabolic equation through non-degenerate parabolic ones, which also fulfill a strong comparison result. The corresponding setting in our optimal control problem consists in adding an $\varepsilon$-noise to the controlled process $X$, by setting:

$$dX_t=-\xi_t+\varepsilon dW_t,$$

where $(W_t)$ is a Brownian motion independent of $(B_t)$, as already mentioned below the SDE (2.5). With this at hand, we can derive the corresponding non-degenerate HJB equation:

$$-V_t+\frac{X^2\sigma^2}{2}V_{rr}+\varepsilon V_{xx}+b\cdot X\,V_r+\sup_{\xi\in\mathbb{R}^d}(\xi\cdot\nabla_x V-f(\xi)V_r).$$

In the d-dimensional framework, things can become more complicated, and we have to use among others Crandall-Ishii's lemma to find the corresponding sub- and superjet associated with the second-order terms in order to prove a comparison result for the non-degenerate parabolic equation. ♢

# Chapter 5

# Numerical approximation

In this chapter, we aim at constructing a numerical scheme in order to approximate the value function of our maximization problem (2.16), which is known to be the unique viscosity solution of (3.23) with initial condition (3.24), as was shown in the previous chapter. This appears to be a very difficult task, since we have to face numerous issues. Let us in the first place enumerate these ones, theoretically. First, we cannot directly apply some well-known convergence result, for instance, à la Barles and Souganidis (1991), for in their work they consider only bounded functions with no singularity. Indeed, in most of the literature, when dealing with monotone numerical schemes to approximate Hamilton-Jacobi-Bellman equations, like in Barles and Jakobsen (2002) (where they discuss the rate of convergence of approximation schemes), or more recently, in Briani et al. (2012) (which is a generalization of the framework of Barles and Souganidis), only *bounded* viscosity solutions are considered. However, slight modifications in the Barles and Souganidis framework permit us to adapt their model to viscosity solutions with linear asymptotic growth. Moreover, a classical change of variables formula will allow us to relax the exponential growth requirement, by introducing an auxiliary HJB equation. Nevertheless, we will still face a polynomial growth and, above all, a singularity at time 0, so that to the best of our knowledge no well-known convergence results for monotone schemes can be directly applied in our case. Fortunately, to deal with the singularity property, we will be able to prove that our auxiliary value function behaves like a predetermined function at time 0, i.e., the quotient of the auxiliary value function and this predetermined function will be close to one, near the initial condition. In this manner, we will be able to transform again our auxiliary HJB equation, by considering a translated version of the latter one, which will permit us to set a zero function as initial condition. However, even with our relaxed conditions, classical results for monotone numerical schemes cannot be directly applied here, since there remains a term which behaves like $Tf(X_0/T)$ and can thus have polynomial growth.

Note that there are other ways to approximate nonlinear parabolic equations. For instance, in Bonnans et al. (2004), analyzing generalized finite difference methods, non-monotone converging schemes are established. In Warin (2013), the convergence is established for some general approximations of the viscosity solutions, provided that a certain optimization problem can be solved at each time step. Unfortunately, here again only bounded viscosity solutions are considered. An alternative approach to approximate nonlinear parabolic PDEs would be to use Monte Carlo methods, combined with the finite difference method, as suggested in Fahim et al. (2011). In their work, they introduce a backward probabilistic scheme that permits to approximate the solution of a nonlinear PDE in two steps. In the first step, the linear part of the PDE is dealt with by using Monte Carlo simulation applied to a conditional expectation operator. The second step applies a finite difference method to the remaining nonlinear part. Moreover, they consider viscosity solutions having polynomial or exponential growth. Nevertheless, the second-order parabolic partial differential equation has to fulfill a Lipschitz condition, uniformly in $t$, which cannot be the case in our framework, due to the Fenchel-Legendre term of the auxiliary HJB equation. Moreover, as argued in their paper, their results do not apply to general degenerate nonlinear parabolic PDEs, and we therefore cannot use directly their method.

In order to remedy to those listed issues, we will have to localize the requirements of building converging monotone schemes; the fact that our second-order term is one-dimensional will be very helpful to us. However, this will lead to some severe Courant-Friedrichs-Lewy (CFL) conditions in the time parameter and, as a consequence, numerical schemes will converge slowly, since the number of time iterations will have to be chosen sufficiently large.

## 5.1 Auxiliary HJB equation, vanishing singularity and comparison result

In this section we consider the following HJB equation:

$$W_t - b \cdot X\, W_r - \frac{X^\top \Sigma X}{2}\left(W_{rr} + (W_r)^2\right) + \sup_{\xi \in \mathbb{R}^d}(\xi \cdot \nabla_x W + f(-\xi) W_r) = 0 \tag{5.1}$$

$$W(0, X, R) = \lim_{T \downarrow 0} W(T, X, R) = \begin{cases} \log(B - u(R)), & \text{if } X = 0, \\ \infty, & \text{otherwise,} \end{cases} \tag{5.2}$$

where $u$ denotes our utility function and $B \geq 0$ is such that $B - u > 0$ on $\mathbb{R}$ (such a $B$ exists, since the utility function considered is bounded from above).

**Proposition 5.1.1.** *$U$ is a viscosity subsolution (resp., $V$ is a viscosity supersolution) of (3.23) if and only if $\log(B-U)$ is a viscosity supersolution (resp., $\log(B-V)$ is a viscosity subsolution) of (5.1).*

*Proof.* We prove the following equivalence: $U$ is a viscosity subsolution of (3.23) if and only if $\log(B-U)$ is a viscosity supersolution of (5.1). The other equivalence, i.e., $V$ is a viscosity supersolution of (3.23) if and only if $\log(B-V)$ is a viscosity subsolution of (5.1), can be treated similarly.

To this end, take $U$ a viscosity subsolution of (3.23), $\varphi \in \mathcal{C}^{1,1,2}([0,T]\times\mathbb{R}^d\times\mathbb{R})$ and $(T-t^*,x^*,r^*)$ such that $(T-t^*,x^*,r^*)$ is a local minimizer of $\log(B-U)-\varphi$. We wish to show that

$$0 \le \Big(\varphi_t - b\cdot x\,(\varphi)_r - \frac{X^\top\Sigma X}{2}\big(\varphi_{rr}+(\varphi_r)^2\big) + \sup_{\xi\in\mathbb{R}^d}\big(\xi\cdot\nabla_x\varphi + f(-\xi)\varphi_r\big)\Big)(T-t^*,x^*,r^*).$$

As argued in Remark 4.1.3, we can w.l.o.g. suppose that $(\log(B-U)-\varphi)(T-t^*,x^*,r^*)=0$. Hence, we have that $B-U=\exp(\varphi)$ at $(T-t^*,x^*,r^*)$, and therefore it follows that $(T-t^*,x^*,r^*)$ is a local maximizer of $U-B+\exp(\varphi)$ (and also of $U+\exp(\varphi)$). We compute now the following derivatives of $\psi := -\exp(\varphi)$ at $(T-t,x,r)$:

$$\begin{aligned} \psi_t &= -\varphi_t\psi, & \psi_r &= \varphi_r\psi, \\ \psi_{rr} &= (\varphi_{rr}+(\varphi_r)^2)\psi, & \nabla_x\psi &= \nabla_x\varphi\psi. \end{aligned}$$

Since $U$ verifies the viscosity subsolution condition (4.3) for (3.23), we can write:

$$\begin{aligned} &\Big(-(\psi)_t + \frac{X^\top\Sigma X}{2}\psi_{rr} + b\cdot X\,(\psi)_r + \sup_{\xi\in\mathbb{R}^d}(\xi\cdot\nabla_x\psi - f(\xi)\psi_r)\Big)(T-t^*,x^*,r^*) \\ &= \psi\Big(-\varphi_t + b\cdot X\,(\varphi)_r + \frac{X^\top\Sigma X}{2}\varphi_{rr} + \frac{X^\top\Sigma X}{2}(\varphi_r)^2 \\ &\quad - \sup_{\xi\in\mathbb{R}^d}\big(-\xi\cdot\nabla_x\varphi + f(\xi)\varphi_r\big)\Big)(T-t^*,x^*,r^*) \\ &\ge 0. \end{aligned}$$

Hence, we get that

$$\varphi_t - b\cdot x\,(\varphi)_r - \frac{X^\top\Sigma X}{2}\big(\varphi_{rr}+(\varphi_r)^2\big) + \sup_{\xi\in\mathbb{R}^d}\big(\xi\cdot\nabla_x\varphi + f(-\xi)\varphi_r\big) \ge 0,$$

at $(T-t^*,x^*,r^*)$, which proves the one direction.

To prove the converse direction, we proceed in a similar way. Suppose that $\log(B-U)$ is a viscosity supersolution of (5.1) and take $\varphi\in\mathcal{C}^{1,1,2}([0,T]\times\mathbb{R}^d\times\mathbb{R})$ and $(T-t^*,x^*,r^*)$ a local maximizer of $U-\varphi$. Then $(T-t^*,x^*,r^*)$ is also a local maximizer of $U-\varphi-B$ and, here again, we can suppose that $(U-B)(T-t^*,x^*,r^*)=\varphi(T-t^*,x^*,r^*)$. In the same manner, we also show that $(T-t^*,x^*,r^*)$ is a local minimizer of $\log(B-U)-\log(-\varphi)$ (note that we locally have that $\varphi<0$, due to the preceding equality and the fact that $B-U>0$). We compute now the following derivatives of $\phi := \log(-\varphi)$:

$$\phi_t = \frac{\varphi_t}{\varphi}, \qquad \phi_r = \frac{\varphi_r}{\varphi},$$

$$\phi_{rr} = \frac{\varphi_{rr}}{\varphi} - \frac{\varphi_r^2}{\varphi^2}, \qquad\qquad \nabla_x \phi = \frac{\nabla_x \varphi}{\varphi}.$$

Since $\log(B-U)$ verifies the viscosity supersolution condition for the equation (5.1), we have:

$$\begin{aligned}
&\phi_t - b\cdot X\,\phi_r - \frac{X^\top \Sigma X}{2}\big(\varphi_{rr} + (\varphi_r)^2\big) + \sup_{\xi\in\mathbb{R}^d}\big(\xi\cdot\nabla_x\phi + f(-\xi)\phi_r\big) \\
&= \frac{\varphi_t}{\varphi} - b\cdot X\,\frac{\varphi_r}{\varphi} - \frac{X^\top\Sigma X}{2}\Big(\frac{\varphi_{rr}}{\varphi} - \frac{\varphi_r^2}{\varphi^2} + \frac{\varphi_r^2}{\varphi^2}\Big) \\
&\quad + \sup_{\xi\in\mathbb{R}^d}\big(\xi\cdot\frac{\nabla_x\varphi}{\varphi} + f(-\xi)\frac{\varphi_r}{\varphi}\big) \\
&= \frac{1}{\varphi}\Big(\varphi_t - b\cdot X\,\varphi_r - \frac{X^\top\Sigma X}{2}\varphi_{rr} + \inf_{\xi\in\mathbb{R}^d}\big(\xi\cdot\nabla_x\varphi + f(-\xi)\varphi_r\big)\Big) \\
&\geq 0.
\end{aligned}$$

Since $\varphi$ is negative in a neighborhood of $(T-t, x^*, r^*)$, we thus have shown that

$$\Big(-\varphi_t + \frac{X^\top\Sigma X}{2}\varphi_{rr} + b\cdot X\,\varphi_r + \sup_{\xi\in\mathbb{R}^d}(\xi\cdot\varphi - f(\xi)\varphi_r)\Big)(T-t^*, x^*, r^*) \geq 0,$$

and thus $U$ verifies the viscosity subsolution condition for (3.23). ■

We show now that a comparison principle also holds for (5.1).

**Proposition 5.1.2.** *Let $W$ (resp., $\widetilde{W}$) be a continuous viscosity subsolution (resp., continuous viscosity supersolution) of (5.1), defined on $]0,T]\times\mathbb{R}^d\times\mathbb{R}$, which satisfies the growth conditions*

$$\log(B-V_2(t,x,r)) \geq v(t,x,r) \geq \log(B-V_1(t,x,r)), \ \textit{for all } (t,x,r)\in\, ]0,T]\times\mathbb{R}^d\times\mathbb{R}, \tag{5.3}$$

*where $v$ can be chosen to be $W$ or $\widetilde{W}$. Further, we suppose that $W$ and $\widetilde{W}$ satisfy the boundary conditions*

$$\limsup_{t\to 0} W(t,x,r) - \widetilde{W}(t,x,r) \leq 0, \qquad \textit{for fixed } x,r\in\mathbb{R}^d\times\mathbb{R}. \tag{5.4}$$

*Then $W\leq\widetilde{W}$ on $]0,T]\times\mathbb{R}^d\times\mathbb{R}$.*

*Proof.* We write $\widetilde{W} = \log(B-\widetilde{U})$ and $W = \log(B-U)$. Then, by applying Proposition 5.1.1 we have that $U$ is a supersolution (resp., $\widetilde{U}$ is a subsolution) of (3.23), and satisfies

$$\limsup_{T\downarrow 0}(U-\widetilde{U})(T, X_0, R_0) \geq 0.$$

Thus we are in the setting of Theorem 4.2.8, and therefore we have that $U\geq\widetilde{U}$, which implies that $W\leq\widetilde{W}$ on $]0,T]\times\mathbb{R}^d\times\mathbb{R}$. ■

The preceding results permit us to relax the exponential growth condition imposed on the value function. By using an affine transform of the preceding HJB equation, with an adequate function, we will also be able to remove the singularity in the initial condition. To this end, we first need to prove the following fundamental proposition.

**Proposition 5.1.3.** *Define* $\tilde{u}(T, X_0, R_0) := \log(B - u(R_0 - Tf(-X_0/T))$, *and let* $V$ *denote the value function of the maximization problem* (2.16) *with initial condition* (3.24). *Then* $\tilde{u} \in \mathcal{C}^{1,1,2}(]0,T] \times \mathbb{R}^d \times \mathbb{R})$ *and verifies*

$$\lim_{T \downarrow 0} \log(B - V(T, X_0, R_0)) - \tilde{u}(T, X_0, R_0) = 0, \tag{5.5}$$

*locally uniformly in* $(X_0, R_0)$.

