Markovian Integral Equations and Path-Dependent Partial Differential Equations

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Abstract

This thesis provides a construction of solutions to Markovian integral equations. By introducing path-dependent diffusion processes, this yields a general existence and uniqueness result for mild solutions to semilinear parabolic path-dependent partial differential equations (PPDEs). In this connection, we verify that mild solutions are also solutions in a viscosity sense.

In the first part of the thesis, we analyze multidimensional Markovian integral equations that are formulated with an underlying time-inhomogeneous progressive Markov process that has Borel measurable transition probabilities. For this purpose, regularity conditions with respect to Borel measures are presented, and relevant facts on Markov processes and additive maps are reviewed. Our goal is to establish uniqueness, stability, existence, and non-extendibility of solutions among a certain class of mappings. By requiring the Feller property of the Markov process, we give weak conditions under which solutions become continuous. Moreover, we prove a multidimensional Feynman-Kac formula and a one-dimensional global existence and uniqueness result.

In the second part, we deal with semilinear parabolic PPDEs that are based on horizontal and vertical derivatives of non-anticipative functionals on path spaces. Within this infinite-dimensional framework, measurable structures and topologies are discussed, and the time and space differential operators are recalled. Then we consider path-dependent diffusion processes, which may fail to be Markov, but whose path processes can be turned into diffusions. We thereby infer that solutions to the associated Markovian integral equations lead to mild solutions to PPDEs. At last, various notions of viscosity solutions are compared and the evidence that every mild solution can indeed be regarded as a viscosity solution is provided under a weak continuity assumption.

Zusammenfassung

Diese Dissertation stellt eine Konstruktion von Lösungen zu Markovschen Integralgleichungen bereit. Indem pfadabhängige Diffusionsprozesse eingeführt werden, ergibt dies ein allgemeines Existenz- und Eindeutigkeitsresultat für milde Lösungen zu semilinearen parabolischen pfadabhängigen partiellen Differentialgleichungen (PPDGLen). In diesem Zusammenhang weisen wir nach, dass milde Lösungen auch Lösungen in einem Viskositätssinne sind.
In dem ersten Teil der Dissertation analysieren wir mehrdimensionale Markovsche Integralgleichungen, die mit einem zugrunde liegenden zeitlich inhomogenen progressiven Markovschen Prozess, der Borel-messbare Übergangswahrscheinlichkeiten besitzt, formuliert werden. Für diesen Zweck werden Regularitätsbedingungen bezüglich Borel-Maße vorgestellt und relevante Tatsachen über Markovsche Prozesse und additive Abbildungen nachgeprüft. Unser Ziel ist es, Eindeutigkeit, Stabilität und Nicht-Erweiterbarkeit von Lösungen innerhalb einer Klasse von Abbildungen festzustellen. Indem man die Feller-Eigenschaft des Markov-Prozesses fordert, geben wir schwache Bedingungen an, unter denen Lösungen stetig sind. Außerdem beweisen wir eine mehrdimensionale Feynman-Kac Formel und ein eindimensionales globales Existenz- und Eindeutigkeitsresultat.
Im zweiten Teil befassen wir uns mit semilinearen parabolischen PPDGLen, die auf den horizontalen und vertikalen Ableitungen nicht-antizipativer Funktionale auf Pfadräumen beruhen. Unter diesen unendlich-dimensionalen Rahmenbedingungen werden messbare Strukturen und Topologien diskutiert, und die Zeit- und Raum-Differentialoperatoren abgerufen. Dann betrachten wir pfadabhängie Diffusionsprozesse, für die die Markov-Eigenschaft fehlschlagen kann, deren Pfadprozesse jedoch zu Diffusionen gemacht werden können. Damit folgern wir, dass Lösungen zu den verbundenen Markovschen Integralgleichungen zu milden Lösungen zu PPDGLen führen. Schließlich werden vielfältige Begriffe von Viskositätslösungen verglichen und der Nachweis, dass jede milde Lösung tatsächlich als Viskositätslösung angesehen werden kann, unter einer schwachen Stetigkeitsannahme erbracht.

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Contents

Chapter 1

Introduction

1.1 Markovian integral equations

Markovian integral equations arise when dealing with diffusion processes and mild solutions to semilinear parabolic partial differential equations (PDEs). This fact was utilized by Dynkin [13, 14] to give probabilistic formulas for mild solutions via the Laplace functionals of superprocesses. In this context, Schied [32] used Markovian integral equations to solve problems of optimal stochastic control in mathematical finance. As we will verify, the connection of Markovian equations to PDEs can also be extended to path-dependent partial differential equations (PPDEs). Inspired by the applications of one-dimensional Markovian equations, the aim of this thesis is to construct solutions in a multidimensional framework.

Let S be a Polish space and $T > 0$. Suppose that $\mathscr{X} = (X, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ is a consistent progressive Markov process on some measurable space $(\Omega, \mathscr{F})$ with state space S that has Borel measurable transition probabilities. We will consider the following *multidimensional Markovian integral equation* coupled with a terminal value condition:

$$
\begin{aligned}
E_{r,x}[u(t, X_t)] &= u(r, x) + E_{r,x}\bigg[\int_r^t f(s, X_s, u(s, X_s))\, \kappa(ds)\bigg],\\
u(T, x) &= g(x)
\end{aligned}
\tag{1.1}
$$

for all $r, t \in [0, T]$ with $r \leq t$ and each $x \in S$. Here, we implicitly assume that $k \in \mathbb{N}$ and κ is an $[-\infty, \infty]^k$-valued map on $\Omega \times \mathscr{B}([0, T])$ whose coordinate functions $\kappa_1, \ldots, \kappa_k$ are continuous additive functionals of $\mathscr{X}$. It is required that $|\kappa_i|([r, t]) \leq c\mu([r, t])$ for all $i \in \{1, \ldots, k\}$, each $r, t \in [0, T]$ with $r \leq t$, some $c \geq 0$, and some Borel measure μ on $[0, T]$ with $\mu(\{t\}) = 0$ for all $t \in [0, T]$. In addition, $D \in \mathscr{B}(\mathbb{R}^k)$ has non-empty interior, f is an $\mathbb{R}^k$-valued measurable map on $[0, T] \times S \times D$, and g is a D-valued Borel measurable bounded map on S.

We first remark that for $D = \mathbb{R}^k$ a Picard iteration and Banach's fixed-point theorem produce existence of solutions to (1.1) locally in time. This can be found, for example, in Pazy [26, Theorem 6.1.4] when $\mathscr{X}$ is a diffusion process. Regarding existence, we will more generally suppose that D is convex. By modifying analytical methods from the classical theory of ordinary differential equations (ODEs), we will derive unique non-extendible solutions to (1.1) that are admissible in an appropriate topological sense. Moreover, weak conditions ensuring the continuity of the derived solutions will be provided. In the particular case when $D = \mathbb{R}^k$ and f is an affine map in the third variable $z \in \mathbb{R}^k$, we will prove a representation for solutions to (1.1). This gives a multidimensional generalization to the Feynman-Kac formula in Dynkin [15, Theorem 4.1.1].

Let us also emphasize that non-negative solutions to one-dimensional Markovian integral equations are well-studied. Namely, for $k = 1$ and $D = \mathbb{R}_+$, solutions to (1.1) have been deduced by a Picard iteration approach. For instance, the classical references are Watanabe [35, Proposition 2.2], Fitzsimmons [17, Proposition 2.3], and Iscoe [21, Theorem A]. In these works, the existence of solutions to (1.1) is used for the construction of superprocesses. Dynkin [11,12,15] establishes superprocesses with probabilistic methods by means of branching particle systems, which in turn yields another existence result to our Markovian integral equations.

These treatments of (1.1) in one dimension require that the function f admits a representation that is related to measure-valued branching processes. To give one of the main examples, the following case is included in [11, 12, 15]:

$$f(t, x, z) = b(t, x)z^\alpha \tag{1.2}$$

for every $(t, x, z) \in [0, T] \times S \times \mathbb{R}_+$ with some Borel measurable bounded function $b : [0, T] \times S \to \mathbb{R}_+$, and some $\alpha \in [1, 2]$. Here, the bound $\alpha \leq 2$ is strict. However, this thesis intends to derive solutions without imposing a specific form of f. Rather, as in the multidimensional case, we will introduce regularity conditions for f with respect to the Borel measure μ like local Lipschitz μ-continuity. This will allow for a more general treatment of (1.1). In particular, our approach includes the case

$$f(t, x, z) = a(t, x) + b(t, x)\varphi(z)$$

for all $(t, x, z) \in [0, T] \times S \times \mathbb{R}_+$, some Borel measurable bounded $a : [0, T] \times S \to \mathbb{R}$, and some locally Lipschitz continuous $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $a \leq 0$ and $\varphi(0) = 0$. Hence, (1.2) is also feasible for $\alpha > 2$. Note that we will not restrict our attention to the case $D = \mathbb{R}_+$. In fact, the one-dimensional global existence and uniqueness result, we will establish, is applicable provided D is a non-degenerate interval, that is, an interval with non-empty interior. In this connection, the same weak conditions as before grant the continuity of solutions to (1.1).

1.2 Path-dependent partial differential equations

Recently, Dupire [10] and Cont and Fournié [6] introduced horizontal and vertical derivatives of non-anticipative functionals on path spaces and proved the functional Itô formula, the path-dependent generalization of the well-known Itô formula. These concepts led to the exciting class of path-dependent partial differential equations. In relevant publications such as Peng [27,28], Peng and Wang [29], Ji and Yang [22], Ekren, Keller, Touzi, and Zhang [16], and Henri-Labordere, Tan, and Touzi [19], the most common approach to construct classical or viscosity solutions to PPDEs is to utilize backward stochastic differential equations (BSDEs). In this thesis we instead rely on Markovian integral equations to derive mild solutions to semilinear parabolic PPDEs. The corresponding terminal value problem reads

$$\begin{cases} (\partial_t + \mathscr{L})(u)(t,x) &= f(t,x,u(t,x)) \quad \text{for } (t,x) \in [0,T) \times C([0,T],\mathbb{R}^d), \\ u(T,x) &= g(x) \quad \text{for } x \in C([0,T],\mathbb{R}^d). \end{cases} \tag{1.3}$$

Here, $T > 0$ and $d \in \mathbb{N}$. We also tacitly suppose that ∂_t denotes the horizontal derivative, a is an $\mathbb{S}_+^d$-valued non-anticipative bounded map on $[0,T] \times C([0,T],\mathbb{R}^d)$, b is an $\mathbb{R}^d$-valued non-anticipative bounded map on $[0,T] \times C([0,T],\mathbb{R}^d)$, and $\mathscr{L}$ is a second-order linear differential operator of the form

$$\mathscr{L}(\varphi)(t,x) := \frac{1}{2}\mathrm{tr}(a(t,x)\partial_{xx}\varphi(t,x)) + \langle b(t,x), \partial_x\varphi(t,x)\rangle \tag{1.4}$$

for all $\varphi \in C_b^{1,2}([0,T) \times C([0,T],\mathbb{R}^d))$, a certain space of bounded continuous test functions that are once horizontally and twice vertically differentiable. We use ∂_x and ∂_{xx} for the first- and second-order vertical derivative, respectively. Finally, D is a non-degenerate interval in $\mathbb{R}$, the inhomogeneity f is a real-valued non-anticipative measurable function on $[0,T] \times C([0,T],\mathbb{R}^d) \times D$, and the terminal value condition g is a Borel measurable bounded function on $C([0,T],\mathbb{R}^d)$.

We choose $C([0,T],\mathbb{R}^d)$ as state space S and let $\mathscr{X}$ be an $\mathscr{L}$-diffusion process on some measurable space $(\Omega,\mathscr{F})$, which we define to be a triple $(X,(\mathscr{F}_t)_{t\in[0,T]},\mathbb{P})$ consisting of a continuous process $X : [0,T] \times \Omega \to \mathbb{R}^d$, a filtration $(\mathscr{F}_t)_{t\in[0,T]}$ of $\mathscr{F}$ to which X is adapted, and a set $\mathbb{P} = \{P_{r,x} \,|\, (r,x) \in [0,T] \times S\}$ of probability measures on $(\Omega,\mathscr{F})$ with $P_{r,x} = P_{r,x^r}$ for all $(r,x) \in [0,T] \times S$ such that for the path process of X given by $\hat{X}_t = X^t$ for all $t \in [0,T]$ the triple

$$\hat{\mathscr{X}} := (\hat{X},(\mathscr{F}_t)_{t\in[0,T]},\mathbb{P})$$

is a diffusion process with state space S satisfying the $\mathscr{L}$-martingale property, which will be explained accurately. Here, as usually, X^t denotes the process X stopped at time $t \in [0,T]$. For example, if $a = \mathbb{I}_d$ and $b = 0$, then $\hat{\mathscr{X}}$ is a historical Brownian motion that was studied by Dawson and Perkins [7] and Dynkin [12] in connection with historical superprocesses. An $\mathscr{L}$-diffusion process allows us to determine mild solutions to (1.3) as solutions to the Markovian terminal value problem (1.1) when X is replaced by its path process $\hat{X}$. This finding entails a general existence and uniqueness result for mild solutions to semilinear parabolic PPDEs.

Furthermore, we compare various notions of viscosity solutions by studying a couple of test function spaces. Under a weak continuity condition on a, b, and f, we then establish that bounded continuous mild solutions can also be seen as viscosity solutions. For this reason, an existence result for viscosity solutions follows, which finishes our treatment of (1.3).

1.3 Structure of the thesis

In Chapter 2 we introduce a variety of regularity conditions for multidimensional measurable mappings relative to an underlying Borel measure. Namely, we make use of local integrability to define dominance with respect to the Borel measure, and combine this concept with notions of boundedness, Lipschitz continuity, and differentiability. Additionally, we justify that the integral functions that show up in the probabilistic construction of superprocesses in Dynkin [11, 12, 15] are within the scope of this theory.

Chapter 3 reviews relevant facts on Markov processes and additive maps in a pseudometric topological setting that allows for path spaces and path-dependent diffusion processes. In this time-space framework, maps that are right-continuous in time and continuous in space are examined. We further give an adjusted definition of a consistent Markov process that is in line with the classical literature. By proving a continuity result for progressive Markov processes that have the (right-hand) Feller property, we get sufficient conditions for the (right-)continuity of solutions to (1.1). In the end, we extend the Markovian Gronwall inequality given in Dynkin [11] that has proven to be useful for estimating solutions.

Chapter 4 analyzes multidimensional Markovian integral equations and includes the main content of the arXiv preprint [23]. At first, we introduce the Markovian terminal value problem (1.1), by defining and describing (approximate) solutions. Then a comparison, a stability result, and a growth estimation follow. Via regularity conditions with respect to a continuous additive functional, we construct solutions locally in time. This allows us to derive solutions that are unique and non-extendible in time among a particular class of maps. In this context, a boundary and growth criterion determines whether the deduced solutions remain non-extendible or become global. Moreover, this criterion yields a Picard iteration result, which in turn leads us to a multidimensional Feynman-Kac representation for global solutions to affine Markovian equations. In one dimension, after studying the boundary and growth behavior of solutions, a global existence and uniqueness result follows from a uniform approximation approach.

Chapter 5 treats semilinear parabolic PPDEs and partially includes the arXiv preprint [24]. We discuss Borel measurable structures and topologies with respect to cylindrical σ-fields, separability, and pseudometrics. In addition, path processes are briefly considered. Then we summarize several facts on horizontal and vertical derivatives, which may be viewed as relaxed time and space differential operators on path spaces. Afterwards, the parabolic terminal value problem (1.3) is presented. By using path-dependent diffusion processes that satisfy a martingale property relative to (1.4), we identify mild solutions as global solutions to the accompanied Markovian integral equations. From this fact we infer a general existence and uniqueness result for bounded mild solutions to semilinear parabolic PPDEs. At last, we prove that bounded right-continuous mild solutions are actually solutions in a viscosity sense. This concludes the thesis.

Chapter 2

Regularity with respect to Borel Measures

In this chapter we present regularity conditions for multidimensional measurable maps with respect to a given Borel measure. This leads us to the class of mappings that will appear in the analysis of Markovian integral equations in Chapter 4. In Section 2.1 we define the concepts of (local) dominance and consistent boundedness, and introduce maps that are affine bounded or locally bounded relative to a Borel measure. In one dimension, functions that are affine bounded from below or from above in this sense are considered as well. In Section 2.2 we familiarize ourselves with the notion of (local) Lipschitz continuity with respect to a Borel measure. In this connection, we show that locally bounded maps that are locally Lipschitz continuous must be Lipschitz continuous on compact sets.

In Section 2.3 we give a meaning to (uniform) differentiability relative to a Borel measure. Under two reasonable topological conditions and a convexity assumption, we prove that uniform differentiability implies Lipschitz continuity in our sense. From this we infer that every map that is differentiable relative to a Borel measure is locally Lipschitz continuous relative to the same measure. Finally, in Section 2.4 we study some of the developed conditions for measurable maps that admit an integral representation. Here, the Bochner integral in finite dimension, provided in Section A.6 of the appendix, applies. This shows that the integral functions that arise in Dynkin [11, 12, 15] are included in our theory.

2.1 Dominance and boundedness

In the sequel, let $J \subset \mathbb{R}_+$ be a non-degenerate closed interval, $(S, \mathscr{S})$ be a measurable space, and μ denote a Borel measure on J. We recall that a real-valued Borel measurable function $\overline{a}$ on an interval $I \subset J$ is locally μ-integrable if and only if

$$\int_K |\overline{a}(t)|\,\mu(dt) < \infty$$

for each compact set K in I. We fix $k \in \mathbb{N}$ and let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^k$. For simplicity, $|\cdot|$ is also used for the Frobenius norm on $\mathbb{R}^{k\times k}$.

2.1 Definition. Let $I \subset J$ be a non-degenerate interval, E be a separable Banach space with complete norm $\|\cdot\|$, and $a : I \times S \to E$ be $\mathscr{B}(I) \otimes \mathscr{S}$-measurable.

(i) The map a is called *(locally) μ-dominated* if there is a (locally) μ-integrable function $\overline{a} \in B(I, \mathbb{R}_+)$ such that $\|a(\cdot, y)\| \leq \overline{a}$ for all $y \in S$ μ-a.s. on I.

(ii) We say that a is *μ-consistently bounded* if for each $r, t \in I$ with $r \leq t$ there is a μ-null set $N \in \mathscr{B}(J)$ such that $\sup_{(s,y)\in(N^c\cap[r,t])\times S} \|a(s, y)\| < \infty$.

(iii) We call *a consistently bounded* if it is locally bounded in $s \in I$, uniformly in $y \in S$. That is, $\sup_{(s,y)\in[r,t]\times S} \|a(s, y)\| < \infty$ for all $r, t \in I$ with $r \leq t$.

By using the notation in the above definition, we immediately see that the set of all E-valued $\mathscr{B}(I) \otimes \mathscr{S}$-measurable locally μ-dominated maps on $I \times S$ is a linear space that contains every E-valued $\mathscr{B}(I) \otimes \mathscr{S}$-measurable μ-consistently bounded map on $I \times S$, since μ is Borel.

2.2 Examples. (i) Suppose that $b : J \times S \to \mathbb{R}^{k\times k}$ and $c : J \times S \to \mathbb{R}^k$ are two $\mathscr{B}(J) \otimes \mathscr{S}$-measurable maps such that c is bounded, and let $a : J \times S \to \mathbb{R}^k$ be defined via

$$a(t, x) := b(t, x)c(t, x),$$

then a is (locally) μ-dominated if b shares this property. This follows from the consistency of the Frobenius and the Euclidean norm, since $|a(t, x)| \leq |b(t, x)||c(t, x)|$ for each $(t, x) \in J \times S$.

(ii) Let $J = [0, 1]$ and μ be the Lebesgue measure on $[0, 1]$. Assume that $\|\cdot\|$ is a norm on S and $\mathscr{S}$ is the Borel σ-field with respect to $\|\cdot\|$. Then $a : [0, 1] \times S \to \mathbb{R}$ given by

$$a(t, x) := \frac{\log(t)}{1 + \|x\|^\alpha}, \quad \text{if } t > 0, \quad \text{and} \quad a(t, x) := 0, \quad \text{if t=0},$$

is μ-dominated, where $\alpha > 0$. Due to the preceding example, this follows from the Lebesgue-integrability of the logarithm function on $(0, 1]$, which is readily inferred from

$$\int_0^1 |\log(t)|\,\mu(dt) = -\lim_{\varepsilon\downarrow 0} \int_\varepsilon^1 \log(t)\,dt = 1 - \lim_{\varepsilon\downarrow 0} \varepsilon(1 - \log(\varepsilon)) = 1.$$

However, as $\lim_{t\downarrow 0} |a(t, x)| = \infty$ for each $x \in S$, the function a is unbounded.

From now on, let S be Polish and $\mathscr{S}$ be its Borel σ-field. We equip $J \times S$ with a topology that is coarser than the product topology and let $\mathscr{B}(J \times S)$ denote the corresponding Borel σ-field, which is included in the product σ-field $\mathscr{B}(J) \otimes \mathscr{S}$, because J is closed and S is Polish. This choice takes the pseudometric setting for path spaces in Section 3.1 into account.

Moreover, we assume that E is a separable Banach space with complete norm $\|\cdot\|$ and Borel σ-field $\mathscr{B}$. Let $D \in \mathscr{B}$ be non-empty and $J \times S \times D$ be endowed with the σ-field $\mathscr{B}(J \times S) \otimes (D \cap \mathscr{B})$. For each map $f : J \times S \times D \to \mathbb{R}^k$, $(t, x, z) \mapsto f(t, x, z)$ that is measurable with respect to this σ-field, we introduce boundedness conditions relative to μ.

2.3 Definition. Let $f : J \times S \times D \to \mathbb{R}^k$ be $\mathscr{B}(J \times S) \otimes (D \cap \mathscr{B})$-measurable.

(i) We call f *affine μ-bounded* if there exist two locally μ-dominated functions $a, b \in B(J \times S, \mathbb{R}_+)$ such that $|f(t,x,z)| \leq a(t,x) + b(t,x)\|z\|$ for every $(t,x,z) \in J \times S \times D$. If one can take $b = 0$, then f is called *μ-bounded.*

(ii) We say that f is *locally μ-bounded* at $\hat{z} \in \overline{D}$ if there is a neighborhood W of $\hat{z}$ in $\overline{D}$ for which $f|(J \times S \times (W \cap D))$ is μ-bounded. The map f is called locally μ-bounded on a set C in $\overline{D}$ if it is locally μ-bounded at each $\hat{z} \in C$. For $C = D$, we simply say that f is locally μ-bounded.

(iii) Let $k = 1$. Then f is said to be *affine μ-bounded from below* (resp. *from above*) if $f(t,x,z) \geq -a(t,x) - b(t,x)\|z\|$ (resp. $f(t,x,z) \leq a(t,x) + b(t,x)\|z\|$) for all $(t,x,z) \in J \times S \times D$ and some locally μ-dominated $a, b \in B(J \times S, \mathbb{R}_+)$. If $b = 0$ is possible, then f is called *μ-bounded from below* (resp. *from above*).

Whenever a mapping $f : J \times S \times D \to \mathbb{R}^k$ that is measurable with respect to $\mathscr{B}(J \times S) \otimes (D \cap \mathscr{B})$ is affine μ-bounded, then it is locally μ-bounded on $\overline{D}$. Suppose instead that f is merely locally μ-bounded, then the Borel measurable map

$$J \times S \to \mathbb{R}^k, \quad (t,x) \mapsto f(t,x,\hat{z})$$

is locally μ-dominated for each $\hat{z} \in D$. Of course, for $k = 1$ the function f is (affine) μ-bounded if and only if it is (affine) μ-bounded from below and from above.

2.4 Examples. (i) Let $a \in B(J \times S, \mathbb{R}^k)$ and $b \in B(J \times S, \mathbb{R}^{k \times k})$ be both locally μ-dominated. Assume that $\varphi \in B(D, \mathbb{R}^k)$ fulfills

$$f(t,x,z) = a(t,x) + b(t,x)\varphi(z)$$

for all $(t,x,z) \in J \times S \times D$. Then f is (affine) μ-bounded whenever φ is (affine) bounded. If instead φ is locally bounded, then f is locally μ-bounded. For $k = 1$ and $b \geq 0$, it follows that f is (affine) μ-bounded from below (resp. from above) if φ is (affine) bounded from below (resp. from above).

(ii) Suppose that $a \in B(J \times S, \mathbb{R}^k)$ is locally μ-dominated, $\varphi : J \times S \times D \to \mathbb{R}_+$ is $\mathscr{B}(J \times S) \otimes (D \cap \mathscr{B})$-measurable, and $A \in \mathbb{S}^k$ is positive semidefinite such that

$$f(t,x,z) = e^{-\varphi(t,x,z)A}a(t,x)$$

for all $(t,x,z) \in J \times S \times D$. Then f is μ-bounded, as we now check. By diagonalizing A, we get a diagonal matrix $\hat{D} \in \mathbb{R}^{k \times k}$ with non-negative entries and an orthogonal matrix $O \in \mathbb{R}^{k \times k}$ such that $A = O\hat{D}O^t$. This yields

$$e^{-\varphi(t,x,z)A} = Oe^{-\varphi(t,x,z)\hat{D}}O^t,$$

which implies that $e^{-\varphi(t,x,z)A}$ is symmetric and each of its eigenvalues is of the form $e^{-\varphi(t,x,z)\lambda}$ for some $\lambda \in \sigma(A)$, where $(t,x,z) \in J \times S \times D$. Hence, as the Frobenius norm satisfies $|B|^2 = \sum_{\lambda \in \sigma(B)} \lambda^2$ for all $B \in \mathbb{S}^k$, we get that

$$\left|e^{-\varphi(t,x,z)A}\right|^2 = \sum_{\lambda \in \sigma(A)} e^{-2\varphi(t,x,z)\lambda} \leq k$$

for all $(t,x,z) \in J \times S \times D$. This clarifies the μ-boundedness of f.

In the end, we relate local μ-boundedness with μ-boundedness on compact sets.

2.5 Lemma. *Let $f : J \times S \times D \to \mathbb{R}^k$ be $\mathscr{B}(J \times S) \otimes (D \cap \mathscr{B})$-measurable and locally μ-bounded on a Borel set C in $\overline{D}$. Then for each compact set K in C there is a neighborhood W of K in C such that $f|(J \times S \times (W \cap D))$ is μ-bounded.*

Proof. By hypothesis, for each $z \in K$ there are a neighborhood W_z of z in $\overline{D}$ and a locally μ-dominated function $a_z \in B(J \times S, \mathbb{R}_+)$ such that

$$|f(t, x, z')| \leq a_z(t, x) \quad \text{for all } (t, x, z') \in J \times S \times (W_z \cap D).$$

Since K is compact, there are $n \in \mathbb{N}$ and $z_1, \ldots, z_n \in K$ such that $\bigcup_{i=1}^n W_{z_i}$ is a neighborhood of K in $\overline{D}$. So, $W := \bigcup_{i=1}^n (W_{z_i} \cap C)$ must be a neighborhood of K in C and we let $a \in B(J \times S, \mathbb{R}_+)$ be defined by $a(t, x) := \max_{i \in \{1,\ldots,n\}} a_{z_i}(t, x)$. Then a is locally μ-dominated and $|f(t, x, z)| \leq a(t, x)$ for all $(t, x, z) \in J \times S \times (W \cap D)$. This shows the claim. □

2.2 Lipschitz continuity

We intend to combine the property of a measurable map being (locally) Lipschitz continuous with local μ-dominance. As before, assume that E is a separable Banach space with complete norm $\|\cdot\|$ and Borel σ-field $\mathscr{B}$, and $D \in \mathscr{B}$ is non-empty.

2.6 Definition. Let $f : J \times S \times D \to \mathbb{R}^k$ be $\mathscr{B}(J \times S) \otimes (D \cap \mathscr{B})$-measurable.

(i) We say that f is *Lipschitz μ-continuous* if there exists a locally μ-dominated function $\lambda \in B(J \times S, \mathbb{R}_+)$ satisfying $|f(t, x, z) - f(t, x, z')| \leq \lambda(t, x)\|z - z'\|$ for all $(t, x) \in J \times S$ and each $z, z' \in D$.

(ii) We call f *locally Lipschitz μ-continuous* at $\hat{z} \in \overline{D}$ if there is a neighborhood W of $\hat{z}$ in $\overline{D}$ such that $f|(J \times S \times (W \cap D))$ is Lipschitz μ-continuous. The map f is locally Lipschitz μ-continuous if it is locally Lipschitz μ-continuous at every $\hat{z} \in D$.

The set of all $\mathbb{R}^k$-valued $\mathscr{B}(J \times S) \otimes (D \cap \mathscr{B})$-measurable maps on $J \times S \times D$ that are locally μ-bounded and locally Lipschitz μ-continuous is denoted by

$$BC_\mu^{1-}(J \times S \times D, \mathbb{R}^k), \tag{2.1}$$

which constitutes a linear space. To simplify notation, we also let $BC_\mu^{1-}(J \times S \times D)$ represent $BC_\mu^{1-}(J \times S \times D, \mathbb{R})$. Clearly, whenever $f : J \times S \times D \to \mathbb{R}^k$ is some $\mathscr{B}(J \times S) \otimes (D \cap \mathscr{B})$-measurable map that is Lipschitz μ-continuous, then it is locally Lipschitz μ-continuous. Furthermore, we see that f is Lipschitz μ-continuous if and only if the map

$$f(t, x, \cdot) : D \to \mathbb{R}^k, \quad z \mapsto f(t, x, z)$$

is Lipschitz continuous with Lipschitz constant $\lambda(t, x)$ for all $(t, x) \in J \times S$ such that the resulting function λ is Borel measurable and locally μ-dominated.

To clarify local Lipschitz μ-continuity, we recall that each neighborhood V in D can be written in the form $V = W \cap D$ for some neighborhood W in $\overline{D}$ and vice versa. Thus, f is locally Lipschitz μ-continuous at $\hat{z} \in D$ if and only if there is a neighborhood V of $\hat{z}$ in D such that $f|(J \times S \times V)$ is Lipschitz μ-continuous. Suppose for the moment that $D \subsetneq \overline{D}$ and decompose $\overline{D}$ in the form

$$\overline{D} = D \cup (\partial D \cap D^c) \quad \text{with} \quad D \cap (\partial D \cap D^c) = \emptyset.$$

Then local Lipschitz μ-continuity of f at $\hat{z} \in \partial D \cap D^c$ yields a neighborhood W of $\hat{z}$ in $\overline{D}$ such that $|f(t,x,z) - f(t,x,z')| \le \lambda(t,x)\|z - z'\|$ for all $(t,x) \in J \times S$, each $z, z' \in W \cap D$, and some locally μ-dominated $\lambda \in B(J \times S, \mathbb{R}_+)$. Hence, it follows from Proposition A.12 that the limit $\lim_{z \to \hat{z}} f(t,x,z)$ exists for each $(t,x) \in J \times S$. Consequently, let $\overline{f}$ be the extension of f to $J \times S \times (D \cup \{\hat{z}\})$ defined by

$$\overline{f}(t,x,\hat{z}) := \lim_{z \to \hat{z}} f(t,x,z),$$

then $\overline{f}$ is locally Lipschitz μ-continuous at $\hat{z}$ in the sense discussed before, as $\hat{z}$ belongs to the domain $D \cup \{\hat{z}\}$ of $\overline{f}(t,x,\cdot)$ for all $(t,x) \in J \times S$. To facilitate access to this continuity concept, we consider the two examples of the previous section.

2.7 Examples. (i) Let $a \in B(J \times S, \mathbb{R}^k)$, $b \in B(J \times S, \mathbb{R}^{k \times k})$, and $\varphi \in B(D, \mathbb{R}^k)$ be such that b is locally μ-dominated. Suppose that

$$f(t,x,z) = a(t,x) + b(t,x)\varphi(z)$$

for all $(t,x,z) \in J \times S \times D$, then from the (local) Lipschitz continuity of φ the (local) Lipschitz μ-continuity of f follows. Due to Examples 2.4, if a is locally μ-dominated and φ is locally Lipschitz continuous, then $f \in BC_\mu^{1-}(J \times S \times D, \mathbb{R}^k)$.

(ii) Let $a \in B_b(J \times S, \mathbb{R}^k)$, $\varphi : J \times S \times D \to \mathbb{R}_+$ be $\mathscr{B}(J \times S) \otimes (D \cap \mathscr{B})$-measurable, and $A \in \mathbb{S}^k$ be positive semidefinite. Assume that f is of the form

$$f(t,x,z) = e^{-\varphi(t,x,z)A} a(t,x)$$

for each $(t,x,z) \in J \times S \times D$. We show that if φ is (locally) Lipschitz μ-continuous, then so is f. From Examples 2.4 we infer that f is bounded. Let $\hat{D} \in \mathbb{R}^{k \times k}$ be a diagonal matrix with non-negative entries and $O \in \mathbb{R}^{k \times k}$ be an orthogonal matrix such that $A = O\hat{D}O^t$. Then $\exp(-\varphi(t,x,z)A) = O\exp(-\varphi(t,x,z)\hat{D})O^t$ and consequently,

$$\begin{aligned} \left| e^{-\varphi(t,x,z)A} - e^{-\varphi(t,x,z')A} \right|^2 &= \sum_{\lambda \in \sigma(A)} \left| e^{-\varphi(t,x,z)\lambda} - e^{-\varphi(t,x,z')\lambda} \right|^2 \\ &\le k\Big(\max_{\lambda \in \sigma(A)} \lambda^2\Big) |\varphi(t,x,z) - \varphi(t,x,z')|^2 \end{aligned}$$

for every $(t,x) \in J \times S$ and all $z, z' \in D$. Here, we have used that the function $\mathbb{R}_+ \to (0,1]$, $x \mapsto e^{-x}$ is Lipschitz continuous with Lipschitz constant 1.

We state some standard properties of (locally) Lipschitz μ-continuous maps. Notice that the proof of the third claim is mainly inferred from Proposition 6.4 in Amann [1].

2.8 Proposition. *Let $f : J \times S \times D \to \mathbb{R}^k$ be $\mathscr{B}(J \times S) \otimes (D \cap \mathscr{B})$-measurable.*

(i) Assume that f is locally Lipschitz μ-continuous and the map $J \times S \to \mathbb{R}^k$, $(t,x) \mapsto f(t,x,\hat{z})$ is locally μ-dominated for every $\hat{z} \in D$. Then f is locally μ-bounded.

(ii) If f is Lipschitz μ-continuous and the map $J \times S \to \mathbb{R}^k$, $(t,x) \mapsto f(t,x,\hat{z})$ is locally μ-dominated just for some $\hat{z} \in D$, then f is affine μ-bounded.

(iii) Let $f \in BC^{1-}_{\mu}(J \times S \times D, \mathbb{R}^k)$ and K be a compact set in D. Then there is a neighborhood W of K in D for which $f|(J \times S \times W)$ is Lipschitz μ-continuous.

Proof. (i) For each $\hat{z} \in D$ there are some neighborhood W of $\hat{z}$ in D and a locally μ-dominated $\lambda \in B(J \times S, \mathbb{R}_+)$ such that $|f(t,x,z) - f(t,x,z')| \leq \lambda(t,x)\|z - z'\|$ for all $(t,x) \in J \times S$ and each $z, z' \in W$. Let $\delta > 0$ and define $a \in B(J \times S, \mathbb{R}_+)$ by

$$a(t,x) := |f(t,x,\hat{z})| + \lambda(t,x)\delta,$$

then $W' := B_\delta(\hat{z}) \cap W$ is another neighborhood of $\hat{z}$ in D and it follows immediately that $|f(t,x,z)| \leq |f(t,x,\hat{z})| + \lambda(t,x)\|z - \hat{z}\| \leq a(t,x)$ for each $(t,x,z) \in J \times S \times W'$. Since a is locally μ-dominated, the assertion is shown.

(ii) Let us choose some locally μ-dominated $b \in B(J \times S, \mathbb{R}_+)$ that satisfies $|f(t,x,z) - f(t,x,z')| \leq b(t,x)\|z - z'\|$ for every $(t,x) \in J \times S$ and all $z, z' \in D$. We define $a \in B(J \times S, \mathbb{R}_+)$ via

$$a(t,x) := |f(t,x,\hat{z})| + b(t,x)\|\hat{z}\|,$$

then we get that $|f(t,x,z)| \leq |f(t,x,\hat{z})| + b(t,x)\|z - \hat{z}\| \leq a(t,x) + b(t,x)\|z\|$ for each $(t,x,z) \in J \times S \times D$. Because a is locally μ-dominated, this justifies that f is affine μ-bounded.

(iii) For each $z \in K$ there are $\delta_z > 0$ and a locally μ-dominated $\lambda_z \in B(J \times S, \mathbb{R}_+)$ such that $|f(t,x,z') - f(t,x,z'')| \leq \lambda_z(t,x)\|z' - z''\|$ for all $(t,x) \in J \times S$ and each $z', z'' \in B_{\delta_z}(z) \cap D$. Since $\{B_{\delta_z/2}(z) \,|\, z \in K\}$ is an open covering of K, there are $n \in \mathbb{N}$ and $z_1, \ldots, z_n \in K$ so that $W := \bigcup_{i=1}^n (B_{\delta_{z_i}/2}(z_i) \cap D)$ is a neighborhood of K in D. We now show that

$$\operatorname{diam}(f(t,x,W)) = \sup_{z,z' \in W} |f(t,x,z) - f(t,x,z')| \leq a(t,x) \tag{2.2}$$

for each $(t,x) \in J \times S$ and some locally μ-dominated $a \in B(J \times S, \mathbb{R}_+)$. For this purpose, let $(t,x) \in J \times S$ and $z, z' \in W$, then there are $i, j \in \{1, \ldots, n\}$ with $\|z - z_i\| < \delta_{z_i}/2$ and $\|z' - z_j\| < \delta_{z_j}/2$. Hence,

$$\begin{aligned} |f(t,x,z) - f(t,x,z')| &\leq |f(t,x,z) - f(t,x,z_i)| + |f(t,x,z_i) - f(t,x,z_j)| \\ &\quad + |f(t,x,z_j) - f(t,x,z')| \\ &\leq \lambda_{z_i}(t,x)\delta_{z_i}/2 + |f(t,x,z_i) - f(t,x,z_j)| + \lambda_{z_j}(t,x)\delta_{z_j}/2. \end{aligned}$$

Thus, we set $a(t,x) := \max_{i\in\{1,\dots,n\}} \lambda_{z_i}(t,x)\delta_{z_i} + \max_{i,j\in\{1,\dots,n\}} |f(t,x,z_i) - f(t,x,z_j)|$ for all $(t,x) \in J \times S$. Then a is readily checked to be Borel measurable and locally μ-dominated, and (2.2) holds.

Next, we let $\delta := (1/2)\min_{i\in\{1,\dots,n\}} \delta_{z_i}$ and define $\lambda \in B(J \times S, \mathbb{R}_+)$ through $\lambda(t,x) := a(t,x)/\delta$, then λ is locally μ-dominated and $\lambda \geq \max_{i\in\{1,\dots,n\}} \lambda_{z_i}$. Choose $(t,x) \in J \times S$ and $z, z' \in W$. Assume initially that $\|z - z'\| < \delta$, then there is $i \in \{1,\dots,n\}$ with $\|z - z_i\| < \delta_{z_i}/2$, which gives

$$\|z' - z_i\| \leq \|z' - z\| + \|z - z_i\| < \delta_{z_i}.$$

Therefore, $|f(t,x,z) - f(t,x,z')| \leq \lambda_{z_i}(t,x)\|z - z'\| \leq \lambda(t,x)\|z - z'\|$. If instead $\|z - z'\| \geq \delta$, then $|f(t,x,z) - f(t,x,z')| \leq a(t,x) \leq \lambda(t,x)\|z - z'\|$, which concludes the proof. □

2.3 Differentiability

In this section we let the separable Banach space E be finite-dimensional and define differentiability with respect to μ. To this end, let $\mathscr{L}(E,\mathbb{R}^k)$ denote the linear space of all $\mathbb{R}^k$-valued linear continuous maps on E. We use the notation

$$\|g\| = \max_{z\in E:\, \|z\|=1} \|g(z)\| \quad \text{for each } g \in \mathscr{L}(E,\mathbb{R}^k)$$

and notice that $\mathscr{L}(E,\mathbb{R}^k)$ equipped with $\|\cdot\|$ is another finite-dimensional Banach space. Moreover, we assume that the set $D \in \mathscr{B}$ has non-empty interior.

2.9 Definition. Let $f : J \times S \times D \to \mathbb{R}^k$ be $\mathscr{B}(J \times S) \otimes (D \cap \mathscr{B})$-measurable.

(i) We call f *uniformly μ-differentiable* if the map $D \to \mathbb{R}^k$, $z \mapsto f(t,x,z)$ is differentiable in D° for all $(t,x) \in J \times S$ such that there is a locally μ-dominated $a \in B(J \times S, \mathbb{R}_+)$ with $\|D_z f(t,x,z)\| \leq a(t,x)$ for each $(t,x,z) \in J \times S \times D^\circ$.

(ii) The map f is said to be *μ-differentiable* around $\hat{z} \in D$ if there is a neighborhood W of $\hat{z}$ in D for which $f|(J \times S \times W)$ is uniformly μ-differentiable. Additionally, f is μ-differentiable if it is μ-differentiable around each $\hat{z} \in D$.

Clearly, if a $\mathscr{B}(J \times S) \otimes (D \cap \mathscr{B})$-measurable map $f : J \times S \times D \to \mathbb{R}^k$ is uniformly μ-differentiable, then it is μ-differentiable. We further see that f is μ-differentiable around $\hat{z} \in D$ if and only if there is a neighborhood W of $\hat{z}$ in D such that $f(t,x,\cdot)$ is differentiable in W°, and

$$\|D_z f(t,x,z)\| \leq a(t,x) \quad \text{for all } (t,x,z) \in J \times S \times W^\circ$$

and some locally μ-dominated function $a \in B(J \times S, \mathbb{R}_+)$. Because $W^\circ = W \cap D^\circ$, the set W° contains no boundary point of D. This justifies that we only consider differentiability at interior points of D.

Although we do not assume that D is an open set, we impose two topological conditions until the end of this section. The first is that D and its interior share the same boundary. That is,

$$\partial D = \partial(D^\circ).$$

For instance, this is true if D is open. By decomposing $\overline{D}$ and $\overline{(D^\circ)}$ in their respective interior and boundary, it follows that D and its interior have the same boundary if and only if the closures of D and its interior coincide. Next, we require that

$$\text{each } \hat{z} \in \partial D \cap D \text{ has a convex neighborhood in } D. \tag{2.3}$$

This requirement becomes redundant if D is open or convex. In the one-dimensional case $E = \mathbb{R}$, we notice that both conditions are met provided D is a non-degenerate interval. In the following, let

$$BC^1_\mu(J \times S \times D, \mathbb{R}^k) \tag{2.4}$$

denote the linear space of all $\mathbb{R}^k$-valued $\mathscr{B}(J \times S) \otimes (D \cap \mathscr{B})$-measurable locally μ-bounded and μ-differentiable maps f on $J \times S \times D$ such that $f(t,x,\cdot) \in C(D,\mathbb{R}^k)$ for every $(t,x) \in J \times S$. We abbreviate $BC^1_\mu(J \times S \times D) := BC^1_\mu(J \times S \times D, \mathbb{R})$ and study our familiar examples.

2.10 Examples. (i) Let $a \in B(J \times S, \mathbb{R}^k)$ and $b \in B(J \times S, \mathbb{R}^{k\times k})$ be so that b is locally μ-dominated. Suppose that $\varphi \in C(D,\mathbb{R}^k)$ is differentiable in D° and

$$f(t,x,z) = a(t,x) + b(t,x)\varphi(z)$$

for all $(t,x,z) \in J \times S \times D$. If $D\varphi$ is bounded, then f is uniformly μ-differentiable. Assume instead that $D\varphi$ is locally bounded on D, that is, $D\varphi$ is locally bounded and to each $\hat{z} \in \partial D \cap D$ there is a neighborhood W of $\hat{z}$ in D such that $\|D\varphi(z)\| \leq c$ for all $z \in W^\circ$ and some $c \geq 0$. Then f is μ-differentiable. Moreover, if a is locally μ-dominated and $D\varphi$ is locally bounded on D, then $f \in BC^1_\mu(J \times S \times D, \mathbb{R}^k)$.

(ii) Suppose that $E = \mathbb{R}^k$ and $\|\cdot\| = |\cdot|$. Let $a \in B_b(J \times S, \mathbb{R}^k)$, $\varphi : J \times S \times D \to \mathbb{R}_+$ be $\mathscr{B}(J \times S) \otimes (D \cap \mathscr{B}(\mathbb{R}^k))$-measurable, and $A \in \mathbb{S}^k$ be positive semidefinite such that

$$f(t,x,z) = e^{-\varphi(t,x,z)A} a(t,x)$$

for all $(t,x,z) \in J \times S \times D$. Then the (uniform) μ-differentiability of φ entails that of f. Indeed, let us write A in the form $A = O\hat{D}O^t$ with a diagonal matrix $\hat{D} \in \mathbb{R}^{k\times k}$ that has non-negative entries and an orthogonal matrix $O \in \mathbb{R}^{k\times k}$. We let $\lambda \in \mathbb{R}^k$ be given by $\lambda_i = \hat{D}_{i,i}$ for all $i \in \{1,\ldots,k\}$. Then $\exp(-\varphi(t,x,z)A)$ $= O\exp(-\varphi(t,x,z)\hat{D})O^t$ and for this reason,

$$\begin{aligned} D_z f(t,x,z) &= -O\lambda D_z\varphi(t,x,z)^t e^{-\varphi(t,x,z)\hat{D}} O^t a(t,x) \\ &= -O\lambda D_z\varphi(t,x,z)^t O^t f(t,x,z), \end{aligned}$$

which in turn gives $|D_z f(t,x,z)| \leq \sqrt{k}c|O\lambda D_z\varphi(t,x,z)^t O^t| = \sqrt{k}c|A||D_z\varphi(t,x,z)|$ for every $(t,x,z) \in J \times S \times D^\circ$, since $|\lambda| = |A|$. Here, $c \geq 0$ is chosen such that $|a(t,x)| \leq c$ for all $(t,x) \in J \times S$. This justifies the claim.

The next proposition ensures that $BC^1_\mu(J \times S \times D, \mathbb{R}^k) \subset BC^{1-}_\mu(J \times S \times D, \mathbb{R}^k)$.

2.11 Proposition. *Let $f : J \times S \times D \to \mathbb{R}^k$ be $\mathscr{B}(J \times S) \otimes (D \cap \mathscr{B})$-measurable and $f(t, x, \cdot) \in C(D, \mathbb{R}^k)$ for all $(t, x) \in J \times S$.*

(i) If f is μ-differentiable and K is a compact set in D, then there exists some neighborhood W of K in D so that $f|(J \times S \times W)$ is uniformly μ-differentiable.

(ii) Suppose that W is a convex neighborhood in D such that $f|(J \times S \times W)$ is uniformly μ-differentiable, then $f|(J \times S \times W)$ is Lipschitz μ-continuous.

(iii) Whenever f is μ-differentiable, then it is locally Lipschitz μ-continuous.

Proof. (i) From the preceding discussion we infer that $f(t, x, \cdot)$ is differentiable in D° for all $(t, x) \in J \times S$ such that the $\mathscr{B}(J \times S) \otimes (D^\circ \cap \mathscr{B})$-measurable function

$$\|D_z f\| : J \times S \times D^\circ \to \mathbb{R}_+, \quad (t, x, z) \mapsto \|D_z f(t, x, z)\|$$

is locally μ-bounded on D. According to Lemma 2.5, there is a neighborhood W of K in D for which the restriction of $\|D_z f\|$ to $J \times S \times (W \cap D^\circ)$ is μ-bounded. Since $W^\circ = W \cap D^\circ$, this is equivalent to the requirement that $f|(J \times S \times W)$ is uniformly μ-differentiable.

(ii) By hypothesis, $f(t, x, \cdot)$ is differentiable in W° for all $(t, x) \in J \times S$ and there exists some locally μ-dominated function $\lambda \in B(J \times S, \mathbb{R}_+)$ such that

$$\|D_z f(t, x, z)\| \leq \lambda(t, x) \quad \text{for every } (t, x, z) \in J \times S \times W^\circ.$$

We pick $(t, x) \in J \times S$ and $z, z' \in W$. Then, since $\partial D = \partial(D^\circ)$, there are two sequences $(z_n)_{n\in\mathbb{N}}$ and $(z'_n)_{n\in\mathbb{N}}$ in W° with $\lim_{n\uparrow\infty} z_n = z$ and $\lim_{n\uparrow\infty} z'_n = z'$. By Lemma A.5, the convexity of W ensures that of W°. Hence, the mean value theorem yields for each $n \in \mathbb{N}$ some $s_n \in (0, 1)$ such that

$$f(t, x, z_n) - f(t, x, z'_n) = D_z f(t, x, s_n z_n + (1 - s_n) z'_n)(z_n - z'_n).$$

This gives us that $|f(t, x, z_n) - f(t, x, z'_n)| \leq \lambda(t, x)\|z_n - z'_n\|$ for all $n \in \mathbb{N}$. By taking the limit $n \uparrow \infty$, we obtain that $|f(t, x, z) - f(t, x, z')| \leq \lambda(t, x)\|z - z'\|$, since $f(t, x, \cdot)$ is continuous.

(iii) Let $\hat{z} \in D$, then there is a neighborhood W in D such that $f|(J \times S \times W)$ is uniformly μ-differentiable. Hence, if we can find a convex neighborhood W' of $\hat{z}$ in D with $W' \subset W$, then (ii) yields that $f|(J \times S \times W')$ is Lipschitz μ-continuous, which is exactly the local Lipschitz μ-continuity of f at $\hat{z}$.

If $\hat{z} \in D^\circ$, then there is some $\delta > 0$ such that $B_\delta(\hat{z}) \subset W$. Thus, here we set $W' := B_\delta(\hat{z})$. If instead $\hat{z} \in \partial D \cap D$, then, by hypothesis (2.3), there is a convex neighborhood W'' of $\hat{z}$ in D. In this case, we choose $\delta > 0$ such that $B_\delta(\hat{z}) \cap D \subset W$, then $W' := W'' \cap B_\delta(\hat{z})$ gives the correct result. $\square$

2.4 Integral maps

We draw our attention to measurable maps that admit an integral representation. Once again, let E be finite-dimensional, the set $D \in \mathscr{B}$ have non-empty interior, and $\mathscr{L}(E, \mathbb{R}^k)$ denote the Banach space of all $\mathbb{R}^k$-valued linear continuous maps on E. We let $(U, \mathscr{U})$ be a measurable space and

$$n : J \times S \times \mathscr{U} \to [0, \infty], \quad (t, x, B) \mapsto n(t, x, B)$$

be a kernel from $J \times S$ to $(U, \mathscr{U})$, where, as usually, $J \times S$ is equipped with the Borel σ-field $\mathscr{B}(J \times S)$. We choose some $\mathscr{U} \otimes (D \cap \mathscr{B})$-measurable map $\varphi : U \times D \to \mathbb{R}^k$ such that the $\mathscr{U}$-measurable map $\varphi(\cdot, z) : U \to \mathbb{R}^k$, $u \mapsto \varphi(u, z)$ is $n(t, x, \cdot)$-integrable for all $(t, x, z) \in J \times S \times D$. In what follows, we are concerned with the mapping $f : J \times S \times D \to \mathbb{R}^k$ defined via

$$f(t, x, z) := \int_U \varphi(u, z)\, n(t, x, du). \tag{2.5}$$

One may question whether the integral map f is indeed measurable. This is answered affirmatively.

2.12 Lemma. *The map f is measurable with respect to $\mathscr{B}(J \times S) \otimes (D \cap \mathscr{B})$.*

Proof. By Lemma A.19, it suffices to show that the i-coordinate function f_i of f is measurable for each $i \in \{1, \ldots, k\}$. We notice that the $n(t, x, \cdot)$-integrability of $\varphi(\cdot, z)$ entails that of its i-th coordinate function $\varphi_i(\cdot, z)$ and

$$f_i(t, x, z) = \int_U \varphi_i(u, z)\, n(t, x, du)$$

for all $(t, x, z) \in J \times S \times D$. Thus, to show that f_i is measurable, we may assume that φ_i is bounded. In the general case, Corollary A.24 yields a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of real-valued $\mathscr{U} \otimes (D \cap \mathscr{B})$-measurable bounded functions on $U \times D$ that converges pointwise to φ_i such that $\sup_{n \in \mathbb{N}} |\varphi_n(u, z)| \leq |\varphi_i(u, z)|$ for each $(u, z) \in U \times D$. Then

$$\lim_{n \uparrow \infty} \int_U \varphi_n(u, z)\, n(t, x, du) = \int_U \varphi_i(u, z)\, n(t, x, du) = f_i(t, x, z)$$

for all $(t, x, z) \in J \times S \times D$, by dominated convergence. This in turn shows that f_i is measurable. Finally, the set of all real-valued $\mathscr{U} \otimes (D \cap \mathscr{B})$-measurable bounded functions ψ on $U \times D$ for which the function

$$J \times S \times D \to \mathbb{R}, \quad (t, x, z) \mapsto \int_U \psi(u, z)\, n(t, x, du)$$

is measurable is a monotone class on $U \times D$, as introduced in Section A.5 of the appendix. If $B \in \mathscr{U}$ and $C \in D \cap \mathscr{B}$, then $\int_U \mathbb{1}_{B \times C}(u, z)\, n(t, x, du) = n(t, x, B) \mathbb{1}_C(z)$ for every $(t, x, z) \in J \times S \times D$. Since n is a kernel and $\mathscr{U} \times (D \cap \mathscr{B})$ is an $\cap$-stable generator of $\mathscr{U} \otimes (D \cap \mathscr{B})$, the Functional Monotone Class Theorem A.29 concludes our verification. $\square$

Let us check under which assumptions f is locally bounded with respect to μ. We let $\hat{z} \in D$ and assume temporarily that there are a neighborhood W of $\hat{z}$ in D and some $\mathscr{U}$-measurable $\alpha : U \to [0, \infty]$ such that $|\varphi(u, z)| \le \alpha(u)$ for all $(u, z) \in U \times W$. Then

$$|f(t, x, z)| \le \int_U \alpha(u)\, n(t, x, du) \quad \text{for all } (t, x, z) \in J \times S \times W,$$

by Proposition A.32. Hence, if the Borel measurable function $\int_U \alpha(u)\, n(\cdot, \cdot, du)$ is finite and locally μ-dominated, then f becomes locally μ-bounded at $\hat{z}$. Let us also study the continuity of $f(t, x, \cdot)$ for each $(t, x) \in J \times S$. To this end, we generalize Lemma 16.1 in Bauer [2], from which the proof ideas originate.

2.13 Lemma. *Let $\hat{z} \in D$ and $\varphi(u, \cdot)$ be continuous at $\hat{z}$ for all $u \in U$. Suppose that there is a neighborhood W of $\hat{z}$ in D and an $\mathscr{U}$-measurable $\alpha : U \to [0, \infty]$ with*

$$|\varphi(u, z)| \le \alpha(u) \quad \textit{for all } (u, z) \in U \times W$$

such that $\int_U \alpha(u)\, n(t, x, du) < \infty$ for each $(t, x) \in J \times S$. Then $f(t, x, \cdot)$ is continuous at $\hat{z}$ for every $(t, x) \in J \times S$.

Proof. Let $(z_n)_{n \in \mathbb{N}}$ be a sequence in D with $\lim_{n\uparrow\infty} z_n = \hat{z}$. Then continuity of $\varphi(u, \cdot)$ at $\hat{z}$ yields that $\lim_{n\uparrow\infty} \varphi(u, z_n) = \varphi(u, \hat{z})$ for all $u \in U$. We choose $n_0 \in \mathbb{N}$ such that $z_n \in W$ for all $n \in \mathbb{N}$ with $n \ge n_0$, then $\sup_{n \in \mathbb{N}:\, n \ge n_0} |\varphi(u, z_n)| \le \alpha(u)$ for each $u \in U$. This entails that

$$\lim_{n\uparrow\infty} f(t, x, z_n) = \lim_{n\uparrow\infty} \int_U \varphi(u, z_n)\, n(t, x, du) = \int_U \varphi(u, \hat{z})\, n(t, x, du) = f(t, x, \hat{z})$$

for each $(t, x) \in J \times S$, by the Dominated Convergence Theorem A.33. □

Under the conditions stated below, the differentiability of $f(t, x, \cdot)$ in an open set $W \subset D$ follows for all $(t, x) \in J \times S$. We also consider the case when $E = \mathbb{R}$ and W is a non-degenerate interval in D that fails to be open. Here, we extend Lemma 16.2 and Corollary 16.3 in Bauer [2].

2.14 Lemma. *Let $W \subset D$ and suppose that either W is open or instead $E = \mathbb{R}$ and W is a non-degenerate interval that fails to be open. Moreover, let $\varphi(u, \cdot)$ be differentiable in W for each $u \in U$ and assume that*

$$\|D_z \varphi(u, z)\| \le \alpha(u) \quad \textit{for all } (u, z) \in U \times W$$

and some $\mathscr{U}$-measurable $\alpha : U \to [0, \infty]$ such that $\int_U \alpha(u)\, n(t, x, du) < \infty$ for every $(t, x) \in J \times S$. Then $f(t, x, \cdot)$ is differentiable in W and

$$D_z f(t, x, z)(z') = \int_U D_z \varphi(u, z)(z')\, n(t, x, du)$$

for each $(t, x, z) \in J \times S \times W$ and every $z' \in E$.

Proof. Let $z \in W$, then in either case there is a convex neighborhood V of z in W. We choose a sequence $(z_n)_{n\in\mathbb{N}}$ in D° with $\lim_{n\uparrow\infty} z_n = z$, then there is $n_0 \in \mathbb{N}$ such that $z_n \in V$ for all $n \in \mathbb{N}$ with $n \geq n_0$. By the mean value theorem,

$$|\varphi(u, z_n) - \varphi(u, z)| \leq \sup_{s\in[0,1]} \|D_z\varphi(u, sz_n + (1-s)z)\|\|z_n - z\| \leq \alpha(u)\|z_n - z\|$$

for each $u \in U$ and every $n \in \mathbb{N}$ with $n \geq n_0$. Additionally, $|D_z\varphi(u,z)(z_n - z)| \leq \|D_z\varphi(u,z)\|\|z_n - z\| \leq \alpha(u)\|z_n - z\|$ for every $u \in U$ and each $n \in \mathbb{N}$ with $n \geq n_0$. In consequence,

$$\begin{aligned}
&\lim_{n\uparrow\infty} \frac{f(t,x,z_n) - f(t,x,z) - \int_U D_z\varphi(u,z)(z_n - z)\, n(t,x,du)}{\|z_n - z\|} \\
&= \lim_{n\uparrow\infty} \int_U \frac{\varphi(u,z_n) - \varphi(u,z) - D_z\varphi(u,z)(z_n - z)}{\|z_n - z\|}\, n(t,x,du) = 0
\end{aligned}$$

for all $(t,x) \in J \times S$, by the Dominated Convergence Theorem A.33, since $\varphi(u,\cdot)$ is differentiable at z for each $u \in U$. $\square$

Now sufficient conditions for the differentiability of f relative to μ can be given without difficulty. Suppose for the time being that $\varphi(u,\cdot)$ is differentiable in D° for every $u \in U$ and let $\hat{z} \in D$. Assume that there exists a neighborhood W of $\hat{z}$ in D such that

$$\|D_z\varphi(u,z)\| \leq \alpha(u) \quad \text{for each } (u,z) \in U \times W^\circ$$

and some $\mathscr{U}$-measurable $\alpha : U \to [0,\infty]$ for which $\int_U \alpha(u)\, n(\cdot,\cdot,du)$ is finite. Then Lemma 2.14 and Proposition A.32 imply that $f(t,x,\cdot)$ is differentiable in W° and

$$\|D_z f(t,x,z)\| \leq \int_U \|D_z\varphi(u,z)\|\, n(t,x,du) \leq \int_U \alpha(u)\, n(t,x,du)$$

for all $(t,x,z) \in J \times S \times W^\circ$. Thus, if we further suppose that $\int_U \alpha(u)\, n(\cdot,\cdot,du)$ is locally μ-dominated, then f becomes μ-differentiable around $\hat{z}$. This has been the final thought regarding regularity with respect to μ. Next, to give an example of an integral map that comes up in the construction of superprocesses, we make the following preparations.

2.15 Lemma. *Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a twice differentiable function with $\psi(0) = 0$, $\psi'(z) \geq 0$, and $\psi''(z) \leq 0$ for all $z \geq 0$. Then $\phi \in B((0,\infty) \times \mathbb{R}_+, \mathbb{R}_+)$ given by*

$$\phi(u,z) := e^{-u\psi(z)} - 1 + u\psi(z)$$

is both increasing and twice differentiable in $z \in \mathbb{R}_+$. Furthermore, for each $r \geq 0$ there is $c \geq 0$ such that

$$0 \leq \max\Big\{\phi(u,z), \frac{\partial\phi}{\partial z}(u,z)\Big\} \leq cu\min\{1,u\}$$

for all $u > 0$ and each $z \in [0,r]$.

Proof. Since ψ is twice differentiable, it is continuous. For this reason, ϕ is Borel measurable and also twice differentiable in $z \in \mathbb{R}_+$. Standard calculations yield

$$\begin{aligned}\frac{\partial \phi}{\partial z}(u,z) &= u\psi'(z)\Big(1-e^{-u\psi(z)}\Big) \in [0, u\psi'(z)],\\ \frac{\partial^2 \phi}{\partial z^2}(u,z) &= u\psi''(z)\Big(1-e^{-u\psi(z)}\Big) + u^2\psi'(z)^2 e^{-u\psi(z)} \le u^2\psi'(z)^2\end{aligned}$$

for each $u > 0$ and every $z \ge 0$. This justifies that ϕ is increasing in $z \in \mathbb{R}_+$. Next, by Taylor's formula, for each $u > 0$ and all $z > 0$ there are $\xi, \eta \in (0,z)$ such that

$$\begin{aligned}\phi(u,z) &= \phi(u,0) + \frac{\partial \phi}{\partial z}(u,0)z + \frac{1}{2}\frac{\partial^2 \phi}{\partial z^2}(u,\xi)z^2 \le \frac{u^2}{2}\psi'(0)^2 z^2,\\ \frac{\partial \phi}{\partial z}(u,z) &= \frac{\partial \phi}{\partial z}(u,0) + \frac{\partial^2 \phi}{\partial z^2}(u,\eta)z \le u^2\psi'(0)^2 z,\end{aligned}$$

because $\phi(u,0) = \frac{\partial \phi}{\partial z}(u,0) = 0$, and ψ' is non-negative and decreasing. For $r \ge 0$ we set $c := \max\{\psi(r), \psi'(0), \psi'(0)^2\max\{1,r\}^2\}$, then we obtain that

$$\begin{aligned}\phi(u,z) &\le \min\Big\{u\psi(z), \frac{u^2}{2}\psi'(0)^2 z^2\Big\} \le cu\min\{1,u\},\\ \frac{\partial \phi}{\partial z}(u,z) &\le \min\{u\psi'(z), u^2\psi'(0)^2 z\} \le cu\min\{1,u\}\end{aligned}$$

for all $u > 0$ and every $z \in [0,r]$. This clarifies the lemma. □

We also need some computations.

2.16 Lemma. *Choose $\alpha \in (1,2)$ and let Γ denote the Gamma function. That is, $\Gamma(x) = \int_0^\infty e^{-v}v^{x-1}\,dv$ for all $x > 0$. Then $\int_0^\infty \min\{1,u\}u^{-\alpha}\,du < \infty$ and*

$$\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}\int_0^\infty (e^{-uz} - 1 + uz)u^{-1-\alpha}\,du = z^\alpha \quad \textit{for all } z \ge 0.$$

Proof. Regarding the first claim,

$$\begin{aligned}\int_0^\infty \min\{1,u\}u^{-\alpha}\,du &= \int_0^1 u^{1-\alpha}\,du + \int_1^\infty u^{-\alpha}\,du\\ &= \frac{1}{2-\alpha} + \lim_{u\uparrow\infty}\frac{1}{1-\alpha}(u^{1-\alpha}-1) = \frac{1}{(2-\alpha)(\alpha-1)} < \infty.\end{aligned}$$

By the substitution rule,

$$\int_0^\infty (e^{-uz}-1+uz)u^{-1-\alpha}\,du = z^\alpha\int_0^\infty (e^{-v}-1+v)v^{-1-\alpha}\,dv$$

for each $z \ge 0$. Integration by parts yields that

$$\int_0^\infty (e^{-v}-1+v)v^{-1-\alpha}\,dv = \frac{1}{\alpha}\int_0^\infty (1-e^{-v})v^{-\alpha}\,dv = \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)},$$

because from L'Hôpital's rule it follows that

$$\lim_{v\uparrow\infty}\frac{e^{-v}-1+v}{v^\alpha} = \frac{1}{\alpha}\lim_{v\uparrow\infty}\frac{1-e^{-v}}{v^{\alpha-1}} = 0 \quad \text{and} \quad \lim_{v\downarrow 0}\frac{e^{-v}-1+v}{v^\alpha} = \frac{1}{\alpha}\lim_{v\downarrow 0}\frac{1-e^{-v}}{v^{\alpha-1}} = 0.$$

Hence, the lemma is verified. □

We conclude with the announced example.

2.17 Example. Let $E = \mathbb{R}$, $D = \mathbb{R}_+$, $k = 1$, $U = (0,\infty)$, and $\mathscr{U}$ be the Borel σ-field of $(0,\infty)$. Suppose that

$$\int_0^\infty u \min\{1,u\}\, n(t,x,du) < \infty$$

for all $(t,x) \in J \times S$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is twice differentiable such that $\psi(0) = 0$, $\psi'(z) \geq 0$, and $\psi''(z) \leq 0$ for every $z \geq 0$. In addition, let φ be of the form $\varphi(u,z) = e^{-u\psi(z)} - 1 + u\psi(z)$ for all $(u,z) \in (0,\infty) \times \mathbb{R}_+$. Then (2.5) becomes

$$f(t,x,z) = \int_0^\infty (e^{-u\psi(z)} - 1 + u\psi(z))\, n(t,x,du) \tag{2.6}$$

for each $(t,x,z) \in J \times S \times \mathbb{R}_+$. For instance, if $n \in \mathbb{N}$, $\alpha_1,\ldots,\alpha_n \in (1,2)$, and $d_1,\ldots,d_n \in B(J \times S, \mathbb{R}_+)$, then Lemma 2.16 shows that n could be of the form

$$n(t,x,B) = \sum_{i=1}^n d_i(t,x) \frac{\alpha_i(\alpha_i - 1)}{\Gamma(2-\alpha_i)} \int_B u^{-1-\alpha_i}\, du$$

for all $(t,x) \in J \times S$ and each Borel set B in $(0,\infty)$, where, as before, Γ is the Gamma function. Moreover, the lemma also implies that

$$f(t,x,z) = \sum_{i=1}^n d_i(t,x)\psi(z)^{\alpha_i}$$

for every $(t,x,z) \in J \times S \times \mathbb{R}_+$. In the general case (2.6), we readily infer from Lemmas 2.13, 2.14, and 2.15 that $f(t,x,\cdot) \in C^1(\mathbb{R}_+)$ with

$$\frac{\partial f}{\partial z}(t,x,z) = \int_0^\infty u\psi'(z)\Big(1 - e^{-u\psi(z)}\Big)\, n(t,x,du)$$

for each $(t,x,z) \in J \times S \times \mathbb{R}_+$. Moreover, if the function $\int_0^\infty u \min\{1,u\}\, n(\cdot,\cdot,du)$ is locally μ-dominated, then $f \in BC^1_\mu(J \times S \times \mathbb{R}_+)$. This follows directly from our discussions to local μ-boundedness and μ-differentiability.

Chapter 3

Markov Processes

This chapter provides an exposition of time-inhomogeneous Markov processes and additive maps in a pseudometric topological setting that allows for path spaces and path-dependent diffusion processes. Throughout, standard notation and basic results from stochastic calculus, summarized in Section A.7 of the appendix, are used. In Section 3.1 we define a specific pseudometric and set up the topological and measurable structure on a time-space Cartesian product. The concept of consistency is introduced and maps that are right-continuous in time and continuous in space are studied. In Section 3.2 we give an adjusted definition of a consistent Markov process that is in accordance with the classical notion. In Section 3.3 we discuss the strong Markov property that requires a progressive Markov process with Borel measurable transition probabilities, and the (right-hand) Feller property, which in combination with continuous paths leads to a diffusion process.

Moreover, in both these sections we enhance the measurability properties of the transition probabilities and the (strong) Markov property, by using monotone class theorems and the Bochner integral in finite dimension, given in Sections A.5 and A.6, respectively. In Section 3.4 we compile several basic properties of additive maps, which are multidimensional maps for which all coordinate functions are additive functionals. Further, we prove a continuity result for progressive Markov processes that possess the (right-hand) Feller property, by using local dominance with respect to a continuous additive functional. As will be shown, this yields conditions that guarantee the continuity of solutions to Markovian integral equations. Eventually, in Section 3.5 two integration by parts formulas are deduced to obtain a general Markovian Gronwall inequality.

3.1 Cartesian products in time and space

In the following, let $J \subset \mathbb{R}_+$ be a non-degenerate closed interval, S be at first a completely metrizable topological space with Borel σ-field $\mathscr{S}$, and ρ denote a complete metric that induces the topology of S. We suppose that

$$\Phi : J \times S \to S, \quad (t, x) \mapsto \Phi_t(x)$$

is a process with càdlàg paths. In other words, $\Phi_t : S \to S$, $x \mapsto \Phi_t(x)$ is Borel measurable for all $t \in J$ and $\Phi(x) : J \to S$, $t \mapsto \Phi_t(x)$ is càdlàg for each $x \in S$. For some results, Φ has to be Lipschitz continuous in $x \in S$, locally uniformly in $t \in J$. That is, for each compact interval I in J there is $L \geq 0$ such that

$$\rho(\Phi_t(x), \Phi_t(y)) \leq L\rho(x, y) \tag{3.1}$$

for all $t \in I$ and each $x, y \in S$. However, it is required that $\Phi_t \circ \Phi_s = \Phi_{s\wedge t}$ for all $s, t \in J$. We let $(\mathscr{S}_t)_{t\in J}$ denote the natural filtration of Φ and equip $J \times S$ with the pseudometric d_S defined via

$$d_S((r, x), (s, y)) := |r - s| + \rho(\Phi_r(x), \Phi_s(y)), \tag{PM}$$

then $J \times S$ becomes a pseudometric space. For instance, if Φ_t is the identity map on S for all $t \in J$, then $\mathscr{S}_t$ coincides with the Borel σ-field $\mathscr{S}$ of S for each $t \in J$ and d_S reduces to the metric d defined by

$$d((r, x), (s, y)) := |r - s| + \rho(x, y),$$

which induces the product topology of $J \times S$. We also notice that d_S is a metric if and only if Φ_t is injective for each $t \in J$. Finally, by $\mathscr{B}(J \times S)$ we denote the Borel σ-field of $J \times S$ with respect to the topology induced by d_S.

Let us point out that in Chapter 5 we will consider path spaces that fit into this general setting. More precisely, let temporarily $T > 0$, $d \in \mathbb{N}$, and $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^d$. The main case we will work with is

$$J = [0, T] \quad \text{and} \quad S = C([0, T], \mathbb{R}^d).$$

We will assume that ρ induces the same topology as the maximum norm and $\rho(x, y) \leq L \max_{t\in[0,T]} |x(t) - y(t)|$ for all $x, y \in C([0, T], \mathbb{R}^d)$ and some $L \geq 0$. Moreover, we will make the choice

$$\Phi_t(x) = x^t$$

for each $(t, x) \in [0, T] \times C([0, T], \mathbb{R}^d)$, where $x^t : [0, T] \to \mathbb{R}^d$, $x^t(s) := x(s \wedge t)$ is the map x stopped at time t. Then $(\mathscr{S}_t)_{t\in[0,T]}$ becomes the natural filtration of the canonical process $\xi : [0, T] \times C([0, T], \mathbb{R}^d) \to \mathbb{R}^d$, $\xi_t(x) := x(t)$ and $\mathscr{S}_T$ agrees with $\mathscr{S}$. With this specific case in mind, we return to our general setting.

3.1 Proposition. *Let S be separable, and Φ satisfy* (3.1) *and have continuous paths, then the following three assertions hold:*

(i) The topology induced by d_S is coarser than the product topology of $J \times S$ and $\mathscr{B}(J \times S) \subset \mathscr{B}(J) \otimes \mathscr{S}$.

(ii) The set S_Φ of all $(t, x) \in J \times S$ with $x = \Phi_t(x)$ is closed with respect to the product topology, $S_\Phi \in \mathscr{B}(J) \otimes \mathscr{S}$, and $S_\Phi \cap \mathscr{B}(J \times S) = S_\Phi \cap (\mathscr{B}(J) \otimes \mathscr{S})$.

(iii) $J \times S$ equipped with d_S is a separable complete pseudometric space.

Proof. (i) To show the first assertion, assume that $O \subset J \times S$ is open with respect to d_S and let $(r,x) \in O$. Then there is $\varepsilon > 0$ such that for all $(s,y) \in J \times S$ with $d_S((s,y),(r,x)) < \varepsilon$ it follows that $(s,y) \in O$. Since Φ is Lipschitz continuous in $y \in S$, locally uniformly in $s \in J$, there are $\delta > 0$ and $L \geq 1$ such that

$$\rho(\Phi_s(x'),\Phi_s(y)) \leq L\rho(x',y)$$

for all $s \in (r-\delta, r+\delta) \cap J$ and every $x', y \in S$. As $\Phi(x)$ is continuous, there exists $\gamma > 0$ such that $\rho(\Phi_s(x),\Phi_r(x)) \leq \varepsilon/2$ for each $s \in (r-\gamma, r+\gamma) \cap J$. We set $\eta := \min\{\gamma, \delta, \varepsilon/(2L)\}$, then

$$\begin{aligned} d_S((s,y),(r,x)) &\leq |s-r| + \rho(\Phi_s(y),\Phi_s(x)) + \rho(\Phi_s(x),\Phi_r(x)) \\ &\leq |s-r| + L\rho(y,x) + \varepsilon/2 \leq Ld((s,y),(r,x)) + \varepsilon/2 < \varepsilon \end{aligned}$$

for all $(s,y) \in J \times S$ with $d((s,y),(r,x)) < \eta$, which yields $(s,y) \in O$. Thus, O is open with respect to d, as claimed. Since J is closed and S equipped with ρ is Polish, the Borel σ-field of $J \times S$ with respect to d coincides with the product σ-field $\mathscr{B}(J) \otimes \mathscr{S}$. Hence, $\mathscr{B}(J \times S) \subset \mathscr{B}(J) \otimes \mathscr{S}$, by what we have just shown.

(ii) We first check that S_Φ is closed with respect to d, which directly implies that $S_\Phi \in \mathscr{B}(J) \otimes \mathscr{S}$. Let $(r_n, x_n)_{n\in\mathbb{N}}$ be a sequence in S_Φ for which there is $(r,x) \in J \times S$ such that $\lim_{n\uparrow\infty} d((r_n,x_n),(r,x)) = 0$. As $x_n = \Phi_{r_n}(x_n)$, we get that

$$\rho(x,\Phi_r(x)) \leq \rho(x,x_n) + \rho(\Phi_{r_n}(x_n),\Phi_{r_n}(x)) + \rho(\Phi_{r_n}(x),\Phi_r(x))$$

for every $n \in \mathbb{N}$. Because Φ is Lipschitz continuous in $y \in S$, locally uniformly in $s \in J$, and $\Phi(x)$ is continuous, we may take the limit $n \uparrow \infty$ to obtain that $\rho(x,\Phi_r(x)) = 0$. That is, $(r,x) \in S_\Phi$. Finally, we observe that

$$d_S((r,x),(s,y)) = |r-s| + \rho(x,y) = d((r,x),(s,y))$$

for all $(r,x),(s,y) \in S_\Phi$. In other words, d_S and the product metric d coincide on $S_\Phi \times S_\Phi$. This proves the last assertion.

(iii) To verify completeness, let $(r_n,x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $J \times S$ with respect to d_S. By the definition of d_S, we see that $(r_n)_{n\in\mathbb{N}}$ and $(\Phi_{r_n}(x_n))_{n\in\mathbb{N}}$ are Cauchy sequences in J and S, respectively. As J is closed and S is complete, there is $(r,x) \in J \times S$ such that

$$\lim_{n\uparrow\infty} d((r_n,\Phi_{r_n}(x_n)),(r,x)) = 0.$$

By assumption, $\Phi_{r_n} \circ \Phi_{r_n} = \Phi_{r_n}$ and hence, $(r_n,\Phi_{r_n}(x_n)) \in S_\Phi$ for all $n \in \mathbb{N}$. So, (ii) implies that $(r,x) \in S_\Phi$. From $d_S((r_n,x_n),(r,x)) = d((r_n,\Phi_{r_n}(x_n)),(r,x))$ for all $n \in \mathbb{N}$ the convergence of $(r_n,x_n)_{n\in\mathbb{N}}$ to (r,x) with respect to d_S follows.

We turn to the separability of $J \times S$ with respect to d_S. At first, as J is closed and S is Polish, $J \times S$ equipped with d instead of d_S is another Polish space. Due to (ii), S_Φ is closed in $J \times S$ with respect to d, which in turn implies that S_Φ endowed with either d_S or d is Polish as well, because $d_S = d$ on $S_\Phi \times S_\Phi$. Let F be a countable dense set in S_Φ. We choose $(r,x) \in J \times S$ and let $\varepsilon > 0$, then, as $(r,\Phi_r(x)) \in S_\Phi$, there is $(s,y) \in F$ with $d_S((s,y),(r,x)) = d((s,y),(r,\Phi_r(x))) < \varepsilon$, as desired. □

If E is a topological space and I is a non-degenerate interval in J, then we will call a map $u : I \times S \to E$ *consistent* if it satisfies $u(t,x) = u(t, \Phi_t(x))$ for each $(t,x) \in I \times S$.

3.2 Lemma. *Suppose that S is separable, E is a topological space with Borel σ-field $\mathscr{B}$, I is a non-degenerate interval in J, and $u : I \times S \to E$. Then the following two assertions hold:*

- *(i) Whenever u is consistent and $\mathscr{B}(I)\otimes\mathscr{S}$-measurable, then it is Borel measurable and progressively measurable.*
- *(ii) Let E be a finite-dimensional Banach space. If u is progressively measurable, then it is consistent and $\mathscr{B}(I) \otimes \mathscr{S}$-measurable.*

Proof. (i) First of all, from (PM) it follows immediately that the time-space process $J \times S \to J \times S$, $(t,x) \mapsto (t, \Phi_t(x))$ is a uniformly continuous map provided the domain is equipped with d_S and the image is equipped with d. In particular, Borel measurability follows. That is,

$$\{(t,x) \in J \times S \,|\, (t, \Phi_t(x)) \in F\} \in \mathscr{B}(J \times S) \quad \text{for all } F \in \mathscr{B}(J) \otimes \mathscr{S}.$$

Hence, $u^{-1}(B) = \{(t,x) \in I \times S \,|\, (t, \Phi_t(x)) \in u^{-1}(B)\} \in \mathscr{B}(I \times S)$ for each $B \in \mathscr{B}$, since u is consistent and $u^{-1}(B) \in \mathscr{B}(I) \otimes \mathscr{S}$. Thus, u is Borel measurable.

Moreover, because Φ has right-continuous paths, Proposition A.38 entails that Φ is progressively measurable with respect to its natural filtration $(\mathscr{S}_t)_{t\in J}$. For this reason, the time-space process

$$J \times S \to J \times S, \quad (t,x) \mapsto (t, \Phi_t(x))$$

is $(\mathscr{S}_t)_{t\in J}$-progressively measurable provided the image is equipped with the product σ-field $\mathscr{B}(J) \otimes \mathscr{S}$. Thus, the $(\mathscr{S}_t)_{t\in I}$-progressive measurability of u follows from $u(t,x) = u(t, \Phi_t(x))$ for all $(t,x) \in I \times S$.

(ii) We note that, as $\Phi_s \circ \Phi_t = \Phi_s$ for all $s,t \in J$ with $s \leq t$, we directly get that $\sigma(\Phi_s) \subset \sigma(\Phi_t)$ and hence, $\mathscr{S}_t = \sigma(\Phi_t)$. Since u is necessarily $(\mathscr{S}_t)_{t\in J}$-adapted, Corollary A.25 provides for each $t \in I$ a map $\phi_t \in B(S,E)$ such that

$$u(t,x) = \phi_t(\Phi_t(x)) \quad \text{for each } x \in S.$$

By using that Φ_t is idempotent for each $t \in I$, we conclude that u is consistent. As the fact that u is $\mathscr{B}(I) \otimes \mathscr{S}$-measurable follows readily from the assumption that u is progressive measurable, the proof is complete. □

Next, let (E, ϱ) be a metric space and I be a non-degenerate interval in J. We call a map $u : I \times S \to E$ *right-continuous* at a point $(r,x) \in I \times S$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\varrho(u(s,y), u(r,x)) < \varepsilon$$

for each $(s,y) \in I \times S$ with $s \geq r$ and $d_S((s,y),(r,x)) < \delta$. We notice that if u is right-continuous, that is, it is right-continuous at each $(r,x) \in I \times S$, then it is automatically consistent. This follows directly from $d_S((t,x),(t,\Phi_t(x))) = 0$ for each $(t,x) \in I \times S$.

3.3 Proposition. *Let* (3.1) *hold,* (E, ϱ) *be a metric space,* I *be a non-degenerate interval in* J*, and* $u : I \times S \to E$*. Then the following three assertions are valid:*

(i) Let u *be right-continuous, then the map* $I \to E$*,* $t \mapsto u(t,x)$ *is right-continuous for each* $x \in S$*. Moreover, if* u *is actually continuous, then* $u(\cdot, x)$ *is càdlàg and left-continuous at each continuity point of* $\Phi(x)$*.*

(ii) Whenever u *is right-continuous at a point* $(t,x) \in I \times S$*, then the map* $S \to E$*,* $y \mapsto u(t,y)$ *is continuous at* x*. In particular, if* u *is right-continuous and* S *is separable, then* u *is progressively measurable.*

(iii) Suppose that Φ *has continuous paths,* I *is compact, and* u *is continuous. Then* u *is continuous in* $x \in S$*, uniformly in* $t \in I$*.*

Proof. (i) We fix $x \in S$, and let $r \in I$ with $r < \sup I$ and $\varepsilon > 0$. Then there is $\delta > 0$ such that $\varrho(u(t,y), u(r,x)) < \varepsilon$ for all $(t,y) \in I \times S$ with $t \geq r$ and $d_S((t,y),(r,x)) < \delta$. Since $\Phi(x)$ is right-continuous, there is $\gamma > 0$ such that

$$\rho(\Phi_t(x), \Phi_r(x)) < \delta/2$$

for all $t \in [r, r+\gamma) \cap J$. We set $\eta := \gamma \wedge (\delta/2)$, then $\varrho(u(t,x), u(r,x)) < \varepsilon$ for all $t \in [r, r+\eta) \cap I$, because $d_S((t,x),(r,x)) < \delta/2 + \rho(\Phi_t(x), \Phi_r(x)) < \delta$.

Next, assume that u is continuous and let $t \in I$ with $t > \inf I$. We prove that $\lim_{s \uparrow t} u(s,x) = u(t, \hat{x})$, where $\hat{x} \in S$ denotes the left-hand limit $\lim_{s\uparrow t} \Phi_s(x)$. Since $\Phi_t(\Phi_s(x)) = \Phi_s(x)$ for all $s \in J$ with $s \leq t$ and Φ_t is Lipschitz continuous, we see that $\Phi_t(\hat{x}) = \lim_{s\uparrow t} \Phi_s(x) = \hat{x}$. Let $\varepsilon > 0$, then there is $\delta > 0$ such that $\varrho(u(s,y), u(t,\hat{x})) < \varepsilon$ for all $(s,y) \in I \times S$ with $d_S((s,y),(t,\hat{x})) < \delta$. Moreover, choose $\gamma > 0$ such that

$$\rho(\Phi_s(x), \hat{x}) < \delta/2$$

for all $s \in (t-\gamma, t) \cap J$. We define $\eta := \gamma \wedge (\delta/2)$, then $\varrho(u(s,x), u(t,\hat{x})) < \varepsilon$ for every $s \in (t-\eta, t) \cap I$, since $d_S((s,x),(t,\hat{x})) < \delta/2 + \rho(\Phi_s(x), \hat{x}) < \delta$. This in fact concludes the proof, because $\hat{x} = \Phi_t(x)$ whenever $\Phi(x)$ is continuous at t.

(ii) Initially, let u be right-continuous at a point $(t,x) \in I \times S$ and $\varepsilon > 0$. Then there is $\delta > 0$ such that $\varrho(u(s,y), (u(t,x)) < \varepsilon$ for each $(s,y) \in I \times S$ with $s \geq r$ and $d_S((s,y),(t,x)) < \delta$. We let $L > 0$ denote a Lipschitz constant of Φ_t and set $\eta := \delta/L$, then we obtain that $\varrho(u(t,y), u(t,x)) < \varepsilon$ for every $y \in S$ with $\rho(y,x) < \eta$, since $d_S((t,y),(t,x)) = \rho(\Phi_t(y), \Phi_t(x)) < \delta$.

Now let u be right-continuous. To infer that u is progressive measurable, we first recall that $u(t,x) = u(t, \Phi_t(x))$ for all $(t,x) \in I \times S$, since u is consistent. By what we have just verified, $u(t, \cdot)$ is continuous and thus, Borel measurable for each $t \in I$. Hence, u is $(\mathscr{S}_t)_{t \in I}$-adapted. For this reason, (i) and Proposition A.38 yield that u is in fact progressively measurable with respect to $(\mathscr{S}_t)_{t\in I}$.

(iii) Let $\varepsilon > 0$ and $x \in S$. We have to show that there is $\eta > 0$ such that $\varrho(u(t,y), u(t,x)) < \varepsilon$ for all $t \in I$ and each $y \in S$ with $\rho(y,x) < \eta$. For each $t \in I$, as u is continuous at (t,x), there is $\delta_t > 0$ such that

$$\varrho(u(s,y), u(t,x)) < \varepsilon/2$$

for all $(s, y) \in I \times S$ with $d_S((s, y), (t, x)) < \delta_t$. In addition, the uniform continuity of the map $I \to S$, $t \mapsto \Phi_t(x)$ gives for each $t \in I$ some $\gamma_t \in (0, \delta_t/3]$ such that

$$\rho(\Phi_r(x), \Phi_s(x)) < \delta_t/3$$

for all $r, s \in I$ with $|r - s| < \gamma_t$. Since $\{(t - \gamma_t, t + \gamma_t) \,|\, t \in I\}$ is an open covering of I, there are $n \in \mathbb{N}$ and $t_1, \ldots, t_n \in I$ with $I \subset \bigcup_{i=1}^n (t_i - \gamma_{t_i}, t_i + \gamma_{t_i})$. Because Φ is Lipschitz continuous in $y \in S$, uniformly in $s \in I$, there is $L > 0$ such that

$$\rho(\Phi_t(x'), \Phi_t(y)) \leq L\rho(x', y)$$

for all $t \in I$ and each $x', y \in S$. We choose $t \in I$ and $y \in S$ with $\rho(y, x) < \eta$, where $\eta := (1/L) \min_{i \in \{1,\ldots,n\}} \gamma_{t_i}$. Let $i \in \{1, \ldots, n\}$ with $t \in (t_i - \gamma_{t_i}, t_i + \gamma_{t_i})$, then $\varrho(u(t, y), u(t, x)) \leq \varrho(u(t, y), u(t_i, x)) + \varrho(u(t_i, x), u(t, x)) < \varepsilon$. Indeed, from $|t - t_i| < \gamma_{t_i}$ we get that $\rho(\Phi_t(x), \Phi_{t_i}(x)) < \delta_{t_i}/3$, which in turn yields that

$$\begin{aligned} d_S((t, y), (t_i, x)) &\leq |t - t_i| + \rho(\Phi_t(y), \Phi_t(x)) + \rho(\Phi_t(x), \Phi_{t_i}(x)) \\ &< \gamma_{t_i} + L\rho(y, x) + \delta_{t_i}/3 < 2\delta_{t_i}/3 + L\eta < \delta_{t_i} \end{aligned}$$

and consequently, $d_S((t, x), (t_i, x)) = |t - t_i| + \rho(\Phi_t(x), \Phi_{t_i}(x)) < \delta_{t_i}$. This establishes the proposition. □

3.2 Consistent stochastic Markov families

In the sequel, we are concerned with time-inhomogeneous Markov processes that are introduced within the pseudometric setting of the previous section. As classical literature, we mainly use Dynkin [11, 12]. Thus, suppose that S is separable, and Φ fulfills (3.1) and has continuous paths, then the assertions of Proposition 3.1 hold. Moreover, for each non-degenerate interval I in J and every topological space E, it follows from $\mathscr{B}(J \times S) \subset \mathscr{B}(J) \otimes \mathscr{S}$ that a consistent map $u : I \times S \to E$ is Borel measurable if and only if it is measurable with respect to the product σ-field $\mathscr{B}(I) \otimes \mathscr{S}$, as Lemma 3.2 asserts.

Let $(\Omega, \mathscr{F})$ be a measurable space. We associate with a *consistent stochastic family* a triple $\mathscr{X} = (X, (\mathscr{F}_t)_{t \in J}, \mathbb{P})$ that is composed of a process $X : J \times \Omega \to S$, a filtration $(\mathscr{F}_t)_{t \in J}$ of $\mathscr{F}$ to which X is adapted, and a set $\mathbb{P} = \{P_{r,x} \,|\, (r, x) \in J \times S\}$ of probability measures on $(\Omega, \mathscr{F})$ such that $P_{r,x} = P_{r,\Phi_r(x)}$ and

$$\Phi_r(X_r) = \Phi_r(x) \quad P_{s,x}\text{-a.s.} \tag{3.2}$$

for all $r, s \in J$ with $r \leq s$ and each $x \in S$. If we want to emphasize the measurable structures, we will refer to a consistent stochastic family on $(\Omega, \mathscr{F})$ with state space S. Additionally, we call $\mathbb{P}$ the *transition probabilities* of $\mathscr{X}$. Note that (3.2) gives

$$d_S((r, X_r), (r, x)) = 0 \quad P_{s,x}\text{-a.s.}$$

while $X_r = x$ $P_{s,x}$-a.s. may fail for each $r, s \in J$ with $r \leq s$ and every $x \in S$. Hence, if Φ_t is the identity map for each $t \in J$, then we will simply call $\mathscr{X}$ a stochastic family.

3.4 Definition. A *consistent Markov process* (on $(\Omega, \mathscr{F})$ with state space S) is a consistent stochastic family $\mathscr{X}$ satisfying the following two properties:

(i) The function $S \to [0,1]$, $x \mapsto P_{s,x}(X_t \in B)$ is Borel measurable for all $s, t \in J$ with $s \leq t$ and each $B \in \mathscr{S}$.

(ii) $P_{r,x}(X_t \in B|\mathscr{F}_s) = P_{s,X_s}(X_t \in B)$ $P_{r,x}$-a.s. for all $r, s, t \in J$ with $r \leq s \leq t$, each $x \in S$, and every $B \in \mathscr{S}$.

In this case, we also say that $\mathscr{X}$ is *Markov*.

If a consistent stochastic family $\mathscr{X}$ fulfills (i), then (ii) is known as the *Markov property* of $\mathscr{X}$. Clearly, from the Monotone Class Theorem A.28 we infer that $\mathscr{X}$ is Markov if there is an $\cap$-stable generator $\mathscr{O}$ of the Borel σ-field $\mathscr{S}$ for which the function $S \to [0,1]$, $x \mapsto P_{s,x}(X_t \in O)$ is Borel measurable and

$$P_{r,x}(X_t \in O|\mathscr{F}_s) = P_{s,X_s}(X_t \in O) \quad P_{r,x}\text{-a.s.}$$

for all $r, s, t \in J$ with $r \leq s \leq t$, each $x \in S$, and every $O \in \mathscr{O}$. Since the topology of S is such a generator, we obtain two sufficient conditions for $\mathscr{X}$ to be Markov. For example, see Chung [4, Section 1.1] in case S is locally compact. These conditions are also necessary, as Proposition 3.7 shows in the end of this section.

3.5 Lemma. *A consistent stochastic family $\mathscr{X}$ is Markov if the function $S \to [0,1]$, $y \mapsto E_{s,y}[\varphi(X_t)]$ is Borel measurable and*

$$E_{r,x}[\varphi(X_t)|\mathscr{F}_s] = E_{s,X_s}[\varphi(X_t)] \quad P_{r,x}\text{-a.s.}$$

for all $r, s, t \in J$ with $r \leq s \leq t$, each $x \in S$, and every Lipschitz continuous $\varphi \in C_b(S, [0,1])$.

Proof. Let $s, t \in J$ satisfy $s \leq t$ and O be an open set in S. Then Lemma A.17 gives an increasing sequence $(\varphi_n)_{n\in\mathbb{N}}$ of $[0,1]$-valued Lipschitz continuous functions on S with $\lim_{n\uparrow\infty} \varphi_n(y) = \mathbb{1}_O(y)$ for each $y \in S$. By monotone convergence,

$$\lim_{n\uparrow\infty} E_{s,y}[\varphi_n(X_t)] = P_{s,y}(X_t \in O) \quad \text{for all } y \in S.$$

Hence, as pointwise limit of a sequence of Borel measurable $[0,1]$-valued functions on S, the function $S \to [0,1]$, $y \mapsto P_{s,y}(X_t \in O)$ is Borel measurable. Therefore, the measurability property (i) of Definition 3.4 holds.

To check the Markov property of $\mathscr{X}$, let $r \in J$ with $r \leq s$ and $x \in S$. Then the assumptions give $E_{r,x}[\varphi_n(X_t)|\mathscr{F}_s] = E_{s,X_s}[\varphi_n(X_t)]$ $P_{r,x}$-a.s. for each $n \in \mathbb{N}$. Thus, monotone convergence for conditional expectations entails that

$$\begin{aligned} P_{r,x}(X_t \in O|\mathscr{F}_s) &= \lim_{n\uparrow\infty} E_{r,x}[\varphi_n(X_t)|\mathscr{F}_s] \\ &= \lim_{n\uparrow\infty} E_{s,X_s}[\varphi_n(X_t)] = P_{s,X_s}(X_t \in O) \quad P_{r,x}\text{-a.s.}, \end{aligned}$$

because $\lim_{n\uparrow\infty} E_{s,X_s(\omega)}[\varphi_n(X_t)] = P_{s,X_s(\omega)}(X_t \in O)$ for each $\omega \in \Omega$, by standard monotone convergence. The claim follows. □

For a consistent stochastic family $\mathscr{X}$, we let $(\hat{\mathscr{F}}'_s)_{s\in J}$ denote the natural backward filtration of X. That is, $\hat{\mathscr{F}}'_s = \sigma(X_t : t \in J \text{ with } t \geq s)$ for each fixed $s \in J$. In addition, let $\mathscr{C}'_s$ be the system of all sets $C' \subset \Omega$ for which there are $n \in \mathbb{N}$, $t_1, \ldots, t_n \in J$ with $s \leq t_1 < \cdots < t_n$, and $B_1, \ldots, B_n \in \mathscr{S}$ such that

$$C' = \{X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n\}.$$

Then $\mathscr{C}'_s$ is an $\cap$-stable generator of $\hat{\mathscr{F}}'_s$. In fact, since $\{X_t \in B\} \in \mathscr{C}'_s$ for all $B \in \mathscr{S}$, we get that $\sigma(X_t) \subset \mathscr{C}'_s$ for each $t \in J$ with $t \geq s$. Hence, $\hat{\mathscr{F}}'_s \subset \sigma(\mathscr{C}'_s)$. Because $\mathscr{C}'_s \subset \hat{\mathscr{F}}'_s$ follows from the definition of $\mathscr{C}'_s$, this clarifies that $\sigma(\mathscr{C}'_s) = \hat{\mathscr{F}}'_s$.

3.6 Lemma. *Assume that $\mathscr{X}$ is a consistent Markov process. Then the subsequent two conditions are valid:*

(i) The function $S \to [0,1]$, $x \mapsto P_{s,x}(A')$ is Borel measurable for all $s \in J$ and each $A' \in \hat{\mathscr{F}}'_s$.

(ii) $P_{r,x}(A'|\mathscr{F}_s) = P_{s,X_s}(A')$ $P_{r,x}$-a.s. for all $r, s \in J$ with $r \leq s$, each $x \in S$, and every $A' \in \hat{\mathscr{F}}'_s$.

Proof. Since $\mathscr{C}'_s$ is an $\cap$-stable generator of $\hat{\mathscr{F}}'_s$ for each $s \in J$, the Monotone Class Theorem A.28 entails that (i) follows once we have shown that the function

$$S \to [0,1], \quad x \mapsto P_{s,x}(X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n)$$

is Borel measurable for all $n \in \mathbb{N}$, each $t_1, \ldots, t_n \in J$ with $s \leq t_1 < \cdots < t_n$, and every $B_1, \ldots, B_n \in \mathscr{S}$. We prove this by induction over $n \in \mathbb{N}$. As the initial induction step $n = 1$ is valid by definition, we may assume that the claim holds for some $n \in \mathbb{N}$. We let $t_1, \ldots, t_{n+1} \in J$ with $s \leq t_1 < \cdots < t_{n+1}$ and define $\mathscr{H}$ to be the set of all $\varphi \in B_b(S^n)$ for which the function

$$S \to \mathbb{R}, \quad x \mapsto E_{s,x}[\varphi(X_{t_1}, \ldots, X_{t_n})]$$

is Borel measurable. Then $\mathscr{H}$ is a monotone class on S^n and $\mathbb{1}_{B_1\times\cdots\times B_n} \in \mathscr{H}$ for all $B_1, \ldots, B_n \in \mathscr{S}$, by induction hypothesis. Since $\mathscr{S}^n$ is an $\cap$-stable generator of the product σ-field $\otimes_{i=1}^n \mathscr{S}$, the Functional Monotone Class Theorem A.29 implies that $\mathscr{H} = B_b(S^n)$. Thus, let $B_1, \ldots, B_{n+1} \in \mathscr{S}$, then from the Markov property of $\mathscr{X}$ we get that

$$\begin{aligned} P_{s,x}(X_{t_1} \in B_1, \ldots, X_{t_{n+1}} \in B_{n+1}) &= E_{s,x}[P_{s,x}(X_{t_1} \in B_1, \ldots, X_{t_{n+1}} \in B_{n+1}|\mathscr{F}_{t_n})] \\ &= E_{s,x}[\mathbb{1}_{\{X_{t_1}\in B_1,\ldots,X_{t_n}\in B_n\}} P_{s,x}(X_{t_{n+1}} \in B_{n+1}|\mathscr{F}_{t_n})] \\ &= E_{s,x}[\mathbb{1}_{\{X_{t_1}\in B_1,\ldots,X_{t_n}\in B_n\}} P_{t_n,X_{t_n}}(X_{t_{n+1}} \in B_{n+1})]. \end{aligned}$$

This calculation completes the induction proof, because the function $S^n \to [0,1]$, $(x_1, \ldots, x_n) \mapsto \mathbb{1}_{B_1\times\cdots\times B_n}(x_1, \ldots, x_n) P_{t_n,x_n}(X_{t_{n+1}} \in B_{n+1})$ is a member of the linear space $B_b(S^n)$. Hence, (i) is proven.

To show (ii), we also choose $r \in J$ with $r \leq s$ and $x \in S$. By the Monotone Class Theorem A.28, we merely have to check that

$$P_{r,x}(X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n | \mathscr{F}_s) = P_{s,X_s}(X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n) \quad P_{r,x}\text{-a.s.}$$

for all $n \in \mathbb{N}$, each $t_1, \ldots, t_n \in J$ with $s \leq t_1 < \cdots < t_n$, and every $B_1, \ldots, B_n \in \mathscr{S}$. This is done by induction over $n \in \mathbb{N}$. The initial induction step $n = 1$ is just the Markov property of $\mathscr{X}$. Therefore, we suppose that the claim is true for some $n \in \mathbb{N}$ and pick $t_1, \ldots, t_{n+1} \in J$ with $s \leq t_1 < \cdots < t_{n+1}$. Then the set $\mathscr{H}$ of all $\varphi \in B_b(S^n)$ satisfying

$$E_{r,x}[\varphi(X_{t_1}, \ldots, X_{t_n}) | \mathscr{F}_s] = E_{s,X_s}[\varphi(X_{t_1}, \ldots, X_{t_n})] \quad P_{r,x}\text{-a.s.}$$

is a monotone class on S^n. The induction hypothesis ensures that $\mathbb{1}_{B_1 \times \cdots \times B_n} \in \mathscr{H}$ for every $B_1, \ldots, B_n \in \mathscr{S}$. Hence, the Functional Monotone Class Theorem A.29 gives $\mathscr{H} = B_b(S^n)$. Now, for each $B_1, \ldots, B_{n+1} \in \mathscr{S}$ we infer from the Markov property of $\mathscr{X}$ that

$$\begin{aligned} P_{r,x}(X_{t_1} \in B_1, \ldots, X_{t_{n+1}} \in B_{n+1} | \mathscr{F}_s) & \\ &= E_{r,x}[\mathbb{1}_{\{X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n\}} P_{t_n, X_{t_n}}(X_{t_{n+1}} \in B_{n+1}) | \mathscr{F}_s] \\ &= E_{s,X_s}[\mathbb{1}_{\{X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n\}} P_{t_n, X_{t_n}}(X_{t_{n+1}} \in B_{n+1})] \quad P_{r,x}\text{-a.s.} \end{aligned}$$

In the last step, we used the previously mentioned fact that the function $S^n \to [0, 1]$, $(x_1, \ldots, x_n) \mapsto \mathbb{1}_{B_1 \times \cdots \times B_n}(x_1, \ldots, x_n) P_{t_n, x_n}(X_{t_{n+1}} \in B_{n+1})$ belongs to $B_b(S^n)$. The Markov property of $\mathscr{X}$ also implies that

$$\begin{aligned} E_{s,X_s(\omega)}[\mathbb{1}_{\{X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n\}} P_{t_n, X_{t_n}}(X_{t_{n+1}} \in B_{n+1})] & \\ &= P_{s,X_s(\omega)}(X_{t_1} \in B_1, \ldots, X_{t_{n+1}} \in B_{n+1}) \end{aligned}$$

for all $\omega \in \Omega$. By combining these two identities, we conclude the induction proof. Thus, the lemma is established. □

Let us further enhance the properties of a consistent Markov process. To this end, we fix $k \in \mathbb{N}$ and let $|\cdot|$ be the Euclidean norm on $\mathbb{R}^k$. For simplicity, we also use $|\cdot|$ for the Frobenius norm on $\mathbb{R}^{k \times k}$. Note that the following one-dimensional assertions are given as statement 0.1.B in Dynkin [11, Appendix].

3.7 Proposition. *Let $\mathscr{X}$ be a consistent Markov process. Then the following two assertions hold:*

(i) Let $s \in J$ and $Y : \Omega \to \mathbb{R}^k$ be $\hat{\mathscr{F}}'_s$-measurable. If $E_{s,x}[|Y|] < \infty$ for all $x \in S$, then the map $S \to \mathbb{R}^k$, $x \mapsto E_{s,x}[Y]$ is Borel measurable.

(ii) Let $r, s \in J$ with $r \leq s$, $x \in S$, and $Y : \Omega \to \mathbb{R}^k$ be $\hat{\mathscr{F}}'_s$-measurable such that $E_{r,x}[|Y|]$ and $E_{s,y}[|Y|]$ are finite for all $y \in S$. Then

$$E_{r,x}[E_{s,X_s}[|Y|]] < \infty \quad \text{and} \quad E_{r,x}[\beta Y] = E_{r,x}[\beta E_{s,X_s}[Y]]$$

for each $\mathscr{F}_s$-measurable bounded map $\beta : \Omega \to \mathbb{R}^{k \times k}$.

Proof. (i) It suffices to consider the case $k = 1$. This causes no loss of generality, since once the claim holds in one dimension, then the fact that $E_{s,x}[|Y_i|] \leq E_{s,x}[|Y|]$ for all $x \in S$ implies that the i-th coordinate function $S \to \mathbb{R}$, $x \mapsto E_{s,x}[Y_i]$ of the map $S \to \mathbb{R}^k$, $x \mapsto E_{s,x}[Y]$ is Borel measurable for each $i \in \{1, \ldots, k\}$. Then from Lemma A.19 the assertion follows.

We may also suppose that Y is bounded. In the general case, Corollary A.24 provides a sequence $(Y_n)_{n \in \mathbb{N}}$ of real-valued $\hat{\mathscr{F}}_s'$-measurable bounded functions on Ω that converges pointwise to Y with $\sup_{n \in \mathbb{N}} |Y_n| \leq |Y|$. Then dominated convergence yields

$$\lim_{n \uparrow \infty} E_{s,x}[Y_n] = E_{s,x}[Y] \quad \text{for all } x \in S,$$

which in turn shows the Borel measurability of the function $S \to \mathbb{R}$, $x \mapsto E_{s,x}[Y]$. Next, the set of all real-valued $\hat{\mathscr{F}}_s'$-measurable bounded functions Z on Ω for which the function $S \to \mathbb{R}$, $x \mapsto E_{s,x}[Z]$ is Borel measurable constitutes a monotone class on Ω. From Lemma 3.6 we know that this linear space contains every indicator function $\mathbb{1}_A$ with $A \in \hat{\mathscr{F}}_s'$. Hence, the Functional Monotone Class Theorem A.29 completes the verification of (i).

(ii) As before, we may let $k = 1$. Indeed, suppose for the moment that the assertion is true in one dimension. To establish the general assertion, we first prove that

$$E_{r,x}[Z|\mathscr{F}_s] = E_{s,X_s}[Z] \quad P_{r,x}\text{-a.s.} \tag{3.3}$$

for each $\hat{\mathscr{F}}_s'$-measurable $Z : \Omega \to \mathbb{R}_+$. It is enough to show this when Z is bounded. In the unbounded case, Corollary A.24 gives us an increasing sequence $(Z_n)_{n \in \mathbb{N}}$ of $\mathbb{R}_+$-valued $\hat{\mathscr{F}}_s'$-measurable bounded functions on Ω that converges pointwise to Z. Then monotone convergence for conditional expectations entails that

$$E_{r,x}[Z|\mathscr{F}_s] = \lim_{n \uparrow \infty} E_{r,x}[Z_n|\mathscr{F}_s] = \lim_{n \uparrow \infty} E_{s,X_s}[Z_n] = E_{s,X_s}[Z] \quad P_{r,x}\text{-a.s.},$$

since $\lim_{n \uparrow \infty} E_{s,X_s(\omega)}[Z_n] = E_{s,X_s(\omega)}[Z]$ for all $\omega \in \Omega$, by monotone convergence. We notice that the set of all real-valued $\hat{\mathscr{F}}_s'$-measurable bounded functions $\hat{Z}$ on Ω with $E_{r,x}[\hat{Z}|\mathscr{F}_s] = E_{s,X_s}[\hat{Z}]$ $P_{r,x}$-a.s. is a monotone class on Ω. Thus, (3.3) is implied by Lemma 3.6 and the Functional Monotone Class Theorem A.29.

Now, to infer the multidimensional case, we make use of (3.3) to obtain that $E_{r,x}[E_{s,X_s}[|Y|]] = E_{r,x}[|Y|] < \infty$, which yields the $P_{r,x}$-integrability of $E_{s,X_s}[|Y|]$. Let us suppose that $\beta : \Omega \to \mathbb{R}^{k \times k}$ is an $\mathscr{F}_s$-measurable bounded map, then the i-th coordinates of $E_{r,x}[\beta Y]$ and $E_{r,x}[\beta E_{s,X_s}[Y]]$ fulfill

$$E_{r,x}[\beta Y]_i = \sum_{j=1}^{k} E_{r,x}[\beta_{i,j} Y_j] = \sum_{j=1}^{k} E_{r,x}[\beta_{i,j} E_{s,X_s}[Y_j]] = E_{r,x}[\beta E_{s,X_s}[Y]]_i$$

for each $i \in \{1, \ldots, k\}$. For these reasons, it suffices to prove the claim for $k = 1$. In one dimension, by writing Y in the form $Y = Y^+ - Y^-$, we get from (3.3) that

$$\begin{aligned} E_{r,x}[\beta Y] &= E_{r,x}[\beta E_{r,x}[Y^+|\mathscr{F}_s]] - E_{r,x}[\beta E_{r,x}[Y^-|\mathscr{F}_s]] \\ &= E_{r,x}[\beta(E_{s,X_s}[Y] - E_{s,X_s}[Y^-])] = E_{r,x}[\beta E_{s,X_s}[Y]], \end{aligned}$$

which completes the proof. $\square$

3.3 The strong Markov and the Feller property

Let us proceed with the study of Markov processes by considering the progressive measurability of the underlying process, the measurability and (right-)continuity properties of the transition probabilities, and the strong Markov property. We refer to Dynkin [11, 12] for definitions that we adjust. In what follows, for each $t \in J$ let J_t be the set of all $s \in J$ with $s \leq t$ and $(\Omega, \mathscr{F})$ be a measurable space.

3.8 Definition. Suppose that $\mathscr{X}$ is a consistent stochastic family.

(i) We call $\mathscr{X}$ *progressive* if the process X is progressively measurable with respect to both its natural filtration and its natural backward filtration.

(ii) We say that $\mathscr{X}$ has *Borel measurable transition probabilities* if the consistent function $J_t \times S \to [0,1]$, $(s,x) \mapsto P_{s,x}(X_t \in B)$ is Borel measurable for all $t \in J$ and each $B \in \mathscr{S}$.

(iii) $\mathscr{X}$ is called *strongly Markov* if it is progressive and has Borel measurable transition probabilities, and the *strong Markov property* holds:

$$P_{r,x}(X_t \in B|\mathscr{F}_\tau) = P_{\tau,X_\tau}(X_t \in B) \quad P_{r,x}\text{-a.s.} \tag{3.4}$$

for all $r,t \in J$ with $r \leq t$, each finite $(\mathscr{F}_s)_{s\in[r,t]}$-stopping time τ, every $x \in S$, and each $B \in \mathscr{S}$.

For a consistent stochastic family $\mathscr{X}$, we set $\hat{\mathscr{F}}_{r,t} := \sigma(X_s : s \in [r,t])$ for each $r,t \in J$ with $r \leq t$. These σ-fields allow for an equivalent description for $\mathscr{X}$ to be progressive.

3.9 Lemma. *A consistent stochastic family $\mathscr{X}$ is progressive if and only if the restriction of X to $[r,t] \times \Omega$ is $\mathscr{B}([r,t]) \otimes \hat{\mathscr{F}}_{r,t}$-measurable for all $r,t \in J$ with $r \leq t$.*

Proof. In the sequel, we denote the natural filtration of X by $(\hat{\mathscr{F}}_s)_{s\in J}$. Let initially $\mathscr{X}$ be progressive and choose $r,t \in J$ with $r \leq t$. Since set of all $D \subset J_t \times \Omega$ with $D \cap ([r,t] \times \Omega) \in \mathscr{B}([r,t]) \otimes \hat{\mathscr{F}}_t$ is a d-system in $J_t \times \Omega$ that includes the $\cap$-stable generator $\{J_s \,|\, s \in J_t\} \times \hat{\mathscr{F}}_t$ of $\mathscr{B}(J_t) \otimes \hat{\mathscr{F}}_t$, we get that

$$F \cap ([r,t] \times \Omega) \in \mathscr{B}([r,t]) \otimes \hat{\mathscr{F}}_t \quad \text{for each } F \in \mathscr{B}(J_t) \otimes \hat{\mathscr{F}}_t,$$

by the Monotone Class Theorem A.28. This gives $\{(s,\omega) \in [r,t] \times \Omega \,|\, X_s(\omega) \in B\} = \{(s,\omega) \in J_t \times \Omega \,|\, X_s(\omega) \in B\} \cap ([r,t] \times \Omega) \in \mathscr{B}([r,t]) \otimes \hat{\mathscr{F}}_t$ for each chosen $B \in \mathscr{S}$. To abbreviate notation, let J_r' be the set of all $s \in J$ with $s \geq r$. Then

$$F \cap ([r,t] \times \Omega) \in \mathscr{B}([r,t]) \otimes \hat{\mathscr{F}}_r' \quad \text{for every } F \in \mathscr{B}(J_r') \otimes \hat{\mathscr{F}}_r',$$

due to the Monotone Class Theorem A.28. Hence, $\{(s,\omega) \in [r,t] \times \Omega \,|\, X_s(\omega) \in B\} = \{(s,\omega) \in J_r' \times \Omega \,|\, X_s(\omega) \in B\} \cap ([r,t] \times \Omega) \in \mathscr{B}([r,t]) \otimes \hat{\mathscr{F}}_r'$. Finally, from the fact that $\mathscr{B}([r,t]) \otimes \hat{\mathscr{F}}_{r,t}$ is the intersection of $\mathscr{B}([r,t]) \otimes \hat{\mathscr{F}}_r'$ and $\mathscr{B}([r,t]) \otimes \hat{\mathscr{F}}_t$, the only if direction is proven.

For if we define $r_0 := \min J$ and $T := \sup J$. First of all, let $t \in J$ and $B \in \mathscr{S}$, then $\{(s,\omega) \in [r_0,t] \times \Omega \,|\, X_s(\omega) \in B\} \in \mathscr{B}([r_0,t]) \otimes \hat{\mathscr{F}}_{r_0,t}$. Because $J_t = [r_0,t]$ and $\hat{\mathscr{F}}_t = \hat{\mathscr{F}}_{r_0,t}$, this shows that X is progressively measurable with respect to its natural filtration. Next, let $r \in J$ and suppose initially that $T \in J$, then $\{(s,\omega) \in [r,T] \times \Omega \,|\, X_s(\omega) \in B\} \in \mathscr{B}([r,T]) \otimes \hat{\mathscr{F}}_{r,T}$. If instead $T \notin J$, then we let $(t_n)_{n\in\mathbb{N}}$ be a sequence in J_r' with $\lim_{n\uparrow\infty} t_n = T$. This entails that

$$\{(s,\omega) \in J_r' \times \Omega \,|\, X_s(\omega) \in B\} = \bigcup_{n\in\mathbb{N}} \{(s,\omega) \in [r,t_n] \times \Omega \,|\, X_s(\omega) \in B\} \in \mathscr{B}(J_r') \otimes \hat{\mathscr{F}}_r',$$

since $\{(s,\omega) \in [r,t_n] \times \Omega \,|\, X_s(\omega) \in B\} \in \mathscr{B}([r,t_n]) \otimes \hat{\mathscr{F}}_{r,t_n}$ and $\hat{\mathscr{F}}_{r,t_n} \subset \hat{\mathscr{F}}_r'$ for each $n \in \mathbb{N}$. Hence, in either case, X is also progressively measurable with respect to its natural backward filtration. □

By the Monotone Class Theorem A.28, a consistent stochastic family $\mathscr{X}$ is Borel, that is, it has Borel measurable transition probabilities, if there is some $\cap$-stable generator $\mathscr{O}$ of $\mathscr{S}$ such that the function $J_t \times S \to [0,1]$, $(s,x) \mapsto P_{s,x}(X_t \in O)$ is Borel measurable for all $t \in J$ and each $O \in \mathscr{O}$. Moreover, let $\mathscr{X}$ be progressive and Borel, then it is strongly Markov if

$$P_{r,x}(X_t \in O|\mathscr{F}_\tau) = P_{\tau,X_\tau}(X_t \in O) \quad P_{r,x}\text{-a.s.}$$

for all $r,t \in J$ with $r \leq t$, every finite $(\mathscr{F}_s)_{s\in[r,t]}$-stopping time τ, all $x \in S$, and each $O \in \mathscr{O}$. Hence, we can give sufficient conditions for $\mathscr{X}$ to be Borel and strongly Markov. The necessity is implied by Proposition 3.13.

3.10 Lemma. *Assume that $\mathscr{X}$ is a consistent stochastic family. Then the following two assertions are valid:*

(i) $\mathscr{X}$ is Borel if the function $J_t \times S \to [0,1]$, $(s,x) \mapsto E_{s,x}[\varphi(X_t)]$ is Borel measurable for each $t \in J$ and every Lipschitz continuous $\varphi \in C_b(S,[0,1])$.

(ii) Let $\mathscr{X}$ be progressive and Borel. Then $\mathscr{X}$ is strongly Markov if

$$E_{r,x}[\varphi(X_t)|\mathscr{F}_\tau] = E_{\tau,X_\tau}[\varphi(X_t)] \quad P_{r,x}\text{-a.s.}$$

for all $r,t \in J$ with $r \leq t$, every finite $(\mathscr{F}_s)_{s\in[r,t]}$-stopping time τ, each $x \in S$, and every Lipschitz continuous $\varphi \in C_b(S,[0,1])$.

Proof. (i) We proceed similarly as in Lemma 3.5. Let $t \in J$ and O be an open set in S. Then Lemma A.17 yields an increasing sequence $(\varphi_n)_{n\in\mathbb{N}}$ of $[0,1]$-valued Lipschitz continuous functions on S that converges pointwise to $\mathbb{1}_O$. By monotone convergence,

$$\lim_{n\uparrow\infty} E_{s,x}[\varphi_n(X_t)] = P_{s,x}(X_t \in O)$$

for all $(s,x) \in J_t \times S$. Hence, as pointwise limit of a sequence of $[0,1]$-valued Borel measurable functions on $J_t \times S$, the function $J_t \times S \to [0,1]$, $(s,x) \mapsto P_{s,x}(X_t \in O)$ is Borel measurable as well.

(ii) Let $r, t \in J$ with $r \leq t$, τ be a finite $(\mathscr{F}_s)_{s\in[r,t]}$-stopping time, and $x \in S$. We once again choose an open set O in S and some increasing sequence $(\varphi_n)_{n\in\mathbb{N}}$ of $[0,1]$-valued Lipschitz continuous functions on S that converges pointwise to $\mathbb{1}_O$, then $E_{r,x}[\varphi_n(X_t)|\mathscr{F}_\tau] = E_{\tau,X_\tau}[\varphi_n(X_t)]$ $P_{r,x}$-a.s. for each $n \in \mathbb{N}$. Monotone convergence for conditional expectations guarantees that

$$\begin{aligned}P_{r,x}(X_t \in O|\mathscr{F}_\tau) &= \lim_{n\uparrow\infty} E_{r,x}[\varphi_n(X_t)|\mathscr{F}_\tau]\\ &= \lim_{n\uparrow\infty} E_{\tau,X_\tau}[\varphi_n(X_t)] = P_{\tau,X_\tau}(X_t \in O) \quad P_{r,x}\text{-a.s.},\end{aligned}$$

since $\lim_{n\uparrow\infty} E_{\tau(\omega),X_\tau(\omega)}[\varphi_n(X_t)] = P_{\tau(\omega),X_\tau(\omega)}(X_t \in O)$ for all $\omega \in \Omega$, due to standard monotone convergence. Therefore, the claim is proven. $\square$

For a consistent stochastic family $\mathscr{X}$ and each $t \in J$, let us use the $\cap$-stable generator $\mathscr{C}'_t$ of $\hat{\mathscr{F}}'_t$, which has been defined to be the system of all sets $C' \subset \Omega$ of the form $C' = \{X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n\}$ for some $n \in \mathbb{N}$, $t_1, \ldots, t_n \in J$ with $t \leq t_1 < \cdots < t_n$, and $B_1, \ldots, B_n \in \mathscr{S}$.

3.11 Lemma. *For every consistent Markov process $\mathscr{X}$ that is Borel the following two assertions hold:*

(i) The function $J_t \times S \to [0,1]$, $(s,x) \mapsto P_{s,x}(A')$ is Borel measurable for each $t \in J$ and all $A' \in \hat{\mathscr{F}}'_t$.

(ii) Assume that $\mathscr{X}$ is strongly Markov, then $P_{r,x}(A'|\mathscr{F}_\tau) = P_{\tau,X_\tau}(A')$ $P_{r,x}$-a.s. for all $r, t \in J$ with $r \leq t$, every finite $(\mathscr{F}_s)_{s\in[r,t]}$-stopping time τ, all $x \in S$, and each $A' \in \hat{\mathscr{F}}'_t$.

Proof. (i) Because the system $\mathscr{C}'_t$ is an $\cap$-stable generator of $\hat{\mathscr{F}}'_t$ for every $t \in J$, it follows from the Monotone Class Theorem A.28 that we only need to verify that the function

$$J_t \times S \to [0,1], \quad (s,x) \mapsto P_{s,x}(X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n)$$

is Borel measurable for each $n \in \mathbb{N}$, all $t_1, \ldots, t_n \in J$ with $t \leq t_1 < \cdots < t_n$, and every $B_1, \ldots, B_n \in \mathscr{S}$. We show this inductively over $n \in \mathbb{N}$. Since the initial induction step $n = 1$ is true by definition, we suppose that the claim holds for some $n \in \mathbb{N}$. Let $t_1, \ldots, t_{n+1} \in J$ with $t \leq t_1 < \cdots < t_{n+1}$, then the set of all $\varphi \in B_b(S^n)$ for which the function $J_t \times S \to \mathbb{R}$, $(s,x) \mapsto E_{s,x}[\varphi(X_{t_1}, \ldots, X_{t_n})]$ is Borel measurable is a monotone class on S^n. Hence, it coincides with $B_b(S^n)$, by the Functional Monotone Class Theorem A.29. Let $B_1, \ldots, B_{n+1} \in \mathscr{S}$, then the Markov property of $\mathscr{X}$ yields that

$$P_{s,x}(X_{t_1} \in B_1, \ldots, X_{t_{n+1}} \in B_{n+1}) = E_{s,x}[\mathbb{1}_{\{X_{t_1}\in B_1,\ldots,X_{t_n}\in B_n\}} P_{t_n,X_{t_n}}(X_{t_{n+1}} \in B_{n+1})]$$

for all $(s,x) \in J_t \times S$. This finishes the induction proof, because, as noted before, the function $S^n \to [0,1]$, $(x_1, \ldots, x_n) \mapsto \mathbb{1}_{B_1\times\cdots\times B_n}(x_1, \ldots, x_n) P_{t_n,x_n}(X_{t_{n+1}} \in B_{n+1})$ is a member of $B_b(S^n)$.

(ii) Let $r, t \in J$ with $r \leq t$, τ be a finite $(\mathscr{F}_s)_{s\in[r,t]}$-stopping time, and $x \in S$. Due to the Monotone Class Theorem A.28, it suffices to show that

$$P_{r,x}(X_{t_1} \in B_1, \dots, X_{t_n} \in B_n|\mathscr{F}_\tau) = P_{\tau,X_\tau}(X_{t_1} \in B_1, \dots, X_{t_n} \in B_n) \quad P_{r,x}\text{-a.s.}$$

for each $n \in \mathbb{N}$, all $t_1, \dots, t_n \in J$ with $t \leq t_1 < \dots < t_n$, and every $B_1, \dots, B_n \in \mathscr{S}$. This is carried out inductively over $n \in \mathbb{N}$. Since $\mathscr{X}$ is strongly Markov, the initial induction step $n = 1$ holds. We assume that the claim is valid for some $n \in \mathbb{N}$. Let $t_1, \dots, t_{n+1} \in J$ with $t \leq t_1 < \dots < t_{n+1}$, then the set of all $\varphi \in B_b(S^n)$ such that

$$E_{r,x}[\varphi(X_{t_1}, \dots, X_{t_n})|\mathscr{F}_\tau] = E_{\tau,X_\tau}[\varphi(X_{t_1}, \dots, X_{t_n})] \quad P_{r,x}\text{-a.s.}$$

is a monotone class on S^n. For this reason, it equals $B_b(S^n)$, by the Functional Monotone Class Theorem A.29. Let $B_1, \dots, B_{n+1} \in \mathscr{S}$, then the Markov property of $\mathscr{X}$ leads us to

$$\begin{aligned} P_{r,x}(X_{t_1} \in B_1, \dots, &X_{t_{n+1}} \in B_{n+1}|\mathscr{F}_\tau) \\ &= E_{r,x}[\mathbb{1}_{\{X_{t_1}\in B_1,\dots,X_{t_n}\in B_n\}} P_{t_n,X_{t_n}}(X_{t_{n+1}} \in B_{n+1})|\mathscr{F}_\tau] \\ &= E_{\tau,X_\tau}[\mathbb{1}_{\{X_{t_1}\in B_1,\dots,X_{t_n}\in B_n\}} P_{t_n,X_{t_n}}(X_{t_{n+1}} \in B_{n+1})] \quad P_{r,x}\text{-a.s.} \end{aligned}$$

The Markov property of $\mathscr{X}$ also implies that

$$\begin{aligned} E_{\tau(\omega),X_\tau(\omega)}[\mathbb{1}_{\{X_{t_1}\in B_1,\dots,X_{t_n}\in B_n\}} &P_{t_n,X_{t_n}}(X_{t_{n+1}} \in B_{n+1})] \\ &= P_{\tau(\omega),X_\tau(\omega)}(X_{t_1} \in B_1, \dots, X_{t_{n+1}} \in B_{n+1}) \end{aligned}$$

for all $\omega \in \Omega$. By putting these two equations together, we complete the induction proof. □

3.12 Remark. Assume that $\mathscr{X}$ is strongly Markov. Let $r, t \in J$ with $r \leq t$, $x \in S$, and $A' \in \hat{\mathscr{F}}_t'$, then every $(\mathscr{F}_s)_{s\in J}$-stopping time τ with $\tau \geq r$ satisfies

$$P_{r,x}(A'|\mathscr{F}_\tau) = P_{\tau,X_\tau}(A') \quad P_{r,x}\text{-a.s. on } \{\tau \leq t\}.$$

Indeed, $P_{r,x}(A'|\mathscr{F}_\tau) = P_{r,x}(A'|\mathscr{F}_{t\wedge\tau})$ $P_{r,x}$-a.s. on $\{\tau \leq t\}$, because

$$E_{r,x}[P_{r,x}(A'|\mathscr{F}_\tau)\mathbb{1}_{B\cap\{\tau\leq t\}}] = P_{r,x}(A' \cap B \cap \{\tau \leq t\}) = E_{r,x}[P_{r,x}(A'|\mathscr{F}_{t\wedge\tau})\mathbb{1}_{B\cap\{\tau\leq t\}}]$$

for each $B \in \mathscr{F}_\tau$. Furthermore, $P_{r,x}(A'|\mathscr{F}_{t\wedge\tau}) = P_{t\wedge\tau,X_{t\wedge\tau}}(A')$ $P_{r,x}$-a.s., since $\mathscr{X}$ is strongly Markov. This gives the result.

Let $\mathscr{X}$ be a consistent stochastic family and I be a non-degenerate interval in J. For each $t \in I$ we denote by I_t the set of all $s \in I$ with $s \leq t$, then the reconstructible σ-field on I is the σ-field generated by the system of all sets F in $I \times \Omega$ of the form $F = I_u \times A'$ for some $u \in I$ and some $A' \in \hat{\mathscr{F}}_u'$, as considered in Section A.7.

If E is a topological space, then a map $Y : I \times \Omega \to E$ that is measurable with respect to this σ-field will be called reconstructible. Note that if Y is $(\hat{\mathscr{F}}_t')_{t\in I}$-adapted and has right-continuous paths, then it is reconstructible, due to Proposition A.40. We now strengthen the measurability properties and the strong Markov property (see statements 0.4 and 0.6.C in [11, Appendix] for the one-dimensional case).

3.13 Proposition. *For every consistent Markov process $\mathscr{X}$ with Borel measurable transition probabilities the subsequent two assertions are valid:*

(i) Let $Y : I \times \Omega \to \mathbb{R}^k$ be reconstructible with $E_{r,x}[|Y_r|] < \infty$ for all $(r,x) \in I \times S$, then the map $I \times S \to \mathbb{R}^k$, $(r,x) \mapsto E_{r,x}[Y_r]$ is consistent and Borel measurable.

(ii) Suppose that $\mathscr{X}$ is strongly Markov. Let $(r,x) \in I \times S$, τ be some finite $(\mathscr{F}_s)_{s\in I}$-stopping time with $\tau \geq r$, and $Y : I \times \Omega \to \mathbb{R}^k$ be reconstructible such that $E_{r,x}[|Y_\tau|]$ and $E_{s,y}[|Y_s|]$ are finite for every $(s,y) \in I \times S$ with $s \geq r$. Then

$$\int_\Omega E_{\tau(\omega),X_\tau(\omega)}[|Y_{\tau(\omega)}|]\, P_{r,x}(d\omega) < \infty \quad \textit{and}$$
$$E_{r,x}[\beta Y_\tau] = \int_\Omega \beta(\omega) E_{\tau(\omega),X_\tau(\omega)}[Y_{\tau(\omega)}]\, P_{r,x}(d\omega)$$

for every $\mathscr{F}_\tau$-measurable bounded map $\beta : \Omega \to \mathbb{R}^{k\times k}$.

Proof. (i) We may let $k = 1$. In fact, assume that the assertion is true in one dimension, then it follows from $E_{r,x}[|Y_r^{(i)}|] \leq E_{r,x}[|Y_r|]$ for all $(r,x) \in I \times S$ that the function $I \times S \to \mathbb{R}$, $(r,x) \mapsto E_{r,x}[Y_r^{(i)}]$ is Borel measurable for each $i \in \{1,\dots,k\}$. As consistency is a direct consequence of the requirement that $P_{r,x} = P_{r,\Phi_r(x)}$ for all $(r,x) \in J \times S$, this yields the claim, by Lemma A.19.

We may also assume that Y is bounded. In the general case, Corollary A.24 yields a sequence $(Y^{(n)})_{n\in\mathbb{N}}$ of real-valued reconstructible bounded processes on $I \times \Omega$ with $\lim_{n\uparrow\infty} Y_t^{(n)} = Y_t$ and $\sup_{n\in\mathbb{N}} |Y_t^{(n)}| \leq |Y_t|$ for every $t \in I$. Dominated convergence entails that

$$\lim_{n\uparrow\infty} E_{r,x}[Y_r^{(n)}] = E_{r,x}[Y_r]$$

for each $(r,x) \in I \times S$. This clarifies that the function $I \times S \to \mathbb{R}$, $(r,x) \mapsto E_{r,x}[Y_r]$ must then be Borel measurable. Moreover, the set of all reconstructible bounded processes $Z : I \times \Omega \to \mathbb{R}$ for which the function $I \times S \to \mathbb{R}$, $(r,x) \mapsto E_{r,x}[Z_r]$ is Borel measurable is a monotone class in $I \times \Omega$. By Lemma 3.11, the process $I \times \Omega \to [0,1]$, $(t,\omega) \mapsto \mathbb{1}_{I_u}(t)\mathbb{1}_{A'}(\omega)$ belongs to this linear space for each $u \in I$ and every $A' \in \hat{\mathscr{F}}'_u$. Hence, the Functional Monotone Class Theorem A.29 leads to the claim.

(ii) It is enough to deduce the assertion for $k = 1$. To infer the multidimensional case, we utilize the fact that

$$E_{r,x}[Z_\tau|\mathscr{F}_\tau](\omega) = E_{\tau(\omega),X_\tau(\omega)}[Z_{\tau(\omega)}] \quad \text{for } P_{r,x}\text{-a.e. } \omega \in \Omega \tag{3.5}$$

for each reconstructible $Z : I \times \Omega \to \mathbb{R}_+$. Indeed, for justifying this, we merely need to consider the case when Z is bounded. In the unbounded case, Corollary A.24 gives an increasing sequence $(Z^{(n)})_{n\in\mathbb{N}}$ of $\mathbb{R}_+$-valued reconstructible bounded processes on $I \times \Omega$ with $\lim_{n\uparrow\infty} Z_t^{(n)} = Z_t$ for all $t \in I$. Then monotone convergence for conditional expectations implies that

$$E_{r,x}[Z_\tau|\mathscr{F}_\tau](\omega) = \lim_{n\uparrow\infty} E_{r,x}[Z_\tau^{(n)}|\mathscr{F}_\tau](\omega) = \lim_{n\uparrow\infty} E_{\tau(\omega),X_\tau(\omega)}[Z_{\tau(\omega)}^{(n)}] = E_{\tau(\omega),X_\tau(\omega)}[Z_{\tau(\omega)}]$$

for $P_{r,x}$-a.e. $\omega \in \Omega$, since $\lim_{n\uparrow\infty} E_{\tau(\omega),X_\tau(\omega)}[Z^{(n)}_{\tau(\omega)}] = E_{\tau(\omega),X_\tau(\omega)}[Z_{\tau(\omega)}]$ follows from standard monotone convergence for every $\omega \in \Omega$. Now, the set of all reconstructible bounded processes $\hat{Z} : I \times \Omega \to \mathbb{R}$ such that

$$E_{r,x}[\hat{Z}_\tau|\mathscr{F}_\tau](\omega) = E_{\tau(\omega),X_\tau(\omega)}[\hat{Z}_{\tau(\omega)}] \quad \text{for } P_{r,x}\text{-a.e. } \omega \in \Omega$$

is a monotone class in $I \times \Omega$. We fix $u \in I$ and $A' \in \hat{\mathscr{F}}'_u$, and let $\hat{Z} : I \times \Omega \to [0,1]$ be given by $\hat{Z}_t(\omega) := \mathbb{1}_{I_u}(t)\mathbb{1}_{A'}(\omega)$. Then Remark 3.12 shows that

$$\begin{aligned} E_{r,x}[\hat{Z}_\tau|\mathscr{F}_\tau](\omega) &= \mathbb{1}_{\{\tau\le u\}}(\omega)P_{r,x}(A'|\mathscr{F}_\tau)(\omega) = \mathbb{1}_{\{\tau\le u\}}(\omega)P_{\tau(\omega),X_\tau(\omega)}(A') \\ &= E_{\tau(\omega),X_\tau(\omega)}[\mathbb{1}_{\{\tau\le u\}}(\omega)\mathbb{1}_{A'}] = E_{\tau(\omega),X_\tau(\omega)}[\hat{Z}_{\tau(\omega)}] \end{aligned}$$

for $P_{r,x}$-a.e. $\omega \in \Omega$. In consequence, the Functional Monotone Class Theorem A.29 establishes the validity of (3.5). This in turn implies that

$$\int_\Omega E_{\tau(\omega),X_\tau(\omega)}[|Y_{\tau(\omega)}|]\, P_{r,x}(d\omega) = E_{r,x}[|Y_\tau|] < \infty,$$

which shows that the function $\Omega \to \mathbb{R}_+$, $\omega \mapsto E_{\tau(\omega),X_\tau(\omega)}[|Y_{\tau(\omega)}|]$ is $P_{r,x}$-integrable. Next, we let $\beta : \Omega \to \mathbb{R}^{k\times k}$ be some $\mathscr{F}_\tau$-measurable bounded map, then the i-th coordinates of $E_{r,x}[\beta Y_\tau]$ and $\int_\Omega \beta(\omega)E_{\tau(\omega),X_\tau(\omega)}[Y_{\tau(\omega)}]\, P_{r,x}(d\omega)$ satisfy

$$\begin{aligned} E_{r,x}[\beta Y_\tau]_i &= \sum_{j=1}^k E_{r,x}[\beta_{i,j}E_{r,x}[Y^{(j)}_\tau|\mathscr{F}_\tau]] = \sum_{j=1}^k \int_\Omega \beta_{i,j}(\omega)E_{\tau(\omega),X_\tau(\omega)}[Y^{(j)}_{\tau(\omega)}]\, P_{r,x}(d\omega) \\ &= \int_\Omega \beta(\omega)E_{\tau(\omega),X_\tau(\omega)}[Y_{\tau(\omega)}]\, P_{r,x}(d\omega)_i \end{aligned}$$

for all $i \in \{1,\dots,k\}$. Therefore, restricting the proof to $k = 1$ leads to no loss of generality. In one dimension, we conclude from $Y = Y^+ - Y^-$ and (3.5) that

$$\begin{aligned} E_{r,x}[\beta Y_\tau] &= E_{r,x}[\beta E_{r,x}[Y^+_\tau|\mathscr{F}_\tau]] - E_{r,x}[\beta E_{r,x}[Y^-_\tau|\mathscr{F}_\tau]] \\ &= \int_\Omega \beta(\omega)E_{\tau(\omega),X_\tau(\omega)}[Y_{\tau(\omega)}]\, P_{r,x}(d\omega). \end{aligned}$$

□

In what follows, we call a consistent stochastic family $\mathscr{X}$ a *diffusion process* if it is strongly Markov and the process X is continuous. Moreover, $\mathscr{X}$ is said to have the *(right-hand) Feller property* if the function

$$J_t \times S \to \mathbb{R}, \quad (r,x) \mapsto E_{r,x}[\varphi(X_t)] \tag{3.6}$$

is (right-)continuous for all $t \in J$ and every $\varphi \in C_b(S)$. We make two observations. First, if $\mathscr{X}$ is (right-hand) Feller, that is, it has the (right-hand) Feller property, then (3.6) is Borel measurable for each $t \in J$ and every $\varphi \in C_b(S)$, by Proposition 3.3 and Lemma 3.2. This in turn implies that $\mathscr{X}$ is Borel, as Lemma 3.10 shows.

Secondly, the strong Markov property holds provided $\mathscr{X}$ is Markov and the finite $(\mathscr{F}_s)_{s\in[r,t]}$-stopping time τ appearing in (3.4) takes finitely many values. More precisely, let $r, t \in J$ with $r \leq t$, $x \in S$, and $B \in \mathscr{S}$, then

$$P_{r,x}(X_t \in B|\mathscr{F}_\tau) = P_{\tau,X_\tau}(X_t \in B) \quad P_{r,x}\text{-a.s.}$$

for each finite $(\mathscr{F}_s)_{s\in[r,t]}$-stopping time τ taking finitely many values. To see this, let $n \in \mathbb{N}$ and $s_1, \ldots, s_n \in [r, t]$ be the pairwise distinct values of τ. Then we obtain that $P_{\tau,X_\tau}(X_t \in B) = \sum_{i=1}^n P_{s_i,X_{s_i}}(X_t \in B)\mathbb{1}_{\{\tau=s_i\}}$. Thus, $P_{\tau,X_\tau}(X_t \in B)$ is actually $\mathscr{F}_\tau$-measurable. Furthermore,

$$P_{r,x}(\{X_t \in B\} \cap A) = \sum_{i=1}^n E_{r,x}[P_{s_i,X_{s_i}}(X_t \in B)\mathbb{1}_{A\cap\{\tau=s_i\}}] = E_{r,x}[P_{\tau,X_\tau}(X_t \in B)\mathbb{1}_A]$$

for all $A \in \mathscr{F}_\tau$, which yields the assertion. These considerations lead to the following conclusion.

3.14 Lemma. *A consistent Markov process $\mathscr{X}$ that is right-hand Feller and for which X has right-continuous paths is strongly Markov. In particular, if X has continuous paths, then $\mathscr{X}$ is a diffusion process.*

Proof. Since X is adapted to its natural filtration and its natural backward filtration, Propositions A.38 and A.40 imply that X is progressively measurable with respect to both filtrations. Thus, $\mathscr{X}$ is progressive and Borel. By Lemma 3.10, to show the strong Markov property, we merely have to prove that

$$E_{r,x}[\varphi(X_t)|\mathscr{F}_\tau] = E_{\tau,X_\tau}[\varphi(X_t)] \quad P_{r,x}\text{-a.s.}$$

for all $r, t \in J$ with $r \leq t$, each finite $(\mathscr{F}_s)_{s\in[r,t]}$-stopping time τ, all $x \in S$, and every $\varphi \in C_b(S)$. To this end, Proposition A.44 gives us a decreasing sequence $(\tau_n)_{n\in\mathbb{N}}$ of finite $(\mathscr{F}_s)_{s\in[r,t]}$-stopping times, each taking only finite many values, such that $\inf_{n\in\mathbb{N}} \tau_n = \tau$. Due to the preceding discussion,

$$E_{r,x}[\varphi(X_t)|\mathscr{F}_{\tau_n}] = E_{\tau_n,X_{\tau_n}}[\varphi(X_t)] \quad P_{r,x}\text{-a.s.}$$

for each $n \in \mathbb{N}$. Since X has right-continuous paths, we get $\lim_{n\uparrow\infty} X_{\tau_n} = X_\tau$, which entails that $\lim_{n\uparrow\infty} d_S((\tau_n, X_{\tau_n}), (\tau, X_\tau)) = 0$. For this reason, the right-hand Feller property of $\mathscr{X}$ implies that

$$\lim_{n\uparrow\infty} E_{\tau_n,X_{\tau_n}}[\varphi(X_t)] = E_{\tau,X_\tau}[\varphi(X_t)].$$

Because $\tau \leq \tau_n$, we must have $\mathscr{F}_\tau \subset \mathscr{F}_{\tau_n}$ for each $n \in \mathbb{N}$. Thus, we obtain that $E_{r,x}[\varphi(X_t)|\mathscr{F}_\tau] = E_{r,x}[E_{r,x}[\varphi(X_t)|\mathscr{F}_{\tau_n}]|\mathscr{F}_\tau] = E_{r,x}[E_{\tau_n,X_{\tau_n}}[\varphi(X_t)]|\mathscr{F}_\tau]$ $P_{r,x}$-a.s. for all $n \in \mathbb{N}$. By dominated convergence for conditional expectations,

$$E_{r,x}[\varphi(X_t)|\mathscr{F}_\tau] = \lim_{n\uparrow\infty} E_{r,x}[E_{\tau_n,X_{\tau_n}}[\varphi(X_t)]|\mathscr{F}_\tau] = E_{\tau,X_\tau}[\varphi(X_t)] \quad P_{r,x}\text{-a.s.}$$

This justifies the claim. □

3.4 Multidimensional additive maps

In this section we are concerned with multidimensional additive maps and several of its properties. Let $\mathscr{X}$ be a consistent progressive Markov process on a measurable space $(\Omega, \mathscr{F})$ with state space S. We utilize the natural backward filtration $(\hat{\mathscr{F}}_t')_{t\in J}$ of X and the σ-fields $\hat{\mathscr{F}}_{r,t} = \sigma(X_s : s \in [r,t])$ for all $r, t \in J$ with $r \leq t$. As before, $k \in \mathbb{N}$, and $|\cdot|$ is the Euclidean norm on $\mathbb{R}^k$ and the Frobenius norm on $\mathbb{R}^{k\times k}$.

3.15 Definition. A *k-dimensional additive map* (of $\mathscr{X}$) is given through a map $\kappa : \Omega \times \mathscr{B}(J) \to [-\infty, \infty]^k$, $(\omega, B) \mapsto \kappa(B)(\omega)$ with the following two properties:

(i) $\kappa_i(\cdot)(\omega) : \mathscr{B}(J) \to [-\infty, \infty]$, $B \mapsto \kappa_i(B)(\omega)$ is a signed Borel measure for all $i \in \{1, \ldots, k\}$ and each $\omega \in \Omega$.

(ii) The map $\kappa([r,t]) : \Omega \to \mathbb{R}^k$, $\omega \mapsto \kappa([r,t])(\omega)$ is $\hat{\mathscr{F}}_{r,t}$-measurable for every $r, t \in J$ with $r \leq t$.

If in addition $\kappa_i \geq 0$ for all $i \in \{1, \ldots, k\}$, then we call κ *non-negative*. Moreover, a k-dimensional additive map κ with $\kappa(\{t\}) = 0$ for all $t \in J$ is said to be *continuous*.

In alignment with the literature, we call every one-dimensional additive map an *additive functional* (cf. Dynkin [11, Appendix], [12, Section 1.1]). From Lemma A.19 we easily draw the conclusion that an $[-\infty, \infty]^k$-valued map κ on $\Omega \times \mathscr{B}(J)$ is a k-dimensional additive map if and only if all its coordinate functions $\kappa_1, \ldots, \kappa_k$ are additive functionals. In this case, κ is non-negative (resp. continuous) if and only if $\kappa_1, \ldots, \kappa_k$ are. Let us prove a multidimensional analogue to statement 0.2.E in [12, Appendix].

3.16 Lemma. *Assume that κ is a non-negative k-dimensional additive map of $\mathscr{X}$. Let $\theta \in B(J \times S, \mathbb{R}_+^k)$ be such that the function $J \to \mathbb{R}_+$, $s \mapsto \theta_i(s, X_s(\omega))$ is locally $\kappa_i(\cdot)(\omega)$-integrable for every $i \in \{1, \ldots, k\}$ and each $\omega \in \Omega$. Then the map $\nu : \Omega \times \mathscr{B}(J) \to [0, \infty]^k$ given by*

$$\nu(B)(\omega) := \int_B \theta(s, X_s(\omega))\, \kappa(ds)(\omega)$$

is another non-negative k-dimensional additive map that is continuous if κ is.

Proof. From above remarks we infer that it suffices to show the claim for $k = 1$. Because $\mathscr{X}$ is progressive and θ is Borel measurable, the process

$$J \times \Omega \to \mathbb{R}_+, \quad (s, \omega) \mapsto \theta(s, X_s(\omega))$$

must be $\mathscr{B}(J) \otimes \mathscr{F}$-measurable. Hence, the function $J \to \mathbb{R}_+$, $s \mapsto \theta(s, X_s(\omega))$ is $\mathscr{B}(J)$-measurable and, by hypothesis, locally $\kappa(\cdot)(\omega)$-integrable for each $\omega \in \Omega$. For this reason, $\nu(\cdot)(\omega)$ is a Borel measure on J.

It remains to verify that $\nu([r,t])$ is $\hat{\mathscr{F}}_{r,t}$-measurable for all $r, t \in J$ with $r \leq t$. Here, we may suppose that θ is bounded. In the general case, Corollary A.24 yields an increasing sequence $(\theta_n)_{n\in\mathbb{N}}$ in $B_b(J \times S, \mathbb{R}_+)$ that converges pointwise to θ. By monotone convergence,

$$\lim_{n\uparrow\infty} \int_r^t \theta_n(s, X_s(\omega))\, \kappa(ds)(\omega) = \int_r^t \theta(s, X_s(\omega))\, \kappa(ds)(\omega) = \nu([r,t])(\omega)$$

for all $\omega \in \Omega$. Thus, as pointwise limit of a sequence of $\mathbb{R}_+$-valued $\hat{\mathscr{F}}_{r,t}$-measurable functions on Ω, it follows that $\nu([r,t])$ is $\hat{\mathscr{F}}_{r,t}$-measurable. This justifies that the boundedness assumption on θ leads to no loss of generality. Now, the set $\mathscr{H}_{r,t}$ of all $\mathscr{B}([r,t]) \otimes \hat{\mathscr{F}}_{r,t}$-measurable bounded processes $Z : [r,t] \times \Omega \to \mathbb{R}$ for which

$$\int_r^t Z_s\, \kappa(ds) \text{ is } \hat{\mathscr{F}}_{r,t}\text{-measurable}$$

is a monotone class on $[r,t] \times \Omega$. For each $s \in [r,t]$ and all $A \in \hat{\mathscr{F}}_{r,t}$, the process $[r,t] \times \Omega \to [0,1]$, $(r',\omega) \mapsto \mathbb{1}_{[r,s]\times A}(r',\omega)$ is a member of $\mathscr{H}_{r,t}$, since

$$\int_r^t \mathbb{1}_{[r,s]\times A}(r',\omega)\, \kappa(dr')(\omega) = \kappa([r,s])(\omega)\mathbb{1}_A(\omega)$$

for all $\omega \in \Omega$. Consequently, the Functional Monotone Class Theorem A.29 implies that $\mathscr{H}_{r,t}$ is the linear space of all real-valued $\mathscr{B}([r,t]) \otimes \hat{\mathscr{F}}_{r,t}$-measurable bounded processes on $[r,t] \times \Omega$. As $\mathscr{X}$ is progressive and θ is assumed to be bounded, the process $[r,t]\times\Omega \to \mathbb{R}_+$, $(s,\omega) \mapsto \theta(s, X_s(\omega))$ belongs to $\mathscr{H}_{r,t}$, by Lemma 3.9. Hence, ν is a non-negative additive functional.