*Proof.* It is sufficient to prove that, for $X_0 \neq 0$, it holds

$$\lim_{T \downarrow 0} \frac{V(T, X_0, R_0)}{u(R_0 - Tf(-X_0/T))} = 1.$$

Toward this end, consider first the linear strategy $\zeta := X_0/T \in \dot{\mathcal{X}}(T, X_0)$. We want to show that

$$\lim_{T \downarrow 0} \frac{\mathbb{E}\big[u(\mathcal{R}_T^\zeta)\big]}{u(R_0 - Tf(-X_0/T))} = 1, \tag{5.6}$$

where

$$\mathcal{R}_T^\zeta = R_0 + X_0 \int_0^T (1 - t/T)\sigma^\top dB_t + \frac{T}{2} b \cdot X_0 - Tf(-X_0/T).$$

But we have

$$\begin{aligned}
\mathbb{E}\big[u(\mathcal{R}_T^\zeta)\big] &= \mathbb{E}\big[u(\mathcal{R}_T^\zeta + A_2/2\langle \mathcal{R}_\cdot^\zeta \rangle_T - A_2/2\langle \mathcal{R}_\cdot^\zeta \rangle_T))\big] \\
&= \mathbb{E}\bigg[u\bigg(R_0 + \frac{T}{2} b \cdot X_0 - Tf(X_0/T) - \frac{A_2}{2} \int_0^T (X_t^\zeta)^\top \Sigma X_t^\zeta \; dt\bigg)\bigg] \\
&\quad + \mathbb{E}\bigg[\int_0^T u'(\mathcal{R}_t^\zeta)\, X_t^\zeta \sigma \, dB_t + \frac{A_2}{2} \int_0^T u'(\mathcal{R}_t^\zeta)(X_t^\zeta)^\top \Sigma X_t^\zeta \; dt \\
&\quad + \frac{1}{2} \int_0^T u''(\mathcal{R}_t^\zeta)\, d\langle \mathcal{R}_\cdot^\zeta \rangle_t\bigg] \\
&\geq u\bigg(R_0 + \frac{T}{2} b \cdot X_0 - Tf(X_0/T) - 2A_2|X_0|^2 T|\Sigma|\bigg) \\
&\quad + \mathbb{E}\bigg[\frac{A_2}{2} \int_0^T u'(\mathcal{R}_t^\zeta)(X_t^\zeta)^\top \Sigma X_t^\zeta \; dt \\
&\quad - \frac{1}{2} \int_0^T u'(\mathcal{R}_t^\zeta) \frac{-u''(\mathcal{R}_t^\zeta)}{u'(\mathcal{R}_t^\zeta)} (X_t^\zeta)^\top \Sigma X_t^\zeta \; dt\bigg] \\
&\geq u\bigg(R_0 + \frac{T}{2} b \cdot X_0 - Tf(X_0/T) - 2A_2|X_0|^2 T|\Sigma|\bigg)
\end{aligned}$$

$$
\begin{aligned}
&+\mathbb{E}\Big[\frac{A_2}{2}\int_0^T u'(\mathcal{R}_t^{\zeta})(X_t^{\zeta})^\top \Sigma X_t^{\zeta}\ dt - \frac{A_2}{2}\int_0^T u'(\mathcal{R}_t^{\zeta})(X_t^{\zeta})^\top \Sigma X_t^{\zeta}\ dt\Big] \\
&= u\Big(R_0 + \frac{T}{2} b\cdot X_0 - Tf(X_0/T) - 2A_2|X_0|^2 T|\Sigma|\Big).
\end{aligned}
$$

And this implies that

$$
\liminf_{T\downarrow 0} \frac{u(R_0 - Tf(-X_0/T))}{\mathbb{E}\big[u(\mathcal{R}_T^{\zeta})\big]} \geq 1. \tag{5.7}
$$

Let now $\xi^*$ be the optimal strategy associated to $V(T, X_0, R_0)$. Observe that applying Jensen's inequality to the convex function $f$ and the concave function $u$ yields the inequality

$$
\mathbb{E}\big[u(\mathcal{R}_T^{\xi^*})\big] \leq u\Big(\mathbb{E}\Big[R_0 + \int_0^T X_t^{\xi^*}\cdot b\, dt - Tf(-X_0/T)\Big]\Big).
$$

Utilizing the requirement (2.6) on strategies belonging to $\dot{\mathcal{X}}^1(T, X_0)$, we can find an $M > 0$ such that $\mathbb{E}[\int_0^T X_t^{\xi^*}\cdot b\, dt] \leq |b|MT$. And therefore we have

$$
\mathbb{E}\big[u(\mathcal{R}_T^{\xi^*})\big] \leq u\Big(R_0 + |b|MT - Tf(-X_0/T)\Big). \tag{5.8}
$$

Since for $T$ close enough to 0 both $V(T, X_0, R_0)$ and $u(R_0 - Tf(-X_0/T))$ are negative, we finally get

$$
\liminf_{T\downarrow 0} \frac{V(T, X_0, R_0)}{u(R_0 - Tf(-X_0/T))} \geq \liminf_{T\downarrow 0} \frac{u(R_0 + |b|MT - Tf(-X_0/T))}{u(R_0 - Tf(-X_0/T))} = 1. \tag{5.9}
$$

Since $\xi^*$ is optimal (and hence $V(T, X_0, R_0) \geq \mathbb{E}\big[u(\mathcal{R}_T^{\zeta})\big]$), we also have (recall that all the quotient involved are positive when $T$ is taken small enough)

$$
\begin{aligned}
1 &\geq \limsup_{T\downarrow 0} \frac{V(T, X_0, R_0)}{\mathbb{E}\big[u(\mathcal{R}_T^{\zeta})\big]} \\
&= \limsup_{T\downarrow 0} \frac{V(T, X_0, R_0) u(R_0 - Tf(-X_0/T))}{\mathbb{E}\big[u(\mathcal{R}_T^{\zeta})\big] u(R_0 - Tf(-X_0/T))} \\
&= \limsup_{T\downarrow 0} \frac{V(T, X_0, R_0)}{u(R_0 - Tf(-X_0/T))} \cdot \liminf_{T\downarrow 0} \frac{u(R_0 - Tf(-X_0/T))}{\mathbb{E}\big[u(\mathcal{R}_T^{\zeta})\big]} \\
&= \limsup_{T\downarrow 0} \frac{V(T, X_0, R_0)}{u(R_0 - Tf(-X_0/T))}.
\end{aligned}
$$

Combining the preceding inequality with (5.9) concludes the proof ■

*Remark* 5.1.4. The preceding proof remains unchanged if we send $|R_0|$ to infinity (instead of sending $T$ to 0), other parameters being fixed. Thus, we have that

$$
\lim_{R_0\to\pm\infty} \log(B - V(T, X_0, R_0)) - \log(B - u(R_0 - Tf(X_0/T)) = 0.
$$

This will later enable us to set $\log(B-u(R_0-Tf(X_0/T))$ as a boundary condition, when taking $|R_0|$ large enough in our scheme (since we will work with a finite grid in the numerical examples); however, in general,

$$\lim_{|X_0|\to\infty} \log(B-V(T,X_0,R_0)) - \log(B-u(R_0-Tf(X_0/T)) \neq 0,$$

for $T \neq 0$. ◇

For $\tilde{u}$ as in the preceding proposition, we consider now the following auxiliary equation for (5.1):

$$\begin{aligned}
(W+\tilde{u})_t - b\cdot X\,(W+\tilde{u})_r - \frac{X^\top \Sigma X}{2}\big((W+\tilde{u})_{rr} + ((W+\tilde{u})_r)^2\big)& \\
+ \sup_{\xi\in\mathbb{R}^d}\big(\xi\cdot\nabla_x(W+\tilde{u}) + f(-\xi)(W+\tilde{u})_r\big) = 0,& \qquad (5.10)\\
\lim_{T\downarrow 0} W(T,X,R) = 0.& \qquad (5.11)
\end{aligned}$$

*Remark* 5.1.5. Note that, as for (3.23), we can rewrite (5.10) in the following way:

$$\begin{aligned}
0 = (W+\tilde{u})_t - b\cdot X\,(W+\tilde{u})_r - \frac{X^\top \Sigma X}{2}\big((W+\tilde{u})_{rr} + ((W+\tilde{u})_r)^2\big)\\
-(W+\tilde{u})_r f^*\left(-\frac{\nabla_x(W+\tilde{u})}{(W+\tilde{u})_r}\right),
\end{aligned}$$

where $f^*$ denotes the Fenchel-Legendre transformation of $f$. ◇

The next proposition states that the notion of viscosity solutions of (5.1) and viscosity solutions of (5.10) is equivalent, and moreover, a comparison result holds.

**Proposition 5.1.6.** *$W$ is a viscosity subsolution (resp., supersolution) of (5.1) with initial condition (5.2) if and only if $W-\tilde{u}$ is a viscosity subsolution (resp., supersolution) of (5.10) with initial condition (5.11). Moreover, a comparison principle holds for (5.10).*

*Proof.* This is a straightforward application of Proposition 5.1.2 and the definition of viscosity solutions: we have that $\varphi$ is a test function for $W$, when applied to (5.1), if and only if $\varphi-\tilde{u}$ is a test function for $W-\tilde{u}$, when applied to (5.10). ■

## 5.2 Numerical schemes and convergence results

In this section, our goal is to prove a convergence result, similar to the one derived in Barles and Souganidis (1991). However, we will have to relax their conditions in order to ensure that finite difference schemes applied to our numerical examples will converge, locally uniformly, to the unique viscosity solution of (5.10). Let us now introduce the definition of a numerical scheme, in our setting.

### 5.2.1 Barles-Souganidis convergence result

**Definition 5.2.1.** A numerical scheme for (5.10) with initial condition (5.11) is an equation of the following form:

$$S(h,t,x,r,w_h(t,x,r),[w_h]_{t,x,r}) = 0, \quad \text{for } (t,x,r) \in \mathbb{G}_h \backslash \{t=0\}, \tag{5.12}$$

$$w_h(0,x,r) = 0, \quad \text{in } \mathbb{G}_h \cap \{t=0\}, \tag{5.13}$$

where $S$ is locally bounded, $h := \max(|\Delta t|, |\Delta x|, |\Delta r|)$ denotes the size of the mesh, and

$$\mathbb{G}_h := \Delta t \cdot \{0,1,\dots,n_T\} \times \Delta x \cdot \mathbb{Z}^d \times \Delta r \cdot \mathbb{Z}.$$

The quantity $w_h$ represents the approximation of $w$, and $[w_h]_{t,x,r}$ stands for the value of $w_h$ close to $(t,x,r)$.

In order to have an analogous result to the Barles-Souganidis convergence theorem that can be applied to our numerical schemes, we need to slightly modify the three conditions required in Barles and Souganidis (1991).

**Definition 5.2.2.** A numerical scheme $S$ is said to be

- *locally $\delta$-monotone* if there exists $\delta > 0$ such that whenever $|w - v| \le \delta$: if $w \ge v$ on an open bounded set $O \subset ]0,T] \times \mathbb{R}^d \times \mathbb{R}$, then

  $$S(h,t,x,r,z,w) \le S(h,t,x,r,z,v),$$

  for all $h > 0$, $(t,x,r) \in O$ and $z \in ]-S_O, S_O[$, where $S_O := \sup_{y\in O} |w(y)| + 1$. Here, $w \ge v$ is to be understood componentwise.

- *consistent* if, for every $\varphi \in \mathcal{C}^{1,1,2}(]0,T] \times \mathbb{R}^d \times \mathbb{R})$ and every $(t,x,r) \in [0,T[ \times \mathbb{R}^d \times \mathbb{R}$, we have

  $$S(h,t,x,r,\varphi(t,x,r),[\varphi+m]_{t,x,r}) \underset{\substack{m\to 0\\ h\to 0}}{\longrightarrow} \Big((\varphi+\widetilde{u})_t - \frac{x^\top \Sigma x}{2}(\varphi+\tilde{u})_r^2 - \inf_{\xi\in\mathbb{R}^d} \widetilde{\mathcal{L}}^\xi(\varphi+\tilde{u})\Big)(t,x,r),$$

  locally uniformly in $(t,x,r)$, with

  $$\begin{aligned}\widetilde{\mathcal{L}}^\xi(\varphi+\tilde{u})\big)(t,x,r) &= \frac{x^\top \Sigma x}{2}(\varphi+\tilde{u})_{rr} + b\cdot x\,(\varphi+\tilde{u})_r \\ &\quad -\big(\xi\cdot\nabla_x(\varphi+\tilde{u}) + f(-\xi)(\varphi+\tilde{u})_r\big)(t,x,r).\end{aligned}$$

- *(locally) stable* if there exists $\delta > 0$ such that, for every $\delta > h > 0$ and every open bounded set $O \subset ]0,T[ \times \mathbb{R}^d \times \mathbb{R}$, there is a locally bounded solution $w_h$ of (5.12) satisfying

  $$\sup_{h>0} |w_h| \le C_O \text{ on } O,$$

  where $C_O$ is a constant depending only on $O$.

*Remark* 5.2.3.
1. In the preceding definition, the monotonicity property as defined in Barles and Souganidis (1991) (i.e., monotonicity of the scheme without requiring an additional control of $|w - v|$) can be replaced by our $\delta$-monotonicity, as mentioned by Tourin (2011).
2. The local stability is equivalent to the one used by Barles and Souganidis, due to the local property of the viscosity solution.
3. Since the viscosity solution of (5.10) is continuous and has a partial derivative in its third variable (Theorem 2.3.4), the approximation $w_h$ can be chosen among the same class of functions. Moreover, as this partial derivative has locally a strictly negative upper bound, we can suppose that the analogous boundedness property also holds for $w_h$.
4. As for the comparison principle, the monotonicity property is crucial, and without this assumption the scheme may fail to converge to the unique viscosity solution, as it can be seen in, e.g., Pooley et al. (2003) or Oberman (2006). This property is in practice the most difficult one to prove, due to the nonlinearity of our HJB equation, as we will see in the next section.

$\diamondsuit$

We can now state and show the fundamental theorem of this chapter.

**Theorem 5.2.4.** *Suppose that the numerical scheme $S$ is $\delta$-monotone, consistent, and locally stable. Then, the solution $w_h$ of (5.12) converges, locally uniformly on the set $]0, T] \times \mathbb{R}^d \times \mathbb{R}$, to the unique continuous viscosity solution of (5.10).*

*Proof.* Take $(t, x, r) \in ]0, T] \times \mathbb{R}^d \times \mathbb{R}$ and let us define $w^*, w_*$ as follows:

$$w^*(t, x, r) := \limsup_{\substack{h \to 0 \\ (t',x',r') \to (t,x,r)}} w_h(t', x', r') \quad \text{and} \quad w_*(t, x, r) := \liminf_{\substack{h \to 0 \\ (t',x',r') \to (t,x,r)}} w_h(t', x', r'). \tag{5.14}$$

These quantities are known as the classical half-relaxed limits and, due to the local stability assumption, $w^*$ and $w_*$ are well-defined. Suppose first that $w^*$ and $w_*$ are viscosity sub- and supersolution of (5.10), respectively, and verify

$$\limsup_{t \to 0} w^*(t, x, r) - w_*(t, x, r) \leq 0, \tag{5.15}$$

whence we can infer (Proposition 5.1.6) that $w^* \leq w_*$. Since we also have that $w^* \geq w_*$, by definition (5.14), we then obtain that $w^* = w_*$ is the unique viscosity solution of (5.10). Hence, it is sufficient to show that $w^*$ and $w_*$ are viscosity sub- and supersolution of (5.10), respectively.

We start by proving that $w^*$ is a subsolution. To this end, take $\varphi \in \mathcal{C}^{1,1,2}(]0, T] \times \mathbb{R}^d \times \mathbb{R})$ such that $w^* - \varphi$ attains its maximum on a bounded open set $O$, at some

$(T-t^*,x^*,r^*)\in\,]0,T]\times\mathbb{R}^d\times\mathbb{R}$. As already argued, by translating $\varphi$ if necessary, we can w.l.o.g suppose that

$$(w^*-\varphi)(T-t^*,x^*,r^*)=0, \tag{5.16}$$

and that this maximum can be taken as strict. Due to the definition of $w^*$, we can find sequences $h_n$ and $(T-t^{h_n},x^{h_n},r^{h_n})\in O$, such that $h_n\downarrow 0$, $(T-t^{h_n},x^{h_n},r^{h_n})\to(T-t^*,x^*,r^*)$ and

$$(w_{h_n}-\varphi)(T-t^{h_n},x^{h_n},r^{h_n})-h_n\uparrow(w^*-\varphi)(T-t^*,x^*,r^*). \tag{5.17}$$

Hence, by taking a subsequence if necessary, we have that $(w_{h_n}-\varphi)$ also attains its maximum on $O$, at some $(T-t^{h_n},x^{h_n},r^{h_n})$, i.e.,

$$w_{h_n}(T-t,x,r)\le\varphi(T-t,x,r)+(w_{h_n}-\varphi)(T-t^{h_n},x^{h_n}). \tag{5.18}$$

Indeed, for $(T-t,x,r)\in O$ we can write on one hand

$$\begin{aligned}(w^*-\varphi)(T-t^*,x^*,r^*)&>(w^*-\varphi)(T-t,x,r)\\&=\limsup_{\substack{h\to 0\\(t',x',r')\to(t,x,r)}} w_h(t',x',r')-\varphi(T-t,x,r)\\&\ge w_{h_n}(T-t,x,r)-\varphi(T-t,x,r)-h_n,\end{aligned}$$

due to (5.17), for all $n$ taken large enough. On the other hand, we can also write (by using again (5.17))

$$\begin{aligned}(w^*-\varphi)(T-t^*,x^*,r^*)&\ge(w_{h_n}-\varphi)(T-t^{h_n},x^{h_n},r^{h_n})-h_n\\&>(w^*-\varphi)(T-t,x,r),\end{aligned}$$

for some $n\in\mathbb{N}$ taken large enough. Further, using (5.16) and the continuity of both $w_{h_n}$ (see preceding remark) and $\varphi$ (taking $O$ smaller if necessary), we have that $|w_{h_n}-(\varphi+m_n)|\le\delta$ on $O$, where

$$m_n:=(w_{h_n}-\varphi)(T-t^{h_n},x^{h_n},r^{h_n}).$$

Applying the $\delta$-monotonicity property of the scheme to $\varphi+m_n$ and using the fact that $w_{h_n}$ is a solution of (5.12) yields:

$$S(h^n,T-t^{h_n},x^{h_n},r^{h_n},\varphi(T-t^{h_n},x^{h_n},r^{h_n}),[\varphi+m_n]_{t,x,r})\le 0.$$

Utilizing moreover the fact that, as $h^n\to 0$, it holds that $m_n\to(w^*-\varphi)(T-t^*,x^*,r^*)$ and the consistency of the scheme, we infer that

$$\left((\varphi+\widetilde{u})_t-\frac{(x^*)^\top\Sigma x^*}{2}(\varphi+\tilde{u})_r^2-\inf_{\xi\in\mathbb{R}^d}\widetilde{\mathcal{L}}^\xi(\varphi+\tilde{u})\right)(T-t^*,x^*,r^*)\le 0,$$

which proves that $w^*$ is a subsolution of (5.10). In the same manner, we can prove that $w_*$ is a viscosity supersolution. Since we also have that (5.15) is verified, due to (5.13), our theorem is established. ∎

In the next step, we are going to apply the preceding results to construct converging numerical schemes. In particular, we will deal with two types of schemes: explicit and implicit schemes. While the first one is easy to apply, it also requires us to take a very small time step, compared to the other step parameters, whereas the second one does not have any restriction at all with the time step. It is however essentially more difficult to numerically apply the implicit scheme. For the sake of simplicity, we will restrict ourselves to the three-dimensional case (i.e., $d = 1$).