Lastly, from the definition of ν we conclude that $\nu(\cdot)(\omega)$ is absolutely continuous with respect to $\kappa(\cdot)(\omega)$ for all $\omega \in \Omega$. This in turn entails that if κ is continuous, then $\{t\}$ is not only a $\kappa(\cdot)(\omega)$-null set, but also a $\nu(\cdot)(\omega)$-null set for each $t \in J$ and every $\omega \in \Omega$. This proves the assertion. □

Let temporarily μ be a signed Borel measure on J. Then the positive part μ^+ and negative part μ^- of μ are given by $\mu^+(B) = \sup\{\mu(A) \mid A \in \mathscr{B}(J) : A \subset B\}$ and $\mu^-(B) = \sup\{-\mu(A) \mid A \in \mathscr{B}(J) : A \subset B\}$ for each $B \in \mathscr{B}(J)$, which are two Borel measures that satisfy the Jordan decomposition

$$\mu = \mu^+ - \mu^-.$$

The variation of μ is another Borel measure given by $|\mu| = \mu^+ + \mu^-$ that fulfills $|\mu(B)| \leq |\mu|(B)$ for all $B \in \mathscr{B}(J)$ (see for instance Cohn [5, Section 4.1]). In what follows, let I be a non-degenerate interval in J.

3.17 Lemma. *Suppose that κ is a continuous k-dimensional additive map of $\mathscr{X}$. Let $Y : I \times \Omega \to \mathbb{R}^k$ be an $(\hat{\mathscr{F}}'_s)_{s\in I}$-progressively measurable process and $t \in I$ such that*

$$\int_r^t |Y_s^{(i)}(\omega)|\, |\kappa_i|(ds)(\omega) < \infty \tag{3.7}$$

for all $i \in \{1,\dots,k\}$ and each $(r,\omega) \in I_t \times \Omega$. Then the map $Z : I_t \times \Omega \to \mathbb{R}^k$ defined via

$$Z_r(\omega) := \int_r^t Y_s(\omega)\, \kappa(ds)(\omega)$$

is a reconstructible continuous process.

Proof. As Y is $(\hat{\mathscr{F}}'_s)_{s\in I}$-progressively measurable, its restriction to $[r,t]\times\Omega$ must be $\mathscr{B}([r,t])\otimes\hat{\mathscr{F}}'_r$-measurable for all $r\in I_t$. Hence, from (3.7) we deduce that Z is well-defined. Moreover, since an $[-\infty,\infty]^k$-valued map on $\Omega\times\mathscr{B}(J)$ is a continuous k-dimensional additive map if and only if all its coordinate functions are continuous additive functionals, it is enough to show the assertion for $k=1$.

Let us prove that $Z(\omega)$ is continuous for each $\omega\in\Omega$. To this end, let $r\in I_t$ and $(r_n)_{n\in\mathbb{N}}$ be a sequence in I_t with $\lim_{n\uparrow\infty} r_n = r$. Then it is readily checked that $\lim_{n\uparrow\infty}\mathbb{1}_{[r_n,t]}(s)=\mathbb{1}_{[r,t]}(s)$ for all $s\in I_t$ with $s\neq r$. Because $\kappa(\{r\})(\omega)=0$, we obtain from dominated convergence that

$$\lim_{n\uparrow\infty} Z_{r_n}(\omega)=\lim_{n\uparrow\infty}\int_{I_t}\mathbb{1}_{[r_n,t]}(s)Y_s(\omega)\,\kappa(ds)(\omega)=\int_{I_t}\mathbb{1}_{[r,t]}(s)Y_s(\omega)\,\kappa(ds)(\omega)=Z_r(\omega).$$

In consequence, if we can verify that Z is adapted to $(\hat{\mathscr{F}}'_s)_{s\in I_t}$, then Proposition A.40 implies that Z is reconstructible, which concludes the proof.

For this purpose, we may suppose that Y is bounded. Indeed, once the claim is true in this case, then Lemma A.39 and Corollary A.24 give us a sequence $(Y^{(n)})_{n\in\mathbb{N}}$ of real-valued $(\hat{\mathscr{F}}'_s)_{s\in I_t}$-progressively measurable bounded processes on $I_t\times\Omega$ with $\lim_{n\uparrow\infty} Y^{(n)}_s = Y_s$ and $\sup_{n\in\mathbb{N}}|Y^{(n)}_s|\leq|Y_s|$ for all $s\in I_t$. Then

$$\lim_{n\uparrow\infty}\int_r^t Y^{(n)}_s(\omega)\,\kappa(ds)(\omega)=\int_r^t Y_s(\omega)\,\kappa(ds)(\omega)=Z_r(\omega)$$

for each $(r,\omega)\in I_t\times\Omega$, by dominated convergence. As pointwise limit of a sequence of real-valued $(\hat{\mathscr{F}}'_s)_{s\in I_t}$-adapted bounded processes on $I_t\times\Omega$, the process Z must also be $(\hat{\mathscr{F}}'_s)_{s\in I_t}$-adapted. This explains the simplification. At last, let $r\in I_t$, then the set $\mathscr{H}_r$ of all $\mathscr{B}([r,t])\otimes\hat{\mathscr{F}}'_r$-measurable bounded processes $\hat{Y}:[r,t]\times\Omega\to\mathbb{R}$ for which

$$\int_r^t \hat{Y}_s\,\kappa(ds)\text{ is }\hat{\mathscr{F}}'_r\text{-measurable}$$

is a monotone class in $[r,t]\times\Omega$. For every $s\in[r,t]$ and each $A'\in\hat{\mathscr{F}}'_r$, the process $[r,t]\times\Omega\to[0,1]$, $(r',\omega)\mapsto\mathbb{1}_{[r,s]\times A'}(r',\omega)$ is a member of $\mathscr{H}_r$. So, the Functional Monotone Class Theorem A.29 entails that $\mathscr{H}_r$ coincides with the linear space of all real-valued $\mathscr{B}([r,t])\otimes\hat{\mathscr{F}}'_r$-measurable bounded processes on $[r,t]\times\Omega$. Because the restriction of Y to $[r,t]\times\Omega$ is $\mathscr{B}([r,t])\otimes\hat{\mathscr{F}}'_r$-measurable and $r\in I_t$ has been arbitrarily chosen, the claim follows. □

Until the end of this section, we suppose that μ is a Borel measure on J with $\mu(\{t\})=0$ for all $t\in J$ and $\theta\in B(J\times S,\mathbb{R}_+)$ is consistently bounded. Then Lemma 3.16 guarantees that the function $\nu:\Omega\times\mathscr{B}(J)\to[0,\infty]$ defined by

$$\nu(B)(\omega):=\int_B\theta(s,X_s(\omega))\,\mu(ds)$$

is a non-negative continuous additive functional of $\mathscr{X}$. By using standard properties of conditional expectations, we show the following integral identity that is used in the upcoming section and in Chapter 4. In one dimension, this is a special case of Theorem 57 in Dellacherie and Meyer [9, Section 6.2].

3.18 Proposition. *Suppose that $\mathscr{X}$ has Borel measurable transition probabilities. Let $b \in B(I \times S, \mathbb{R}^{k\times k})$ be locally μ-dominated and $Y : I \times \Omega \to \mathbb{R}^k$ be reconstructible and consistently bounded. Then*

$$E_{r,x}\bigg[\int_r^t b(s, X_s)Y_s\,\nu(ds)\bigg] = E_{r,x}\bigg[\int_r^t b(s, X_s)E_{s,X_s}[Y_s]\,\nu(ds)\bigg] \tag{3.8}$$

for all $r, t \in I$ with $r \leq t$ and each $x \in S$.

Proof. Because $\mathscr{X}$ is progressive, b is Borel measurable, and Y is reconstructible, it follows from Lemma 3.9 and Proposition 3.13 that the processes

$$I \times \Omega \to \mathbb{R}^k, \quad (s, \omega) \mapsto b(s, X_s(\omega))Y_s(\omega)$$

and $I \times \Omega \to \mathbb{R}^k$, $(s, \omega) \mapsto b(s, X_s(\omega))E_{s,X_s(\omega)}[Y_s]$ must be $(\hat{\mathscr{F}}'_s)_{s\in I}$-progressively measurable. Let $\bar{b} \in B(I, \mathbb{R}_+)$ be a locally μ-integrable function with $|b(\cdot, y)| \leq \bar{b}$ for all $y \in S$ μ-a.s. on I. For each $r, t \in I$ with $r \leq t$, let $c_{r,t} \geq 0$ and $\bar{\theta}_{r,t} \geq 0$ be such that $|Y_s(\omega)| \leq c_{r,t}$ and $|\theta(s, y)| \leq \bar{\theta}_{r,t}$ for all $s \in [r, t]$, each $\omega \in \Omega$, and every $y \in S$. Then

$$\begin{aligned}\int_r^t |b(s, X_s)|(|Y_s| \vee E_{s,X_s}[|Y_s|])\,\nu(ds) &\leq c_{r,t}\int_r^t \bar{b}(s)|\theta(s, X_s)|\,\mu(ds)\\ &\leq c_{r,t}\bar{\theta}_{r,t}\int_r^t \bar{b}(s)\,\mu(ds) < \infty\end{aligned}$$

for each $r, t \in I$ with $r \leq t$. Hence, Lemma 3.17 implies that for fixed $t \in I$ the processes

$$I_t \times \Omega \to \mathbb{R}^k, \quad (s, \omega) \mapsto \int_r^t b(s, X_s(\omega))Y_s(\omega)\,\nu(ds)(\omega)$$

and $I_t \times \Omega \to \mathbb{R}^k$, $(s, \omega) \mapsto \int_r^t b(s, X_s(\omega))E_{s,X_s(\omega)}[Y_s]\,\nu(ds)(\omega)$ are reconstructible, consistently bounded, and continuous. This also justifies that the two expectations appearing in (3.8) are well-defined for all $r \in I_t$ and each $x \in S$.

Let us show that these two expectations coincide. We pick a μ-null set $N \in \mathscr{B}(J)$ such that $|b(s, y)| \leq \bar{b}(s)$ for all $(s, y) \in (N^c \cap I) \times S$, then from Proposition 3.7 and the standard properties of conditional expectations we infer that

$$\begin{aligned}E_{r,x}[b_{i,j}(s, X_s)Y_s^{(j)}\theta(s, X_s)] &= E_{r,x}[b_{i,j}(s, X_s)E_{r,x}[Y_s^{(j)}|\mathscr{F}_s]\theta(s, X_s)]\\ &= E_{r,x}[b_{i,j}(s, X_s)E_{s,X_s}[Y_s^{(j)}]\theta(s, X_s)]\end{aligned}$$

for each $s \in N^c \cap [r, t]$ and every $i, j \in \{1, \ldots, k\}$. In combination with Fubini's theorem, this gives

$$\begin{aligned}E_{r,x}\bigg[\int_r^t b_{i,j}(s, X_s)Y_s^{(j)}\,\nu(ds)\bigg] &= \int_{N^c\cap[r,t]} E_{r,x}[b_{i,j}(s, X_s)Y_s^{(j)}\theta(s, X_s)]\,\mu(ds)\\ &= \int_{N^c\cap[r,t]} E_{r,x}[b_{i,j}(s, X_s)E_{s,X_s}[Y_s^{(j)}]\theta(s, X_s)]\,\mu(ds)\\ &= E_{r,x}\bigg[\int_r^t b_{i,j}(s, X_s)E_{s,X_s}[Y_s^{(j)}]\,\nu(ds)\bigg]\end{aligned}$$

for each $i, j \in \{1, \ldots, k\}$. Consequently, we compute that

$$\begin{aligned} E_{r,x}\left[\int_r^t b(s, X_s) Y_s \, \nu(ds)\right]_i &= \sum_{j=1}^k E_{r,x}\left[\int_r^t b_{i,j}(s, X_s) Y_s^{(j)} \, \nu(ds)\right] \\ &= \sum_{j=1}^k E_{r,x}\left[\int_r^t b_{i,j}(s, X_s) E_{s,X_s}[Y_s^{(j)}] \, \nu(ds)\right] \\ &= E_{r,x}\left[\int_r^t b(s, X_s) E_{s,X_s}[Y_s] \, \nu(ds)\right]_i \end{aligned}$$

for all $i \in \{1, \ldots, k\}$. Hence, the proof is complete. □

We conclude with a (right-)continuity result for consistent progressive Markov processes that are (right-hand) Feller. The idea to use dominated convergence comes from Professor Dr. Schied.

3.19 Proposition. *Assume that $\mathscr{X}$ is (right-hand) Feller. Let $\varphi \in B(I \times S, \mathbb{R}^k)$ be locally μ-dominated and $t \in I$. If there is a μ-null set $N \in \mathscr{B}(J)$ such that θ and φ are right-continuous at each point of $(N^c \cap I_t) \times S$, then the map*

$$\psi : I_t \times S \to \mathbb{R}^k, \quad \psi(r, x) := E_{r,x}\left[\int_r^t \varphi(s, X_s) \, \nu(ds)\right]$$

is consistently bounded and (right-)continuous.

Proof. Since φ is Borel measurable, the process $I \times \Omega \to \mathbb{R}^k$, $(s, \omega) \mapsto \varphi(s, X_s(\omega))$ is $(\hat{\mathscr{F}}_s')_{s \in I}$-progressively measurable. Let $\overline{a} \in B(I, \mathbb{R}_+)$ be some locally μ-integrable function with $|\varphi(\cdot, y)| \leq \overline{a}$ for all $y \in S$ μ-a.s. on I, and for each $r \in I_t$ choose $\overline{\theta}_r \geq 0$ such that $|\theta(s, y)| \leq \overline{\theta}_r$ for every $(s, y) \in [r, t] \times S$. Then

$$\int_r^t |\varphi(s, X_s)| \, \nu(ds) = \int_r^t |\varphi(s, X_s)||\theta(s, X_s)| \, \mu(ds) \leq \overline{\theta}_r \int_r^t \overline{a}(s) \, \mu(ds) < \infty$$

for each $r \in I_t$. These considerations in combination with Lemma 3.17 imply that the process $I_t \times \Omega \to \mathbb{R}^k$, $(r, \omega) \mapsto \int_r^t \varphi(s, X_s(\omega)) \, \nu(ds)(\omega)$ is in fact reconstructible, consistently bounded, and continuous. So, we have clarified that ψ is well-defined and consistently bounded.

To show that ψ is (right-)continuous, let $(r, x) \in I_t \times S$ and $(r_n, x_n)_{n \in \mathbb{N}}$ be a sequence in $I_t \times S$ (with $r_n \geq r$ for all $n \in \mathbb{N}$) that converges to (r, x). First, we consider the case $r = t$, then $\psi(t, x) = 0$, since $\mu(\{t\}) = 0$. Let $q \in I_t$ be such that $q \leq r_n$ for almost all $n \in \mathbb{N}$. Then

$$|\psi(r_n, x_n)| \leq \overline{\theta}_q \int_{r_n}^t \overline{a}(s) \, \mu(ds)$$

for almost each $n \in \mathbb{N}$. By dominated convergence, $\lim_{n \uparrow \infty} \int_{r_n}^t \overline{a}(s) \, \mu(ds) = 0$, which gives $\lim_{n \uparrow \infty} \psi(r_n, x_n) = \psi(t, x)$.

Let now $r < t$ and choose a μ-null set $L \in \mathscr{B}(J)$ such that $|\varphi(s,y)| \leq \overline{a}(s)$ for all $(s,y) \in (L^c \cap I) \times S$. For each $n \in \mathbb{N}$ we define the map $\phi_n : I_t \to \mathbb{R}^k$ through

$$\phi_n(s) := E_{r_n,x_n}[\varphi(s,X_s)\theta(s,X_s)], \quad \text{if } s \in L^c \cap [r_n,t],$$

and $\phi_n(s) := 0$, otherwise. In a similar way, we let $\phi : I_t \to \mathbb{R}^k$ be defined through $\phi(s) := E_{r,x}[\varphi(s,X_s)\theta(s,X_s)]$, if $s \in L^c \cap [r,t]$, and $\phi(s) := 0$, otherwise. Then Fubini's theorem entails that

$$\psi(r_n,x_n) = \int_{L^c\cap[r_n,t]} E_{r_n,x_n}[\varphi(s,X_s)\theta(s,X_s)]\,\mu(ds) = \int_{I_t} \phi_n(s)\,\mu(ds),$$
$$\psi(r,x) = \int_{L^c\cap[r,t]} E_{r,x}[\varphi(s,X_s)\theta(s,X_s)]\,\mu(ds) = \int_{I_t} \phi(s)\,\mu(ds)$$

for all $n \in \mathbb{N}$. Let $s \in N^c \cap L^c \cap (r,t]$, then the map $S \to \mathbb{R}^k$, $x \mapsto \varphi(s,x)\theta(s,x)$ belongs to $C_b(S,\mathbb{R}^k)$, by Proposition 3.3. Hence, the (right-hand) Feller property of $\mathscr{X}$ entails that the map

$$I_s \times S \to \mathbb{R}^k, \quad (r',x') \mapsto E_{r',x'}[\varphi(s,X_s)\theta(s,X_s)]$$

is (right-)continuous. We pick $n_0 \in \mathbb{N}$ such that $r_n < s$ for every $n \in \mathbb{N}$ with $n > n_0$ and set $(r_n^{(s)}, x_n^{(s)}) := (r_{n+n_0}, x_{n+n_0})$ for all $n \in \mathbb{N}$, then $(r_n^{(s)}, x_n^{(s)})_{n\in\mathbb{N}}$ is a sequence in $I_s \times S$ that converges to (r,x). Thus,

$$\lim_{n\uparrow\infty} \phi_n(s) = \lim_{n\uparrow\infty} E_{r_n^{(s)},x_n^{(s)}}[\varphi(s,X_s)\theta(s,X_s)] = E_{r,x}[\varphi(s,X_s)\theta(s,X_s)] = \phi(s).$$

For $s \in I$ with $s < r$ it follows that $\phi_n(s) = 0$ for almost each $n \in \mathbb{N}$. As $\mu(\{r\}) = 0$, we have shown that $\lim_{n\uparrow\infty} \phi_n(s) = \phi(s)$ for μ-a.e. $s \in I_t$. Let us choose $q \in I$ and $n_0 \in \mathbb{N}$ such that $q \leq r_n$ for all $n \in \mathbb{N}$ with $n \geq n_0$. This in turn gives $\sup_{n\in\mathbb{N}:\, n\geq n_0} |\phi_n(s)| \leq \mathbb{1}_{[q,t]}(s)\overline{a}(s)\overline{\theta}_q$ for μ-a.e. $s \in I_t$. For this reason, the Dominated Convergence Theorem A.33 yields that

$$\lim_{n\uparrow\infty} \psi(r_n,x_n) = \lim_{n\uparrow\infty} \int_{I_t} \phi_n(s)\,\mu(ds) = \int_{I_t} \phi(s)\,\mu(ds) = \psi(r,x).$$

This proves the proposition. □

3.5 A Markovian Gronwall inequality

By using the concepts of consistent boundedness and local dominance, we aim to give a general Markovian Gronwall inequality. A well-known result in this direction is provided by Dynkin [11, Lemma 3.2]. First, two integration by parts formulas are derived on a non-degenerate interval I in J with $T := \sup I \in I$. We initially let ν be a Borel measure on J with $\nu(\{t\}) = 0$ for all $t \in J$.

3.20 Lemma. *Let $a \in C(I)$ be locally of bounded variation, $b \in B(I)$ be locally ν-integrable, and $\varphi \in C^1(\mathbb{R})$. Then*

$$\int_r^T \varphi'\bigg(\int_r^t b(s)\,\nu(ds)\bigg) b(t)a(t)\,\nu(dt) = \varphi\bigg(\int_r^T b(s)\,\nu(ds)\bigg) a(T) \\ - \varphi(0)a(r) - \int_r^T \varphi\bigg(\int_r^t b(s)\,\nu(ds)\bigg)\, a(dt)$$

for each $r \in I$.

Proof. Clearly, the function $B : [r,T] \to \mathbb{R}$ defined via $B(t) := \int_r^t b(s)\,\nu(ds)$ is continuous and of bounded variation. For this reason, the Fundamental Theorem of Calculus for Lebesgue-Stieltjes integrals entails that $C : [r,T] \to \mathbb{R}$ given by $C(t) := \varphi(B(t))$ is also continuous and of bounded variation, and satisfies

$$C(t) - C(r) = \int_r^t \varphi'(B(s))\,B(ds) = \int_r^t \varphi'(B(s))b(s)\,\nu(ds)$$

for all $t \in [r,T]$. Moreover, the integration by parts formula for Lebesgue-Stieltjes integrals yields that

$$\int_r^T \varphi'(B(t))b(t)a(t)\,\nu(dt) = \int_r^T a(t)\,C(dt) \\ = C(T)a(T) - C(r)a(r) - \int_r^T C(t)\,a(dt).$$

By inserting the definition of B and C, we obtain the claim. □

In the context of the lemma, there are two cases for the function a that we are mainly interested in. First, suppose that $a(r) = \int_r^T \overline{a}(t)\,\nu(dt)$ for every $r \in I$ and some locally ν-integrable $\overline{a} \in B(I)$. Then we infer that

$$\int_r^T \varphi'\bigg(\int_r^t b(s)\,\nu(ds)\bigg) b(t) \int_t^T \overline{a}(t')\,\nu(dt')\,\nu(dt) \\ = \int_r^T \bigg(\varphi\bigg(\int_r^t b(s)\,\nu(ds)\bigg) - \varphi(0)\bigg)\overline{a}(t)\,\nu(dt)$$

for each $r \in I$. Secondly, let $a(t) = 1$ for all $t \in I$. Then the assertion of the lemma reduces to

$$\int_r^T \varphi'\bigg(\int_r^t b(s)\,\nu(ds)\bigg) b(t)\,\nu(dt) = \varphi\bigg(\int_r^T b(s)\,\nu(ds)\bigg) - \varphi(0)$$

for every $r \in I$. After having considered these specific cases, let us prove another integration by parts formula.

3.21 Lemma. *Let $a \in B(I)$ be locally bounded, $b \in B(I)$ be locally ν-integrable, and $\varphi \in C^1(\mathbb{R})$. Then*

$$\int_r^T b(s) \int_s^T \varphi'\bigg(\int_s^t b(s')\,\nu(ds')\bigg) a(t)\,\nu(dt)\,\nu(ds) \\ = \int_r^T \bigg(\varphi\bigg(\int_r^t b(s)\,\nu(ds)\bigg) - \varphi(0)\bigg) a(t)\,\nu(dt)$$

for all $r \in I$.

Proof. From Fubini's theorem and the fact that $\mathbb{1}_{[s,T]}(t) = \mathbb{1}_{[r,t]}(s)$ for all $s, t \in [r, T]$ we get that

$$\int_r^T b(s) \int_s^T \varphi'\bigg(\int_s^t b(s')\,\nu(ds')\bigg) a(t)\,\nu(dt)\,\nu(ds) \\ = \int_r^T \int_r^t b(s)\varphi'\bigg(\int_s^t b(s')\,\nu(ds')\bigg)\,\nu(ds)a(t)\,\nu(dt).$$

Let $t \in [r, T]$, then $B : [r, t] \to \mathbb{R}$ defined by $B(s) := \int_s^t b(s')\,\nu(ds')$ is continuous and of bounded variation. Therefore, the Fundamental Theorem of Calculus for Lebesgue-Stieltjes integrals implies that

$$\varphi(B(t)) - \varphi(B(r)) = \int_r^t \varphi'(B(s))\,B(ds) = -\int_r^t \varphi'(B(s))b(s)\,\nu(ds).$$

Together with the last equation, this yields the correct result. $\square$

Under the hypotheses of the lemma, let us make the choice $\varphi(x) = x^{n+1}/(n+1)!$ for all $x \in \mathbb{R}$ and some $n \in \mathbb{N}_0$, then

$$\int_r^T b(s) \int_s^T \bigg(\int_s^t b(s')\,\nu(ds')\bigg)^n \frac{a(t)}{n!}\,\nu(dt)\,\nu(ds) \\ = \int_r^T \bigg(\int_r^t b(s)\,\nu(ds)\bigg)^{n+1} \frac{a(t)}{(n+1)!}\,\nu(dt)$$

for each $r \in I$. This is the main case that is of our interest. From now on we suppose more generally that ν is a non-negative continuous additive functional of $\mathscr{X}$ that is of the form

$$\nu(B)(\omega) = \int_B \theta(s, X_s(\omega))\,\mu(ds)$$

for all $(\omega, B) \in \Omega \times \mathscr{B}(J)$, some consistently bounded $\theta \in B(J \times S, \mathbb{R}_+)$, and some Borel measure μ on J with $\mu(\{t\}) = 0$ for all $t \in J$.

3.22 Proposition. *Let $\alpha : I \times \Omega \to \mathbb{R}_+$ be reconstructible and consistently bounded, $b \in B(I \times S, \mathbb{R}_+)$ be locally μ-dominated, and $c \in B_b(I, \mathbb{R}_+)$. Suppose that $(u_n)_{n\in\mathbb{N}_0}$ is a sequence of μ-consistently bounded functions in $B(I \times S, \mathbb{R}_+)$ subject to*

$$u_n(r, x) \le E_{r,x}[\alpha_r] + c(r)E_{r,x}\bigg[\int_r^T b(s, X_s)u_{n-1}(s, X_s)\,\nu(ds)\bigg]$$

for each $n \in \mathbb{N}$ and every $(r,x) \in I \times S$. Then

$$\begin{aligned} u_n(r,x) \leq{}& E_{r,x}[\alpha_r] \\ &+ c(r)\sum_{i=0}^{n-2} E_{r,x}\bigg[\int_r^T \bigg(\int_r^t b(s,X_s)c(s)\,\nu(ds)\bigg)^i \frac{b(t,X_t)}{i!}\alpha_t\,\nu(dt)\bigg] \\ &+ c(r)E_{r,x}\bigg[\int_r^T \bigg(\int_r^t b(s,X_s)c(s)\,\nu(ds)\bigg)^{n-1} \frac{(b\cdot u_0)}{(n-1)!}(t,X_t)\,\nu(dt)\bigg] \end{aligned}$$

for all $n \in \mathbb{N}$ with $n \geq 2$ and each $(r,x) \in I \times S$.

Proof. Let us prove the assertion by induction over $n \in \mathbb{N}$ with $n \geq 2$. To abbreviate notation, we set $\gamma(r,x) := b(r,x)c(r)$ for all $(r,x) \in I \times S$. In the initial induction step $n = 2$ Proposition 3.18 yields that

$$\begin{aligned} u_2(r,x) \leq{}& E_{r,x}[\alpha_r] + c(r)E_{r,x}\bigg[\int_r^T b(s,X_s)\alpha_s\,\nu(ds)\bigg] \\ &+ c(r)E_{r,x}\bigg[\int_r^T \gamma(s,X_s)\int_s^T (b\cdot u_0)(t,X_t)\,\nu(dt)\,\nu(ds)\bigg] \end{aligned}$$

for each $(r,x) \in I \times S$. For the last expectation Lemma 3.21 gives us that

$$\begin{aligned} &E_{r,x}\bigg[\int_r^T \gamma(s,X_s)\int_s^T (b\cdot u_0)(t,X_t)\,\nu(dt)\,\nu(ds)\bigg] \\ &\qquad = E_{r,x}\bigg[\int_r^T \bigg(\int_r^t \gamma(s,X_s)\,\nu(ds)\bigg)(b\cdot u_0)(t,X_t)\,\nu(dt)\bigg]. \end{aligned}$$

This concludes the initial induction step. Let us now suppose that the claim is valid for some $n \in \mathbb{N}$ with $n \geq 2$ and fix $(r,x) \in I \times S$. Then Proposition 3.18 implies that

$$\begin{aligned} &u_{n+1}(r,x) \leq E_{r,x}[\alpha_r] + c(r)E_{r,x}\bigg[\int_r^T b(s,X_s)\alpha_s\,\nu(ds)\bigg] \\ &+ c(r)E_{r,x}\bigg[\sum_{i=0}^{n-2}\int_r^T \gamma(s,X_s)\int_s^T \bigg(\int_s^t \gamma(s',X_{s'})\,\nu(ds')\bigg)^i \frac{b(t,X_t)}{i!}\alpha_t\,\nu(dt)\,\nu(ds)\bigg] \\ &+ c(r)E_{r,x}\bigg[\int_r^T \gamma(s,X_s)\int_s^T \bigg(\int_s^t \gamma(s',X_{s'})\,\nu(ds')\bigg)^{n-1} \frac{(b\cdot u_0)}{(n-1)!}(t,X_t)\,\nu(dt)\,\nu(ds)\bigg]. \end{aligned}$$

Consequently, we deduce from Lemma 3.21 that

$$\begin{aligned} u_{n+1}(r,x) \leq{}& E_{r,x}[\alpha_r] + c(r)E_{r,x}\bigg[\int_r^T b(s,X_s)\alpha_s\,\nu(ds)\bigg] \\ &+ c(r)\sum_{i=0}^{n-2} E_{r,x}\bigg[\int_r^T \bigg(\int_r^t \gamma(s,X_s)\,\nu(ds)\bigg)^{i+1} \frac{b(t,X_t)}{(i+1)!}\alpha_t\,\nu(dt)\bigg] \\ &+ c(r)E_{r,x}\bigg[\int_r^T \bigg(\int_r^t \gamma(s,X_s)\,\nu(ds)\bigg)^{n} \frac{(b\cdot u_0)}{n!}(t,X_t)\,\nu(dt)\bigg]. \end{aligned}$$

Hence, the assertion follows. □

By using the estimate in Amann [1, Lemma 6.1], the announced inequality can be proven inductively.

3.23 Markovian Gronwall Inequality. Let $\alpha : I \times \Omega \to \mathbb{R}_+$ be reconstructible and consistently bounded, $b \in B(I \times S, \mathbb{R}_+)$ be locally μ-dominated, and in addition $c \in B_b(I, \mathbb{R}_+)$. If $u \in B(I \times S, \mathbb{R}_+)$ is μ-consistently bounded and satisfies

$$u(r,x) \leq E_{r,x}[\alpha_r] + c(r)E_{r,x}\bigg[\int_r^T b(s,X_s)u(s,X_s)\,\nu(ds)\bigg]$$

for each $(r,x) \in I \times S$, then

$$u(r,x) \leq E_{r,x}[\alpha_r] + c(r)E_{r,x}\bigg[\int_r^T \exp\bigg(\int_r^t b(s,X_s)c(s)\,\nu(ds)\bigg)b(t,X_t)\alpha_t\,\nu(dt)\bigg]$$

for every $(r,x) \in I \times S$.

Proof. Proposition 3.22 implies that

$$\begin{aligned} u(r,x) &\leq E_{r,x}[\alpha_r] \\ &\quad + c(r)\sum_{i=0}^{n-1} E_{r,x}\bigg[\int_r^T \bigg(\int_r^t b(s,X_s)c(s)\,\nu(ds)\bigg)^i \frac{b(t,X_t)}{i!}\alpha_t\,\nu(dt)\bigg] \\ &\quad + c(r)E_{r,x}\bigg[\int_r^T \bigg(\int_r^t b(s,X_s)c(s)\,\nu(ds)\bigg)^n \frac{(b\cdot u)}{n!}(t,X_t)\,\nu(dt)\bigg] \end{aligned}$$

for every $(r,x) \in I \times S$ and each $n \in \mathbb{N}$. Monotone convergence entails that

$$\begin{aligned} \lim_{n\uparrow\infty}\sum_{i=0}^{n-1} E_{r,x}\bigg[\int_r^T \bigg(\int_r^t b(s,X_s)c(s)\,\nu(ds)\bigg)^i \frac{b(t,X_t)}{i!}\alpha_t\,\nu(dt)\bigg] \\ = E_{r,x}\bigg[\int_r^T \exp\bigg(\int_r^t b(s,X_s)c(s)\,\nu(ds)\bigg)b(t,X_t)\alpha_t\,\nu(dt)\bigg]. \end{aligned}$$

Moreover, since u is μ-consistently bounded, dominated convergence gives

$$\lim_{n\uparrow\infty} E_{r,x}\bigg[\int_r^T \bigg(\int_r^t b(s,X_s)c(s)\,\nu(ds)\bigg)^n \frac{(b\cdot u)}{n!}(t,X_t)\,\nu(dt)\bigg] = 0.$$

This establishes the claim. □

We make another estimation that rests on Amann [1, Corollary 6.2].

3.24 Corollary. *Let $\alpha : I \times \Omega \to \mathbb{R}_+$ be reconstructible and consistently bounded, and $a, b \in B(I \times S, \mathbb{R}_+)$ be locally μ-dominated. Suppose that $c \in B_b(I, \mathbb{R}_+)$, and $u \in B(I \times S, \mathbb{R}_+)$ is μ-consistently bounded and fulfills*

$$u(r,x) \leq E_{r,x}[\alpha_r] + c(r)E_{r,x}\bigg[\int_r^T a(s,X_s) + b(s,X_s)u(s,X_s)\,\nu(ds)\bigg]$$

for all $(r,x) \in I \times S$. If α has decreasing paths and c is decreasing, then

$$u(r,x) \leq E_{r,x}\Big[e^{c(r)\int_r^T b(s,X_s)\,\nu(ds)}\alpha_r\Big] + c(r)E_{r,x}\bigg[\int_r^T e^{\int_r^t b(s,X_s)c(s)\,\nu(ds)}a(t,X_t)\,\nu(dt)\bigg]$$

for each $(r,x) \in I \times S$.

Proof. First, $\eta : I \times \Omega \to \mathbb{R}_+$ given by $\eta_r(\omega) := \alpha_r(\omega) + c(r)\int_r^T a(s, X_s(\omega))\,\nu(ds)(\omega)$ is a reconstructible consistently bounded process, as Lemma 3.17 asserts. Hence, the Markovian Gronwall Inequality 3.23 in combination with Lemma 3.20 yield

$$\begin{aligned}
u(r,x) &\leq E_{r,x}[\eta_r] + c(r)E_{r,x}\left[\int_r^T e^{c(r)\int_r^t b(s,X_s)\,\nu(ds)} b(t,X_t)\,\nu(dt)\alpha_r\right] \\
&\quad + c(r)E_{r,x}\left[\int_r^T e^{\int_r^t b(s,X_s)c(s)\,\nu(ds)} b(t,X_t)c(t)\int_t^T a(t',X_{t'})\,\nu(dt')\,\nu(dt)\right] \\
&= E_{r,x}\left[e^{c(r)\int_r^T b(s,X_s)\,\nu(ds)}\alpha_r\right] + c(r)E_{r,x}\left[\int_r^T e^{\int_r^t b(s,X_s)c(s)\,\nu(ds)} a(t,X_t)\,\nu(dt)\right]
\end{aligned}$$

for all $(r,x) \in I \times S$, since, by assumption, $\alpha_t(\omega) \leq \alpha_r(\omega)$ and $c(t) \leq c(r)$ for each $(t,\omega) \in [r,T] \times \Omega$. $\square$

Chapter 4

Markovian Integral Equations

This chapter contains an analysis of multidimensional Markovian integral equations that are formulated with a consistent progressive Markov process that has Borel measurable transition probabilities. In Section 4.1 we first give a precise meaning to Markovian terminal value problems by defining (approximate) solutions. Under the hypothesis that the strong Markov property holds, we prove a characterization of solutions that turns out to be useful for mild solutions to PPDEs in Chapter 5. In Section 4.2 we compare (approximate) solutions, prove their stability, and investigate their growth behavior. In Section 4.3 we construct solutions locally in time via local boundedness and local Lipschitz continuity with respect to a certain non-negative continuous additive functional.

In Section 4.4 we are concerned with the derivation of a solution that is unique and non-extendible in time among a certain class of maps. In this connection, a boundary and growth criterion decides whether the non-extendible solution turns into a global solution. By requiring the (right-hand) Feller property of the underlying Markov process, we give weak conditions under which the derived solution becomes continuous. Furthermore, a Picard iteration result ensures global solutions to affine Markovian equations, the type of integral equations we treat in Section 4.5. There, we verify the absolute convergence of an intrinsic matrix series. This produces a matrix-valued operator that leads us to a multidimensional Feynman-Kac formula. The aim of Section 4.6 is to establish a global existence and uniqueness result for one-dimensional Markovian equations. To this end, we represent the difference of two solutions via the Feynman-Kac formula, deduce one-sided bounds for solutions, and study the boundary behavior of solutions.

4.1 The Markovian terminal value problem

In what follows, we use the pseudometric topological setting of Section 3.1 for the choice $J = [0,T]$ with $T > 0$. More precisely, let S be a Polish space with Borel σ-field $\mathscr{S}$ and ρ be a complete metric that induces the topology of S. We require that

$$\Phi : [0,T] \times S \to S, \quad (t,x) \mapsto \Phi_t(x)$$

is a continuous process such that Φ is Lipschitz continuous in $x \in S$, uniformly in $t \in [0,T]$, and $\Phi_t \circ \Phi_s = \Phi_{s\wedge t}$ for all $s,t \in [0,T]$. We let $\Phi_T(x) = x$ for all $x \in S$ and endow $[0,T]\times S$ with the pseudometric d_S given by (PM). Then $[0,T]\times S$ is a separable complete pseudometric space whose Borel σ-field $\mathscr{B}([0,T]\times S)$ is included in $\mathscr{B}([0,T])\otimes\mathscr{S}$.

We also assume that $\mathscr{X} = (X,(\mathscr{F}_t)_{t\in[0,T]},\mathbb{P})$ is a consistent progressive Markov process on a measurable space $(\Omega,\mathscr{F})$ with state space S that is Borel. Let $k \in \mathbb{N}$, and $|\cdot|$ denote both the Euclidean norm on $\mathbb{R}^k$ and the Frobenius norm on $\mathbb{R}^{k\times k}$. Moreover, let κ be a continuous k-dimensional additive map of $\mathscr{X}$ for which there are a right-continuous $\theta \in B_b([0,T]\times S,\mathbb{R}_+)$ and a Borel measure μ on $[0,T]$ with $\mu(\{t\}) = 0$ for all $t\in[0,T]$ such that

$$|\kappa_i|([r,t]) \leq \int_r^t \theta(s,X_s)\,\mu(ds) \tag{4.1}$$

for all $i \in \{1,\dots,k\}$ and each $r,t\in[0,T]$ with $r \leq t$. Eventually, we note that $\nu : \Omega\times\mathscr{B}([0,T]) \to [0,\infty]$ defined via $\nu(B)(\omega) := \int_B \theta(s,X_s(\omega))\,\mu(ds)$ is itself a non-negative continuous additive functional, by Lemma 3.16. Hence, let us call κ of *standard form* if $\kappa_i = \nu$ for each $i \in \{1,\dots,k\}$.

We let $D \in \mathscr{B}(\mathbb{R}^k)$ have non-empty interior and $f : [0,T]\times S\times D \to \mathbb{R}^k$ be measurable with respect to $\mathscr{B}([0,T]\times S)\otimes(D\cap\mathscr{B}(\mathbb{R}^k))$. The mapping f is called right-continuous if for each $(r,x,z)\in[0,T]\times S\times D$ and every $\varepsilon > 0$ there is $\delta > 0$ such that

$$|f(s,y,z') - f(r,x,z)| < \varepsilon$$

for all $(s,y,z') \in [r,T]\times S\times D$ with $d_S((s,y),(r,x)) + |z'-z| < \delta$. Clearly, if this is the case, then f is consistent in the sense that $f(r,x,z) = f(r,\Phi_r(x),z)$ for all $(r,x,z)\in[0,T]\times S\times D$. After these preparations, let us introduce the *Markovian integral equation*

$$\mathbb{E}[du(t,X_t)] = \mathbb{E}[f(t,X_t,u(t,X_t))\,\kappa(dt)] \quad \text{for } t\in[0,T]. \tag{4.2}$$

Namely, for each $\varepsilon \in B([0,T]\times S,\mathbb{R}_+)$ for which $\int_r^T \varepsilon(s,X_s)\,\nu(ds)$ is finite and $P_{r,x}$-integrable for all $(r,x)\in[0,T]\times S$, we introduce the notion of ε-approximate solutions.

4.1 Definition. An *ε-approximate solution* to (4.2) on a non-degenerate interval I in $[0,T]$ is a consistent map $u \in B(I\times S,D)$ such that

$$|u(t,X_t)| + \max_{i\in\{1,\dots,k\}} \int_r^t |f_i(s,X_s,u(s,X_s))|\,|\kappa_i|(ds)$$

is a finite $P_{r,x}$-integrable function and

$$\left|E_{r,x}[u(t,X_t)] - u(r,x) - E_{r,x}\left[\int_r^t f(s,X_s,u(s,X_s))\,\kappa(ds)\right]\right| \leq E_{r,x}\left[\int_r^t \varepsilon(s,X_s)\,\nu(ds)\right]$$

for all $r,t\in I$ with $r \leq t$ and each $x \in S$. Every 0-approximate solution to (4.2) on I is called a *solution* to (4.2) on I. If in addition $I = [0,T]$, then we will speak about a *global* solution.

Let us discuss several facts on approximate solutions. We temporarily let I be a non-degenerate interval in $[0,T]$ and $u \in B(I \times S, D)$. Then the $\hat{\mathscr{F}}_{r,t}$-measurable function $\max_{i\in\{1,\dots,k\}} \int_r^t |f_i(s,X_s,u(s,X_s))|\,|\kappa_i|(ds)$ is finite for all $r,t \in I$ with $r \leq t$ if and only if the Borel measurable function

$$I \to \mathbb{R}, \quad s \mapsto f_i(s, X_s(\omega), u(s, X_s(\omega)))$$

is locally $|\kappa_i|(\cdot)(\omega)$-integrable for every $i \in \{1,\dots,k\}$ and each $\omega \in \Omega$. In this case, it follows readily that

$$\begin{aligned} \left|\int_r^t f(s,X_s,u(s,X_s))\,\kappa(ds)\right|^2 &= \sum_{i=1}^k \left|\int_r^t f_i(s,X_s,u(s,X_s))\,\kappa_i(ds)\right|^2 \\ &\leq \sum_{i=1}^k \left|\int_r^t |f_i(s,X_s,u(s,X_s))|\,|\kappa_i|(ds)\right|^2 \end{aligned} \tag{4.3}$$

for each $r,t \in I$ with $r \leq t$. Hence, if $\max_{i\in\{1,\dots,k\}} \int_r^t |f_i(s,X_s,u(s,X_s))|\,|\kappa_i|(ds)$ is finite and $P_{r,x}$-integrable, then $\int_r^t f(s,X_s,u(s,X_s))\,\kappa(ds)$ is well-defined and also $P_{r,x}$-integrable.

Assume now that u is an ε-approximate solution to (4.2) on I. Then for each non-degenerate interval H in I, the restriction of u to $H \times S$ is an ε-approximate solution to (4.2) on H. Furthermore,

$$\begin{aligned} &\left|E_{r,x}[u(t,X_t)] - E_{r,x}[u(s,X_s)] - E_{r,x}\left[\int_s^t f(s',X_{s'},u(s',X_{s'}))\,\kappa(ds')\right]\right| \\ &\quad\leq E_{r,x}\left[\left|E_{s,X_s}[u(t,X_t)] - u(s,X_s) - E_{s,X_s}\left[\int_s^t f(s',X_{s'},u(s',X_{s'}))\,\kappa(ds')\right]\right|\right] \\ &\quad\leq E_{r,x}\left[E_{s,X_s}\left[\int_s^t \varepsilon(s',X_{s'})\,\nu(ds')\right]\right] = E_{r,x}\left[\int_s^t \varepsilon(s',X_{s'})\,\nu(ds')\right] \end{aligned}$$

for all $r,s,t \in I$ with $r \leq s \leq t$ and each $x \in S$, by Propositions A.32 and 3.7. In particular, if u is a solution to (4.2), then this shows us that

$$E_{r,x}[u(t,X_t)] = E_{r,x}[u(s,X_s)] + E_{r,x}\left[\int_s^t f(s',X_{s'},u(s',X_{s'}))\,\kappa(ds')\right]$$

for each $r,s,t \in I$ with $r \leq s \leq t$ and every $x \in S$. Finally, under an integrability condition, the concatenation of two approximate solutions in time yields another approximate solution.

4.2 Lemma. *Let u and v be two ε-approximate solutions to* (4.2) *on non-degenerate intervals H and I in $[0,T]$, respectively, with $s_0 := \sup H \in H$ and $s_0 = \inf I \in I$. Assume that $u(s_0,\cdot) = v(s_0,\cdot)$ and*

$$|v(t,X_t)| + \max_{i\in\{1,\dots,k\}} \int_{s_0}^t |f_i(s,X_s,v(s,X_s))|\,|\kappa_i|(ds)$$

is a finite $P_{r,x}$-integrable function for all $r,t \in H \cup I$ with $r < s_0 < t$ and each $x \in S$. Then $w : (H \cup I) \times S \to D$ given by $w(r,x) := u(r,x)$ for $r < s_0$ and $w(r,x) := v(r,x)$ for $r \geq s_0$ is another ε-approximate solution to (4.2) *on $H \cup I$.*

Proof. First, since $u \in B(H \times S, D)$ and $v \in B(I \times S, D)$, we directly obtain that $w \in B((H \cup I) \times S, D)$, and the consistency of w is easily seen. Let $r, t \in H \cup I$ with $r \leq t$. If either $t \leq s_0$ or $r \geq s_0$, then the definition of w implies that

$$|w(t, X_t)| + \max_{i \in \{1,\ldots,k\}} \int_r^t |f_i(s, X_s, w(s, X_s))| \, |\kappa_i|(ds) \tag{4.4}$$

is a finite $P_{r,x}$-integrable function and

$$\left| E_{r,x}\left[w(t, X_t) - w(r, X_r) - \int_r^t f(s, X_s, w(s, X_s)) \, \kappa(ds) \right] \right| \leq E_{r,x}\left[\int_r^t \varepsilon(s, X_s) \, \nu(ds) \right]$$

for each $x \in S$, as u and v are two ε-approximate solutions to (4.2) on H and I, respectively. Now, let $r < s_0 < t$, then the splitting

$$\begin{aligned} \int_r^t |f_i(s, X_s, w(s, X_s))| \, |\kappa_i|(ds) = &\int_r^{s_0} |f_i(r', X_{r'}, u(r', X_{r'}))| \, |\kappa_i|(dr') \\ &+ \int_{s_0}^t |f_i(s, X_s, v(s, X_s))| \, |\kappa_i|(ds) \end{aligned}$$

for all $i \in \{1, \ldots, k\}$ in combination with the assumptions of the lemma show that (4.4) is once again a finite $P_{r,x}$-integrable function. By Proposition 3.7,

$$\begin{aligned} &\left| E_{r,x}[w(t, X_t)] - w(r, x) - E_{r,x}\left[\int_r^t f(s, X_s, w(s, X_s)) \, \kappa(ds) \right] \right| \\ &\quad \leq \left| E_{r,x}[u(s_0, X_{s_0})] - u(r, x) - E_{r,x}\left[\int_r^{s_0} f(r', X_{r'}, u(r', X_{r'})) \, \kappa(dr') \right] \right| \\ &\qquad + \left| E_{r,x}[v(t, X_t)] - E_{r,x}[v(s_0, X_{s_0})] - E_{r,x}\left[\int_{s_0}^t f(s, X_s, v(s, X_s)) \, \kappa(ds) \right] \right| \\ &\quad \leq E_{r,x}\left[\int_r^{s_0} \varepsilon(s, X_s) \, \nu(ds) \right] + E_{r,x}\left[E_{s_0, X_{s_0}}\left[\int_{s_0}^t \varepsilon(s, X_s) \, \nu(ds) \right] \right] \\ &\quad = E_{r,x}\left[\int_r^t \varepsilon(s, X_s) \, \nu(ds) \right], \end{aligned}$$

because $u(s_0, \cdot) = v(s_0, \cdot)$. This completes the proof. □

On a non-degenerate interval I in $[0, T]$, we introduce notions of admissibility of a map $u \in B(I \times S, D)$. We call u (*weakly*) *μ-admissible* if for each $r, t \in I$ with $r \leq t$ there is a μ-null set $N \in \mathscr{B}([0, T])$ such that

$$u((N^c \cap [r, t]) \times S) \text{ is relatively compact in } D^\circ \text{ (resp. } D).$$

We say that u is *(weakly) admissible* if $u([r, t] \times S)$ is relatively compact in D° (resp. D) for all $r, t \in I$ with $r \leq t$. For example, let for the moment f be locally μ-bounded and u be weakly μ-admissible. Then for each $r, t \in I$ with $r \leq t$ there is a compact set K in D such that

$$u(\cdot, y) \in K \quad \text{for all } y \in S \quad \mu\text{-a.s. on } [r, t].$$

By Lemma 2.5, we can choose a μ-dominated function $a \in B([0,T] \times S, \mathbb{R}_+)$ so that $|f(t,x,z)| \leq a(t,x)$ for all $(t,x,z) \in [0,T] \times S \times K$. Then from (4.1) we get

$$\max_{i \in \{1,\dots,k\}} \int_r^t |f_i(s, X_s, u(s, X_s))| \, |\kappa_i|(ds) \leq \int_r^t a(s, X_s) \, \nu(ds).$$

This implies that the left-hand expression is finite and $P_{r,x}$-integrable. Hence, u is an ε-approximate solution if and only if it is consistent and the inequality appearing in Definition 4.1 holds.

Let us add a *terminal value condition* to our Markovian integral equation. We choose a map $g \in B(S, D)$ for which $E_{r,x}[|g(X_T)|]$ is finite for all $(r,x) \in [0,T] \times S$, and introduce the *Markovian terminal value problem*

$$\mathbb{E}[du(t, X_t)] = \mathbb{E}[f(t, X_t, u(t, X_t)) \, \kappa(dt)] \quad \text{for } t \in [0,T], \quad u(T, \cdot) = g. \tag{M}$$

To this end, we define a non-degenerate interval I in $[0,T]$ to be *admissible* whenever $\max I = T$. In other words, I is admissible if it is of the form $I = (t,T]$ or $I = [t,T]$ for some $t \in [0,T)$. By an *ε-approximate solution* to (M) on an admissible interval I, we mean an ε-approximate solution u to (4.2) on I that satisfies

$$u(T,x) = g(x) \quad \text{for all } x \in S.$$

Correspondingly, every 0-approximate solution to (M) on I is a *solution* to (M) on I, and we refer to a *global solution* provided $I = [0,T]$. Additionally, an admissible solution u to (M) on I is said to be *extendible* if there is an admissible solution v to (M) on another admissible interval J such that

$$I \subsetneq J \quad \text{and} \quad u = v \quad \text{on } I \times S.$$

Otherwise, we say that u is *non-extendible* and I is called a *maximal interval of existence*. In Section 4.4 we derive a non-extendible admissible solution such that the maximal interval of existence is open in $[0,T]$. To this end, a characterization of solutions is required.

4.3 Lemma. *Let I be an admissible interval. Then a mapping $u \in B(I \times S, D)$ solves* (M) *on I if and only if $\max_{i \in \{1,\dots,k\}} \int_r^T |f_i(s, X_s, u(s, X_s))| \, |\kappa_i|(ds)$ is a finite $P_{r,x}$-integrable function such that*

$$E_{r,x}[g(X_T)] = u(r,x) + E_{r,x}\left[\int_r^T f(s, X_s, u(s, X_s)) \, \kappa(ds)\right]$$

for each $(r,x) \in I \times S$.

Proof. The only if direction is covered by the definition of a solution to (4.2) on I, because $u(T,x) = g(x)$ for each $x \in S$. For if we first notice that u is automatically consistent, as $P_{r,x} = P_{r,\Phi_r(x)}$ for all $(r,x) \in [0,T] \times S$. Let $r, t \in I$ with $r \leq t$ and $x \in S$, then

$$\max_{i \in \{1,\dots,k\}} \int_r^t |f_i(s, X_s, u(s, X_s))| \, |\kappa_i|(ds)$$

is clearly finite and $P_{r,x}$-integrable. From Propositions A.32 and 3.7 in combination with (4.3) we get that

$$\begin{aligned}
E_{r,x}[|u(t,X_t)|] &\leq E_{r,x}[E_{t,X_t}[|g(X_T)|]] \\
&\quad + \sqrt{k}E_{r,x}\bigg[E_{t,X_t}\bigg[\max_{i\in\{1,\dots,k\}}\int_t^T |f_i(t',X_{t'},u(t',X_{t'}))|\,|\kappa_i|(dt')\bigg]\bigg] \\
&= E_{r,x}[|g(X_T)|] + \sqrt{k}E_{r,x}\bigg[\max_{i\in\{1,\dots,k\}}\int_t^T |f_i(t',X_{t'},u(t',X_{t'}))|\,|\kappa_i|(dt')\bigg]
\end{aligned}$$

and the last term is finite. Another application of Proposition 3.7 yields that

$$\begin{aligned}
E_{r,x}[u(t,X_t)] &= E_{r,x}[E_{t,X_t}[g(X_T)]] - E_{r,x}\bigg[E_{t,X_t}\bigg[\int_t^T f(t',X_{t'},u(t',X_{t'}))\,\kappa(dt')\bigg]\bigg] \\
&= E_{r,x}[g(X_T)] - E_{r,x}\bigg[\int_t^T f(t',X_{t'},u(t',X_{t'}))\,\kappa(dt')\bigg] \\
&= u(r,x) + E_{r,x}\bigg[\int_r^t f(s,X_s,u(s,X_s))\,\kappa(ds)\bigg],
\end{aligned}$$

as desired. Now, the claim follows. □

In the end, we give an equivalent description of solutions to Markovian integral equations under the hypothesis that $\mathscr{X}$ is strongly Markov.

4.4 Lemma. *Suppose that $\mathscr{X}$ is strongly Markov and let I be a non-degenerate interval in $[0,T]$. Then a map $u \in B(I\times S,D)$ solves (4.2) on I if and only if*

$$|u(t\wedge\tau,X_{t\wedge\tau})| + \max_{i\in\{1,\dots,k\}}\int_r^{t\wedge\tau} |f_i(s,X_s,u(s,X_s))|\,|\kappa_i|(ds)$$

is a finite $P_{r,x}$-integrable function and

$$E_{r,x}[u(t\wedge\tau,X_{t\wedge\tau})] = u(r,x) + E_{r,x}\bigg[\int_r^{t\wedge\tau} f(s,X_s,u(s,X_s))\,\kappa(ds)\bigg]$$

for all $r,t\in I$ with $r\leq t$, each $(\mathscr{F}_s)_{s\in I}$-stopping time τ with $\tau\geq r$, and every $x\in S$.

Proof. By choosing $\tau=\infty$, we immediately see that the stated conditions are sufficient. To show the necessity, let $t\in I$, then from Lemma 3.17 we learn that the process $Y: I_t\times\Omega\to\mathbb{R}^k$ defined through

$$Y_r(\omega) := u(t,X_t(\omega)) - \int_r^t f(s,X_s(\omega),u(s,X_s(\omega)))\,\kappa(ds)(\omega)$$

is reconstructible and continuous. Here, as usually, I_t denotes the set of all $s\in I$ with $s\leq t$. We let $(r,x)\in I_t\times S$ and fix an $(\mathscr{F}_s)_{s\in I}$-stopping time with τ with $\tau\geq r$. By (4.3),

$$\bigg|\int_s^t f(s',X_{s'},u(s',X_{s'}))\,\kappa(ds')\bigg| \leq \sqrt{k}\max_{i\in\{1,\dots,k\}}\int_r^t |f_i(s',X_{s'},u(s',X_{s'}))|\,|\kappa_i|(ds')$$

for all $s \in [r,t]$. Hence, $E_{r,x}[|Y_{t\wedge\tau}|]$ and $E_{s,y}[|Y_s|]$ are finite for every $(s,y) \in [r,t] \times S$. In consequence, Proposition 3.13 implies that $u(t\wedge\tau, X_{t\wedge\tau})$ is $P_{r,x}$-integrable and $E_{r,x}[u(t\wedge\tau, X_{t\wedge\tau})] = E_{r,x}[Y_{t\wedge\tau}]$. Thus,

$$\begin{aligned} E_{r,x}[u(t\wedge\tau, X_{t\wedge\tau})] &= E_{r,x}[u(t,X_t)] - E_{r,x}\bigg[\int_{t\wedge\tau}^t f(s,X_s,u(s,X_s))\,\kappa(ds)\bigg] \\ &= u(r,x) + E_{r,x}\bigg[\int_r^{t\wedge\tau} f(s,X_s,u(s,X_s))\,\kappa(ds)\bigg]. \end{aligned}$$

This proves the assertion. □

4.2 Comparison, stability, and growth behavior

We first intend to compare (approximate) solutions, which yields an uniqueness result for weakly μ-admissible solutions. Afterwards a stability result is proven that is used to construct solutions in Sections 4.3 and 4.6. Furthermore, we deduce a growth estimate that plays a major role in the proof of Proposition 4.17, where we show the uniform convergence of Picard iterations. We recall that $D \in \mathscr{B}(\mathbb{R}^k)$ and $f : [0,T] \times S \times D \to \mathbb{R}^k$ is $\mathscr{B}([0,T]\times S) \otimes (D \cap \mathscr{B}(\mathbb{R}^k))$-measurable. Let in addition $g \in B_b(S,D)$ and note that

$$\bigg|\int_r^t a(s,X_s)\,\kappa(ds)\bigg| \leq \int_r^t |a(s,X_s)|\,\nu(ds) \tag{4.5}$$

for each admissible interval I, every locally μ-dominated map $a \in B(I\times S, \mathbb{R}^k)$, and all $r,t \in I$ with $r \leq t$, by Proposition A.32 and (4.1). We use this fact in what follows.

4.5 Proposition. *Assume that $f|([0,T]\times S\times W)$ is Lipschitz μ-continuous for some set $W \subset D$. That is, there is a μ-dominated $\lambda \in B([0,T]\times S, \mathbb{R}_+)$ such that*

$$|f(t,x,z) - f(t,x,z')| \leq \lambda(t,x)|z-z'| \quad \textit{for all } (t,x) \in [0,T]\times S$$

and each $z,z' \in W$. Let $\delta, \varepsilon \in B([0,T]\times S, \mathbb{R}_+)$ be μ-dominated, $h \in B_b(S,D)$, and I be an admissible interval. Then every δ-approximate solution u to (M) *on I and each ε-approximate solution v to* (M) *on I, where g is replaced by h, satisfy*

$$|u-v|(r,x) \leq E_{r,x}\bigg[\exp\bigg(\int_r^T \lambda(s,X_s)\,\nu(ds)\bigg)\bigg(|g-h|(X_T) + \int_r^T (\delta+\varepsilon)(s,X_s)\,\nu(ds)\bigg)\bigg]$$

for all $(r,x) \in I\times S$ provided u, v are μ-consistently bounded and $u(\cdot,y), v(\cdot,y) \in W$ for each $y \in S$ μ-a.s. on I.

Proof. Let $N \in \mathscr{B}([0,T])$ be some μ-null set such that $u(s,y), v(s,y) \in W$ for every $(s,y) \in (N^c \cap I) \times S$. Then the triangle inequality and (4.5) yield

$$\begin{aligned}|u-v|(r,x) &\leq E_{r,x}\bigg[\int_r^T (\delta+\varepsilon)(s,X_s)\,\nu(ds)\bigg] + |E_{r,x}[(g-h)(X_T)]| \\ &\quad + \bigg|E_{r,x}\bigg[\int_r^T (f(s,X_s,u(s,X_s)) - f(s,X_s,v(s,X_s)))\,\kappa(ds)\bigg]\bigg| \\ &\leq E_{r,x}[|g-h|(X_T)] + E_{r,x}\bigg[\int_r^T (\delta+\varepsilon)(s,X_s)\,\nu(ds)\bigg] \\ &\quad + E_{r,x}\bigg[\int_r^T \lambda(s,X_s)|u-v|(s,X_s)\,\nu(ds)\bigg]\end{aligned}$$

for all $(r,x) \in I \times S$, since $|f(s,y,u(s,y)) - f(s,y,v(s,y))| \leq \lambda(s,y)|u-v|(s,y)$ for each $(s,y) \in (N^c \cap [r,T]) \times S$. By Lemma 3.17, the process $\alpha : I \times \Omega \to \mathbb{R}_+$ defined by

$$\alpha_r(\omega) := |g-h|(X_T(\omega)) + \int_r^T (\delta+\varepsilon)(s,X_s(\omega))\,\nu(ds)(\omega)$$

is reconstructible and consistently bounded, and has decreasing continuous paths. Hence, Corollary 3.24 leads us to the asserted estimate. □

From the comparison we get an uniqueness result, by using the linear space (2.1). Note that the procedure of the proof originates from Theorem 6.7 in Amann [1].

4.6 Corollary. *Assume that $f \in BC_\mu^{1-}([0,T] \times S \times D, \mathbb{R}^k)$. Then there is at most a unique weakly μ-admissible solution to* (M) *on every admissible interval I.*

Proof. Suppose that u and v are two weakly μ-admissible solutions to (M) on I and let $r \in I$. Then there is a compact set K in D such that $u(\cdot,y)$, $v(\cdot,y) \in K$ for all $y \in S$ μ-a.s. on $[r,T]$. Proposition 2.8 yields a neighborhood W of K in D such that $f|([0,T] \times S \times W)$ is Lipschitz μ-continuous. Hence, $u = v$ on $[r,T] \times S$, by Proposition 4.5. The assertion follows. □

Now, we consider stability.

4.7 Proposition. *Let $f \in BC_\mu^{1-}([0,T] \times S \times D, \mathbb{R}^k)$ and I be an admissible interval. For all $n \in \mathbb{N}$ let $\varepsilon_n \in B([0,T] \times S, \mathbb{R}_+)$ be μ-dominated, $g_n \in B_b(S,D)$, and u_n be an ε_n-approximate solution to* (M) *on I with g replaced by g_n. Assume that the following two conditions hold:*

- *(i) The sequences $(g_n)_{n\in\mathbb{N}}$ and $\left(\int_0^T \varepsilon_n(t,X_t)\,\nu(dt)\right)_{n\in\mathbb{N}}$ converge uniformly to g and 0, respectively.*
- *(ii) For each $r \in I$ there is a compact set K in D with $\bigcup_{n\in\mathbb{N}} u_n([r,T] \times S) \subset K$.*

Then $(u_n)_{n\in\mathbb{N}}$ converges locally uniformly in $t \in I$ and uniformly in $x \in S$ to the unique weakly admissible solution to (M) *on I.*

Proof. As uniqueness is covered by Corollary 4.6, we directly turn to the existence claim. Let $r \in I$ and K be a compact set in D such that $u_n([r,T] \times S) \subset K$ for all $n \in \mathbb{N}$. By Proposition 2.8, there exist some neighborhood W of K in D and a μ-dominated $\lambda \in B([r,T] \times S, \mathbb{R}_+)$ with $|f(t,x,z) - f(t,x,z')| \leq \lambda(t,x)|z-z'|$ for all $(t,x) \in [r,T] \times S$ and each $z, z' \in W$. Thus, Proposition 4.5 ensures that

$$|u_m - u_n|(s,x) \leq E_{s,x}\bigg[e^{\int_s^T \lambda(t,X_t)\,\nu(dt)}\bigg(|g_m - g_n|(X_T) + \int_s^T (\varepsilon_m + \varepsilon_n)(t,X_t)\,\nu(dt)\bigg)\bigg]$$

for all $m, n \in \mathbb{N}$ and every $(s,x) \in [r,T] \times S$. Since $(g_n)_{n\in\mathbb{N}}$ is uniformly convergent, it is a uniformly Cauchy sequence. That is, $\lim_{n\uparrow\infty} \sup_{m\in\mathbb{N}:\, m\geq n} \sup_{x\in S} |g_m - g_n|(x) = 0$. As $\lim_{n\uparrow\infty} \sup_{\omega\in\Omega} \int_0^T \varepsilon_n(t, X_t(\omega))\,\nu(dt)(\omega) = 0$, it follows that

$$\lim_{n\uparrow\infty} \sup_{m\in\mathbb{N}:\, m\geq n} \sup_{(s,x)\in[r,T]\times S} |u_m - u_n|(s,x) = 0.$$

In other words, $(u_n)_{n\in\mathbb{N}}$ is a uniformly Cauchy sequence on $[r,T] \times S$. As $r \in I$ has been arbitrarily chosen, this shows that $(u_n)_{n\in\mathbb{N}}$ converges locally uniformly in $t \in I$ and uniformly in $x \in S$ to some consistently bounded $u \in B(I \times S, \mathbb{R}^k)$.

We now check that u is a weakly admissible solution to (M) on I. To this end, let $r \in I$ and K be a compact set in D fulfilling $u_n([r,T] \times S) \subset K$ for all $n \in \mathbb{N}$, which in turn gives $u([r,T] \times S) \subset K$. Let us also pick some μ-dominated $\lambda \in B([r,T] \times S, \mathbb{R}_+)$ such that $|f(t,x,z) - f(t,x,z')| \leq \lambda(t,x)|z - z'|$ for all $(t,x) \in [r,T] \times S$ and every $z, z' \in K$. Then

$$\begin{aligned}
&\bigg|u_n(s,x) - E_{s,x}[g(X_T)] - E_{s,x}\bigg[\int_s^T f(t,X_t,u(t,X_t))\,\kappa(dt)\bigg]\bigg| \\
&\qquad \leq E_{s,x}[|g_n - g|(X_T)] + E_{s,x}\bigg[\int_s^T \lambda(t,X_t)|u_n - u|(t,X_t) + \varepsilon_n(t,X_t)\,\nu(dt)\bigg]
\end{aligned}$$

for all $n \in \mathbb{N}$ and each $(s,x) \in [r,T] \times S$. This entails that $(u_n)_{n\in\mathbb{N}}$ also converges locally uniformly in $t \in I$ and uniformly in $x \in S$ to the map

$$I \times S \to \mathbb{R}^k, \quad (r,x) \mapsto E_{r,x}[g(X_T)] - E_{r,x}\bigg[\int_r^T f(s,X_s,u(s,X_s))\,\kappa(ds)\bigg].$$

This proves the proposition. □

We conclude with a growth estimate.

4.8 Proposition. *Assume that f is affine μ-bounded. In other words, there are two μ-dominated $a, b \in B([0,T] \times S, \mathbb{R}_+)$ with $|f(t,x,z)| \leq a(t,x) + b(t,x)|z|$ for all $(t,x,z) \in [0,T] \times S \times D$. Let $\varepsilon \in B([0,T] \times S, \mathbb{R}_+)$ be μ-dominated and I be an admissible interval, then*

$$|u(r,x)| \leq E_{r,x}\bigg[\exp\bigg(\int_r^T b(s,X_s)\,\nu(ds)\bigg)\bigg(|g(X_T)| + \int_r^T (a+\varepsilon)(s,X_s)\,\nu(ds)\bigg)\bigg]$$

for every μ-consistently bounded ε-approximate solution u to (M) *on I and each $(r,x) \in I \times S$. In particular, u is bounded.*

Proof. From the triangle inequality and (4.5) we get that

$$\begin{aligned}|u(r,x)| &\leq E_{r,x}\left[\int_r^T \varepsilon(s,X_s)\,\nu(ds)\right] + |E_{r,x}[g(X_T)]| \\ &\quad + \left|E_{r,x}\left[\int_r^T f(s,X_s,u(s,X_s))\,\kappa(ds)\right]\right| \\ &\leq E_{r,x}[|g(X_T)|] + E_{r,x}\left[\int_r^T (a+\varepsilon)(s,X_s)\,\nu(ds)\right] \\ &\quad + E_{r,x}\left[\int_r^T b(s,X_s)|u(s,X_s)|\,\nu(ds)\right]\end{aligned}$$

for every $(r,x) \in I \times S$, as $|f(s,X_s,u(s,X_s))| \leq a(s,X_s) + b(s,X_s)|u(s,X_s)|$ for all $s \in [r,T]$. From Lemma 3.17 we know that the process $\alpha : I \times \Omega \to \mathbb{R}_+$ defined by

$$\alpha_r(\omega) := |g(X_T(\omega))| + \int_r^T (a+\varepsilon)(s,X_s(\omega))\,\nu(ds)(\omega)$$

is reconstructible and consistently bounded, and has decreasing continuous paths. In consequence, Corollary 3.24 gives the claimed estimate. To justify that u is actually bounded, let $\overline{a}, \overline{b}, \overline{\varepsilon} \in B([0,T],\mathbb{R}_+)$ be three μ-integrable functions such that $a(\cdot,y) \leq \overline{a}$, $b(\cdot,y) \leq \overline{b}$, and $\varepsilon(\cdot,y) \leq \overline{\varepsilon}$ for all $y \in S$ μ-a.s. Then

$$|u(r,x)| \leq \exp\left(\overline{\theta}\int_0^T \overline{b}(s)\,\mu(ds)\right)\left(\sup_{y\in S}|g(y)| + \overline{\theta}\int_0^T \overline{a}(s) + \overline{\varepsilon}(s)\,\mu(ds)\right)$$

for every $(r,x) \in I \times S$ with $\overline{\theta} := \sup_{(s,y)\in[0,T]\times S}\theta(s,y)$. □

4.3 Local existence in time

We aim to construct an approximate solution locally in time as concatenation of approximate solutions. Once this is achieved, we apply the stability result of the previous section to deduce a solution locally in time as uniform limit of a sequence of approximate solutions. This is a common approach in the classical theory of ODEs (see for instance Amann [1, Section 7]). Furthermore, we are concerned with the continuity of the deduced solution. For every mapping $g \in B_b(S,D)$ and each $\beta > 0$, we define $N_{\mathscr{X},\beta}(g)$ to be the set of all $z \in \mathbb{R}^k$ such that

$$|z - E_{r,x}[g(X_T)]| < \beta$$

for some $(r,x) \in [0,T] \times S$. Because we are dealing with transition probabilities $\mathbb{P}$, convexity of D should be required, as the lemma below indicates. Here, we use the notation $\mathrm{dist}(z,C) = \inf_{z'\in C}|z-z'|$ for all $z \in \mathbb{R}^k$ and each $C \subset \mathbb{R}^k$, as introduced in Section A.3 of the appendix.

4.9 Lemma. *Let D be convex and $g \in B_b(S,D)$ be bounded away from ∂D, that means, there is $\varepsilon > 0$ such that* $\mathrm{dist}(g(x),\partial D) \geq \varepsilon$ *for all $x \in S$. Then there exists $\beta > 0$ such that*

$$N_{\mathscr{X},\beta}(g) \text{ is relatively compact in } D^\circ. \tag{4.6}$$

Proof. Since g is bounded away from ∂D, Lemma A.13 entails that the image of g is relatively compact in D°. So, let K be a compact set in D° with $g(S) \subset K$. Recall that for each $C \subset \mathbb{R}^k$, the convex hull of C, denoted by $\mathrm{conv}(C)$, is the set of all convex combinations of points of C, as considered in Section A.1. Because K is compact, Proposition A.34 implies that

$$\int_S g(x)\, P(dx) \in \mathrm{conv}(K) \tag{4.7}$$

for each probability measure P on $(S, \mathscr{S})$. Since D is convex, Lemma A.5 ensures that its interior is convex as well. In consequence, from $K \subset D^\circ$ we obtain that $\mathrm{conv}(K) \subset \mathrm{conv}(D^\circ) = D^\circ$. By Corollary A.4, along with K the convex hull of K is compact.

Next, we recall that for each $C \subset \mathbb{R}^k$ and every $\beta > 0$, the set $\bigcup_{z \in C} B_\beta(z)$ is the β-neighborhood of C, as introduced in Section A.3. Due to our considerations, (4.7) in combination with Lemma A.13 give some $\beta > 0$ such that

$$\inf_{(r,x) \in [0,T] \times S} \mathrm{dist}(E_{r,x}[g(X_T)], \partial D) > \beta.$$

Since $N_{\mathscr{X},\beta}(g)$ is simply the β-neighborhood of $\{E_{r,x}[g(X_T)] \,|\, (r,x) \in [0,T] \times S\}$, Corollary A.16 ensures the validity of (4.6). □

Until the end of this section, let D be convex, f be locally μ-bounded, and $g \in B_b(S, D)$ be a map that is bounded away from ∂D. Due to the lemma, we can choose $\beta > 0$ satisfying (4.6). Let $a \in B([0,T] \times S, \mathbb{R}_+)$ be μ-dominated such that $|f(t,x,z)| \leq a(t,x)$ for all $(t,x) \in [0,T] \times S$ and each $z \in \overline{N}_{\mathscr{X},\beta}(g)$, the closure of $N_{\mathscr{X},\beta}(g)$. Then there exists $\alpha \in (0,T]$ such that

$$E_{r,x}\left[\int_r^T a(s, X_s)\, \nu(ds)\right] \leq \beta \tag{4.8}$$

for all $(r,x) \in [T-\alpha, T] \times S$. Indeed, we can pick a μ-integrable $\overline{a} \in B([0,T], \mathbb{R}_+)$ with $a(\cdot, y) \leq \overline{a}$ for each $y \in S$ μ-a.s., then $\lim_{r \uparrow T} \int_r^T \overline{a}(s)\, \mu(ds) = 0$, by dominated convergence, which yields the result. The choices of β and α such that (4.6) and (4.8) hold, respectively, are used to construct a solution to (M) on $[T-\alpha, T]$ with values in $\overline{N}_{\mathscr{X},\beta}(g)$.

4.10 Proposition. *Suppose that $\varepsilon \in B([T-\alpha, T] \times S, \mathbb{R}_+)$ is μ-dominated and there is $\delta > 0$ such that $|f(t,x,z) - f(t,x,z')| \leq \varepsilon(t,x)$ for all $(t,x) \in [T-\alpha, T] \times S$ and each $z, z' \in \overline{N}_{\mathscr{X},\beta}(g)$ with $|z - z'| < \delta$. Then there is an ε-approximate solution u to* (M) *on $[T-\alpha, T]$ such that*

$$u([T-\alpha, T] \times S) \subset \overline{N}_{\mathscr{X},\beta}(g).$$

In addition, whenever $\mathscr{X}$ has the (right-hand) Feller property, κ is of standard form, f is right-continuous, and $g \in C_b(S, \mathbb{R}^k)$, then u is (right-)continuous.

Proof. At first, the function $[T-\alpha,T]\to\mathbb{R}_+$, $s\mapsto\int_s^T\overline{a}(t)\,\mu(dt)$ must be uniformly continuous. Hence, we let $\overline{\theta}:=\sup_{(s,y)\in[0,T]\times S}\theta(s,y)$, then there exists $\eta\in(0,\alpha]$ such that

$$E_{r,x}\bigg[\int_r^t a(s,X_s)\,\nu(ds)\bigg]\leq\overline{\theta}\int_r^t\overline{a}(s)\,\mu(ds)<\delta \tag{4.9}$$

for all $r,t\in[T-\alpha,T]$ with $r\leq t<r+\eta$ and each $x\in S$. Given η, we choose $n\in\mathbb{N}$ and $t_0,\ldots,t_n\in[T-\alpha,T]$ such that $T-\alpha=t_n<\cdots<t_0=T$ and $\max_{i\in\{1,\ldots,n\}}(t_{i-1}-t_i)<\eta$.

Starting with $u_0:[T-\alpha,T]\times S\to\overline{N}_{\mathscr{X},\beta}(g)$ given by $u_0(r,x):=E_{r,x}[g(X_T)]$, we recursively introduce a sequence $(u_i)_{i\in\{1,\ldots,n\}}$ of consistent Borel measurable maps, by letting for each $i\in\{0,\ldots,n-1\}$ the map $u_{i+1}:[t_{i+1},t_i]\times S\to\overline{N}_{\mathscr{X},\beta}(g)$ be defined via

$$u_{i+1}(r,x):=E_{r,x}[u_i(t_i,X_{t_i})]-E_{r,x}\bigg[\int_r^{t_i}f(s,X_s,E_{s,X_s}[u_i(t_i,X_{t_i})])\,\kappa(ds)\bigg]. \tag{4.10}$$

We verify by induction over $i\in\{1,\ldots,n\}$ that u_i is indeed a well-defined consistent Borel measurable map taking all its values in $\overline{N}_{\mathscr{X},\beta}(g)$ such that

$$|E_{r,x}[u_i(t,X_t)]-u_0(r,x)|\leq E_{r,x}\bigg[\int_t^{t_0}a(t',X_{t'})\,\nu(dt')\bigg] \tag{4.11}$$

and

$$|E_{r,x}[u_i(t,X_t)]-u_i(r,x)|\leq E_{r,x}\bigg[\int_r^t a(s,X_s)\,\nu(ds)\bigg] \tag{4.12}$$

for all $r,t\in[t_i,t_{i-1}]$ with $r\leq t$ and each $x\in S$. In the initial induction step $i=1$ it follows from Proposition 3.7 that $E_{s,y}[u_0(t_0,X_{t_0})]=u_0(s,y)\in\overline{N}_{\mathscr{X},\beta}(g)$ for every $(s,y)\in[t_1,t_0]\times S$. Due to Lemma 3.17 and (4.5), the process $[t_1,t_0]\times\Omega\to\mathbb{R}^k$,

$$(r,\omega)\mapsto g(X_{t_0}(\omega))-\int_r^{t_0}f(s,X_s(\omega),u_0(s,X_s(\omega)))\,\kappa(ds)(\omega)$$

is reconstructible, bounded, and continuous. Thus, from Proposition 3.13 it follows that u_1 is well-defined, consistent, and Borel measurable. In addition, we infer from Proposition 3.7 that

$$E_{r,x}[u_1(t,X_t)]=u_0(r,x)-E_{r,x}\bigg[\int_t^{t_0}f(t',X_{t'},u_0(t',X_{t'}))\,\kappa(dt')\bigg]$$

for all $r,t\in[t_1,t_0]$ with $r\leq t$ and each $x\in S$, which in turn yields (4.11) and (4.12). This concludes the initial induction step. We now assume that the claim holds for some $i\in\{1,\ldots,n-1\}$, then (4.11) and (4.8) entail that

$$|E_{s,y}[u_i(t_i,X_{t_i})]-u_0(s,y)|\leq E_{s,y}\bigg[\int_{t_i}^{t_0}a(t',X_{t'})\,\nu(dt')\bigg]\leq\beta$$

for all $(s,y)\in[t_{i+1},t_i]\times S$, which gives us that $E_{s,y}[u_i(t_i,X_{t_i})]\in\overline{N}_{\mathscr{X},\beta}(g)$. For this reason, we infer from Lemma 3.17 and (4.5) that the process $[t_{i+1},t_i]\times\Omega\to\mathbb{R}^k$,

$$(r,\omega)\mapsto u_i(t_i,X_{t_i}(\omega))-\int_r^{t_i}f(s,X_s(\omega),E_{s,X_s(\omega)}[u_i(t_i,X_{t_i})])\,\kappa(ds)(\omega)$$

is reconstructible, bounded, and continuous. According to Proposition 3.13, u_{i+1} is a well-defined consistent Borel measurable map. By Proposition 3.7,

$$E_{r,x}[u_{i+1}(t,X_t)] = E_{r,x}[u_i(t_i,X_{t_i})] - E_{r,x}\bigg[\int_t^{t_i} f(t',X_{t'},E_{t',X_{t'}}[u_i(t_i,X_{t_i})])\,\kappa(dt')\bigg]$$

for all $r,t \in [t_{i+1},t_i]$ with $r \leq t$ and every $x \in S$, which immediately entails that (4.12) is valid with u_{i+1} instead of u_i. We observe that (4.11) guarantees

$$|u_i(t_i,y) - u_0(t_i,y)| \leq E_{t_i,y}\bigg[\int_{t_i}^{t_0} a(t',X_{t'})\,\nu(dt')\bigg]$$

for each $y \in S$. Eventually, by using that $u_0(r,x) = E_{r,x}[u_0(t_i,X_{t_i})]$, we obtain from Proposition 3.7 that

$$\begin{aligned}|E_{r,x}[u_{i+1}(t,X_t)] - u_0(r,x)| \leq{}& |E_{r,x}[u_{i+1}(t,X_t)] - E_{r,x}[u_i(t_i,X_{t_i})]| \\ &+ E_{r,x}\bigg[\int_{t_i}^{t_0} a(t',X_{t'})\,\nu(dt')\bigg]\end{aligned}$$

for each $r,t \in [t_{i+1},t_i]$ with $r \leq t$ and every $x \in S$, which shows that (4.11) holds when u_i is replaced by u_{i+1}. Hence, the induction proof is complete.

The crucial outcome of this construction is that u_i is an ε-approximate solution to (4.2) on $[t_i,t_{i-1}]$ for each $i \in \{1,\dots,n\}$. To see this, note that

$$\begin{aligned}&\bigg|E_{r,x}[u_i(t,X_t)] - u_i(r,x) - E_{r,x}\bigg[\int_r^t f(s,X_s,u_i(s,X_s))\,\kappa(ds)\bigg]\bigg| \\ &\quad= \bigg|E_{r,x}\bigg[\int_r^t (f(s,X_s,E_{s,X_s}[u_{i-1}(t_{i-1},X_{t_{i-1}})]) - f(s,X_s,u_i(s,X_s)))\,\kappa(ds)\bigg]\bigg| \\ &\quad\leq E_{r,x}\bigg[\int_r^t \varepsilon(s,X_s)\,\nu(ds)\bigg]\end{aligned}$$

for all $r,t \in [t_i,t_{i-1}]$ with $r \leq t$ and each $x \in S$, since $u_{i-1}(t_{i-1},\cdot) = u_i(t_{i-1},\cdot)$ and from $t_{i-1} - t_i < \eta$ in combination with (4.12) and (4.9) we infer that

$$|E_{s,y}[u_i(t_{i-1},X_{t_{i-1}})] - u_i(s,y)| \leq E_{s,y}\bigg[\int_s^{t_{i-1}} a(s',X_{s'})\,\nu(ds')\bigg] < \delta$$

for all $(s,y) \in [t_i,t_{i-1}] \times S$. As a result, if we define $u : [T-\alpha,T] \times S \to \overline{N}_{\mathscr{X},\beta}(g)$ by $u(r,x) := u_i(r,x)$ with $i \in \{1,\dots,n\}$ such that $r \in [t_i,t_{i-1}]$, then Lemma 4.2 ensures that u is an ε-approximate solution to (M) on $[T-\alpha,T]$. Hence, the first assertion follows.

Let us now suppose that $\mathscr{X}$ is (right-hand) Feller, κ is of standard form, f is right-continuous, and $g \in C_b(S,\mathbb{R}^k)$. Then we easily see that for each non-degenerate interval I in $[0,T]$ and every right-continuous $v \in B(I \times S, D)$, the map

$$I \times S \to \mathbb{R}^k, \quad (r,x) \mapsto f(r,x,v(r,x))$$

is right-continuous. By using this fact, we justify inductively over $i \in \{1,\dots,n\}$ that u_i is (right-)continuous. As soon as this is shown, the (right-)continuity of u follows, because $u_{i+1}(t_i,\cdot) = u_i(t_i,\cdot)$ for each $i \in \{0,\dots,n-1\}$. In the initial induction step $i = 1$ the (right-hand) Feller property of $\mathscr{X}$ directly implies that u_0 is (right-)continuous. From Proposition 3.19 and the representation

$$u_1(r,x) = u_0(r,x) - E_{r,x}\bigg[\int_r^{t_0} f(s,X_s,u_0(s,X_s))\,\kappa(ds)\bigg]$$

for every $(r,x) \in [t_1,t_0] \times S$ we infer that u_1 is (right-)continuous, which completes the initial induction step. Next, let us assume that u_i is (right-)continuous for some $i \in \{1,\dots,n-1\}$. Then $u(t_i,\cdot) \in C_b(S,\mathbb{R}^k)$, by Proposition 3.3. Therefore, the (right-hand) Feller property of $\mathscr{X}$ makes sure that the map

$$[t_{i+1},t_i] \times S \to \mathbb{R}^k, \quad (r,x) \mapsto E_{r,x}[u(t_i,X_{t_i})]$$

is (right-)continuous. Finally, from (4.10), which is just the definition of u_{i+1}, and Proposition 3.19 it follows that u_{i+1} is (right-)continuous. Thus, the induction proof is complete. □

By constructing a suitable sequence of approximate solutions, a local existence result can be derived.

4.11 Proposition. *Let $f \in BC_\mu^{1-}([0,T] \times S \times D,\mathbb{R}^k)$, then there exists a unique admissible solution u to* (M) *on $[T-\alpha,T]$, which only takes its values in $\overline{N}_{\mathscr{X},\beta}(g)$. Moreover, if $\mathscr{X}$ is (right-hand) Feller, κ is of standard form, f is right-continuous, and $g \in C_b(S,\mathbb{R}^k)$, then u is (right-)continuous.*

Proof. The uniqueness assertion follows directly from Corollary 4.6. To establish existence, we note that, as $\overline{N}_{\mathscr{X},\beta}(g)$ is compact, Proposition 2.8 yields a μ-dominated $\lambda \in B([T-\alpha,T] \times S,\mathbb{R}_+)$ such that

$$|f(t,x,z) - f(t,x,z')| \leq \lambda(t,x)|z-z'|$$

for all $(t,x) \in [T-\alpha,T] \times S$ and each $z,z' \in \overline{N}_{\mathscr{X},\beta}(g)$. Thus, for each $n \in \mathbb{N}$ Proposition 4.10 provides some (λ/n)-approximate solution u_n to (M) on $[T-\alpha,T]$ with

$$u_n([T-\alpha,T] \times S) \subset \overline{N}_{\mathscr{X},\beta}(g).$$

Additionally, if $\mathscr{X}$ is (right-hand) Feller, κ is of standard form, f is right-continuous, and $g \in C_b(S,\mathbb{R}^k)$, then u_n must be (right-)continuous for every $n \in \mathbb{N}$. Because λ is μ-dominated,

$$\lim_{n\uparrow\infty} \frac{1}{n} \sup_{\omega\in\Omega} \int_{T-\alpha}^T \lambda(s,X_s(\omega))\,\nu(ds)(\omega) = 0.$$

Hence, Proposition 4.7 entails that $(u_n)_{n\in\mathbb{N}}$ converges uniformly to a solution u to (M) on $[T-\alpha,T]$ that merely takes all its values in $\overline{N}_{\mathscr{X},\beta}(g)$. This proves the first claim. Since Lemma A.9 implies that the uniform limit of a sequence of $\mathbb{R}^k$-valued (right-)continuous maps on $[T-\alpha,T] \times S$ is again (right-)continuous, the second assertion follows directly from what we have just shown. □

Now, we prove a fixed-point result, which we need later on.

4.12 Proposition. *Suppose that I is a compact admissible interval, $\mathscr{H}$ is a closed set in $B_b(I \times S, \mathbb{R}^k)$, and $\Psi : \mathscr{H} \to \mathscr{H}$ a map for which there is a μ-dominated $\lambda \in B(I \times S, \mathbb{R}_+)$ such that*

$$|\Psi(u) - \Psi(v)|(r,x) \leq E_{r,x}\bigg[\int_r^T \lambda(s,X_s)|u-v|(s,X_s)\,\nu(ds)\bigg] \tag{4.13}$$

for all $u, v \in \mathscr{H}$ and each $(r,x) \in I \times S$. Then for every $u_0 \in \mathscr{H}$ the sequence $(u_n)_{n\in\mathbb{N}_0}$, recursively given by $u_n := \Psi(u_{n-1})$ for all $n \in \mathbb{N}$, converges uniformly to the unique fixed-point of Ψ.

Proof. Because the uniqueness assertion can be readily inferred from Corollary 3.24, we just show that $(u_n)_{n\in\mathbb{N}_0}$ converges uniformly to some fixed-point of Ψ. To this end, note that

$$|u_{n+1} - u_n|(r,x) \leq E_{r,x}\bigg[\int_r^T \lambda(s,X_s)|u_n - u_{n-1}|(s,X_s)\,\nu(ds)\bigg]$$

for all $n \in \mathbb{N}$ and every $(r,x) \in I \times S$. By Proposition 3.22,

$$|u_{n+1} - u_n|(r,x) \leq E_{r,x}\bigg[\int_r^T \bigg(\int_r^t \lambda(s,X_s)\,\nu(ds)\bigg)^{n-1}\frac{\lambda(t,X_t)}{(n-1)!}\Delta(t,X_t)\,\nu(dt)\bigg]$$

for all $n \in \mathbb{N}$ and each $(r,x) \in I \times S$, where $\Delta := |\Psi(u_0) - u_0|$. From the triangle inequality we get that $|u_m - u_n| \leq \sum_{i=n}^{m-1}|u_{i+1} - u_i|$, which in combination with Lemma 3.20 gives

$$\begin{aligned}|u_m - u_n|(r,x) &\leq \sum_{i=n}^{m-1} E_{r,x}\bigg[\int_r^T \bigg(\int_r^t \lambda(s,X_s)\,\nu(ds)\bigg)^{i-1}\frac{\lambda(t,X_t)}{(i-1)!}\Delta(t,X_t)\,\nu(dt)\bigg]\\ &\leq \sum_{i=n}^{m-1}\frac{1}{i!}E_{r,x}\bigg[\bigg(\int_r^T \lambda(s,X_s)\,\nu(ds)\bigg)^{i}\bigg]\sup_{(s,y)\in[r,T]\times S}\Delta(s,y)\end{aligned}$$

for all $m, n \in \mathbb{N}$ with $m > n$ and each $(r,x) \in I \times S$. This in turn shows that $(u_n)_{n\in\mathbb{N}_0}$ is a uniformly Cauchy sequence. Since $\mathscr{H}$ is closed in $B_b(I \times S, \mathbb{R}^k)$, it converges uniformly to some $u \in \mathscr{H}$. In the end, as

$$|\Psi(u) - u_{n+1}|(r,x) \leq E_{r,x}\bigg[\int_r^T \lambda(s,X_s)|u - u_n|(s,X_s)\,\nu(ds)\bigg]$$

for all $n \in \mathbb{N}_0$ and each $(r,x) \in I \times S$, the sequence $(u_{n+1})_{n\in\mathbb{N}_0}$ converges uniformly to $\Psi(u)$. Hence, we conclude that $u = \Psi(u)$. □

4.13 Remark. The sequence $(u_n)_{n\in\mathbb{N}_0}$ fulfills the following *error estimate*:

$$|u - u_n|(r,x) \leq \sum_{i=n}^{\infty} E_{r,x}\bigg[\int_r^T \bigg(\int_r^t \lambda(s,X_s)\,\nu(ds)\bigg)^{i-1}\frac{\lambda(t,X_t)}{(i-1)!}\Delta(t,X_t)\,\nu(dt)\bigg]$$

for each $n \in \mathbb{N}$ and every $(r,x) \in I \times S$ with $\Delta = |\Psi(u_0) - u_0|$. This follows directly by taking the limit $m \uparrow \infty$ in the estimate for $|u_m - u_n|$, appearing in above proof.

Let us indicate another local existence approach.

4.14 Remark. Since $\overline{N}_{\mathscr{X},\beta}(g)$ is compact, the set $\mathscr{H} := B_b([T-\alpha,T]\times S, \overline{N}_{\mathscr{X},\beta}(g))$ is closed in $B_b([T-\alpha,T]\times S,\mathbb{R}^k)$. Furthermore, (4.8) guarantees that the mapping $\Psi : \mathscr{H} \to B([T-\alpha,T]\times S,\mathbb{R}^k)$ defined via

$$\Psi(u)(r,x) := E_{r,x}[g(X_T)] - E_{r,x}\left[\int_r^T f(s,X_s,u(s,X_s))\,\kappa(ds)\right]$$

maps $\mathscr{H}$ into itself. So, let f be locally Lipschitz μ-continuous, then Proposition 2.8 gives some μ-dominated $\lambda \in B([T-\alpha,T]\times S,\mathbb{R}_+)$ such that (4.13) holds for all $u,v\in\mathscr{H}$ and each $(r,x)\in[T-\alpha,T]\times S$. For this reason, Proposition 4.12 implies that Ψ has a unique fixed-point, which is exactly the unique admissible solution to (M) on $[T-\alpha,T]$ that takes all its values in $\overline{N}_{\mathscr{X},\beta}(g)$.

4.4 Non-extendibility and global existence

After having constructed solutions locally in time, we derive a unique non-extendible admissible solution and provide conditions ensuring its continuity. As it turns out, a boundary and growth criterion determines whether the derived solution is actually global. In this regard, the proof of Theorem 7.6 in Amann [1] has been a good source for ideas. In second part of this section, we approximate solutions uniformly by Picard iterations for $D=\mathbb{R}^k$.

4.15 Non-Extendibility Theorem. Let $f\in BC_\mu^{1-}([0,T]\times S\times D,\mathbb{R}^k)$, D be convex, and $g\in B_b(S,D)$ be bounded away from ∂D. Then there exists a unique non-extendible admissible solution u_g to (M) on a maximal interval of existence I_g that is open in $[0,T]$. With $t_g^- := \inf I_g$ either $I_g=[0,T]$ or

$$\lim_{t\downarrow t_g^-}\inf_{x\in S}\min\left\{\operatorname{dist}(u_g(t,x),\partial D), \frac{1}{1+|u_g(t,x)|}\right\} = 0. \tag{B}$$

Moreover, suppose that $\mathscr{X}$ has the (right-hand) Feller property, κ is of standard form, f is right-continuous, and $g\in C_b(S,\mathbb{R}^k)$, then u_g is (right-)continuous.

Proof. We begin with the first claim and define I_g to be the set consisting of $\{T\}$ and of all $t\in[0,T)$ for which (M) admits an admissible solution on $[t,T]$. By Proposition 4.11, we have $\{T\}\subsetneq I_g$ and hence, $t_g^- = \inf I_g < T$. Let $t\in(t_g^-,T]$, then there is $s\in I_g$ with $s<t$, which means that there is an admissible solution u to (M) on $[s,T]$. As the restriction $u|([t,T]\times S)$ is an admissible solution to (M) on $[t,T]$, we get that $t\in I_g$. Thus, I_g is an admissible interval.

To verify that I_g is open in $[0,T]$, we have to show that if $I_g\neq[0,T]$, then $t_g^-\notin I_g$. On the contrary, assume that $I_g\neq[0,T]$ but $t_g^-\in I_g$. Then $t_g^->0$ and there is an admissible solution u to (M) on $[t_g^-,T]$. Since $u(t_g^-,\cdot)$ belongs to $B_b(S,D)$ and is bounded away from ∂D, Proposition 4.11 entails that the Markovian terminal value problem

$$\mathbb{E}[dv(s,X_s)] = \mathbb{E}[f(s,X_s,v(s,X_s))\,\kappa(ds)] \quad \text{for } s\in[0,t_g^-], \quad v(t_g^-,\cdot)=u(t_g^-,\cdot)$$

has an admissible solution v on $[t_g^- - \alpha, t_g^-]$ for some $\alpha \in (0, t_g^-]$. Consequently, the map $w : [t_g^- - \alpha, T] \times S \to D^\circ$ given by $w(r,x) := u(r,x)$, if $r \geq t_g^-$, and $w(r,x) := v(r,x)$, if $r < t_g^-$, is another solution to (M) on $[t_g^- - \alpha, T]$, by Lemma 4.2. From the fact that w is admissible, we conclude that $t_g^- - \alpha \in I_g$, which contradicts the definition of t_g^-.

Let us now introduce the unique non-extendible admissible solution. We recall that if $r, t \in I_g$ satisfy $r \leq t$, and u, v are two admissible solutions to (M) on $[r,T]$ and $[t,T]$, respectively, then $u = v$ on $[t,T] \times S$, due to Corollary 4.6. So, we can mark for each $r \in I_g$ the unique admissible solution to (M) on $[r,T]$ by u_r. Then

$$u_g : I_g \times S \to D^\circ, \quad u_g(r,x) := u_r(r,x)$$

is well-defined and the unique non-extendible admissible solution to (M). In fact, if $t_g^- \in I_g$, which occurs if and only if $t_g^- = 0$ and $I_g = [0,T]$, then $u_g(r,x) = u_{t_g^-}(r,x)$ for all $(r,x) \in [0,T] \times S$. This in turn implies that u_g is well-defined and a global admissible solution to (M). Now, let instead $t_g^- \notin I_g$, then $I_g = (t_g^-, T]$. We pick a strictly decreasing sequence $(t_n)_{n\in\mathbb{N}}$ in I_g with $\lim_{n\uparrow\infty} t_n = t_g^-$, then

$$u_g^{-1}(B) = \bigcup_{n\in\mathbb{N}} u_{t_n}^{-1}(B) \in \mathscr{B}(I_g \times S)$$

for all $B \in D \cap \mathscr{B}(\mathbb{R}^k)$, since $u_g^{-1}(B) \cap ([t_n,T] \times S) = u_{t_n}^{-1}(B) \in \mathscr{B}([t_n,T] \times S)$ for each $n \in \mathbb{N}$. Thus, u_g is Borel measurable. The representation $u_g|([r,T] \times S) = u_r$ for each $r \in I_g$ entails that u_g is an admissible solution to (M) on I_g. Finally, suppose that J is an admissible interval with $I_g \subsetneq J$ and v is an admissible solution to (M) on J, then there is $t \in J$ with $t \leq t_g^-$. Since the restriction of v to $[t,T] \times S$ is an admissible solution to (M) on $[t,T]$, we obtain that $t \in I_g$, which is a contradiction to $I_g = (t_g^-, T]$. This justifies that u_g is non-extendible.

We turn to the second claim. By way of contradiction, assume that $I_g \neq [0,T]$ but (B) fails. Then $I_g = (t_g^-, T]$ and there exist $\varepsilon \in (0, 1/\sqrt{2})$ and some sequence $(t_n)_{n\in\mathbb{N}}$ in I_g with $\lim_{n\uparrow\infty} t_n = t_g^-$ such that

$$\inf_{x\in S} \min\left\{\mathrm{dist}(u_g(t_n,x), \partial D), \frac{1}{1 + |u_g(t_n,x)|}\right\} \geq 2\varepsilon$$

for every $n \in \mathbb{N}$. As the intersection of two convex sets in $\mathbb{R}^k$ is convex, it follows from Lemma A.16 that $D_\eta := \{z \in D \,|\, \mathrm{dist}(z, \partial D) \geq \eta \text{ and } |z| \leq 1/\eta\}$ is a convex compact set in D° for each $\eta \in (0, 2\varepsilon]$. Hence, Proposition A.34 guarantees that

$$E_{r,x}[u_g(t_n, X_{t_n})] \in D_{2\varepsilon} \tag{4.14}$$

for all $n \in \mathbb{N}$ and each $(r,x) \in [0,t_n] \times S$. Since f is locally μ-bounded, Lemma 2.5 yields a μ-dominated $a \in B([t_g^-, T] \times S, \mathbb{R}_+)$ fulfilling $|f(t,x,z)| \leq a(t,x)$ for every $(t,x,z) \in [t_g^-, T] \times S \times D_\varepsilon$. In addition, there exists $\delta \in (0, T - t_g^-]$ such that

$$\sup_{x\in S} E_{r,x}\left[\int_r^t a(s, X_s)\,\nu(ds)\right] < \varepsilon \tag{4.15}$$

for all $r, t \in [t_g^-, T]$ with $r \leq t < r + \delta$. This entails that

$$u_g(t, S) \text{ is relatively compact in } D_\varepsilon^\circ \tag{4.16}$$

for every $n \in \mathbb{N}$ and each $t \in (t_n - \delta_n, t_n]$, where $\delta_n := \delta \wedge (t_n - t_g^-)$. Indeed, suppose this is false, then there is $n \in \mathbb{N}$ for which $u_g(t, S)$ fails to be relatively compact in D_ε° for at least one $t \in (t_n - \delta_n, t_n]$. We set

$$s_n := \sup\{t \in (t_n - \delta_n, t_n] \,|\, u_g(t, S) \text{ is not relatively compact in } D_\varepsilon^\circ\}.$$

Let us show that $u_g(s_n, S)$ is not relatively compact in D_ε°, which implies $s_n < t_n$, since $D_{2\varepsilon} \subset D_\varepsilon^\circ$ and (4.14) gives $u_g(t_n, S) \subset D_{2\varepsilon}$. On the contrary, suppose that $u_g(s_n, S)$ is relatively compact in D_ε°. As the restriction $\overline{f} := f|([0, T] \times S \times D_\varepsilon^\circ)$ belongs to $BC_\mu^{1-}([0, T] \times S \times D_\varepsilon^\circ)$, Proposition 4.11 ensures that the Markovian terminal value problem

$$\mathbb{E}[dv(s, X_s)] = \mathbb{E}[\overline{f}(s, X_s, v(s, X_s))\, \kappa(ds)] \quad \text{for } s \in [0, s_n], \quad v(s_n, \cdot) = u_g(s_n, \cdot)$$

admits a solution v to (M) on $[s_n - \alpha, s_n]$ for some $\alpha \in (0, s_n - t_g^-)$ such that the image of v is relatively compact in D_ε°. From Corollary 4.6 we get that $u_g = v$ on $[s_n - \alpha, s_n] \times S$, which contradicts the definition of s_n. Hence, $u_g(s_n, S)$ cannot be relatively compact in D_ε°. In particular, $s_n < t_n$ and $u_g(t, S)$ is relatively compact in D_ε° for each $t \in (s_n, t_n]$. These considerations imply that

$$\begin{aligned} |E_{s_n,x}[u_g(t_n, X_{t_n})] - u_g(s_n, x)| &= \left| E_{s_n,x}\left[\int_{s_n}^{t_n} f(s, X_s, u_g(s, X_s))\, \kappa(ds) \right] \right| \\ &\leq E_{s_n,x}\left[\int_{s_n}^{t_n} a(s, X_s)\, \nu(ds) \right] < \varepsilon \end{aligned}$$

for every $x \in S$, since $\nu(\{s_n\}) = 0$ and $t_n - s_n < \delta_n \leq \delta$. From (4.14) and $\varepsilon^2 < 1/2$ it follows that $|u_g(s_n, x)| < |E_{s_n,x}[u_g(t_n, X_{t_n})]| + \varepsilon \leq 1/(2\varepsilon) + \varepsilon < 1/\varepsilon$ for all $x \in S$. Moreover, by Lemma A.13,

$$\begin{aligned} \operatorname{dist}(u_g(s_n, x), \partial D) &\geq \operatorname{dist}(E_{s_n,x}[u_g(t_n, X_{t_n})], \partial D) - |E_{s_n,x}[u_g(t_n, X_{t_n})] - u_g(s_n, x)| \\ &\geq 2\varepsilon - |E_{s_n,x}[u_g(t_n, X_{t_n})] - u_g(s_n, x)| > \varepsilon \end{aligned}$$

for all $x \in S$. In consequence, due to (4.15), it follows that $u_g(s_n, S)$ is relatively compact in D_ε°, which is a contradiction. Therefore, condition (4.16) is valid.

Next, since $\lim_{n\uparrow\infty} t_n = t_g^-$, there is $n_0 \in \mathbb{N}$ such that $t_n - t_g^- \leq \delta$ and hence, $t_n - \delta_n = t_g^-$ for all $n \in \mathbb{N}$ with $n \geq n_0$. Thus, (4.16) leads us to

$$\begin{aligned} |E_{t_g^-,x}[u_g(r, X_r)] - E_{t_g^-,x}[u_g(t, X_t)]| &= \left| E_{t_g^-,x}\left[\int_r^t f(s, X_s, u_g(s, X_s))\, \kappa(ds) \right] \right| \\ &\leq E_{t_g^-,x}\left[\int_r^t a(s, X_s)\, \nu(ds) \right] < \varepsilon \end{aligned} \tag{4.17}$$

for every $r, t \in (t_g^-, t_{n_0}]$ with $r \leq t$ and each $x \in S$. For this reason, the map $(t_g^-, T] \times S \to D^\circ$, $(t, x) \mapsto E_{t_g^-,x}[u_g(t, X_t)]$ is uniformly continuous in $t \in (t_g^-, T]$, uniformly in $x \in S$. By Proposition A.12, there exists a unique map $\hat{z} \in B(S, D_\varepsilon)$ such that

$$\lim_{t \downarrow t_g^-} E_{t_g^-,x}[u_g(t, X_t)] = \hat{z}(x), \quad \text{uniformly in } x \in S. \tag{4.18}$$

At the same time, it follows from (4.16) together with the Dominated Convergence Theorem A.33 that

$$\begin{aligned} \lim_{r \downarrow t_g^-} E_{t_g^-,x}\left[\int_r^T f(s, X_s, u_g(s, X_s))\,\kappa(ds)\right] \\ = E_{t_g^-,x}\left[\int_{(t_g^-,T]} f(s, X_s, u_g(s, X_s))\,\kappa(ds)\right] \end{aligned} \tag{4.19}$$

for every $x \in S$. Furthermore, from (4.17) we see that the map $(t_g^-, T] \times S \to \mathbb{R}^k$,

$$(r, x) \mapsto E_{t_g^-,x}\left[\int_r^T f(s, X_s, u_g(s, X_s))\,\kappa(ds)\right]$$

is uniformly continuous in $r \in (t_g^-, T]$, uniformly in $x \in S$. So, another application of Proposition A.12 implies that the limit (4.19) holds uniformly in $x \in S$.

Thus, we define $v : [t_g^-, T] \times S \to D^\circ$ by $v(t, x) := u_g(t, x)$, if $t > t_g^-$, and $v(t, x) := \hat{z}(x)$, if $t = t_g^-$, then v is an admissible solution to (M) on $[t_g^-, T]$. In fact, from (4.18) we infer that v is consistent, and its Borel measurability is readily checked. Since $u_g((t_g^-, T] \times S)$ and $\hat{z}(S)$ are relatively compact in D°, we see that v is admissible. Finally, let $(t, x) \in (t_g^-, T] \times S$, then

$$E_{t_g^-,x}[v(t, X_t)] = E_{t_g^-,x}[v(r, X_r)] + E_{t_g^-,x}\left[\int_r^t f(s, X_s, v(s, X_s))\,\kappa(ds)\right]$$

for every $r \in (t_g^-, T]$ with $r \leq t$. By taking the limit $r \downarrow t_g^-$, we obtain from (4.18) and (4.19) that

$$E_{t_g^-,x}[v(t, X_t)] = v(t_g^-, x) + E_{t_g^-,x}\left[\int_{t_g^-}^t f(s, X_s, v(s, X_s))\,\kappa(ds)\right].$$

This is the last piece of information to conclude that v solves (M) on $[t_g^-, T]$. Hence, $t_g^- \in I_g$, which contradicts that I_g is open in $[0, T]$. This concludes the verification of the second claim.

At last, let us assume that $\mathscr{X}$ is (right-hand) Feller, κ is of standard form, f is right-continuous, and $g \in C_b(S, \mathbb{R}^k)$. We define $\hat{I}_g$ to be the set consisting of $\{T\}$ and of all $t \in [0, T)$ for which (M) admits an admissible (right-)continuous solution on $[t, T]$. We set $\hat{t}_g^- := \inf \hat{I}_g$, then Proposition 4.11 makes sure that $\{T\} \subsetneq \hat{I}_g$ and thus, $\hat{t}_g^- < T$. Using similar arguments as before, it follows that $\hat{I}_g$ is an admissible interval that is open in $[0, T]$.

By Corollary 4.6, the proof is complete, once we have shown that $\hat{t}_g^- = t_g^-$. Since $\hat{t}_g^- \geq t_g^-$, let us suppose that $\hat{t}_g^- > t_g^-$. Then $\hat{I}_g \neq [0,T]$ and hence, $\hat{I}_g = (\hat{t}_g^-, T]$. As the restriction of u_g to $[t,T] \times S$ is (right-)continuous for each $t \in \hat{I}_g$, we see that $u_g|(\hat{I}_g \times S)$ is (right-)continuous. Because $\kappa(\{\hat{t}_g^-\}) = 0$ and

$$u_g(r,x) = E_{r,x}[g(X_T)] - E_{r,x}\left[\int_r^T f(s, X_s, u_g(s, X_s))\, \kappa(ds)\right]$$

for all $(r,x) \in [\hat{t}_g^-, T] \times S$, the (right-hand) Feller property of $\mathscr{X}$ and Proposition 3.19 imply that u_g is (right-)continuous on $[\hat{t}_g^-, T] \times S$. For this reason, we must face the contradiction that $\hat{t}_g^- \in \hat{I}_g$. This completes the proof. $\square$

4.16 Remarks. (i) Assume that u_g is bounded away from ∂D. That is, there is $\varepsilon > 0$ such that $\mathrm{dist}(u_g(t,x), \partial D) \geq \varepsilon$ for all $(t,x) \in I_g \times S$. Let $I_g \neq [0,T]$, then from (B) it follows that

$$\limsup_{t \downarrow t_g^-} \sup_{x \in S} |u_g(t,x)| = \infty. \tag{4.20}$$

In fact, for each $\eta > 0$ with $1/(1+\eta) < \varepsilon$ there must be some $\delta \in (0, T - t_g^-)$ such that $\inf_{x \in S} \min\{\mathrm{dist}(u_g(t,x), \partial D), 1/(1 + |u_g(t,x)|)\} < 1/(1+\eta)$ for all $t \in (t_g^-, t_g^- + \delta)$. Thus, $\sup_{x \in S} |u_g(t,x)| > \eta$ for each $t \in (t_g^-, t_g^- + \delta)$, which shows (4.20).

(ii) Let us instead assume that u_g is bounded. For instance, this occurs whenever f is affine μ-bounded, by Proposition 4.8. Then the preceding theorem states that either u_g is a global solution or

$$\liminf_{t \downarrow t_g^-} \inf_{x \in S} \mathrm{dist}(u_g(t,x), \partial D) = 0. \tag{4.21}$$

This follows readily from the fact that if $c \geq 0$ satisfies $|u_g(t,x)| \leq c$ for every $(t,x) \in I_g \times S$, then $\inf_{x \in S} 1/(1 + |u_g(t,x)|) \geq 1/(1+c)$ for all $t \in I_g$.

(iii) In particular, suppose that u_g is not only bounded, but also its image is relatively compact in D°, then

$$I_g = [0,T].$$

To clarify this, note that $u_g(I_g \times S)$ is included in a compact set in D° if and only if u_g is both bounded and bounded away from ∂D, by Lemma A.13. Hence, the assertion follows from (i) and (ii), because neither $\lim_{t \downarrow t_g^-} \sup_{x \in S} |u_g(t,x)| = \infty$ nor $\lim_{t \downarrow t_g^-} \inf_{x \in S} \mathrm{dist}(u_g(t,x), \partial D) = 0$.

In the special case $D = \mathbb{R}^k$, we combine these considerations with a Picard iteration to obtain the following approximation result.

4.17 Proposition. *Let $D = \mathbb{R}^k$, $f \in BC_\mu^{1-}([0,T] \times S \times \mathbb{R}^k, \mathbb{R}^k)$, and $g \in B_b(S, \mathbb{R}^k)$. Assume that f is affine μ-bounded, then $I_g = [0,T]$ and the sequence $(u_n)_{n \in \mathbb{N}_0}$ in $B_b([0,T] \times S, \mathbb{R}^k)$, recursively defined by $u_0(r,x) := E_{r,x}[g(X_T)]$ and*

$$u_n(r,x) := u_0(r,x) - E_{r,x}\left[\int_r^T f(s, X_s, u_{n-1}(s, X_s))\, \kappa(ds)\right]$$

for all $n \in \mathbb{N}$, converges uniformly to u_g, the unique global bounded solution to (M).

Proof. To establish the claim, we invoke Proposition 4.12. First, since f is affine μ-bounded, Proposition 4.8 implies that u_g is bounded. As (4.21) cannot hold, we get from Remarks 4.16 that $I_g = [0,T]$. We also notice that a solution to (M) on an admissible interval is admissible if and only if it is consistently bounded. Hence, u_g is the unique global bounded solution to (M), by the preceding theorem.

We choose two μ-dominated functions $a, b \in B([0,T]\times S, \mathbb{R}_+)$ such that $|f(t,x,z)| \leq a(t,x) + b(t,x)|z|$ for all $(t,x,z) \in [0,T] \times S \times \mathbb{R}^k$ and let $\mathscr{H}$ be the set of all $u \in B([0,T] \times S, \mathbb{R}^k)$ satisfying the estimate in Proposition 4.8 for $\varepsilon = 0$, which amounts to

$$|u(r,x)| \leq E_{r,x}\bigg[\exp\bigg(\int_r^T b(s,X_s)\,\nu(ds)\bigg)\bigg(|g(X_T)| + \int_r^T a(s,X_s)\,\nu(ds)\bigg)\bigg]$$

for all $(r,x) \in [0,T] \times S$. Then $\mathscr{H}$ is closed in $B_b([0,T] \times S, \mathbb{R}^k)$ and $u_0, u_g \in \mathscr{H}$. Let us be more specific about the fact that every $u \in \mathscr{H}$ is bounded. We pick two μ-integrable $\overline{a}, \overline{b} \in B([0,T], \mathbb{R}_+)$ with $a(\cdot,y) \leq \overline{a}$ and $b(\cdot,y) \leq \overline{b}$ for all $y \in S$ μ-a.s., and set

$$c := \exp\bigg(\overline{\theta}\int_0^T \overline{b}(s)\,\mu(ds)\bigg)\bigg(\sup_{y\in S}|g(y)| + \overline{\theta}\int_0^T \overline{a}(s)\,\mu(ds)\bigg),$$

where, as usually, $\overline{\theta}$ denotes $\sup_{(s,y)\in[0,T]\times S}\theta(s,y)$. Then each map $u \in \mathscr{H}$ satisfies $|u(r,x)| \leq c$ for each $(r,x) \in [0,T] \times S$. In addition, we introduce the mapping $\Psi : \mathscr{H} \to B_b([0,T] \times S, \mathbb{R}^k)$ defined via

$$\Psi(u)(r,x) := u_0(r,x) - E_{r,x}\bigg[\int_r^T f(s,X_s,u(s,X_s))\,\kappa(ds)\bigg],$$

then a map $u \in \mathscr{H}$ is a global solution to (M) if and only if it coincides with u_g, the unique fixed-point of Ψ. Furthermore, Ψ maps $\mathscr{H}$ into itself. Indeed, let $u \in \mathscr{H}$, then Proposition 3.18 and Lemma 3.20 yield

$$\begin{aligned}|\Psi(u)(r,x)| &\leq E_{r,x}\bigg[\bigg(1 + \int_r^T b(s,X_s)e^{\int_s^T b(t,X_t)\,\nu(dt)}\,\nu(ds)\bigg)|g(X_T)|\bigg] \\ &\quad + E_{r,x}\bigg[\bigg(1 + \int_r^T b(s,X_s)e^{\int_s^T b(t,X_t)\,\nu(dt)}\,\nu(ds)\bigg)\int_r^T a(s,X_s)\,\nu(ds)\bigg] \\ &= E_{r,x}\bigg[\exp\bigg(\int_r^T b(s,X_s)\,\nu(ds)\bigg)\bigg(|g(X_T)| + \int_r^T a(s,X_s)\,\nu(ds)\bigg)\bigg]\end{aligned}$$

for all $(r,x) \in [0,T] \times S$. Hence, $\Psi(u) \in \mathscr{H}$, which justifies that $\Psi(\mathscr{H}) \subset \mathscr{H}$. By Proposition 2.8, there exists some μ-dominated function $\lambda \in B([0,T] \times S, \mathbb{R}_+)$ such that

$$|f(t,x,z) - f(t,x,z')| \leq \lambda(t,x)|z - z'|$$

for every $(t,x) \in [0,T] \times S$ and each $z, z' \in \mathbb{R}^k$ with $|z| \vee |z'| \leq c$. This guarantees that (4.13) is valid for all $u, v \in \mathscr{H}$ and each $(r,x) \in [0,T] \times S$. As this was the last condition we had to check, the claim follows from Proposition 4.12. □

4.18 Remark. The sequence $(u_n)_{n\in\mathbb{N}_0}$ is subject to the following *growth estimate*:

$$\begin{aligned}|u_n(r,x)| \leq \sum_{i=0}^{n} E_{r,x}\bigg[\frac{1}{i!}\bigg(\int_r^T b(s,X_s)\,\nu(ds)\bigg)^i |g(X_T)|\bigg]\\ + \sum_{i=0}^{n-1} E_{r,x}\bigg[\int_r^T \frac{1}{i!}\bigg(\int_r^t b(s,X_s)\,\nu(ds)\bigg)^i a(t,X_t)\,\nu(dt)\bigg]\end{aligned} \tag{4.22}$$

for all $n \in \mathbb{N}$ and each $(r,x) \in [0,T]\times S$. Certainly, $\alpha : [0,T]\times\Omega \to \mathbb{R}_+$ defined via $\alpha_r(\omega) := |g(X_T(\omega))| + \int_r^T a(s,X_s(\omega))\,\nu(ds)(\omega)$ is a reconstructible consistently bounded continuous process, due to Lemma 3.17. Hence,

$$\begin{aligned}|u_n(r,x)| \leq E_{r,x}[\alpha_r] + \sum_{i=0}^{n-2} E_{r,x}\bigg[\int_r^T \bigg(\int_r^t b(s,X_s)\,\nu(ds)\bigg)^i \frac{b(t,X_t)}{i!}\alpha_t\,\nu(dt)\bigg]\\ + E_{r,x}\bigg[\int_r^T \bigg(\int_r^t b(s,X_s)\,\nu(ds)\bigg)^{n-1} \frac{b(t,X_t)}{(n-1)!}|u_0(t,X_t)|\,\nu(dt)\bigg]\end{aligned}$$

for all $n \in \mathbb{N}$ with $n \geq 2$ and every $(r,x) \in [0,T]\times S$, by Proposition 3.22. Thus, (4.22) follows from Proposition 3.18 and Lemma 3.20.

4.5 Affine Markovian equations

We now focus on Markovian integral equations that are affine or actually linear. By constructing a matrix-valued operator, which reduces to a matrix exponential under a commutation condition, we inductively derive a multidimensional Feynman-Kac formula. Let $D = \mathbb{R}^k$ and assume that there are two maps $a : [0,T]\times S \to \mathbb{R}^k$ and $b : [0,T]\times S \to \mathbb{R}^{k\times k}$ such that f is of the form

$$f(t,x,z) = a(t,x) + b(t,x)z \quad \text{for all } (t,x,z) \in [0,T]\times S\times\mathbb{R}^k.$$

Note that a and b are necessarily Borel measurable, since the map $[0,T]\times S \to \mathbb{R}^k$, $(t,x) \mapsto f(t,x,z)$ is Borel measurable for all $z \in \mathbb{R}^k$. Then every Markovian integral equation (4.2) is called *affine* and it can be written in the form

$$\mathbb{E}[du(t,X_t)] = \mathbb{E}[a(t,X_t) + b(t,X_t)u(t,X_t)\,\kappa(dt)] \quad \text{for } t \in [0,T]. \tag{4.23}$$

For $a = 0$ we call (4.23) *homogeneous*. Otherwise, it is said to be *non-homogeneous*. Throughout the section, we assume that a and b are μ-dominated. Then f is affine μ-bounded and uniformly μ-differentiable, as discussed in Examples 2.4 and 2.10. In particular, $f \in BC^1_\mu([0,T]\times S\times\mathbb{R}^k,\mathbb{R}^k)$, which can be recalled from (2.4). Hence, Proposition 4.17 implies that for each $g \in B_b(S,\mathbb{R}^k)$ there is a unique global bounded solution u_g to the corresponding Markovian terminal value problem

$$\begin{aligned}\mathbb{E}[du(t,X_t)] &= \mathbb{E}[a(t,X_t) + b(t,X_t)u(t,X_t)\,\kappa(dt)] \quad \text{for } t \in [0,T],\\ u(T,\cdot) &= g.\end{aligned} \tag{AM}$$

After these preliminaries, we begin with the homogeneous case. It is treated similarly as in the theory of ODEs (cf. Theorem 11.2 in Amann [1]).

4.19 Lemma. *Let $a = 0$. Then for each non-degenerate interval I in $[0,T]$ the set V_I of all admissible solutions to (4.23) on I is a linear space. Moreover, the map $B_b(S,\mathbb{R}^k) \to V_{[0,T]}$, $g \mapsto u_g$ is an isomorphism.*

Proof. Let $\alpha, \beta \in \mathbb{R}$ and assume that u, v are two admissible solutions to (4.23) on a non-degenerate interval I in $[0,T]$. Then $\alpha u + \beta v$ is admissible, because the set of all $\mathbb{R}^k$-valued Borel measurable consistently bounded maps on $I \times S$ is a linear space. Moreover, linearity of f in the third variable $z \in \mathbb{R}^k$ gives

$$\begin{aligned} E_{r,x}[(\alpha u + \beta v)(t,X_t)] &= \alpha u(r,x) + \alpha E_{r,x}\left[\int_r^t b(s,X_s)u(s,X_s)\,\kappa(ds)\right] \\ &\quad + \beta v(r,x) + \beta E_{r,x}\left[\int_r^t b(s,X_s)v(s,X_s)\,\kappa(ds)\right] \\ &= (\alpha u + \beta v)(r,x) + E_{r,x}\left[\int_r^t b(s,X_s)(\alpha u + \beta v)(s,X_s)\,\kappa(ds)\right] \end{aligned}$$

for all $r,t \in I$ with $r \leq t$ and each $x \in S$. Thus, $\alpha u + \beta v$ is another admissible solution to (4.23) on I. In other words, $\alpha u + \beta v \in V_I$.

Now, let $g,h \in B_b(S,\mathbb{R}^k)$, then $\alpha u_g + \beta u_h \in V_{[0,T]}$, by what we have just shown. In addition, $(\alpha u_g + \beta u_h)(T,x) = (\alpha g + \beta h)(x) = u_{\alpha g + \beta h}(T,x)$ for every $x \in S$. So, $\alpha u_g + \beta u_h$ is a global bounded solution to (AM). Corollary 4.6 entails that $\alpha u_g + \beta u_h = u_{\alpha g + \beta h}$. Hence, the map $B_b(S,\mathbb{R}^k) \to V_{[0,T]}$, $g \mapsto u_g$ is linear. It is also injective, because if $u_g = 0$ for some $g \in B_b(S,\mathbb{R}^k)$, then

$$g(x) = u_g(T,x) = 0 \quad \text{for all } x \in S.$$

Finally, let $u \in V_{[0,T]}$. Then, since $u(T,\cdot) \in B_b(S,\mathbb{R}^k)$, Corollary 4.6 yields that $u = u_{u(T,\cdot)}$. So, every $u \in V_{[0,T]}$ is of the form $u = u_g$ for some $g \in B_b(S,\mathbb{R}^k)$. This shows that the map $B_b(S,\mathbb{R}^k) \to V_{[0,T]}$, $g \mapsto u_g$ is onto as well. □

We turn to the non-homogeneous case.

4.20 Lemma. *For every non-degenerate interval I in $[0,T]$ the set of all admissible solutions to (4.23) on I is given by the affine space $v + V_I$, where v is an arbitrary admissible solution to (4.23) on I.*

Proof. As before, we suppose that u and v are two admissible solutions to (4.23) on a non-degenerate interval I in $[0,T]$, then $u - v$ is admissible. Since f is affine in $z \in \mathbb{R}^k$, we get that

$$\begin{aligned} E_{r,x}[(u-v)(t,X_t)] &= u(r,x) + E_{r,x}\left[\int_r^t a(s,X_s) + b(s,X_s)u(s,X_s)\,\kappa(ds)\right] \\ &\quad - v(r,x) - E_{r,x}\left[\int_r^t a(s,X_s) + b(s,X_s)v(s,X_s)\,\kappa(ds)\right] \\ &= (u-v)(r,x) + E_{r,x}\left[\int_r^t b(s,X_s)(u-v)(s,X_s)\,\kappa(ds)\right] \end{aligned}$$

for each $r,t \in I$ with $r \leq t$ and all $x \in S$. Thus, $u - v$ is an admissible solution to (4.23) on I for $a = 0$. Put differently, $u - v \in V_I$. □

We next consider an integral sequence of $\mathbb{R}^{k\times k}$-valued maps. To this end, we use the conventions that $[r,t] := [t,r]$,

$$\int_r^t \bar{b}(s)\,\nu(ds) := -\int_t^r \bar{b}(s)\,\nu(ds),$$

and $\hat{\mathscr{F}}_{r,t} := \hat{\mathscr{F}}_{t,r}$ for all $r,t \in [0,T]$ with $t < r$, each $d \in \mathbb{N}$, and every μ-integrable map $\bar{b} \in B([0,T],\mathbb{R}^{d\times d})$.

4.21 Lemma. *Let the sequence $(\Sigma_n)_{n\in\mathbb{N}_0}$ of $\mathbb{R}^{k\times k}$-valued maps on $[0,T]\times[0,T]\times\Omega$ be recursively given by $\Sigma_0(r,t)(\omega) := \mathbb{I}_k$ and*

$$\Sigma_n(r,t)(\omega) := \int_r^t b(s,X_s(\omega))\Sigma_{n-1}(s,t)(\omega)\,\nu(ds)(\omega)$$

for all $n \in \mathbb{N}$. Then $\Sigma_n(r,t)$ is $\hat{\mathscr{F}}_{r,t}$-measurable and bounded, and satisfies

$$|\Sigma_n(r,t)| \leq \frac{\sqrt{k}}{n!}\bigg(\bigg|\int_r^t |b(s,X_s)|\,\nu(ds)\bigg|\bigg)^n \tag{4.24}$$

for all $n \in \mathbb{N}_0$ and each $r,t \in [0,T]$. Moreover, $\Sigma_n(\cdot,\cdot)(\omega) \in C([0,T]\times[0,T],\mathbb{R}^{k\times k})$ for every $n \in \mathbb{N}_0$ and each $\omega \in \Omega$.

Proof. We prove the lemma by induction over $n \in \mathbb{N}_0$. In the initial induction step $n = 0$ the assignment $\Sigma_0 = \mathbb{I}_k$ gives all results, since $|\mathbb{I}_k| = \sqrt{k}$. Therefore, let us suppose that the claims are true for some $n \in \mathbb{N}_0$ and pick $r,t \in [0,T]$. Then, since $\mathscr{X}$ is progressive and b is Borel measurable, Lemma 3.9 ensures that the map

$$[r,t]\times\Omega \to \mathbb{R}^{k\times k}, \quad (s,\omega) \mapsto b(s,X_s(\omega))\Sigma_n(s,t)(\omega)$$

is $\mathscr{B}([r,t])\otimes\hat{\mathscr{F}}_{r,t}$-measurable. As the Frobenius norm on $\mathbb{R}^{k\times k}$ is submultiplicative in the sense that $|AB| \leq |A||B|$ for all $A, B \in \mathbb{R}^{k\times k}$, it follows from (4.24), Lemma 3.20, and the Fundamental Theorem of Calculus for Lebesgue-Stieltjes integrals that

$$\begin{aligned}\bigg|\int_r^t |b(s,X_s)\Sigma_n(s,t)|\,\nu(ds)\bigg| &\leq \sqrt{k}\bigg|\int_r^t \frac{|b(s,X_s)|}{n!}\bigg(\bigg|\int_s^t |b(s',X_{s'})|\,\nu(ds')\bigg|\bigg)^n\,\nu(ds)\bigg|\\ &= \frac{\sqrt{k}}{(n+1)!}\bigg(\bigg|\int_r^t |b(s,X_s)|\,\nu(ds)\bigg|\bigg)^{n+1}.\end{aligned}$$

Thus, $\Sigma_{n+1}(r,t)$ is well-defined and (4.24) holds when n is replaced by $n+1$, which implies that $\Sigma_{n+1}(r,t)$ is also bounded. In addition, an application of Fubini's theorem to each coordinate ensures that $\Sigma_{n+1}(r,t)$ is $\hat{\mathscr{F}}_{r,t}$-measurable.

To show that $\Sigma_{n+1}(\cdot,\cdot)(\omega)$ is continuous for all $\omega \in \Omega$, let $(r,t) \in [0,T]\times[0,T]$ and $(r_m,t_m)_{m\in\mathbb{N}}$ be a sequence in $[0,T]\times[0,T]$ that converges to (r,t). Then (4.24) and the Dominated Convergence Theorem A.33 give us

$$\begin{aligned}\lim_{m\uparrow\infty}\Sigma_{n+1}(r_m,t_m)(\omega) &= \lim_{m\uparrow\infty}\int_0^T \mathbb{1}_{[r_m,t_m]}(s)b(s,X_s(\omega))\Sigma_n(s,t_m)(\omega)\,\nu(ds)(\omega)\\ &= \int_0^T \mathbb{1}_{[r,t]}(s)b(s,X_s(\omega))\Sigma_n(s,t)(\omega)\,\nu(ds)(\omega) = \Sigma_{n+1}(r,t)(\omega),\end{aligned}$$

as $\Sigma_n(\cdot,\cdot)(\omega)$ is continuous and from $\mu(\{r,t\}) = 0$ it follows that $\lim_{m\uparrow\infty}\mathbb{1}_{[r_m,t_m]}(s) = \mathbb{1}_{[r,t]}(s)$ for μ-a.e. $s \in [0,T]$. ☐

The recursive definition of $(\Sigma_n)_{n\in\mathbb{N}_0}$ allows for an explicit formula. Namely, for every $n \in \mathbb{N}$ and each $r,t \in [0,T]$ with $r \leq t$, we define $C_n(r,t)$ to be the set of all $(s_1,\dots,s_n) \in [r,t]^n$ with $s_1 \leq \cdots \leq s_n$, then

$$\Sigma_n(r,t) = \int_{C_n(r,t)} b(s_1, X_{s_1}) \cdots b(s_n, X_{s_n})\, d\nu^n(s_1,\dots,s_n). \tag{4.25}$$

Here, $\nu^n(\cdot)(\omega)$ denotes the product measure $\otimes_{i=1}^n \nu(\cdot)(\omega)$ for all $\omega \in \Omega$. Let us check the formula inductively. For $n = 1$ its validity follows immediately from $\Sigma_0 = \mathbb{I}_k$. If (4.25) also holds for some $n \in \mathbb{N}$, then Fubini's theorem yields that

$$\begin{aligned}\Sigma_{n+1}(r,t) &= \int_r^t b(s_1, X_{s_1})\Sigma_n(s_1,t)\,\nu(ds_1)\\ &= \int_r^t \int_{C_n(s_1,t)} b(s_1,X_{s_1})\cdots b(s_{n+1}, X_{s_{n+1}})\, d\nu^n(s_2,\dots,s_{n+1})\,\nu(ds_1)\\ &= \int_{C_{n+1}(r,t)} b(s_1,X_{s_1})\cdots b(s_{n+1}, X_{s_{n+1}})\, d\nu^{n+1}(s_1,\dots,s_{n+1})\end{aligned}$$

for all $r,t \in [0,T]$ with $r \leq t$. This concludes the induction proof. We continue with an absolute convergence result for an intrinsic matrix series.

4.22 Proposition. *The series map $\sum_{n=0}^\infty (-1)^n \Sigma_n$ converges absolutely, uniformly in $(r,t,\omega) \in [0,T]\times[0,T]\times\Omega$, and the limit map $\Sigma := \sum_{n=0}^\infty (-1)^n\Sigma_n$ satisfies the following three properties:*

(i) *$\Sigma(r,t)$ is $\hat{\mathscr{F}}_{r,t}$-measurable and bounded, and fulfills*

$$|\Sigma(r,t)| \leq \sqrt{k}\exp\left(\left|\int_r^t |b(s,X_s)|\,\nu(ds)\right|\right) \tag{4.26}$$

for all $r,t \in [0,T]$. Moreover, $\Sigma(\cdot,\cdot)(\omega) \in C([0,T]\times[0,T], \mathbb{R}^{k\times k})$ for all $\omega \in \Omega$.

(ii) *$\Sigma(r,r) = \mathbb{I}_k$, $\Sigma(r,s)\Sigma(s,t) = \Sigma(r,t)$, and $\Sigma(r,t)(\omega)$ is an invertible matrix such that $\Sigma(r,t)(\omega)^{-1} = \Sigma(t,r)(\omega)$ for all $r,s,t \in [0,T]$ and every $\omega \in \Omega$.*

(iii) *If $b(r,x)b(s,y) = b(s,y)b(r,x)$ for each $(r,x),(s,y) \in [0,T]\times S$, then*

$$\Sigma(r,t) = \exp\left(-\int_r^t b(s,X_s)\,\nu(ds)\right) \quad \textit{for all } r,t \in [0,T].$$

Proof. From (4.24) we immediately get that

$$\sum_{n=0}^\infty |(-1)^n\Sigma_n(r,t)| \leq \sqrt{k}\exp\left(\left|\int_r^t |b(s,X_s)|\,\nu(ds)\right|\right) \tag{4.27}$$

for each $r,t \in [0,T]$. Because b is μ-dominated, there is a μ-integrable function $\bar{b} \in B([0,T],\mathbb{R}_+)$ such that $|b(\cdot,y)| \leq \bar{b}$ for all $y \in S$ μ-a.s. Hence,

$$\sup_{(r,t,\omega)\in[0,T]\times[0,T]\times\Omega} \sum_{n=0}^\infty |(-1)^n\Sigma_n(r,t)(\omega)| \leq \sqrt{k}\exp\left(\bar{\theta}\int_0^T \bar{b}(s)\,\mu(ds)\right), \tag{4.28}$$

where $\overline{\theta} := \sup_{(s,y)\in[0,T]\times S}\theta(s,y)$. This justifies the first assertion. From (4.27) we infer that $\Sigma(r,t)$ satisfies (4.26) and hence, is bounded for each $r,t\in[0,T]$. The $\hat{\mathscr{F}}_{r,t}$-measurability of $\Sigma(r,t)$ for each $r,t\in[0,T]$ and the continuity of $\Sigma(\cdot,\cdot)(\omega)$ for all $\omega\in\Omega$ follow from Lemma 4.21 and the first claim. This is because (4.28) gives

$$\lim_{n\uparrow\infty}\sup_{(r,t,\omega)\in[0,T]\times[0,T]\times\Omega}\left|\sum_{i=0}^{n}(-1)^i\Sigma_i(r,t)(\omega)-\Sigma(r,t)(\omega)\right|=0.$$

Hence, Σ fulfills (i). Let us verify that (ii) holds as well. From $\Sigma_0(r,r)=\mathbb{I}_k$ and $\Sigma_n(r,r)=0$ for all $n\in\mathbb{N}$ we get that $\Sigma(r,r)=\mathbb{I}_k$ for each $r\in[0,T]$. To verify that $\Sigma(r,s)\Sigma(s,t)=\Sigma(r,t)$ for all $r,s,t\in[0,T]$, it is enough to show that

$$\sum_{i=0}^{n}\Sigma_i(r,s)\Sigma_{n-i}(s,t)=\Sigma_n(r,t)\tag{4.29}$$

for all $n\in\mathbb{N}_0$. Indeed, we first note that $\mathbb{R}^{k\times k}$ equipped with the Frobenius norm $|\cdot|$ is a Banach algebra. Consequently, once (4.29) is established, it follows from the Cauchy product for absolutely convergent matrix series that

$$\begin{aligned}\Sigma(r,s)\Sigma(s,t)&=\left(\sum_{n=0}^{\infty}(-1)^n\Sigma_n(r,s)\right)\left(\sum_{n=0}^{\infty}(-1)^n\Sigma_n(s,t)\right)\\&=\sum_{n=0}^{\infty}(-1)^n\left(\sum_{i=0}^{n}\Sigma_i(r,s)\Sigma_{n-i}(s,t)\right)=\sum_{n=0}^{\infty}(-1)^n\Sigma_n(r,t)=\Sigma(r,t)\end{aligned}$$

for every $r,s,t\in[0,T]$. Thus, let us justify (4.29) by induction over $n\in\mathbb{N}_0$. In the initial induction step $n=0$ we directly obtain the assertion from $\Sigma_0=\mathbb{I}_k$. So, let us assume that (4.29) is true for some $n\in\mathbb{N}_0$. Then

$$\begin{aligned}\sum_{i=0}^{n+1}\Sigma_i(r,s)\Sigma_{n+1-i}(s,t)&=\Sigma_0(r,s)\Sigma_{n+1}(s,t)+\sum_{i=0}^{n}\Sigma_{i+1}(r,s)\Sigma_{n-i}(s,t)\\&=\Sigma_{n+1}(s,t)+\sum_{i=0}^{n}\int_r^s b(r',X_{r'})\Sigma_i(r',s)\,\nu(dr')\Sigma_{n-i}(s,t)\\&=\int_s^t b(s',X_{s'})\Sigma_n(s',t)\,\nu(ds')+\int_r^s b(r',X_{r'})\Sigma_n(r',t)\,\nu(dr')\\&=\int_r^t b(s',X_{s'})\Sigma_n(s',t)\,\nu(ds')=\Sigma_{n+1}(r,t)\end{aligned}$$

for every $r,s,t\in[0,T]$. This completes the induction proof. Furthermore, from $\Sigma(r,t)\Sigma(t,r)=\Sigma(r,r)=\mathbb{I}_k$ we draw the conclusion that the matrix $\Sigma(r,t)(\omega)$ is invertible and $\Sigma(r,t)(\omega)^{-1}=\Sigma(t,r)(\omega)$ for all $r,t\in[0,T]$ and each $\omega\in\Omega$.

Regarding (iii), assume that b fulfills $b(r,x)b(s,y)=b(s,y)b(r,x)$ for every $(r,x),(s,y)\in[0,T]\times S$. Then the proposition follows as soon as we have proven that

$$\Sigma_n(r,t)=\frac{1}{n!}\left(\int_r^t b(s,X_s)\,\nu(ds)\right)^n\tag{4.30}$$

for every $n \in \mathbb{N}$ and each $r, t \in [0,T]$ with $r \leq t$. In fact, the definition of Σ would then yield

$$\Sigma(r,t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\bigg(\int_r^t b(s,X_s)\,\nu(ds)\bigg)^n = \exp\bigg(-\int_r^t b(s,X_s)\,\nu(ds)\bigg)$$

for all $r,t \in [0,T]$ with $r \leq t$, which together with $\Sigma(t,r)(\omega) = \Sigma(r,t)(\omega)^{-1}$ for each $\omega \in \Omega$ would give the complete result. Hence, we let $n \in \mathbb{N}$ and and write S_n for the set of all permutations of $\{1,\dots,n\}$. For each $\sigma \in S_n$ we define $\varphi_\sigma : [0,T]^n \to [0,T]^n$ by

$$\varphi_\sigma(s_1,\dots,s_n) := (s_{\sigma(1)},\dots,s_{\sigma(n)})$$

and set $C_n^\sigma(r,t) := \{(s_1,\dots,s_n) \in [r,t]^n \,|\, s_{\sigma(1)} \leq \cdots \leq s_{\sigma(n)}\}$ for each $r,t \in [0,T]$ with $r \leq t$. Then φ_σ is a homeomorphism such that $\varphi_\sigma^{-1} = \varphi_{\sigma^{-1}}$ and $\varphi_\sigma(C_n^\sigma(r,t)) = C_n(r,t)$ for each $\sigma \in S_n$. Let $\nu_\sigma^n(\cdot)(\omega)$ denote the image measure of $\nu^n(\cdot)(\omega)$ under φ_σ, that is, $\nu_\sigma^n(B)(\omega) := \nu^n(\varphi_\sigma^{-1}(B))(\omega)$ for all $\omega \in \Omega$ and each $B \in \mathscr{B}([0,T]^n)$. Then

$$\begin{aligned}\nu_\sigma^n(B_1 \times \cdots \times B_n) &= \nu(B_{\sigma^{-1}(1)}) \cdots \nu(B_{\sigma^{-1}(n)}) \\ &= \nu(B_1)\cdots\nu(B_n) = \nu^n(B_1 \times \cdots \times B_n)\end{aligned}$$

for all $B_1,\dots,B_n \in \mathscr{B}([0,T])$. Therefore, $\nu_\sigma^n(B) = \nu^n(B)$ for each $B \in \mathscr{B}([0,T]^n)$, by the Monotone Class Theorem A.28. From the measure transformation formula and the representation (4.25) we now obtain that

$$\begin{aligned}&\int_{C_n^\sigma(r,t)} b(s_1,X_{s_1})\cdots b(s_n,X_{s_n})\,d\nu^n(s_1,\dots,s_n)\\ &\qquad= \int_{C_n^\sigma(r,t)} b(s_{\sigma(1)},X_{s_{\sigma(1)}})\cdots b(s_{\sigma(n)},X_{s_{\sigma(n)}})\,d\nu^n(s_1,\dots,s_n)\\ &\qquad= \int_{C_n(r,t)} b(s_1,X_{s_1})\cdots b(s_n,X_{s_n})\,d\nu_\sigma^n(s_1,\dots,s_n) = \Sigma_n(r,t).\end{aligned}$$

In the end, we use that $[r,t]^n = \bigcup_{\sigma\in S_n} C_n^\sigma(r,t)$. Since the continuity of ν yields that $\nu^n(C_n^\sigma(r,t) \cap C_n^\tau(r,t)) = 0$ for all $\sigma,\tau \in S_n$ with $\sigma \neq \tau$, Fubini's theorem leads to

$$\begin{aligned}\bigg(\int_r^t b(s,X_s)\,\nu(ds)\bigg)^n &= \int_{[r,t]^n} b(s_1,X_{s_1})\cdots b(s_n,X_{s_n})\,d\nu^n(s_1,\dots,s_n)\\ &= \sum_{\sigma\in S_n}\int_{C_n^\sigma(r,t)} b(s_1,X_{s_1})\cdots b(s_n,X_{s_n})\,d\nu^n(s_1,\dots,s_n)\\ &= n!\Sigma_n(r,t).\end{aligned}$$

That is, (4.30) is justified and the claim follows. □

Let us now prove a multidimensional Feynman-Kac representation.

4.23 Proposition. *Let κ be of standard form and $g \in B_b(S,\mathbb{R}^k)$. Then the unique global bounded solution u_g to* (AM) *is of the form*

$$u_g(r,x) = E_{r,x}[\Sigma(r,T)g(X_T)] - E_{r,x}\bigg[\int_r^T \Sigma(r,t)a(t,X_t)\,\nu(dt)\bigg]$$

for all $(r,x) \in [0,T] \times S$.

Proof. Let us denote the unique global bounded solution to (AM) for $a = 0$ by v_g. Then Proposition 4.17 entails that the sequence $(v_n)_{n\in\mathbb{N}_0}$ in $B_b([0,T]\times S,\mathbb{R}^k)$, recursively given by $v_0(r,x) := E_{r,x}[g(X_T)]$ and

$$v_n(r,x) := v_0(r,x) - E_{r,x}\bigg[\int_r^T b(s,X_s)v_{n-1}(s,X_s)\,\nu(ds)\bigg]$$

for every $n \in \mathbb{N}$, converges uniformly to v_g. We show by induction that v_n is of the form

$$v_n(r,x) = \sum_{i=0}^{n}(-1)^i E_{r,x}[\Sigma_i(r,T)g(X_T)]$$

for each $n \in \mathbb{N}_0$ and all $(r,x) \in [0,T]\times S$. The initial induction step $n = 0$ is valid due to $\Sigma_0 = \mathbb{I}_k$. We assume that the claim is true for some $n \in \mathbb{N}_0$. By Lemma 4.21 and Proposition A.40, the process $[0,T]\times\Omega \to \mathbb{R}^k$, $(r,\omega) \mapsto \Sigma_i(r,T)(\omega)g(X_T(\omega))$ is reconstructible, bounded, and continuous for all $i \in \mathbb{N}_0$. Hence, Proposition 3.18 entails that

$$\begin{aligned}
v_{n+1}(r,x) &= v_0(r,x) - \sum_{i=0}^{n}(-1)^i E_{r,x}\bigg[\int_r^T b(s,X_s)\Sigma_i(s,T)g(X_T)\,\nu(ds)\bigg]\\
&= E_{r,x}[\Sigma_0(r,T)g(X_T)] + \sum_{i=0}^{n}(-1)^{i+1}E_{r,x}[\Sigma_{i+1}(r,T)g(X_T)]\\
&= \sum_{i=0}^{n+1}(-1)^i E_{r,x}[\Sigma_i(r,T)g(X_T)]
\end{aligned}$$

for each $(r,x) \in [0,T]\times S$. This concludes the induction proof. Moreover, another application of Proposition 4.17 yields that the sequence $(w_n)_{n\in\mathbb{N}_0}$ in $B_b([0,T]\times S,\mathbb{R}^k)$, recursively given via $w_0(r,x) := 0$ and

$$w_n(r,x) := -E_{r,x}\bigg[\int_r^T a(s,X_s) + b(s,X_s)w_{n-1}(s,X_s)\,\nu(ds)\bigg]$$

for each $n \in \mathbb{N}$, converges uniformly to u_0, the unique global bounded solution to (AM) when $g = 0$. We prove inductively that w_n can be written in the form

$$w_n(r,x) = -\sum_{i=0}^{n-1}(-1)^i E_{r,x}\bigg[\int_r^T \Sigma_i(r,t)a(t,X_t)\,\nu(dt)\bigg]$$

for all $n \in \mathbb{N}$ and each $(r,x) \in [0,T]\times S$. The initial induction step $n = 1$ is certainly true, since $w_0 = 0$. So, let us assume that the formula holds for some $n \in \mathbb{N}$. Then it is readily seen that the process $[0,T]\times\Omega \to \mathbb{R}^k$,

$$(r,\omega) \mapsto \int_r^T \Sigma_i(r,t)(\omega)a(t,X_t(\omega))\,\nu(dt)(\omega)$$

is reconstructible, bounded, and continuous for each $i \in \mathbb{N}_0$. For this reason, from

Proposition 3.18 and Fubini's theorem it follows that

$$\begin{aligned}
w_{n+1}(r,x) &= -E_{r,x}\bigg[\int_r^T a(s,X_s)\,\nu(ds)\bigg] \\
&\quad + \sum_{i=0}^{n-1}(-1)^i E_{r,x}\bigg[\int_r^T b(s,X_s)\int_s^T \Sigma_i(s,t)a(t,X_t)\,\nu(dt)\,\nu(ds)\bigg] \\
&= -E_{r,x}\bigg[\int_r^T \Sigma_0(r,t)a(t,X_t)\,\nu(dt)\bigg] \\
&\quad + \sum_{i=0}^{n-1}(-1)^i E_{r,x}\bigg[\int_r^T \int_r^t b(s,X_s)\Sigma_i(s,t)\,\nu(ds)a(t,X_t)\,\nu(dt)\bigg] \\
&= -\sum_{i=0}^{n}(-1)^i E_{r,x}\bigg[\int_r^T \Sigma_i(r,t)a(t,X_t)\,\nu(dt)\bigg]
\end{aligned}$$

for each $(r,x) \in [0,T] \times S$. This completes the induction proof. Now the decisive observation comes from Lemmas 4.19 and 4.20. Namely, the unique global bounded solution u_g to (AM) is of the form $u_g = v_g + u_0$. Moreover,

$$v_g(r,x) = \lim_{n\uparrow\infty} E_{r,x}\bigg[\bigg(\sum_{i=0}^{n}(-1)^i\Sigma_i(r,T)\bigg)g(X_T)\bigg] = E_{r,x}[\Sigma(r,T)g(X_T)]$$

and

$$\begin{aligned}
u_0(r,x) &= -\lim_{n\uparrow\infty} E_{r,x}\bigg[\int_r^T \bigg(\sum_{i=0}^{n-1}(-1)^i\Sigma_i(r,t)\bigg)a(t,X_t)\,\nu(dt)\bigg] \\
&= -E_{r,x}\bigg[\int_r^T \Sigma(r,t)a(t,X_t)\,\nu(dt)\bigg]
\end{aligned}$$

for every $(r,x) \in [0,T] \times S$, by Proposition 4.22 and the Dominated Convergence Theorem A.33. The asserted representation follows. □

We note that if there are a μ-dominated $c \in B([0,T]\times S)$ and a matrix $B \in \mathbb{R}^{k\times k}$ such that b is of the form $b(r,x) = c(r,x)B$ for each $(r,x) \in [0,T] \times S$, then the commutation condition in Proposition 4.22 holds. More precisely,

$$b(r,x)b(s,y) = c(r,x)c(s,y)B^2 = b(s,y)b(r,x)$$

for all $(r,x),(s,y) \in [0,T]\times S$. Let us further suppose that B is diagonalizable, that is, there are a diagonal matrix $\hat{D} \in \mathbb{R}^{k\times k}$ and an invertible matrix $U \in \mathbb{R}^{k\times k}$ such that $B = U\hat{D}U^{-1}$. Then

$$\begin{aligned}
\Sigma(r,t) &= \exp\bigg(-\int_r^t c(s,X_s)\,\nu(ds)B\bigg) \\
&= U\exp\bigg(-\int_r^t c(s,X_s)\,\nu(ds)\hat{D}\bigg)U^{-1}
\end{aligned} \tag{4.31}$$

for every $r,t \in [0,T]$ with $r \le t$, because $(\alpha B)^n = \alpha^n U\hat{D}^n U^{-1}$ for each $n \in \mathbb{N}$ and all $\alpha \in \mathbb{R}$. Hence, we may consider an example that involves trigonometric functions.