### 5.2.2 Construction of a converging explicit scheme

Establishing the local $\delta$-monotonicity property of a scheme can be very challenging, in general, even in linear cases. This is mostly the case for explicit schemes for the equation (5.10), which shows that the Barles-Souganidis convergence result is quite difficult to apply, here. Before we construct such a scheme, we first need to make the following assumptions:

**Assumption 5.2.5.** We restrict ourselves to the situation where the solution of (5.10) is locally Lipschitz-continuous in the second parameter $x$, i.e., for every bounded set $O \subset ]0, T[\times\mathbb{R}^d\times\mathbb{R}$, there exists $L_O > 0$ such that, for every $(t, x, r) \in O$ we have

$$\limsup_{h\to 0}\left|\frac{W(t, x+h, r) - W(t, x, r)}{h}\right| \leq L_O.$$

We suppose that this is also the case for the partial derivative $W_r$, i.e., for every bounded set $O \subset ]0, T[\times\mathbb{R}^d\times\mathbb{R}$, there exists $\overline{K}'_O > 0$ such that, for every $(t, x, r) \in O$

$$\limsup_{h\to 0}\left|\frac{W_r(t, x, r+h) - W_r(t, x, r)}{h}\right| \leq \overline{K}'_O.$$

*Remark* 5.2.6. Since $W_r$ is continuous, we automatically have that $W$ is locally Lipschitz-continuous in its third parameter, $r$. Hence, there exists $\overline{K}_O > 0$ such that, for every $(t, x, r) \in O$

$$\limsup_{h\to 0}\left|\frac{W(t, x, r+h) - W(t, x, r)}{h}\right| \leq \overline{K}_O.$$

$\diamondsuit$

Denoting $\tilde{u}^n_{i,k} := \tilde{u}(n\Delta t, i\Delta x, k\Delta r)$, let us consider the following standard explicit scheme:

$$\begin{aligned}&\frac{w^{n+1}_{i,k} + \tilde{u}^{n+1}_{i,k} - (w^n_{i,k} + \tilde{u}^n_{i,k})}{\Delta t}\\
&\quad = \frac{(i\Delta x)^2\sigma^2}{2}\left(\frac{w^n_{i,k+1} + \tilde{u}^n_{i,k+1} + w^n_{i,k-1} + \tilde{u}^n_{i,k-1} - 2(w^n_{i,k} + \tilde{u}^n_{i,k})}{\Delta^2 r}\right.\\
&\qquad \left. + \left(\frac{w^n_{i,k+1} + \tilde{u}^n_{i,k+1} - (w^n_{i,k} + \tilde{u}^n_{i,k})}{\Delta r}\right)^2\right) + b(i\Delta x)\,\frac{w^n_{i,k+1} + \tilde{u}^n_{i,k+1} - (w^n_{i,k} + \tilde{u}^n_{i,k})}{\Delta r}\end{aligned}$$

$$+\frac{w^n_{i,k+1}+\tilde u^n_{i,k+1}-(w^n_{i,k}+\tilde u^n_{i,k})}{\Delta r}f^*\left(\frac{\Delta r}{\Delta x}\frac{w^n_{i+1,k}+\tilde u^n_{i+1,k}-(w^n_{i,k}+\tilde u^n_{i,k})}{w^n_{i,k+1}+\tilde u^n_{i,k+1}-(w^n_{i,k}+\tilde u^n_{i,k})}\right),$$

$$w^0_{i,k}=0,$$

which can be rewritten in the following way, by setting $\tilde w^n_{i,k} := w^n_{i,k}+\tilde u^n_{i,k}$:

$$\begin{aligned}\frac{\tilde w^{n+1}_{i,k}-\tilde w^n_{i,k}}{\Delta t} &= \frac{(i\Delta x)^2\sigma^2}{2}\left(\frac{\tilde w^n_{i,k+1}+\tilde w^n_{i,k-1}+-2\tilde w^n_{i,k}}{\Delta^2 r}+\left(\frac{\tilde w^n_{i,k+1}-\tilde w^n_{i,k}}{\Delta r}\right)^2\right)\\ &\quad +b(i\Delta x)\,\frac{\tilde w^n_{i,k+1}-\tilde w^n_{i,k}}{\Delta r}+\frac{\tilde w^n_{i,k+1}-\tilde w^n_{i,k}}{\Delta r}f^*\left(\frac{\Delta r}{\Delta x}\frac{\tilde w^n_{i+1,k}-\tilde w^n_{i,k}}{\tilde w^n_{i,k+1}-\tilde w^n_{i,k}}\right),\\ \tilde w^0_{i,k}-\tilde u^0_{i,k}&=0.\end{aligned}$$

The corresponding scheme $S$ is then defined as follows:

$$\begin{aligned}&S(h,\Delta t,\Delta x,\Delta r,\tilde w^{n+1}_{i,k},[\tilde w^n_{i,k+1},\tilde w^n_{i+1,k},\tilde w^n_{i,k},\tilde w^n_{i,k-1}])\\ &:=\frac{\tilde w^{n+1}_{i,k}-\tilde w^n_{i,k}}{\Delta t}-\frac{(i\Delta x)^2\sigma^2}{2}\left(\frac{\tilde w^n_{i,k+1}+\tilde w^n_{i,k-1}-2\tilde w^n_{i,k}}{\Delta_r^2}+\left(\frac{\tilde w^n_{i,k+1}-\tilde w^n_{i,k}}{\Delta r}\right)^2\right)\\ &\quad -b\cdot(i\Delta x)\,\frac{\tilde w^n_{i,k+1}-\tilde w^n_{i,k}}{\Delta r}-\frac{\tilde w^n_{i,k+1}-\tilde w^n_{i,k}}{\Delta r}f^*\left(\frac{\Delta r}{\Delta x}\frac{\tilde w^n_{i+1,k}-\tilde w^n_{i,k}}{\tilde w^n_{i,k+1}-\tilde w^n_{i,k}}\right).\end{aligned}$$

In the following we will see that the monotonicity property is difficult to establish, even in a simple example. Take $b=0$ and $f(x)=\lambda x^2$, where $\lambda>0$. We have then that $f^*(x)=x^2/(4\lambda)$ and can expess $S$ in the following way:

$$\begin{aligned}&S(h,\Delta t,\Delta x,\Delta r,\tilde w^{n+1}_{i,k},[\tilde w^n_{i,k+1},\tilde w^n_{i+1,k},\tilde w^n_{i,k},\tilde w^n_{i,k-1}])\\ &=\frac{\tilde w^{n+1}_{i,k}-\tilde w^n_{i,k}}{\Delta t}-\frac{(i\Delta x)^2\sigma^2}{2(\Delta r)^2}\left(\tilde w^n_{i,k+1}+\tilde w^n_{i,k-1}-2\tilde w^n_{i,k}+\left(\tilde w^n_{i,k+1}-\tilde w^n_{i,k}\right)^2\right)\\ &\quad -\frac{\Delta r}{4\lambda(\Delta x)^2}\cdot\frac{(\tilde w^n_{i+1,k}-\tilde w^n_{i,k})^2}{\tilde w^n_{i,k+1}-\tilde w^n_{i,k}}.\end{aligned}$$

In order to prove the monotonicity property of the scheme, we have to show that if

$$(\tilde w_1,\tilde w_2,\tilde w_3,\tilde w_4)\le(w'_1,w'_2,w'_3,w'_4),$$

the scheme fulfills

$$S(h,\Delta t,\Delta x,\Delta r,\tilde w,[\tilde w_1,\tilde w_2,\tilde w_3,\tilde w_4])\ge S(h,\Delta t,\Delta x,\Delta r,\tilde w,[w'_1,w'_2,w'_3,w'_4]),$$

for all $\tilde w_i, w'_i\in\mathbb{R}, i=1,\dots,4$. Note first that the scheme is unconditionally nonincreasing in $\tilde w^n_{i,k-1}$. However, when focusing on the term $\tilde w^n_{i,k}$, it seems to be difficult, even impossible, to establish a condition on $\Delta t,\Delta x,\Delta r$ such that this scheme is nonincreasing in $\tilde w^n_{i,k}$. We thus need to modify our preceding scheme by taking into account the following facts:

1. Starting from the upwind schemes for $\tilde{w}_x$,

$$\tilde{w}_x = \tilde{w}_i - \tilde{w}_{i-1} \quad \text{and} \quad -\tilde{w}_x = \tilde{w}_i - \tilde{w}_{i+1},$$

   and using $|x| = \max(x, -x)$, $x^2 = |x|^2$, we can obtain the following scheme for $\tilde{w}_x^2$:

$$\tilde{w}_x^2 = \frac{1}{\Delta x} \max(\tilde{w}_i - \tilde{w}_{i-1}, \tilde{w}_i - \tilde{w}_{i+1}, 0)^2,$$

   in which we omit the index of the non-concerned terms.

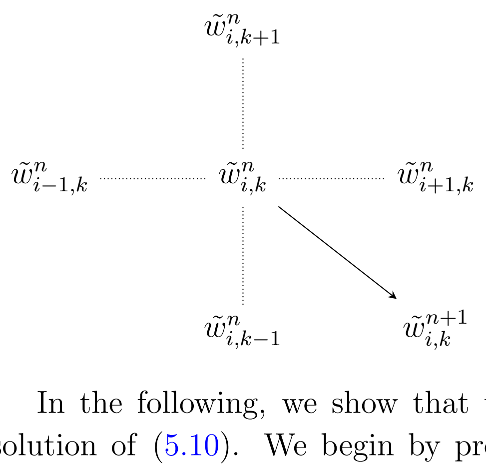
2. Since $W_r$ is continuous, we can approximate it by either $(\tilde{w}_k - \tilde{w}_{k-1})/\Delta r$ or $(\tilde{w}_{k+1} - \tilde{w}_k)/\Delta r$. Since $V_r$ is strictly positive on $]0,T] \times \mathbb{R}^d \times \mathbb{R}$, we have that $W_r = \log(B - V)_r$ is strictly negative and hence, on every bounded set $O \subset ]0,T] \times \mathbb{R}^d \times \mathbb{R}$ there exists $K_O > 0$ such that $W_r < -K_O$ on $O$. Thus, we can suppose that

$$\max\left\{(\tilde{w}_{k+1} - \tilde{w}_k)/\Delta r, (\tilde{w}_k - \tilde{w}_{k-1})/\Delta r\right\} < -K_O. \tag{5.19}$$

These considerations show that we may have to consider the following explicit scheme:

$$\begin{aligned}
&S(h, \Delta t, \Delta x, \Delta r, \tilde{w}_{i,k}^{n+1}, [\tilde{w}_{i+1,k}^n, \tilde{w}_{i-1,k}^n, \tilde{w}_{i,k+1}^n, \tilde{w}_{i,k-1}^n, \tilde{w}_{i,k}^n]) \\
&= \frac{\tilde{w}_{i,k}^{n+1} - \tilde{w}_{i,k}^n}{\Delta t} + \frac{1}{2}\left(\frac{i\Delta x \sigma}{\Delta r}\right)^2 \left(\tilde{w}_{i,k}^n - \tilde{w}_{i,k-1}^n + \tilde{w}_{i,k}^n - \tilde{w}_{i,k+1}^n - \left(\tilde{w}_{i,k}^n - \tilde{w}_{i,k+1}^n\right)^2\right) \\
&\quad - \frac{\Delta r}{4\lambda(\Delta x)^2} \cdot \frac{\max\left(\tilde{w}_{i,k}^n - \tilde{w}_{i-1,k}^n, \tilde{w}_{i,k}^n - \tilde{w}_{i+1,k}^n, 0\right)^2}{\tilde{w}_{i,k}^n - \tilde{w}_{i,k-1}^n}.
\end{aligned}$$

Its stencil is represented below:




In the following, we show that this scheme converges to the unique viscosity solution of (5.10). We begin by proving the local $\delta$-monotonicity of the scheme, where it is moreover shown that $\delta$ can be taken as $1/2$. To this end, take an open bounded set $O \subset ]0,T] \times \mathbb{R}^d \times \mathbb{R}$. First, note that our scheme $S$ is unconditionally decreasing in $\tilde{w}_{i,k-1}$, since $x \longmapsto -1/x$ is increasing for $x < 0$. It is also nonincreasing in $\tilde{w}^n_{i+1,k}$ and in $\tilde{w}^n_{i-1,k}$ (recall that $(\tilde{w}_k - \tilde{w}_{k-1})/\Delta r < 0$). Further, $S$ is nonincreasing in $\tilde{w}^n_{i,k+1}$ for $|\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k+1}| \leq 1/2$, because the function $x - x^2$ is nondecreasing for $-1/2 \leq x \leq 1/2$.

We now prove that $S$ is nonincreasing in $\tilde{w}^n_{i,k}$. This is the most difficult part of proving the monotonicity property of $S$, and we will only give a sufficient condition for it (CFL-type condition).

<u>First case:</u> $\max\left(\tilde{w}^n_{i,k} - \tilde{w}^n_{i-1,k}, \tilde{w}^n_{i,k} - \tilde{w}^n_{i+1,k}, 0\right) = 0$.

Consider the function

$$\psi_1 : \tilde{w}^n_{i,k} \longmapsto -\tilde{w}^n_{i,k} + \frac{\Delta t}{2}\left(\frac{i\Delta x\sigma}{\Delta r}\right)^2 \left(2\tilde{w}^n_{i,k} - (\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k+1})^2\right),$$

whose derivative is given by

$$\psi_1' : \tilde{w}^n_{i,k} \longmapsto -1 + \frac{\Delta t}{2}\left(\frac{i\Delta x\sigma}{\Delta r}\right)^2 \left(2 - 2(\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k+1})\right).$$

Then, for $|\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k+1}| \leq 1/2$ and $3\Delta t/2(i\Delta x\sigma/\Delta r)^2 \leq 1$, we have that $\psi_1' \leq 0$, and $S$ is hence nonincreasing in $\tilde{w}^n_{i,k}$.

<u>Second case:</u> $\max\left(\tilde{w}^n_{i,k} - \tilde{w}^n_{i-1,k}, \tilde{w}^n_{i,k} - \tilde{w}^n_{i+1,k}, 0\right) \neq 0$.

We can suppose w.l.o.g. that $\max\left(\tilde{w}^n_{i,k} - \tilde{w}^n_{i-1,k}, \tilde{w}^n_{i,k} - \tilde{w}^n_{i+1,k}, 0\right) = \tilde{w}^n_{i,k} - \tilde{w}^n_{i-1,k}$.
Consider now the following function:

$$\psi_2 : \tilde{w}^n_{i,k} \longmapsto -\tilde{w}^n_{i,k} + \frac{\Delta t}{2}\left(\frac{i\Delta x\sigma}{\Delta r}\right)^2 \left(2\tilde{w}^n_{i,k} - (\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k+1})^2\right) - \frac{\Delta r \Delta t}{4\lambda(\Delta x)^2}\frac{(\tilde{w}^n_{i,k} - \tilde{w}^n_{i-1,k})^2}{\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k-1}},$$

whose derivative is given by

$$\begin{aligned}\psi_2' : \tilde{w}^n_{i,k} \longmapsto & -1 + \frac{\Delta t}{2}\left(\frac{i\Delta x\sigma}{\Delta r}\right)^2 \left(2 - 2(\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k+1})\right) \\ & - \frac{\Delta r \Delta t}{4\lambda(\Delta x)^2}\frac{(\tilde{w}^n_{i,k} - \tilde{w}^n_{i-1,k})(\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k+1} + \tilde{w}^n_{i-1,k} - \tilde{w}^n_{i,k} + \tilde{w}^n_{i,k} - \tilde{w}^n_{i,k+1})}{(\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k-1})^2}.\end{aligned}$$

As $W$ is known to be continuous, it is uniformly continuous on any bounded set $O \subset ]0,T] \times \mathbb{R} \times \mathbb{R}$ (where $\overline{O} \subset ]0,T] \times \mathbb{R} \times \mathbb{R}$) and thus, there exists $h > 0$ such that $|\tilde{w}^m_{j,l} - \tilde{w}^{m'}_{j',l'}| \leq 1/2$, for $|(m,j,l) - (m',j',l')| \leq h$. We denote by $X_O$ the maximum value of $|i\Delta x|$ on $O \cap \mathbb{R}$. Using the fact that $|(\tilde{w}^m_{j,l} - \tilde{w}^m_{j+1,l})/\Delta r| \geq K_O$ on $O$ (due to (5.19)), we infer

$$\begin{aligned}\psi_2'(\tilde{w}^n_{i,k}) &\leq -1 + \frac{3\Delta t}{2}\left(\frac{i\Delta x\sigma}{\Delta r}\right)^2 + \frac{\Delta r \Delta t}{4\lambda(\Delta x)^2}\frac{(1/2)(3/2)}{(\Delta r)^2 K_O^2} \\ &= -1 + \frac{3\Delta t}{8\lambda}\left(\frac{(i\Delta x\sigma)(\Delta x)^2 + \Delta r}{(\Delta x)^2(\Delta r)^2 K_O^2}\right) \\ &\leq 0,\end{aligned}$$

for
$$\frac{3\Delta t}{8\lambda}\left(\frac{(X_O\sigma)^2(\Delta x)^2K_O^2+\Delta r}{(\Delta r)^2(\Delta x)^2K_O^2}\right)\leq 1. \tag{5.20}$$
The condition (5.20) can be regarded as the Courant Friedrichs Lewy (CFL) condition for this explicit scheme.
It remains to prove the consistency and local stability of the scheme. Classical computations using the Taylor expansion yield:
$$\begin{aligned}
\frac{\tilde{w}^n_{i,k+1}+\tilde{w}^n_{i,k-1}-2\tilde{w}^n_{i,k}}{(\Delta r)^2} &= \tilde{w}_{rr}(n\Delta t, i\Delta x, k\Delta r)\\
&\quad+\frac{1}{12}\tilde{w}_{rrrr}(n\Delta t, i\Delta x, k\Delta r)(\Delta r)^2+o(\Delta r)^2,\\
\frac{\tilde{w}^n_{i,k+1}-\tilde{w}^n_{i,k}}{\Delta r} &= \tilde{w}_r(n\Delta t, i\Delta x, k\Delta r)+\frac{1}{2}\tilde{w}_{rr}(n\Delta t, i\Delta x, k\Delta_r)\Delta r\\
&\quad+o(\Delta r),\\
\frac{\tilde{w}^n_{i,k}-\tilde{w}^n_{i,k-1}}{\Delta r} &= \tilde{w}_r(n\Delta t, i\Delta x, (k-1)\Delta r)\\
&\quad+\frac{1}{2}\tilde{w}_{rr}(n\Delta t, i\Delta x, (k-1)\Delta_r)\Delta r+o(\Delta r),\\
\frac{\tilde{w}^n_{i+1,k}-\tilde{w}^n_{i,k}}{\Delta x} &= \tilde{w}_x(n\Delta t, i\Delta x, k\Delta r)+\frac{1}{2}\tilde{w}_{xx}(n\Delta t, i\Delta x, k\Delta r)\Delta x\\
&\quad+o(\Delta x),\\
\frac{\tilde{w}^n_{i,k}-\tilde{w}^n_{i-1,k}}{\Delta x} &= \tilde{w}_x(n\Delta t, (i-1)\Delta x, k\Delta r)\\
&\quad+\frac{1}{2}\tilde{w}_{xx}(n\Delta t, (i-1)\Delta x, k\Delta r)\Delta x+o(\Delta x),\\
\frac{\tilde{w}^{n+1}_{i,k}-\tilde{w}^n_{i,k}}{\Delta_t} &= \tilde{w}_t((n+1)\Delta t, i\Delta x, k\Delta r)+\frac{1}{2}\tilde{w}_{tt}(n\Delta t, i\Delta x, k\Delta_r)\Delta t\\
&\quad+o(\Delta t).
\end{aligned}$$

Hence, the consistency of the scheme follows from the continuity of the auxiliary HJB operator (note that the truncation error is at most of order one in each parameter, for the approximation of the first derivatives).

We now prove the local stability. To this end, set $\mathcal{I}_O := \{-p,\ldots,p\}\times\{-q,\ldots,q\}$, where $p, q\in\mathbb{N}$ are the largest possible natural numbers such that
$$[-p\Delta x, p\Delta x]\times[-q\Delta r, q\Delta r]\subset P_r(O),$$
with $P_r$ denoting the orthogonal projection of $]0,T]\times\mathbb{R}\times\mathbb{R}$ on $\mathbb{R}\times\mathbb{R}$. Using Assumption 5.2.5, we can write
$$\begin{aligned}
\left|\tilde{w}^{n+1}_{i,k}\right| = \Bigg|&\tilde{w}^n_{i,k}-\frac{\Delta t}{2}\left(\frac{i\Delta x\sigma}{\Delta r}\right)^2\left(\tilde{w}^n_{i,k}-\tilde{w}^n_{i,k-1}+\tilde{w}^n_{i,k}-\tilde{w}^n_{i,k+1}-\left(\tilde{w}^n_{i,k}-\tilde{w}^n_{i,k+1}\right)^2\right)\\
&+\frac{\Delta r\Delta t}{(\Delta x)^2}\frac{\max\left(\tilde{w}^n_{i,k}-\tilde{w}^n_{i-1,k}, \tilde{w}^n_{i,k}-\tilde{w}^n_{i+1,k}, 0\right)^2}{\tilde{w}^n_{i,k}-\tilde{w}^n_{i,k-1}}\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\le \left|\tilde{w}_{i,k}^{n}\right| + \frac{\Delta t}{2}\big(i\Delta x\sigma\big)^2 \left| \frac{\tilde{w}_{i,k}^{n} - \tilde{w}_{i,k-1}^{n} + \tilde{w}_{i,k}^{n} - \tilde{w}_{i,k+1}^{n}}{\Delta r} - \left(\frac{\tilde{w}_{i,k}^{n} - \tilde{w}_{i,k+1}^{n}}{\Delta r}\right)^2 \right| \\
&\quad - \frac{\Delta r \Delta t}{(\Delta x)^2} \frac{\max\left(\tilde{w}_{i,k}^{n} - \tilde{w}_{i-1,k}^{n}, \tilde{w}_{i,k}^{n} - \tilde{w}_{i+1,k}^{n}, 0\right)^2}{\tilde{w}_{i,k}^{n} - \tilde{w}_{i,k-1}^{n}} \\
&\le \left|\tilde{w}_{i,k}^{n}\right| + \Delta t\big(X_O\sigma\big)^2(\overline{K}'_O + \overline{K}_O^2) \\
&\quad + \frac{\Delta t}{(\Delta x)^2} \frac{\max\left(\tilde{w}_{i,k}^{n} - \tilde{w}_{i-1,k}^{n}, \tilde{w}_{i,k}^{n} - \tilde{w}_{i+1,k}^{n}, 0\right)^2}{K_O} \\
&\le \left|\tilde{w}_{i,k}^{n}\right| + \Delta t\big(X_O\sigma\big)^2(\overline{K}'_O + \overline{K}_O^2) + \frac{\Delta t L_O^2}{K_O},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\max_{i,k\in\mathcal{I}_O} \left|\tilde{w}_{i,k}^{n+1}\right| &\le \max_{i,k\in\mathcal{I}_O} \left|\tilde{w}_{i,k}^{0}\right| + n\Delta t\big(X_O\sigma\big)^2(\overline{K}'_O + \overline{K}_O^2) + n\frac{\Delta t L_O^2}{K_O} \\
&= \max_{i,k\in\mathcal{I}_O} \left|\tilde{w}_{i,k}^{0}\right| + T\big(X_O\sigma\big)^2(\overline{K}'_O + \overline{K}_O^2) + \frac{T L_O^2}{K_O} \\
&< \infty,
\end{aligned}
$$

and this proves the stability of the scheme. We have thus established that this explicit scheme converges to the viscosity solution of (5.10).

Let us now consider the more general case. We will need the following lemma.

**Lemma 5.2.7.** *Take $X \in \mathbb{R}^d$. Then the map*

$$
\begin{array}{rcl}
]0,\infty[ & \longrightarrow & \mathbb{R} \\
\widetilde{f}_X^* : T & \longmapsto & T f^*\big(-\frac{X}{T}\big)
\end{array}
\tag{5.21}
$$

*is strictly decreasing in $T$.*

*Proof.* First note that, due to the strict convexity of $f$, $f^*$ is also strictly convex and hence fulfills the following subgradient inequality,

$$
f^*(b) - f^*(a) > (b-a)\cdot\nabla f^*(a).
$$

Setting now $b = 0$ in the preceding inequality, we get

$$
a\cdot\nabla f^*(a) > f^*(a) \ge 0, \tag{5.22}
$$

because $f^*(0) = 0$. Computing the derivative of $\widetilde{f}_X^*$ with respect to $T$ we obtain

$$
\widetilde{f^*}'_X(T) = f^*\Big(-\frac{X}{T}\Big) - \frac{X}{T}\nabla_x f^*\Big(-\frac{X}{T}\Big),
$$

which is strictly negative, due to the preceding subgradient inequality. ∎

Suppose that $f$ is symmetric (i.e., $f(x) = f(-x)$, $\forall x \in \mathbb{R}$) and $b \neq 0$. (Note that this symmetry also holds for $f^*$). Since $f^*(w_x) = f^*(|w_x|)$, we obtain the following expression (scheme) for the term $f^*(w_x/w_r)$:

$$f^*\left(\frac{\Delta r}{\Delta x} \cdot \frac{\max\left(\tilde{w}^n_{i,k} - \tilde{w}^n_{i-1,k}, \tilde{w}^n_{i,k} - \tilde{w}^n_{i+1,k}\right)}{\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k-1}}\right).$$

Therefore we can derive the following generalization of the preceding scheme:

$$\begin{aligned}
&S(h, \Delta t, \Delta x, \Delta r, \tilde{w}^{n+1}_{i,k}, [\tilde{w}^n_{i+1,k}, \tilde{w}^n_{i-1,k}, \tilde{w}^n_{i,k+1}, \tilde{w}^n_{i,k-1}, \tilde{w}^n_{i,k}]) \\
&\quad = \frac{\tilde{w}^{n+1}_{i,k} - \tilde{w}^n_{i,k}}{\Delta t} + \frac{1}{2}\left(\frac{i\Delta x \sigma}{\Delta r}\right)^2 \big(\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k-1} + \tilde{w}^n_{i,k} - \tilde{w}^n_{i,k+1} \\
&\quad\quad - \left(\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k+1}\right)^2\big) - b \cdot i \, \Delta x \, F_{b,k}(\tilde{w}^n_{i,k}) \\
&\quad\quad - \frac{\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k-1}}{\Delta r} f^*\left(\frac{\Delta r}{\Delta x} \frac{\max\left(\tilde{w}^n_{i,k} - \tilde{w}^n_{i-1,k}, \tilde{w}^n_{i,k} - \tilde{w}^n_{i+1,k}, 0\right)}{\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k-1}}\right), \\
&\tilde{w}^0_{i,k} = 0,
\end{aligned}$$

where

$$F_{b,k}(\tilde{w}^n_{i,k}) = \begin{cases} \frac{\tilde{w}^n_{i,k+1} - \tilde{w}^n_{i,k}}{\Delta r}, & \text{if } \operatorname{sgn}(b \cdot i) > 0, \\ \frac{\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k-1}}{\Delta r}, & \text{if } \operatorname{sgn}(b \cdot i) \leq 0. \end{cases}$$

Using Lemma 5.2.7, we have

$$-\frac{\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k-1}}{\Delta r} f^*\left(\frac{\Delta r}{\Delta x} \frac{\max\left(\tilde{w}^n_{i,k} - \tilde{w}^n_{i-1,k}, \tilde{w}^n_{i,k} - \tilde{w}^n_{i+1,k}, 0\right)}{\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k-1}}\right)$$

is nonincreasing in $\tilde{w}^n_{i,k-1}$. Due to the definition of $F_{b,k}$, it is also nonincreasing in $\tilde{w}^n_{i,k-1}$, and the scheme is hence unconditionally nonincreasing in this parameter. Noting that $\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k-1} < 0$, and using the fact that $f^*$ is decreasing on $]-\infty, 0]$ and increasing on $[0, \infty[$ (due to its positivity, convexity and the fact that $f^*(0) = 0$), it follows that the scheme is nonincreasing in both $\tilde{w}^n_{i-1,k}$ and $\tilde{w}^n_{i+1,k}$. Again, the definition of $F_{b,k}$ and the same argumentation as before (for $|\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k+1}| \leq 1/2$, as seen above) allow us to deduce that the scheme is nonincreasing in $\tilde{w}^n_{i,k+1}$. We now present a sufficient condition under which $S$ is nonincreasing in $\tilde{w}^n_{i,k}$.
<u>First case:</u> $\max\left(\tilde{w}^n_{i,k} - \tilde{w}^n_{i-1,k}, \tilde{w}^n_{i,k} - \tilde{w}^n_{i+1,k}, 0\right) = 0$.
Consider the function

$$\psi_3 : \tilde{w}^n_{i,k} \longmapsto -\tilde{w}^n_{i,k} + \frac{\Delta t}{2}\left(\frac{i\Delta x \sigma}{\Delta r}\right)^2 \left(\tilde{w}^n_{i,k} - (\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k+1})^2\right) - b \cdot i \, \Delta x \Delta t \, F_{b,k}(\tilde{w}^n_{i,k}).$$

Its derivative is given by

$$\psi_3' : \tilde{w}^n_{i,k} \longmapsto -1 + \frac{\Delta t}{2}\left(\frac{i\Delta x \sigma}{\Delta r}\right)^2 \left(1 - 2(\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k+1})\right) + |b \cdot i \Delta x| \, \frac{\Delta t}{\Delta r}.$$