4.24 Example. Let $k = 2$, $a = 0$, and κ be of standard form. Suppose that there are some μ-dominated function $c \in B([0,T] \times S)$ and $\delta, \varepsilon \in \mathbb{R}\backslash\{0\}$ such that

$$b(r,x) = c(r,x)\begin{pmatrix} 0 & \delta \\ \varepsilon & 0 \end{pmatrix} \quad \text{for all } (r,x) \in [0,T] \times S. \tag{4.32}$$

We set $\rho := 1$ for $\delta\varepsilon > 0$ and $\rho := i \in \mathbb{C}$ for $\delta\varepsilon < 0$. Then for every $g \in B_b(S, \mathbb{R}^k)$ the unique global bounded solution u_g to (AM) is of the form

$$\begin{aligned}
(u_g)_1(r,x) &= E_{r,x}\bigg[\cosh\bigg(-\rho\sqrt{|\delta\varepsilon|}\int_r^T c(s,X_s)\,\nu(ds)\bigg)g_1(X_T)\bigg] \\
&\quad + \rho\frac{\sqrt{|\delta\varepsilon|}}{\varepsilon}E_{r,x}\bigg[\sinh\bigg(-\rho\sqrt{|\delta\varepsilon|}\int_r^T c(s,X_s)\,\nu(ds)\bigg)g_2(X_T)\bigg], \\
(u_g)_2(r,x) &= \rho\frac{\sqrt{|\delta\varepsilon|}}{\delta}E_{r,x}\bigg[\sinh\bigg(-\rho\sqrt{|\delta\varepsilon|}\int_r^T c(s,X_s)\,\nu(ds)\bigg)g_1(X_T)\bigg] \\
&\quad + E_{r,x}\bigg[\cosh\bigg(-\rho\sqrt{|\delta\varepsilon|}\int_r^T c(s,X_s)\,\nu(ds)\bigg)g_2(X_T)\bigg]
\end{aligned}$$

for each $(r,x) \in [0,T] \times S$. In particular, for $\delta = -1$, $\varepsilon = 1$, and $g_2 = 0$ we recover from the trigonometric formulas $\cosh(iz) = \cos(z)$ and $\sinh(iz) = i\sin(z)$, where $z \in \mathbb{C}$, that

$$\begin{aligned}
(u_g)_1(r,x) &= E_{r,x}\bigg[\cos\bigg(-\int_r^T c(s,X_s)\,\nu(ds)\bigg)g_1(X_T)\bigg], \\
(u_g)_2(r,x) &= E_{r,x}\bigg[\sin\bigg(-\int_r^T c(s,X_s)\,\nu(ds)\bigg)g_1(X_T)\bigg]
\end{aligned}$$

for all $(r,x) \in [0,T] \times S$. To see this, note that the two distinct eigenvalues of the matrix appearing in (4.32) are $\rho\sqrt{|\delta\varepsilon|}$ and $-\rho\sqrt{|\delta\varepsilon|}$ with respective eigenvectors

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \delta \\ \rho\sqrt{|\delta\varepsilon|} \end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} \delta \\ -\rho\sqrt{|\delta\varepsilon|} \end{pmatrix}.$$

Hence, this matrix is diagonalizable and it admits the representation

$$\begin{pmatrix} 0 & \delta \\ \varepsilon & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \delta & \delta \\ \rho\sqrt{|\delta\varepsilon|} & -\rho\sqrt{|\delta\varepsilon|} \end{pmatrix}\begin{pmatrix} \rho\sqrt{|\delta\varepsilon|} & 0 \\ 0 & -\rho\sqrt{|\delta\varepsilon|} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\delta} & \frac{1}{\rho\sqrt{|\delta\varepsilon|}} \\ \frac{1}{\delta} & -\frac{1}{\rho\sqrt{|\delta\varepsilon|}} \end{pmatrix}.$$

For this reason, the solution formulas follow from Proposition 4.23 and (4.31) after standard computations.

4.6 Global existence in one dimension

We restrict our attention to one-dimensional Markovian equations and prove global existence and uniqueness in a general setting. Conditions granting the continuity of the derived solution are also given. Let $k = 1$, κ be of standard form, and $g \in B_b(S, D)$. First, we use the Feynman-Kac formula to represent the difference of two solutions. This idea is essentially based on Proposition 3.1 in Schied [32].

4.25 Lemma. *Let $f \in BC_\mu^{1-}([0,T] \times S \times D)$, $\psi : [0,T] \times S \times D \to \mathbb{R}$ be measurable with respect to $\mathscr{B}([0,T] \times S) \otimes (D \cap \mathscr{B}(\mathbb{R}))$ and locally μ-bounded, $h \in B_b(S, D)$, and I be some admissible interval. Suppose that u is a weakly μ-admissible solution to* (M) *on I and v is a weakly μ-admissible solution to* (M) *on I with f and g replaced by ψ and h, respectively. Define $\delta, \varepsilon \in B(I \times S)$ by $\delta(r,x) := (f - \psi)(r, x, v(r,x))$ and*

$$\varepsilon(r,x) := \frac{f(r,x,u(r,x)) - f(r,x,v(r,x))}{(u-v)(r,x)}, \quad \text{if } u(r,x) \neq v(r,x),$$

and $\varepsilon(r,x) := 0$, if $u(r,x) = v(r,x)$. Then δ, ε are locally μ-dominated and

$$\begin{aligned}(u-v)(r,x) = E_{r,x}\bigg[\exp\bigg(-\int_r^T \varepsilon(s,X_s)\,\nu(ds)\bigg)(g-h)(X_T)\bigg] \\ - E_{r,x}\bigg[\int_r^T \exp\bigg(-\int_r^t \varepsilon(s,X_s)\,\nu(ds)\bigg)\delta(t,X_t)\,\nu(dt)\bigg]\end{aligned}$$

for each $(r,x) \in I \times S$. In particular, if $f \leq \psi$ and $g \geq h$, then $u \geq v$.

Proof. The second claim is a direct consequence of the first, since $\delta \leq 0$ whenever $f \leq \psi$. Thus, we merely have to prove the first assertion. The fact that δ and ε are Borel measurable follows from the $\mathscr{B}([0,T] \times S) \otimes (D \cap \mathscr{B}(\mathbb{R}))$-measurability of f and ψ. To check that δ and ε are locally μ-dominated, it suffices to show that for each $r \in I$ there is a μ-dominated $\gamma \in B([r,T] \times S, \mathbb{R}_+)$ such that

$$|\delta(\cdot,y)| \vee |\varepsilon(\cdot,y)| \leq \gamma(\cdot,y) \quad \text{for all } y \in S \quad \mu\text{-a.s. on } [r,T].$$

Let K be a compact set in D such that $u(\cdot,y), v(\cdot,y) \in K$ for each $y \in S$ μ-a.s. on $[r,T]$. According to Lemma 2.5 and Proposition 2.8, there exist two μ-dominated $a, \lambda \in B([r,T] \times S, \mathbb{R}_+)$ such that

$$|f(t,x,z)| \vee |\psi(t,x,z)| \leq a(t,x) \quad \text{and} \quad |f(t,x,z) - f(t,x,z')| \leq \lambda(t,x)|z - z'|$$

for all $(t,x) \in [r,T] \times S$ and each $z, z' \in K$. This in turn gives $|\delta(\cdot,y)| \leq 2a(\cdot,y)$ and $|\varepsilon(\cdot,y)| \leq \lambda(\cdot,y)$ for every $y \in S$ μ-a.s. on $[r,T]$. Hence, all that remains is to set $\gamma := (2a) \vee \lambda$. Next, we note that

$$\delta(t,x) + \varepsilon(t,x)(u-v)(t,x) = f(t,x,u(t,x)) - \psi(t,x,v(t,x)) \tag{4.33}$$

for each $(t,x) \in I \times S$. Let $r \in I$ and define $\delta_r, \varepsilon_r \in B([0,T] \times S)$ by $\delta_r(t,x) := \delta(t,x)$, if $t \geq r$, and $\delta_r(t,x) := 0$, otherwise, and $\varepsilon_r(t,x) := \varepsilon(t,x)$, if $t \geq r$, and $\varepsilon_r(t,x) := 0$, otherwise. Then the function $[0,T] \times S \times \mathbb{R} \to \mathbb{R}$, $(t,x,z) \mapsto \delta_r(t,x) + \varepsilon_r(t,x)z$ belongs to $BC_\mu^1([0,T] \times S \times \mathbb{R})$. By (4.33), the restriction of $u - v$ to $[r,T] \times S$ is a μ-admissible solution to the linear Markovian terminal value problem

$$\mathbb{E}[dw(t,X_t)] = \mathbb{E}[\delta_r(t,X_t) + \varepsilon_r(t,X_t)w(t,X_t)\,\nu(dt)] \quad \text{for } t \in [0,T], \quad w(T,\cdot) = g - h.$$

Thus, from Proposition 4.23, Corollary 4.6, and the arbitrariness of $r \in I$ we infer the assertion. $\square$

As the only convex sets in $\mathbb{R}$ are intervals, we suppose in the sequel that D is an interval. Additionally, we set $\underline{d} := \inf D$ and $\overline{d} := \sup D$, then $\partial D = \{\underline{d}, \overline{d}\} \cap \mathbb{R}$.

4.26 Lemma. *Let $\underline{d} > -\infty$ (resp. $\overline{d} < \infty$) and f be affine μ-bounded from below (resp. from above). That is, there are two μ-dominated $a, b \in B([0,T] \times S, \mathbb{R}_+)$ with*

$$f(t,x,z) \geq -a(t,x) - b(t,x)|z| \quad (\textit{resp. } f(t,x,z) \leq a(t,x) + b(t,x)|z|)$$

for every $(t,x,z) \in [0,T] \times S \times D$. Then every μ-consistently bounded solution u to (M) *on an admissible interval I fulfills either*

$$u(r,x) - \underline{d} \leq -|\underline{d}| + E_{r,x}\left[e^{\int_r^T b(s,X_s)\,\nu(ds)}\left(g(X_T) + 2\underline{d}^- + \int_r^T a(s,X_s)\,\nu(ds)\right)\right]$$

for all $(r,x) \in I \times S$ or

$$\overline{d} - u(r,x) \leq -|\overline{d}| + E_{r,x}\left[e^{\int_r^T b(s,X_s)\,\nu(ds)}\left(2\overline{d}^+ - g(X_T) + \int_r^T a(s,X_s)\,\nu(ds)\right)\right]$$

for each $(r,x) \in I \times S$, respectively.

Proof. First of all, it is enough to show the assertion in the first case. In fact, suppose temporarily that $\overline{d} < \infty$ and write $-D$ for the set $\{-z \,|\, z \in D\}$. Then $-\overline{d} = \inf -D$ and $-\underline{d} = \sup -D$. In addition, the function $\psi : [0,T] \times S \times (-D) \to \mathbb{R}$ defined via $\psi(t,x,z) := -f(t,x,-z)$ is $\mathscr{B}([0,T] \times S) \otimes (D \cap \mathscr{B}(\mathbb{R}))$-measurable and affine μ-bounded from below.

We also observe that if I is an admissible interval, then a map $u \in B(I \times S, D)$ solves (M) if and only if the map $I \times S \to -D$, $(r,x) \mapsto -u(r,x)$ solves (M) when f and g are replaced by ψ and $-g$, respectively. Thus, an application of the lemma in the case $\inf -D > -\infty$ to ψ yields the asserted estimate in the case $\sup D < \infty$ for f. For this reason, let $\underline{d} > -\infty$ and suppose that u is a μ-consistently bounded solution to (M) on an admissible interval I. Then

$$\begin{aligned} u(r,x) - \underline{d} \leq{}& E_{r,x}[g(X_T) - \underline{d}] + E_{r,x}\left[\int_r^T (a + b|\underline{d}|)(s,X_s)\,\nu(ds)\right] \\ &+ E_{r,x}\left[\int_r^T b(s,X_s)(u(s,X_s) - \underline{d})\,\nu(ds)\right] \end{aligned}$$

for each $(r,x) \in I \times S$, because $|u(s,X_s)| \leq (u(s,X_s) - \underline{d}) + |\underline{d}|$ for all $s \in [r,T]$. By Corollary 3.24, we obtain that

$$\begin{aligned} u(r,x) - \underline{d} \leq{}& E_{r,x}\left[e^{\int_r^T b(s,X_s)\,\nu(ds)}(g(X_T) - \underline{d})\right] \\ &+ E_{r,x}\left[\int_r^T e^{\int_r^t b(s,X_s)\,\nu(ds)}(a + b|\underline{d}|)(t,X_t)\,\nu(dt)\right] \end{aligned}$$

for all $(r,x) \in I \times S$. Because $|\underline{d}| - \underline{d} = 2\underline{d}^-$, the asserted estimate follows immediately from Lemma 3.20. $\square$

4.27 Remark. Let $\underline{d} > -\infty$ and f be affine μ-bounded from below. Then there is $c > \underline{d}$ such that every μ-consistently bounded solution u to (M) on an admissible interval I satisfies

$$u(I \times S) \subset [\underline{d}, c] \cap D.$$

In fact, let $\overline{a}, \overline{b} \in B([0,T], \mathbb{R}_+)$ be two μ-integrable functions with $a(\cdot, y) \leq \overline{a}$ and $b(\cdot, y) \leq \overline{b}$ for all $y \in S$ μ-a.s. Then

$$c := \exp\left(\overline{\theta} \int_0^T \overline{b}(s)\,\mu(ds)\right)\left(\sup_{y \in S} g(y) + 2\underline{d}^- + \overline{\theta} \int_0^T \overline{a}(s)\,\mu(ds)\right)$$

with $\overline{\theta} := \sup_{(s,y) \in [0,T] \times S} \theta(s,y)$ clarifies the claim. Of course, a similar remark holds when $\overline{d} < \infty$ and f is affine μ-bounded from above.

Next, we study the boundary behavior of solutions in comparison to the terminal value condition.

4.28 Proposition. *Let $f \in BC_\mu^{1-}([0,T] \times S \times D)$ and $\underline{d} > -\infty$ (resp. $\overline{d} < \infty$). Suppose that f is both locally μ-bounded and locally Lipschitz μ-continuous at $\underline{d}$ (resp. $\overline{d}$) with $\lim_{z \downarrow \underline{d}} f(\cdot, x, z) \leq 0$ (resp. $\lim_{z \uparrow \overline{d}} f(\cdot, x, z) \geq 0$) for all $x \in S$ μ-a.s. Morever, let one of the following two conditions hold:*

(i) f is μ-bounded from above (resp. from below).

(ii) $\overline{d} = \infty$ (resp. $\underline{d} = -\infty$) and f is affine μ-bounded from below (resp. from above).

Then there is $c \in (0,1]$ such that each weakly μ-admissible solution u to (M) *on an admissible interval I is subject to*

$$u(r,x) - \underline{d} \geq c(E_{r,x}[g(X_T)] - \underline{d}) \quad \textit{(resp. } \overline{d} - u(r,x) \geq c(\overline{d} - E_{r,x}[g(X_T)])\textit{)}$$

for all $(r,x) \in I \times S$.

Proof. Once again, it suffices to prove the claim in the case $\underline{d} > -\infty$. Indeed, assume for the moment that $\overline{d} < \infty$ and let $\psi \in BC_\mu^{1-}([0,T] \times S \times (-D))$ be given by $\psi(t,x,z) := -f(t,x,-z)$. Then ψ is both locally μ-bounded and locally Lipschitz μ-continuous at $-\overline{d}$ with $\lim_{z \downarrow -\overline{d}} \psi(\cdot, x, z) \leq 0$ for all $x \in S$ μ-a.s. In addition, if I is an admissible interval, then a map $u \in B(I \times S, D)$ solves (M) exactly if the map $I \times S \to -D$, $(r,x) \mapsto -u(r,x)$ is a solution to (M) when f and g are replaced by ψ and $-g$, respectively. Hence, an application of the proposition in the case $\inf -D > -\infty$ to ψ establishes the claim in the case $\sup D < \infty$ for f.

Thus, let $\underline{d} > -\infty$. Whenever $\underline{d} \notin D$, then we define the extension $\overline{f}$ of f to $[0,T] \times S \times (D \cup \{\underline{d}\})$ through $\overline{f}(t,x,\underline{d}) := \lim_{z \downarrow \underline{d}} f(t,x,z)$. Otherwise, we simply set $\overline{f} := f$. This gives $\overline{f} \in BC_\mu^{1-}([0,T] \times S \times (D \cup \{\underline{d}\}))$. Furthermore, we see that the constant function $[0,T] \times S \to \{\underline{d}\}$, $(r,x) \mapsto \underline{d}$ is a global weakly admissible solution to the Markovian terminal value problem

$$\mathbb{E}[dv(t,X_t)] = \mathbb{E}[(\overline{f}(t,X_t,v(t,X_t)) - \overline{f}(t,X_t,\underline{d}))\,\nu(dt)] \quad \text{for } t \in [0,T], \quad v(T,\cdot) = \underline{d}.$$

Now, let u be a weakly μ-admissible solution to (M) on an admissible interval I. Then, as u is D-valued, it is also a weakly μ-admissible solution to (M) when f is replaced by $\overline{f}$. Hence, Lemma 4.25 implies that $\varepsilon_u \in B(I \times S)$ given by

$$\varepsilon_u(r,x) := \frac{\overline{f}(r,x,u(r,x)) - \overline{f}(r,x,\underline{d})}{u(r,x) - \underline{d}}, \quad \text{if } u(r,x) > \underline{d},$$

and $\varepsilon_u(r,x) := 0$, otherwise, is locally μ-dominated and satisfies

$$u(r,x) - \underline{d} \geq E_{r,x}\bigg[\exp\bigg(-\int_r^T \varepsilon_u(s,X_s)\,\nu(ds)\bigg)(g(X_T) - \underline{d})\bigg]$$

for each $(r,x) \in I \times S$, since $\overline{f}(t,X_t,\underline{d}) \leq 0$ for μ-a.e. $t \in [r,T]$. We derive some μ-dominated $n \in B([0,T] \times S, \mathbb{R}_+)$ such that every weakly μ-admissible solution u to (M) on an admissible interval I satisfies $\varepsilon_u(r,x) \leq n(r,x)$ for each $(r,x) \in I \times S$. Once this is shown, the assertion follows. Indeed, the only remaining task is to choose a μ-integrable $\overline{n} \in B([0,T], \mathbb{R}_+)$ such that $n(\cdot, y) \leq \overline{n}$ for all $y \in S$ μ-a.s. and set

$$c := \exp\bigg(-\overline{\theta}\int_0^T \overline{n}(s)\,\mu(ds)\bigg) \quad \text{with} \quad \overline{\theta} := \sup_{(s,y)\in[0,T]\times S} \theta(s,y).$$

So, let us at first assume that (i) holds. Then there exists some μ-dominated $a \in B([0,T] \times S, \mathbb{R}_+)$ with $\overline{f}(t,x,z) \leq a(t,x)$ for each $(t,x,z) \in [0,T] \times S \times D$. Since f is locally Lipschitz μ-continuous at $\underline{d}$, there are $\delta > 0$ and a μ-dominated $\lambda \in B([0,T] \times S, \mathbb{R}_+)$ fulfilling $|\overline{f}(t,x,z) - \overline{f}(t,x,z')| \leq \lambda(t,x)|z - z'|$ for every $(t,x) \in [0,T] \times S$ and all $z, z' \in [\underline{d}, \underline{d} + \delta) \cap D$. Hence, for every weakly μ-admissible solution u to (M) on an admissible interval I we obtain that

$$\varepsilon_u(r,x) \leq \lambda(r,x)\mathbb{1}_{[\underline{d},\underline{d}+\delta)}(u(r,x)) + \frac{a(r,x) - \overline{f}(r,x,\underline{d})}{\delta}\mathbb{1}_{[\underline{d}+\delta,\infty)}(u(r,x)) \leq n(r,x)$$

for all $(r,x) \in I \times S$ with $n(r,x) := \max\{\lambda(r,x), (a(r,x) - \overline{f}(r,x,\underline{d}))/\delta\}$ for each $(r,x) \in [0,T] \times S$. Note that the function $[0,T] \times S \to \mathbb{R}$, $(t,x) \mapsto \overline{f}(t,x,\underline{d})$ is μ-dominated, as f is locally μ-bounded at $\underline{d}$. For this reason, n is μ-dominated, as desired.

In place of assuming that f is μ-bounded from above, let (ii) be true. That is, $\overline{d} = \infty$ and f is affine μ-bounded from below. Then Remark 4.27 yields $c > \underline{d}$ such that

$$u(I \times S) \subset [\underline{d}, c] \cap D$$

for each weakly μ-admissible solution u to (M) on an admissible interval I. Because $[\underline{d}, c]$ is compact, there is a μ-dominated function $\lambda \in B([0,T] \times S, \mathbb{R}_+)$ such that $|\overline{f}(t,x,z) - \overline{f}(t,x,z')| \leq \lambda(t,x)|z - z'|$ for all $(t,x) \in [0,T] \times S$ and each $z, z' \in [\underline{d}, c]$. Hence, each weakly μ-admissible solution u to (M) on an admissible interval I fulfills $|\varepsilon_u(r,x)| \leq n(r,x)$ for every $(r,x) \in I \times S$ with $n := \lambda$. □

Eventually, we are ready to establish the main result of this section. Namely, the one-dimensional global existence and uniqueness theorem.

4.29 Theorem. *Let $f \in BC^{1-}_\mu([0,T] \times S \times D)$ and suppose that the following two conditions hold:*

(i) If $\underline{d} > -\infty$ (resp. $\overline{d} < \infty$), then f is both locally μ-bounded and locally Lipschitz μ-continuous at $\underline{d}$ (resp. $\overline{d}$) with $\lim_{z\downarrow\underline{d}} f(\cdot,x,z) \leq 0$ (resp. $\lim_{z\uparrow\overline{d}} f(\cdot,x,z) \geq 0$) for all $x \in S$ μ-a.s.

(ii) If $\underline{d} = -\infty$ (resp. $\overline{d} = \infty$), then f is affine μ-bounded from above (resp. from below).

Then there exists a unique global bounded solution $\overline{u}_g$ to (M) *that coincides with u_g if g is bounded away from $\{\underline{d},\overline{d}\} \cap \mathbb{R}$. Moreover, if $\mathscr{X}$ is (right-hand) Feller, f is right-continuous, and $g \in C_b(S)$, then $\overline{u}_g$ is (right-)continuous.*

Proof. Let us verify the first claim. We begin with the case $\underline{d} > -\infty$ and $\overline{d} < \infty$. Proposition 4.28 and Remarks 4.16 yield that $I_h = [0,T]$ for each $h \in B_b(S,(\underline{d},\overline{d}))$ that is bounded away from $\{\underline{d},\overline{d}\}$. Thus, for all $n \in \mathbb{N}$ we define

$$g_n := (g \vee (\underline{d} + (\overline{d} - \underline{d})2^{-n})) \wedge (\overline{d} - (\overline{d} - \underline{d})2^{-n}), \tag{4.34}$$

then $g_n \in B_b(S,(\underline{d},\overline{d}))$ and $\mathrm{dist}(g_n,\{\underline{d},\overline{d}\}) \geq (\overline{d} - \underline{d})2^{-n}$, which guarantees that $I_{g_n} = [0,T]$. Because $|g_n - g| \leq (\overline{d} - \underline{d})2^{-n}$ for all $n \in \mathbb{N}$, the sequence $(g_n)_{n\in\mathbb{N}}$ converges uniformly to g. If $D \subsetneq [\underline{d},\overline{d}]$, then we let $\overline{f}$ denote the unique extension of f to $[0,T] \times S \times [\underline{d},\overline{d}]$ such that

$$\overline{f} \in BC^{1-}_\mu([0,T] \times S \times [\underline{d},\overline{d}]).$$

Otherwise, we just set $\overline{f} := f$. According to Proposition 4.7, the sequence $(u_{g_n})_{n\in\mathbb{N}}$ converges uniformly to the unique global bounded solution to (M) with $\overline{f}$ instead of f, which we denote by $\overline{u}_g$. By uniqueness, $\overline{u}_g = u_g$ whenever g is bounded away from $\{\underline{d},\overline{d}\}$. Since Proposition 4.28 also shows that $\overline{u}_g$ does not attain the value $\underline{d}$ (resp. $\overline{d}$) if the same is true for g, the function $\overline{u}_g$ is D-valued. Hence, $\overline{u}_g$ is the unique global bounded solution to (M).

Let us turn to the case $\underline{d} > -\infty$ and $\overline{d} = \infty$. Lemma 4.26 and Proposition 4.28 entail that $I_h = [0,T]$ for every $h \in B_b(S,(\underline{d},\infty))$ that is bounded away from $\underline{d}$. For each $n \in \mathbb{N}$ we set

$$g_n := g \vee (\underline{d} + 2^{-n}), \tag{4.35}$$

then $g_n \in B_b(S,(\underline{d},\infty))$ and $\mathrm{dist}(g_n,\underline{d}) \geq 2^{-n}$, which implies that $I_{g_n} = [0,T]$. In addition, $|g_n - g| \leq 2^{-n}$ and $g_n(x) - \underline{d} \leq (g(x) - \underline{d}) \vee (1/2)$ for all $n \in \mathbb{N}$ and each $x \in S$. We can now infer from Remark 4.27 and Proposition 4.7 that $(u_{g_n})_{n\in\mathbb{N}}$ converges uniformly to the unique global bounded solution to (M), which we denote by $\overline{u}_g$. Of course, uniqueness forces $\overline{u}_g = u_g$ if g is bounded away from $\underline{d}$. From Proposition 4.28 we see that $\overline{u}_g$ cannot attain the value $\underline{d}$ if $g(x) > \underline{d}$ for all $x \in S$. For this reason, $\overline{u}_g$ is D-valued, which concludes the case $\underline{d} > -\infty$ and $\overline{d} = \infty$. The case $\underline{d} = -\infty$ and $\overline{d} < \infty$ is a consequence of the last case by using the transformed function $\psi \in BC^{1-}_\mu([0,T] \times S \times (-D))$ defined through $\psi(t,x,z) := -f(t,x,-z)$, as the considerations in the beginning of the proof of Proposition 4.28 show.

In the end, we note that for each $n \in \mathbb{N}$ the function g_n given either by (4.34) or (4.35), depending on which case occurs, is continuous if $g \in C_b(S)$. Hence, because Lemma A.9 guarantees that the uniform limit of a sequence of real-valued (right-)continuous functions on $[0,T]\times S$ is (right-)continuous, the Non-Extendibility Theorem 4.15 implies the second assertion. □

In the case that $f(t,x,z) = a(t,x) + b(t,x)z$ for all $(t,x,z) \in [0,T] \times S \times D$ and some $a, b \in B([0,T] \times S)$, we get a one-dimensional Feynman-Kac formula, which for $D = \mathbb{R}$, $a = 0$, and $b \geq 0$ can also be inferred from Theorem 4.1.1 in Dynkin [15].

4.30 Corollary. *Suppose that there are two μ-dominated $a, b \in B([0,T] \times S)$ with $f(t,x,z) = a(t,x) + b(t,x)z$ for all $(t,x,z) \in [0,T] \times S \times D$ such that if $\underline{d} > -\infty$ (resp. $\overline{d} < \infty$), then*

$$a(\cdot,x) + b(\cdot,x)\underline{d} \leq 0 \quad (\textit{resp. } a(\cdot,x) + b(\cdot,x)\overline{d} \geq 0) \tag{4.36}$$

for each $x \in S$ μ-a.s. Then the unique global bounded solution $\overline{u}_g$ to (M) *admits the representation*

$$\overline{u}_g(r,x) = E_{r,x}\left[e^{-\int_r^T b(s,X_s)\,\nu(ds)} g(X_T)\right] - E_{r,x}\left[\int_r^T e^{-\int_r^t b(s,X_s)\,\nu(ds)} a(t,X_t)\,\nu(dt)\right]$$

for all $(r,x) \in [0,T] \times S$. Furthermore, if $\mathscr{X}$ is (right-hand) Feller, a and b are right-continuous, and $g \in C_b(S)$, then $\overline{u}_g$ is (right-)continuous.

Proof. Clearly, f is affine μ-bounded and uniformly μ-differentiable. In particular, $f \in BC^1_\mu([0,T]\times S\times D)$. Condition (4.36) translates to requirement that if $\underline{d} > -\infty$ (resp. $\overline{d} < \infty$), then

$$\lim_{z\downarrow\underline{d}} f(\cdot,x,z) \leq 0 \quad \Big(\text{resp. } \lim_{z\uparrow\overline{d}} f(\cdot,x,z) \geq 0\Big)$$

for each $x \in S$ μ-a.s. We observe that f is right-continuous provided a and b are right-continuous. Hence, Theorem 4.29 entails that (M) admits the unique global bounded solution $\overline{u}_g$, which is (right-)continuous if $\mathscr{X}$ is (right-hand) Feller, a and b are right-continuous, and $g \in C_b(S)$. Let us set

$$\overline{f}(t,x,z) := a(t,x) + b(t,x)z \quad \text{for all } (t,x,z) \in [0,T] \times S \times \mathbb{R},$$

then Proposition 4.23 implies that the unique global bounded solution v_g to (M) with f replaced by $\overline{f}$ admits the required representation

$$v_g(r,x) = E_{r,x}\left[e^{-\int_r^T b(s,X_s)\,\nu(ds)} g(X_T)\right] - E_{r,x}\left[\int_r^T e^{-\int_r^t b(s,X_s)\,\nu(ds)} a(t,X_t)\,\nu(dt)\right]$$

for every $(r,x) \in [0,T] \times S$. However, as $\overline{u}_g$ is D-valued, it is also a global bounded solution to (M) with f replaced by $\overline{f}$. Uniqueness, manifested in Corollary 4.6, gives $\overline{u}_g = v_g$, which concludes the proof. □

In the specific case $D = \mathbb{R}_+$, global bounded solutions to (M) can be represented via the log-Laplace functionals of superprocesses.

4.31 Example. Let $D = \mathbb{R}_+$ and $a, b, c \in B([0,T] \times S, \mathbb{R}_+)$ be μ-dominated. We let $\varphi_a, \varphi_b, \varphi_c : \mathbb{R}_+ \to \mathbb{R}$ be locally Lipschitz continuous and n be a kernel from $[0,T] \times S$ to $(0,\infty)$ such that

$$\int_0^\infty u \min\{1,u\}\, n(\cdot,\cdot,du)$$

is finite and μ-dominated. In addition, let $\varphi_d : \mathbb{R}_+ \to \mathbb{R}_+$ be twice differentiable with $\varphi_d(0) = 0$, $\varphi_d'(z) \geq 0$, and $\varphi_d''(z) \leq 0$ for all $z \geq 0$. Assume that f is of the form

$$\begin{aligned} f(t,x,z) = {}& a(t,x)\varphi_a(z) + b(t,x)\varphi_b(z) + c(t,x)\varphi_c(z) \\ & + \int_0^\infty (e^{-u\varphi_d(z)} - 1 + u\varphi_d(z))\, n(t,x,du) \end{aligned} \tag{4.37}$$

for all $(t,x,z) \in [0,T] \times S \times \mathbb{R}_+$. Then $f \in BC_\mu^{1-}([0,T] \times S \times \mathbb{R}_+)$, by Example 2.17. Moreover, if we had $\varphi_a, \varphi_b, \varphi_c \in C^1(\mathbb{R}_+)$, then $f \in BC_\mu^1([0,T] \times S \times \mathbb{R}_+)$ would follow. We suppose that $\varphi_a(z) \geq -c_a$, $\varphi_b(z) \geq -c_b|z|$, and $\varphi_c(z) \geq 0$ for all $z \geq 0$ and some $c_a, c_b \geq 0$. Then f is affine μ-bounded from below, because

$$f(t,x,z) \geq -a(t,x)c_a - b(t,x)c_b|z|$$

for every $(t,x,z) \in [0,T] \times S \times \mathbb{R}_+$. Let in addition $\varphi_a(0) \leq 0$, $\varphi_b(0) \leq 0$, and $\varphi_c(0) = 0$, then $f(t,x,0) \leq 0$ for each $(t,x) \in [0,T] \times S$. Hence, under these conditions, Theorem 4.29 yields a unique global bounded solution to (M).

Let us now specifically suppose that $a = 0$ and that the functions b, c, and $[0,T] \times S \to \mathbb{R}_+$, $(t,x) \mapsto \int_0^\infty u \min\{1,u\}\, n(t,x,du)$ are bounded. Furthermore, let $\varphi_b(z) = \varphi_d(z) = z$ and $\varphi_c(z) = z^2$ for all $z \geq 0$. Then (4.37) becomes

$$f(t,x,z) = b(t,x)z + c(t,x)z^2 + \int_0^\infty (e^{-uz} - 1 + uz)\, n(t,x,du)$$

for every $(t,x,z) \in [0,T] \times S \times \mathbb{R}_+$ and $f \in BC_\mu^1([0,T] \times S \times \mathbb{R}_+)$. In this case, by Theorem 1.1 in Dynkin [12] there exists an $(\mathscr{X}, \nu, f)$-superprocess, which is a progressive Markov process $\mathscr{Z} = (Z, (\mathscr{G}_t)_{t\in[0,T]}, \mathbb{Q})$ with state space $\mathscr{M}_f(S)$, the set of all finite Borel measures on S, such that for each $t \in (0,T]$ and every $h \in B_b(S, \mathbb{R}_+)$, the function

$$[0,t] \times S \to \mathbb{R}_+, \quad (r,x) \mapsto -\log\left(E_{r,\delta_x}^Q\left[\exp\left(-\int_S h(x)\, dZ_t(x)\right)\right]\right)$$

is Borel measurable and a global solution to (M) when T and g are replaced by t and h, respectively. Here, $\mathbb{Q}$ is of the form $\mathbb{Q} = \{Q_{r,\lambda} \,|\, (r,\lambda) \in [0,T] \times \mathscr{M}_f(S)\}$ and $E_{r,x}^Q$ denotes the expectation with respect to Q_{r,δ_x} for all $(r,x) \in [0,T] \times S$. Thus,

$$\overline{u}_g(r,x) = -\log\left(E_{r,\delta_x}^Q\left[\exp\left(-\int_S g(x)\, dZ_T(x)\right)\right]\right)$$

for each $(r,x) \in [0,T] \times S$, due to Theorem 4.29.

Chapter 5

Path-Dependent PDEs

In the final chapter we deal with semilinear parabolic PPDEs that rest on horizontal and vertical derivatives of non-anticipative functionals on path spaces. In Section 5.1 we set up our notation, discuss measurable structures and topologies, and recover the familiar pseudometric topological setting. A short overview of path processes is given as well. In Section 5.2 the definitions of horizontal and vertical derivatives of non-anticipative maps on path spaces are recalled. Furthermore, we compile some facts on these relaxed time and space differential operators. Section 5.3 introduces the parabolic terminal value problem that is formulated with a linear differential operator and an inhomogeneity that depends on the solution.

By making use of path-dependent diffusion processes that fulfill a martingale property, we relate classical solutions to mild solutions and conclude that the latter are in fact global solutions to the associated Markovian integral equations. After this, we compare various notions of viscosity solutions in Section 5.4. To this end, we study several test function spaces and discuss the relations between them. The goal of Section 5.5 is to provide the evidence that each bounded right-continuous mild solution may also be viewed as a viscosity solution. In the case that the PPDE is affine, we can show that right-continuity is redundant. Moreover, we verify that the right-hand semicontinuous envelopes of a bounded mild solution, introduced in Section A.2, fulfill a right-hand viscosity property. This concludes our work.

5.1 Path spaces and path processes

Throughout the chapter, assume that $T > 0$, $d \in \mathbb{N}$, and $|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$. Let $\widetilde{S}$ denote the linear space of all $\mathbb{R}^d$-valued càdlàg maps on $[0,T]$ and set $S := C([0,T],\mathbb{R}^d)$. We work with the canonical process

$$\widetilde{\xi} : [0,T] \times \widetilde{S} \to \mathbb{R}^d, \quad \widetilde{\xi}_t(x) := x(t)$$

and its restriction ξ to $[0,T] \times S$. By $(\widetilde{\mathscr{S}_t})_{t\in[0,T]}$ we denote the natural filtration of $\widetilde{\xi}$ and set $\mathscr{S}_t := S \cap \widetilde{\mathscr{S}_t}$ for all $t \in [0,T]$, which in turn gives the natural filtration $(\mathscr{S}_t)_{t\in[0,T]}$ of ξ. To keep notation simple, we set $\widetilde{\mathscr{S}} := \widetilde{\mathscr{S}_T}$ and $\mathscr{S} := S \cap \widetilde{\mathscr{S}}$. Next, for each $t \in [0,T]$ and all $x \in \widetilde{S}$, we define $\|x\| := \sup_{s\in[0,T]} |x(s)|$ and let $x^t \in \widetilde{S}$ be the map x stopped at time t. That is, $x^t(s) = x(s \wedge t)$ for every $s \in [0,T]$. Of course, $\widetilde{S}$ equipped with $\|\cdot\|$ becomes a Banach space, which, however, fails to be separable, and S is a separable closed set in $\widetilde{S}$.

Due to the non-separability of $\widetilde{S}$ under the supremum norm and the fact that the Borel σ-field of $\widetilde{S}$ with respect to $\|\cdot\|$ is strictly larger than the cylindrical σ-field $\widetilde{\mathscr{S}}$, we equip $\widetilde{S}$ with a complete metric ρ that induces the Skorohod topology and which satisfies

$$\rho(x,y) \leq L\|x-y\| \tag{5.1}$$

for all $x,y \in \widetilde{S}$ and some $L > 0$. Then $\widetilde{S}$ endowed with ρ turns into a Polish space and the Borel σ-field of $\widetilde{S}$ with respect to ρ is exactly $\widetilde{\mathscr{S}}$. Moreover, ρ and $\|\cdot\|$ induce the same topology on S.

5.1 Example. In Billingsley [3, Section 12] such a metric is defined for $T = 1$, which can be generalized to arbitrarily $T > 0$. Namely, let Λ be the set of all strictly increasing homeomorphisms from $[0,1]$ onto $[0,1]$ and set

$$|\lambda|_\Lambda := \sup_{s,t\in[0,1]:\, s<t} \left| \log\left(\frac{\lambda(t)-\lambda(s)}{t-s}\right)\right| \quad \text{for all } \lambda \in \Lambda,$$

then $|\lambda|_\Lambda = 0$ if and only if λ is the identity map on $[0,T]$. The metric ρ on $\widetilde{S}$ is then defined via $\rho(x,y) := \inf_{\lambda\in\Lambda}(|\lambda|_\Lambda \vee \|x - y\circ\lambda\|)$. Billingsley verifies that ρ is a complete metric that is equivalent to another metric which induces the Skorohod topology. Moreover, $\rho(x,y) \leq \|x-y\|$ for all $x,y \in \widetilde{S}$.

We intend to use the pseudometric topological setting introduced in Section 3.1 for the choice $J = [0,T]$ and $\Phi : [0,T] \times \widetilde{S} \to \widetilde{S}$, $\Phi_t(x) = x^t$. Then $(x^s)^t = x^{s\wedge t}$ for all $s,t \in [0,T]$ and each $x \in \widetilde{S}$, which entails that $\sigma(\Phi_t) = \widetilde{\mathscr{S}}_t$ for each $t \in [0,T]$. Let us also verify that Φ, regarded as a process, has càdlàg paths.

5.2 Lemma. *For each $x \in \widetilde{S}$ the map $[0,T] \to \widetilde{S}$, $t \mapsto x^t$ is càdlàg and also left-continuous at each continuity point of x.*

Proof. Initially, we show that the map $[0,T] \to \widetilde{S}$, $r \mapsto x^r$ is right-continuous. Let $r \in [0,T)$, then there is $\delta \in (0, T-r)$ such that $|x(s) - x(r)| < \varepsilon/L$ for all $s \in [r, r+\delta)$, since x is right-continuous. By (5.1),

$$\rho(x^t, x^r) \leq L\|x^t - x^r\| = L \sup_{s\in[r,t]} |x(s) - x(r)| \leq \varepsilon$$

for all $t \in [r, r+\delta)$. Now let $t \in (0,T]$ and denote the left-hand limit $\lim_{s\uparrow t} x(s)$ by $x(t-)$. We define $x_t \in \widetilde{S}$ via $x_t(s) := x(s)\mathbb{1}_{[0,t)}(s) + x(t-)\mathbb{1}_{[t,T]}(s)$, then $x_t = x^t$ whenever x is continuous at t. Hence, the proof is complete, once we have shown that

$$\lim_{r\uparrow t} \rho(x^r, x_t) = 0.$$

We pick $\delta \in (0,t)$ such that $|x(s) - x(t-)| < \varepsilon/(2L)$ for each $s \in (t-\delta, t)$, then $\rho(x^r, x_t) \leq L\|x^r - x_t\| \leq L|x(r) - x(t-)| + L\sup_{s\in[r,t)} |x(t-) - x(s)| < \varepsilon$ for every $r \in (t-\delta, t)$. $\square$

We recall that if (E, ϱ) is a metric space and I is a non-degenerate interval in $[0,T]$, then a map $u : I \times \widetilde{S} \to E$ is called *non-anticipative* if $u(t,x) = u(t,x^t)$ for all $(t,x) \in I \times \widetilde{S}$. This coincides with our notion of consistency and also works if the domain of u is merely $I \times S$. Following Cont and Fournié [6], and using the Cartesian setting in Ekren, Keller, Touzi, and Zhang [16], we consider the pseudometric d_∞ on $[0,T] \times \widetilde{S}$ given by

$$d_\infty((r,x),(s,y)) := |r-s| + \|x^r - y^s\|.$$

Then $d_\infty((r,x),(s,y)) = 0$ exactly if $r = s$ and $x^r = y^r$ for all $r,s \in [0,T]$ and each $x,y \in \widetilde{S}$. Thus, if u is continuous with respect to d_∞, then it is non-anticipative. However, there is no reason that u is $(\widetilde{\mathscr{S}_t})_{t \in I}$-progressively measurable or at least $\mathscr{B}(I) \otimes \widetilde{\mathscr{S}}$-measurable. To circumvent this difficulty, we endow $[0,T] \times \widetilde{S}$ with the pseudometric d_S defined via

$$d_S((r,x),(s,y)) := |r-s| + \rho(x^r, y^s).$$

From (5.1) we infer that if u is continuous with respect to d_S, then continuity relative to d_∞ follows. The same is true for right-continuity, in which case u is progressively measurable. Indeed, $u(\cdot,x)$ is right-continuous for each $x \in \widetilde{S}$, by Proposition 3.3, and non-anticipation yields that $u(t,\cdot)$ is $\widetilde{\mathscr{S}_t}$-measurable for every $t \in I$. Thus, Proposition A.38 applies. Finally, a combination of Lemma 3.2 with Proposition 3.3 summarizes two more facts.

5.3 Corollary. *Let (E, ϱ) be a metric space, I be a non-degenerate interval in $[0,T]$, and $u : I \times \widetilde{S} \to E$. Then the following two assertions hold:*

(i) If u is non-anticipative and $\mathscr{B}(I) \otimes \widetilde{\mathscr{S}}$-measurable, then it is progressively measurable. Then converse is true provided E is a finite-dimensional linear space and ϱ is induced by a norm.

(ii) Let u be right-continuous with respect to d_∞, then $u(\cdot,x)$ is right-continuous for each $x \in \widetilde{S}$ and $u(t,\cdot)$ is continuous relative to $\|\cdot\|$ for all $t \in I$. Further, if u is actually continuous, then $u(\cdot,x)$ is càdlàg and left-continuous at each continuity point of x.

From Section 5.3 onwards, when mild and viscosity solutions to path-dependent PDEs are considered, we work with S rather than $\widetilde{S}$. Certainly, Proposition 3.1 ensures that $[0,T] \times S$ equipped with the corresponding restriction of d_S is a separable complete pseudometric space and its Borel σ-field fulfills

$$\mathscr{B}([0,T] \times S) \subset \mathscr{B}([0,T]) \otimes \mathscr{S}.$$

In addition, for each non-degenerate interval I in $[0,T]$ and every topological space E, a non-anticipative map $u : I \times S \to E$ is Borel measurable if and only if it is measurable with respect to $\mathscr{B}(I) \otimes \mathscr{S}$. We conclude the introduction to path spaces with a continuity statement that follows from Proposition 3.3.

5.4 Corollary. *Let (E, ϱ) be a metric space, I be a non-degenerate interval in $[0,T]$, and $u : I \times S \to E$. Then the subsequent two assertions are valid:*

(i) Whenever u is (right-)continuous, then it is progressively measurable, $u(\cdot, x)$ is (right-)continuous for each $x \in S$, and $u(t,\cdot)$ is continuous for all $t \in I$.

(ii) Suppose that I is compact and u is continuous. Then u is continuous in $x \in S$, uniformly in $t \in I$.

We now give a concise overview to path processes that are used in Section 5.3 for presenting path-dependent diffusion processes. Let $(\Omega, \mathscr{F})$ be a measurable space and assume that $X : [0,T] \times \Omega \to \mathbb{R}^d$ is a continuous process whose natural filtration we denote by $(\hat{\mathscr{F}}_t)_{t\in[0,T]}$. By the *path process* of X, we mean the process

$$\hat{X} : [0,T] \times \Omega \to S, \quad \hat{X}_t(\omega) := X^t(\omega).$$

For instance, whenever $(\Omega, \mathscr{F}) = (S, \mathscr{S})$ and $X = \xi$, then $\hat{\xi}_t(x) = x^t$ for every $(t,x) \in [0,T] \times S$. The natural filtration of $\hat{X}$ is easily identified. Let $t \in [0,T]$, then $\{\hat{X}_t \in \xi_s^{-1}(B)\} = \{X_{s\wedge t} \in B\}$ for all $s \in [0,T]$ and each $B \in \mathscr{B}(\mathbb{R}^d)$. On the one hand, this implies that $\sigma(X_s) \subset \sigma(\hat{X}_t)$ for all $s \in [0,t]$. On the other hand, as $\{\xi_s^{-1}(B) \,|\, s \in [0,T],\, B \in \mathscr{B}(\mathbb{R}^d)\}$ is a generator of $\mathscr{S}$, this shows that $\hat{X}_t$ is in fact $\hat{\mathscr{F}}_t$-measurable. For this reason,

$$\sigma(\hat{X}_t) = \hat{\mathscr{F}}_t \quad \text{and} \quad \sigma(\hat{X}_s) \subset \sigma(\hat{X}_t) \quad \text{for all } s \in [0,t]. \tag{5.2}$$

Consequently, X is adapted to a filtration of $\mathscr{F}$ if and only if $\hat{X}$ is. Regarding the natural backward filtration of $\hat{X}$, we simply note that $\sigma(\hat{X}_u : u \in [t,T]) = \hat{\mathscr{F}}_T$ for every $t \in [0,T]$. Moreover,

$$\|\hat{X}_r - \hat{X}_t\| = \max_{s\in[0,T]} |X_s^r - X_s^t| = \max_{s\in[r,t]} |X_r - X_s|$$

for each $r, t \in [0,T]$ with $r \leq t$. Thus, $\hat{X}$ has continuous paths. In conclusion, for each continuous process $X : [0,T] \times \Omega \to \mathbb{R}^d$, the path process $\hat{X} : [0,T] \times \Omega \to S$ is another continuous process that fulfills (5.2) for each $t \in [0,T]$.

5.2 Differential calculus on path spaces

Let us review several of the standard facts on horizontal and vertical derivatives of non-anticipative maps on path spaces that were introduced by Dupire [10] and Cont and Fournié [6]. Again, the Cartesian setting in [16] is used. We fix $r \in [0,T)$ and $k \in \mathbb{N}$, and let $\{e_1, \dots, e_d\}$ denote the standard basis of $\mathbb{R}^d$.

5.5 Definition. A non-anticipative map $u : [r,T) \times \widetilde{S} \to \mathbb{R}^k$ is said to be *horizontally differentiable* at $(t,x) \in [r,T) \times \widetilde{S}$ if the map

$$[0, T-t) \to \mathbb{R}^k, \quad h \mapsto u(t+h, x^t)$$

is differentiable at 0. Its derivative at 0 is called the *the horizontal derivative* of u at (t,x) and is denoted by $\partial_t u(t,x)$. We call u horizontally differentiable if it is horizontally differentiable at each $(t,x) \in [r,T) \times \widetilde{S}$.

Suppose that $u : [r,T) \times \widetilde{S} \to \mathbb{R}^k$ is non-anticipative and $(t,x) \in [r,T) \times \widetilde{S}$. Clearly, $u(t+h,(x^t)^t) = u(t+h,x^t)$ for all $h \in [0,T-t)$. Thus, if u is horizontally differentiable at (t,x), then it is also horizontally differentiable at (t,x^t) and

$$\partial_t u(t,x) = \lim_{h\downarrow 0} \frac{u(t+h,x^t) - u(t,x)}{h} = \partial_t u(t,x^t).$$

In particular, whenever u is horizontally differentiable, then its horizontal derivative $\partial_t u : [r,T) \times \widetilde{S} \to \mathbb{R}^k$ is automatically non-anticipative.

5.6 Definition. Let $u : [r,T) \times \widetilde{S} \to \mathbb{R}^k$ be non-anticipative.

(i) We call u *vertically differentiable* at $(t,x) \in [r,T) \times \widetilde{S}$ if the map

$$\mathbb{R}^d \to \mathbb{R}^k, \quad h \mapsto u(t, x + h\mathbb{1}_{[t,T]})$$

is differentiable at 0. Its derivative at 0, called the *vertical derivative* of u at (t,x), is represented by $\partial_x u(t,x)$.

(ii) Let $i \in \{1,\ldots,d\}$. Then u is said to be *partially vertically differentiable in the i-th direction* at $(t,x) \in [r,T) \times \widetilde{S}$ if the map

$$\mathbb{R} \to \mathbb{R}^k, \quad h \mapsto u(t, x + he_i\mathbb{1}_{[t,T]})$$

is differentiable at 0. Its derivative at 0, called the *i-th partial vertical derivative* of u at (t,x), is denoted by $\partial_{x_i} u(t,x)$.

(iii) We say that u is *partially vertically differentiable* at $(t,x) \in [r,T) \times \widetilde{S}$ if it is partially vertically differentiable in every direction at this point.

By saying that u is vertically differentiable or partially vertically differentiable (in the i-th direction for some $i \in \{1,\ldots,d\}$), we demand that u fulfills the corresponding property at each $(t,x) \in [r,T) \times \widetilde{S}$.

Let $u : [r,T) \times \widetilde{S} \to \mathbb{R}^k$ be non-anticipative, $i \in \{1,\ldots,d\}$, and $(t,x) \in [r,T) \times \widetilde{S}$, then $x^t(s) + he_i\mathbb{1}_{[t,T]}(s) = x(s) + he_i\mathbb{1}_{[t,T]}(s)$ for all $s \in [r,t]$ and each $h \in \mathbb{R}$. So, if u is partially vertically differentiable in the i-th direction at (t,x), then the same is true at (t,x^t) and

$$\partial_{x_i} u(t,x) = \lim_{h\to 0} \frac{u(t, x^t + he_i\mathbb{1}_{[t,T]}) - u(t,x)}{h} = \partial_{x_i} u(t,x^t).$$

Hence, if u is partially vertically differentiable in the i-th direction, then its i-th partial vertical derivative $\partial_{x_i} u : [r,T) \times \widetilde{S} \to \mathbb{R}^k$ is non-anticipative. Let us assume that u is vertically differentiable at (t,x). Then the i-th column of $\partial_x u(t,x) \in \mathbb{R}^{k\times d}$, which is exactly $\partial_x u(t,x)e_i$, satisfies

$$\lim_{h\to 0} \frac{u(t, x + he_i\mathbb{1}_{[t,T]}) - u(t,x)}{h} = \partial_x u(t,x)e_i.$$

This forces u to be partially vertically differentiable in the i-th direction at (t,x) and $\partial_{x_i}u(t,x) = \partial_x u(t,x)e_i$. As $i \in \{1,\dots,d\}$ has been arbitrarily chosen, u must be partially vertically differentiable at (t,x) and

$$\partial_x u(t,x) = (\partial_{x_1}u(t,x),\dots,\partial_{x_d}u(t,x)) \in \mathbb{R}^{k\times d}.$$

For this reason, if u is vertically differentiable, then $\partial_x u$ is non-anticipative. Next, let $i,j \in \{1,\dots,d\}$ and suppose that u is partially vertically differentiable in the i-th direction such that $\partial_{x_i}u$ is partially vertically differentiable in the j-th direction at (t,x), then we set

$$\partial_{x_j x_i}u(t,x) := \partial_{x_j}(\partial_{x_i}u)(t,x).$$

For $k=1$ the function u is said to be twice vertically differentiable at (t,x) if its vertical derivative $\partial_x u$ is vertically differentiable there. In this case, we define

$$\partial_{xx}u(t,x) := \partial_x(\partial_x u)(t,x).$$

At last, we call u twice vertically differentiable if it is twice vertically differentiable at each $(t,x) \in [r,T)\times\widetilde{S}$. From Schwarz's lemma, we obtain a rule for interchanging the order of partial vertical derivatives.

5.7 Lemma. *Let $u : [r,T)\times\widetilde{S} \to \mathbb{R}$ be non-anticipative and partially vertically differentiable in the i-th and in the j-th direction for some $i,j \in \{1,\dots,d\}$ with $i \neq j$. If $\partial_{x_i}u$ is partially vertically differentiable in the j-th direction and $\partial_{x_j x_i}u$ is right-continuous with respect to d_∞, then $\partial_{x_j}u$ is partially vertically differentiable in the i-th direction and*

$$\partial_{x_i x_j}u = \partial_{x_j x_i}u.$$

Proof. We choose $(t,x) \in [r,T)\times\widetilde{S}$ and set $x_{a,b} := x + (ae_i + be_j)\mathbb{1}_{[t,T]}$ for all $a,b \in \mathbb{R}$. Let $\varphi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be defined via $\varphi(a,b) := u(t,x_{a,b})$. Then φ is partially differentiable with

$$\frac{\partial\varphi}{\partial a}(a,b) = \partial_{x_i}u(t,x_{a,b}) \quad \text{and} \quad \frac{\partial\varphi}{\partial b}(a,b) = \partial_{x_j}u(t,x_{a,b})$$

for every $a,b \in \mathbb{R}$, because $x_{a+h,b} = x_{a,b} + he_i\mathbb{1}_{[t,T]}$ and $x_{a,b+h} = x_{a,b} + he_j\mathbb{1}_{[t,T]}$ for each $h \in \mathbb{R}$. Moreover, $\frac{\partial\varphi}{\partial a}$ is partially differentiable in the second direction and

$$\frac{\partial^2\varphi}{\partial b\partial a}(a,b) = \partial_{x_j x_i}u(t,x_{a,b}).$$

Due to Corollary 5.3, right-continuity of $\partial_{x_j x_i}u$ with respect to d_∞ implies continuity of $\partial_{x_j x_i}u(t,\cdot)$ relative to $\|\cdot\|$. For this reason, $\frac{\partial^2\varphi}{\partial b\partial a}$ is continuous. By Schwarz's lemma, $\frac{\partial\varphi}{\partial b}$ is partially differentiable in the first direction and

$$\frac{\partial^2\varphi}{\partial a\partial b}(a,b) = \frac{\partial^2\varphi}{\partial b\partial a}(a,b) \quad \text{for all } a,b \in \mathbb{R}.$$

Hence, $\partial_{x_j}u$ is partially vertically differentiable in the i-th direction at $(t,x_{a,b})$ with $\partial_{x_i x_j}u(t,x_{a,b}) = \frac{\partial^2\varphi}{\partial a\partial b}(a,b)$ for all $a,b \in \mathbb{R}$. This yields that $\partial_{x_i x_j}u(t,x) = \frac{\partial^2\varphi}{\partial a\partial b}(0,0)$ $= \frac{\partial^2\varphi}{\partial b\partial a}(0,0) = \partial_{x_j x_i}u(t,x)$. □

As a matter of fact, the lemma entails that for every non-anticipative twice vertically differentiable function $u : [r,T) \times \widetilde{S} \to \mathbb{R}$ whose second-order vertical derivative $\partial_{xx} u$ is right-continuous with respect to d_∞,

$$\partial_{x_j x_i} u = \partial_{x_i x_j} u \quad \text{for all } i,j \in \{1,\dots,d\}.$$

Put differently, in this case, $\partial_{xx} u$ is $\mathbb{S}^d$-valued. In what follows, let us define $C_b^{1,2}([r,T) \times \widetilde{S})$ to be the linear space of all functions $u \in C_b([r,T) \times \widetilde{S})$ that are once horizontally differentiable and twice vertically differentiable such that

$$\partial_t u,\, \partial_{x_i} u,\, \partial_{x_i x_j} u \in C_b([r,T) \times \widetilde{S}) \quad \text{for all } i,j \in \{1,\dots,d\}.$$

Moreover, let $C_b^{1,2}([r,T) \times S)$ denote the linear space of all $u : [r,T) \times S \to \mathbb{R}$ for which there is $\tilde{u} \in C_b^{1,2}([r,T) \times \widetilde{S})$ satisfying $u = \tilde{u}$ on $[r,T) \times S$. The motivation of the latter space comes from the following fact. Let $u \in C_b^{1,2}([r,T) \times S)$ and suppose that $\tilde{u} \in C_b^{1,2}([r,T) \times \widetilde{S})$ is an extension of u to $[r,T) \times \widetilde{S}$. Then it follows from Theorem 2.3 in Fournié [18] and the functional Itô formula in Cont and Fournié [6] that the definitions

$$\partial_t u := \partial_t \tilde{u}, \quad \partial_x u := \partial_x \tilde{u}, \quad \text{and} \quad \partial_{xx} u := \partial_{xx} \tilde{u} \quad \text{on } [r,T) \times S$$

are independent of the choice of the extension $\tilde{u}$. This has already been noted in Ekren, Keller, Touzi, and Zhang [16, Theorem 2.4]. Moreover, we let $C_b^{1,2}([r,T) \times \mathbb{R}^d)$ denote the set of all $v \in C_b([r,T) \times \mathbb{R}^d)$ that are once differentiable in the time variable and twice differentiable in the space variable such that

$$\frac{\partial v}{\partial t}, \frac{\partial v}{\partial x_i}, \frac{\partial^2 v}{\partial x_i \partial x_j} \in C_b([r,T) \times \mathbb{R}^d) \quad \text{for all } i,j \in \{1,\dots,d\}.$$

To conclude this section, we investigate three examples of a non-anticipative function $u : [r,T) \times \widetilde{S} \to \mathbb{R}$ with respect to horizontal and vertical differentiability.

5.8 Examples. (i) Suppose that there is a function $v : [r,T) \times \mathbb{R}^d \to \mathbb{R}$ such that u is of the form

$$u(t,x) = v(t,x(t))$$

for all $(t,x) \in [r,T) \times \widetilde{S}$. If u is horizontally differentiable at a point $(t,x) \in [r,T) \times \widetilde{S}$, then the right-hand time derivative of v at $(t,x(t))$ exists and vice versa. In this case,

$$\partial_t u(t,x) = \lim_{h \downarrow 0} \frac{v(t+h,x(t)) - v(t,x(t))}{h} = \frac{\partial^+ v}{\partial t}(t,x(t)).$$

Moreover, $u(t, x + h\mathbb{1}_{[t,T]}) = v(t, x(t) + h)$ for each $h \in \mathbb{R}^d$. Thus, u is vertically differentiable at (t,x) if and only if the space derivative of v at $(t,x(t))$ exists. In this case,

$$\partial_x u(t,x) = D_x v(t,x(t)) \quad \text{and} \quad \partial_{x_i} u(t,x) = \frac{\partial v}{\partial x_i}(t,x(t))$$

for all $i \in \{1,\dots,d\}$. Therefore, if $v \in C_b^{1,2}([r,T) \times \mathbb{R}^d)$, then $u \in C_b^{1,2}([r,T) \times \widetilde{S})$.

(ii) Let $\alpha, \beta \in C([0,T])$ and assume that there is a locally Lebesgue-integrable function $\varphi \in B(\mathbb{R}^d)$ such that

$$u(t,x) = \int_r^t \alpha(s) + \beta(s)\varphi(x(s))\,ds$$

for every $(t,x) \in [r,T) \times \widetilde{S}$. Then u is horizontally differentiable at every point $(t,x) \in [r,T) \times \widetilde{S}$ and

$$\partial_t u(t,x) = \alpha(t) + \beta(t)\varphi(x(t)).$$

This follows immediately from $u(t+h, x^t) = u(t,x) + \int_t^{t+h} \alpha(s) + \beta(s)\varphi(x(t))\,ds$ for all $h \in [0, T-t)$. In addition, u is twice vertically differentiable with

$$\partial_x u(t,x) = 0 \quad \text{and} \quad \partial_{xx} u(t,x) = 0$$

for each $(t,x) \in [r,T) \times \widetilde{S}$, since $u(t, x + h\mathbb{1}_{[t,T]}) = u(t,x)$ for every $h \in \mathbb{R}^d$. Thus, if we also suppose that $\varphi \in C_b(\mathbb{R}^d)$, then $u \in C_b^{1,2}([r,T) \times \widetilde{S})$.

(iii) Assume that $\varphi \in C(\mathbb{R}^d)$ fails to be differentiable at some point $\overline{x} \in \mathbb{R}^d$ such that $\varphi(\overline{x} + h) > \varphi(\overline{x})$ for all $h \in \mathbb{R}^d \backslash \{0\}$ and let u admit the representation

$$u(t,x) = \sup_{s \in [r,t]} \varphi(x(s))$$

for all $(t,x) \in [r,T) \times \widetilde{S}$. For instance, we could take $\varphi(\overline{y}) = |\overline{y}|$ for all $\overline{y} \in \mathbb{R}^d$, then φ is not differentiable at 0 and $\varphi(h) > 0$ for each $h \in \mathbb{R}^d \backslash \{0\}$. We readily see that u is horizontally differentiable with

$$\partial_t u(t,x) = 0$$

for all $(t,x) \in [r,T) \times \widetilde{S}$. However, u fails to be vertically differentiable at each $(t,x) \in [r,T) \times \widetilde{S}$ with $x(s) = \overline{x}$ for all $s \in [r,t]$, because $u(t, x + h\mathbb{1}_{[t,T]}) = \varphi(\overline{x} + h)$ for all $h \in \mathbb{R}^d$. In consequence, $u \notin C_b^{1,2}([r,T) \times \widetilde{S})$.

5.3 The parabolic terminal value problem

In the sequel, we assume that $a \in B_b([0,T] \times S, \mathbb{S}_+^d)$ and $b \in B_b([0,T] \times S, \mathbb{R}^d)$ are non-anticipative. Here, $\mathbb{S}_+^d$ represents the set of all positive definite matrices in $\mathbb{S}^d$. To the mappings a and b we always associate the linear differential operator $\mathscr{L} : C_b^{1,2}([0,T) \times S) \to B_b([0,T) \times S)$ defined via

$$\mathscr{L}(\varphi)(t,x) := \frac{1}{2}\mathrm{tr}(a(t,x)\partial_{xx}\varphi(t,x)) + \langle b(t,x), \partial_x \varphi(t,x)\rangle. \tag{L}$$

Let $D \subset \mathbb{R}$ be a non-degenerate interval, $f : [0,T] \times S \times D \to \mathbb{R}$ be non-anticipative and $\mathscr{B}([0,T] \times S) \otimes (D \cap \mathscr{B}(\mathbb{R}))$-measurable, and $g \in B_b(S, D)$. In what follows, we analyze the following *semilinear parabolic path-dependent PDE* combined with a terminal value condition:

$$\begin{cases} (\partial_t + \mathscr{L})(u)(t,x) &= f(t,x,u(t,x)) \quad \text{for } (t,x) \in [0,T) \times S, \\ u(T,x) &= g(x) \qquad\qquad\quad \text{for } x \in S. \end{cases} \tag{P}$$

Initially, we recall those classical solutions which together with their horizontal derivatives and their first- and second-order vertical derivatives are bounded. That is, a *classical subsolution* (resp. *supersolution*) to (P) in $C_b^{1,2}([0,T)\times S)$ is a function $u \in C_b^{1,2}([0,T)\times S)\cap C([0,T]\times S, D)$ such that

$$(\partial_t + \mathscr{L})(\varphi)(t,x) \geq \text{(resp. } \leq \text{)}\ f(t,x,u(t,x)) \quad \text{and} \quad u(T,x) \leq \text{(resp. } \geq \text{)}\ g(x)$$

for each $(t,x) \in [0,T)\times S$. Hence, a *classical solution* to (P) in $C_b^{1,2}([0,T)\times S)$ is a function $u \in C_b^{1,2}([0,T)\times S)\cap C([0,T]\times S, D)$ that is a classical sub- and supersolution to (P) in the same space. For existence and uniqueness results for classical solutions, the reader may consult Peng and Wang [29] and Ji and Yang [22]. We intend to utilize classical solutions only to introduce mild solutions.

In this regard, we require the notion of an $\mathscr{L}$-diffusion process that is based on path-dependency. At first, a *path-dependent diffusion process* on some measurable space $(\Omega,\mathscr{F})$ is a triple $\mathscr{X} = (X,(\mathscr{F}_t)_{t\in[0,T]},\mathbb{P})$ that consists of a continuous process $X : [0,T]\times\Omega \to \mathbb{R}^d$, a filtration $(\mathscr{F}_t)_{t\in[0,T]}$ of $\mathscr{F}$ to which X is adapted, and a set $\mathbb{P} = \{P_{r,x}\,|\,(r,x)\in[0,T]\times S\}$ of probability measures on $(\Omega,\mathscr{F})$ such that for the path process of X given by $\hat{X}_t = X^t$ for all $t\in[0,T]$ the triple

$$\hat{\mathscr{X}} := (\hat{X},(\mathscr{F}_t)_{t\in[0,T]},\mathbb{P})$$

is a non-anticipative diffusion process on $(\Omega,\mathscr{F})$ with state space S, as introduced in Section 3.3. Since $\hat{X}$ has continuous paths, this results in the additional requirement that $\hat{\mathscr{X}}$ is a non-anticipative stochastic family that is Borel and satisfies the strong Markov property. That means, the subsequent three conditions hold:

(i) $P_{r,x} = P_{r,x^r}$ and $\hat{X}_r = x^r$ $P_{r,x}$-a.s. for each $(r,x)\in[0,T]\times S$.

(ii) The function $[0,t]\times S \to [0,1]$, $(s,y)\mapsto P_{s,y}(\hat{X}_t\in B)$ is Borel measurable for all $t\in[0,T]$ and each $B\in\mathscr{S}$.

(iii) $P_{r,x}(\hat{X}_t\in B|\mathscr{F}_\tau) = P_{\tau,\hat{X}_\tau}(\hat{X}_t\in B)$ $P_{r,x}$-a.s. for all $r,t\in[0,T]$ with $r\leq t$, each finite $(\mathscr{F}_s)_{s\in[r,t]}$-stopping time τ, every $x\in S$, and all $B\in\mathscr{S}$.

This notion includes in particular the class of path or historical processes used by Dawson and Perkins [7] and Dynkin [12] for constructing historical superprocesses. Furthermore, an *$\mathscr{L}$-diffusion process* is a path-dependent diffusion process $\mathscr{X}$ such that the following additional condition holds: the process $[r,T)\times\Omega\to\mathbb{R}$,

$$(t,\omega)\mapsto \varphi(t,X^t(\omega)) - \int_r^t (\partial_s+\mathscr{L})(\varphi)(s,X^s(\omega))\,ds$$

must be an $(\mathscr{F}_t)_{t\in[r,T)}$-martingale under $P_{r,x}$ for each $(r,x)\in[0,T)\times S$ and every $\varphi\in C_b^{1,2}([0,T)\times S)$. This is what we call the *$\mathscr{L}$-martingale property* of $\mathscr{X}$. The following example explains how standard diffusion processes fit into our framework of $\mathscr{L}$-diffusion processes.

5.9 Example. Assume that there are two mappings $\overline{a} \in B_b([0,T] \times \mathbb{R}^d, \mathbb{S}_+^d)$ and $\overline{b} \in B_b([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ such that $a(t,x) = \overline{a}(t, x(t))$ and $b(t,x) = \overline{b}(t,x(t))$ for every $(t,x) \in [0,T] \times S$. Then with $\overline{a}$ and $\overline{b}$ we can link the linear differential operator $\overline{\mathscr{L}} : C_b^{1,2}([0,T) \times \mathbb{R}^d) \to B_b([0,T) \times \mathbb{R}^d)$ given by

$$\overline{\mathscr{L}}(\varphi)(t,\overline{x}) := \frac{1}{2}\mathrm{tr}(\overline{a}(t,\overline{x})D_x^2\varphi(t,\overline{x})) + \langle \overline{b}(t,\overline{x}), D_x\varphi(t,\overline{x})\rangle.$$

Suppose that there is a set $\overline{\mathbb{P}} = \{\overline{P}_{r,\overline{x}} \,|\, (r,\overline{x}) \in [0,T] \times \mathbb{R}^d\}$ of probability measures on $(S,\mathscr{S})$ for which $(\xi, (\mathscr{S}_t)_{t\in[0,T]}, \overline{\mathbb{P}})$ becomes a canonical $\overline{\mathscr{L}}$-diffusion process in the standard sense. In other words, it is a diffusion process on $(S,\mathscr{S})$ with state space $\mathbb{R}^d$ for which the $\overline{\mathscr{L}}$-martingale property holds, that is, the process $[r,T) \times S \to \mathbb{R}$,

$$(t,x) \mapsto \varphi(t,x(t)) - \int_r^t \left(\frac{\partial}{\partial s} + \overline{\mathscr{L}}\right)(\varphi)(s,x(s))\,ds$$

is always an $(\mathscr{S}_t)_{t\in[r,T)}$-martingale under $\overline{P}_{r,\overline{x}}$ for each $(r,\overline{x}) \in [0,T) \times \mathbb{R}^d$ and every $\varphi \in C_b^{1,2}([0,T) \times \mathbb{R}^d)$. Then for each $(r,x) \in [0,T] \times S$, we let $P_{r,x}$ denote the unique probability measure on $(S,\mathscr{S})$ with $\xi^r = x^r$ $P_{r,x}$-a.s. such that the law of ξ restricted to $[r,T] \times S$ under $\overline{P}_{r,x(r)}$ remains the same under $P_{r,x}$ (cf. Lemma 6.1.1 in Stroock and Varadhan [33]). By setting

$$\mathbb{P} := \{P_{r,x} \,|\, (r,x) \in [0,T] \times S\},$$

it follows that $(\xi, (\mathscr{S}_t)_{t\in[0,T]}, \mathbb{P})$ is a path-dependent diffusion process on $(S,\mathscr{S})$. This procedure appears for instance in the construction of historical superprocesses (see [7], [11], and [12]). Next, since $\overline{a}$ is $\mathbb{S}_+^d$-valued, there is $\overline{\sigma} \in B_b([0,T] \times \mathbb{R}^d, \mathbb{R}^{d\times d})$ such that $\overline{\sigma}(t,\overline{x})$ is an invertible matrix and

$$\overline{a}(t,\overline{x}) = \overline{\sigma}(t,\overline{x})\overline{\sigma}(t,\overline{x})^t$$

for every $(t,\overline{x}) \in [0,T] \times \mathbb{R}^d$. We choose $(r,x) \in [0,T) \times S$ and let $W : [r,T] \times S \to \mathbb{R}^d$ be an $(\mathscr{S}_t)_{t\in[r,T]}$-adapted right-continuous process with $P_{r,x}$-a.s. continuous paths such that

$$W_t = \int_r^t \overline{\sigma}(s,\xi_s)^{-1}\,d\xi_s - \int_r^t \overline{\sigma}(s,\xi_s)^{-1}\overline{b}(s,\xi_s)\,ds \quad \text{for all } t \in [r,T] \quad P_{r,x}\text{-a.s.},$$

then W becomes a standard d-dimensional $(\mathscr{S}_t)_{t\in[r,T]}$-Brownian motion under $P_{r,x}$ (cf. Theorem 4.5.1 in [33]). This simply amounts to $W_r = 0$ $P_{r,x}$-a.s., and $W_t - W_s$ is independent of $\mathscr{S}_s$ and $\mathscr{N}(0,(t-s)\mathbb{I}_d)$-distributed under $P_{r,x}$ for each $s,t \in [r,T]$ with $s \leq t$. Moreover, we obtain that

$$\xi_t = x(r) + \int_r^t \overline{b}(s,\xi_s)\,ds + \int_r^t \overline{\sigma}(s,\xi_s)\,dW_s \quad \text{for all } t \in [r,T] \quad P_{r,x}\text{-a.s.}$$

Consequently, it follows from the functional Itô formula in Cont and Fournié [6] that $(\xi, (\mathscr{S}_t)_{t\in[0,T]}, \mathbb{P})$ is an $\mathscr{L}$-diffusion process on $(S,\mathscr{S})$.