Then, for

$$|\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k+1}| \leq 1/2 \quad \text{and} \quad \Delta t\Big(\frac{3}{2}\Big(\frac{i\Delta x\sigma}{\Delta r}\Big)^2 + \frac{|b\cdot i\Delta x|}{\Delta r}\Big) \leq 1,$$

we have that $\psi_3' \leq 0$, and $S$ is hence nonincreasing in $\tilde{w}^n_{i,k}$.
<u>Second case:</u> $\max\big(\tilde{w}^n_{i,k} - \tilde{w}^n_{i-1,k}, \tilde{w}^n_{i,k} - \tilde{w}^n_{i+1,k}, 0\big) \neq 0$.
We can suppose w.l.o.g. that $\max\big(\tilde{w}^n_{i,k} - \tilde{w}^n_{i-1,k}, \tilde{w}^n_{i,k} - \tilde{w}^n_{i+1,k}, 0\big) = \tilde{w}^n_{i,k} - \tilde{w}^n_{i-1,k}$.
Consider now the following function

$$\psi_4 : \tilde{w}^n_{i,k} \longmapsto -\tilde{w}^n_{i,k} + \frac{\Delta t}{2}\left(\frac{i\Delta x\sigma}{\Delta r}\right)^2 \big(\tilde{w}^n_{i,k} - (\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k+1})^2\big) - b\cdot i\,\Delta x\Delta t\, F_{b,k}(\tilde{w}^n_{i,k})$$
$$- \Delta t\frac{\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k-1}}{\Delta r} f^*\left(\frac{\Delta r}{\Delta x}\frac{\tilde{w}^n_{i,k} - \tilde{w}^n_{i-1,k}}{\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k-1}}\right),$$

whose derivative is given by

$$\psi_4' : \tilde{w}^n_{i,k} \longmapsto -1 + \frac{\Delta t}{2}\left(\frac{i\Delta x\sigma}{\Delta r}\right)^2 \big(1 - 2(\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k+1})\big) + |b\cdot i\Delta x|\,\frac{\Delta t}{\Delta r}$$
$$- \frac{\Delta t}{\Delta r}\left(f^*\left(\frac{\Delta r}{\Delta x}\frac{\tilde{w}^n_{i,k} - \tilde{w}^n_{i-1,k}}{\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k-1}}\right)\right.$$
$$\left. + \frac{\Delta r}{\Delta x}\frac{\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k-1} - \tilde{w}^n_{i,k} + \tilde{w}^n_{i-1,k}}{\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k-1}}(f^*)'\left(\frac{\Delta r}{\Delta x}\frac{\tilde{w}^n_{i,k} - \tilde{w}^n_{i-1,k}}{\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k-1}}\right)\right).$$

As in the preceding special case, taking $h > 0$ such that $|\tilde{w}^m_{j,l} - \tilde{w}^{m'}_{j',l'}| \leq 1/2$, for $|(m,j,l) - (m',j',l')| \leq h$, and using the fact that $(f^*)'$ is negative on $]-\infty, 0[$, nonnegative otherwise and decreasing on the whole of $\mathbb{R}$, we can write

$$\begin{aligned}
& f^*\left(\frac{\Delta r}{\Delta x}\frac{\tilde{w}^n_{i,k} - \tilde{w}^n_{i-1,k}}{\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k-1}}\right) \\
&\quad + \frac{\Delta r}{\Delta x}\frac{\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k-1} - \tilde{w}^n_{i,k} + \tilde{w}^n_{i-1,k}}{\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k-1}}(f^*)'\left(\frac{\Delta r}{\Delta x}\frac{\tilde{w}^n_{i,k} - \tilde{w}^n_{i-1,k}}{\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k-1}}\right) \\
&\geq \frac{\Delta r}{\Delta x}\frac{\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k-1} - \tilde{w}^n_{i,k} + \tilde{w}^n_{i-1,k}}{\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k-1}}(f^*)'\left(\frac{\Delta r}{\Delta x}\frac{\tilde{w}^n_{i,k} - \tilde{w}^n_{i-1,k}}{\tilde{w}^n_{i,k} - \tilde{w}^n_{i,k-1}}\right) \\
&\geq -\frac{1}{\Delta x K_O}(f^*)'\Big(\frac{1}{2\Delta x K_O}\Big).
\end{aligned}$$

Finally, we get with the CFL condition

$$\Delta t\Big(\frac{3}{2}\Big(\frac{i\Delta x\sigma}{\Delta r}\Big)^2 + \frac{|b\cdot i\Delta x|}{\Delta r} + \frac{1}{\Delta x\Delta r K_O}(f^*)'\Big(\frac{1}{2\Delta x K_O}\Big)\Big) \leq 1$$

that $\psi_4'(w^n_{i,k}) \leq 0$, and the scheme is therefore locally $\delta$-monotone.
The consistency of the scheme can be proved in an analogous manner as above, using the preceding Taylor expansions and the fact that both max and $f^*$ are continuous functions.

We have now left to prove the local stability. But here again, using Assumption 5.2.5 we get

$$
\begin{aligned}
\left|\tilde{w}_{i,k}^{n+1}\right| &= \Bigg|\tilde{w}_{i,k}^{n} - \frac{\Delta t}{2}\left(\frac{i\Delta x\sigma}{\Delta r}\right)^2 \left(\tilde{w}_{i,k}^{n} - \tilde{w}_{i,k-1}^{n} + \tilde{w}_{i,k}^{n} - \tilde{w}_{i,k+1}^{n} - \ \left(\tilde{w}_{i,k}^{n} - \tilde{w}_{i,k+1}^{n}\right)^2\right) \\
&\quad - b \cdot i\,\Delta x \Delta t\, F_{b,k}(\tilde{w}_{i,k}^{n}) \\
&\quad - \Delta t \frac{\tilde{w}_{i,k}^{n} - \tilde{w}_{i,k-1}^{n}}{\Delta r} f^*\left(\frac{\Delta r}{\Delta x}\frac{\max\left(\tilde{w}_{i,k}^{n} - \tilde{w}_{i-1,k}^{n}, \tilde{w}_{i,k}^{n} - \tilde{w}_{i+1,k}^{n}, 0\right)}{\tilde{w}_{i,k}^{n} - \tilde{w}_{i,k-1}^{n}}\right) \\
&\leq \left|\tilde{w}_{i,k}^{n}\right| + \frac{\Delta t}{2}\left(\frac{i\Delta x\sigma}{\Delta r}\right)^2 \left|\tilde{w}_{i,k}^{n} - \tilde{w}_{i,k-1}^{n} + \tilde{w}_{i,k}^{n} - \tilde{w}_{i,k+1}^{n} - \ \left(\tilde{w}_{i,k}^{n} - \tilde{w}_{i,k+1}^{n}\right)^2\right| \\
&\quad - |b \cdot i\,\Delta x| \Delta t\, F_{b,k}(\tilde{w}_{i,k}^{n}) \\
&\quad - \Delta t \frac{\tilde{w}_{i,k}^{n} - \tilde{w}_{i,k-1}^{n}}{\Delta r} f^*\left(\frac{\Delta r}{\Delta x}\frac{\max\left(\tilde{w}_{i,k}^{n} - \tilde{w}_{i-1,k}^{n}, \tilde{w}_{i,k}^{n} - \tilde{w}_{i+1,k}^{n}, 0\right)}{\tilde{w}_{i,k}^{n} - \tilde{w}_{i,k-1}^{n}}\right) \\
&\leq \left|\tilde{w}_{i,k}^{n}\right| + \Delta t\left(X_O\sigma\right)^2(\overline{K}'_O + \overline{K}_O^2) + |b|\left|X_O\right|\Delta t\overline{K}_O + \overline{K}_O\Delta t f^*\left(\frac{L_O}{K_O}\right),
\end{aligned}
$$

which gives us recursively

$$
\begin{aligned}
\max_{i,k\in\mathcal{I}_O} |\tilde{w}_{i,k}^{n+1}| &\leq \max_{i,k\in\mathcal{I}_0} |\tilde{w}_{i,k}^{0}| + T\left(\left(X_O\sigma\right)^2(\overline{K}'_O + \overline{K}_O^2) + |b|\left|X_O\right|\overline{K}_O + \overline{K}_O f^*\left(\frac{L_O}{K_O}\right)\right) \\
&< \infty.
\end{aligned}
$$

Thus, the local stability is proved. This establishes that the preceding explicit scheme indeed converges to the viscosity solution.

### 5.2.3 Construction of a converging implicit scheme

Proving the $\delta$-monotonicity will turn out to be more obvious for the following implicit scheme than for the preceding explicit one. Moreover, the following implicit scheme will be unconditionally stable. Nevertheless, there will be two main issues which restrict its use. The first one follows from the fact that terms must be obtained by implicit computations, which implies that we have to find them before using them in the scheme (by applying in general a Newton-Raphson method). In this nonlinear case, this will result in an implementation error, which will be combined with the approximation error. The second issue follows from the fact that the local stability is difficult to obtain in practice (due to the appearance of a quotient term and the difficulty of computing the constants $\overline{K_O}$ and $L_O$, which will moreover impose restrictions on $\Delta x$ and $\Delta r$), as we will see below.

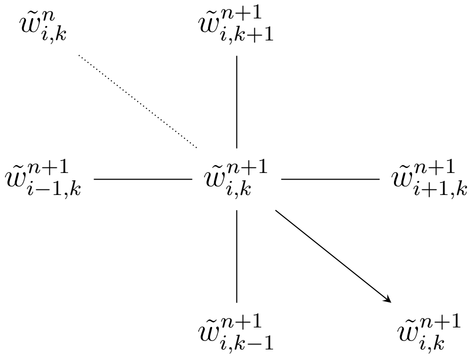
Let us consider the following scheme, where $b = 0$, $f(x) = \lambda x^2$, and $\lambda > 0$.

$$
\begin{aligned}
&S(h, \Delta t, \Delta x, \Delta r, \tilde{w}_{i,k}^{n+1}, [\tilde{w}_{i+1,k}^{n+1}, \tilde{w}_{i-1,k}^{n+1}, \tilde{w}_{i,k+1}^{n+1}, \tilde{w}_{i,k-1}^{n+1}, \tilde{w}_{i,k}^{n}]) \\
&\quad = \frac{\tilde{w}_{i,k}^{n+1} - \tilde{w}_{i,k}^{n}}{\Delta t} + \frac{1}{2}\left(\frac{i\Delta x\sigma}{\Delta r}\right)^2 \left(\tilde{w}_{i,k}^{n+1} - \tilde{w}_{i,k-1}^{n+1} + \tilde{w}_{i,k}^{n+1} - \tilde{w}_{i,k+1}^{n+1}\right.
\end{aligned}
$$

$$- \left(\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i,k+1}\right)^2\Big) - \frac{\Delta r}{4\lambda(\Delta x)^2} \cdot \frac{\max\left(\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i-1,k}, \tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i+1,k}, 0\right)^2}{\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i,k-1}},$$

$$\tilde{w}^{0}_{i,k} = 0,$$

whose stencil is represented below as:




First, note that

$$-\frac{\Delta r}{4\lambda(\Delta x)^2} \cdot \frac{\max\left(\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i-1,k}, \tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i+1,k}, 0\right)^2}{\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i,k-1}}$$

is nonincreasing in both $\tilde{w}^{n+1}_{i,k-1}, \tilde{w}^{n+1}_{i-1,k}$ and $\tilde{w}^{n+1}_{i+1,k}$. Take now $h > 0$ small enough such that $|\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i,k+1}| \leq 1/2$. Then,

$$\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i,k+1} - \left(\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i,k+1}\right)^2$$

is nonincreasing in $\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i,k-1}$, because the function $x - x^2$ is increasing in $x$ for $-1/2 \leq x \leq 1/2$. Since the first term is also nonincreasing in $\tilde{w}^{n}_{i,k}$, we have thus proved that the scheme is (unconditionally) monotone, for $|w^{n+1}_{i,k} - w^{n+1}_{i,k+1}| \leq 1/2$. It remains to prove its consistency and local stability. Classical computations using the Taylor expansion again yield:

$$\begin{aligned}
\frac{\tilde{w}^{n+1}_{i,k+1} + \tilde{w}^{n+1}_{i,k-1} - 2\tilde{w}^{n+1}_{i,k}}{(\Delta r)^2} &= \tilde{w}_{rr}((n+1)\Delta t, i\Delta x, k\Delta r) \\
&\quad + \frac{1}{12}\tilde{w}_{rrrr}((n+1)\Delta t, i\Delta x, k\Delta r)(\Delta r)^2 + o(\Delta r)^2, \\
\frac{\tilde{w}^{n+1}_{i,k+1} - \tilde{w}^{n+1}_{i,k}}{\Delta r} &= \tilde{w}_{r}((n+1)\Delta t, i\Delta x, k\Delta r) + \frac{1}{2}\tilde{w}_{rr}((n+1)\Delta t, i\Delta x, k\Delta_r)\Delta r \\
&\quad + o(\Delta r), \\
\frac{\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i,k-1}}{\Delta r} &= \tilde{w}_{r}((n+1)\Delta t, i\Delta x, (k-1)\Delta r) \\
&\quad + \frac{1}{2}\tilde{w}_{rr}((n+1)\Delta t, i\Delta x, (k-1)\Delta_r)\Delta r + o(\Delta r), \\
\frac{\tilde{w}^{n+1}_{i+1,k} - \tilde{w}^{n+1}_{i,k}}{\Delta x} &= \tilde{w}_{x}((n+1)\Delta t, i\Delta x, k\Delta r) + \frac{1}{2}\tilde{w}_{xx}((n+1)\Delta t, i\Delta x, k\Delta r)\Delta x
\end{aligned}$$

$$
\begin{aligned}
&\quad + \; o(\Delta x),\\
\frac{\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i-1,k}}{\Delta x} &= \tilde{w}_x((n+1)\Delta t, (i-1)\Delta x, k\Delta r)\\
&\quad + \; \frac{1}{2}\tilde{w}_{xx}((n+1)\Delta t, (i-1)\Delta x, k\Delta r)\Delta x + o(\Delta x),\\
\frac{\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n}_{i,k}}{\Delta_t} &= \tilde{w}_t((n+1)\Delta t, i\Delta x, k\Delta r) + \frac{1}{2}\tilde{w}_{tt}((n+1)\Delta t, i\Delta x, k\Delta_r)\Delta t\\
&\quad + \; o(\Delta t).
\end{aligned}
$$

Note that here again the truncation error is only of order one in each parameter for the approximation of the first derivatives. However, this order will be weakened because of implicit computation of the corresponding terms. Hence, the consistency of the scheme follows from the continuity of the auxiliary HJB operator. To prove its local stability, we have to require that $\sigma\Delta x/\Delta r$ is bounded. We use the fact that

$$
\max\left\{\left|\frac{\tilde{w}^{n+1}_{i,k+1} - \tilde{w}^{n+1}_{i,k}}{\Delta r}\right|, \left|\frac{\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i,k-1}}{\Delta r}\right|\right\} \leq \overline{K}_O,
$$

on a bounded open set $O$. Due to Assumption 5.2.5, we also have that

$$
\max\left\{\left|\frac{\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i-1,k}}{\Delta x}\right|, \left|\frac{\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i+1,k}}{\Delta x}\right|\right\} \leq L_O.
$$

Hence, expressing the differences as follows:

$$
\begin{aligned}
\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n}_{i,k} = \frac{\Delta t}{2}\left(\frac{i\Delta x\sigma}{\Delta r}\right)^2 & \left(\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i,k-1} + \tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i,k+1}\right.\\
&\left. - \left(\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i,k+1}, 0\right)^2\right)\\
&- \frac{\Delta t \Delta r}{(\Delta x)^2}\frac{\max\left(\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i-1,k}, \tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i+1,k}, 0\right)^2}{\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i,k-1}},
\end{aligned}
$$

we finally deduce

$$
\begin{aligned}
\max_{i,k\in\mathcal{I}_O}|\tilde{w}^{n+1}_{i,k}| &\leq \max_{i,k\in\mathcal{I}_0}|\tilde{w}^{0}_{i,k}| + \frac{3n\Delta t}{8}\left(\frac{i\Delta x\sigma}{\Delta r}\right)^2 + n\Delta t\frac{L_O^2}{\overline{K}_O}\\
&= \max_{i,k\in\mathcal{I}_0}|\tilde{w}^{0}_{i,k}| + \frac{3T}{8}\left(\frac{i\Delta x\sigma}{\Delta r}\right)^2 + T\frac{L_O^2}{\overline{K}_O}\\
&\leq \max_{i,k\in\mathcal{I}_0}|\tilde{w}^{0}_{i,k}| + \frac{3T|I_O|}{8}\left(\frac{\Delta x\sigma}{\Delta r}\right)^2 + T\frac{L_O^2}{\overline{K}_O}\\
&\leq \max_{i,k\in\mathcal{I}_0}|\tilde{w}^{0}_{i,k}| + \frac{3T|I_O|}{8}C^2 + T\frac{L_O^2}{\overline{K}_O},
\end{aligned}
$$

where $C \geq \sigma\Delta x/\Delta r$. Hence, this proves the stability of the scheme. Thus, the implicit scheme considered converges to the viscosity solution of (5.10).