From now on, we suppose that $\mathscr{X}$ is an $\mathscr{L}$-diffusion process on some measurable space $(\Omega, \mathscr{F})$. Then we can accomplish the following. Let u be a classical subsolution (resp. supersolution) to (P) in $C_b^{1,2}([0,T) \times S)$, then

$$E_{r,x}[u(t \wedge \tau, X^{t\wedge\tau})] - u(r,x) = E_{r,x}\bigg[\int_r^{t\wedge\tau} (\partial_s + \mathscr{L})(u)(s, X^s)\,ds\bigg]$$
$$\geq (\text{resp.} \leq)\ E_{r,x}\bigg[\int_r^{t\wedge\tau} f(s, X^s, u(s, X^s))\,ds\bigg]$$

for all $r, t \in [0,T)$ with $r \leq t$, every $(\mathscr{F}_s)_{s\in[r,T]}$-stopping time τ, and each $x \in S$, due to the $\mathscr{L}$-martingale property of $\mathscr{X}$ and optional sampling. Hence, if τ is finite, then we may take the limit $t \uparrow T$ to obtain that

$$E_{r,x}[u(\tau, X^\tau)] - u(r,x) \geq (\text{resp.} \leq)\ E_{r,x}\bigg[\int_r^\tau f(s, X^s, u(s, X^s))\,ds\bigg],$$

by dominated convergence. This motivates notions of mild sub- and supersolutions as well as mild solutions to (P).

5.10 Definition. A *mild subsolution* (resp. *supersolution*) to the parabolic terminal value problem (P) is a non-anticipative function $u \in B([0,T] \times S, D)$ for which

$$|u(\tau, X^\tau)| + \int_r^\tau |f(s, X^s, u(s, X^s))|\,ds$$

is finite and $P_{r,x}$-integrable such that

$$E_{r,x}[u(\tau, X^\tau)] - u(r,x) \geq (\text{resp.} \leq)\ E_{r,x}\bigg[\int_r^\tau f(s, X^s, u(s, X^s))\,ds\bigg]$$

for all $(r,x) \in [0,T] \times S$ and each finite $(\mathscr{F}_t)_{t\in[r,T]}$-stopping time τ. In addition, we require that $u(T,x) \leq (\text{resp.} \geq)\ g(x)$ for all $x \in S$. A *mild solution* to (P) is a function $u \in B([0,T] \times S, D)$ that is a mild sub- and supersolution to (P).

Because $\hat{\mathscr{X}}$ is a non-anticipative diffusion process on $(\Omega, \mathscr{F})$ with state space S, it follows immediately from Lemmas 4.3 and 4.4 that a function $u \in B([0,T] \times S, D)$ is a mild solution to (P) if and only if it is a global solution to the Markovian terminal value problem

$$\mathbb{E}[du(t, \hat{X}_t)] = \mathbb{E}[f(t, \hat{X}_t, u(t, \hat{X}_t))\,dt] \quad \text{for } t \in [0,T], \quad u(T,\cdot) = g.$$

Put differently, u is a mild solution to (P) if and only if $\int_r^T |f(s, X^s, u(s, X^s))|\,ds$ is a finite $P_{r,x}$-integrable function and

$$u(r,x) = E_{r,x}[g(X^T)] - E_{r,x}\bigg[\int_r^T f(s, X^s, u(s, X^s))\,ds\bigg]$$

for all $(r,x) \in [0,T] \times S$. In addition, u is automatically non-anticipative as soon as these two conditions hold. We also get a representation for mild solutions.

5.11 Lemma. *Let u be a mild solution to* (P)*, then $\phi : [0,T] \times S \to \mathbb{R}$ defined via*

$$\phi(r,x) := g(x) - \int_r^T f(s, x^s, u(s,x^s))\, ds$$

is $\mathscr{B}([0,T]) \otimes \mathscr{S}$-measurable and the function $[0,T] \to \mathbb{R}$, $r \mapsto \phi(r,x)$ is continuous for each $x \in S$. Moreover, $E_{r,x}[|\phi(r,X)|] < \infty$ and

$$u(r,x) = E_{r,x}[\phi(r,X)] \quad \text{for all } (r,x) \in [0,T] \times S.$$

Proof. Let $x \in S$, then, since $X = x$ $P_{T,x}$-a.s., there is at least one $\omega \in \Omega$ such that $X(\omega) = x$. By the characterization of a mild solution,

$$\int_r^T |f(s, x^s, u(s,x^s))|\, ds = \int_r^T |f(s, X^s(\omega), u(s, X^s(\omega)))|\, ds < \infty$$

for each $r \in [0,T]$. Thus, ϕ is well-defined and the function $[0,T] \to \mathbb{R}$, $r \mapsto \phi(r,x)$ is continuous for all $x \in S$, according to dominated convergence. As the function

$$[0,T] \times S \to \mathbb{R}, \quad (s,x) \mapsto f(s,x,u(s,x))$$

is in particular $\mathscr{B}([0,T]) \otimes \mathscr{S}$-measurable, Fubini's theorem implies that the function $S \to \mathbb{R}$, $x \mapsto \phi(r,x)$ is Borel measurable for every $r \in [0,T]$. Consequently, ϕ must be $\mathscr{B}([0,T]) \otimes \mathscr{S}$-measurable, by Proposition A.38. Finally, the facts that

$$E_{r,x}[|\phi(r,X)|] < \infty \quad \text{and} \quad u(r,x) = E_{r,x}[\phi(r,X)]$$

for all $(r,x) \in [0,T] \times S$ also follow from the characterization of a mild solution, which concludes the proof. □

We let λ be the Lebesgue measure on $[0,T]$, and set $\underline{d} := \inf D$ and $\overline{d} := \sup D$, then Theorem 4.29 directly entails an existence and uniqueness result for bounded mild solutions.

5.12 Corollary. *Let $f \in BC_\lambda^{1-}([0,T] \times S \times D)$ and suppose that the following two conditions hold:*

(i) If $\underline{d} > -\infty$ (resp. $\overline{d} < \infty$), then f is both locally λ-bounded and locally Lipschitz λ-continuous at $\underline{d}$ (resp. $\overline{d}$) with $\lim_{z \downarrow \underline{d}} f(\cdot,x,z) \leq 0$ (resp. $\lim_{z \uparrow \overline{d}} f(\cdot,x,z) \geq 0$) for all $x \in S$ λ-a.s.

(ii) If $\underline{d} = -\infty$ (resp. $\overline{d} = \infty$), then f is affine λ-bounded from above (resp. from below).

Then there is a unique bounded mild solution u to (P)*. Moreover, whenever $\hat{\mathscr{X}}$ has the (right-hand) Feller property, f is right-continuous, and $g \in C_b(S)$, then u must be (right-)continuous.*

At last, Corollary 4.30 yields a Feynman-Kac formula for bounded mild solutions.

5.13 Corollary. *Suppose that there are two λ-dominated $\alpha, \beta \in B([0,T] \times S)$ with $f(t,x,z) = \alpha(t,x) + \beta(t,x)z$ for all $(t,x,z) \in [0,T] \times S \times D$ such that if $\underline{d} > -\infty$ (resp. $\overline{d} < \infty$), then*

$$\alpha(\cdot, x) + \beta(\cdot, x)\underline{d} \leq 0 \quad (\textit{resp. } \alpha(\cdot, x) + \beta(\cdot, x)\overline{d} \geq 0)$$

for each $x \in S$ λ-a.s. Then the unique bounded mild solution u to (P) *admits the representation*

$$u(r,x) = E_{r,x}\left[e^{-\int_r^t \beta(s,X^s)\,ds} g(X^T)\right] - E_{r,x}\left[\int_r^T e^{-\int_r^t \beta(s,X^s)\,ds} \alpha(t, X^t)\,dt\right]$$

for each $(r,x) \in [0,T] \times S$. Furthermore, if $\hat{\mathscr{X}}$ is (right-hand) Feller, α and β are right-continuous, and $g \in C_b(S)$, then u is (right-)continuous.

5.4 Notions of viscosity solutions

Since we intend to compare a variety of notions of viscosity solutions, we present several test function spaces. Let $\mathscr{T}$ denote the set of all $(\mathscr{F}_t)_{t\in[0,T]}$-stopping times τ for which there exists a lower semicontinuous function $\phi : S \to [0,T]$ such that

$$\tau(\omega) = \phi(X(\omega)) \quad \text{for all } \omega \in \Omega. \tag{5.3}$$

It follows from the lemma below that if $\mathscr{X}$ is canonical, that is, $(\Omega, \mathscr{F}) = (S, \mathscr{S})$, $X = \xi$, and $\mathscr{F}_t = \mathscr{S}_t$ for all $t \in [0,T]$, then our definition of $\mathscr{T}$ reduces to that in Ekren, Keller, Touzi, and Zhang [16]. Let use here $\hat{X}_T$ instead of X, which should make clear that $\{\hat{X}_T \in B\} = \{\omega \in \Omega \,|\, X(\omega) \in B\}$ for all $B \in \mathscr{S}$.

5.14 Lemma. *A finite $(\mathscr{F}_t)_{t\in[0,T]}$-stopping time τ is a member of $\mathscr{T}$ if and only if for each $t \in [0,T)$ there is an open set O_t in S such that $\{\tau > t\} = \{\hat{X}_T \in O_t\}$.*

Proof. Suppose first that $\tau \in \mathscr{T}$, then $\tau(\omega) = \phi(\hat{X}_T(\omega))$ for each $\omega \in \Omega$ and some lower semicontinuous function $\phi : S \to [0,T]$. We choose $t \in [0,T)$ and see that $\{\tau > t\} = \{\hat{X}_T \in O_t\}$ with $O_t := \phi^{-1}((t,T])$. Since $(t,T]$ is open in $[0,T]$, it follows from Lemma A.11 that O_t is open in S.

Conversely, assume that for each $t \in [0,T)$ there is an open set O_t in S with $\{\tau > t\} = \{\hat{X}_T \in O_t\}$. Then τ is measurable with respect to the σ-field $\sigma(\hat{X}_T)$. Hence, Corollary A.25 provides a function $\phi \in B(S, [0,T])$ satisfying (5.3). Clearly, for each $x \in S$ there is at least one $\omega \in \Omega$ such that $\hat{X}_T(\omega) = x$, since $\hat{X}_T = x$ $P_{T,x}$-a.s. Thus, the mapping $\hat{X}_T : \Omega \to S$ is onto and therefore,

$$\phi^{-1}((t,T]) = \hat{X}_T(\{\tau > t\}) = \hat{X}_T(\{\hat{X}_T \in O_t\}) = O_t$$

for all $t \in [0,T)$, which entails the lower semicontinuity of ϕ, by Lemma A.11. □

Our main example of a stopping time in $\mathscr{T}$ is a hitting time. Therefore, some considerations with respect to lower semicontinuity have to be made.

5.15 Lemma. *Let $r \in [0,T]$, $u \in C([r,T]\times S)$, and I be a closed interval in $\mathbb{R}$. Then the function $\phi : S \to [r,T] \cup \{\infty\}$ defined via $\phi(x) := \inf\{t \in [r,T] \,|\, u(t,x^t) \in I\}$ is lower semicontinuous.*

Proof. Let $x \in S$ and $\varepsilon > 0$. For $\phi(x) = r$ it holds that $\phi(y) \geq r > \phi(x) - \varepsilon$ for all $y \in S$. Suppose instead that $r < \phi(x) < \infty$. If $\varepsilon > \phi(x) - r$, then $\phi(y) \geq r > \phi(x) - \varepsilon$ for each $y \in S$. Hence, let $\varepsilon \leq \phi(x) - r$. By Corollary 5.4, the function $[r, \phi(x) - \varepsilon] \to \mathbb{R}$, $t \mapsto u(t,x^t)$ is continuous and so,

$$K := \{u(t,x^t) \,|\, t \in [r, \phi(x) - \varepsilon]\}$$

is a compact set in I^c. A combination of Lemma A.13 with Corollary A.16 gives $\eta > 0$ such that the η-neighborhood $N_\eta(K)$ of K is relatively compact in the open set I^c. Now, Corollary 5.4 also yields $\delta > 0$ such that

$$|u(t,y^t) - u(t,x^t)| < \eta$$

for all $t \in [r, \phi(x) - \varepsilon]$ and each $y \in S$ with $\rho(y,x) < \delta$. So, we choose $y \in S$ with $\rho(y,x) < \delta$, then $u(t,y^t) \in N_\eta(K)$ for all $t \in [r, \phi(x) - \varepsilon]$. As $\phi(y) = \phi(x) - \varepsilon$ would yield a contradiction, we get that $\phi(y) > \phi(x) - \varepsilon$.

Eventually, let $\phi(x) = \infty$, then $u(t,x^t) \notin I$ for all $t \in [r,T]$. Similarly as before, because $K := \{u(t,x^t) \,|\, t \in [r,T]\}$ is a compact set in I^c, there is $\eta > 0$ such that $N_\eta(K)$ is relatively compact in I^c. We let $\delta > 0$ be such that $|u(t,y^t) - u(t,x^t)| < \eta$ for all $t \in [r,T]$ and each $y \in S$ with $\rho(y,x) < \delta$. Then $\phi(y) = \phi(x) = \infty$ for all $y \in S$ with $\rho(y,x) < \delta$. This verifies the lemma. □

5.16 Example. Let $r,t \in [0,T]$ with $r \leq t$, $u \in C([r,T] \times S)$, and I be a closed interval in $\mathbb{R}$. Then

$$\tau := \inf\{s \in [r,T] \,|\, u(s,X^s) \in I\} \wedge t \in \mathscr{T}.$$

In fact, as the process $[r,T] \times \Omega \to \mathbb{R}$, $(s,\omega) \mapsto u(s,X^s(\omega))$ is $(\mathscr{F}_s)_{s\in[r,T]}$-adapted and continuous, Proposition A.42 entails that τ is a finite $(\mathscr{F}_t)_{t\in[r,T]}$-stopping time. Hence, the assertion follows from Lemmas 5.15 and A.11.

For every $(r,x) \in [0,T) \times S$ and each non-anticipative function $u \in B_b([0,T) \times S)$, we define $\underline{\mathscr{SP}}\, u(r,x)$ to be the set of all $\varphi \in C_b^{1,2}([0,T) \times S)$ for which there is an $(\mathscr{F}_t)_{t\in[r,T]}$-stopping time τ with $P_{r,x}(\tau > r) > 0$ such that

$$(u - \varphi)(r,x) \geq E_{r,x}[(u - \varphi)(\tilde{\tau} \wedge \tau, X^{\tilde{\tau}\wedge\tau})]$$

for every $\tilde{\tau} \in \mathscr{T}$ with $\tilde{\tau} \in [r, r+\delta)$ and some $\delta \in (0, T-r)$. In addition, we set $\overline{\mathscr{SP}}\, u(r,x) := -\underline{\mathscr{SP}}\,(-u)(r,x)$. Let $\underline{\mathscr{P}}\, u(r,x)$ be the set of all $\varphi \in C_b^{1,2}([0,T) \times S)$ such that $u - \varphi$ has a right-hand local maximum at (r,x) in the sense that

$$(u - \varphi)(r,x) \geq (u - \varphi)(s,y)$$

for all $(s,y) \in [r,T) \times S$ with $d_S((s,y),(r,x)) < \delta$ and some $\delta \in (0, T-r)$. Moreover, we set $\overline{\mathscr{P}}\, u(r,x) := -\underline{\mathscr{P}}(-u)(r,x)$.

5.17 Definition. Let $u \in B_b([0,T] \times S, D)$ be non-anticipative.

(i) We call u a *stochastic viscosity subsolution* (resp. *supersolution*) to (P) if for every $(r,x) \in [0,T) \times S$ and each $\varphi \in \underline{\mathscr{S}\mathscr{P}}\, u(r,x)$ (resp. $\varphi \in \overline{\mathscr{S}\mathscr{P}}\, u(r,x)$),

$$(\partial_r + \mathscr{L})(\varphi)(r,x) \geq \text{(resp. } \leq) \; f(r,x,u(r,x)) \quad \text{and} \quad u(T,x) \leq \text{(resp. } \geq) \; g(x).$$

Moreover, u is said to be a *stochastic viscosity solution* to (P) if it is a stochastic viscosity sub- and supersolution to (P).

(ii) We say that u is a *right-hand viscosity subsolution* (resp. *supersolution*) to (P) if for all $(r,x) \in [0,T) \times S$ and each $\varphi \in \underline{\mathscr{P}}\, u(r,x)$ (resp. $\varphi \in \overline{\mathscr{P}}\, u(r,x)$),

$$(\partial_r + \mathscr{L})(\varphi)(r,x) \geq \text{(resp. } \leq) \; f(r,x,u(r,x)) \quad \text{and} \quad u(T,x) \leq \text{(resp. } \geq) \; g(x).$$

Furthermore, u is a *right-hand viscosity solution* to (P) if it is a right-hand viscosity sub- and supersolution to (P).

As we will show, each stochastic viscosity subsolution (resp. supersolution) is a right-hand viscosity subsolution (resp. supersolution). To discuss the relations between the notion of a viscosity solution in [16] and the above definition, we fix $(r,x) \in [0,T) \times S$ and $L \geq 0$, and let $\mathscr{U}_r^L$ denote the set of all $(\mathscr{F}_t)_{t\in[r,T]}$-progressively measurable processes $\beta : [r,T] \times \Omega \to \mathbb{R}^d$ for which each coordinate function is bounded by L.

For every $\beta \in \mathscr{U}_r^L$, we choose an $(\mathscr{F}_t)_{t\in[r,T]}$-progressively measurable process $M^{r,\beta} : [r,T] \times \Omega \to (0,\infty)$ that has right-continuous and $P_{r,x}$-a.s. continuous paths such that

$$M_t^{r,\beta} = \exp\left(\int_r^t \beta_s \, dX_s - \int_r^t \langle b(s,X^s), \beta_s\rangle \, ds - \frac{1}{2}\int_r^t \langle \beta_s, a(s,X^s)\beta_s\rangle \, ds\right)$$

for all $t \in [r,T]$ $P_{r,x}$-a.s., then $M^{r,\beta}$ becomes an $(\mathscr{F}_t)_{t\in[r,T]}$-martingale under $P_{r,x}$ with $E_{r,x}[M_T^{r,\beta}] = 1$, due to Itô's formula. For each non-anticipative $u \in B_b([0,T) \times S)$, we let $\underline{\mathscr{A}}^L u(r,x)$ represent the set of all $\varphi \in C_b^{1,2}([0,T) \times S)$ for which there are $\delta \in (0, T-r)$ and $\tau \in \mathscr{T}$ with $\tau > r$ $P_{r,x}$-a.s. such that

$$(u-\varphi)(r,x) \geq E_{r,x}[M_T^{r,\beta}(u-\varphi)(\tilde{\tau}\wedge\tau, X^{\tilde{\tau}\wedge\tau})]$$

for each $\tilde{\tau} \in \mathscr{T}$ with $\tilde{\tau} \in [r, r+\delta)$ and all $\beta \in \mathscr{U}_r^L$. Let $\overline{\mathscr{A}}^L u(r,x) := -\underline{\mathscr{A}}^L(-u)(r,x)$, as before. This translates the concepts and spaces of test functions used for the definition of a viscosity solution in [16] to our current setting. So, a non-anticipative $u \in B_b([0,T] \times S, D)$ is a viscosity subsolution (resp. supersolution) to (P) in the sense of [16] if there is $L \geq 0$ such that

$$(\partial_r + \mathscr{L})(\varphi)(r,x) \geq \text{(resp. } \leq) \; f(r,x,u(r,x)) \quad \text{and} \quad u(T,x) \leq \text{(resp. } \geq) \; g(x)$$

for all $(r,x) \in [0,T) \times S$ and each $\varphi \in \underline{\mathscr{A}}^L u(r,x)$ (resp. $\varphi \in \overline{\mathscr{A}}^L u(r,x)$). Accordingly, u is a viscosity solution to (P) in the sense of the paper if it is a viscosity sub- and supersolution to (P) in the same sense.

In comparison to [16], where d_∞ is used and only continuous functions u are considered, the choice of d_S should have a negligible effect on the sizes of the test functions spaces that we use for our Definition 5.17 of viscosity solutions. Finally, the lemma below concludes our discussion on the notions of viscosity solutions. Note that the second assertion remains true if solution is either replaced by sub- or supersolution.

5.18 Lemma. *Let $(r,x) \in [0,T) \times S$, $u \in B_b([0,T) \times S)$ be non-anticipative, and $L \geq 0$, then $\underline{\mathscr{P}}\, u(r,x) \subset \underline{\mathscr{A}}^L u(r,x) \subset \underline{\mathscr{SP}}\, u(r,x)$. In particular, each stochastic viscosity solution to* (P) *is a viscosity solution in the sense of [16] and every such solution is a right-hand viscosity solution.*

Proof. As the second assertion is an immediate consequence of the first, we only show the first claim. The inclusion $\underline{\mathscr{A}}^0\, u(r,x) \subset \underline{\mathscr{SP}}\, u(r,x)$ follows from $M_T^{r,0} = 1$ $P_{r,x}$-a.s. We notice that if $L' \geq 0$ is such that $L' \leq L$, then $\mathscr{U}_r^{L'} \subset \mathscr{U}_r^L$, which in turn gives us that $\underline{\mathscr{A}}^L\, u(r,x) \subset \underline{\mathscr{A}}^{L'}\, u(r,x)$. Hence,

$$\underline{\mathscr{A}}^L\, u(r,x) \subset \underline{\mathscr{A}}^0\, u(r,x) \subset \underline{\mathscr{SP}}\, u(r,x).$$

It remains to verify that $\underline{\mathscr{P}}\, u(r,x) \subset \underline{\mathscr{A}}^L\, u(r,x)$. Thus, let $\varphi \in \underline{\mathscr{P}}\, u(r,x)$. Then $(u-\varphi)(r,x) \geq (u-\varphi)(s,y)$ for all $(s,y) \in [r,T) \times S$ with $d_S((s,y),(r,x)) < \delta$ and some $\delta \in (0, T-r)$. We define $\tau := \inf\{t \in [r,T] \,|\, \|X^t - x^r\| \geq \delta/3\} \wedge (r + \delta/2)$, then $\tau \in \mathscr{T}$ and $\tau > r$ $P_{r,x}$-a.s., by Example 5.16. Let $\tilde{\tau} \in \mathscr{T}$ with $\tilde{\tau} \geq r$, then

$$d_S((\tilde{\tau} \wedge \tau, X^{\tilde{\tau}\wedge\tau}), (r,x)) \leq d_S((\tau, X^\tau), (r,x)) \leq \delta/2 + \|X^\tau - x^r\| < \delta$$

on $\{X^r = x^r\}$. Hence, $(u-\varphi)(r,x) \geq (u-\varphi)(\tilde{\tau} \wedge \tau, X^{\tilde{\tau}\wedge\tau})$ on the same set. Let $\beta \in \mathscr{U}_r^L$, then, as $M^{r,\beta}$ is positive and $E_{r,x}[M_T^{r,\beta}] = 1$, we get that

$$(u-\varphi)(r,x) = E_{r,x}[M_T^{r,\beta}(u-\varphi)(r,x)] \geq E_{r,x}[M_T^{r,\beta}(u-\varphi)(\tilde{\tau} \wedge \tau, X^{\tilde{\tau}\wedge\tau})].$$

This justifies that $\varphi \in \underline{\mathscr{A}}^L\, u(r,x)$, which completes the proof. □

5.5 Relation between mild and viscosity solutions

Here, we prove that if the maps a and b, and the function f are right-continuous, then every bounded mild solution to (P) that is right-continuous on $[0,T) \times S$ is a stochastic viscosity solution. In this connection, we look more closely at the case that

$$f(t,x,z) = \alpha(t,x) + \beta(t,x)z$$

for all $(t,x,z) \in [0,T] \times S \times D$ and some right-continuous $\alpha, \beta \in B([0,T] \times S)$ such that $\alpha(\cdot,x)$ and $\beta(\cdot,x)$ are Lebesgue-integrable for all $x \in S$. In this specific case, each bounded mild solution is a stochastic viscosity solution regardless of whether it is right-continuous on $[0,T) \times S$. If additionally D is closed and $\mathscr{X}$ satisfies a reasonable topological condition, we can identify the upper and lower right-hand semicontinuous envelope of a bounded mild solution to (P) as right-hand viscosity sub- and supersolution, respectively. Thus, let us begin with a crucial limit inequality.

5.19 Lemma. *Let $(r,x) \in [0,T) \times S$ and τ be an $(\mathscr{F}_t)_{t\in[r,T]}$-stopping time. Assume that $\varphi \in B([r,T) \times S)$ is non-anticipative and the following two conditions hold:*

(i) $\int_r^{t\wedge\tau} |\varphi(s, X^s)|\, ds$ *is finite and* $P_{r,x}$*-integrable for all* $t \in [r,T)$.

(ii) There are $\zeta \in (0, T-r)$ *and* $c \geq 0$ *so that* $|\varphi(s,X^s)| \leq c$ *for all* $s \in [r,(r+\zeta)\wedge\tau]$ $P_{r,x}$*-a.s.*

If φ is upper right-hand semicontinuous at (r,x), then

$$\limsup_{t\downarrow r} E_{r,x}\bigg[\int_r^{t\wedge\tau} \frac{\varphi(s,X^s)}{t-r}\, ds\bigg] \leq \varphi(r,x) P_{r,x}(\tau > r).$$

Proof. Let $\varepsilon > 0$ and $\omega \in \{X^r = x^r\} \cap \{\tau > r\}$. Then there exists $\delta > 0$ such that $\varphi(s,y) < \varphi(r,x) + \varepsilon$ for every $(s,y) \in [r,T) \times S$ with $d_S((s,y),(r,x)) < \delta$. Since $X(\omega)$ is right-continuous, there is $\gamma \in (0, T-r)$ such that $\|X^s(\omega) - x^r\| < \delta/2$ for each $s \in [r, r+\gamma)$. We define $\eta := \gamma \wedge (\delta/2) \wedge (\tau(\omega) - r)$, then

$$\int_r^{t\wedge\tau(\omega)} \frac{\varphi(s,X^s(\omega))}{t-r}\, ds \leq \varphi(r,x) + \varepsilon$$

for every $t \in (r, r+\eta)$, because we can use that $t < \tau(\omega)$ and $d_S((s,X^s(\omega)),(r,x))$ $= (s-r) + \|X^s(\omega) - x^r\| < \delta$ for all $s \in [r,t]$. Thus, we have shown that

$$\limsup_{t\downarrow r} \int_r^{t\wedge\tau} \frac{\varphi(s,X^s)}{t-r}\, ds \leq \varphi(r,x) \quad P_{r,x}\text{-a.s. on } \{\tau > r\}.$$

Because $\int_r^{t\wedge\tau} |\varphi(s,X^s)|\, ds \leq c(t-r)$ for each $t \in [r, r+\zeta]$ $P_{r,x}$-a.s., the claim follows from Fatou's lemma. □

This produces our first announced result.

5.20 Theorem. *Suppose that a, b, and f are right-continuous. Then every bounded mild subsolution (resp. supersolution) u to* (P) *that is right-continuous on* $[0,T) \times S$ *is a stochastic viscosity subsolution (resp. supersolution) to* (P).

Proof. We consider the case that u is a mild subsolution. Let $(r,x) \in [0,T) \times S$ and $\varphi \in \underline{\mathscr{SP}}\, u(r,x)$. Then there are $\delta \in (0, T-r)$ and some $(\mathscr{F}_t)_{t\in[r,T]}$-stopping time τ with $P_{r,x}(\tau > r) > 0$ such that

$$(u-\varphi)(r,x) \geq E_{r,x}[(u-\varphi)(\tilde{\tau}\wedge\tau, X^{\tilde{\tau}\wedge\tau})] \tag{5.4}$$

for each $\tilde{\tau} \in \mathscr{T}$ with $\tilde{\tau} \in [r, r+\delta)$. We note that, as the function $[r,T) \times S \to \mathbb{R}$, $(s,y) \mapsto f(s,y^s,u(s,y^s))$ is right-continuous, it must be right-hand locally bounded at (r,x). That is, there are $c \geq 0$ and $\gamma \in (0,\delta]$ such that $|f(s,y,u(s,y))| \leq c$ for each $(s,y) \in [r,T) \times S$ with $d_S((s,y),(r,x)) < \gamma$. Then

$$\tilde{\tau} := \inf\{t \in [r,T] \,|\, \|X^t - x^r\| \geq \gamma/2\} \wedge T \in \mathscr{T},$$

by Example 5.16, and $d_S((s, X^s(\omega)), (r, x)) < \gamma$ for all $\omega \in \{X^r = x^r\}$ and each $s \in [r, (r + \gamma/3) \wedge \tilde{\tau}(\omega)]$. We set $\hat{\tau} := \tilde{\tau} \wedge \tau$, then the $\mathscr{L}$-martingale property of $\mathscr{X}$ and optional sampling entail that the stopped process $[r, T) \times \Omega \to \mathbb{R}$,

$$(t, \omega) \mapsto \varphi(t \wedge \hat{\tau}(\omega), X^{t\wedge\hat{\tau}}(\omega)) - \int_r^{t\wedge\hat{\tau}(\omega)} (\partial_s + \mathscr{L})(\varphi)(s, X^s(\omega))\, ds$$

is an $(\mathscr{F}_t)_{t\in[r,T)}$-martingale under $P_{r,x}$. Moreover, because u is a mild subsolution to (P), it follows that

$$E_{r,x}[(u - \varphi)(t \wedge \hat{\tau}, X^{t\wedge\hat{\tau}})] \geq (u - \varphi)(r, x) + E_{r,x}\left[\int_r^{t\wedge\hat{\tau}} f(s, X^s, u(s, X^s))\, ds\right] - E_{r,x}\left[\int_r^{t\wedge\hat{\tau}} (\partial_s + \mathscr{L})(\varphi)(s, X^s)\, ds\right]$$

for all $t \in [r, T)$. Hence, we obtain from (5.4) that

$$\frac{1}{t-r} E_{r,x}\left[\int_r^{t\wedge\hat{\tau}} (\partial_s + \mathscr{L})(\varphi)(s, X^s)\, ds\right] \geq \frac{1}{t-r} E_{r,x}\left[\int_r^{t\wedge\hat{\tau}} f(s, X^s, u(s, X^s))\, ds\right]$$

for each $t \in (r, r+\gamma/3)$. Since the Borel measurable bounded function $[r, T)\times S \to \mathbb{R}$, $(s, y) \mapsto (\partial_s + \mathscr{L})(\varphi)(s, y^s)$ is right-continuous and $|f(s, X^s, u(s, X^s))| \leq c$ for all $s \in [r, (r + \gamma/3) \wedge \hat{\tau}]$ $P_{r,x}$-a.s., Lemma 5.19 allows us to take the limit $t \downarrow r$, which establishes that

$$(\partial_r + \mathscr{L})(\varphi)(r, x) \geq f(r, x, u(r, x)).$$

Thus, u is a stochastic viscosity subsolution to (P). Eventually, if u is a mild supersolution, then similar arguments yield that it is also a stochastic viscosity supersolution. □

A combination of Theorem 5.20 with Corollary 5.12 gives an existence result for stochastic viscosity solutions. Here, as usually, λ is the Lebesgue measure on $[0, T]$.

5.21 Corollary. *Assume that $\hat{\mathscr{X}}$ has the (right-hand) Feller property, and a, b, and f are right-continuous. Let $f \in BC_\lambda^{1-}([0, T] \times S \times D)$ and $g \in C_b(S)$, and suppose that the following two conditions hold:*

(i) If $\underline{d} > -\infty$ (resp. $\overline{d} < \infty$), then f is both locally λ-bounded and locally Lipschitz λ-continuous at $\underline{d}$ (resp. $\overline{d}$) with $\lim_{z\downarrow\underline{d}} f(\cdot, x, z) \leq 0$ (resp. $\lim_{z\uparrow\overline{d}} f(\cdot, x, z) \geq 0$) for all $x \in S$ λ-a.s.

(ii) If $\underline{d} = -\infty$ (resp. $\overline{d} = \infty$), then f is affine λ-bounded from above (resp. from below).

Then there is a bounded (right-)continuous stochastic viscosity solution to (P).

From now on, we let $\alpha, \beta \in B([0, T] \times S)$ be two non-anticipative functions such that $\alpha(\cdot, x)$ and $\beta(\cdot, x)$ are Lebesgue-integrable for each $x \in S$ and f is of the form

$$f(t, x, z) = \alpha(t, x) + \beta(t, x)z \quad \text{for all } (t, x, z) \in [0, T] \times S \times D.$$

Then we can verify another limit equality without assuming right-continuity of the mild solution in question.

5.22 Lemma. *Let $(r,x) \in [0,T) \times S$ and τ be an $(\mathscr{F}_t)_{t\in[r,T]}$-stopping time. Suppose that β is right-continuous at (r,x), and there are $\zeta \in (0,T-r)$ and $c \geq 0$ such that $|\beta(s,X^s)| \leq c$ for all $s \in [r,(r+\zeta)\wedge\tau]$ $P_{r,x}$-a.s. Then each mild solution u to* (P) *fulfills*

$$\lim_{t\downarrow r} E_{r,x}\bigg[\int_r^{t\wedge\tau} \frac{\beta(s,X^s)}{t-r} u(s,X^s)\,ds\bigg] = \beta(r,x)u(r,x)P_{r,x}(\tau > r).$$

Proof. By Lemma 5.11, the $\mathscr{B}([0,T])\otimes\mathscr{S}$-measurable function $\phi : [0,T]\times S \to \mathbb{R}$ defined by

$$\phi(s,y) := g(y) - \int_s^T \alpha(t,y^t) + \beta(t,y^t)u(t,y^t)\,dt$$

satisfies $E_{s,y}[|\phi(s,X)|] < \infty$ and $u(s,y) = E_{s,y}[\phi(s,X)]$ for each $(s,y) \in [0,T]\times S$. In addition, the function $[0,T] \to \mathbb{R}$, $s \mapsto \phi(s,y)$ is continuous for all $y \in S$. For this reason, the Borel measurable function $[0,T] \to \mathbb{R}$, $s \mapsto \beta(s,X^s(\omega))\phi(s,X(\omega))$ is Lebesgue-integrable for each $\omega \in \Omega$. Moreover,

$$\begin{aligned}\int_r^{t\wedge\tau} |\beta(s,X^s)\phi(s,X)|\,ds &\leq c(t-r)|g(X^T)| \\ &\quad + c(t-r)\int_r^T |\alpha(s,X^s) + \beta(s,X^s)u(s,X^s)|\,ds \end{aligned} \tag{5.5}$$

for all $t \in [r,r+\zeta]$ $P_{r,x}$-a.s. As the right-hand expression is finite and $P_{r,x}$-integrable, and $E_{r,x}[|\phi(s,X)|] < \infty$ for every $s \in [r,T]$, it follows from Fubini's theorem and Proposition 3.7 that

$$\begin{aligned} E_{r,x}\bigg[\int_r^{t\wedge\tau} |\beta(s,X^s)|E_{s,X^s}[|\phi(s,X)|]\,ds\bigg] &= \int_r^t E_{r,x}[|\beta(s,X^s)||\phi(s,X)|\mathbb{1}_{\{\tau>s\}}]\,ds \\ &= E_{r,x}\bigg[\int_r^{t\wedge\tau} |\beta(s,X^s)\phi(s,X)|\,ds\bigg] < \infty \end{aligned}$$

for each $t \in [r,r+\zeta]$. Because $|u(s,X^s)| \leq E_{s,X^s}[|\phi(s,X)|]$ for all $s \in [r,T]$, another application of Fubini's theorem and Proposition 3.7 yield that

$$\begin{aligned} E_{r,x}\bigg[\int_r^{t\wedge\tau} \beta(s,X^s)u(s,X^s)\,ds\bigg] &= \int_r^t E_{r,x}[\beta(s,X^s)\phi(s,X)\mathbb{1}_{\{\tau>s\}}]\,ds \\ &= E_{r,x}\bigg[\int_r^{t\wedge\tau} \beta(s,X^s)\phi(s,X)\,ds\bigg] \end{aligned}$$

for every $t \in [r,r+\zeta]$. The next step of the proof is to choose some $P_{r,x}$-null set $N \in \mathscr{F}$ such that $|\beta(s,X^s(\omega))| \leq c$ for all $\omega \in N^c$ and each $s \in [r,(r+\zeta)\wedge\tau(\omega)]$. We let $\varepsilon > 0$ and $\omega \in N^c \cap \{X^r = x^r\} \cap \{\tau > r\}$. Then the right-continuity of β at (r,x) yields some $\delta > 0$ such that $|\phi(r,X(\omega))||\beta(s,y) - \beta(r,x)| < \varepsilon/2$ for all $(s,y) \in [r,T)\times S$ with $d_S((s,y),(r,x)) < \delta$. Since $X(\omega)$ and the function $[0,T] \to \mathbb{R}$, $s \mapsto \phi(s,X(\omega))$ are right-continuous, we get $\gamma \in (0,T-r)$ such that

$$\|X^s(\omega) - x^r\| < \delta/2 \quad \text{and} \quad c|\phi(s,X(\omega)) - \phi(r,X(\omega))| < \varepsilon/2$$

for each $s \in [r, r+\gamma)$. Consequently, $|\beta(s, X^s(\omega))\phi(s, X(\omega)) - \beta(r,x)\phi(r, X(\omega))| \leq c|\phi(s, X(\omega)) - \phi(r, X(\omega))| + |\phi(r, X(\omega))||\beta(s, X^s(\omega)) - \beta(r,x)| < \varepsilon$ for every $s \in [r, r+\eta)$, where $\eta := \gamma \wedge (\delta/2) \wedge \zeta \wedge (\tau(\omega) - r)$. If in addition we use that

$$\beta(r,x)\phi(r, X(\omega)) = \int_r^{t\wedge\tau(\omega)} \frac{\beta(r,x)}{t-r}\phi(r, X(\omega))\, ds$$

for each $t \in (r, r+\eta)$, then our considerations show that

$$\left| \int_r^{t\wedge\tau(\omega)} \frac{\beta(s, X^s(\omega))}{t-r}\phi(s, X(\omega))\, ds - \beta(r,x)\phi(r, X(\omega)) \right| < \varepsilon$$

for all $t \in (r, r+\eta)$. Therefore, we have proven that

$$\lim_{t\downarrow r} \int_r^{t\wedge\tau} \frac{\beta(s, X^s)}{t-r}\phi(s, X)\, ds = \beta(r,x)\phi(r, X) \quad P_{r,x}\text{-a.s. on } \{\tau > r\}.$$

Because $E_{r,x}[\phi(r,X)\mathbb{1}_{\{\tau>r\}}] = E_{r,x}[E_{r,x}[\phi(r,X)|\mathscr{F}_r]\mathbb{1}_{\{\tau>r\}}] = u(r,x)P_{r,x}(\tau > r)$ and (5.5) holds, the claim follows from dominated convergence. □

We are now in a position to drop the right-continuity assumption for bounded mild solutions to become stochastic viscosity solutions.

5.23 Proposition. *Suppose that a, b, α, and β are right-continuous. Then each bounded mild solution u to* (P) *is a stochastic viscosity solution to* (P).

Proof. We proceed similarly as in the proof of Theorem 5.20. Let $(r,x) \in [0,T) \times S$ and $\varphi \in \underline{\mathscr{SP}}\, u(r,x)$. Then there exist $\delta \in (0, T-r)$ and an $(\mathscr{F}_t)_{t\in[r,T]}$-stopping time τ with $P_{r,x}(\tau > r) > 0$ such that

$$(u - \varphi)(r,x) \geq E_{r,x}[(u-\varphi)(\tilde{\tau} \wedge \tau, X^{\tilde{\tau}\wedge\tau})]$$

for every $\tilde{\tau} \in \mathscr{T}$ with $\tilde{\tau} \in [r, r+\delta)$. Let us choose $c \geq 0$ and $\gamma \in (0, \delta]$ such that $|\alpha(s,y)| \vee |\beta(s,y)| \leq c$ for all $(s,y) \in [r,T) \times S$ with $d_S((s,y),(r,x)) < \gamma$. We set $\tilde{\tau} := \inf\{t \in [r,T]\,|\, \|X^t - x^r\| \geq \gamma/2\} \wedge T$ and $\hat{\tau} := \tilde{\tau} \wedge \tau$, then, as $\mathscr{X}$ fulfills the $\mathscr{L}$-martingale property and u is a mild subsolution to (P), it follows that

$$\begin{aligned} \frac{1}{t-r}E_{r,x}\left[\int_r^{t\wedge\hat{\tau}} (\partial_s + \mathscr{L})(\varphi)(s, X^s)\, ds\right] \geq \frac{1}{t-r}E_{r,x}\left[\int_r^{t\wedge\hat{\tau}} \alpha(s, X^s)\, ds\right] \\ + \frac{1}{t-r}E_{r,x}\left[\int_r^{t\wedge\hat{\tau}} \beta(s, X^s)u(s, X^s)\, ds\right] \end{aligned}$$

for each $t \in (r, r+\gamma/3)$. Due to Lemmas 5.19 and 5.22, regardless of whether u is right-continuous on $[0,T) \times S$, we may take the limit $t \downarrow r$ to obtain that

$$(\partial_r + \mathscr{L})(\varphi)(r,x) \geq \alpha(r,x) + \beta(r,x)u(r,x),$$

since $|\alpha(s, X^s)| \vee |\beta(s, X^s)| \leq c$ for all $s \in [r, (r+\gamma/3) \wedge \hat{\tau}]$ $P_{r,x}$-a.s. For this reason, u is a stochastic viscosity subsolution to (P). The fact that u is also a stochastic viscosity supersolution can be proven with similar reasoning. □

We turn to the final aim of this section. Notice that, according to Lemma A.10, the upper and lower right-hand semicontinuous envelopes of a right-hand locally bounded function $u : [0,T] \times S \to \mathbb{R}$ are given by

$$u^{\leftarrow}(r,x) = \limsup_{(s,y)\to(r,x):\, s\geq r} u(s,y) \quad \text{and} \quad u_{\leftarrow}(r,x) = \liminf_{(s,y)\to(r,x):\, s\geq r} u(s,y)$$

for all $(r,x) \in [0,T] \times S$, respectively. Indeed, $u^{\leftarrow}$ (resp. $u_{\leftarrow}$) is of this form as soon as u is merely right-hand locally bounded from above (resp. from below). We are now concerned with another decisive limit inequality.

5.24 Lemma. *Let $(r,x) \in [0,T) \times S$ and $\varphi \in B([r,T) \times S)$ be non-anticipative. Suppose that $(r_n, x_n)_{n\in\mathbb{N}}$ is a sequence in $[r,T) \times S$, $(t_n)_{n\in\mathbb{N}}$ is a sequence in $[r,T)$, and $(\tau_n)_{n\in\mathbb{N}}$ is a sequence of $(\mathscr{F}_t)_{t\in[0,T]}$-stopping times such that the following three conditions hold:*

(i) $\tau_n > r_n$ P_{r_n,x_n}*-a.s.,* $\tau_n \geq r_n$*, and* $r_n < t_n$ *for each* $n \in \mathbb{N}$*. In addition,* $\lim_{n\uparrow\infty} d_S((r_n,x_n),(r,x)) = 0$ *and* $\lim_{n\uparrow\infty} t_n = r$*.*

(ii) $\int_{r_n}^{t_n\wedge\tau_n} |\varphi(s,X^s)|\,ds$ *is finite for all* $n \in \mathbb{N}$ *and there exists* $c \geq 0$ *such that* $|\varphi(s,X^s)| \leq c$ *for all* $s \in [r_n, t_n \wedge \tau_n]$ P_{r_n,x_n}*-a.s. for every* $n \in \mathbb{N}$*.*

(iii) $\lim_{n\uparrow\infty} P_{r_n,x_n}(\tau_n \leq t_n) = 0$ *and* $\lim_{n\uparrow\infty} P_{r_n,x_n}(\|X^{t_n} - x_n^{r_n}\| \geq \gamma) = 0$ *for each* $\gamma > 0$*.*

If φ is upper right-hand semicontinuous at (r,x), then

$$\limsup_{n\uparrow\infty} E_{r_n,x_n}\left[\int_{r_n}^{t_n\wedge\tau_n} \frac{\varphi(s,X^s)}{t_n - r_n}\,ds\right] \leq \varphi(r,x).$$

Proof. Let $\varepsilon > 0$, then there is some $\delta > 0$ such that $\varphi(s,y) < \varphi(r,x) + \varepsilon/4$ for each $(s,y) \in [r,T) \times S$ with $d_S((s,y),(r,x)) < \delta$. By (i), we can choose $n_0 \in \mathbb{N}$ such that

$$(t_n - r_n) + d_S((r_n,x_n),(r,x)) < \delta/2$$

for all $n \in \mathbb{N}$ with $n \geq n_0$. Moreover, for each $n \in \mathbb{N}$ we let $Y^{(n)} : [r_n,T] \times \Omega \to \mathbb{R}_+$ be given by $Y_s^{(n)}(\omega) := \|X^s(\omega) - x_n^{r_n}\|$ and set $\sigma_n := \inf\{t \in [r_n,T] \,|\, \|X^t - x_n^{r_n}\| \geq \delta/2\}$, then $Y^{(n)}$ is an $(\mathscr{F}_t)_{t\in[r_n,T]}$-adapted process with increasing continuous paths and σ_n is an $(\mathscr{F}_t)_{t\in[r_n,T]}$-stopping time with $\sigma_n > r_n$ P_{r_n,x_n}-a.s. and $\{\sigma_n > s\} = \{Y_s^{(n)} < \delta/2\}$ for all $s \in [r_n,T]$. This yields that

$$E_{r_n,x_n}\left[\int_{r_n}^{t_n\wedge\tau_n} \frac{\varphi(s,X^s)}{t_n - r_n}\mathbb{1}_{\left\{Y_s^{(n)}<\delta/2\right\}}\,ds\right] \leq \frac{\varphi(r,x)}{t_n - r_n}E_{r_n,x_n}[(t_n \wedge \tau_n \wedge \sigma_n - r_n)] + \varepsilon/4$$

for every $n \in \mathbb{N}$ with $n \geq n_0$, because $d_S((s,X^s(\omega)),(r,x)) \leq (s - r_n) + Y_s^{(n)}(\omega)$ $+\, d_S((r_n,x_n),(r,x)) < \delta$ for all $\omega \in \{X^{r_n} = x_n^{r_n}\}$ and each $s \in [r_n, t_n \wedge \tau_n(\omega) \wedge \sigma_n(\omega)]$. We observe that

$$\frac{1}{t_n - r_n}E_{r_n,x_n}[(t_n - t_n \wedge \tau_n \wedge \sigma_n)] \leq P_{r_n,x_n}(\tau_n \leq t_n) + P_{r_n,x_n}(Y_{t_n}^{(n)} \geq \delta/2)$$

for each $n \in \mathbb{N}$, since it holds that $(t_n - t_n \wedge \tau_n \wedge \sigma_n) = (t_n - \tau_n \wedge \sigma_n)\mathbb{1}_{\{\tau_n \wedge \sigma_n \leq t_n\}} \leq (t_n - r_n)\mathbb{1}_{\{\tau_n \wedge \sigma_n \leq t_n\}}$. At the same time it follows from (ii) that

$$E_{r_n,x_n}\left[\int_{r_n}^{t_n \wedge \tau_n} \frac{|\varphi(s, X^s)|}{t_n - r_n}\mathbb{1}_{\left\{Y_s^{(n)} \geq \delta/2\right\}}\, ds\right] \leq cP_{r_n,x_n}(Y_{t_n}^{(n)} \geq \delta/2)$$

for all $n \in \mathbb{N}$. For $c' := c \vee |\varphi(r,x)|$ there is $n_1 \in \mathbb{N}$ such that $c'P_{r_n,x_n}(\tau_n \leq t_n) < \varepsilon/4$ and $c'P_{r_n,x_n}(Y_{t_n}^{(n)} \geq \delta/2) < \varepsilon/4$ for all $n \in \mathbb{N}$ with $n \geq n_1$, due to (iii). Hence, we set $n_2 := n_0 \vee n_1$, then we obtain that

$$\begin{aligned} E_{r_n,x_n}\left[\int_{r_n}^{t_n \wedge \tau_n} \frac{\varphi(s, X^s)}{t_n - r_n}\, ds\right] &\leq \varphi(r,x) + \frac{c'}{t_n - r_n}E_{r_n,x_n}[(t_n - t_n \wedge \tau_n \wedge \sigma_n)] + \varepsilon/2 \\ &< \varphi(r,x) + \varepsilon \end{aligned}$$

for each $n \in \mathbb{N}$ with $n \geq n_2$. This entails the assertion. □

To clarify the way we proceed, note that for every function $u : [0,T] \times S \to \mathbb{R}$ that is right-hand locally bounded from above and each $(r,x) \in [0,T) \times S$, there exists a sequence $(r_n,x_n)_{n\in\mathbb{N}}$ in $[r,T) \times S$ with $\lim_{n\uparrow\infty} d_S((r_n,x_n),(r,x)) = 0$ and $\lim_{n\uparrow\infty} u(r_n,x_n) = u^{\leftarrow}(r,x)$, by Lemma A.7. This technique is well-known in the literature of viscosity solutions (see for example Pham [30, Section 4.3]).

5.25 Lemma. *Let $(r,x) \in [0,T) \times S$, β be right-continuous at (r,x), and u be a right-hand locally bounded mild solution to* (P). *Suppose that $(r_n,x_n)_{n\in\mathbb{N}}$ is a sequence in $[r,T) \times S$, $(t_n)_{n\in\mathbb{N}}$ is a sequence in $[r,T)$, and $(\tau_n)_{n\in\mathbb{N}}$ is some sequence of $(\hat{\mathscr{F}}_t)_{t\in[0,T]}$-stopping times such that the following three conditions hold:*

(i) $\tau_n > r_n$ P_{r_n,x_n}-a.s., $\tau_n \geq r_n$, and $r_n < t_n$ for every $n \in \mathbb{N}$. Furthermore, $\lim_{n\uparrow\infty} d_S((r_n,x_n),(r,x)) = 0$, $\lim_{n\uparrow\infty} u(r_n,x_n) = u^{\leftarrow}(r,x)$, and $\lim_{n\uparrow\infty} t_n = r$.

(ii) There is $c \geq 0$ such that $|\alpha(s,X^s)| \vee |\beta(s,X^s)| \vee |u(s,X^s)| \leq c$ for each $s \in [r_n, t_n \wedge \tau_n]$ P_{r_n,x_n}-a.s. for every $n \in \mathbb{N}$.

(iii) $\lim_{n\uparrow\infty} P_{r_n,x_n}(\tau_n \leq t_n) = 0$ and $\lim_{n\uparrow\infty} P_{r_n,x_n}(\|X^{t_n} - x_n^{r_n}\| \geq \gamma) = 0$ for all $\gamma > 0$.

Then

$$\lim_{n\uparrow\infty} E_{r_n,x_n}\left[\int_{r_n}^{t_n \wedge \tau_n} \frac{\beta(s,X^s)}{t_n - r_n}u(s,X^s)\, ds\right] = \beta(r,x)u^{\leftarrow}(r,x).$$

Proof. Since u is a mild solution to (P), it holds that

$$\begin{aligned} E_{r_n,x_n}[u(t_n \wedge \tau_n, X^{t_n \wedge \tau_n})] &= u(r_n,x_n) \\ &\quad + E_{r_n,x_n}\left[\int_{r_n}^{t_n \wedge \tau_n} \alpha(s,X^s) + \beta(s,X^s)u(s,X^s)\, ds\right] \end{aligned}$$

for all $n \in \mathbb{N}$. Together with (ii), this gives $|E_{r_n,x_n}[u(t_n \wedge \tau_n, X^{t_n \wedge \tau_n})] - u(r_n,x_n)| \leq c(1+c)(t_n - r_n)$ for each $n \in \mathbb{N}$. Hence,

$$\lim_{n\uparrow\infty} E_{r_n,x_n}[u(t_n \wedge \tau_n, X^{t_n \wedge \tau_n})] = \lim_{n\uparrow\infty} u(r_n,x_n) = u^{\leftarrow}(r,x). \tag{5.6}$$

We note that, because the function $[0,T] \times S \to \mathbb{R}_+$, $(s,y) \mapsto |\beta(s,y) - \beta(r,x)|$ is right-continuous at (r,x), Lemma 5.24 implies that

$$\lim_{n\uparrow\infty} E_{r_n,x_n}\bigg[\int_{r_n}^{t_n\wedge\tau_n} \frac{|\beta(s,X^s) - \beta(r,x)|}{t_n - r_n}\,ds\bigg] = 0.$$

So, from the hypothesis that $|u(t_n \wedge \tau_n, X^{t_n\wedge\tau_n})| \leq c\ P_{r_n,x_n}$-a.s. for all $n \in \mathbb{N}$ and the fact that

$$\beta(r,x) = \int_{r_n}^{t_n\wedge\tau_n} \frac{\beta(r,x)}{t_n - r_n}\,ds + \frac{\beta(r,x)}{t_n - r_n}(t_n - t_n \wedge \tau_n)$$

for each $n \in \mathbb{N}$, we readily infer from (5.6) that

$$\lim_{n\uparrow\infty} E_{r_n,x_n}\bigg[\int_{r_n}^{t_n\wedge\tau_n} \frac{\beta(s,X^s)}{t_n - r_n} u(t_n \wedge \tau_n, X^{t_n\wedge\tau_n})\,ds\bigg] = \beta(r,x)u^{\leftarrow}(r,x).$$

Here, similarly as in the proof of Lemma 5.24, we can utilize that

$$\lim_{n\uparrow\infty} \frac{1}{t_n - r_n} E_{r_n,x_n}[(t_n - t_n \wedge \tau_n)] = \lim_{n\uparrow\infty} P_{r_n,x_n}(\tau_n \leq t_n) = 0,$$

since $(t_n - t_n\wedge\tau_n) = (t_n - \tau_n)\mathbb{1}_{\{\tau_n \leq t_n\}} \leq (t_n - r_n)\mathbb{1}_{\{\tau_n\leq t_n\}}$ for all $n \in \mathbb{N}$. Consequently, the claim follows once we have shown that

$$\lim_{n\uparrow\infty} E_{r_n,x_n}\bigg[\int_{r_n}^{t_n\wedge\tau_n} \frac{\beta(s,X^s)}{t_n - r_n}(u(t_n \wedge \tau_n, X^{t_n\wedge\tau_n}) - u(s,X^s))\,ds\bigg] = 0. \tag{5.7}$$

To this end, we let $n \in \mathbb{N}$ and set $\tau_{n,s} := \tau_n \vee s$ for each $s \in [r_n, t_n]$, then $\tau_{n,s}$ is an $(\hat{\mathscr{F}}_t)_{t\in[s,T]}$-stopping time. As u is a mild solution to (P), we get that

$$\begin{aligned} E_{s,y}[u(t_n \wedge \tau_{n,s}, X^{t_n\wedge\tau_{n,s}})] &= u(s,y) \\ &\quad + E_{s,y}\bigg[\int_s^{t_n\wedge\tau_{n,s}} \alpha(s', X^{s'}) + \beta(s', X^{s'})u(s', X^{s'})\,ds'\bigg] \end{aligned}$$

for each $(s,y) \in [r_n, t_n] \times S$. Hence, Fubini's theorem and Propositions 3.7, 3.13 yield that

$$\begin{aligned} &\bigg|E_{r_n,x_n}\bigg[\int_{r_n}^{t_n\wedge\tau_n} \frac{\beta(s,X^s)}{t_n - r_n}(u(t_n\wedge\tau_n, X^{t_n\wedge\tau_n}) - u(s,X^s))\,ds\bigg]\bigg| \\ &= \bigg|\int_{r_n}^{t_n} E_{r_n,x_n}\bigg[\frac{\beta(s,X^s)}{t_n - r_n}(E_{s,X^s}[u(t_n\wedge\tau_{n,s}, X^{t_n\wedge\tau_{n,s}})] - u(s,X^s))\mathbb{1}_{\{\tau_n > s\}}\bigg]\,ds\bigg| \\ &= \bigg|\int_{r_n}^{t_n} E_{r_n,x_n}\bigg[\frac{\beta(s,X^s)}{t_n - r_n}\int_s^{t_n\wedge\tau_{n,s}} \alpha(s',X^{s'}) + \beta(s',X^{s'})u(s',X^{s'})\,ds'\mathbb{1}_{\{\tau_n>s\}}\bigg]\,ds\bigg| \\ &\leq c(1+c)\int_{r_n}^{t_n} E_{r_n,x_n}[|\beta(s,X^s)|\mathbb{1}_{\{\tau_n>s\}}]\,ds \leq c^2(1+c)(t_n - r_n), \end{aligned}$$

since $\tau_n = \tau_{n,s}$ on $\{\tau_n > s\}$ for all $s \in [r_n, t_n]$. As $n \in \mathbb{N}$ has been arbitrarily chosen, we may take the limit $n \uparrow \infty$ to obtain (5.7), which proves the assertion. $\square$

We recall that, as $\hat{\mathscr{X}}$ is in particular a non-anticipative Markov process, it follows for each $\gamma > 0$ and every $(r,x) \in [0,T) \times S$ that

$$\lim_{t \downarrow r} P_{r,x}(\|X^t - x^r\| \geq \gamma) = 0.$$

However, let us now require that for all $\gamma > 0$, every $(r,x) \in [0,T) \times S$, and each sequence $(r_n, x_n, t_n)_{n\in\mathbb{N}}$ in $[r,T) \times S \times [r,T)$ such that $t_n \geq r_n$ for all $n \in \mathbb{N}$ and $\lim_{n\uparrow\infty}(t_n - r) + d_S((r_n,x_n),(r,x)) = 0$ it holds that

$$\lim_{n\uparrow\infty} P_{r_n,x_n}(\|X^{t_n} - x_n^{r_n}\| \geq \gamma) = 0. \tag{5.8}$$

This slightly stronger condition leads us to our final result, which establishes the relation between the upper and lower right-hand semicontinuous envelopes of a mild solution and right-hand viscosity sub- and supersolutions.

5.26 Proposition. *Suppose that $\mathscr{X}$ satisfies* (5.8)*, D is closed, and a, b, α, β are right-continuous. If u is a bounded mild solution to* (P)*, then $u^{\leftarrow}$ (resp. $u_{\leftarrow}$) must be a right-hand viscosity subsolution (resp. supersolution) to* (P)*.*

Proof. To verify that $u^{\leftarrow}$ is a right-hand viscosity subsolution, let $(r,x) \in [0,T) \times S$ and $\varphi \in \underline{\mathscr{P}}\, u^{\leftarrow}(r,x)$. Then there is $\delta \in (0, T-r)$ such that

$$(u^{\leftarrow} - \varphi)(r,x) \geq (u^{\leftarrow} - \varphi)(s,y) \tag{5.9}$$

for each $(s,y) \in [r,T) \times S$ fulfilling $d_S((s,y),(r,x)) < \delta$. Due to Lemma A.7, there exists a sequence $(r_n,x_n)_{n\in\mathbb{N}}$ in $[r,T) \times S$ such that $\lim_{n\uparrow\infty} d_S((r_n,x_n),(r,x)) = 0$ and $\lim_{n\uparrow\infty} u(r_n,x_n) = u^{\leftarrow}(r,x)$. We set

$$\eta_n := (u^{\leftarrow} - \varphi)(r,x) - (u - \varphi)(r_n,x_n) \quad \text{for all } n \in \mathbb{N}.$$

Then, since $\lim_{n\uparrow\infty} \eta_n = 0$, there is a sequence $(t_n)_{n\in\mathbb{N}}$ in $[r,T)$ such that $r_n < t_n$ for each $n \in \mathbb{N}$, $\lim_{n\uparrow\infty} t_n = r$, and $\lim_{n\uparrow\infty} \eta_n/(t_n - r_n) = 0$. For instance, we could have set $t_n := r_n + (1/2)\min\left\{\sqrt{|\eta_n|} + 1/n, T - r_n\right\}$ for each $n \in \mathbb{N}$.

As α and β are right-continuous at (r,x), there are $c > 0$ and $\gamma \in (0,\delta]$ satisfying $|\alpha(s,y)| \vee |\beta(s,y)| \leq c$ for all $(s,y) \in [r,T) \times S$ with $d_S((s,y),(r,x)) < \gamma$. We set

$$\tau_n := \inf\{t \in [r_n,T] \,|\, \|X^t - x_n^{r_n}\| \geq \gamma/2\} \quad \text{for each } n \in \mathbb{N},$$

then τ_n is an $(\hat{\mathscr{F}}_t)_{t\in[r_n,T]}$-stopping time with $\tau_n > r_n$ P_{r_n,x_n}-a.s. In addition, let $n_0 \in \mathbb{N}$ be such that $(t_n - r_n) + d_S((r_n,x_n),(r,x)) < \gamma/2$ for all $n \in \mathbb{N}$ with $n \geq n_0$. Then from (5.9) and $u^{\leftarrow} \geq u$ we infer that

$$(u^{\leftarrow} - \varphi)(r,x) \geq E_{r_n,x_n}[(u - \varphi)(t_n \wedge \tau_n, X^{t_n\wedge\tau_n})]$$

for every $n \in \mathbb{N}$ such that $n \geq n_0$, because it holds that $d_S((t_n \wedge \tau_n, X^{t_n\wedge\tau_n}),(r,x)) \leq d_S((t_n \wedge \tau_n, X^{t_n\wedge\tau_n}),(r_n,x_n)) + d_S((r_n,x_n),(r,x)) < \gamma$ on $\{X^{r_n} = x_n^{r_n}\}$. Moreover, since u is a mild subsolution to (P) and the stopped process $[r_n,T) \times \Omega \to \mathbb{R}$,

$$(t,\omega) \mapsto \varphi(t \wedge \tau_n(\omega), X^{t\wedge\tau_n}(\omega)) - \int_{r_n}^{t\wedge\tau_n(\omega)} (\partial_s + \mathscr{L})(\varphi)(s, X^s(\omega))\, ds$$

is an $(\mathscr{F}_t)_{t\in[r_n,T)}$-martingale under P_{r_n,x_n}, it follows that

$$\begin{aligned}E_{r_n,x_n}[(u-\varphi)(t_n\wedge\tau_n, X^{t_n\wedge\tau_n})] \geq (u-\varphi)(r_n,x_n)\\ + E_{r_n,x_n}\left[\int_{r_n}^{t_n\wedge\tau_n}\alpha(s,X^s)+\beta(s,X^s)u(s,X^s)\,ds\right]\\ - E_{r_n,x_n}\left[\int_{r_n}^{t_n\wedge\tau_n}(\partial_s+\mathscr{L})(\varphi)(s,X^s)\,ds\right]\end{aligned}$$

for each $n\in\mathbb{N}$ with $n\geq n_0$. By recalling the definition of η_n, this implies that

$$\begin{aligned}\frac{\eta_n}{t_n-r_n}+\frac{1}{t_n-r_n}E_{r_n,x_n}\left[\int_{r_n}^{t_n\wedge\tau_n}(\partial_s+\mathscr{L})(\varphi)(s,X^s)\,ds\right]\\ \geq \frac{1}{t_n-r_n}E_{r_n,x_n}\left[\int_{r_n}^{t_n\wedge\tau_n}\alpha(s,X^s)+\beta(s,X^s)u(s,X^s)\,ds\right]\end{aligned}$$

for all $n\in\mathbb{N}$ with $n\geq n_0$. Hence, Lemmas 5.24 and 5.25 ensure that we may take the limit $n\uparrow\infty$, which yields that

$$(\partial_r+\mathscr{L})(\varphi)(r,x)\geq\alpha(r,x)+\beta(r,x)u^{\leftarrow}(r,x).$$

Here, we have used that the function $[r,T)\times S\to\mathbb{R}$, $(s,y)\mapsto(\partial_s+\mathscr{L})(\varphi)(s,y^s)$ and α, β are right-continuous. Moreover, the fact that $\{\tau_n\leq t_n\}=\{\|X^{t_n}-x_n^{r_n}\|\geq\gamma/2\}$ for all $n\in\mathbb{N}$ and the hypothesis that $\mathscr{X}$ fulfills (5.8) ensure that

$$\lim_{n\uparrow\infty}P_{r_n,x_n}(\tau_n\leq t_n)=\lim_{n\uparrow\infty}P_{r_n,x_n}(\|X^{t_n}-x_n^{r_n}\|\geq\gamma/2)=0.$$

This shows that $u^{\leftarrow}$ is a right-hand viscosity subsolution to (P). Since the verification that $u_{\leftarrow}$ is a right-hand viscosity supersolution can be handled in much the same way, the claim is proven. □

In conclusion, in the first part of the thesis, we derived unique non-extendible admissible solutions to multidimensional Markovian integral equations that involve a progressive Markov process with Polish state space and Borel measurable transition probabilities. Then a boundary and growth analysis led us to unique global bounded solutions to one-dimensional Markovian integral equations. In the second part, by using path-dependent diffusion processes, we were able to identify mild solutions to semilinear parabolic PPDEs as global solutions to the associated Markovian integral equations. Consequently, existence and uniqueness for bounded mild solutions were inferred. In the end, under weak continuity conditions, we verified that bounded right-continuous mild solutions are also solutions in a viscosity sense, which in turn yielded existence for bounded (right-)continuous viscosity solutions.

Appendix

A.1 Convex sets

Here, we review some standard facts on convex sets and consider Carathéodory's Convex Hull Theorem. To this end, we follow [31, Section 2.1] mainly. Let E be a linear space, then a set $D \subset E$ is *convex* if $\alpha z + (1-\alpha)z' \in D$ for all $z, z' \in D$ and each $\alpha \in (0,1)$. A point $z \in E$ is said to be a *convex combination of points* of D if there are $n \in \mathbb{N}$, $z_1, \dots, z_n \in D$, and $\alpha_1, \dots, \alpha_n \in [0,1]$ such that

$$z = \alpha_1 z_1 + \cdots + \alpha_n z_n \quad \text{and} \quad \alpha_1 + \cdots + \alpha_n = 1.$$

A.1 Lemma. *Every convex set $D \subset E$ contains all convex combinations of its points. That is, $\alpha_1 z_1 + \cdots + \alpha_n z_n \in D$ for all $n \in \mathbb{N}$ with $n \geq 2$, each $z_1, \dots, z_n \in D$, and every $\alpha_1, \dots, \alpha_n \in [0,1]$ with $\alpha_1 + \cdots + \alpha_n = 1$.*

Proof. We verify the claim by induction over $n \in \mathbb{N}$ with $n \geq 2$. In the initial induction step $n = 2$, the claim reduces to the convexity of D. Thus, we may assume that the claim holds for some $n \in \mathbb{N}$ with $n \geq 2$. Let $z_1, \dots, z_{n+1} \in D$ and $\alpha_1, \dots, \alpha_{n+1} \in [0,1]$ be such that $\alpha_1 + \cdots + \alpha_{n+1} = 1$. For $\alpha_{n+1} = 1$ there is nothing to show. So, let $\alpha_{n+1} < 1$, then the induction hypothesis entails that

$$z := \alpha_1/(1-\alpha_{n+1})z_1 + \cdots + \alpha_n/(1-\alpha_{n+1})z_n \in D,$$

since $\alpha_1/(1-\alpha_{n+1}) + \cdots + \alpha_n/(1-\alpha_{n+1}) = (\alpha_1 + \cdots + \alpha_n)/(1-\alpha_{n+1}) = 1$. For this reason, $\alpha_1 z_1 + \cdots + \alpha_{n+1} z_{n+1} = (1-\alpha_{n+1})z + \alpha_{n+1} z_{n+1} \in D$, which completes the induction proof. $\square$

Note that if $\mathbb{D}$ is a family of convex sets in E, then the intersection $\bigcap_{D \in \mathbb{D}} D$ is another convex set in E. Indeed, for all $z, z' \in \bigcap_{D \in \mathbb{D}} D$ and each $\alpha \in (0,1)$ we have that $\alpha z + (1-\alpha)z' \in D$ for every $D \in \mathbb{D}$, which directly yields that

$$\alpha z + (1-\alpha)z' \in \bigcap_{D \in \mathbb{D}} D.$$

So, for a set $D \subset E$, the *convex hull* of D, denoted by $\operatorname{conv}(D)$, is defined to be the smallest convex set in E including D in the sense that $\operatorname{conv}(D)$ is a convex set in E that includes D and which is included in every convex set in E including D. Clearly, the convex hull of D must be unique. Regarding existence, let $\mathbb{D}$ be the family of all convex sets in E which include D, then $\operatorname{conv}(D) = \bigcap_{D' \in \mathbb{D}} D'$. This is because $D \subset \bigcap_{D' \in \mathbb{D}} D'$ and $\bigcap_{D' \in \mathbb{D}} D' \subset D''$ for each $D'' \in \mathbb{D}$.

A.2 Lemma. *Let $D \subset E$, then* conv(D) *is the set of all convex combinations of points of D.*

Proof. We have to check that the set of all convex combinations of points of D is convex and included in each convex set in E which includes D. Thus, let $z, z' \in E$ be convex combinations of points of D and $\gamma \in (0,1)$. Then

$$z = \alpha_1 z_1 + \cdots + \alpha_m z_m \quad \text{and} \quad z' = \beta_1 z'_1 + \cdots + \beta_n z'_n$$

for some $m, n \in \mathbb{N}$, $z_1, \ldots, z_m, z'_1, \ldots, z'_n \in E$, and $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \in [0,1]$ with $\alpha_1 + \cdots + \alpha_m = \beta_1 + \cdots + \beta_n = 1$. The representation

$$\gamma z + (1-\gamma) z' = \gamma\alpha_1 z_1 + \cdots + \gamma\alpha_m z_m + (1-\gamma)\beta_1 z'_1 + \cdots + (1-\gamma)\beta_n z'_n$$

shows that $\gamma z + (1-\gamma)z'$ is also a convex combination of points of D, because $\gamma\alpha_1 + \cdots + \gamma\alpha_m + (1-\gamma)\beta_1 + \cdots + (1-\gamma)\beta_n = 1$. Finally, let D' be a convex set in E which includes D. Then the set of all convex combinations of points of D is included in the set of all convex combinations of points of D'. As D' is convex, Lemma A.1 yields that D' agrees with the latter set, which is the desired conclusion. □

If E is finite-dimensional, then we can bound the number of points of a set D in E that are needed to represent a point in conv(D).

A.3 Carathéodory's Convex Hull Theorem. Let $d := \dim(E) < \infty$ and $D \subset E$. Then each point $z \in \operatorname{conv}(D)$ is a convex combination of at most $d+1$ points of E. That means, z can be written in the form

$$z = \alpha_1 z_1 + \cdots + \alpha_{d+1} z_{d+1}$$

for some $z_1, \ldots, z_{d+1} \in D$ and $\alpha_1, \ldots, \alpha_{d+1} \in [0,1]$ with $\alpha_1 + \cdots + \alpha_{d+1} = 1$.

Proof. By Lemma A.2, it suffices to show that if $n \in \mathbb{N}$ is such that $n > d+1$ and z is a convex combination of n points of D, say $z = \beta_1 z_1 + \cdots + \beta_n z_n$ for some $z_1, \ldots, z_n \in D$ and some $\beta_1, \ldots, \beta_n \in [0,1]$ with $\beta_1 + \cdots + \beta_n = 1$, then n can be reduced by one in the sense that there are $\alpha_1, \ldots, \alpha_n \in [0,1]$ such that

$$z = \alpha_1 z_1 + \cdots + \alpha_n z_n \quad \text{and} \quad \alpha_1 + \cdots + \alpha_n = 1,$$

but $\alpha_i = 0$ for at least one $i \in \{1, \ldots, n\}$. If $\beta_i = 0$ for some $i \in \{1, \ldots, n\}$, then this is certainly true. Thus, let us assume that $\beta_1, \ldots, \beta_n$ are positive. We notice that $z_1 - z_n, \ldots, z_{n-1} - z_n$ are linearly dependent, since $n > d+1$. Hence, there are $\gamma_1, \ldots, \gamma_{n-1} \in \mathbb{R}$ such that

$$\gamma_1(z_1 - z_n) + \cdots + \gamma_{n-1}(z_{n-1} - z_n) = 0$$

and $\gamma_i \neq 0$ for at least one $i \in \{1, \ldots, n-1\}$. By setting $\gamma_n := -\gamma_1 - \cdots - \gamma_{n-1}$, we obtain that $\gamma_1 z_1 + \cdots + \gamma_n z_n = 0$ and $\gamma_1 + \cdots + \gamma_n = 0$. For the second equation to be valid, we must have $\gamma_i > 0$ for at least one $i \in \{1, \ldots, n\}$. This entails that $\tau := \min\{\beta_i/\gamma_i \,|\, i \in \{1,\dots,n\} : \gamma_i > 0\}$ is well-defined and positive. We set $\alpha_i := \beta_i - \tau\gamma_i$, then $\alpha_i \geq 0$ for each $i \in \{1,\dots,n\}$. In addition,

$$\begin{aligned}\alpha_1 z_1 + \cdots + \alpha_n z_n &= \beta_1 z_1 + \cdots + \beta_n z_n - \tau(\gamma_1 z_1 + \cdots + \gamma_n z_n)\\ &= \beta_1 z_1 + \cdots + \beta_n z_n = z\end{aligned}$$

and $\alpha_1 + \cdots + \alpha_n = \beta_1 + \cdots + \beta_n - \tau(\gamma_1 + \cdots + \gamma_n) = 1$. Eventually, let $i \in \{1,\dots,n\}$ be such that $\gamma_i > 0$ and $\tau = \beta_i/\gamma_i$, then $\alpha_i = \beta_i - \tau\gamma_i = 0$. This shows the claim. □

Until the end of this section, we require that E is equipped with a norm $\|\cdot\|$.

A.4 Corollary. *Assume that $d := \dim(E) < \infty$, then the convex hull of each compact set $D \subset E$ must be compact.*

Proof. Let K be set of all $(\alpha_1,\dots,\alpha_{d+1}) \in [0,1]^{d+1}$ with $\alpha_1 + \cdots + \alpha_{d+1} = 1$. As the function $[0,1]^{d+1} \to [0,1]$, $(\alpha_1,\dots,\alpha_{d+1}) \mapsto \alpha_1 + \cdots + \alpha_{d+1}$ is continuous, K is closed in the compact set $[0,1]^{d+1}$. Therefore, K is compact. This in turn ensures that $K \times D^{d+1}$ is compact as well. The map $\varphi : \mathbb{R}^{d+1} \times E^{d+1} \to E$ defined via

$$\varphi(\alpha_1,\dots,\alpha_{d+1}, z_1,\dots,z_{d+1}) := \alpha_1 z_1 + \cdots + \alpha_{d+1} z_{d+1}$$

is readily seen to be continuous and Carathéodory's Convex Hull Theorem A.3 yields the representation $\operatorname{conv}(D) = \varphi(K \times D^{d+1})$. This establishes the claim. □

Let us note at this point that the union of each increasing sequence $(D_n)_{n\in\mathbb{N}}$ of convex sets in E is convex. Indeed, let $z, z' \in \bigcup_{n\in\mathbb{N}} D_n$ and $\alpha \in (0,1)$, then there is $m \in \mathbb{N}$ with $z, z' \in D_m$. Since D_m is convex, $\alpha z + (1-\alpha)z' \in D_m$, which shows the convexity of $\bigcup_{n\in\mathbb{N}} D_n$. We conclude with topological properties for convex sets.