In a more general framework (i.e. $b \neq 0$ and $f$ symmetric), as it was the case with the explicit scheme above, we can consider the following scheme:

$$
\begin{aligned}
&S(h, \Delta t, \Delta x, \Delta r, \tilde{w}^{n+1}_{i,k}, [\tilde{w}^{n+1}_{i+1,k}, \tilde{w}^{n+1}_{i-1,k}, \tilde{w}^{n+1}_{i,k+1}, \tilde{w}^{n+1}_{i,k-1}, \tilde{w}^{n}_{i,k}]) \\
&\quad = \frac{\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n}_{i,k}}{\Delta t} + \frac{1}{2}\left(\frac{i\Delta x\sigma}{\Delta r}\right)^2 \big(\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i,k-1} + \tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i,k+1} \\
&\qquad - \big(\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i,k+1}\big)^2\big) - b \cdot i\,\Delta x\, F_{b,k}(\tilde{w}^{n+1}_{i,k}) \\
&\qquad - \frac{\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i,k-1}}{\Delta r} f^*\left(\frac{\Delta r}{\Delta x} \frac{\max\big(\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i-1,k}, \tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i+1,k}, 0\big)}{\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i,k-1}}\right), \\
&\tilde{w}^0_{i,k} = 0,
\end{aligned}
$$

where

$$
F_{b,k}(\tilde{w}^{n+1}_{i,k}) = \begin{cases} \frac{\tilde{w}^{n+1}_{i,k+1} - \tilde{w}^{n+1}_{i,k}}{\Delta r}, & \text{if } \operatorname{sgn}(b \cdot i) > 0, \\ \frac{\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i,k-1}}{\Delta r}, & \text{if } \operatorname{sgn}(b \cdot i) \leq 0. \end{cases}
$$

In analogy to the previous argumentation, we can prove that this scheme is again (unconditionally) nonincreasing in $\tilde{w}^{n+1}_{i-1,k}$, $\tilde{w}^{n+1}_{i+1,k}$, $\tilde{w}^{n+1}_{i,k-1}$, $\tilde{w}^{n+1}_{i,k+1}$ and $\tilde{w}^{n}_{i,k}$ (when taking $h > 0$ small enough such that $|\tilde{w}^{n+1}_{i,k} - \tilde{w}^{n+1}_{i,k+1}| \leq 1/2$), and it is therefore monotone.
The consistency of the scheme can be proved in the same manner as beforehand, by using the preceding Taylor expansions and the fact that max and $f^*$ are both continuous.
Using step by step the arguments and computations used to prove the local stability of the explicit version of this scheme also yields its local stability (where here again we have to impose suitable restrictions on $\Delta x$ and $\Delta r$). Hence, this scheme converges to the unique viscosity solution, provided that a method to compute the implicit terms is given.

## 5.3 Numerical examples

In this section, we provide an application of the preceding results. Implementing our implicit schemes is a challenging task, due to mainly the following two reasons. First, classical computations in the spirit of the Newton-Raphson method would become rather involved in our case (because of the nonlinear part). This is due to the fact that, although the quadratic term can be linearized in order to make the task easier, there is still a quotient term to be dealt with. Second, the number of implicit variables to compute at each stage (five terms, as it can be seen in its corresponding stencil above) is another reason why we shall consider here only explicit schemes to visualize the value function of our maximization problem.

Nevertheless, even in the case of explicit schemes, we still face some issues in our modeling. For example, our initial condition involves exponential growth, which means that taking $T$ small leads to large terms in the exponent. Since in most of the available computer programs we cannot use values larger than $\exp(1000)$, no reasonable results are displayed. For instance, Matlab displays "`Inf`" for $\log(\exp(1000))$, instead of displaying 1000. Moreover, as we consider only bounded domains for the schemes, we have to impose boundary conditions, which results in approximation errors. As mentioned above (see Remark 5.1.4), we will use the approximated value of $W$ with $R_0$ taken large enough. However, we cannot take it as large as one wants to (see previous argumentation). Last but not least, the evaluation of the lower bound $K_O$ of the partial derivative $W_r$ presents another issue, since the latter one, which is in general difficult to obtain, is necessary to impose a CFL condition on the grid parameters.

### 5.3.1 Exponential value function

Let us start with approximating a known solution. In particular, we will thus show the accuracy of our scheme. In Schied and Schöneborn (2007), we have the following explicit formula for the value function of the problem when considering the one-dimensional case with $f(x) = \lambda x^2$, $\lambda > 0$, and $u(x) = -\exp(-Ax)$, $A > 0$:

$$V(T, X_0, R_0) = -\exp\left( - AR_0 + X_0^2 \sqrt{\frac{\lambda A^3 \sigma^2}{2}} \coth\left( T \sqrt{\frac{A\sigma^2}{2\lambda}} \right) \right).$$

In Figure 5.1, we show $\log(-V)$ for $R_0 = 1$, $\lambda = 0.1$, $A = 1$ and $\sigma = 0.1$. We now wish to approximate

$$w(T, X_0, R_0) := \log(-V)(T, X_0, R_0) = -AR_0 + X_0^2 \sqrt{\frac{\lambda A^3 \sigma^2}{2}} \coth\left( T \sqrt{\frac{A\sigma^2}{2\lambda}} \right),$$

with the help of the following explicit scheme:

$$\begin{aligned}
&S(h, \Delta t, \Delta x, \Delta r, \tilde{w}_{i,k}^{n+1}, [\tilde{w}_{i+1,k}^n, \tilde{w}_{i-1,k}^n, \tilde{w}_{i,k+1}^n, \tilde{w}_{i,k-1}^n, \tilde{w}_{i,k}^n]) \\
&\qquad = \frac{\tilde{w}_{i,k}^{n+1} - \tilde{w}_{i,k}^n}{\Delta t} + \frac{1}{2}\left( \frac{i\Delta x \sigma}{\Delta r} \right)^2 \Big( \tilde{w}_{i,k}^n - \tilde{w}_{i,k-1}^n + \tilde{w}_{i,k}^n - \tilde{w}_{i,k+1}^n \\
&\qquad\quad - \left( \tilde{w}_{i,k}^n - \tilde{w}_{i,k+1}^n \right)^2 - \frac{\Delta r}{4\lambda (\Delta x)^2} \cdot \frac{\max\left( \tilde{w}_{i,k}^n - \tilde{w}_{i-1,k}^n, \tilde{w}_{i,k}^n - \tilde{w}_{i+1,k}^n, 0 \right)^2}{\tilde{w}_{i,k}^n - \tilde{w}_{i,k-1}^n}, \\
&\tilde{w}_{i,k}^0 - \tilde{u}_{i,k}^0 = 0.
\end{aligned}$$

We cannot directly start with $n = 1$ as proposed above, since both $\tilde{w}_{i,k}^0$ and $\tilde{u}_{i,k}^0$ are undefined ($= \infty$), only their differences being defined and equal to 0. Moreover, we will need to impose some boundary conditions.

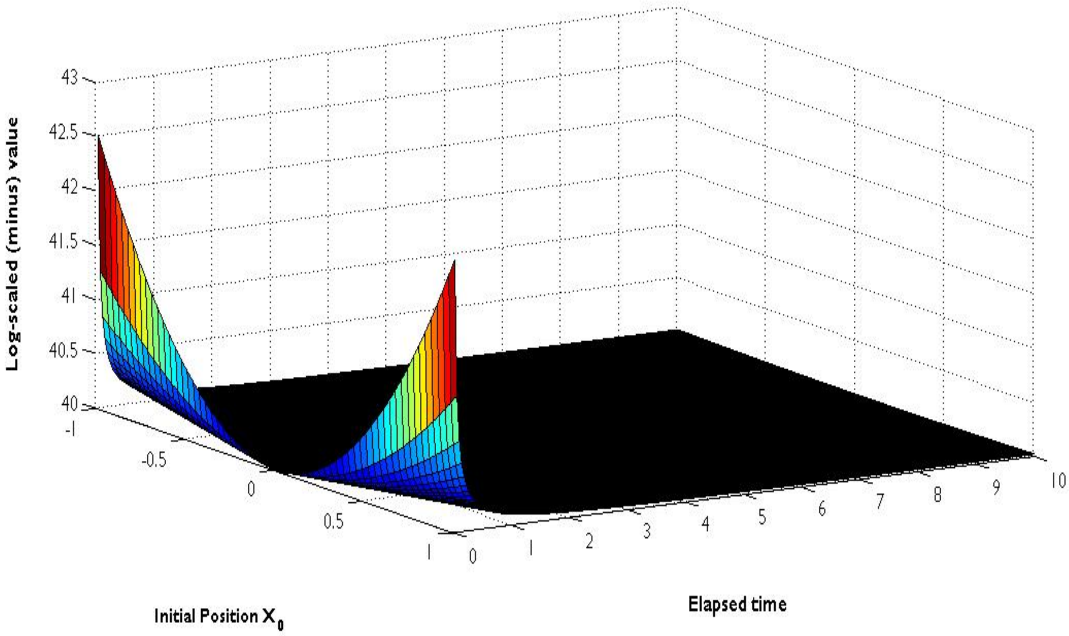



Figure 5.1: Logarithmic representation of the value function (negative values)

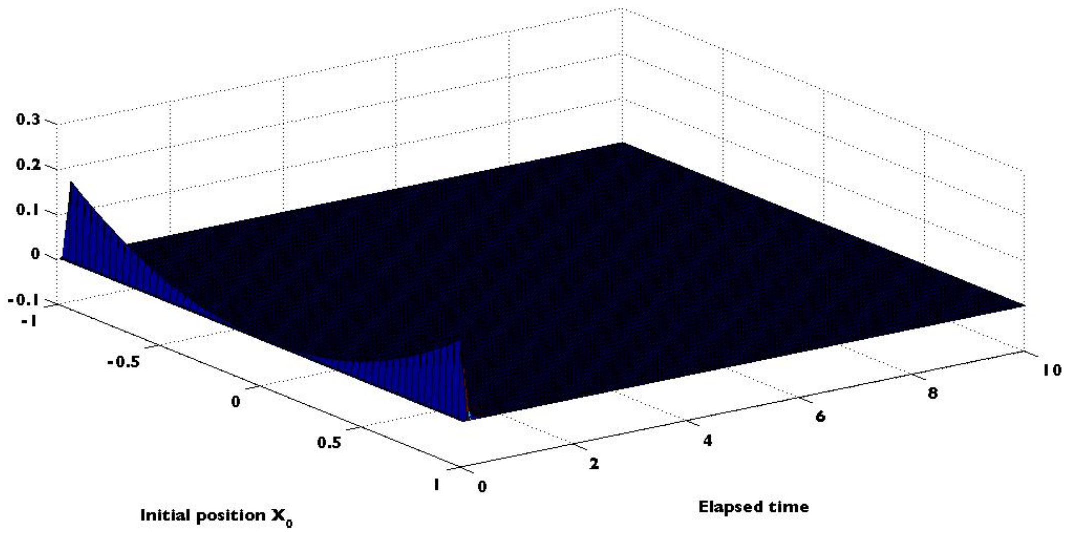



Figure 5.2: Implementation of the real solution in our scheme

First, note that in this simple case $w_r = -A$, and hence $K_O = A$. Therefore, our CFL condition (5.20) is here given by

$$\frac{3\Delta t}{8\lambda}\left(\frac{(X_O\sigma)^2(\Delta x)^2A^2 + \Delta r}{(\Delta r)^2(\Delta x)^2A^2}\right) \leq 1. \tag{5.23}$$

In the following, we will show that the preceding CFL condition was taken rather too restrictive, and our scheme does not need to necessarily fulfill it in order to converge. Subsequently we set

$$O = ]0.04, 10] \times ]-1, 1[ \times ]-50, 50[,\ dr = 0.833,\ dt = 0.04 \text{ and } dx = 0.0333.$$

We show the consistency of the scheme by implementing the real solution of (5.1), as shown in figure 5.2. With an absolute value of at most 0.18, this scheme seems to be very consistent. Using Proposition 5.1.3, we set the following initial condition

$$w^1_{i,k} = \log(B - u(k\Delta r - (i\Delta x)^2/n\Delta t)) = -A(k\Delta r + \lambda(i\Delta x)^2/(n\Delta t)).$$

We also have to add boundary conditions in our scheme. We define them as follows: denoting by $\pm x_{\max} := \pm i_m \cdot \Delta x$ and $\pm r_{\max} := \pm k_m \cdot \Delta r$ the extreme values taken by $x$ and $r$, respectively, on the grid, we have to set for $n \geq 1$:

$$\begin{aligned} w^n_{\pm i_m,k} &= \log(B - u(k\Delta r - (x_{\max})^2/n\Delta t)), \\ w^n_{i,\pm k_m} &= \log(B - u(\pm r_{\max} - (i\Delta x)^2/n\Delta t)). \end{aligned}$$

As already argued in Remark 5.1.4, this setting could only work out for large values of $R_0$, not for large values of $X_0$. However, in this particular case, this represents a very good setting of the boundary conditions (see figure 5.3). We also display the approximation error (figure 5.4). With at most 2.5 % error for small time $T$ (and at most 0.03% from time $T$ larger than 2), our scheme seems to give a very good approximation in this particular case, even if our CFL condition is not satisfied (the left-hand side term of (5.23) yields here 162.0043).

Nevertheless, things are not working so well when $B$ is not any longer supposed to be equal to zero, since we now have to deal with the second partial derivative of $w$ in its third parameter (whereas before it was equal to zero), and since the CFL condition can "explode", due to exponential terms. Let us fix it. To this end, we start with computing a strictly negative upper bound $K_O$, on a bounded set $O$, for $w_r$ (in order to set a CFL condition in our scheme). We compute

$$w_r(T, X_0, R_0) = \frac{-A\exp\left(-AR_0 + X_0^2\sqrt{\frac{\lambda A^3\sigma^2}{2}}\coth\left(T\sqrt{\frac{A\sigma^2}{2\lambda}}\right)\right)}{1 + \exp\left(-AR_0 + X_0^2\sqrt{\frac{\lambda A^3\sigma^2}{2}}\coth\left(T\sqrt{\frac{A\sigma^2}{2\lambda}}\right)\right)}.$$

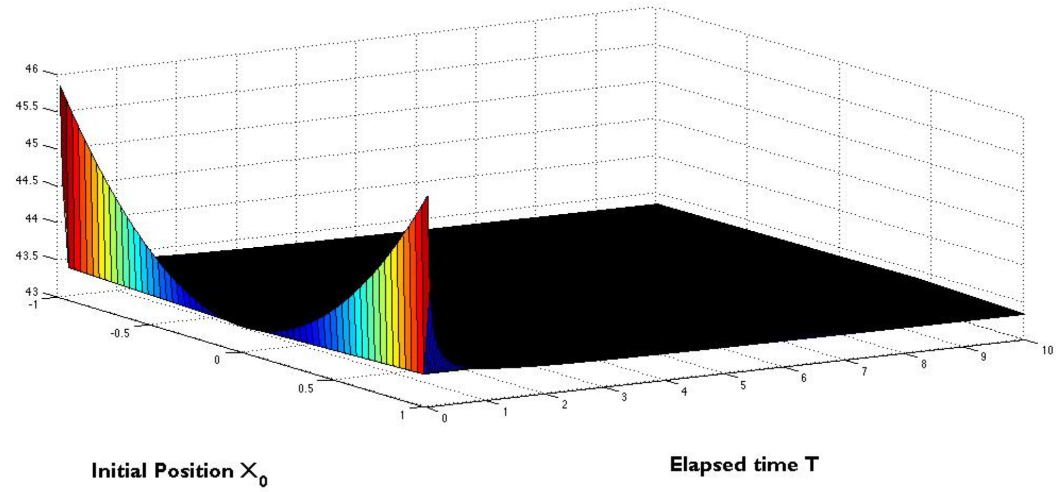



Figure 5.3: Value returned by the scheme

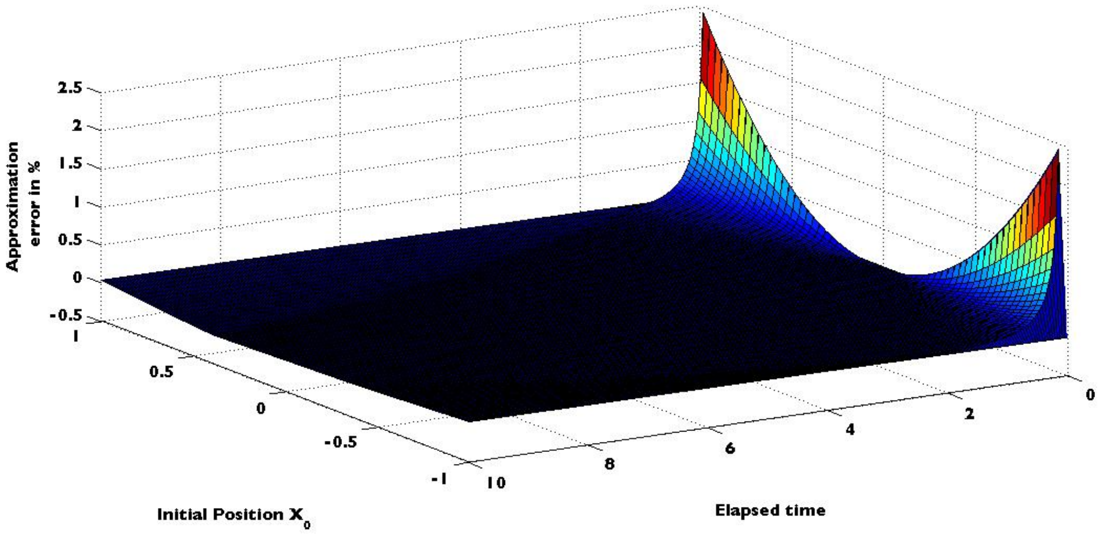



Figure 5.4: Approximation error for $\lambda = 0.1, \sigma = 0.1, R_0 = -43.3333$ and $A = 1$

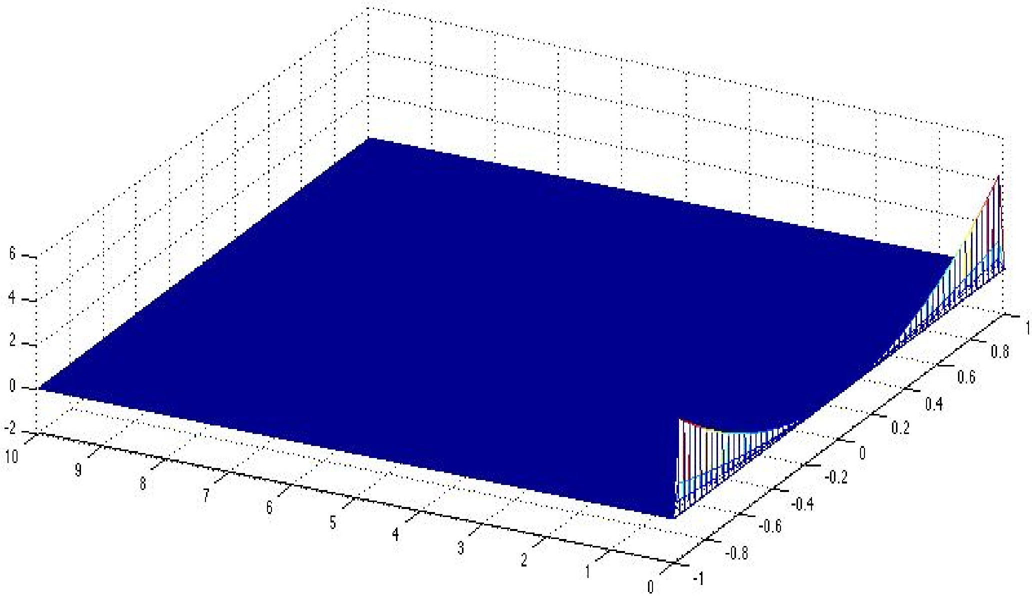



Figure 5.5: Implementation of the real solution in our scheme for $B = 1$.

Taking $O = ]\Delta t; T] \times ]X_{\min}; X_{\max}[ \times ]R_{\min}; R_{\max}[$, and using the fact that

$$x \longmapsto -\frac{x}{1+x}$$

is strictly decreasing for $x > 0$, we infer the following upper bound:

$$-K_O := \frac{-A\exp\left(-AR_{\max} + x_{\min}^2\sqrt{\frac{\lambda A^3\sigma^2}{2}}\coth\left(T\sqrt{\frac{A\sigma^2}{2\lambda}}\right)\right)}{1+\exp\left(-AR_{\max} + x_{\min}^2\sqrt{\frac{\lambda A^3\sigma^2}{2}}\coth\left(T\sqrt{\frac{A\sigma^2}{2\lambda}}\right)\right)} \geq w_r(T, X_0, R_0),$$

where $x_{\min}^2 := \inf_{x\in ]X_{\min};X_{\max}[} x^2$. Calculating this value of $K_O$ for our parameters $R_{\max} = 50,\ A = 5,\ \lambda = 0.1,\ x_{\min}^2 = 0$ gives us $K_O \leq 10^{-108}$ and a value of the left-hand side of (5.23) larger than $10^{217}$! To remedy to this issue, while maintaining our parameters $\lambda, \sigma$ and $A$, we have to allow only negative values for $R_0$. For instance, we may take $R_0 \in ]-50, -40[$. In order to set the CFL condition, we take moreover $\Delta t = 1/1250$. When implementing the real value in our scheme, we get at most the value 4 for a time $T$ smaller than one quarter. After this, things are getting better and we have values much closer to zero, more precisely, whose orders are at most $10^{-3}$ (see figure 5.5). Further, the approximation error of the real solution seems to be higher here, as represented in figure 5.6. With the preceding parameters, the left-hand side of (5.23) is equal to 0.9009.

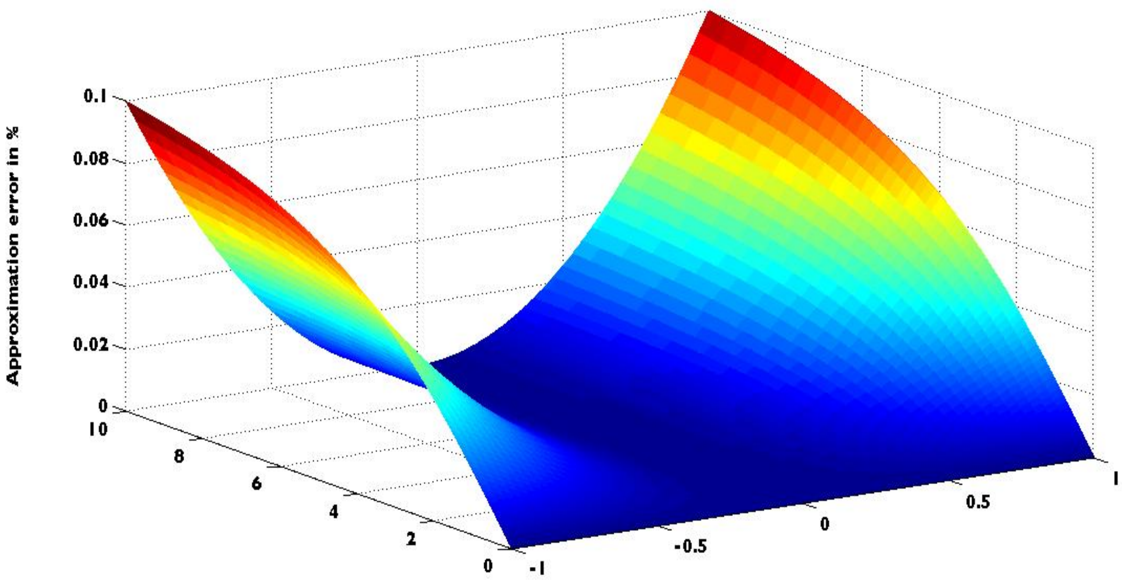



Figure 5.6: Approximation error for $B = 1$.

### 5.3.2 Convex combinations of exponential utility functions

In this subsection, we suppose that there exist $A_2 > 1 > A_1 > 0$ and $\mu \in ]0,1[$ such that

$$u(x) = \mu\big(1/A_1 - \exp(-A_1 x)\big) - (1-\mu)\exp\big(-A_2 x\big).$$

With this formulation of $u$, no well-known explicit formula for the associated value function (and hence for the solution of the associated auxiliary equation) can be given. Note that taking the corresponding convex combination of exponential value functions gives us only a supersolution of the corresponding HJB equation, as argued in Remark 3.3.2. Our goal in this section is to give an approximated value of the viscosity solution of (5.10). As discussed previously, we are going to use the explicit scheme to achieve this. Let us start by finding a lower bound $K_O$ for $w_r$. To this end, we use inequalities (2.9), (2.10) and (2.14) to infer

$$\begin{aligned} w_r = \frac{-V_r}{B-V} = & \frac{\mathbb{E}\Big[-u'\Big(\mathcal{R}_T^{\xi^*}\Big)\Big]}{B - \mathbb{E}\Big[u\Big(\mathcal{R}_T^{\xi^*}\Big)\Big]} \\ & \leq \frac{-1 - \mathbb{E}\Big[\exp(-A_1\Big(\mathcal{R}_T^{\xi^*}\Big)\Big]}{B - V_2(T, X_0, R_0)} \\ & \leq \frac{V_1(T, X_0, R_0) - (1 + 1/A_1)}{B - V_2(T, X_0, R_0)}, \end{aligned}$$

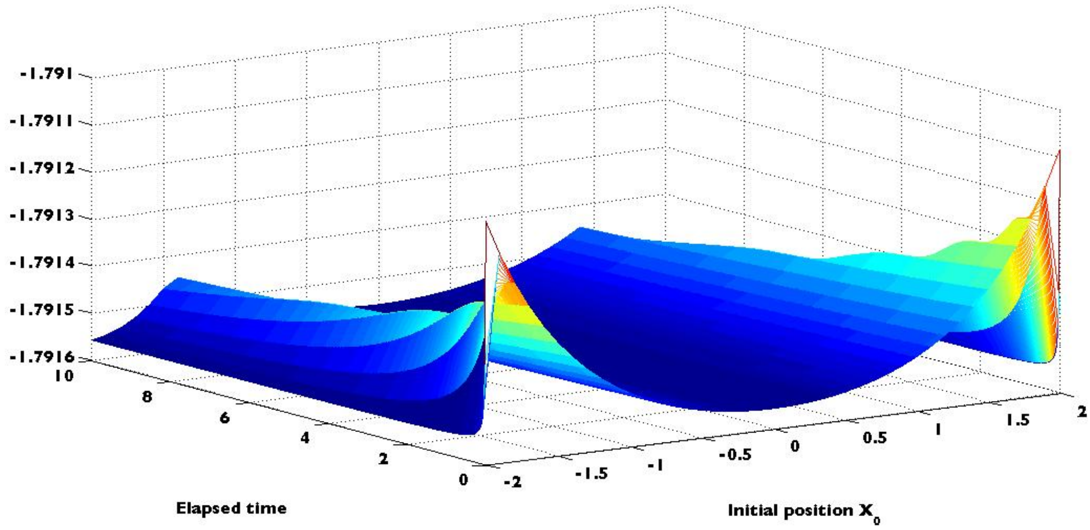



Figure 5.7: Approximated value of the solution of (5.1)

and in the case where $f(x) = \lambda x^2$, we get

$$w_r(T, X_0, R_0) = -\frac{1 + \exp\left( - A_1 R_0 + X_0^2 \sqrt{\frac{\lambda A_1^3 \sigma^2}{2}} \coth\left( T \sqrt{\frac{A_1 \sigma^2}{2\lambda}} \right) \right)}{B + \exp\left( - A_2 R_0 + X_0^2 \sqrt{\frac{\lambda A_2^3 \sigma^2}{2}} \coth\left( T \sqrt{\frac{A_2 \sigma^2}{2\lambda}} \right) \right)}.$$

For the sake of simplicity, take $B = 1$ (consequently, we will have to take $\mu/A_1 < 1$ in order for $\log(B - u)$ to be well-defined), then we obtain with

$$O = ]\Delta t; T] \times ] - X_{\max}; X_{\max}[ \times ]0; R_{\max}[,$$

the following lower bound:

$$w_r(T, X_0, R_0) \leq -\frac{1 + \exp(-A_1 R_{\max})}{1 + \exp\left( - A_2 R_{\max} + X_{\max}^2 \sqrt{\frac{\lambda A_2^3 \sigma^2}{2}} \coth\left( T \sqrt{\frac{A_2 \sigma^2}{2\lambda}} \right) \right)} =: -K_O.$$

In the sequel we set:

$$O = ]0.04, 10] \times ] - 2, 2[ \times ]0, 20[, \; dr = 0.8, \; dx = 0.1, \; dt = 0.013.$$

In Figure 5.7, the approximate value of the solution of (5.1) is displayed. In Figure 5.8, we give an approximated representation of the value function of (3.23). Note that the approximate displayed value function is concave for a fixed time when $x$ takes values far enough from the boundaries (e.g., $x \in [-1.45; 1.45]$), which is in concordance with Proposition 2.2.2.

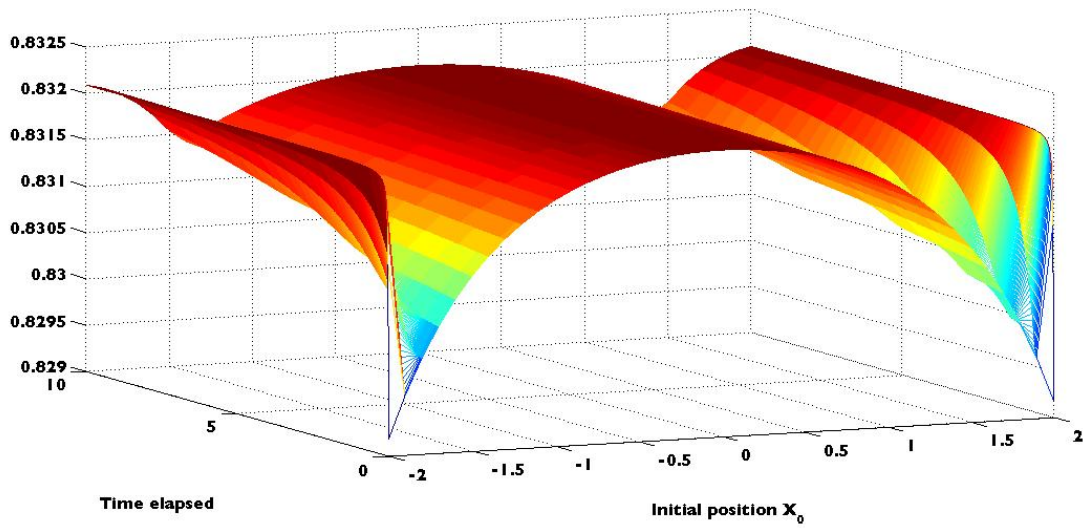



Figure 5.8: Approximated value of the solution of (3.23)

# Appendix A

# Matlab code

Source code for Figure 5.2, 5.3 and 5.4:

```
lambda=0.1;T=10;X=1;R=50;N=250;I=15;K=30;sigma=0.1;A=1;
dt=T/N;dx=X/(2*I);dr=R/(2*K);
x=(-X:dx:X);r=(-R:dr:R);t=(dt:dt:T);
evf=zeros(length(t),length(x),length(r));
for n=1:length(t)
for i=1:length(x)
for k=1:length(r)
evf(n,i,k)=-A*r(k)+(x(i))^2*sqrt((lambda*A^3*sigma^2)/2)*...
coth(t(n)*sqrt((A*sigma^2)/(2*lambda)));
end
end
end
% definition of tilde_u
tilde_u=zeros(length(t),length(x),length(r));
for n=1:length(t)
for i=1:length(x)
for k=1:length(r)
tilde_u(n,i,k)=A*(-r(k)+lambda*(x(i))^2/(t(n)));
end
end
end
% Initialisation of w for n=1
for i=1:length(x)
for k=1:length(r)
w(1,i,k)=tilde_u(1,i,k);
end
end
%Set up of the boundary condition w(n,i,1)
```

```
for n=2:length(t)
for i=1:length(x)
w(n,i,1)=tilde_u(n,i,1);
end
end
%Set up of the boundary condition w(n,1,k)
for n=2:length(t)
for k=1:length(r)
w(n,1,k)=tilde_u(n,1,k);
end
end
%Set up of the boundary condition w(n,i,length(r))
for n=2:length(t)
for i=1:length(x)
w(n,i,length(r))=tilde_u(n,i,length(r));
end
end
%Set up of the boundary condition w(n,length(x),k)
for n=2:length(t)
for k=1:length(r)
w(n,length(x),k)=tilde_u(n,length(x),k);
end
end
% Implementation of the explicit scheme for n>1
for n=1:length(t)-1,
for i=2:length(x)-1,
for k=2:length(r)-1,
w(n+1,i,k)= w(n,i,k)-dt/2*(i*dx*sigma)^2*((2*w(n,i,k)-w(n,i,k-1)...
-w(n,i,k+1))/(dr)^2-((w(n,i,k)-w(n,i,k+1))/dr)^2) ...
+dt/(4*lambda)*(max([(w(n,i,k)-w(n,i-1,k))/dx;...
(w(n,i,k)-w(n,i+1,k))/dx;0])^2)...
/((w(n,i,k)-w(n,i,k-1))/dr);
end
end
end
surf(x,t,w(:,:,9))
% Implementation of the real solution in the explicit scheme for n>1
s=zeros(length(t),length(x),length(r));
for n=2:length(t)-1,
for i=2:length(x)-1,
for k=2:length(r)-1,
```

```
s(n+1,i,k)=evf(n+1,i,k)-evf(n,i,k)...
-0.5*dt*(i*dx*sigma)^2*((evf(n,i,k)-evf(n,i,k+1))/dr)^2 ...
-dt/(4*lambda)*max([(evf(n,i,k)-evf(n,i-1,k))/dx...
;(evf(n,i,k)-evf(n,i+1,k))/dx;0])^2/((evf(n,i,k)-evf(n,i,k-1))/dr);
end
end
end
surf(x,t,s(:,:,9))
% Approximation error in %
h=zeros(length(t),length(x),length(r));
for n=1:length(t),
for i=1:length(x)
for k=2:length(r),
h(n,i,k)=100*(evf(n,i,k)-w(n,i,k))/evf(n,i,k);
end
end
end
surf(x,t,h(:,:,9))
```