A.5 Lemma. *Let $D \subset E$ be convex, then $D^\varepsilon := \{z \in D \,|\, B_\varepsilon(z) \subset D\}$ is convex for each $\varepsilon > 0$. Moreover, the interior D° and the closure $\overline{D}$ of D are convex.*

Proof. At first, let $\varepsilon > 0$, $z, z' \in D^\varepsilon$, and $\alpha \in (0,1)$. We fix $\hat{z} \in B_\varepsilon(\alpha z + (1-\alpha)z')$ and show that $\hat{z} \in D$, which then yields the first claim. To this end, let us set

$$z_0 := z + \hat{z} - \alpha z - (1-\alpha)z' \quad \text{and} \quad z_0' := z' + \hat{z} - \alpha z - (1-\alpha)z',$$

then $\|z_0 - z\| = \|z_0' - z'\| = \|\hat{z} - \alpha z - (1-\alpha)z'\| < \varepsilon$. Thus, $z_0 \in B_\varepsilon(z)$ and $z_0' \in B_\varepsilon(z')$. Furthermore, $z_0 = (1-\alpha)(z-z') + \hat{z}$ and $z_0' = -\alpha(z-z') + \hat{z}$, which entails that $\hat{z} = \alpha z_0 + (1-\alpha)z_0' \in D$, as desired.

Next, we observe that $D^\varepsilon \subset D^\delta$ for all $\delta, \varepsilon > 0$ with $\delta \leq \varepsilon$. Therefore, $(D^{1/n})_{n\in\mathbb{N}}$ is an increasing sequence of convex sets in E. From the preceding remark and fact that $D^\circ = \bigcup_{n\in\mathbb{N}} D^{1/n}$, we obtain the convexity of D°. To show the convexity of $\overline{D}$, let $z, z' \in \overline{D}$ and $\alpha \in (0,1)$. Then there exist two sequences $(z_n)_{n\in\mathbb{N}}$ and $(z_n')_{n\in\mathbb{N}}$ in D such that $\lim_{n\uparrow\infty} z_n = z$ and $\lim_{n\uparrow\infty} z_n' = z'$. Because D is convex, $\alpha z_n + (1-\alpha)z_n' \in D$ for all $n \in \mathbb{N}$. From $\lim_{n\uparrow\infty} \alpha z_n + (1-\alpha)z_n' = \alpha z + (1-\alpha)z'$ we infer that $\alpha z + (1-\alpha)z' \in \overline{D}$. Hence, the lemma is proven. □

A.2 Right-hand semicontinuity

The purpose of this section is to present the notion of right-hand semicontinuity and verify a representation for right-hand semicontinuous envelopes. In the end, some basic results on semicontinuity and uniform continuity are provided as well.

We suppose that $J \subset \mathbb{R}$ is a non-degenerate interval, S is a non-empty set, and d_S is a pseudometric on $J \times S$. Let $J \times S$ be endowed with the topology induced by d_S and for each $(r,x) \in J \times S$ we define $\mathscr{U}(r,x)$ to be the system of all neighborhoods of (r,x) in $J \times S$. If in addition $\delta > 0$, then $B_\delta(r,x)$ denotes the set of all $(s,y) \in J \times S$ with $d_S((s,y),(r,x)) < \delta$.

Let $F \subset J \times S$ be non-empty, then a function $u : F \to [-\infty, \infty]$ is said to be *right-hand locally bounded from above* (resp. *from below*) at a point $(r,x) \in F$ if there is $U \in \mathscr{U}(r,x)$ such that

$$\sup_{(s,y)\in U\cap F:\, s\geq r} u(s,y) < \infty \quad \Big(\text{resp. } \inf_{(s,y)\in U\cap F:\, s\geq r} u(s,y) > -\infty\Big).$$

We call u *right-hand locally bounded* at (r,x) if it is right-hand locally bounded from above and from below there. At last, u is right-hand locally bounded (from above or from below) if it fulfills the corresponding property at all $(r,x) \in F$.

Whenever u is right-hand locally bounded from above at a point $(r,x) \in F$, then the *right-hand limit superior* of u at (r,x) is given by

$$\limsup_{(s,y)\to(r,x):\, s\geq r} u(s,y) := \inf_{U\in\mathscr{U}(r,x)} \sup_{(s,y)\in U\cap F:\, s\geq r} u(s,y).$$

In a similar way, if u is right-hand locally bounded from below at (r,x), then the *right-hand limit inferior* of u at (r,x) is defined by

$$\liminf_{(s,y)\to(r,x):\, s\geq r} u(s,y) := \sup_{U\in\mathscr{U}(r,x)} \inf_{(s,y)\in U\cap F:\, s\geq r} u(s,y).$$

In what follows, to each fact on right-hand local boundedness from above there is a dual fact on right-hand local boundedness from below. So, we omit the former.

A.6 Lemma. *Suppose that $u : F \to [-\infty, \infty)$ is right-hand locally bounded from above at a point $(r,x) \in F$, then the right-hand limit superior of u at (r,x) coincides with $\inf_{\delta\in\Delta} \sup_{(s,y)\in B_\delta(r,x)\cap F:\, s\geq r} u(s,y)$ for every set Δ in $(0,\infty)$ with $\inf \Delta = 0$.*

Proof. Let us first assume that there is $U \in \mathscr{U}(r,x)$ such that $\sup_{(s,y)\in U\cap F:\, s\geq r} u(s,y) < \inf_{\delta\in\Delta} \sup_{(s,y)\in B_\delta(r,x)\cap F:\, s\geq r} u(s,y)$. As U is open, there is $\varepsilon > 0$ with $B_\varepsilon(r,x) \subset U$. Since $\inf \Delta = 0$, there is $\gamma \in \Delta$ with $\gamma \leq \varepsilon$. Thus,

$$\sup_{(s,y)\in B_\gamma(r,x)\cap F:\, s\geq r} u(s,y) < \inf_{\delta\in\Delta} \sup_{(s,y)\in B_\delta(r,x)\cap F:\, s\geq r} u(s,y),$$

a contradiction. In combination with the fact that $B_\delta(r,x) \in \mathscr{U}(r,x)$ for every $\delta > 0$, this gives the claim. $\square$

Let $u : F \to [-\infty, \infty]$ and $(r, x) \in F$. If there is $z \in \mathbb{R}$ such that for each $\varepsilon > 0$ there is $U \in \mathscr{U}(r, x)$ with $|u(s, y) - z| < \varepsilon$ for all $(s, y) \in U \cap F$ with $s \geq r$, then we say that z is the *right-hand limit* of u at (r, x) and denote it by $\lim_{(s,y)\to(r,x):\, s\geq r} u(s, y)$.

A.7 Lemma. *Let $u : F \to [-\infty, \infty)$ be right-hand locally bounded from above at a point $(r, x) \in F$. Then the following three assertions hold:*

(i) Every sequence $(r_n, x_n)_{n\in\mathbb{N}}$ in F that converges to (r, x) with $r_n \geq r$ for almost all $n \in \mathbb{N}$ satisfies $\limsup_{n\uparrow\infty} u(r_n, x_n) \leq \limsup_{(s,y)\to(r,x):\, s\geq r} u(s, y)$.

(ii) There exists a sequence $(r_n, x_n)_{n\in\mathbb{N}}$ in F that converges to (r, x) with $r_n \geq r$ for all $n \in \mathbb{N}$ and $\lim_{n\uparrow\infty} u(r_n, x_n) = \limsup_{(s,y)\to(r,x):\, s\geq r} u(s, y)$.

(iii) Let u be right-hand locally bounded from below at (r, x). Then the right-hand limit $\lim_{(s,y)\to(r,x):\, s\geq r} u(s, y)$ exists if and only if

$$\liminf_{(s,y)\to(r,x):\, s\geq r} u(s, y) = \limsup_{(s,y)\to(r,x):\, s\geq r} u(s, y). \tag{A.1}$$

In this case, it coincides with the common value.

Proof. (i) Suppose that there is a sequence $(r_n, x_n)_{n\in\mathbb{N}}$ in F that converges to (r, x) such that $r_n \geq r$ for almost all $n \in \mathbb{N}$ and for which the asserted inequality fails. Then there is $U \in \mathscr{U}(r, x)$ such that

$$\sup_{n\in\mathbb{N}:\, n\geq n_0} u(r_n, x_n) > \sup_{(s,y)\in U\cap F:\, s\geq r} u(s, y)$$

for each $n_0 \in \mathbb{N}$. Since $(r_n, x_n)_{n\in\mathbb{N}}$ converges to (r, x), there is some $n_1 \in \mathbb{N}$ such that $(r_n, x_n) \in U \cap F$ and $r_n \geq r$ for each $n \in \mathbb{N}$ with $n \geq n_1$. This implies that $\sup_{n\in\mathbb{N}:\, n\geq n_1} u(r_n, x_n) \leq \sup_{(s,y)\in U\cap F:\, s\geq r} u(s, y)$, which is a contradiction.

(ii) First of all, we choose a strictly decreasing sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that $\lim_{n\uparrow\infty} \alpha_n = \limsup_{(s,y)\to(r,x):\, s\geq r} u(s, y) \in [-\infty, \infty)$. By Lemma A.6, for each $n \in \mathbb{N}$ there is $\nu_n \in \mathbb{N}$ with $\nu_n \geq n$ such that

$$\sup_{(s,y)\in B_{1/\nu_n}(r,x)\cap F:\, s\geq r} u(s, y) < \alpha_n. \tag{A.2}$$

Moreover, for every $n \in \mathbb{N}$ there exists some $(r_n, x_n) \in B_{1/\nu_n}(r, x) \cap F$ with $r_n \geq r$ and $\sup_{(s,y)\in B_{1/\nu_n}(r,x)\cap F:\, s\geq r} u(s, y) < u(r_n, x_n) + 1/n$. From $\lim_{n\uparrow\infty} \nu_n = \infty$ and (A.2) we infer that the resulting sequence $(r_n, x_n)_{n\in\mathbb{N}}$ converges to (r, x) and satisfies $\lim_{n\uparrow\infty} u(r_n, x_n) = \limsup_{(s,y)\to(r,x):\, s\geq r} u(s, y)$.

(iii) To show the only if direction, suppose that (A.1) fails. By (ii), there are two sequences $(\underline{r}_n, \underline{x}_n)_{n\in\mathbb{N}}$ and $(\overline{r}_n, \overline{x}_n)_{n\in\mathbb{N}}$ in F that converge to (r, x) such that $\underline{r}_n \wedge \overline{r}_n \geq r$ for all $n \in \mathbb{N}$ and

$$\lim_{n\uparrow\infty} u(\underline{r}_n, \underline{x}_n) = \liminf_{(s,y)\to(r,x):\, s\geq r} u(s, y) < \limsup_{(s,y)\to(r,x):\, s\geq r} u(s, y) = \lim_{n\uparrow\infty} u(\overline{r}_n, \overline{x}_n),$$

which is a contradiction. For the converse direction, let $(r_n, x_n)_{n\in\mathbb{N}}$ be a sequence in F that converges to (r, x) with $r_n \geq r$ for all $n \in \mathbb{N}$, then (i) ensures that

$$\liminf_{(s,y)\to(r,x):\, s\geq r} u(s, y) \leq \liminf_{n\uparrow\infty} u(r_n, x_n) \leq \limsup_{n\uparrow\infty} u(r_n, x_n) \leq \limsup_{(s,y)\to(r,x)} u(s, y).$$

Thus, $(u(r_n, x_n))_{n\in\mathbb{N}}$ converges to the common value of $\liminf_{(s,y)\to(r,x):\, s\geq r} u(s, y)$ and $\limsup_{(s,y)\to(r,x):\, s\geq r} u(s, y)$. This in turn concludes the proof. □

Now, a function $u : F \to [-\infty, \infty]$ is called *upper right-hand semicontinuous* at a point $(r, x) \in F$ if $u(r, x) < \infty$ and for each $\varepsilon > 0$ there is $U \in \mathscr{U}(r, x)$ such that

$$u(s, y) < \begin{cases} u(r, x) + \varepsilon, & \text{if } u(r, x) > -\infty, \\ -\varepsilon, & \text{if } u(r, x) = -\infty, \end{cases}$$

for all $(s, y) \in U\cap F$ with $s \geq r$. We say that u is lower right-hand semicontinuous at (r, x) if $-u$ is upper right-hand semicontinuous there. Hence, u is upper (resp. lower) right-hand semicontinuous if it is upper (resp. lower) right-hand semicontinuous at each $(r, x) \in F$. As before, to each fact on upper right-hand semicontinuity there is a dual fact on lower right-hand semicontinuity.

A.8 Lemma. *A function $u : F \to [-\infty, \infty)$ is upper right-hand semicontinuous at a point $(r, x) \in F$ if and only if one of the following three equivalent conditions hold:*

(i) For each $\alpha \in \mathbb{R}$ with $u(r, x) < \alpha$ there is $U \in \mathscr{U}(r, x)$ such that $u(s, y) < \alpha$ for all $(s, y) \in U \cap F$ with $s \geq r$.

(ii) $\limsup_{(s,y)\to(r,x):\, s\geq r} u(s, y) \leq u(r, x)$.

(iii) For every sequence $(r_n, x_n)_{n\in\mathbb{N}}$ in F that converges to (r, x) with $r_n \geq r$ for almost all $n \in \mathbb{N}$ it holds that $\limsup_{n\uparrow\infty} u(r_n, x_n) \leq u(r, x)$.

Proof. We at first show that if u is upper right-hand semicontinuous at (r, x), then (i) holds. Let $\alpha \in \mathbb{R}$ satisfy $u(r, x) < \alpha$. If $u(r, x) > -\infty$, then for $\varepsilon := \alpha - u(r, x)$ there is $U \in \mathscr{U}(r, x)$ such that $u(s, y) < u(r, x) + \varepsilon = \alpha$ for all $(s, y) \in U \cap F$ with $s \geq r$. Otherwise, there exists some $U \in \mathscr{U}(r, x)$ with $u(s, y) < -|\alpha| \leq \alpha$ for every $(s, y) \in U \cap F$ with $s \geq r$, as desired.

(i) ⇒ (ii): Suppose the claimed inequality fails. Then we can pick $\alpha \in \mathbb{R}$ with $u(r, x) < \alpha < \limsup_{(s,y)\to(r,x):\, s\geq r} u(s, y)$. According to (i), there is $U \in \mathscr{U}(r, x)$ such that $u(s, y) < \alpha$ for all $(s, y) \in U \cap F$ with $s \geq r$. However, this is in conflict with $\sup_{(s,y)\in U\cap F:\, s\geq r} u(s, y) \geq \limsup_{(s,y)\to(r,x):\, s\geq r} u(s, y)$.

(ii) ⇒ (iii): Let $(r_n, x_n)_{n\in\mathbb{N}}$ be a sequence in F that converges to (r, x) and fulfills $r_n \geq r$ for almost all $n \in \mathbb{N}$. Then $\limsup_{n\uparrow\infty} u(r_n, x_n) \leq \limsup_{(s,y)\to(r,x):\, s\geq r} u(s, y)$ $\leq u(r, x)$, by Lemma A.7, which is the correct conclusion.

Finally, assume that (iii) holds but u fails to be upper right-hand semicontinuous at (r, x). We first let $u(r, x) > -\infty$. Then there exists $\varepsilon > 0$ such that for every

$n \in \mathbb{N}$ there is $(r_n, x_n) \in B_{1/n}(r,x) \cap F$ with $r_n \geq r$ and $u(r_n, x_n) \geq u(r,x) + \varepsilon$. The resulting sequence $(r_n, x_n)_{n\in\mathbb{N}}$ converges to (r,x) and satisfies

$$\limsup_{n\uparrow\infty} u(r_n, x_n) \geq u(r,x) + \varepsilon,$$

which is a contradiction to (iii). Suppose now that $u(r,x) = -\infty$, then there is some $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ there is $(r_n, x_n) \in B_{1/n}(r,x) \cap F$ with $r_n \geq r$ and $u(r_n, x_n) \geq -\varepsilon$. Thus, $(r_n, x_n)_{n\in\mathbb{N}}$ converges to (r,x) and $\limsup_{n\uparrow\infty} u(r_n, x_n) \geq -\varepsilon$, which is impossible. This shows the lemma. $\square$

Let us denote the set of all $[-\infty,\infty)$-valued upper right-hand semicontinuous functions on F by $USC^{\leftarrow}(F)$ and the set of all $(-\infty,\infty]$-valued lower right-hand semicontinuous functions on F by $LSC_{\leftarrow}(F)$. Then we can state two crucial facts. The proof of the first one is based on Theorem 7.22 in [20].

A.9 Lemma. *The following two assertions hold:*

(i) If $\mathscr{H} \subset USC^{\leftarrow}(F)$ is non-empty, then the function $v : F \to [-\infty,\infty)$ defined via $v(r,x) := \inf_{u\in\mathscr{H}} u(r,x)$ belongs to $USC^{\leftarrow}(F)$.

(ii) Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real-valued functions in $USC^{\leftarrow}(F)$ that converges locally uniformly to some function $u : F \to \mathbb{R}$, then $u \in USC^{\leftarrow}(F)$.

Proof. (i) Let $(r,x) \in F$ and $\alpha \in \mathbb{R}$ with $v(r,x) < \alpha$. Then the definition of v yields $u \in \mathscr{H}$ with $v(r,x) \leq u(r,x) < \alpha$. As u is upper right-hand semicontinuous, Lemma A.8 gives $U \in \mathscr{U}(r,x)$ such that $u(s,y) < \alpha$ for all $(s,y) \in U \cap F$ with $s \geq r$. Thus, $v(s,y) \leq u(s,y) < \alpha$ for each $(s,y) \in U \cap F$ with $s \geq r$, which shows the upper right-hand semicontinuity of v.

(ii) Let $(r,x) \in F$ and $\varepsilon > 0$, then there is $U \in \mathscr{U}(r,x)$ such that $(u_n)_{n\in\mathbb{N}}$ converges uniformly to u on $U \cap F$. We choose $n_0 \in \mathbb{N}$ fulfilling

$$|u_n(s,y) - u(s,y)| < \varepsilon/3$$

for every $(s,y) \in U \cap F$ and each $n \in \mathbb{N}$ with $n \geq n_0$. We let $n \in \mathbb{N}$ satisfy $n \geq n_0$, then the upper right-hand semicontinuity of u_n yields $V \in \mathscr{U}(r,x)$ such that $u_n(s,y) < u_n(r,x) + \varepsilon/3$ for all $(s,y) \in V \cap F$ with $s \geq r$. Thus,

$$u(s,y) < u_n(s,y) + \frac{\varepsilon}{3} < u_n(r,x) + \frac{2\varepsilon}{3} < u(r,x) + \varepsilon$$

for all $(s,y) \in U \cap V \cap F$ with $s \geq r$. This concludes the proof. $\square$

Our considerations motivate right-hand semicontinuous envelopes. So, let us fix a function $u : F \to [-\infty,\infty]$. First, if u is right-hand locally bounded from above, then the function

$$u^{\leftarrow} : F \to [-\infty,\infty), \quad u^{\leftarrow}(r,x) := \inf\{v(r,x) \,|\, v \in USC^{\leftarrow}(F) : u \leq v\}$$

is called the *upper right-hand semicontinuous envelope* of u. Similarly, whenever u is right-hand locally bounded from below, then

$$u_{\leftarrow} : F \to (-\infty, \infty], \quad u_{\leftarrow}(r,x) := \sup\{v(r,x) \,|\, v \in LSC_{\leftarrow}(F) : u \geq v\}$$

is the *lower right-hand semicontinuous envelope* of u. Let us emphasize that there cannot exist $v \in USC^{\leftarrow}(F)$ with $u \leq v$ as soon as u fails to be right-hand locally bounded from above. Indeed, in this case, there is $(r,x) \in F$ such that for each $n \in \mathbb{N}$ there is $\nu_n \in \mathbb{N}$ with $\nu_n \geq n$ and

$$\sup_{(s,y) \in B_{1/\nu_n}(r,x) \cap F:\, s \geq r} u(s,y) > n.$$

For each $n \in \mathbb{N}$ we choose $(r_n, x_n) \in B_{1/\nu_n}(r,x) \cap F$ with $r_n \geq r$ and $u(r_n, x_n) > n$, then the resulting sequence $(r_n, x_n)_{n \in \mathbb{N}}$ converges to (r,x) and $\lim_{n\uparrow\infty} u(r_n, x_n) = \infty$. Therefore, if there was $v \in USC^{\leftarrow}(F)$ with $u \leq v$, then

$$v(r,x) \geq \limsup_{n\uparrow\infty} v(r_n, x_n) \geq \limsup_{n\uparrow\infty} u(r_n, x_n) = \infty,$$

due to Lemma A.8, which is a contradiction. A similar remark holds for the lower right-hand semicontinuous envelope.

A.10 Lemma. *Let $u : F \to [-\infty, \infty)$ be right-hand locally bounded from above. Then $u^{\leftarrow} \in USC^{\leftarrow}(F)$ and the representation*

$$u^{\leftarrow}(r,x) = \limsup_{(s,y)\to(r,x):\, s \geq r} u(s,y)$$

holds for every $(r,x) \in F$. In particular, u is upper right-hand semicontinuous at a point $(r,x) \in F$ if and only if $u(r,x) = u^{\leftarrow}(r,x)$.

Proof. Let us first of all prove that the function $w : F \to [-\infty, \infty)$ defined via $w(r,x) := \limsup_{(s,y)\to(r,x):\, s\geq r} u(s,y)$ is upper right-hand semicontinuous. We fix $(r,x) \in F$ and $\alpha \in \mathbb{R}$ with $w(r,x) < \alpha$. Then there is $U \in \mathscr{U}(r,x)$ fulfilling

$$\sup_{(s,y) \in U \cap F:\, s \geq r} u(s,y) < \alpha.$$

We see that for each $(s,y) \in U$ it holds that $U \in \mathscr{U}(s,y)$. Thus, the definition of w gives $w(s,y) \leq \sup_{(s',y') \in U \cap F:\, s' \geq r} u(s',y') < \alpha$ for each $(s,y) \in U$ with $s \geq r$. By Lemma A.8, the upper right-hand semicontinuity of w is verified. Let us now choose a function $v \in USC^{\leftarrow}(F)$ with $u \leq v$. Then we obtain

$$w(r,x) = \limsup_{(s,y)\to(r,x):\, s\geq r} u(s,y) \leq \limsup_{(s,y)\to(r,x):\, s\geq r} v(s,y) \leq v(r,x)$$

from Lemma A.8, because v is upper right-hand semicontinuous at (r,x). Hence, as $u(r,x) \leq w(r,x)$ and $(r,x) \in F$ has been arbitrarily chosen, both claims follow. $\square$

Let us now suppose that ρ is a metric on S and for each $x \in S$ let $\mathscr{U}(x)$ be the system of all neighborhoods of x in S. We let $R \subset S$ be non-empty and recall that a function $u : R \to [-\infty, \infty]$ is upper semicontinuous if $u < \infty$ and for each $x \in R$ and every $\varepsilon > 0$ there is $U \in \mathscr{U}(x)$ such that

$$u(y) < \begin{cases} u(x) + \varepsilon, & \text{if } u(x) > -\infty, \\ -\varepsilon, & \text{if } u(x) = -\infty, \end{cases}$$

for all $y \in U$. Moreover, u is lower semicontinuous if $-u$ is upper semicontinuous (cf. Definition 7.20 in [20]).

A.11 Lemma. *Let $u : R \to [-\infty, \infty)$, then the following three assertions hold:*

(i) u is upper semicontinuous if and only if $u^{-1}([-\infty, \alpha))$ is open in R for all $\alpha \in \mathbb{R}$.

(ii) Whenever u is upper semicontinuous, then it attains its maximum on every compact set K in R.

(iii) Let $D \subset \mathbb{R}$ be a non-degenerate interval, $u(R) \subset D$, and $\Phi : D \to \mathbb{R}$ be increasing and right-continuous. If u is upper semicontinuous, then so is $\Phi \circ u$.

Proof. (i) For the only if direction let $\alpha \in \mathbb{R}$ and $x \in u^{-1}([-\infty, \alpha))$. If $u(x) > -\infty$, then for $\varepsilon := \alpha - u(x)$ there is $U \in \mathscr{U}(x)$ such that $u(y) < u(x) + \varepsilon = \alpha$ for all $y \in U \cap R$. If instead $u(x) = -\infty$, then there is $U \in \mathscr{U}(x)$ with $u(y) < -|\alpha| \leq \alpha$ for each $y \in U \cap R$. In either case, $U \subset u^{-1}([-\infty, \alpha))$, as desired.

For if pick $x \in R$ and $\varepsilon > 0$. If $u(x) = -\infty$, then, as $x \in u^{-1}([-\infty, -\varepsilon))$, there is $U \in \mathscr{U}(x)$ with $u(y) < -\varepsilon$ for all $y \in U \cap R$. Otherwise, we see from $x \in u^{-1}([-\infty, u(x) + \varepsilon))$ that there is $U \in \mathscr{U}(x)$ with $u(y) < u(x) + \varepsilon$ for every $y \in U \cap R$, which concludes the proof of (i).

(ii) Let us suppose the contrary. That is, $u(x) < \sup_{y \in K} u(y)$ for each $x \in K$. According to (i), for all $x \in K$ we can pick $\alpha_x \in \mathbb{R}$ and $U_x \in \mathscr{U}(x)$ such that

$$u(y) < \alpha_x < \sup_{y' \in K} u(y')$$

for each $y \in U_x \cap R$. As $\{U_x \,|\, x \in K\}$ is an open covering of K, there are $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in K$ such that $K \subset \bigcup_{i=1}^n U_{x_i}$. Let $y \in K$, then there is $i \in \{1, \ldots, n\}$ with $y \in U_{x_i}$ and hence, $u(y) < \alpha_{x_i}$. So, $\sup_{y \in K} u(y) \leq \max_{i \in \{1,\ldots,n\}} \alpha_{x_i}$, a contradiction. Therefore, there is $x \in K$ with $u(x) = \sup_{y \in K} u(y)$, which is the claim.

(iii) Let $x \in R$ and $\alpha \in \mathbb{R}$ be such that $\Phi(u(x)) < \alpha$. If $\Phi(z) < \alpha$ for every $z \in D$, then $\Phi(u(y)) < \alpha$ for each $y \in R$. Otherwise, there is at least one $z' \in D$ with $\Phi(z') \geq \alpha$. We set $q_\alpha := \sup\{z \in D \,|\, \Phi(z) < \alpha\}$, then $u(x) \leq q_\alpha \leq z'$, as Φ is increasing. This also shows that $q_\alpha \in D$, since D is an interval. Right-continuity of Φ at q_α gives $\Phi(q_\alpha) \geq \alpha$. Hence, $u(x) < q_\alpha$ and there is $U \in \mathscr{U}(x)$ such that $u(y) < q_\alpha$ for all $y \in U \cap R$. Because for each $y \in U \cap R$ there is $\hat{z} \in D$ with $\Phi(\hat{z}) < \alpha$ and $u(y) < \hat{z} < q_\alpha$, the proof is complete. ☐

Finally, we change the setting, and suppose more generally that J is merely a metric space and S is a non-empty set. We denote the underlying metric on J by ρ and let D be a non-empty closed set in a Banach space E with complete norm $\|\cdot\|$.

A.12 Proposition. *Let I be a dense set in J and $u : I \times S \to D$ be uniformly continuous in $r \in I$, uniformly in $x \in S$. Then there is a unique extension $\overline{u}$ of u to $J \times S$ such that $\overline{u}(\cdot, x)$ is continuous for each $x \in S$. Moreover, $\overline{u}(J \times S) \subset D$,*

$$\lim_{r\to t} u(r,x) = \overline{u}(t,x), \quad \textit{uniformly in } x \in S, \tag{A.3}$$

for each $t \in J$, and $\overline{u}$ is actually uniformly continuous in $t \in J$, uniformly in $x \in S$.

Proof. To verify uniqueness, assume that v and w are two extensions of u to $J \times S$ such that $v(\cdot,x)$ and $w(\cdot,x)$ are continuous for all $x \in S$. Let $t \in J$, then there is a sequence $(r_n)_{n\in\mathbb{N}}$ in I such that $\lim_{n\uparrow\infty} r_n = t$. We also choose $x \in S$, then $v(r_n,x) = u(r_n,x) = w(r_n,x)$ for all $n \in \mathbb{N}$. By taking the limit $n \uparrow \infty$, we obtain that $v(t,x) = \lim_{n\uparrow\infty} v(r_n,x) = \lim_{n\uparrow\infty} w(r_n,x) = w(t,x)$. Hence, $v = w$.

Let us establish the existence of $\overline{u}$. We first note that if $(r_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in I, then $(u(r_n,\cdot))_{n\in\mathbb{N}}$ is a uniformly Cauchy sequence. Indeed, for each $\varepsilon > 0$ there is $\delta > 0$ such that $\|u(q,x) - u(r,x)\| < \varepsilon$ for all $q, r \in I$ with $\rho(q,r) < \delta$ and each $x \in S$. Thus, if $n_0 \in \mathbb{N}$ is such that $\rho(r_m, r_n) < \delta$ for all $m, n \in \mathbb{N}$ with $m \wedge n \geq n_0$, then $\|u(r_m,x) - u(r_n,x)\| < \varepsilon$ for each $m, n \in \mathbb{N}$ with $m \wedge n \geq n_0$ and every $x \in S$, as claimed.

We let $t \in J$ and $(r_n)_{n\in\mathbb{N}}$ be a sequence in I with $\lim_{n\uparrow\infty} r_n = t$, then $(u(r_n,\cdot))_{n\in\mathbb{N}}$ converges uniformly to some map $z : S \to D$, since D is closed and E is complete. If we can show that for each sequence $(q_n)_{n\in\mathbb{N}}$ in I with $\lim_{n\uparrow\infty} q_n = t$ it follows that $(u(q_n,\cdot))_{n\in\mathbb{N}}$ also converges uniformly to z, then, by defining $\overline{u}(t,x) := z(x)$ for each $x \in S$, the existence of $\overline{u}$ and (A.3) follow. So, let $\varepsilon > 0$, then there is $\delta > 0$ such that $\|u(q,x) - u(r,x)\| < \varepsilon/2$ for all $q, r \in I$ with $\rho(q,r) < \delta$ and every $x \in S$. We choose $n_0 \in \mathbb{N}$ such that

$$\rho(q_n,t) \vee \rho(r_n,t) < \delta/2 \quad \text{and} \quad \|u(r_n,x) - z(x)\| < \varepsilon/2$$

for all $n \in \mathbb{N}$ with $n \geq n_0$ and each $x \in S$. Then $\rho(q_n, r_n) < \delta$, which gives $\|u(q_n,x) - z(x)\| \leq \|u(q_n,x) - u(r_n,x)\| + \|u(r_n,x) - z(x)\| < \varepsilon$ for each $n \in \mathbb{N}$ with $n \geq n_0$ and every $x \in S$. Thus, $(u(q_n,\cdot))_{n\in\mathbb{N}}$ converges uniformly to z as well.

Finally, let us show that $\overline{u}$ is uniformly continuous in $t \in J$, uniformly in $x \in S$. We pick $\varepsilon > 0$, then there is some $\delta > 0$ such that $\|u(q,x) - u(r,x)\| < \varepsilon/3$ for all $q, r \in I$ with $\rho(q,r) < \delta$ and each $x \in S$. We choose $s, t \in J$ with $\rho(s,t) < \delta/3$, then (A.3) gives $\delta_s > 0$ and $\delta_t > 0$ such that

$$\|u(q,x) - \overline{u}(s,x)\| < \varepsilon/3 \quad \text{and} \quad \|u(r,x) - \overline{u}(t,x)\| < \varepsilon/3$$

for all $q, r \in I$ with $\rho(q,s) < \delta_s$ and $\rho(r,t) < \delta_t$ and every $x \in S$. We pick $q, r \in I$ such that $\rho(q,s) < (\delta/3) \wedge \delta_s$ and $\rho(r,t) < (\delta/3) \wedge \delta_t$, then we conclude that

$$\begin{aligned}\|\overline{u}(s,x) - \overline{u}(t,x)\| \leq{}& \|\overline{u}(s,x) - u(q,x)\| \\ &+ \|u(q,x) - u(r,x)\| + \|u(r,x) - \overline{u}(t,x)\| < \varepsilon,\end{aligned}$$

because $\rho(q,r) \leq \rho(q,s) + \rho(s,t) + \rho(t,r) < \delta$. $\square$

A.3 Distance functions

The purpose of this section is to summarize relevant facts on distance functions and neighborhoods of sets. Furthermore, we approximate open sets in a metric space pointwise by a sequence of Lipschitz continuous functions.

Let (E, ϱ) be a metric space. For $\hat{z} \in E$ and $D \subset E$ we recall that the number $\operatorname{dist}(\hat{z}, D) := \inf_{z \in D} \varrho(\hat{z}, z)$ is called the *distance from $\hat{z}$ to D*. If in addition $C \subset E$, then $\operatorname{dist}(C, D) := \inf_{(z,z') \in C \times D} \varrho(z, z')$ is said to be the distance from C to D. We first state several standard properties of distance functions (cf. Exercises 6.86 and 6.87 in [20]).

A.13 Lemma. *Let $C, D \subset E$, then the following four assertions hold:*

(i) $\overline{D} = \{z \in E \,|\, \operatorname{dist}(z, D) = 0\}$ *and* $\operatorname{dist}(C, D) = \operatorname{dist}(C, \overline{D})$.

(ii) The function $\operatorname{dist}(\cdot, D) : E \to \mathbb{R}_+$, $z \mapsto \operatorname{dist}(z, D)$ *is Lipschitz continuous with Lipschitz constant* 1.

(iii) If D is relatively compact, then there is $\hat{z} \in \overline{D}$ with $\operatorname{dist}(C, D) = \operatorname{dist}(C, \hat{z})$.

(iv) Let $C \subset \overline{D}$. If $\operatorname{dist}(C, \partial D) > 0$, *then $\overline{C} \subset D^\circ$. Conversely, if C or ∂D is relatively compact and $\overline{C} \subset D^\circ$, then* $\operatorname{dist}(C, \partial D) > 0$.

Proof. (i) Let $\hat{z} \in E$, then, by definition, $\operatorname{dist}(\hat{z}, D) = 0$ if and only if for each $\varepsilon > 0$ there exists some $z \in D$ such that $\varrho(z, \hat{z}) < \varepsilon$, which is equivalent to $\hat{z} \in \overline{D}$. Thus, $\overline{D} = \{z \in E \,|\, \operatorname{dist}(z, D) = 0\}$. Next, from $D \subset \overline{D}$ we directly obtain that $\operatorname{dist}(C, D) \geq \operatorname{dist}(C, \overline{D})$. Contrary to the claim, assume that

$$\operatorname{dist}(C, D) > \operatorname{dist}(C, \overline{D}).$$

Then there is $(z, z') \in C \times \overline{D}$ with $\operatorname{dist}(C, D) > \varrho(z, z')$. Since z' is a limit point of D, there is $z'' \in D$ such that $\varrho(z', z'') < \operatorname{dist}(C, D) - \varrho(z, z')$. This gives us that $\varrho(z, z'') \leq \varrho(z, z') + \varrho(z', z'') < \operatorname{dist}(C, D)$, a contradiction.

(ii) As ϱ is symmetric, that is, $\varrho(z, z') = \varrho(z', z)$ for all $z, z' \in E$, it is enough to show that $\operatorname{dist}(z, D) - \operatorname{dist}(z', D) \leq \varrho(z, z')$ for each $z, z' \in E$. By contradiction, suppose that there are $z, z' \in E$ for which this inequality fails. Then there is $z'' \in D$ such that $\varrho(z', z'') < \operatorname{dist}(z, D) - \varrho(z, z')$. Consequently, $\varrho(z, z'') \leq \varrho(z, z') + \varrho(z', z'')$ $< \operatorname{dist}(z, D)$, which is impossible.

(iii) We note that $\operatorname{dist}(C, D) = \inf_{z \in D} \operatorname{dist}(C, z)$, even if D is not relatively compact. By (ii) and Lemma A.11, the function $\operatorname{dist}(C, \cdot) : E \to \mathbb{R}_+$, $z \mapsto \operatorname{dist}(C, z)$ attains a minimum over $\overline{D}$, say at $\hat{z}$. In consequence, from (i) we conclude that $\operatorname{dist}(C, D) = \operatorname{dist}(C, \overline{D}) = \min_{z \in \overline{D}} \operatorname{dist}(C, z) = \operatorname{dist}(C, \hat{z})$.

(iv) If there exists $\hat{z} \in \overline{C} \cap \partial D$, then (i) gives us that $\operatorname{dist}(C, \partial D) = \operatorname{dist}(\overline{C}, \partial D)$ $\leq \varrho(\hat{z}, \partial D) = 0$, a contradiction. Conversely, let C or ∂D be relatively compact and suppose that $\operatorname{dist}(C, \partial D) = 0$. If C is relatively compact, then (iii) yields $\hat{z} \in \overline{C}$ with $\operatorname{dist}(\hat{z}, \partial D) = 0$. But then (i) implies that $\hat{z} \in \partial D$, since ∂D is closed. Similarly, if ∂D is relatively compact, then there is some $\hat{z} \in \partial D$ with $\operatorname{dist}(C, \hat{z}) = 0$, which entails that $\hat{z} \in \overline{C}$. In either case, a contradiction follows. $\square$

A.14 Example. Let $E = \mathbb{R}^2$, $C = \{(x,y) \in (0,\infty)^2 \,|\, y > 1/x\}$, and $D = \mathbb{R}^2_+$. Then C and ∂D are unbounded and $\overline{C} \subset D^\circ$. However, $\operatorname{dist}(C, \partial D) = 0$, because $\lim_{x\uparrow\infty} 1/x = 0$.

We notice that for each $D \subset E$ the interior of D^c is exactly $(\overline{D})^c$. More precisely, $\overline{D}$ is closed and hence, $(\overline{D})^c$ is an open set included in D^c. Moreover,

$$D^c \backslash (\overline{D})^c = D^c \cap \overline{D} = D^c \cap \partial D.$$

But if $\hat{z} \in D^c \cap \partial D$, then for each $\varepsilon > 0$ there is $z \in D$ with $\varrho(z, \hat{z}) < \varepsilon$. So, $\hat{z}$ cannot be an interior point of D^c, and $(D^c)^\circ = (\overline{D})^c$ follows. Next, we recall that for each $z, \hat{z} \in E$ a *path* from z to $\hat{z}$ is a map $\gamma \in C([0,1], E)$ with $\gamma(0) = z$ and $\gamma(1) = \hat{z}$.

A.15 Lemma. *Let $D \subset E$ and $z \in D^\circ$. If for $\hat{z} \in E$ there is a path γ from z to $\hat{z}$ such that $\varrho(z, \gamma(t)) < \operatorname{dist}(z, \partial D)$ for all $t \in [0,1]$, then $\hat{z} \in D^\circ$.*

Proof. If $\hat{z} \in \partial D$, then $\varrho(z, \gamma(1)) = \varrho(z, \hat{z}) \geq \operatorname{dist}(z, \partial D)$, which is impossible. Thus, since E is the union of the disjoint sets D°, ∂D and $(\overline{D})^c$, we either have $\hat{z} \in D^\circ$ or $\hat{z} \in (\overline{D})^c$. Contrary to our assertion, let us assume that $\hat{z} \in (\overline{D})^c$. By hypothesis, $\gamma(0) = z \in D$ and $\gamma(1) = \hat{z} \in (\overline{D})^c$. Therefore, $\gamma^{-1}(D)$ is not empty and included in $[0,1)$.

In addition, since z and $\hat{z}$ are interior points of D and D^c, respectively, there is $\varepsilon > 0$ with $B_\varepsilon(z) \subset D$ and $B_\varepsilon(\hat{z}) \subset D^c$. As γ is right-continuous at 0 and left-continuous at 1, there is $\delta \in (0, 1/2]$ with

$$[0,\delta) \subset \gamma^{-1}(B_\varepsilon(z)) \quad \text{and} \quad (1-\delta, 1] \subset \gamma^{-1}(B_\varepsilon(\hat{z})).$$

Hence, $\gamma(t) \in D$ for all $t \in [0,\delta)$ and $\gamma(t) \in D^c$ for all $t \in (1-\delta, 1]$. Consequently, $t^* := \sup \gamma^{-1}(D)$ is subject to $0 < \delta \leq t^* \leq 1 - \delta < 1$. Eventually, we verify that $\gamma(t^*) \in \partial D$, which contradicts $\varrho(z, \gamma(t^*)) < \operatorname{dist}(z, \partial D)$.

For this purpose, let $\eta > 0$, then the continuity of γ at t^* yields $\delta' \in (0, t^* \wedge (1-t^*))$ such that $\gamma(t) \in B_\eta(\gamma(t^*))$ for all $t \in (t^* - \delta', t^* + \delta')$. At the same time the definition of t^* gives $s \in [0, t^*]$ with $\gamma(s) \in D$ and $t^* < s + \delta'$, which implies that $\gamma(s) \in B_\eta(\gamma(t^*))$. However, the definition of t_* also entails that $\gamma(t) \in D^c$ for all $t \in (t^*, 1]$. That is, $B_\eta(\gamma(t^*)) \cap D$ and $B_\eta(\gamma(t^*)) \cap D^c$ are not empty. Hence, as $\eta > 0$ has been arbitrarily chosen, $\gamma(t^*) \in \partial D$. □

As we know, for $D \subset E$ and $\varepsilon > 0$, the set $N_\varepsilon(D) := \bigcup_{z \in D} B_\varepsilon(z)$ is called the *ε-neighborhood* of D. Since the union of arbitrarily many open sets is open, $N_\varepsilon(D)$ is open. Moreover,

$$N_\varepsilon(D) = \{z \in E \,|\, \operatorname{dist}(z, D) < \varepsilon\}.$$

Indeed, $z \in E$ belongs to $N_\varepsilon(D)$ if and only if there is $\hat{z} \in D$ with $\varrho(z, \hat{z}) < \varepsilon$, which is equivalent to $\operatorname{dist}(z, D) < \varepsilon$. By Lemma A.13, $\operatorname{dist}(z, D) = \operatorname{dist}(z, \overline{D})$ for all $z \in E$. This implies that

$$N_\varepsilon(D) = \{z \in E \,|\, \operatorname{dist}(z, \overline{D}) < \varepsilon\} = N_\varepsilon(\overline{D}).$$

From $\overline{D} \subset N_\varepsilon(\overline{D})$ we also get that $\overline{D} \subset N_\varepsilon(D)$. Finally, to keep notation simple, we denote the closure of $N_\varepsilon(D)$ by $\overline{N}_\varepsilon(D)$.

A.16 Corollary. *Suppose that E is a linear space and ϱ is induced by a norm $\|\cdot\|$. That is, $\varrho(z,z') = \|z - z'\|$ for all $z, z' \in E$. In addition, let $D \subset E$.*

(i) Let $C \subset \overline{D}$ be such that $\mathrm{dist}(C, \partial D) > 0$. Then $\overline{N}_\varepsilon(C) \subset D^\circ$ for every $\varepsilon \in (0, \mathrm{dist}(C, \partial D)]$.

(ii) If D is convex, then $D_\varepsilon := \{z \in D \,|\, \mathrm{dist}(z, \partial D) \geq \varepsilon\}$ is convex for each $\varepsilon > 0$.

Proof. (i) By the preceding discussion, $N_\varepsilon(\overline{C}) = \{z \in E \,|\, \mathrm{dist}(z, C) < \varepsilon\}$. Thus, let $\hat{z} \in N_\varepsilon(\overline{C})$, then there exists $z \in C$ with $\|z - \hat{z}\| < \varepsilon$. Now, the map $\gamma : [0,1] \to E$, $\gamma(t) := t\hat{z} + (1-t)z$ is a path from z to $\hat{z}$ and it holds that

$$\varrho(z, \gamma(t)) = t\|z - \hat{z}\| \leq \|z - \hat{z}\| < \varepsilon$$

for all $t \in [0,1]$. We notice that $z \in D^\circ$, by Lemma A.13. Since $\varepsilon \leq \mathrm{dist}(C, \partial D)$ $\leq \mathrm{dist}(z, \partial D)$, we conclude from Lemma A.15 that $\hat{z} \in D^\circ$.

(ii) To show the claim, it suffices to verify that $D_\varepsilon = \{z \in D \,|\, B_\varepsilon(z) \subset D\}$, by Lemma A.5. Let $z \in D_\varepsilon$, then $z \in D^\circ$, since $\mathrm{dist}(z, \partial D) > 0$. Moreover,

$$\|z - \hat{z}\| < \varepsilon \leq \mathrm{dist}(z, \partial D) \quad \text{for all } \hat{z} \in B_\varepsilon(z).$$

For this reason, Lemma A.15 shows that $B_\varepsilon(z) \subset D^\circ$. For the converse inclusion, let $z \in D$ fulfill $B_\varepsilon(z) \subset D$, then, as $B_\varepsilon(z)$ is an open set included in D, we must have $B_\varepsilon(z) \subset D^\circ$. Suppose that $\mathrm{dist}(z, \partial D) < \varepsilon$, then there is $\hat{z} \in \partial D$ with $\|z - \hat{z}\| < \varepsilon$, which yields the contradiction $B_\varepsilon(z) \cap \partial D \neq \emptyset$. Thus, the claim is verified. $\square$

We conclude this section with the pointwise approximation of indicator functions of open sets in E.

A.17 Lemma. *For each open set O in E there is an increasing sequence $(\varphi_n)_{n\in\mathbb{N}}$ of $[0,1]$-valued Lipschitz continuous functions on E that converges pointwise to $\mathbb{1}_0$.*

Proof. Let us choose an increasing Lipschitz continuous function $\varphi : \mathbb{R}_+ \to [0,1]$ with $\varphi(0) = 0$ and $\lim_{x\uparrow\infty} \varphi(x) = 1$. For instance, $\varphi(x) = 1 - e^{-x}$ for all $x \geq 0$. Indeed, $\lim_{x\uparrow\infty} e^{-x} = 0$ and the Lipschitz continuity follows from the mean value theorem, which gives $|\varphi(x) - \varphi(y)| \leq \max_{t\in[0,1]} e^{-tx-(1-t)y}|x - y| \leq |x - y|$ for each $x, y \geq 0$. We define a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of $[0,1]$-valued functions on E through

$$\varphi_n(z) := \varphi(n\mathrm{dist}(z, O^c)) \quad \text{for all } n \in \mathbb{N}.$$

Then $\varphi_n(z) \leq \varphi_{n+1}(z)$ for every $n \in \mathbb{N}$ and each $z \in E$, because φ is increasing. Let $L \geq 0$ be a Lipschitz constant of φ, then Lemma A.13 yields that

$$|\varphi_n(z) - \varphi_n(z')| \leq Ln|\mathrm{dist}(z, O^c) - \mathrm{dist}(z', O^c)| \leq Ln\varrho(z, z')$$

for all $n \in \mathbb{N}$ and every $z, z' \in E$, which entails that φ_n has Lipschitz constant Ln. Since O is open, we know from Lemma A.13 that each $z \in E$ fulfills $\mathrm{dist}(z, O^c) > 0$ if and only if $z \in O$. Hence, $\lim_{n\uparrow\infty} \varphi_n(z) = 1$ for all $z \in O$. As $\varphi_n(z) = \varphi(0) = 0$ for all $n \in \mathbb{N}$ and each $z \in O^c$, the claim is proven. $\square$

A.4 Approximation of measurable maps

The main content of this section is the pointwise approximation of a measurable map taking values in a finite-dimensional Banach space by a suitable sequence of simple maps. As we will see, in one dimension, this reduces to the classical result that every non-negative measurable function is the pointwise limit of an increasing sequence of non-negative simple functions (see for instance Theorem 11.6 in [2]).

We start with a measurable space $(\Omega, \mathscr{F})$ and a metric space (E, ϱ). The Borel σ-field of E is denoted by $\mathscr{B}$. In our setting, a map $f : \Omega \to E$ is *$\mathscr{F}$-measurable* if $f^{-1}(B) \in \mathscr{F}$ for all $B \in \mathscr{B}$. Since the set of all $B \in \mathscr{B}$ such that $f^{-1}(B) \in \mathscr{F}$ is a σ-field in E and the topology of E generates $\mathscr{B}$, it follows that f is $\mathscr{F}$-measurable if and only if $f^{-1}(O) \in \mathscr{F}$ for each open set O in E. Let us consider two results on measurability, stated as Lemma 8.1.9 and Proposition 8.1.10 in [5]. First, f is $\mathscr{F}$-measurable if and only if

$$\varphi \circ f \text{ is } \mathscr{F}\text{-measurable for all } \varphi \in C(E, \mathbb{R}_+).$$

Since every $\mathbb{R}_+$-valued continuous function on E is Borel measurable, we only have to check the if direction. To this end, let O be an open set in E. Then Lemma A.13 yields that $\varphi : E \to \mathbb{R}_+$ defined by $\varphi(z) := \operatorname{dist}(z, O^c)$ is Lipschitz continuous and satisfies $O = \{z \in E \,|\, \varphi(z) > 0\} = \varphi^{-1}((0,\infty))$. Hence, $f^{-1}(O) = f^{-1}(\varphi^{-1}((0,\infty)))$ $= (\varphi \circ f)^{-1}((0,\infty)) \in \mathscr{F}$, as desired. The second result deals with pointwise limits of sequences of measurable maps.

A.18 Lemma. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of E-valued $\mathscr{F}$-measurable maps on Ω that converges pointwise to some map $f : \Omega \to E$, then f is $\mathscr{F}$-measurable.*

Proof. By the preceding discussion, it is enough to show that $\varphi \circ f$ is $\mathscr{F}$-measurable for each $\varphi \in C(E, \mathbb{R}_+)$. As f_n is $\mathscr{F}$-measurable, so is $\varphi \circ f_n$ for all $n \in \mathbb{N}$. Continuity of φ entails that the sequence $(\varphi \circ f_n)_{n\in\mathbb{N}}$ of $\mathbb{R}_+$-valued $\mathscr{F}$-measurable functions on Ω converges pointwise to $\varphi \circ f$. Hence, $\varphi \circ f$ is $\mathscr{F}$-measurable. □

From here on, let E be a finite-dimensional linear space and ρ be induced by a complete norm $\|\cdot\|$ on E. That is, $\varrho(z, z') = \|z - z'\|$ for each $z, z' \in E$. We set $k := \dim(E)$, then there is an isomorphism $\phi : E \to \mathbb{R}^k$, which is necessarily bimeausurable in the sense that ϕ and its inverse ϕ^{-1} are Borel measurable.

A.19 Lemma. *A map $f : \Omega \to E$ is $\mathscr{F}$-measurable if and only if the i-th coordinate function $\phi_i \circ f$ of the map $\phi \circ f : \Omega \to \mathbb{R}^k$ is $\mathscr{F}$-measurable for all $i \in \{1, \ldots, k\}$.*

Proof. The only if direction is valid, since the composition of two measurable maps is measurable. For if it suffices to show that $\phi \circ f$ is $\mathscr{F}$-measurable, because from $f = \phi^{-1} \circ (\phi \circ f)$ the $\mathscr{F}$-measurability of f follows. We note that

$$(\phi \circ f)^{-1}(B) = (\phi_1 \circ f)^{-1}(B_1) \cap \cdots \cap (\phi_k \circ f)^{-1}(B_k) \in \mathscr{F}$$

for each $B \in \mathscr{B}(\mathbb{R}^k)$ of the form $B = B_1 \times \cdots \times B_k$ for some $B_1, \ldots, B_k \in \mathscr{B}(\mathbb{R})$. Since the set of all $B \in \mathscr{B}(\mathbb{R}^k)$ with $(\phi \circ f)^{-1}(B) \in \mathscr{F}$ is a σ-field in $\mathbb{R}^k$ and $\times_{i=1}^k \mathscr{B}(\mathbb{R})$ generates $\mathscr{B}(\mathbb{R}^k)$, the assertion follows. □

We recall that a map $f : \Omega \to E$ is *$\mathscr{F}$-simple* if it is $\mathscr{F}$-measurable and takes finitely many values. In this case, there exists $m \in \mathbb{N}$ such that $z_1, \dots, z_m \in E$ are the pairwise distinct values of f. The sets $A_1 := \{f = z_1\}, \dots, A_m := \{f = z_m\}$ belong to $\mathscr{F}$ and form a *decomposition* of Ω. That means, $A_1, \dots, A_m$ are pairwise disjoint and $\bigcup_{i=1}^m A_i = \Omega$. Furthermore,

$$f = \sum_{i=1}^{m} z_i \mathbb{1}_{A_i}. \tag{A.4}$$

Conversely, assume that $m \in \mathbb{N}$, $z_1, \dots, z_m \in E$, and $A_1, \dots, A_m \in \mathscr{F}$ are such that above representation holds. Then f is $\mathscr{F}$-measurable and takes at most m pairwise distinct values. Hence, f is $\mathscr{F}$-simple if and only if (A.4) is valid for some $m \in \mathbb{N}$, $z_1, \dots, z_m \in E$, and $A_1, \dots, A_m \in \mathscr{F}$. Every representation (A.4) for f in which the sets $A_1, \dots, A_m$ form a decomposition of Ω is called *normal*. As we wish to approximate every E-valued $\mathscr{F}$-measurable map on Ω by an appropriate sequence of E-valued $\mathscr{F}$-simple maps on Ω, we introduce set partitions.

A.20 Definition. Let $C, D \in \mathscr{B}$ be non-empty with $C \subset D$ and $(C_n)_{n\in\mathbb{N}}$ be an increasing sequence in $\mathscr{B}$ with $C_1 \neq \emptyset$.

(i) A *set partition* of C is a countable system $\mathbb{T}$ of bounded and pairwise disjoint Borel sets in C with $\bigcup_{B\in\mathbb{T}} B = C$. If C is compact, then we also require that $\mathbb{T}$ has finitely many elements.

(ii) Let $\mathbb{T}$ be a set partition of C. Then $|\mathbb{T}| := \sup_{B\in\mathbb{T}} \operatorname{diam}(B)$ is called the *mesh* of $\mathbb{T}$.

(iii) Let $\mathbb{S}$ and $\mathbb{T}$ be two set partitions of C and D, respectively. We say that $\mathbb{T}$ *refines* $\mathbb{S}$ if for each $B \in \mathbb{S}$ there are $n \in \mathbb{N}$ and pairwise distinct sets $B_1, \dots, B_n \in \mathbb{T}$ such that $\bigcup_{i=1}^n B_i = B$.

(iv) A *refining sequence of set partitions* of $(C_n)_{n\in\mathbb{N}}$ is a sequence $(\mathbb{T}_n)_{n\in\mathbb{N}}$, where $\mathbb{T}_n$ is a set partition of C_n for all $n \in \mathbb{N}$, such that $\mathbb{T}_{n+1}$ refines $\mathbb{T}_n$ for each $n \in \mathbb{N}$ and $\lim_{n\uparrow\infty} |\mathbb{T}_n| = 0$.

We justify the existence of set partitions and refining sequences of set partitions.

A.21 Lemma. *Let $C \in \mathscr{B}$ be non-empty and $\delta > 0$, then there is a set partition $\mathbb{T}$ of C with $|\mathbb{T}| \leq \delta$ that has finitely many elements if C is bounded. Moreover, for each increasing sequence $(C_n)_{n\in\mathbb{N}}$ in $\mathscr{B}$ with $C_1 \neq \emptyset$ there is a refining sequence $(\mathbb{T}_n)_{n\in\mathbb{N}}$ of set partitions of $(C_n)_{n\in\mathbb{N}}$.*

Proof. Let first assume that C is bounded. Since $\overline{C}$ is compact and $\{B_{\delta/2}(z) \,|\, z \in \overline{C}\}$ is an open covering of $\overline{C}$, there are $n \in \mathbb{N}$ and pairwise distinct $z_1, \dots, z_n \in \overline{C}$ such that $\bigcup_{i=1}^n B_{\delta/2}(z_i)$ includes $\overline{C}$ while

$$\bigcup_{j=1}^{m} B_{\delta/2}(z_{i_j}) \text{ fails to include } \overline{C}$$

for each $m \in \{1, \dots, n-1\}$ and every $i_1, \dots, i_m \in \{1, \dots, n\}$ with $i_1 < \cdots < i_m$. We let $B_1 := B_{\delta/2}(z_1) \cap C$ and define recursively $B_i := (B_{\delta/2}(z_i) \cap C) \setminus B_{i-1}$ for all $i \in \{2, \dots, n\}$. Then $B_1, \dots, B_n$ are non-empty and pairwise disjoint Borel sets in C with $\bigcup_{i=1}^n B_i = \bigcup_{i=1}^n (B_{\delta/2}(z_i) \cap C) = C$. Thus, $\mathbb{T} := \{B_1, \dots, B_n\}$ is a set partition of C and $|\mathbb{T}| \leq \sup_{i \in \{1,\dots,n\}} \operatorname{diam}(B_{\delta/2}(z_i)) = \delta$.

Now, let C be unbounded. We define a sequence $(\nu_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ recursively by $\nu_0 := 0$ and $\nu_n := \min\{m \in \mathbb{N} \,|\, m > \nu_{n-1},\, m-1 \leq \|z\| < m \text{ for some } z \in C\}$ for each $n \in \mathbb{N}$, then $(\nu_n)_{n\in\mathbb{N}}$ is strictly increasing. Moreover, we set

$$C_n := \{z \in C \,|\, \nu_n - 1 \leq \|z\| < \nu_n\} \quad \text{for all } n \in \mathbb{N}.$$

Then $(C_n)_{n\in\mathbb{N}}$ is a sequence of non-empty, bounded, and pairwise disjoint Borel sets in C with $\bigcup_{n\in\mathbb{N}} C_n = C$. By what we have shown, for each $n \in \mathbb{N}$ there exists a set partition $\mathbb{T}_n$ of C_n with finitely many elements such that $|\mathbb{T}_n| \leq \delta$. Consequently,

$$\mathbb{T} := \bigcup_{n\in\mathbb{N}} \mathbb{T}_n$$

is a set partition of C. Clearly, $\mathbb{T}$ must be countable. In addition, for each $A, B \in \mathbb{T}$ there are $m, n \in \mathbb{N}$ such that $A \in \mathbb{T}_m$ and $B \in \mathbb{T}_n$. If $m = n$, then $A \cap B = \emptyset$, as $\mathbb{T}_n$ is a set partition of C_n. Otherwise, it follows from $A \subset C_m$, $B \subset C_n$, and $C_m \cap C_n = \emptyset$ that $A \cap B = \emptyset$. Moreover,

$$\bigcup_{A\in\mathbb{T}} A = \bigcup_{n\in\mathbb{N}} \bigcup_{B\in\mathbb{T}_n} B = \bigcup_{n\in\mathbb{N}} C_n = C.$$

In the same manner, $|\mathbb{T}| = \sup_{n\in\mathbb{N}} \sup_{B\in\mathbb{T}_n} \operatorname{diam}(B) = \sup_{n\in\mathbb{N}} |\mathbb{T}_n| \leq \delta$. Hence, the first assertion is proven.

We turn to the second claim. Let $(C_n)_{n\in\mathbb{N}}$ be an increasing sequence in $\mathscr{B}$ with $C_1 \neq \emptyset$. We use the first assertion to construct a sequence of set partitions $(\mathbb{T}_n)_{n\in\mathbb{N}}$ of $(C_n)_{n\in\mathbb{N}}$ recursively as follows:

(i) Let $\mathbb{T}_1$ be a set partition of C_1 with $|\mathbb{T}_1| \leq 1$.

(ii) For $n \in \mathbb{N}$ suppose that $\mathbb{T}_n$ is a set partition of C_n with $|\mathbb{T}_n| \leq 1/n$. For each $B \in \mathbb{T}_n$ we choose a set partition $\mathbb{T}_B$ of B with $|\mathbb{T}_B| \leq 1/(n+1)$ and let $\mathbb{S}_{n+1}$ be a set partition of $C_{n+1} \setminus C_n$ with $|\mathbb{S}_{n+1}| \leq 1/(n+1)$ provided $C_{n+1} \setminus C_n \neq \emptyset$, otherwise let $\mathbb{S}_{n+1} := \emptyset$. Finally, we set $\mathbb{T}_{n+1} := \Big(\bigcup_{B\in\mathbb{T}_n} \mathbb{T}_B\Big) \cup \mathbb{S}_{n+1}$.

This yields the correct result. To see this, let $n \in \mathbb{N}$, then $\mathbb{T}_{n+1}$ contains only bounded and pairwise disjoint Borel sets in C_{n+1}, by construction. This system must be countable, and has finitely many elements whenever C_{n+1} is bounded. We also observe that

$$\bigcup_{B\in\mathbb{T}_{n+1}} = \bigcup_{B\in\mathbb{T}_n} \Big(\bigcup_{A\in\mathbb{T}_B} A\Big) \cup \Big(C_{n+1} \setminus C_n) = C_{n+1}.$$

Finally, let $B \in \mathbb{T}_n$ and write $\mathbb{T}_B = \{A_1, \dots, A_m\}$ for some $m \in \mathbb{N}$ and some bounded and pairwise disjoint Borel sets $A_1, \dots, A_m$ in B, then $B = \bigcup_{i=1}^m A_i$. Since $A_1, \dots, A_m \in \mathbb{T}_{n+1}$, this shows that $\mathbb{T}_{n+1}$ refines $\mathbb{T}_n$. Hence, we set $|\emptyset| := 0$, then from $|\mathbb{T}_{n+1}| = \max\{\sup_{B\in\mathbb{T}_n} |\mathbb{T}_B|, |\mathbb{S}_{n+1}|\} \leq 1/(n+1)$ we infer the claim. $\square$

A.22 Example. Let $E = \mathbb{R}^k$ and $\|\cdot\|$ be the Euclidean norm $|\cdot|$, then $\mathscr{B} = \mathscr{B}(\mathbb{R}^k)$. We set $C_n := [-n,n)^k$ for each $n \in \mathbb{N}$, which gives an increasing sequence $(C_n)_{n\in\mathbb{N}}$ of bounded sets in $\mathscr{B}(\mathbb{R}^k)$ with $\bigcup_{n\in\mathbb{N}} C_n = \mathbb{R}^k$, and for each compact set K in $\mathbb{R}^k$ there is $n \in \mathbb{N}$ with $K \subset C_n$. We readily see that

$$\max_{z\in\overline{C}_n} |z| = \sqrt{k}n = \min_{z\in\overline{C}_{n+1}\setminus C_n^\circ} |z| \quad \text{for all } n \in \mathbb{N}.$$

Furthermore, for each $n \in \mathbb{N}$ we define $\mathbb{T}_n$ to be the system of all sets $B \subset C_n$ that can be written in the form $B = 2^{-n}([i_1, i_1+1) \times \cdots \times [i_k, i_k+1))$ for some $i_1, \ldots, i_k \in \{-n2^n, -n2^n+1, \ldots, n2^n-1\}$, then $\mathbb{T}_n$ is a set partition of C_n with

$$\operatorname{diam}(B) = \sqrt{k}2^{-n} \quad \text{for each } B \in \mathbb{T}_n.$$

The definition entails that $\mathbb{T}_n \subset \mathbb{T}_{n+1}$ for all $n \in \mathbb{N}$. Hence, $(\mathbb{T}_n)_{n\in\mathbb{N}}$ is a refining sequence of set partitions of $(C_n)_{n\in\mathbb{N}}$.

Set partitions allow for a local uniform approximation of the identity map on a closed set by a suitable sequence.

A.23 Proposition. *Let D be a non-empty closed set in E. Then there is a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of E-valued $\mathscr{B}$-simple maps on E with $\varphi_n(D) \subset D$ for all $n \in \mathbb{N}$ that converges locally uniformly to the identity map $E \to E$, $z \mapsto z$ on D such that*

$$\|\varphi_n(z)\| \leq \|\varphi_{n+1}(z)\| \leq \|\varphi(z)\| \quad \text{for all } n \in \mathbb{N} \text{ and each } z \in D.$$

Proof. We choose an increasing sequence $(C_n)_{n\in\mathbb{N}}$ of bounded Borel sets in D with $C_1 \neq \emptyset$ satisfying the following two properties:

(i) $\bigcup_{n\in\mathbb{N}} C_n = D$ and for each compact set K in D there is $n \in \mathbb{N}$ with $K \subset C_n$.

(ii) $\max_{z\in\overline{C}_n} \|z\| \leq \min_{z\in\overline{C}_{n+1}\setminus C_n^\circ} \|z\|$ for all $n \in \mathbb{N}$.

For instance, we could choose $c > 0$ such that $\|z\| \leq c$ for at least one $z \in D$ and let $C_n = \{z \in D \,|\, \|z\| \leq cn\}$ for each $n \in \mathbb{N}$. By Lemma A.21, there is a refining sequence of set partitions $(\mathbb{T}_n)_{n\in\mathbb{N}}$ of $(C_n)_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$ we choose $z_n \in \overline{C}_n$ such that $\|z_n\| = \max_{z\in\overline{C}_n} \|z\|$ and for each $B \in \mathbb{T}_n$ we let $z_B \in \overline{B}$ satisfy $\|z_B\| = \min_{z\in\overline{B}} \|z\|$. Then $\varphi_n : E \to E$ defined by

$$\varphi_n(z) := \sum_{B\in\mathbb{T}_n} z_B \mathbb{1}_B(z) + z_n \mathbb{1}_{C_n^c}(z)$$

is $\mathscr{B}$-simple and fulfills $\varphi_n(D) \subset D$. In fact, each set $B \in \mathbb{T}_n$ is Borel and $C_n^c \in \mathscr{B}$, which implies that φ_n is $\mathscr{B}$-simple. Next, let $z \in D$. If $z \in C_n$, then there is a unique $B \in \mathbb{T}_n$ with $z \in B$, which gives $\varphi_n(z) = z_B \in \overline{B}$. If instead $z \in C_n^c$, then $\varphi_n(z) = z_n \in \overline{C}_n$. Hence, from the fact that $\overline{B} \subset \overline{C}_n \subset D$ for all $B \in \mathbb{T}_n$ we get that $\varphi_n(z) \in D$ for all $z \in D$, as desired.

Now, let K be a compact set in D. Then (i) gives $n_0 \in \mathbb{N}$ such that $K \subset C_n$ for all $n \in \mathbb{N}$ with $n \geq n_0$. We choose such an $n \in \mathbb{N}$, then for each $z \in K$ there is a unique set $B \in \mathbb{T}_n$ with $z \in B$, which entails that

$$\|\varphi_n(z) - z\| = \|z_B - z\| \leq \operatorname{diam}(\overline{B}) = \operatorname{diam}(B) \leq |\mathbb{T}_n|.$$

So, $\lim_{n\uparrow\infty} \sup_{z\in K} \|\varphi_n(z) - z\| = 0$. As E is locally compact, this shows that $(\varphi_n)_{n\in\mathbb{N}}$ converges locally uniformly to the identity map $E \to E$, $z \mapsto z$ on D.

Let us prove that $\|\varphi_n(z)\| \leq \|\varphi_{n+1}(z)\|$ for all $n \in \mathbb{N}$ and each $z \in D$. Assume initially that $z \in C_n$, then there is a unique $B \in \mathbb{T}_n$ with $z \in B$. Since $\mathbb{T}_{n+1}$ refines $\mathbb{T}_n$, there are a unique $m \in \mathbb{N}$ and unique pairwise distinct sets $B_1, \ldots, B_m \in \mathbb{T}_{n+1}$ with $B = \bigcup_{i=1}^m B_i$. We choose the unique $i \in \{1, \ldots, m\}$ such that $z \in B_i$, then

$$\|\varphi_n(z)\| = \|z_B\| = \min_{z'\in\overline{B}} \|z'\| \leq \min_{z'\in\overline{B}_i} \|z'\| = \|z_{B_i}\| = \|\varphi_{n+1}(z)\|.$$

For $z \in C_{n+1}\backslash C_n$ there is a unique $B \in \mathbb{T}_{n+1}$ with $B \subset C_{n+1}\backslash C_n$ and $z \in B$, because $\mathbb{T}_{n+1}\backslash\mathbb{T}_n$ is a set partition of $C_{n+1}\backslash C_n$. Thus, (ii) yields that

$$\|\varphi_n(z)\| = \|z_n\| = \max_{z'\in\overline{C}_n} \|z'\| \leq \min_{\overline{C_{n+1}\backslash C_n^\circ}} \|z'\| \leq \min_{z'\in\overline{B}} \|z'\| = \|z_B\| = \|\varphi_{n+1}(z)\|.$$

In the last case $z \notin C_{n+1}$, we have that $\|\varphi_n(z)\| = \|z_n\| \leq \|z_{n+1}\| = \|\varphi_{n+1}(z)\|$. Hence, the claim is established. □

This gives the main result of this section.

A.24 Corollary. *Let D be a non-empty closed set in E. Then to each $\mathscr{F}$-measurable map $f : \Omega \to D$ there is a sequence $(f_n)_{n\in\mathbb{N}}$ of D-valued $\mathscr{F}$-simple maps on Ω that converges pointwise to f such that*

$$\|f_n(\omega)\| \leq \|f_{n+1}(\omega)\| \leq \|f(\omega)\| \quad \textit{for all } n \in \mathbb{N} \textit{ and every } \omega \in \Omega.$$

Moreover, if f is bounded, then the convergence is uniform.

Proof. Proposition A.23 yields a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of E-valued $\mathscr{B}$-simple maps on E with $\varphi_n(D) \subset D$ for all $n \in \mathbb{N}$ that converges locally uniformly to the identity map $E \to E$, $z \mapsto z$ on D such that

$$\|\varphi_n(z)\| \leq \|\varphi_{n+1}(z)\| \leq \|\varphi(z)\| \quad \text{for all } n \in \mathbb{N} \text{ and each } z \in D.$$

We set $f_n := \varphi_n \circ f$ for all $n \in \mathbb{N}$. Then f_n is a D-valued $\mathscr{F}$-simple map on Ω, since $f(\Omega) \subset D$. We also see that $\lim_{n\uparrow\infty} f_n(\omega) = \lim_{n\uparrow\infty} \varphi_n(f(\omega)) = f(\omega)$ and

$$\|f_n(\omega)\| = \|\varphi_n(f(\omega))\| \leq \|\varphi_{n+1}(f(\omega))\| = \|f_{n+1}(\omega)\|$$

for all $n \in \mathbb{N}$ and each $\omega \in \Omega$. To justify the second claim, let f be bounded. Then there is $c \geq 0$ with $\|f(\omega)\| \leq c$ for each $\omega \in \Omega$. Since $K := \{z \in D | \, \|z\| \leq c\}$ is a compact set in D and

$$\sup_{\omega\in\Omega} \|f_n(\omega) - f(\omega)\| \leq \sup_{z\in K} \|\varphi_n(z) - z\|$$

for each $n \in \mathbb{N}$, we obtain that $\lim_{n\uparrow\infty} \sup_{\omega\in\Omega} \|f_n(\omega) - f(\omega)\| = 0$, which completes the proof. □

The pointwise approximation of measurable maps by simple maps leads us to a classical statement (cf. Theorem 18 in [8, Section 1.2]).

A.25 Corollary. *Let $(\Omega', \mathscr{F}')$ be another measurable space and $h : \Omega \to \Omega'$ be a $\mathscr{F}$-$\mathscr{F}'$-measurable map. Then a $\mathscr{F}$-measurable map $f : \Omega \to E$ is measurable with respect to $\sigma(h)$ if and only if there is a $\mathscr{F}'$-measurable map $g : \Omega' \to E$ such that*

$$f(\omega) = g(h(\omega)) \quad \textit{for all } \omega \in \Omega.$$

Proof. As the composition of two measurable maps is measurable, the condition is clearly sufficient. To prove its necessity, we first assume that f is $\mathscr{F}$-simple. Then there are $m \in \mathbb{N}$ and pairwise distinct $z_1, \dots, z_m \in E$ with $f(\Omega) = \{z_1, \dots, z_m\}$. The sets $A_1 := \{f = z_1\}, \dots, A_m := \{f = z_m\}$ belong to $\mathscr{F}$, form a decomposition of Ω, and satisfy

$$f = \sum_{i=1}^{m} z_i \mathbb{1}_{A_i}.$$

Because $\sigma(h) = \{h^{-1}(A') \,|\, A' \in \mathscr{F}'\} \subset \mathscr{F}$ and f is $\sigma(h)$-measurable, there must exist $A_1', \dots, A_m' \in \mathscr{F}'$ such that $A_i = \{h \in A_i'\}$ for all $i \in \{1, \dots, m\}$. Hence, the map

$$g := \sum_{i=1}^{m} z_i \mathbb{1}_{A_i'}$$

is $\mathscr{F}'$-simple and satisfies $f(\omega) = g(h(\omega))$ for each $\omega \in \Omega$, which follows directly from $\mathbb{1}_{A_i'}(h(\omega)) = \mathbb{1}_{A_i}(\omega)$ for all $i \in \{1, \dots, m\}$. We turn to the general case. The preceding corollary provides a sequence $(f_n)_{n\in\mathbb{N}}$ of E-valued $\sigma(h)$-simple maps on Ω that converges pointwise to f and fulfills $\|f_n(\omega)\| \leq \|f_{n+1}(\omega)\| \leq \|f(\omega)\|$ for each $n \in \mathbb{N}$ and all $\omega \in \Omega$. By what we have shown, for each $n \in \mathbb{N}$ there is a $\mathscr{F}'$-simple map $g_n : \Omega' \to E$ such that

$$f_n(\omega) = g_n(h(\omega)) \quad \text{for all } \omega \in \Omega.$$

Let A' be the set of all $\omega' \in \Omega'$ for which the sequence $(g_n(\omega'))_{n\in\mathbb{N}}$ converges, then $A' \in \mathscr{F}'$ and $h(\Omega) \subset A'$. For this reason, we can conclude that $g : \Omega' \to E$ defined by $g(\omega') := \lim_{n\uparrow\infty} g_n(\omega')$, if $\omega' \in A'$, and $g(\omega') := 0$, if $\omega' \notin A'$, is the demanded map. □

A.5 Monotone class theorems

In this section, we recall the standard and the functional monotone class theorem. For this purpose, [2, Section 2] and [5, Section 1.6] are used as references. Let Ω be a non-empty set, then a system $\mathscr{D}$ of sets in Ω with $\Omega \in \mathscr{D}$ is a *d-system* (in Ω) if it satisfies $D\backslash C \in \mathscr{D}$ for all $C, D \in \mathscr{D}$ with $C \subset D$ and $\bigcup_{n\in\mathbb{N}} D_n \in \mathscr{D}$ for each increasing sequence $(D_n)_{n\in\mathbb{N}}$ in $\mathscr{D}$. We also make use of another characterization of d-systems.