Source code for Figure 5.5 and 5.6:

```
lambda=0.1;T=10;X=1;R=-50;N=1250;I=15;K=30;sigma=0.1;A=5;
dt=T/N;dx=X/(2*I);dr=-R/(2*K);x=(-X:dx:X);r=(R:dr:R+10);t=(dt:dt:T);
% Computation of the right hand-side of (5.17)
k_o=A*exp(-A*R)/(1+exp(-A*R));
CFL=3*dt/(8*lambda)*(2*X*sigma^2*dx^2*k_o^2+dr^3)/(dr^2*dx^2*k_o^2);
% definition of tilde_u
for n=1:length(t)
for i=1:length(x)
for k=1:length(r)
tilde_u(n,i,k)=log(1+exp(A*(-r(k)+lambda*(x(i))^2/(t(n)))));
end
end
end
%definition of the exponential value function
evf=zeros(length(t),length(x),length(r));
for n=1:length(t)
for i=1:length(x)
for k=1:length(r)
evf(n,i,k)=log(1+exp(-A*r(k)+(x(i))^2*sqrt((lambda*A^3*sigma^2)/2)*...
coth(t(n)*sqrt((A*sigma^2)/(2*lambda)))));
end
```

```
end
end
% Initialisation of w for n=1
for i=1:length(x)
for k=1:length(r)
w(1,i,k)=tilde_u(1,i,k);
end
end
%Set up of the boundary condition w(n,i,1)
for n=2:length(t)
for i=1:length(x)
w(n,i,1)=tilde_u(n,i,1);
end
end
%Set up of the boundary condition w(n,1,k)
for n=2:length(t)
for k=1:length(r)
w(n,1,k)=tilde_u(n,1,k);
end
end
%Set up of the boundary condition w(n,i,length(r))
for n=2:length(t)
for i=1:length(x)
w(n,i,length(r))=tilde_u(n,i,length(r));
end
end
%Set up of the boundary condition w(n,length(x),k)
for n=2:length(t)
for k=1:length(r)
w(n,length(x),k)=tilde_u(n,length(x),k);
end
end
% Implementation of the explicit scheme for n>1
for n=1:length(t)-1,
for i=2:length(x)-1,
for k=2:length(r)-1,
w(n+1,i,k)= w(n,i,k)-dt/2*(i*dx*sigma)^2*((2*w(n,i,k)-w(n,i,k-1)...
-w(n,i,k+1))/(dr)^2-((w(n,i,k)-w(n,i,k+1))/dr)^2) ...
+dt/(4*lambda)*(max([(w(n,i,k)-w(n,i-1,k))/dx;...
(w(n,i,k)-w(n,i+1,k))/dx;0])^2)...
/((w(n,i,k)-w(n,i,k-1))/dr);
```

```
end
end
end
% Approximation error in %
g=zeros(length(t),length(x),length(r));
for n=1:length(t),
for i=1:length(x)
for k=2:length(r),
g(n,i,k)=100*(evf(n,i,k)-w(n,i,k))/evf(n,i,k);
end
end
end
mesh(x,t,g(:,:,13))
% Implementation of the exact value in our scheme
s=zeros(length(t),length(x),length(r));
for n=2:length(t)-1,
for i=2:length(x)-1,
for k=2:length(r)-1,
s(n+1,i,k)=evf(n+1,i,k)-evf(n,i,k)...
-0.5*dt*(i*dx*sigma)^2*((evf(n,i,k)-evf(n,i,k+1))/dr)^2 ...
-dt/(4*lambda)*max([(evf(n,i,k)-evf(n,i-1,k))/dx;...
(evf(n,i,k)-evf(n,i+1,k))/dx;0])^2/((evf(n,i,k)-evf(n,i,k-1))/dr);
end
end
end
mesh(x,t,s(:,:,13))
```

# Bibliography


R. Almgren. Optimal execution with nonlinear impact functions and trading-enhanced risk. *Applied Mathematical Finance 10*, pages 1–18, 2003.

R. Almgren and N. Chriss. Optimal execution of portfolio transactions. *Journal of Risk*, 3:5–40, 2001.

R. Almgren and J. Lorenz. Adaptive arrival price. *Trading*, 2007(1):59–66, 2007.

Peter Bank and Dietmar Baum. Hedging and portfolio optimization in financial markets with a large trader. *Math. Finance*, 14(1):1–18, 2004. ISSN 0960-1627. doi: 10.1111/j.0960-1627.2004.00179.x. URL http://dx.doi.org/10.1111/j.0960-1627.2004.00179.x.

G. Barles. An introduction to the theory of viscosity solutions for first-order Hamilton-Jacobi equations and applications. In *Hamilton-Jacobi equations: approximations, numerical analysis and applications*, volume 2074 of *Lecture Notes in Math.*, pages 49–109. Springer, Heidelberg, 2013. doi: 10.1007/978-3-642-36433-4_2. URL http://dx.doi.org/10.1007/978-3-642-36433-4_2.

G. Barles and P. E. Souganidis. Convergence of approximation schemes for fully nonlinear second order equations. *Asymptotic Anal.*, 4(3):271–283, 1991.

Guy Barles and Espen Robstad Jakobsen. On the convergence rate of approximation schemes for Hamilton-Jacobi-Bellman equations. *M2AN Math. Model. Numer. Anal.*, 36(1):33–54, 2002. ISSN 0764-583X. doi: 10.1051/m2an:2002002. URL http://dx.doi.org/10.1051/m2an:2002002.

D.P. Bertsekas and S.E. Shreve. *Stochastic optimal control: The discrete time case*, volume 139 of *Mathematics in Science and Engineering*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978. ISBN 0-12-093260-1.

D. Bertsimas and A.W. Lo. Optimal control of execution costs. *Journal of Financial Markets*, 1(1):1–50, 1998.

J. Frédéric Bonnans, Élisabeth Ottenwaelter, and Housnaa Zidani. A fast algorithm for the two dimensional HJB equation of stochastic control. *M2AN Math. Model. Numer. Anal.*, 38(4):723–735, 2004. ISSN 0764-583X. doi: 10.1051/m2an:2004034. URL http://dx.doi.org/10.1051/m2an:2004034.

B. Bouchard and M. Nutz. Weak dynamic programming for generalized state constraints. *SIAM J. Control Optim.*, 50(6):3344–3373, 2012. ISSN 0363-0129. URL http://dx.doi.org/10.1137/110852942.

B. Bouchard and N. Touzi. Weak dynamic programming principle for viscosity solutions. *SIAM J. Control Optim.*, 49(3):948–962, 2011. ISSN 0363-0129. URL http://dx.doi.org/10.1137/090752328.

Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011. ISBN 978-0-387-70913-0.

Ariela Briani, Fabio Camilli, and Hasnaa Zidani. Approximation schemes for monotone systems of nonlinear second order partial differential equations: convergence result and error estimate. *Differ. Equ. Appl.*, 4(2):297–317, 2012. ISSN 1847-120X. doi: 10.7153/dea-04-18. URL http://dx.doi.org/10.7153/dea-04-18.

Umut Çetin, Robert A. Jarrow, and Philip Protter. Liquidity risk and arbitrage pricing theory. *Finance Stoch.*, 8(3):311–341, 2004. ISSN 0949-2984. doi: 10.1007/s00780-004-0123-x. URL http://dx.doi.org/10.1007/s00780-004-0123-x.

P. Cheridito, H. M. Soner, N. Touzi, and N. Victoir. Second-order backward stochastic differential equations and fully nonlinear parabolic pdes. *Communications on Pure and Applied Mathematics*, 60(7):1081–1110, 2007.

M.G. Crandall, H. Ishii, and P.L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc*, 27(1):1–67, 1992.

William F. Donoghue, Jr. *Monotone matrix functions and analytic continuation.* Springer-Verlag, New York-Heidelberg, 1974. Die Grundlehren der mathematischen Wissenschaften, Band 207.

N. Dunford and J. T. Schwartz. *Linear operators. Part I.* Wiley Classics Library. John Wiley & Sons, Inc., New York, 1988. ISBN 0-471-60848-3. General theory, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1958 original, A Wiley-Interscience Publication.

Arash Fahim, Nizar Touzi, and Xavier Warin. A probabilistic numerical method for fully nonlinear parabolic PDEs. *Ann. Appl. Probab.*, 21(4):1322–1364, 2011. ISSN 1050-5164. URL http://dx.doi.org/10.1214/10-AAP723.

W.H. Fleming and H.M. Soner. *Controlled Markov processes and viscosity solutions*, volume 25. Springer Verlag, 2006.

H. Föllmer and A. Schied. *Stochastic finance. An introduction in discrete time*. Walter de Gruyter & Co., Berlin, 3rd revised and extended edition, 2011. ISBN 978-3-11-021804-6.

G. Huberman and W. Stanzl. Price manipulation and quasi-arbitrage. *Econometrica*, 72(4):1247–1275, 2004. ISSN 0012-9682. URL http://dx.doi.org/10.1111/j.1468-0262.2004.00531.x.

P. J. Hunt and J. E. Kennedy. *Financial derivatives in theory and practice*. Wiley Series in Probability and Statistics. John Wiley & Sons, Ltd., Chichester, revised edition, 2004. ISBN 0-470-86358-7. URL http://dx.doi.org/10.1002/0470863617.

Robert A Jarrow. Derivative security markets, market manipulation, and option pricing theory. *Journal of Financial and Quantitative Analysis*, 29(02):241–261, 1994.

S. Koike and O. Ley. Comparison principle for unbounded viscosity solutions of degenerate elliptic PDEs with gradient superlinear terms. *J. Math. Anal. Appl.*, 381(1):110–120, 2011. ISSN 0022-247X. URL http://dx.doi.org/10.1016/j.jmaa.2011.03.009.

Holger Kraft and Christoph Kühn. Large traders and illiquid options: hedging vs. manipulation. *J. Econom. Dynam. Control*, 35(11):1898–1915, 2011. ISSN 0165-1889. doi: 10.1016/j.jedc.2011.06.001. URL http://dx.doi.org/10.1016/j.jedc.2011.06.001.

S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.

N. V. Krylov. *Controlled diffusion processes*, volume 14 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin, 2009. ISBN 978-3-540-70913-8. Translated from the 1977 Russian original by A. B. Aries, Reprint of the 1980 edition.

A. S. Kyle. Continuous auctions and insider trading. *Econometrica: Journal of the Econometric Society*, pages 1315–1335, 1985.

B. Louvel. Cassation partielle avec renvoi r 12-87.416 fp-p+b+r+i, March 2014. URL https://www.courdecassation.fr/IMG/CC_crim_arret1193_140319.pdf.

P.-A. Meyer. *Probability and potentials*. Blaisdell Publishing Co. Ginn and Co., Waltham, Mass.-Toronto, Ont.-London, 1966.

A. M. Oberman. Convergent difference schemes for degenerate elliptic and parabolic equations: Hamilton-Jacobi equations and free boundary problems. *SIAM J. Numer. Anal.*, 44(2):879–895 (electronic), 2006. ISSN 0036-1429. URL http://dx.doi.org/10.1137/S0036142903435235.

H. Pham. *Continuous-time stochastic control and optimization with financial applications*, volume 61 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin, 2009. ISBN 978-3-540-89499-5. URL http://dx.doi.org/10.1007/978-3-540-89500-8.

D. M. Pooley, P. A. Forsyth, and K. R. Vetzal. Numerical convergence properties of option pricing PDEs with uncertain volatility. *IMA J. Numer. Anal.*, 23(2): 241–267, 2003. ISSN 0272-4979. URL http://dx.doi.org/10.1093/imanum/23.2.241.

P. E. Protter. *Stochastic integration and differential equations*, volume 21 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, second edition, 2004. ISBN 3-540-00313-4. Stochastic Modelling and Applied Probability.

D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999. ISBN 3-540-64325-7. doi: 10.1007/978-3-662-06400-9. URL http://dx.doi.org/10.1007/978-3-662-06400-9.

U. Rieder. Measurable selection theorems for optimization problems. *Manuscripta Math.*, 24(1):115–131, 1978. ISSN 0025-2611.

R. T. Rockafellar. *Convex analysis*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997. Reprint of the 1970 original, Princeton Paperbacks.

A. Schied and T. Schöneborn. Optimal portfolio liquidation for cara investors. *Munich Personal RePEc Archive*, 2007. URL http://mpra.ub.uni-muenchen.de/5075/1/MPRA_paper_5075.pdf.

A. Schied and T. Schöneborn. Optimal basket liquidation with finite time horizon for CARA investors. *Preprint, TU Berlin*, 2008. URL http://www.alexschied.de/BasketCARA.pdf.

A. Schied and T. Schöneborn. Risk aversion and the dynamics of optimal liquidation strategies in illiquid markets. *Finance and Stochastics*, 13(2):181–204, 2009.

A. Schied, T. Schöneborn, and M. Tehranchi. Optimal basket liquidation for CARA investors is deterministic. *Appl. Math. Finance*, 17(6):471–489, 2010. URL http://dx.doi.org/10.1080/13504860903565050.

T. Schöneborn. *Trade execution in illiquid markets. Optimal stochastic control and multi-agent equilibria.* PhD thesis, TU Berlin, 2008.

A. Tourin. An introduction to finite difference methods for pdes in finance. *Book Chapter: Nizar Touzi, Optimal Stochastic Target problems, and Backward SDE, Fields Institute Monographs*, 29:201–212, 2011.

N. Touzi. *Optimal stochastic control, stochastic target problems, and backward SDE*, volume 29 of *Fields Institute Monographs*. Springer, New York; Fields Institute for Research in Mathematical Sciences, Toronto, ON, 2013. ISBN 978-1-4614-4285-1; 978-1-4614-4286-8. With Chapter 13 by Angès Tourin.

Nizar Touzi. *Stochastic control problems, viscosity solutions and application to finance.* Scuola normale superiore, 2004.

D. H. Wagner. Survey of measurable selection theorems: an update. In *Measure theory, Oberwolfach 1979 (Proc. Conf., Oberwolfach, 1979)*, volume 794 of *Lecture Notes in Math.*, pages 176–219. Springer, Berlin-New York, 1980.

Xavier Warin. Some non monotone schemes for hamilton-jacobi-bellman equations. *arXiv preprint arXiv:1312.5052*, 2013.

David Vernon Widder. *The Laplace Transform.* Princeton Mathematical Series, v. 6. Princeton University Press, Princeton, N. J., 1941.

D. Williams. *Probability with martingales.* Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991. ISBN 0-521-40455-X; 0-521-40605-6.

J. Yong and X. Y. Zhou. *Stochastic controls: Hamiltonian systems and HJB equations*, volume 43 of *Applications of Mathematics (New York).* Springer-Verlag, New York, 1999. ISBN 0-387-98723-1.