A.26 Lemma. *A system $\mathscr{D}$ of sets in Ω with $\Omega \in \mathscr{D}$ is a d-system if and only if the following two properties hold:*

(i) $D^c \in \mathscr{D}$ *for all* $D \in \mathscr{D}$.

(ii) $\bigcup_{n\in\mathbb{N}} D_n \in \mathscr{D}$ *for each sequence* $(D_n)_{n\in\mathbb{N}}$ *of pairwise disjoint sets in* $\mathscr{D}$.

Proof. Assume that $\mathscr{D}$ is a d-system, then $D^c = \Omega\backslash D \in \mathscr{D}$ for all $D \in \mathscr{D}$, since $\Omega \in \mathscr{D}$. Hence, (i) is valid. Now let $(D_n)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint sets in $\mathscr{D}$. We set

$$C_n := \bigcup_{i=1}^{n} D_i \quad \text{for all } n \in \mathbb{N}.$$

Then $C_1 = D_1 \in \mathscr{D}$ and $C_{n+1} = C_n \cup D_{n+1} = (D_{n+1}^c \backslash C_n)^c$ with $C_n \subset D_{n+1}^c$ for all $n \in \mathbb{N}$. Thus, it follows inductively from (i) that $(C_n)_{n\in\mathbb{N}}$ is an increasing sequence in $\mathscr{D}$. We conclude from $\bigcup_{n\in\mathbb{N}} D_n = \bigcup_{n\in\mathbb{N}} C_n \in \mathscr{D}$ that (ii) holds as well.

Conversely, suppose that $\mathscr{D}$ fulfills the stated properties. Let $C, D \in \mathscr{D}$ with $C \subset D$, then C and D^c are disjoint, which yields that $D^c \cup C \in \mathscr{D}$. So,

$$D\backslash C = D \cap C^c = (D^c \cup C)^c \in \mathscr{D}.$$

Next, let $(D_n)_{n\in\mathbb{N}}$ be an increasing sequence in $\mathscr{D}$. By using the property of $\mathscr{D}$ that we have just shown, we define a sequence $(C_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets in $\mathscr{D}$ through $C_1 := D_1$ and $C_{n+1} := D_{n+1}\backslash D_n$ for all $n \in \mathbb{N}$. Hence, the fact that $\bigcup_{n\in\mathbb{N}} D_n = \bigcup_{n\in\mathbb{N}} C_n \in \mathscr{D}$ shows the lemma. $\square$

As verified below, d-systems are plainly related to σ-fields.

A.27 Lemma. *A system $\mathscr{F}$ of sets in Ω is a σ-field if and only if it is an $\cap$-stable d-system.*

Proof. Since every σ-field is σ-$\cap$-stable, the only if direction follows directly from Lemma A.26. For the converse implication, it suffices to show that $\mathscr{F}$ is σ-$\cup$-stable. First, let $A, B \in \mathscr{F}$, then $A^c, B^c \in \mathscr{F}$ and hence, $A \cup B = A^c \cap B^c \in \mathscr{F}$, which shows that $\mathscr{F}$ is $\cup$-stable. Now let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{F}$. We set

$$B_n := \bigcup_{i=1}^{n} A_i \quad \text{for all } n \in \mathbb{N}.$$

Then $(B_n)_{n\in\mathbb{N}}$ is an increasing sequence in $\mathscr{F}$. Thus, $\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} B_n \in \mathscr{F}$. $\square$

We note that if $\mathbb{D}$ is a family of d-systems, then the intersection $\bigcap_{\mathscr{D}\in\mathbb{D}} \mathscr{D}$ is another a d-system. Clearly, let $D \in \bigcap_{\mathscr{D}\in\mathbb{D}} \mathscr{D}$, then $D^c \in \mathscr{D}$ for all $\mathscr{D} \in \mathbb{D}$, and whenever $(D_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint sets in $\bigcap_{\mathscr{D}\in\mathbb{D}} \mathscr{D}$, then

$$\bigcup_{n\in\mathbb{N}} D_n \in \mathscr{D} \quad \text{for each } \mathscr{D} \in \mathbb{D}.$$

Thus, for a system $\mathscr{C}$ of sets in Ω, the d-system generated by $\mathscr{C}$, denoted by $d(\mathscr{C})$, is defined to be the smallest d-system including $\mathscr{C}$ in the sense that $d(\mathscr{C})$ is a d-system that includes $\mathscr{C}$ and which is included in every d-system including $\mathscr{C}$. This directly implies that the d-system generated by $\mathscr{C}$ is necessarily unique. To verify existence, let $\mathbb{E}$ be the family of all d-systems which include $\mathscr{C}$, then $d(\mathscr{C}) = \bigcap_{\mathscr{D}\in\mathbb{E}} \mathscr{D}$. This is due to $\mathscr{C} \subset \bigcap_{\mathscr{D}\in\mathbb{E}} \mathscr{D}$ and $\bigcap_{\mathscr{D}\in\mathbb{E}} \mathscr{D} \subset \mathscr{D}'$ for each $\mathscr{D}' \in \mathbb{E}$.

A.28 Monotone Class Theorem. Let $\mathscr{C}$ be an $\cap$-stable system of sets in Ω, then $d(\mathscr{C}) = \sigma(\mathscr{C})$. In particular, if $\mathscr{D}$ is a d-system that includes $\mathscr{C}$, then $\sigma(\mathscr{C}) \subset \mathscr{D}$.

Proof. We merely have to show the first assertion, since the second follows directly from the definition of $d(\mathscr{C})$. By Lemma A.27, every σ-field is a d-system. Thus, $d(\mathscr{C}) \subset \sigma(\mathscr{C})$. To justify the converse inclusion, it is enough to check that $d(\mathscr{C})$ is $\cap$-stable. Let $D \in d(\mathscr{C})$, then

$$\mathscr{D}_D := \{C \subset \Omega \,|\, C \cap D \in d(\mathscr{C})\}$$

is a d-system. Indeed, $\Omega \in \mathscr{D}_D$, and if $B, C \in \mathscr{D}_D$ satisfy $B \subset C$, then $B\cap D \subset C\cap D$, which gives $(C\backslash B) \cap D = (C \cap D)\backslash(B \cap D) \in d(\mathscr{C})$. So, $C\backslash B \in \mathscr{D}_D$. If $(C_n)_{n\in\mathbb{N}}$ is an increasing sequence in $\mathscr{D}_D$ whose union is C, then $(C_n \cap D)_{n\in\mathbb{N}}$ is an increasing sequence in $d(\mathscr{C})$. Hence, $C \cap D = \bigcup_{n\in\mathbb{N}}(C_n \cap D) \in d(\mathscr{C})$.

We notice that $\mathscr{D}_D$ includes $\mathscr{C}$ for each $D \in \mathscr{C}$, because $\mathscr{C}$ is $\cap$-stable. Hence, $d(\mathscr{C}) \subset \mathscr{D}_D$ for every $D \in \mathscr{C}$. This yields that $C \cap D \in d(\mathscr{C})$ for all $C \in d(\mathscr{C})$ and each $D \in \mathscr{C}$. But then $\mathscr{D}_D$ is a d-system including $\mathscr{C}$ for every $D \in d(\mathscr{C})$, which entails that $d(\mathscr{C})$ is $\cap$-stable. □

We recall that a linear space $\mathscr{H}$ of real-valued bounded functions on Ω is a *monotone class* if it contains the constant function $\mathbb{1}_\Omega$ and if it fulfills $\sup_{n\in\mathbb{N}} h_n \in \mathscr{H}$ for each $\mathbb{R}_+$-valued increasing bounded sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathscr{H}$.

A.29 Functional Monotone Class Theorem. Let $\mathscr{H}$ be a monotone class and $\mathscr{C}$ be an $\cap$-stable system of sets in Ω. If $\mathbb{1}_C \in \mathscr{H}$ for all $C \in \mathscr{C}$, then $\mathscr{H}$ contains all real-valued $\sigma(\mathscr{C})$-measurable bounded functions on Ω.

Proof. First, we show that $\mathbb{1}_A \in \mathscr{H}$ for all $A \in \sigma(\mathscr{C})$. Let $\mathscr{D} := \{D \subset \Omega \,|\, \mathbb{1}_D \in \mathscr{H}\}$, then $\mathscr{D}$ is a d-system including $\mathscr{C}$. Clearly, $\Omega \in \mathscr{D}$, and for each $C, D \in \mathscr{D}$ such that $C \subset D$, $\mathbb{1}_{D\backslash C} = \mathbb{1}_D - \mathbb{1}_C \in \mathscr{H}$. If $(D_n)_{n\in\mathbb{N}}$ is an increasing sequence in $\mathscr{D}$ whose union is D, then $(\mathbb{1}_{D_n})_{n\in\mathbb{N}}$ is a non-negative increasing bounded sequence in $\mathscr{H}$. Thus, $\mathbb{1}_D = \sup_{n\in\mathbb{N}} \mathbb{1}_{D_n} \in \mathscr{H}$. Consequently, the Monotone Class Theorem A.28 implies that $\sigma(\mathscr{C}) \subset \mathscr{D}$.

The next step of the proof is to check that if $h : \Omega \to \mathbb{R}_+$ is $\sigma(\mathscr{C})$-measurable and bounded, then $h \in \mathscr{H}$. By Corollary A.24, there is an increasing sequence $(h_n)_{n\in\mathbb{N}}$ of $\mathbb{R}_+$-valued $\sigma(\mathscr{C})$-simple functions on Ω with $\sup_{n\in\mathbb{N}} h_n = h$. Since $\mathbb{1}_A \in \mathscr{H}$ for all $A \in \sigma(\mathscr{C})$ and $\mathscr{H}$ is a linear space, $h_n \in \mathscr{H}$ for all $n \in \mathbb{N}$. This in turn gives $h \in \mathscr{H}$. If more generally h is real-valued, then h^+ and h^- belong to $\mathscr{H}$. For this reason, $h = h^+ - h^- \in \mathscr{H}$, which completes the proof. □

A.6 The Bochner integral in finite dimension

This section contains a concise introduction of the Bochner integral in a Banach space of finite dimension. The presentation is mainly based on [5, Appendix E]. Additionally, under the hypothesis that the underlying measure is a probability measure, we verify that if a measurable integrable map takes all its values in a closed convex set, then so does its integral.

Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and E be a finite-dimensional Banach space with complete norm $\|\cdot\|$ and Borel σ-field $\mathscr{B}$. A $\mathscr{F}$-measurable map $f : \Omega \to E$ is said to be *μ-integrable* if $\|f\|$ is μ-integrable in the usual sense. First, let f be $\mathscr{F}$-simple, as introduced in Section A.4. If f is μ-integrable, then the set $\{f \neq 0\}$ has finite μ-measure. In fact, let $m \in \mathbb{N}$, $z_1, \dots, z_m \in E$, and $A_1, \dots, A_m \in \mathscr{F}$ form a decomposition of Ω such that $f = \sum_{i=1}^m z_i \mathbb{1}_{A_i}$. Then

$$\mu(A_i) \leq \mu(f = z_i) \leq \mu(\|f\| = \|z_i\|) \leq \frac{1}{\|z_i\|} \int_\Omega \|f(\omega)\| \, \mu(d\omega) < \infty$$

for each $i \in \{1, \dots, m\}$ with $z_i \neq 0$. From $\{f \neq 0\} = \bigcup_{i=1:\, z_i \neq 0}^m A_i$ we see that $\mu(f \neq 0) = \sum_{i=1:\, z_i \neq 0}^m \mu(A_i) < \infty$. For this reason, the *(Bochner) μ-integral* of f (over Ω) can be defined by

$$\int_\Omega f(\omega) \, \mu(d\omega) := \sum_{i=1:\, z_i \neq 0}^m z_i \mu(A_i).$$

This definition does not dependent on the choice of the normal representation for f. To see this, let $n \in \mathbb{N}$, $z_1', \dots, z_n' \in E$, and $B_1, \dots, B_n \in \mathscr{F}$ form a decomposition of Ω such that $f = \sum_{j=1}^n z_j' \mathbb{1}_{B_j}$. Then

$$\mu(A_i) = \sum_{l=1}^n \mu(A_i \cap B_l) \quad \text{and} \quad \mu(B_j) = \sum_{l=1}^m \mu(A_l \cap B_j)$$

for all $i \in \{1, \dots, m\}$ and each $j \in \{1 \dots, n\}$ with $z_i \neq 0$ and $z_j' \neq 0$. We observe that whenever $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ are such that $A_i \cap B_j \neq \emptyset$, which in particular follows from $\mu(A_i \cap B_j) > 0$, then $z_i = z_j'$. Consequently,

$$\begin{aligned}
\sum_{i=1:\, z_i \neq 0}^m z_i \mu(A_i) &= \sum_{i=1:\, z_i \neq 0}^m \sum_{j=1}^n z_i \mu(A_i \cap B_j) \\
&= \sum_{i=1}^m \sum_{j=1:\, z_j' \neq 0}^n z_j' \mu(A_i \cap B_j) = \sum_{j=1:\, z_j' \neq 0}^n z_j' \mu(B_j),
\end{aligned}$$

which is the correct result. In the sequel, we let $S(\Omega, \mathscr{F})$ denote the linear space of all E-valued $\mathscr{F}$-simple μ-integrable maps on Ω. Then the basic properties of the μ-integral of each $f \in S(\Omega, \mathscr{F})$ can be summarized as follows.

A.30 Lemma. *The map $S(\Omega, \mathscr{F}) \to E$, $f \mapsto \int_\Omega f(\omega) \, \mu(d\omega)$ is linear and satisfies*

$$\int_\Omega z \mathbb{1}_A(\omega) \, \mu(d\omega) = z\mu(A) \quad \textit{and} \quad \left\| \int_\Omega f(\omega) \, \mu(d\omega) \right\| \leq \int_\Omega \|f(\omega)\| \, \mu(d\omega)$$

for all $A \in \mathscr{F}$ with $\mu(A) < \infty$, each $z \in E$, and every $f \in S(\Omega, \mathscr{F})$.

Proof. Let $\alpha \in \mathbb{R}$ and $f \in S(\Omega, \mathscr{F})$. Then there are $m \in \mathbb{N}$, $z_1, \ldots, z_m \in E$, and $A_1, \ldots, A_m \in \mathscr{F}$ that form a decomposition of Ω such that $f = \sum_{i=1}^m z_i \mathbb{1}_{A_i}$. By definition,

$$\alpha \int_\Omega f(\omega)\,\mu(d\omega) = \sum_{i=1:\, z_i \neq 0}^{m} \alpha z_i \mu(A_i) = \int_\Omega \alpha f(\omega)\,\mu(d\omega).$$

Let moreover $g \in S(\Omega, \mathscr{F})$. We choose $n \in \mathbb{N}$, $z_1', \ldots, z_n' \in E$, and $B_1, \ldots, B_n \in \mathscr{F}$ forming a decomposition of Ω such that $g = \sum_{j=1}^n z_j' \mathbb{1}_{B_j}$. Then it follows that

$$\begin{aligned}
\int_\Omega (f+g)(\omega)\,\mu(d\omega) &= \sum_{i=1}^m \sum_{\substack{j=1:\\ z_i + z_j' \neq 0}}^{n} (z_i + z_j')\mu(A_i \cap B_j) \\
&= \sum_{i=1:\, z_i \neq 0}^{m} \sum_{j=1}^{n} z_i \mu(A_i \cap B_j) + \sum_{i=1}^{m} \sum_{j=1:\, z_j' \neq 0}^{n} z_j' \mu(A_i \cap B_j) \\
&= \int_\Omega f(\omega)\,\mu(d\omega) + \int_\Omega g(\omega)\,\mu(d\omega).
\end{aligned}$$

This shows the linearity of the map $S(\Omega, \mathscr{F}) \to E$, $f \mapsto \int_\Omega f(\omega)\,\mu(d\omega)$. Now let $A \in \mathscr{F}$ with $\mu(A) < \infty$ and $z \in E$. Then A, A^c form a decomposition of Ω and $z\mathbb{1}_A = z\mathbb{1}_A + 0\mathbb{1}_{A^c}$. Thus,

$$\int_\Omega z\mathbb{1}_A(\omega)\,\mu(d\omega) = z\mu(A).$$

Eventually, the triangle inequality and the standard definition of the μ-integral of $\|f\|$ imply that

$$\left\| \int_\Omega f(\omega)\,\mu(d\omega) \right\| = \left\| \sum_{i=1:\, z_i \neq 0}^{m} z_i \mu(A_i) \right\| \leq \sum_{i=1:\, z_i \neq 0}^{m} \|z_i\| \mu(A_i) = \int_\Omega \|f(\omega)\|\,\mu(d\omega).$$

This proves the lemma. □

For a $\mathscr{F}$-measurable μ-integrable map $f : \Omega \to E$ that may not be $\mathscr{F}$-simple, Corollary A.24 yields a sequence $(f_n)_{n \in \mathbb{N}}$ in $S(\Omega, \mathscr{F})$ that converges pointwise to f such that the function $\Omega \to [0, \infty]$, $\omega \mapsto \sup_{n \in \mathbb{N}} \|f_n(\omega)\|$ is μ-integrable. Because $(f_n(\omega))_{n \in \mathbb{N}}$ is a Cauchy sequence,

$$\lim_{n \uparrow \infty} \sup_{m \in \mathbb{N}:\, m \geq n} \|f_m(\omega) - f_n(\omega)\| = 0 \quad \text{for all } \omega \in \Omega.$$

It is clear that $\sup_{n \in \mathbb{N}} \sup_{m \in \mathbb{N}:\, m \geq n} \|f_m(\omega) - f_n(\omega)\| \leq 2 \sup_{n \in \mathbb{N}} \|f_n(\omega)\|$ for each $\omega \in \Omega$. Hence, from dominated convergence it follows readily that

$$\lim_{n \uparrow \infty} \sup_{m \in \mathbb{N}:\, m \geq n} \int_\Omega \|f_m(\omega) - f_n(\omega)\|\,\mu(d\omega) = 0.$$

Due to Lemma A.30, this implies that $(\int_\Omega f_n(\omega)\,\mu(d\omega))_{n \in \mathbb{N}}$ is a Cauchy sequence. Since E is complete, it must converge. The *(Bochner) μ-integral* of f (over Ω) is defined to be this limit, that is,

$$\int_\Omega f(\omega)\,\mu(d\omega) := \lim_{n \uparrow \infty} \int_\Omega f_n(\omega)\,\mu(d\omega).$$

This definition is independent of the choice of the sequence $(f_n)_{n\in\mathbb{N}}$. More precisely, suppose that $(g_n)_{n\in\mathbb{N}}$ is another sequence in $S(\Omega,\mathscr{F})$ that converges pointwise to f such that the function $\Omega\to[0,\infty]$, $\omega\mapsto\sup_{n\in\mathbb{N}}\|g_n(\omega)\|$ is μ-integrable. Then from $\lim_{n\uparrow\infty}\|f_n(\omega)-g_n(\omega)\|=0$ for all $\omega\in\Omega$ we infer that

$$\lim_{n\uparrow\infty}\int_\Omega \|f_n(\omega)-g_n(\omega)\|\,\mu(d\omega)=0,$$

by dominated convergence. In view of Lemma A.30, this gives us that

$$\lim_{n\uparrow\infty}\int_\Omega g_n(\omega)\,\mu(d\omega)=\lim_{n\uparrow\infty}\int_\Omega f_n(\omega)\,\mu(d\omega),$$

which is the desired conclusion. Clearly, the extended μ-integral remains linear, as we shortly verify.

A.31 Lemma. *Let $\alpha,\beta\in\mathbb{R}$ and f,g be two E-valued $\mathscr{F}$-measurable μ-integrable maps on Ω. Then $\alpha f+\beta g$ is μ-integrable and*

$$\int_\Omega(\alpha f+\beta g)(\omega)\,\mu(d\omega)=\alpha\int_\Omega f(\omega)\,\mu(d\omega)+\beta\int_\Omega g(\omega)\,\mu(d\omega).$$

Proof. Let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ be two sequences of E-valued $\mathscr{F}$-simple maps on Ω that converge to f and g, respectively, such that the functions $\Omega\to[0,\infty]$, $\omega\mapsto\sup_{n\in\mathbb{N}}\|f_n(\omega)\|$ and $\Omega\to[0,\infty]$, $\omega\mapsto\sup_{n\in\mathbb{N}}\|g_n(\omega)\|$ are μ-integrable. In Lemma A.30 we have shown that $\alpha f_n+\beta g_n$ is a $\mathscr{F}$-simple μ-integrable map that satisfies

$$\int_\Omega(\alpha f_n+\beta g_n)(\omega)\,\mu(d\omega)=\alpha\int_\Omega f_n(\omega)\,\mu(d\omega)+\beta\int_\Omega g_n(\omega)\,\mu(d\omega)$$

for every $n\in\mathbb{N}$. Clearly, $\lim_{n\uparrow\infty}(\alpha f_n+\beta g_n)(\omega)=(\alpha f+\beta g)(\omega)$ and $\|\alpha f_n+\beta g_n\|(\omega)\leq|\alpha|\|f_n(\omega)\|+|\beta|\|g_n(\omega)\|$ for all $n\in\mathbb{N}$ and each $\omega\in\Omega$. For this reason,

$$\begin{aligned}\int_\Omega(\alpha f+\beta g)(\omega)\,\mu(d\omega)&=\lim_{n\uparrow\infty}\int_\Omega(\alpha f_n+\beta g_n)(\omega)\,\mu(d\omega)\\&=\alpha\lim_{n\uparrow\infty}\int_\Omega f_n(\omega)\,\mu(d\omega)+\beta\lim_{n\uparrow\infty}\int_\Omega g_n(\omega)\,\mu(d\omega)\\&=\alpha\int_\Omega f(\omega)\,\mu(d\omega)+\beta\int_\Omega g(\omega)\,\mu(d\omega).\end{aligned}$$

□

The norm inequality for the μ-integral in Lemma A.30 remains valid as well.

A.32 Proposition. *Let $f:\Omega\to E$ be a $\mathscr{F}$-measurable μ-integrable map, then*

$$\left\|\int_\Omega f(\omega)\,\mu(d\omega)\right\|\leq\int_\Omega\|f(\omega)\|\,\mu(d\omega).$$

Proof. By Corollary A.24, we can choose a sequence $(f_n)_{n\in\mathbb{N}}$ of E-valued $\mathscr{F}$-simple maps on Ω such that $\lim_{n\uparrow\infty} f_n(\omega) = f(\omega)$ and $\sup_{n\in\mathbb{N}} \|f_n(\omega)\| \leq \|f(\omega)\|$ for each $\omega \in \Omega$. According to Lemma A.30,

$$\left\| \int_\Omega f_n(\omega)\,\mu(d\omega) \right\| \leq \int_\Omega \|f_n(\omega)\|\,\mu(d\omega) \leq \int_\Omega \|f(\omega)\|\,\mu(d\omega)$$

for all $n \in \mathbb{N}$. Hence,

$$\left\| \int_\Omega f(\omega)\,\mu(d\omega) \right\| = \lim_{n\uparrow\infty} \left\| \int_\Omega f_n(\omega)\,\mu(d\omega) \right\| \leq \int_\Omega \|f(\omega)\|\,\mu(d\omega).$$

□

Finally, we derive the multidimensional version of dominated convergence.

A.33 Dominated Convergence Theorem. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of E-valued $\mathscr{F}$-measurable maps on Ω and $f : \Omega \to E$ be $\mathscr{F}$-measurable. Suppose that

$$\lim_{n\uparrow\infty} f_n(\omega) = f(\omega) \quad \text{and} \quad \sup_{n\in\mathbb{N}} \|f_n(\omega)\| \leq g(\omega)$$

for μ-a.e. $\omega \in \Omega$ and some $\mathscr{F}$-measurable μ-integrable function $g : \Omega \to [0,\infty]$. Then f_n and f are μ-integrable for all $n \in \mathbb{N}$ and

$$\lim_{n\uparrow\infty} \int_\Omega f_n(\omega)\,\mu(d\omega) = \int_\Omega f(\omega)\,\mu(d\omega).$$

Proof. Since $\|f(\omega)\| = \lim_{n\uparrow\infty} \|f_n(\omega)\| \leq g(\omega)$ for μ-a.e. $\omega \in \Omega$, it follows that $\|f_n\|$ and $\|f\|$ are μ-integrable for each $n \in \mathbb{N}$. Furthermore,

$$\lim_{n\uparrow\infty} \|f_n(\omega) - f(\omega)\| = 0 \quad \text{and} \quad \sup_{n\in\mathbb{N}} \|f_n(\omega) - f(\omega)\| \leq 2g(\omega)$$

for μ-a.e. $\omega \in \Omega$. By standard dominated convergence,

$$\lim_{n\uparrow\infty} \int_\Omega \|f_n(\omega) - f(\omega)\|\,\mu(d\omega) = 0.$$

Hence, from Proposition A.32 the claim follows. □

After this introduction to multidimensional integration theory, we suppose that μ is a probability measure. In the one-dimensional case $E = \mathbb{R}$,

$$\int_\Omega f(\omega)\,\mu(d\omega) \in D$$

for each non-degenerate closed interval D in $\mathbb{R}$ and every $\mathscr{F}$-measurable μ-integrable function $f : \Omega \to D$. Let us prove the multidimensional generalization.

A.34 Proposition. *Let μ be a probability measure and D be a non-empty closed convex set in E. Then every $\mathscr{F}$-measurable μ-integrable map $f : \Omega \to D$ satisfies $\int_\Omega f(\omega)\,\mu(d\omega) \in D$.*

Proof. At first, we assume that f is $\mathscr{F}$-simple. Thus, let $m \in \mathbb{N}$, $z_1, \ldots, z_m \in D$, and $A_1, \ldots, A_m \in \mathscr{F}$ form a decomposition of Ω such that $f = \sum_{i=1}^m z_i \mathbb{1}_{A_i}$. Then from Lemma A.1 we obtain that

$$\int_\Omega f(\omega)\,\mu(d\omega) = \sum_{i=1}^m z_i \mu(A_i) \in D,$$

since $\mu(A_1), \ldots, \mu(A_m) \in [0,1]$ and $\sum_{i=1}^m \mu(A_i) = 1$. In other words, $\int_\Omega f(\omega)\,\mu(d\omega)$ is a convex combination of points of D. Next, Corollary A.24 provides a sequence $(f_n)_{n\in\mathbb{N}}$ of D-valued $\mathscr{F}$-simple maps on Ω such that $\lim_{n\uparrow\infty} f_n(\omega) = f(\omega)$ and $\sup_{n\in\mathbb{N}} \|f_n(\omega)\| \leq \|f(\omega)\|$ for all $\omega \in \Omega$. By what we have just shown,

$$\int_\Omega f_n(\omega)\,\mu(d\omega) \in D \quad \text{for all } n \in \mathbb{N}.$$

Since D is closed, we conclude that

$$\int_\Omega f(\omega)\,\mu(d\omega) = \lim_{n\uparrow\infty} \int_\Omega f_n(\omega)\,\mu(d\omega) \in D.$$

□

A.7 Stochastic processes and stopping times

Here, we summarize the relevant material on adapted and progressively measurable stochastic processes and on stopping times. In particular, we look more closely at hitting times. To this end, [25, Sections 1.1 and 1.2] and [33, Section 1.5] are mainly used as references. Reconstructibility that originates from the classical theory of Markov processes is also studied (cf. [11, Appendix]).

Let J be a non-degenerate interval in $\mathbb{R}_+$ and $(\Omega, \mathscr{F})$, $(S, \mathscr{S})$ be two measurable spaces. We assume that $(\mathscr{F}_t)_{t\in J}$ is a family of sub-σ-fields of $\mathscr{F}$. In our context, a process is a map

$$X : J \times \Omega \to S, \quad (t, \omega) \mapsto X_t(\omega)$$

such that the map $X_t : \Omega \to S$, $\omega \mapsto X_t(\omega)$ is $\mathscr{F}$-measurable for each $t \in J$. The map $J \to S$, $t \mapsto X_t(\omega)$ is called a *path* of X for each $\omega \in \Omega$. In case there is no reason of ambiguity, we say that X is (right-)continuous if all its paths are (right-)continuous. Next, if X_t is $\mathscr{F}_t$-measurable for all $t \in J$, then X is said to be $(\mathscr{F}_t)_{t\in J}$*-adapted*.

Let us further suppose that $(\mathscr{F}_t)_{t\in J}$ is a *filtration* of $\mathscr{F}$, that is, $\mathscr{F}_s \subset \mathscr{F}_t$ for all $s, t \in J$ with $s \leq t$. To abbreviate notation, we set $J_t := \{s \in J \,|\, s \leq t\}$ for each $t \in J$. Then X is called $(\mathscr{F}_t)_{t\in J}$*-progressively measurable* if the map

$$J_t \times \Omega \to S, \quad (s, \omega) \mapsto X_s(\omega)$$

is $\mathscr{B}(J_t) \otimes \mathscr{F}_t$-measurable for all $t \in J$. Put differently, X is progressively measurable with respect to $(\mathscr{F}_t)_{t\in J}$ if $\{(s, \omega) \in J_t \times \Omega \,|\, X_s(\omega) \in B\} \in \mathscr{B}(J_t) \otimes \mathscr{F}_t$ for all $t \in J$ and each $B \in \mathscr{S}$.

Let us now consider a *backward filtration* of $\mathscr{F}$, which is a family $(\mathscr{F}'_t)_{t\in J}$ of sub-σ-fields of $\mathscr{F}$ with $\mathscr{F}'_t \subset \mathscr{F}'_s$ for each $s,t \in J$ with $s \leq t$. For convenience, we write $J'_t := \{u \in J \,|\, u \geq t\}$ for all $t \in J$. Then X is said to be $(\mathscr{F}'_t)_{t\in J}$-progressively measurable if the map

$$J'_t \times \Omega \to S, \quad (u,\omega) \mapsto X_u(\omega)$$

is $\mathscr{B}(J'_t)\otimes\mathscr{F}'_t$-measurable for all $t \in J$. In other words, X is progressively measurable with respect to $(\mathscr{F}'_t)_{t\in J}$ if $\{(u,\omega) \in J'_t \times \Omega \,|\, X_u(\omega) \in B\} \in \mathscr{B}(J'_t) \otimes \mathscr{F}'_t$ for all $t \in J$ and every $B \in \mathscr{S}$.

A.35 Example. Suppose that $(\mathscr{F}_t)_{t\in J}$ and $(\mathscr{F}'_t)_{t\in J}$ are of the form

$$\mathscr{F}_t = \sigma(X_s :\, s \in J_t) \quad \text{and} \quad \mathscr{F}'_t = \sigma(X_u :\, u \in J'_t) \quad \text{for all } t \in J.$$

Then $(\mathscr{F}_t)_{t\in J}$ and $(\mathscr{F}'_t)_{t\in J}$ are called the *natural filtration* and the *natural backward filtration* of X, respectively.

We say that a set F in $J \times \Omega$ is $(\mathscr{F}_t)_{t\in J}$-progressively measurable if the indicator function $\mathbb{1}_F : J \times \Omega \to \{0,1\}$, $(t,\omega) \mapsto \mathbb{1}_F(t,\omega)$ is $(\mathscr{F}_t)_{t\in J}$-progressively measurable. This leads us to a characterization of progressive measurability of processes (see Exercise 1.5.11 in [33]).

A.36 Lemma. *The system of all $(\mathscr{F}_t)_{t\in J}$-progressively measurable sets in $J \times \Omega$ is a σ-field. Moreover, a process $X : J \times \Omega \to S$ is $(\mathscr{F}_t)_{t\in J}$-progressively measurable if and only if it is measurable with respect to this σ-field.*

Proof. It is readily seen that a set F in $J \times \Omega$ is $(\mathscr{F}_t)_{t\in J}$-progressively measurable if and only if the set $\{(s,\omega) \in J_t \times \Omega \,|\, \mathbb{1}_F(s,\omega) = 1\}$, which coincides with $F \cap (J_t \times \Omega)$, is a member of $\mathscr{B}(J_t) \otimes \mathscr{F}_t$ for all $t \in J$.

Since $(J \times \Omega) \cap (J_t \times \Omega) = J_t \times \Omega \in \mathscr{B}(J_t) \otimes \mathscr{F}_t$ for each $t \in J$, the set $J \times \Omega$ is $(\mathscr{F}_t)_{t\in J}$-progressively measurable. Let F be an $(\mathscr{F}_t)_{t\in J}$-progressively measurable set in $J \times \Omega$ and choose $t \in J$, then

$$F^c \cap (J_t \times \Omega) = \{(s,\omega) \in J_t \times \Omega \,|\, (s,\omega) \notin F\}.$$

Thus, $F^c \cap (J_t \times \Omega)$ is the complement of $F \cap (J_t \times \Omega)$ in $J_t \times \Omega$, which implies that $F^c \cap (J_t \times \Omega) \in \mathscr{B}(J_t) \otimes \mathscr{F}_t$. So, F^c is another $(\mathscr{F}_t)_{t\in J}$-progressively measurable set in $J \times \Omega$. If $(F_n)_{n\in\mathbb{N}}$ is a sequence of $(\mathscr{F}_t)_{t\in J}$-progressively measurable sets in $J \times \Omega$, then its union $F := \bigcup_{n\in\mathbb{N}} F_n$ satisfies $F \cap (J_t \times \Omega) = \bigcup_{n\in\mathbb{N}}(F_n \cap (J_t \times \Omega))$. For this reason, F is $(\mathscr{F}_t)_{t\in J}$-progressively measurable. This clarifies the first claim.

Now, let $X : J \times \Omega \to S$ be a process and $B \in \mathscr{S}$. Then $X^{-1}(B) \cap (J_t \times \Omega)$ agrees with $\{(s,\omega) \in J_t \times \Omega \,|\, X_s(\omega) \in B\}$ for all $t \in J$. Consequently, $X^{-1}(B)$ is $(\mathscr{F}_t)_{t\in J}$-progressively measurable if and only if $\{(s,\omega) \in J_t \times \Omega \,|\, X_s(\omega) \in B\}$ belongs to $\mathscr{B}(J_t) \otimes \mathscr{F}_t$ for each $t \in J$. The lemma is established. $\square$

The σ-field of all $(\mathscr{F}_t)_{t\in J}$-progressively measurable sets in $J \times \Omega$ is called the *$(\mathscr{F}_t)_{t\in J}$-progressive σ-field.* Of course, if $\mathscr{F}_t = \mathscr{F}$ for every $t \in J$, then it reduces to the product σ-field $\mathscr{B}(J) \otimes \mathscr{F}$. At this place, we recall partitions, which are used to approximate right-continuous processes and stopping times.

A.37 Definition. Let $H, I \subset \mathbb{R}$ be two non-degenerate intervals with $H \subset I$ and $(I_n)_{n\in\mathbb{N}}$ be an increasing sequence of non-degenerate intervals in $\mathbb{R}$.

(i) A *partition* of I is a countable set $\mathbb{T} \subset I$ so that $\inf I \in \mathbb{T}$, if $\inf I \in I$, and $\sup I \in \mathbb{T}$, if $\sup I \in I$. Moreover, if I is compact, then $\mathbb{T}$ is required to be finite.

(ii) Let $\mathbb{T}$ be a partition of I. The *successor* of a point $t \in \mathbb{T}$ with respect to $\mathbb{T}$ is defined by $t' := \min\{u \in \mathbb{T} \,|\, u > t\}$, if $t < \sup I$, and $t' := t$, if $t = \sup I$. In addition, $|\mathbb{T}| := \sup_{t\in\mathbb{T}}(t' - t)$ is called the *mesh* of $\mathbb{T}$.

(iii) Let $\mathbb{S}$ and $\mathbb{T}$ be two partitions of H and I, respectively. We say that $\mathbb{T}$ *refines* $\mathbb{S}$ if $\mathbb{S} \subset \mathbb{T}$.

(iv) A *refining sequence of partitions* of $(I_n)_{n\in\mathbb{N}}$ is a sequence $(\mathbb{T}_n)_{n\in\mathbb{N}}$, where $\mathbb{T}_n$ is a partition of I_n for all $n \in \mathbb{N}$, such that $\mathbb{T}_{n+1}$ refines $\mathbb{T}_n$ for each $n \in \mathbb{N}$ and $\lim_{n\uparrow\infty} |\mathbb{T}_n| = 0$. If $I = I_n$ for all $n \in \mathbb{N}$, then we will speak about a refining sequence of partitions of I.

We now relate $(\mathscr{F}_t)_{t\in J}$-adapted and $(\mathscr{F}_t)_{t\in J}$-progressively measurable processes, which partially extends Proposition 1.13 in [25].

A.38 Proposition. *Assume that ρ is some metric on S for which $\mathscr{S}$ is the Borel σ-field of S with respect to ρ. Then every $(\mathscr{F}_t)_{t\in J}$-adapted right-continuous process $X : J \times \Omega \to S$ is $(\mathscr{F}_t)_{t\in J}$-progressively measurable.*

Proof. Let $t \in J$ and $(\mathbb{T}_n)_{n\in\mathbb{N}}$ be a refining sequence of partitions of J_t. Then for all $r \in J$ with $r < t$ and each $n \in \mathbb{N}$ there is a unique $s \in \mathbb{T}_n$ such that $r \in [s, s')$. So, for each $n \in \mathbb{N}$ we may define an $\mathscr{B}(J_t) \otimes \mathscr{F}_t$-measurable process $X^{(n)} : J_t \times \Omega \to S$ by $X_r^{(n)}(\omega) := X_{s'}(\omega)$, if $r < t$ and with $s \in \mathbb{T}_n$ satisfying $r \in [s, s')$, and $X_r^{(n)}(\omega) := X_t(\omega)$, if $r = t$. The $\mathscr{B}(J_t) \otimes \mathscr{F}_t$-measurability of $X^{(n)}$ is easily checked. Let $B \in \mathscr{S}$, then

$$\{(r,\omega) \in J_t \times \Omega \,|\, X_r^{(n)}(\omega) \in B\} = \left(\bigcup_{s\in\mathbb{T}_n} [s, s') \times \{X_{s'} \in B\}\right) \cup (\{t\} \times \{X_t \in B\}),$$

as J_t is the union of $\bigcup_{s\in\mathbb{T}_n}[s, s')$ and $\{t\}$. From $[s, s') \in \mathscr{B}(J_t)$ and $\{X_{s'} \in B\} \in \mathscr{F}_t$ we conclude that $[s, s') \times \{X_{s'} \in B\} \in \mathscr{B}(J_t) \otimes \mathscr{F}_t$ for all $s \in \mathbb{T}_n$. Since $\{t\} \times \{X_t \in B\}$ also belongs to $\mathscr{B}(J_t) \otimes \mathscr{F}_t$ and $\mathbb{T}_n$ is countable, the $\mathscr{B}(J_t) \otimes \mathscr{F}_t$-measurability of $X^{(n)}$ is clarified.

We now show that $(X^{(n)})_{n\in\mathbb{N}}$ converges pointwise to the restriction of X to $J_t \times \Omega$, which then completes the proof, by Lemma A.18. Let $(r, \omega) \in J_t \times \Omega$ and $\varepsilon > 0$. If $r = t$, then $X_t^{(n)}(\omega) = X_t(\omega)$ for all $n \in \mathbb{N}$. Otherwise, the right-continuity of $X(\omega)$ gives $\delta > 0$ with

$$\rho(X_s(\omega), X_r(\omega)) < \varepsilon$$

for all $s \in [r, r + \delta) \cap J$. We choose $n_0 \in \mathbb{N}$ such that $|\mathbb{T}_n| < \delta$ for all $n \in \mathbb{N}$ with $n \geq n_0$. Then $\rho(X_r^{(n)}(\omega), X_r(\omega)) < \varepsilon$ for every such $n \in \mathbb{N}$, since $X_r^{(n)}(\omega) = X_{s'}(\omega)$ and $0 < s' - r \leq |\mathbb{T}_n| < \delta$ with $s \in \mathbb{T}_n$ fulfilling $r \in [s, s')$. $\square$

Similarly, we define a set F in $J \times \Omega$ to be $(\mathscr{F}_t')_{t\in J}$-progressively measurable if the indicator function $\mathbb{1}_F$ shares this property. By proceeding as in Lemma A.36, we readily see that the system of all $(\mathscr{F}_t')_{t\in J}$-progressively measurable sets in $J \times \Omega$ is a σ-field and we call it the $(\mathscr{F}_t')_{t\in J}$-progressive σ-field. As before, if $\mathscr{F}_t' = \mathscr{F}$ for all $t \in J$, then it is simply the product σ-field $\mathscr{B}(J) \otimes \mathscr{F}$. Furthermore, a process $X : J \times \Omega \to S$ is $(\mathscr{F}_t')_{t\in J}$-progressively measurable if and only if it is measurable with respect to this σ-field.

Let us also introduce the *reconstructible σ-field* as the σ-field generated by the system of all sets F in $J \times \Omega$ of the form $F = J_u \times A'$ for some $u \in J$ and some $A' \in \mathscr{F}_u'$. Correspondingly, a process $X : J \times \Omega \to S$ is called *reconstructible* if it is measurable with respect to the reconstructible σ-field.

A.39 Lemma. *The $(\mathscr{F}_t')_{t\in J}$-progressive σ-field includes the reconstructible σ-field. In particular, every reconstructible process $X : J \times \Omega \to S$ is $(\mathscr{F}_t')_{t\in J}$-progressively measurable.*

Proof. For the first claim, it suffices to show that $J_u \times A'$ is $(\mathscr{F}_t')_{t\in J}$-progressively measurable for all $u \in J$ and each $A' \in \mathscr{F}_u'$. We readily see that $(J_u \times A') \cap (J_t' \times \Omega) = [t, u] \times A'$, if $t \leq u$, and $(J_u \times A') \cap (J_t' \times \Omega) = \emptyset$, if $t > u$, for all $t \in J$. Hence, $(J_u \times A') \cap (J_t' \times \Omega) \in \mathscr{B}(J_t') \otimes \mathscr{F}_t'$ for all $t \in J$, as desired.

The check the second claim, let $X : J \times \Omega \to S$ be a reconstructible process. By what we have shown, X must be measurable with respect to the $(\mathscr{F}_t')_{t\in J}$-progressive σ-field. For this reason, the preceding discussion concludes the proof. $\square$

We also relate $(\mathscr{F}_t')_{t\in J}$-adapted and reconstructible processes.

A.40 Proposition. *Suppose that ρ is some metric on S for which $\mathscr{S}$ is the Borel σ-field of S with respect to ρ. Then every $(\mathscr{F}_t')_{t\in J}$-adapted right-continuous process $X : J \times \Omega \to S$ is reconstructible and $(\mathscr{F}_t')_{t\in J}$-progressively measurable.*

Proof. Let $(\mathbb{T}_n)_{n\in\mathbb{N}}$ be a refining sequence of partitions of J. We set $T := \sup J$, then for all $t \in J$ with $t < T$ and each $n \in \mathbb{N}$ there is a unique $u \in \mathbb{T}_n$ such that $t \in [u, u')$. Thus, for each $n \in \mathbb{N}$ we may define a reconstructible process $X^{(n)} : J \times \Omega \to S$ by $X_t^{(n)}(\omega) := X_{u'}(\omega)$, if $t < T$ and with $u \in \mathbb{T}_n$ fulfilling $t \in [u, u')$, and $X_t^{(n)}(\omega) := X_T(\omega)$, if $t = T$. To justify that $X^{(n)}$ is reconstructible, we let $B \in \mathscr{S}$. For $T \notin J$ it holds that

$$\{(t, \omega) \in J \times \Omega \,|\, X_t^{(n)}(\omega) \in B\} = \bigcup_{u\in\mathbb{T}_n} [u, u') \times \{X_{u'} \in B\},$$

because $J = \bigcup_{u\in\mathbb{T}_n} [u, u')$. Suppose instead that $T \in J$, then

$$\{(t, \omega) \in J \times \Omega \,|\, X_t^{(n)}(\omega) \in B\} = \left(\bigcup_{u\in\mathbb{T}_n} [u, u') \times \{X_{u'} \in B\}\right) \cup (\{T\} \times \{X_T \in B\}).$$

Since $\{X_{u'} \in B\} \in \mathscr{F}_u'$, the set $[u, u') \times \{X_{u'} \in B\}$ belongs to the reconstructible σ-field for each $u \in \mathbb{T}_n$. If $T \in J$, then $\{T\} \times \{X_T \in B\}$ is also a member of the reconstructible σ-field. Hence, as $\mathbb{T}_n$ is countable, $X^{(n)}$ is reconstructible.

Finally, we verify that $(X^{(n)})_{n\in\mathbb{N}}$ converges pointwise to X, which then establishes that X is reconstructible, by Lemma A.18. Once this is shown, Lemma A.39 completes the proof. Let $(t,\omega)\in J\times\Omega$ and $\varepsilon>0$. If $t=T$, then $X_T^{(n)}(\omega)=X_T(\omega)$ for all $n\in\mathbb{N}$. Otherwise, the right-continuity of $X(\omega)$ gives $\delta>0$ such that

$$\rho(X_u(\omega),X_t(\omega))<\varepsilon$$

for all $u\in[t,t+\delta)\cap J$. We choose $n_0\in\mathbb{N}$ satisfying $|\mathbb{T}_n|<\delta$ for all $n\in\mathbb{N}$ with $n\geq n_0$. Then $\rho(X_t^{(n)}(\omega),X_t(\omega))<\varepsilon$ for every such $n\in\mathbb{N}$, since $X_t^{(n)}(\omega)=X_{u'}(\omega)$ and $0<u'-t\leq|\mathbb{T}_n|<\delta$ with $u\in\mathbb{T}_n$ such that $t\in[u,u')$. □

Let es return to the filtration $(\mathscr{F}_t)_{t\in J}$ of $\mathscr{F}$. We recall that an $(\mathscr{F}_t)_{t\in J}$-*optional time* is a function $\tau:\Omega\to J\cup\{\infty\}$ such that $\{\tau<t\}\in\mathscr{F}_t$ for all $t\in J$. If in addition $\tau(\Omega)\subset J$, then τ is called *finite*. An $(\mathscr{F}_t)_{t\in J}$-*stopping time* is a function $\tau:\Omega\to J\cup\{\infty\}$ such that $\{\tau\leq t\}\in\mathscr{F}_t$ for all $t\in J$.

Let τ be an $(\mathscr{F}_t)_{t\in J}$-stopping time. Since $\{\tau<t\}=\bigcup_{n\in\mathbb{N}}\{\tau\leq t-1/n\}\in\mathscr{F}_t$ for all $t\in J$, it is clear that τ must be an $(\mathscr{F}_t)_{t\in J}$-optional time. Moreover, with τ we associate the system $\mathscr{F}_\tau:=\{A\in\mathscr{F}\,|\,A\cap\{\tau\leq t\}\in\mathscr{F}_t$ for all $t\in J$, which constitutes a sub-σ-field of $\mathscr{F}$. Of course, if $\tau=t$ for some $t\in J$, then $\mathscr{F}_\tau=\mathscr{F}_t$. Let $X:J\times\Omega\to S$ be a process, then the *stopped process* of X with respect to τ is given by

$$X^\tau:J\times\Omega\to S,\quad X_t^\tau(\omega):=X_{t\wedge\tau(\omega)}(\omega).$$

Suppose temporarily that τ is finite, then the map $X_\tau:\Omega\to S$ is defined via $X_\tau(\omega):=X_{\tau(\omega)}(\omega)$. We notice that if X is $\mathscr{B}(J)\otimes\mathscr{F}$-measurable, then X_τ is $\mathscr{F}$-measurable. Indeed, let $\Phi:\Omega\to J\times\Omega$ be given by $\Phi(\omega):=(\tau(\omega),\omega)$, then

$$X_\tau=X\circ\Phi\quad\text{and}\quad\{X_\tau\in B\}=\{\Phi\in X^{-1}(B)\}\in\mathscr{F}\quad\text{for all } B\in\mathscr{S},$$

because $X^{-1}(B)\in\mathscr{B}(J)\otimes\mathscr{F}$ and $\Phi^{-1}(J_t\times A)=\{\tau\leq t\}\cap A\in\mathscr{F}$ for each $t\in J$ and every $A\in\mathscr{F}$. We verify two basic facts on these concepts for stopping times (cf. Exercise 1.5.12 in [33]).

A.41 Lemma. *Let $X:J\times\Omega\to S$ be an $(\mathscr{F}_t)_{t\in J}$-progressively measurable process and τ be an $(\mathscr{F}_t)_{t\in J}$-stopping time. Then the following two assertions hold:*

(i) The stopped process X^τ is $(\mathscr{F}_t)_{t\in J}$-progressively measurable.

(ii) If τ is finite, then the map X_τ is $\mathscr{F}_\tau$-measurable.

Proof. (i) Let $\Phi:J\times\Omega\to J\times\Omega$ be defined by $\Phi(t,\omega):=(t\wedge\tau(\omega),\omega)$. Then $X^\tau=X\circ\Phi$ and it holds that

$$\begin{aligned}\{(r,\omega)\in J_t\times\Omega\,|\,X_r^\tau(\omega)\in B\}&=\{(r,\omega)\in J_t\times\Omega\,|\,\Phi(r,\omega)\in X^{-1}(B)\}\\&=\{(r,\omega)\in J_t\times\Omega\,|\,\Phi(r,\omega)\in X^{-1}(B)\cap(J_t\times\Omega)\}\end{aligned}$$

for all $B\in\mathscr{S}$ and each $t\in J$. Hence, if we can show that

$$\{(r,\omega)\in J_t\times\Omega\,|\,\Phi(r,\omega)\in C\}\in\mathscr{B}(J_t)\otimes\mathscr{F}_t$$

for each $C \in \mathscr{B}(J_t) \otimes \mathscr{F}_t$, then the claim follows. According to the Monotone Class Theorem A.28, we may assume that C is of the form $C = J_s \times A$ for some $s \in J_t$ and some $A \in \mathscr{F}_t$. In this case, we conclude that

$$\begin{aligned}\{(r,\omega) \in J_t \times \Omega \,|\, \Phi(r,\omega) \in C\} &= \{(r,\omega) \in J_t \times A \,|\, r \wedge \tau(\omega) \leq s\}\\ &= (J_s \times A) \cup ((s,t] \times (A \cap \{\tau \leq s\})) \in \mathscr{B}(J_t) \otimes \mathscr{F}_t.\end{aligned}$$

(ii) We observe that, since X^τ is $(\mathscr{F}_t)_{t\in J}$-progressively measurable, it must be $\mathscr{B}(J) \otimes \mathscr{F}$-measurable and $(\mathscr{F}_t)_{t\in J}$-adapted. In consequence, X_τ is $\mathscr{F}$-measurable and we obtain that $\{X_\tau \in B\} \cap \{\tau \leq t\} = \{X_t^\tau \in B\} \cap \{\tau \leq t\} \in \mathscr{F}_t$ for all $t \in J$ and each $B \in \mathscr{S}$. This shows the lemma. □

We now provide a standard result on hitting times, which generalizes Problems 2.6 and 2.7 in [25]. Note that the proof of the fourth assertion is mainly concluded from the solution in this book. For convenience, we set $r_0 := \inf J$ and $T := \sup J$.

A.42 Proposition. *Suppose that $r_0 \in J$ and ρ is a metric on S for which $\mathscr{S}$ is the Borel σ-field of S with respect to ρ. Let $X : J \times \Omega \to S$ be an $(\mathscr{F}_t)_{t\in J}$-progressively measurable process, $B \in \mathscr{S}$, and $\tau := \inf\{t \in J \,|\, X_t \in B\}$. Then the following four assertions hold:*

(i) $X_s \notin B$ on $\{\tau > s\}$ for all $s \in J$, and $X_\tau \in B$ on $\{\tau = T\}$ provided $T \in J$.

(ii) Let X be right-continuous, then $X_\tau \in \overline{B}$ on $\{\tau < \infty\}$. Assume in addition that X is left-continuous, then $X_\tau \in \partial B$ on $\{r_0 < \tau < \infty\}$.

(iii) If B is open and X is right-continuous, then τ is an $(\mathscr{F}_t)_{t\in J}$-optional time.

(iv) If B is closed and X is continuous, then τ is an $(\mathscr{F}_t)_{t\in J}$-stopping time.

Proof. (i) If we had $X_s(\omega) \in B$ for some $(s,\omega) \in J \times \Omega$ with $\tau(\omega) > s$, then we would get that $s \geq \inf\{t \in J \,|\, X_t(\omega) \in B\} = \tau(\omega)$, a contradiction. Let $T \in J$ and suppose that $\omega \in \{\tau = T\}$ fulfills $X_\tau(\omega) \notin B$. Then, by what we have just shown, $\{t \in J \,|\, X_t(\omega) \in B\} = \emptyset$. Hence, $\tau(\omega) = \infty$, which is impossible.

(ii) If $T \in J$ and $\omega \in \{\tau = T\}$, then (i) already gives $X_\tau(\omega) \in B$. Thus, let us suppose that $\omega \in \{\tau < T\}$ satisfies $X_\tau(\omega) \notin \overline{B}$. Then, as $(\overline{B})^c$ is open, there is $\varepsilon > 0$ with $B_\varepsilon(X_\tau(\omega)) \subset (\overline{B})^c$. By the right-continuity of $X(\omega)$, there is $\delta > 0$ such that

$$X_s(\omega) \in B_\varepsilon(X_\tau(\omega)) \quad \text{for all } s \in [\tau(\omega), \tau(\omega) + \delta) \cap J.$$

However, the definition of $\tau(\omega)$ yields $s \in [\tau(\omega), \tau(\omega) + \delta) \cap J$ with $X_s(\omega) \in B$, a contradiction. Now, let X be left-continuous. We suppose that $\omega \in \{r_0 < \tau < \infty\}$ fulfills $X_\tau(\omega) \notin \partial B$. Since $X_\tau(\omega) \in (\overline{B})^c$ cannot occur, we must have $X_\tau(\omega) \in B^\circ$. As B° is open, there is $\varepsilon > 0$ with $B_\varepsilon(X_\tau(\omega)) \subset B^\circ$. Left-continuity of $X(\omega)$ gives $\delta \in (0, \tau(\omega) - r_0)$ such that

$$X_s(\omega) \in B_\varepsilon(X_\tau(\omega)) \quad \text{for each } s \in (\tau(\omega) - \delta, \tau(\omega)].$$

Thus, we obtain the contradiction that $\tau(\omega) - \delta/2 \in \{t \in J \,|\, X_t(\omega) \in B\}$ while $\tau(\omega) - \delta/2 < \tau(\omega) = \inf\{t \in J \,|\, X_t(\omega) \in B\}$.

(iii) Let $t \in J$, then for each $\omega \in \{\tau < t\}$ there exists $s \in J$ with $X_s(\omega) \in B$ and $\tau(\omega) \leq s < t$. If instead $\omega \in \{X_s \in B\}$ for some $s \in J$ with $s < t$, then $s \geq \inf\{t \in J \,|\, X_t(\omega) \in B\} = \tau(\omega)$. Hence, $\{\tau < t\} = \bigcup_{s\in J:\, s<t}\{X_s \in B\}$. If we can show that

$$\bigcap_{s\in J: s<t} \{X_s \in B^c\} = \bigcap_{s\in J\cap\mathbb{Q}:\, s<t} \{X_s \in B^c\},$$

then, as the rational numbers $\mathbb{Q}$ are countable, we get that $\{\tau \geq t\} \in \mathscr{F}_t$, which proves the claim. Let $\omega \in \bigcap_{s\in J\cap\mathbb{Q}:\, s<t}\{X_s \in B^c\}$ and $s \in J$ with $s < t$. Since $[s,t)\cap\mathbb{Q}$ is dense in $[s,t)$, there is a sequence $(s_n)_{n\in\mathbb{N}}$ in $[s,t)\cap\mathbb{Q}$ such that $\lim_{n\uparrow\infty} s_n = s$. As $X_{s_n}(\omega) \in B^c$ for all $n \in \mathbb{N}$ and B^c is closed, we conclude from the right-continuity of $X(\omega)$ that $X_s(\omega) = \lim_{n\uparrow\infty} X_{s_n}(\omega) \in B^c$.

(iv) We set $B_n := \{x \in S \,|\, \mathrm{dist}(x,B) < 1/n\}$ and $\tau_n := \inf\{t \in J \,|\, X_t \in B_n\}$ for all $n \in \mathbb{N}$, where we use the notation $\mathrm{dist}(x,C) = \inf_{y\in C}\rho(x,y)$ for all $x \in S$ and each $C \subset S$. Then B_n is the $(1/n)$-neighborhood of B, as introduced in Section A.3, and for this reason, it is open. Hence, (iii) implies that τ_n is an $(\mathscr{F}_t)_{t\in J}$-optional time. From $B \subset B_{n+1} \subset B_n$ we infer that

$$\tau_n \leq \tau_{n+1} \leq \tau \quad \text{for all } n \in \mathbb{N}.$$

As B is closed, $B = \bigcap_{n\in\mathbb{N}} B_n$. In fact, if $x \in S$ satisfies $\mathrm{dist}(x,B) < 1/n$ for all $n \in \mathbb{N}$, then $\mathrm{dist}(x,B) = 0$, which is equivalent to $x \in B$, by Lemma A.13. The next step of the proof is to show the following two conditions:

(a) $\tau_n = r_0$ for each $n \in \mathbb{N}$ on $\{\tau = r_0\}$ and $\tau_n > r_0$ for almost all $n \in \mathbb{N}$ on $\{\tau > r_0\}$.

(b) $r_0 < \tau_n < \tau_{n+1} < \tau$ for almost each $n \in \mathbb{N}$ on $\{r_0 < \tau < \infty\}$.

Clearly, from $\tau_n \leq \tau$ we obtain that $\tau_n = r_0$ on $\{\tau = r_0\}$ for every $n \in \mathbb{N}$. We let $\omega \in \{\tau > r_0\}$ and suppose that $\tau_n(\omega) = r_0$ for infinitely many $n \in \mathbb{N}$. Then there exists a strictly increasing sequence $(\nu_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ and a sequence $(t_n)_{n\in\mathbb{N}}$ in J with $\mathrm{dist}(X_{t_n}(\omega), B) < 1/\nu_n$ for each $n \in \mathbb{N}$ and $\lim_{n\uparrow\infty} t_n = r_0$. Lemma A.13 and the right-continuity of $X(\omega)$ yield that

$$\mathrm{dist}(X_{r_0}(\omega), B) = \lim_{n\uparrow\infty} \mathrm{dist}(X_{t_n}(\omega), B) = 0.$$

Hence, $X_{r_0}(\omega) \in B$, which gives the contradiction $\tau(\omega) = r_0$. This verifies (a). Next, let $\omega \in \{r_0 < \tau < \infty\}$ and assume that $r_0 < \tau_n(\omega) = \tau_{n+1}(\omega)$ for some $n \in \mathbb{N}$. Then (ii) implies that $X_{\tau_n}(\omega) \in \partial B_n \cap \partial B_{n+1}$, which is impossible, as $\partial B_n \cap \partial B_{n+1} = \emptyset$. Simlarly, if $\tau_n(\omega) = \tau(\omega)$ for some $n \in \mathbb{N}$, then $X_{\tau_n}(\omega) \in \partial B_n \cap \partial B$, which is another contradiction, since $\partial B_n \cap \partial B = \emptyset$. So, (b) holds.

We now check that $\sigma := \sup_{n\in\mathbb{N}} \tau_n$ agrees with τ on $\{\tau < \infty\}$. Since $\sigma \leq \tau$, we merely have to prove that $\sigma \geq \tau$ on $\{\sigma < \infty\} \cap \{r_0 < \tau < \infty\}$. Let us choose $\omega \in \{\sigma < \infty\} \cap \{r_0 < \tau < \infty\}$. From (b) and (ii) we get $n_0 \in \mathbb{N}$ such

$\operatorname{dist}(X_{\tau_n}(\omega), B) = 1/n$ for all $n \in \mathbb{N}$ with $n \geq n_0$, since Lemma A.13 entails that $\partial B_n = \{x \in S \,|\, \operatorname{dist}(x, B) = 1/n\}$. Hence, left-continuity of $X(\omega)$ implies that

$$\operatorname{dist}(X_\sigma(\omega), B) = \lim_{n\uparrow\infty} \operatorname{dist}(X_{\tau_n}(\omega), B) = 0,$$

which shows that $X_\sigma(\omega) \in B$. Thus, $\sigma(\omega) \geq \inf\{t \in J \,|\, X_t(\omega) \in B\} = \tau(\omega)$. So, $\sup_{n\in\mathbb{N}} \tau_n = \tau$ on $\{\tau < \infty\}$. Finally, (b) entails that $\{\tau \leq t\} = \bigcap_{n\in\mathbb{N}}\{\tau_n < t\} \in \mathscr{F}_t$ for all $t \in J$ with $t > r_0$. Since $\{\tau = r_0\} = \{X_{r_0} \in B\} \in \mathscr{F}_{r_0}$, the proposition is established. $\square$

We are interested in a certain construction of stopping times taking finitely many values. As before, we use partitions and let $r_0 = \inf J$.

A.43 Lemma. *Let $r_0 \in J$ and τ be an $(\mathscr{F}_t)_{t\in J}$-stopping time. Then for each $t \in J$ and every partition $\mathbb{T}$ of $[r_0, t]$, the function $\tau_{t,\mathbb{T}} : \Omega \to (r_0, t]$ defined by*

$$\tau_{t,\mathbb{T}}(\omega) := \sum_{s\in\mathbb{T}} s' \mathbb{1}_{\{s\leq\tau<s'\}}(\omega) + t\mathbb{1}_{\{\tau\geq t\}}(\omega)$$

is an $(\mathscr{F}_t)_{t\in J}$-stopping time that takes finitely many values such that $0 < \tau_{t,\mathbb{T}} - \tau \leq |\mathbb{T}|$ on $\{\tau < t\}$ and $\tau_{t,\mathbb{T}} = \tau$ on $\{\tau = t\}$.

Proof. From $\tau_{t,\mathbb{T}}(\Omega) = \mathbb{T}\backslash\{r_0\}$ we see that $\tau_{t,\mathbb{T}}(\Omega)$ is finite. For $s \in \mathbb{T}$ with $s' < t$ it holds that $\{\tau_{t,\mathbb{T}} = s'\} = \{s \leq \tau < s'\} = \{\tau < s'\} \cap \{\tau < s\}^c \in \mathscr{F}_{s'}$. In addition,

$$\{\tau_{t,\mathbb{T}} = t\} = \{s \leq \tau < t\} \cup \{\tau \geq t\} = \{\tau \geq s\} \in \mathscr{F}_t$$

with the unique $s \in \mathbb{T}\backslash\{t\}$ satisfying $s' = t$. Consequently, $\tau_{t,\mathbb{T}}$ is an $(\mathscr{F})_{t\in J}$-stopping time. We choose $\omega \in \{\tau \leq t\}$. If $\tau(\omega) = t$, then $\tau_{t,\mathbb{T}}(\omega) = t$. Otherwise, there is a unique $s \in \mathbb{T}$ with $s \leq \tau(\omega) < s'$. In this case, $\tau_{t,\mathbb{T}}(\omega) = s'$ and $\tau_{t,\mathbb{T}}(\omega) - \tau(\omega) \leq |\mathbb{T}|$. This verifies the lemma. $\square$

We conclude with the pointwise approximation of a stopping time by a decreasing sequence of stopping times that take only finitely many values.

A.44 Proposition. *Let $r_0 \in J$ and τ be an $(\mathscr{F}_t)_{t\in J}$-stopping time τ. Then there is a decreasing sequence $(\tau_n)_{n\in\mathbb{N}}$ of $(\mathscr{F}_t)_{t\in J}$-stopping times, each taking only finitely many values, that converges pointwise to τ. Moreover, if $\tau \leq t$ for some $t \in J$, then $(\tau_n)_{n\in\mathbb{N}}$ can be chosen such that the convergence is uniform and* $\sup_{n\in\mathbb{N}} \tau_n \leq t$.

Proof. First, let $r, t \in J$ with $r \leq t$, $\mathbb{S}$ be a partition of $[r_0, r]$, and $\mathbb{T}$ be a partition of $[r_0, t]$ that refines $\mathbb{S}$. Using the notation of the preceding lemma, we seek to show that

$$\tau_{r,\mathbb{S}} \geq \tau_{t,\mathbb{T}} \quad \text{on } \{\tau < r\}. \tag{A.5}$$

Let $\omega \in \{\tau < r\}$, then there is a unique $q \in \mathbb{S}$ such that $q \leq \tau(\omega) < q'$. Here, q' denotes the successor of q with respect to $\mathbb{S}$, that is, $q' = \min\{s \in \mathbb{S} \,|\, s > q\}$. Since $\mathbb{T}$ refines $\mathbb{S}$, there are $n \in \mathbb{N}$ and $s_1, \ldots, s_n \in \mathbb{T}$ with $n \geq 2$ and $q = s_1 < \cdots < s_n = q'$.

We choose the unique $i \in \{1, \ldots, n-1\}$ with $s_i \leq \tau(\omega) < s_{i+1}$, then we draw the conclusion that $\tau_{r,\mathbb{S}}(\omega) = q' \geq s_{i+1} = \tau_{t,\mathbb{T}}(\omega)$. Thus, (A.5) holds.

Now we can justify the first assertion. Let initially $T \in J$, then $J = [r_0, T]$ and $\{\tau < \infty\} = \{\tau \leq T\}$. We choose a refining sequence $(\mathbb{T}_n)_{n\in\mathbb{N}}$ of partitions of $[r_0, T]$ and for each $n \in \mathbb{N}$ we define $\tau_n : \Omega \to [r_0, T] \cup \{\infty\}$ by

$$\tau_n(\omega) := \tau_{T,\mathbb{T}_n}(\omega) \quad \text{for } \omega \in \{\tau < \infty\} \quad \text{and} \quad \tau_n(\omega) := \infty \quad \text{for } \omega \in \{\tau = \infty\}.$$

Then τ_n is an $(\mathscr{F}_t)_{t\in[r_0,T]}$-stopping time taking finitely many values such that $\tau_n \geq \tau$, as $\tau_{T,\mathbb{T}_n} \geq \tau$ on $\{\tau < \infty\}$ and $\{\tau_n \leq t\} = \{\tau_{T,\mathbb{T}_n} \leq t\} \cap \{\tau \leq t\}$ for all $t \in [r_0, T]$. From (A.5) and the fact that $\tau_{T,\mathbb{T}_n} = T$ on $\{\tau = T\}$ for all $n \in \mathbb{N}$ we infer that $(\tau_n)_{n\in\mathbb{N}}$ is decreasing. If $\omega \in \{\tau = \infty\}$, then $\tau_n(\omega) = \tau(\omega)$ for each $n \in \mathbb{N}$. Suppose instead that $\omega \in \{\tau < \infty\}$, then $0 \leq \tau_n(\omega) - \tau(\omega) \leq |\mathbb{T}_n|$ for all $n \in \mathbb{N}$, which gives $\lim_{n\uparrow\infty} \tau_n(\omega) = \tau(\omega)$, as claimed.

Now assume that $T \notin J$, then $J = [r_0, T)$. Let $(t_n)_{n\in\mathbb{N}}$ be an increasing sequence in $[r_0, T)$ such that $\lim_{n\uparrow\infty} t_n = T$ and $(\mathbb{T}_n)_{n\in\mathbb{N}}$ be a refining sequence of partitions of $([r_0, t_n])_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$ we let $\tau_n : \Omega \to [r_0, T) \cup \{\infty\}$ be defined by

$$\tau_n(\omega) := \tau_{t_n,\mathbb{T}_n}(\omega) \quad \text{for } \omega \in \{\tau < t_n\} \quad \text{and} \quad \tau_n(\omega) := \infty \quad \text{for } \omega \in \{\tau \geq t_n\}.$$

Then once again τ_n is an $(\mathscr{F}_t)_{t\in[r_0,T)}$-stopping time taking finitely many values such that $\tau_n \geq \tau$, since $\tau_{t_n,\mathbb{T}_n} > \tau$ on $\{\tau < t_n\}$ and $\{\tau_n \leq t\} = \{\tau_{t_n,\mathbb{T}_n} \leq t\} \cap \{\tau < t\}$ for all $t \in [r_0, t_n]$. Next, we pick $n \in \mathbb{N}$. For $\omega \in \{\tau < t_n\}$ we infer from (A.5) and $\{\tau < t_n\} \subset \{\tau < t_{n+1}\}$ that

$$\tau_n(\omega) = \tau_{t_n,\mathbb{T}_n}(\omega) \geq \tau_{t_{n+1},\mathbb{T}_{n+1}}(\omega) = \tau_{n+1}(\omega).$$

For $\omega \in \{\tau \geq t_n\}$ we have that $\tau_n(\omega) = \infty \geq \tau_{n+1}(\omega)$. Thus, $(\tau_n)_{n\in\mathbb{N}}$ is decreasing. Suppose that $\omega \in \{\tau < T\}$, then $\tau_n(\omega) - \tau(\omega) = \tau_{t_n,\mathbb{T}_n}(\omega) - \tau(\omega) \leq |\mathbb{T}_n|$ for almost all $n \in \mathbb{N}$. Hence, $\lim_{n\uparrow\infty} \tau_n(\omega) = \tau(\omega)$. Since $\{\tau \geq T\} = \{\tau = \infty\}$, we get that $\tau_n(\omega) = \tau(\omega) = \infty$ on $\{\tau \geq T\}$ for every $n \in \mathbb{N}$. This completes the verification of the first claim.

To prove the second assertion, let $t \in J$ be such that $\tau(\omega) \leq t$ for all $\omega \in \Omega$. We choose a refining sequence $(\mathbb{T}_n)_{n\in\mathbb{N}}$ of partitions of $[r_0, t]$ and set $\tau_n := \tau_{t,\mathbb{T}_n}$ for each $n \in \mathbb{N}$. Then $(\tau_n)_{n\in\mathbb{N}}$ is decreasing and $\tau_n(\Omega) \subset [r_0, t]$ as well as $0 \leq \tau_n - \tau \leq |\mathbb{T}_n|$ for all $n \in \mathbb{N}$, since $\{\tau \leq t\} = \Omega$. This concludes the proof. $\square$

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List of Symbols

Here, the main symbols used in the thesis are described. We choose $d, k \in \mathbb{N}$, a metric space D, and a pseudometric space S. Moreover, F is a non-empty set in some Banach space E, and $(\Omega, \mathscr{F})$, $(\Omega', \mathscr{F}')$ are two measurable spaces.

Sets and elements

$\mathbb{N}$	set of all natural numbers
$\mathbb{Q}$	field of all rational numbers
$\mathbb{1}_A$	indicator function of a set $A \in \mathscr{F}$
$\mathbb{R}$	real line
$\mathbb{R}_+$	semiring of all non-negative real numbers
$\mathbb{C}$	complex plane
$\mathbb{R}^d$	d-dimensional Euclidean space
$\mathbb{R}^{k \times d}$	linear space of all real $k \times d$ matrices
$\mathbb{I}_d$	identity matrix in $\mathbb{R}^{d \times d}$
$\mathbb{S}^d$	linear space of all symmetric matrices in $\mathbb{R}^{d \times d}$
$\mathbb{S}^d_+$	linear space of all positive definite matrices in $\mathbb{S}^d$
$B_\delta(x)$	open ball in S with center $x \in S$ and radius $\delta > 0$

Number operations

$x \wedge y$	minimum of two real numbers x and y
$x \vee y$	maximum of x and y
x^+	positive part of x
x^-	negative part of x
$\|x\|$	absolute value of x

Vector and matrix operations

$\langle \overline{x}, \overline{y} \rangle$	dot product of two vectors $\overline{x}$ and $\overline{y}$ in $\mathbb{R}^d$
$\lvert\overline{x}\rvert$	Euclidean norm of $\overline{x}$
A^t	transpose of a matrix $A \in \mathbb{R}^{d\times k}$
$\mathrm{tr}(B)$	trace of a matrix $B \in \mathbb{R}^{d\times d}$
$\lvert B\rvert$	Frobenius norm of B
$\sigma(B)$	set of all complex eigenvalues of B
B^{-1}	inverse of B provided B is invertible

Map spaces

$B(S,D)$	set of all D-valued Borel measurable maps on S and $B(S) := B(S,\mathbb{R})$
$B_b(S,F)$	set of all bounded maps in $B(S,F)$ and $B_b(S) := B(S,\mathbb{R})$
$C(S,D)$	set of all D-valued continuous maps on S and $C(S) := C(S,\mathbb{R})$
$C_b(S,F)$	set of all bounded maps in $C(S,F)$ and $C_b(S) := C_b(S,\mathbb{R})$

System of sets operations

$\sigma(\mathscr{C})$	σ-field generated by a system of sets $\mathscr{C}$ in Ω
$\sigma(X)$	σ-field generated by a $\mathscr{F}$-$\mathscr{F}'$-measurable map $X : \Omega \to \Omega'$
$\mathscr{B}(S)$	Borel σ-field of S

Topological set operations

D°	interior of D
∂D	boundary of D
$\overline{D}$	closure of D
$\mathrm{diam}(D)$	diameter of D